Homework N° 6

Chapter 3

A7

Conditional distribution.

Solution:

1.
$$P(Y = 1|X = 1) = P(Y = 2|X = 1) = 0.5$$ ✓

2.
$$P(X = 0|Y = 1) = \frac{1}{3}, P(X = 1|Y = 1) = \frac{2}{3}$$ ✓

A8

Joint distribution and conditional distribution.

Solution:

X	Y		
	0	1	2
0	0.2	0.1	0.1
1	0	0.4	0.2

✓

$$P(Y = 0|X = 0) = 0.5, P(Y = 1|X = 0) = 0.25, P(Y = 2|X = 0) = 0.25$$

✓ ✓ ✓

A10

Joint distribution of a broken machine and its lost.

Solution:

X	Y		
	0	a	2a
0	0.6	0	0
1	$0.3(1-p)$	$0.3p$	0
2	$0.1(1-p)^2$	$0.2p(1-p)$	$0.1p^2$

✓

$$P(Y=0|X=1)=1-p, P(Y=a|X=1)=p$$

✓

A12

Joint distribution.

Solution:

X	Y	
	0	1
1	0.1	0.2
2	0.3	0.4

✓

$$P(X=1|Y=0)=0.25, P(X=2|Y=0)=0.75$$

Hence,

$$F_{X|Y}(x|0)=\begin{cases} 0 & ,x<1 \\ 0.25 & ,1\leq x<2 \\ 1 & ,x\geq 2 \end{cases}$$

✓

A13

Joint, marginal and conditional distribution.

Solution:

X	Y	
	0	1
0	0.35	0.35
1	0.25	0.05

Hence,

$$F_X(x) = \begin{cases} 0 & , x < 0 \\ 0.7 & , 0 \leq x < 1 \\ 1 & , x \geq 1 \end{cases}$$

And also,

$$F_{Y|X}(y|1) = \begin{cases} 0 & , y < 0 \\ \frac{5}{6} & , 0 \leq y < 1 \\ 1 & , y \geq 1 \end{cases}$$

A14

Solve several problems with joint PDF given.

Solution:

Normalization condition,

$$\int_0^1 \int_0^1 (c + xy)\, dx\, dy = 1.$$

With,

$$\begin{aligned} \int_0^1 (c + xy) dx &= \int_0^1 c\, dx + \int_0^1 xy\, dx \\ &= c \int_0^1 dx + y \int_0^1 x dx \\ &= c(1-0) + y \cdot \frac{1}{2}(1^2 - 0^2) = c + \frac{y}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^1 \left(c + \frac{y}{2}\right) dy &= c \int_0^1 dy + \frac{1}{2} \int_0^1 y dy \\ &= c(1-0) + \frac{1}{2} \cdot \frac{1}{2}(1^2 - 0^2) \\ &= c + \frac{1}{4} = 1 \end{aligned}$$

$$c = \frac{3}{4}$$

$$
\begin{aligned}
P(X \leq 0.5, Y \leq 0.5) &= \int_0^{0.5} \int_0^{0.5} f(x, y) dx dy \\
&= \int_0^{0.5} \int_0^{0.5} \left(\frac{3}{4} + xy \right) dx dy \\
&= \int_0^{0.5} (\frac{3}{8} + \frac{1}{8} y) dy \\
&= \frac{13}{64}
\end{aligned}
$$

✓

$$
\begin{aligned}
P(X + Y \leq 1) &= \int_0^1 \int_0^{1-x} \left(\frac{3}{4} + xy \right) dy dx \\
&= \int_0^1 (\frac{3}{4}(1-x) + x \int_0^{1-x} y dy) dx \\
&= \int_0^1 (\frac{3}{4} - \frac{3}{4}x + \frac{x}{2} - x^2 + \frac{x^3}{2}) dx \\
&= (\frac{3}{4}x - \frac{3}{8}x^2 + \frac{1}{4}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4)\Big|_0^1 \\
&= \frac{5}{12}
\end{aligned}
$$

✓

$$
\begin{aligned}
P(X > 0.5) &= \int_{0.5}^1 \int_0^1 f(x, y) dy dx \\
&= \int_{0.5}^1 \int_0^1 \left(\frac{3}{4} + xy \right) dy dx \\
&= \int_{0.5}^1 \left[\frac{3}{4} \int_0^1 dy + x \int_0^1 y dy \right] dx \\
&= \int_{0.5}^1 \left(\frac{3}{4} \times 1 + x \times \frac{1}{2} \right) dx \\
&= \int_{0.5}^1 \left(\frac{3}{4} + \frac{x}{2} \right) dx \\
&= (\frac{3}{4}x + \frac{1}{4}x^2)\Big|_{0.5}^1 \\
&= \frac{9}{16}
\end{aligned}
$$

✓

A16

Deduct marginal distribution from joint distribution.

Solution:

$$\begin{aligned}\int_1^2\int_x^{4-x} c(x-1)dydx &= \int_1^2 c(x-1)(4-2x)dx \\ &= \int_1^2 c(-2x^2+6x-4)dx \\ &= \frac{1}{3}c = 1\end{aligned}$$

$$c = 3$$

$$f_X(x) = \begin{cases} \int_x^{4-x} 3(x-1)dy = 6(x-1)(2-x), 1<x<2 \\ 0, x_{others} \end{cases}$$

$$f_Y(y) = \begin{cases} 3\int_1^{4-y}(x-1)dx = 3\times(\frac{1}{2}x^2-x)\Big|_1^{4-y} = \frac{3}{2}(y-3)^2 & ,2<y<3 \\ 3\int_1^{4-y}(x-1)dx = \frac{3}{2}(y-1)^2 & ,1<y<2 \\ 0 & ,y_{others} \end{cases}$$

A17

Rinse and repeat.

Solution:

$$f_X(x) = \begin{cases} \int_0^x e^{-x}dy = xe^{-x} & ,x>0 \\ 0 & ,x<0 \end{cases}$$

$$f_Y(y) = \begin{cases} \int_y^{+\infty} e^{-x}dx = e^{-y} & ,y>0 \\ 0 & ,y<0 \end{cases}$$

Thus, when $x > 0$,

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & ,0<y<x \\ 0 & ,y_{other} \end{cases}$$

So yes, it's marginal distribution is uniform.

A19

Solve joint distribution with marginal PDF and conditional PDF given.

Solution:

$$f(x,y) = f_X(x) f_{Y|X}(y|x) = \begin{cases} \lambda^2 e^{-\lambda x - \frac{y}{x}} & , x > 0, y > 0 \\ 0 & , (x,y)_{others} \end{cases}$$

$$F_{Y|X}(y|x) = \int_{-\infty}^{y} \frac{1}{x} e^{-\frac{y}{x}} dy = \begin{cases} 1 - e^{-\frac{y}{x}} & , y \geq 0 \\ 0 & , y < 0 \end{cases}$$

$$P(Y > 1 | X = 1) = 1 - F_{Y|X}(1|1) = e^{-1}$$

A21

Uniform distribution in triangular region.

Solution:

Set,

$$f(x,y) = c, 0 < x < 2, 0 < y < x$$

The distribution must satisfy the normalization condition.

$$\int_0^2 \int_0^x c dy dx = 2c = 1 \rightarrow c = \frac{1}{2}$$

Thus,

$$f(x,y) = \begin{cases} \frac{1}{2} & , 0 < x < 2, 0 < y < x \\ 0 & , (x,y)_{others} \end{cases}$$

$$P(X + Y > 2) = \int_1^2 \int_{2-x}^{x} \frac{1}{2} dy dx = 0.5$$

$$P(X < 1) = \int_0^1 \int_0^x \frac{1}{2} dy dx = 0.25$$

B4

Joint distribution given.

Solution:

$$\int_0^1 \int_x^1 c(y - x) dy dx = \frac{1}{6} c = 1 \rightarrow c = 6$$

$$P(X+Y \le 1) = \int_0^{0.5} \int_x^{1-x} 6(y-x)dydx = \int_0^{0.5} 12(x-0.5)^2 dx = \frac{1}{2}$$ ✓

$$P(X < 0.5) = \int_0^{0.5} \int_x^{1} 6(y-x)dydx = \frac{7}{8}$$

✓

B6

Road problem.

Solution:

One could get,

$$Z = |X - Y|$$

As well as,

$$f(x,y) = \frac{1}{m^2}, 0 < x < m, 0 < y < m$$

Thus, it is easy to deduce that,

$$F_Z(z) = \int_S f(x,y)dS = \frac{(2m-z)z}{m^2}, 0 < z < m$$

Which implies its PDF,

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{2(m-z)}{m^2} & , 0 < z < m \\ 0 & , z_{others} \end{cases}$$ ✓