HOMEWORK N° 6

Chapter 3

A24

Judge whether X and Y are independent.

Solution:

Not independent. For instance,

$$P(X=0,Y=0)=0.1\neq P(X=0)P(Y=0)=0.12$$

A25

Solve a,b,c

Solution:

$$0.4\times(0.15+a)=0.1\rightarrow a=0.1$$

$$0.4\times(0.3+b)=0.2\rightarrow b=0.2$$

$$0.4\times(0.15+c)=0.1\rightarrow c=0.1$$

A27

Distribution of the coordinate created when darting a cyclic-shaped region.

Solution:

For the dart position is uniformly distributed, the PDF of which could be described using following equations,

$$f(x,y) = \begin{cases} \frac{2}{\pi} & ,(x,y) \in \mathcal{D} \\ 0 & ,others \end{cases}$$

Therefore,

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \begin{cases} \frac{4\sqrt{1-x^2}}{\pi} & ,x \in (0,1) \\ 0 & ,x_{others} \end{cases}$$

and

$$P(X < \frac{1}{2}) = \int_0^{\frac{1}{2}} f_X(x)dx = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}$$

For the shape is symmetrical, it is blatantly evident that the $f_Y(y)$ must follow suit,

$$f_Y(y) = \begin{cases} \frac{\sqrt{1-y^2}}{\pi} & ,y \in (-1,1) \\ 0 & ,x_{others} \end{cases}$$

Obviously,

$$f(x,y) \neq f_X(x) \cdot f_Y(y)$$

Thus, X and Y is not independent.

A28

(X,Y) marginal distribution in two-dimensional normal distribution.

Solution:

A very direct response would emerge.

$$X \sim N(0,2), Y \sim N(1,4)$$

Therefore, their PDFs yield,

$$f_X(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}}, x \in \mathcal{R}$$

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(y-1)^2}{8}}, y \in \mathcal{R}$$

And since $\rho = 0$, it is safe to say that X is not correlated with Y.

A30

$X \sim B(1, 0.4), Y \sim B(2, 0.4), Z = X + Y.$

Solution:

The sum of two independent binomial distributions with same p obtains such rule,

$$P(Z=k) = \sum_{i=0}^{k} [\mathbf{C}_n^i p^i (1-p)^{n-i}][\mathbf{C}_m^{k-i} p^{k-i} (1-p)^{m-i+k}] = p^k (1-p)^{m+n-k} \sum_{i=0}^{k} \mathbf{C}_n^i \mathbf{C}_m^{k-i}$$

And, according to Vandermonde's identity,

$$\sum_{i=0}^{k} \mathbf{C}_n^i \mathbf{C}_m^{k-i} = \mathbf{C}_{m+n}^k$$

Therefore, for all $Z = 0, 1, ...m+n$,

$$P(Z=k) = \mathbf{C}_{m+n}^k p^k (1-p)^{m+n-k}$$

It is the equivalent of saying,

$$Z \sim B(m+n, p) \to Z \sim B(3, 0.4)$$

✓

A32

Solve $Z = X + Y, M = max(X, Y), N = min(X, Y)$ respectively.

Solution:

1. For $Z = X + Y$,

$$P(Z=1) = 0.2 \times 0.2 = 0.04, P(Z=2) = 0.2 \times 0.4 + 0.3 \times 0.2 = 0.14$$

$$P(Z=3) = 0.08 + 0.12 + 0.10 = 0.3, P(Z=4) = 0.12 + 0.2 = 0.32$$

$$P(Z=5) = 0.2$$

✓

2. For $M = max(X, Y)$,

$$P(M=1) = 0.5 \times 0.2 = 0.1, P(M=2) = 0.5 \times 0.6 + 0.4 \times 0.5 = 0.5$$

$$P(M=3) = 0.4$$

3. For $N = min(X, Y)$,

✓

$$P(N=0) = 0.2, P(N=1) = 0.3 + 0.1 = 0.4, P(N=2) = 0.4$$

✓

A35

$X \sim N(0,1), Y \sim N(0,1)$, independent.

Solution:

For the convolution property of a normal distribution, $Z = X + Y$ yields,

$$Z = N(0, 1+1) = N(0,2)$$

Therefore,

$$f_Z(z) = \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}}, z \in \mathcal{R} \quad \checkmark$$

The same goes on $M = max(X,Y), N = min(X,Y)$, each having its unique property.

$$P(N < 1) = F_N(1) = 1 - (1 - \Phi(1))^2 = 0.975$$

$$P(M < 1) = F_M(1) = \Phi(1)^2 = 0.708 \quad \checkmark$$

A36

$f(x,y)$ known, solve $Z = 2X - Y$.

Solution:

Given:
Joint PDF of (X,Y):

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}, & 0 < x < 2,\ 0 < y < 2x \\ 0, & \text{otherwise} \end{cases}$$

Define new variable:

$$Z = 2X - Y$$

Let $W = X$, then:

$$X = W, \quad Y = 2W - Z$$

Jacobian determinant:

$$J = \left| \frac{\partial(x,y)}{\partial(z,w)} \right| = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 1$$

Support of $f_Z(z)$:
From:

$$0 < w < 2, \quad 0 < y = 2w - z < 2w \Rightarrow 0 < z < 2w$$

Compute marginal PDF of Z**:**

$$f_Z(z) = \int_{z/2}^{2} f_{X,Y}(w, 2w - z) \cdot |J| \, dw = \int_{z/2}^{2} \frac{1}{4} \, dw = \frac{1}{4}(2 - z/2) = \frac{1}{2} - \frac{z}{8}$$

Final result:

$$f_Z(z) = \begin{cases} \frac{1}{2} - \frac{z}{8}, & 0 < z < 4 \\ 0, & \text{otherwise} \end{cases}$$

A38

$X \sim U(0,1)$ and $f_Y(y)$ are independent and known.

Solution:

The CDFs are,

$$F_X(x) = \begin{cases} x & , x \geq 0 \\ 0 & , otherwise \end{cases}$$

$$F_Y(y) = \begin{cases} y^2 & , y \geq 0 \\ 0 & , otherwise \end{cases}$$

Therefore,

$$F_M(m) = F_X(m)F_Y(m) = m^3, m \geq 0, F_N(n) = 1-(1-F_X(n))(1-F_Y(n)) = -n^3+n^2+n, n \geq 0$$

With the final PDFs.

$$f_M(m) = \begin{cases} 3m^2 & , m \in (0,1) \\ 0 & , otherwise \end{cases}, f_N(n) = \begin{cases} -3n^2+2n+1 & , n \in (0,1) \\ 0 & , otherwise \end{cases}$$

B5

This was due to a error in the assignment of last week. One shall find B6 at the assignment before.

Solution:

$$f(x,y) = f_{Y|X}(y|x)f_X(x) = \begin{cases} \frac{2(4-y)}{(3-x)^2} & , 1 < x < 2, x+1 < y < 4 \\ 0 & , otherwise \end{cases}$$

$$P(Y<3) = \int_2^3 \int_1^{y-1} \frac{2(4-y)}{(3-x)^2} dx = \int_2^3 2(4-y)(\frac{1}{4-y} - \frac{1}{2})dx = \int_2^3 (y-2)dx = \frac{1}{2}$$

From the deduction aforementioned,

$$f_Y(y) = \begin{cases} y-2 & ,y \in (2,3) \\ 4-y & ,y \in (3,4) \\ 0 & ,otherwise \end{cases}$$

And the probability that X does not exceed 1.5 hours when Y equals 3 is,

$$P(X < 1.5|Y=3) = F_{X|Y}(1.5|3) = \int_1^{1.5} f_{X|Y}(x|3)dx = \int_1^{1.5} \frac{f(x,y)}{f_Y(y)}\Big|_{y=3} dx$$
$$= \int_1^{1.5} \frac{2}{(3-x)^2}dx = \frac{1}{3}$$

B7

$X_i \sim Exp(\lambda)$.

Solution:

T yields,

$$T = min(max(X_1, X_2), X_3)$$

Therefore,

$$F_T(t) = 1-(1-F_3(t))(1-F_1(t)F_2(t)) = 1-e^{-\lambda t}(2e^{-\lambda t} - e^{-2\lambda t}) = 1-2e^{-2\lambda t} + e^{-3\lambda t}, t > 0$$

And,

$$f_T(t) = \begin{cases} 4\lambda e^{-2\lambda t} - 3\lambda e^{-3\lambda t} & ,t > 0 \\ 0 & ,otherwise \end{cases}$$

B9

$X \sim U(-a,a), Y \sim N(\mu, \sigma^2)$, solve $Z = X+Y$.

Solution:

$$f_X(x) = \begin{cases} \frac{1}{2a} & ,x \in (-a,a) \\ 0 & ,otherwise \end{cases}$$

Therefore,

$$f_Z(z) = \int_{-a}^{+a} f_X(x) f_Y(z-x)dx = \frac{1}{2a}[\Phi(\frac{z+a-\mu}{\sigma}) - \Phi(\frac{z-a-\mu}{\sigma})], z \in \mathcal{R}$$

B10

Solve $Z = X + Y$.

Solution:

Set a transformation where $W = X$, $Z = X + Y$. In such a case,

$$0 < w < 1, w < z < 2 + w$$

Therefore,

$$f_Z(z) = \int_{-\infty}^{+\infty} f(w, z-w)dw = \int_{-\infty}^{+\infty} \frac{3-z}{3} dw$$

It comes at four stages where the upper bound of integral and lower bound is different.

$$f_Z(z) = \begin{cases} \frac{z(3-z)}{3} & , z \in (0,1) \\ \frac{3-z}{3} & , z \in (1,2) \\ \frac{(3-z)^2}{3} & , z \in (2,3) \\ 0 & , otherwise \end{cases}$$

B12

$X_i \sim Pois(\lambda)$

Solution:

According to PGF, the sum of Poisson distributions is still a Poisson distribution.

$$S = \sum_{i=1}^{10} X_i \sim Pois(10\lambda)$$

Therefore,

$$P(S \geq 2) = 1 - P(S = 0) - P(S = 1) = 1 - e^{-10\lambda} - 10\lambda e^{-10\lambda}$$

The same goes on

$$M = \max_{1 \leq i \leq 10}(X_i)$$

$$P(M \geq 2) = 1 - P(M < 2) = 1 - P(X_i < 2)^{10} = 1 - e^{-10\lambda}(1+\lambda)^{10}$$