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HOMEWORK N° 7

The 7th Assignment of Robot Modeling and Control

7-3

Prove the parallel axis theorem.

Solution:

The moment of inertia tensor about a point O is defined as:

$$\mathbf{I}_O = \int_{\mathcal{B}} \left(\|\mathbf{r}\|^2 \mathbf{I} - \mathbf{r}\mathbf{r}^T \right) dm$$

where $\mathbf{r}$ is the position vector of an infinitesimal mass element dm relative to O, and $\mathcal{B}$ represents the rigid body.

Let C be the center of mass, and let $\mathbf{r}_C$ and $\mathbf{r}_P$ be the position vectors of dm relative to C and P, respectively:

$$\mathbf{r}_P = \mathbf{r}_C + \mathbf{d}$$

where $\mathbf{d}$ is the displacement from C to P.

The inertia tensor about P is:

$$\mathbf{I}_P = \int_{\mathcal{B}} \left(\|\mathbf{r}_P\|^2 \mathbf{I} - \mathbf{r}_P \mathbf{r}_P^T \right) dm.$$

Substituting $\mathbf{r}_P = \mathbf{r}_C + \mathbf{d}$:

$$\mathbf{I}_P = \int_{\mathcal{B}} \left(\|\mathbf{r}_C + \mathbf{d}\|^2 \mathbf{I} - (\mathbf{r}_C + \mathbf{d})(\mathbf{r}_C + \mathbf{d})^T \right) dm.$$

Expanding the squared term:

$$\|\mathbf{r}_C + \mathbf{d}\|^2 = \|\mathbf{r}_C\|^2 + 2\mathbf{d}^T \mathbf{r}_C + \|\mathbf{d}\|^2.$$

Expanding the outer product:

$$(\mathbf{r}_C + \mathbf{d})(\mathbf{r}_C + \mathbf{d})^T = \mathbf{r}_C\mathbf{r}_C^T + \mathbf{r}_C\mathbf{d}^T + \mathbf{d}\mathbf{r}_C^T + \mathbf{d}\mathbf{d}^T.$$

Thus,

$$\mathbf{I}_P = \int_{\mathcal{B}} \left((\|\mathbf{r}_C\|^2 + 2\mathbf{d}^T\mathbf{r}_C + \|\mathbf{d}\|^2)\mathbf{I} - \mathbf{r}_C\mathbf{r}_C^T - \mathbf{r}_C\mathbf{d}^T - \mathbf{d}\mathbf{r}_C^T - \mathbf{d}\mathbf{d}^T\right) dm.$$

Rearranging:

$$\mathbf{I}_P = \int_{\mathcal{B}} \left(\|\mathbf{r}_C\|^2\mathbf{I} - \mathbf{r}_C\mathbf{r}_C^T\right) dm + \int_{\mathcal{B}} \left(\|\mathbf{d}\|^2\mathbf{I} - \mathbf{d}\mathbf{d}^T\right) dm + \int_{\mathcal{B}} \left(2\mathbf{d}^T\mathbf{r}_C\mathbf{I} - \mathbf{r}_C\mathbf{d}^T - \mathbf{d}\mathbf{r}_C^T\right) dm.$$

The first integral is the moment of inertia tensor about the center of mass:

$$\mathbf{I}_C = \int_{\mathcal{B}} \left(\|\mathbf{r}_C\|^2\mathbf{I} - \mathbf{r}_C\mathbf{r}_C^T\right) dm.$$

The second integral simplifies because $\|\mathbf{d}\|^2\mathbf{I} - \mathbf{d}\mathbf{d}^T$ is independent of $\mathbf{r}_C$:

$$M\left(\|\mathbf{d}\|^2\mathbf{I} - \mathbf{d}\mathbf{d}^T\right).$$

The third integral vanishes because C is the center of mass:

$$\int_{\mathcal{B}} \mathbf{r}_C dm = \mathbf{0}.$$

Thus, we obtain:

$$\mathbf{I}_P = \mathbf{I}_C + M\left(\|\mathbf{d}\|^2\mathbf{I} - \mathbf{d}\mathbf{d}^T\right).$$

7-4

Formulate the dynamical equation of a **RP** robot arm by iterating the Newton-Euler equations.

Solution:

This is dumb.

PS: I have deliberately chosen a non-DH coordinate to define the robot arm. For $\mathbf{P}_1^0 \neq constant$, the first iteration on $\dot{v}_1^1$ and the last torque propelling equation is a little bit tricky, not using the chain formula provided by the textbook.

$$\mathbf{R}_0^1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Assuming the gravitation field exists,

$$\omega_0^0 = \mathbf{0}, \dot{\omega}_0^0 = \mathbf{0}, \dot{v}_0^0 = \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix}$$

$$\omega_1^1 = \mathbf{R}_0^1 \omega_0^0 + \dot{\theta}_1 \hat{\mathbf{Z}}_1^1 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix}, \dot{\omega}_1^1 = \mathbf{R}_0^1 \dot{\omega}_0^0 + \mathbf{R}_0^1 \omega_0^0 \times \dot{\theta}_1 \hat{\mathbf{Z}}_1^1 + \ddot{\theta}_1 \hat{\mathbf{Z}}_1^1 = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix}$$

$$\dot{v}_1^1 = \mathbf{R}_0^1 [\dot{\mathbf{V}}_1^0 + 2\omega_0^0 \times \mathbf{V}_1^0 + \dot{\omega}_0^0 \times \mathbf{P}_1^0 + \omega_0^0 \times (\omega_0^0 \times \mathbf{P}_1^0) + \dot{v}_0^0] = \begin{pmatrix} g \sin\theta_1 \\ l_1 \ddot{\theta}_1 + g\cos\theta_1 \\ 0 \end{pmatrix}$$

$$\dot{v}_{C_1}^1 = \dot{\omega}_1^1 \times \mathbf{P}_{C_1}^1 + \omega_1^1 \times (\omega_1^1 \times \mathbf{P}_{C_1}^1) + \dot{v}_1^1 = \begin{pmatrix} g \sin\theta_1 \\ l_1 \ddot{\theta}_1 + g\cos\theta_1 \\ 0 \end{pmatrix}$$

$$\omega_2^2 = \mathbf{R}_1^2 \omega_1^1 + \dot{\theta}_2 \hat{\mathbf{Z}}_2^2 = \begin{pmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{pmatrix}, \dot{\omega}_2^2 = \mathbf{R}_1^2 \dot{\omega}_1^1 = \begin{pmatrix} 0 \\ -\ddot{\theta}_1 \\ 0 \end{pmatrix}$$

$$\dot{v}_2^2 = \mathbf{R}_1^2 [\dot{\omega}_1^1 \times \mathbf{P}_2^1 + \omega_1^1 \times (\omega_1^1 \times \mathbf{P}_2^1) + \dot{v}_1^1] + 2\omega_2^2 \times \dot{d}_2 \hat{\mathbf{Z}}_2^2 + \ddot{d}_2 \hat{\mathbf{Z}}_2^2 = \begin{pmatrix} -\ddot{\theta}_1 d_2 - 2\dot{\theta}_1 \dot{d}_2 + g\sin\theta_1 \\ 0 \\ l_1 \ddot{\theta}_1 - \dot{\theta}_1^2 d_2 + \ddot{d}_2 + g\cos\theta_1 \end{pmatrix}$$

$$\dot{v}_{C_2}^2 = \dot{\omega}_2^2 \times \mathbf{P}_{C_2}^2 + \omega_2^2 \times (\omega_2^2 \times \mathbf{P}_{C_2}^2) + \dot{v}_2^2 = \begin{pmatrix} -\ddot{\theta}_1 d_2 - 2\dot{\theta}_1 \dot{d}_2 + g\sin\theta_1 \\ 0 \\ l_1 \ddot{\theta}_1 - \dot{\theta}_1^2 d_2 + \ddot{d}_2 + g\cos\theta_1 \end{pmatrix}$$

Therefore,

$$F_2^2 = -m_2 \dot{v}_{C_2}^2 = \begin{pmatrix} -m_2(-\ddot{\theta}_1 d_2 - 2\dot{\theta}_1 \dot{d}_2 + g\sin\theta_1) \\ 0 \\ -m_2(l_1 \ddot{\theta}_1 - \dot{\theta}_1^2 d_2 + \ddot{d}_2 + g\cos\theta_1) \end{pmatrix}$$

$$N_2^2 = I_2^{C_2} \dot{\omega}_2^2 + \omega_2^2 \times I_2^{C_2} \omega_2^2 = \mathbf{0}$$

$$F_1^1 = -m_1 \dot{v}_{C_1}^1 = \begin{pmatrix} -m_1 g \sin\theta_1 \\ -m_1(l_1 \ddot{\theta}_1 + g\cos\theta_1) \\ 0 \end{pmatrix}$$

$$N_1^1 = -I_1^{C_1} \dot{\omega}_1^1 - \omega_1^1 \times I_1^{C_1} \omega_1^1 = \mathbf{0}$$

Therefore,

$$f_2^2 = -F_2^2 = \begin{pmatrix} m_2(-\ddot{\theta}_1 d_2 - 2\dot{\theta}_1 \dot{d}_2 + g\sin\theta_1) \\ 0 \\ m_2(l_1 \ddot{\theta}_1 - \dot{\theta}_1^2 d_2 + \ddot{d}_2 + g\cos\theta_1) \end{pmatrix}$$

$$n_2^2 = \mathbf{P}_2^2 \times f_2^2 = \begin{pmatrix} 0 \\ m_2 d_2(-\ddot{\theta}_1 d_2 - 2\dot{\theta}_1 \dot{d}_2 + g\sin\theta_1) \\ 0 \end{pmatrix}$$

$$f_1^1 = \mathbf{R}_2^1 f_2^2 - F_1^1 = \begin{pmatrix} m_2(-\ddot{\theta}_1 d_2 - 2\dot{\theta}_1 \dot{d}_2 + g\sin\theta_1) + m_1 g \sin\theta_1 \\ m_2(l_1\ddot{\theta}_1 - \dot{\theta}_1^2 d_2 + \ddot{d}_2 + g\cos\theta_1) + m_1(l_1\ddot{\theta}_1 + g\cos\theta_1) \\ 0 \end{pmatrix}$$

$$n_1^1 = \mathbf{R}_2^1 n_2^2 + \mathbf{P}_1^1 \times f_1^1 = \begin{pmatrix} 0 \\ 0 \\ m_2(l_1\ddot{\theta}_1 - \dot{\theta}_1^2 d_2 + \ddot{d}_2 + g\cos\theta_1) + m_1(l_1\ddot{\theta}_1 + g\cos\theta_1) - m_2 d_2(-\ddot{\theta}_1 d_2 - 2\dot{\theta}_1 \dot{d}_2 + g\sin\theta_1) \end{pmatrix}$$

Thus, we obtain its dynamic equation.

$$\tau_1 = {n_1^1}^{\mathrm{T}} \hat{\mathbf{Z}}_1^1 = (m_1+m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 \ddot{d}_2 + 2 m_2 d_2 \dot{\theta}_1 \dot{d}_2 + m_2 \ddot{\theta}_1 d_2^2 + (m_1+m_2) g l_1 \cos\theta_1 - m_2 g d_2 \sin\theta_1$$

$$\tau_2 = {f_2^2}^{\mathrm{T}} \hat{\mathbf{Z}}_2^2 = m_2(l_1\ddot{\theta}_1 - \dot{\theta}_1^2 d_2 + \ddot{d}_2 + g\cos\theta_1)$$

7-5

Prove that the rotational energy of a rigid body in 7-25 is $E_{rot} = \frac{1}{2}\omega^{c\mathrm{T}} I^c \omega^c$

Solution:

Evident.

The component is:

$$\frac{1}{2}\int_{V_{l_i}} r_i^T S^T(\omega_i) S(\omega_i) r_i \rho dV = \frac{1}{2}\omega_i^T \left(\int_{V_{l_i}} S^T(r_i) S(r_i) \rho dV \right) \omega_i$$

where the property $S(\omega_i) r_i = -S(r_i)\omega_i$ is used. Due to this property, since the matrix operator $S(\cdot)$ is

$$S(r_i) = \begin{bmatrix} 0 & -r_{iz} & r_{iy} \\ r_{iz} & 0 & -r_{ix} \\ -r_{iy} & r_{ix} & 0 \end{bmatrix}$$

Thus, we have:

$$\frac{1}{2}\int_{V_{l_i}} r_i^T S^T(\omega_i) S(\omega_i) r_i \rho dV = \frac{1}{2}\omega_i^T I_{l_i} \omega_i$$

For any coordinate denoted, it must obtain a certain transformation, for instance, its tensor of moment of inertia,

$$I_{l_i}^c = \mathbf{R}^c I_{l_i} \mathbf{R}^{c\mathrm{T}}$$

Substituting it back would one obtain the desired equation.

$$E_{rot} = \frac{1}{2}\omega_i^T I_{l_i} \omega_i = \frac{1}{2}\omega_i^T \mathbf{R}^c I_{l_i} \mathbf{R}^{c\mathrm{T}} \omega_i = \frac{1}{2}\omega^{c\mathrm{T}} I^c \omega^c$$

7-7

Check the answer in 7-4 by Lagrangian mechanics.

Solution:

Its Lagrangian components is,

$$T = \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \frac{1}{2}m_2[(\dot{\theta}_1 l_1 + \dot{d}_2)^2 + (\dot{\theta}_1 d_2)^2]$$

$$V = m_1 g l_1 \sin\theta_1 + m_2 g(l_1 \sin\theta_1 + d_2 \cos\theta_1)$$

$$L = T - V$$

Therefore, under the condition of Lagrangian equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \tau$$

Would one obtain,

$$\tau_1 = (m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 \ddot{d}_2 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 + m_2 \ddot{\theta}_1 d_2^2 + (m_1 + m_2) g l_1 \cos\theta_1 - m_2 g d_2 \sin\theta_1$$

$$\tau_2 = m_2 l_1 \ddot{\theta}_1 + m_2 \ddot{d}_2 - m_2 \dot{\theta}_1^2 d_2 + m_2 g \cos\theta_1$$