HOMEWORK N° 14

Chapter 8

A2

The quality standard for television picture tubes specifies an average lifespan of 15,000 hours. A television manufacturer claims that the average lifespan of its picture tubes significantly exceeds this standard. To verify this claim, a random sample of 100 picture tubes produced by the manufacturer was tested, yielding an average lifespan of 15,525 hours. Assume that the lifespan X of the picture tubes produced by the manufacturer follows a normal distribution $N(\mu, 1500^2)$. Use hypothesis testing to determine whether there is sufficient evidence to conclude that the manufacturer's picture tubes have a significantly higher average lifespan than the specified standard (significance level $\alpha = 0.05$).

(1) State the null hypothesis, alternative hypothesis, test statistic, and rejection region. Then, make a decision based on the sample data.

(2) Calculate the P-value and make an inference. Is the conclusion consistent with the result from part (1)?

Solution:

1. Null Hypothesis (H_0): The average lifespan of the picture tubes is equal to the standard. $H_0 : \mu = 15,000$ hours.

2. Alternative Hypothesis (H_1): The average lifespan of the picture tubes is significantly higher than the standard. $H_1 : \mu > 15,000$ hours.

Since the population standard deviation is known ($\sigma = 1,500$) and the sample size is large ($n = 100$), we use the z-test. The test statistic is:

$$z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{15,525 - 15,000}{1,500/\sqrt{100}} = \frac{525}{150} = 3.5$$

For a one-tailed test at $\alpha = 0.05$, the critical z-value is $z_{0.05} = 1.645$. Reject H_0 if $z > 1.645$.

Since $z = 3.5 > 1.645$, we reject the null hypothesis. There is sufficient evidence at the 0.05 significance level to conclude that the manufacturer's picture tubes have a significantly higher average lifespan than the standard.

The P-value is the probability of observing a test statistic as extreme as or more extreme than the one observed, assuming H_0 is true. For $z = 3.5$, the P-value is:

$$P = P(Z > 3.5) = 1 - \Phi(3.5) \approx 0.0002$$

(Here, Φ is the cumulative distribution function of the standard normal distribution.)

Since $P = 0.0002 < \alpha = 0.05$, we reject H_0. This conclusion is consistent with the result from part (1).

Both methods lead to the rejection of the null hypothesis, providing strong evidence that the manufacturer's picture tubes have a significantly higher average lifespan than the specified standard of 15,000 hours.

A3

A food factory uses an automatic canning machine to fill cans with food. The standard weight per can is 500 g. To check whether the machine is working properly, random sampling is conducted periodically. A sample of 10 cans is randomly selected, yielding an average weight of 498 g and a standard deviation of 6.5 g. Assuming the weight of the cans follows a normal distribution, use hypothesis testing to determine whether the machine is functioning correctly (significance level $\alpha = 0.02$).

(1) State the null hypothesis, alternative hypothesis, test statistic, and rejection region. Then, make a decision based on the sample data.

(2) Calculate the P-value and make an inference. Is the conclusion consistent with the result from part (1)?

Solution:

1. Null Hypothesis (H_0): The machine is working normally, and the average weight of the cans is equal to the standard. $H_0 : \mu = 500$ g.
2. Alternative Hypothesis (H_1): The machine is not working normally, and the average weight of the cans is different from the standard. $H_1 : \mu \neq 500$ g.

Since the population standard deviation is unknown and the sample size is small ($n = 10$), we use the t-test. The test statistic is:

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{498 - 500}{6.5/\sqrt{10}} = \frac{-2}{2.055} \approx -0.973$$

Here, $s = 6.5$ g is the sample standard deviation, and $n = 10$ is the sample size.

For a two-tailed test at $\alpha = 0.02$ and degrees of freedom $df = n - 1 = 9$, the critical t-values are $t_{0.01,9} \approx \pm 2.821$. Reject H_0 if $|t| > 2.821$.

Since $|t| = 0.973 < 2.821$, we fail to reject the null hypothesis. There is insufficient evidence at the 0.02 significance level to conclude that the machine is not working normally.

The P-value is the probability of observing a test statistic as extreme as or more extreme than the one observed, assuming H_0 is true. For $t = -0.973$ with $df = 9$, the P-value for the two-tailed test is:

$$P = 2 \cdot P(T < -0.973) \approx 2 \cdot 0.178 = 0.356$$

(Here, T follows the t-distribution with 9 degrees of freedom.)

Since $P = 0.356 > \alpha = 0.02$, we fail to reject H_0. This conclusion is consistent with the result from part (1).

Both methods lead to the same conclusion: there is no significant evidence to suggest that the machine is not working normally at the 0.02 significance level. The observed deviation in the average weight could be due to random sampling variability.

✓

A4

The administration of a university conducted a survey to understand the online food delivery spending habits of its students. They randomly sampled 100 students and recorded their spending on food delivery last month. The sample mean was 478 yuan with a standard deviation of 85 yuan. Assuming the students' spending follows a normal distribution $N(\mu, \sigma^2)$, test the following hypotheses at a significance level of $\alpha = 0.05$:

$$H_0 : \mu \leq 450, \quad H_1 : \mu > 450.$$

Provide the test statistic, calculate the P-value, and draw a conclusion.

Solution:

1. Null Hypothesis (H_0): The average spending on food delivery is less than or equal to 450 yuan. $H_0 : \mu \leq 450$.

2. Alternative Hypothesis (H_1): The average spending on food delivery is greater than 450 yuan. $H_1 : \mu > 450$.

Since the sample size is large ($n = 100$) and the population standard deviation is unknown, we use the t-test (approximated by the sample standard deviation). The test statistic is calculated as:

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{478 - 450}{85/\sqrt{100}} = \frac{28}{8.5} \approx 3.294$$

Here, $\bar{X} = 478$ is the sample mean, $\mu_0 = 450$ is the hypothesized population mean, $s = 85$ is the sample standard deviation, and $n = 100$ is the sample size.

The P-value is the probability of observing a test statistic as extreme as or more extreme than the one observed, assuming H_0 is true. For $t \approx 3.294$ with $df = 99$:

$$P = P(T > 3.294) \approx 0.000685$$

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(Here, Φ is the cumulative distribution function of the standard normal distribution.)

Since $P \approx 0.000685 < \alpha = 0.05$, we reject H_0. This conclusion is consistent with the rejection of H_0 based on the test statistic.

There is strong evidence at the 0.05 significance level to conclude that the average spending on food delivery among the university students is significantly higher than 450 yuan. The administration may consider this result when evaluating student spending habits or making policy decisions.

A7

According to the "Report on Nutrition and Chronic Diseases of Chinese Residents (2015)," the average height of adult males aged 18 and above in China is 1.67 m. A random sample of 400 adult males from a certain region was selected, and their average height was measured as 1.69 m with a standard deviation of 0.042 m. Assuming the height of males in this region follows a normal distribution $N(\mu, \sigma^2)$, determine whether the average height in this region is significantly higher than the national average (significance level $\alpha = 0.05$).

Solution:

1. Null Hypothesis (H_0): The average height of males in the region is less than or equal to the national average. $H_0 : \mu \leq 1.67$ m.

2. Alternative Hypothesis (H_1): The average height of males in the region is significantly higher than the national average. $H_1 : \mu > 1.67$ m.

Since the sample size is large ($n = 400$) and the population standard deviation is unknown, we use the z-test (approximated by the sample standard deviation). The test statistic is calculated as:

$$z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{1.69 - 1.67}{0.042/\sqrt{400}} = \frac{0.02}{0.0021} \approx 9.524$$

Here, $\bar{X} = 1.69$ m is the sample mean, $\mu_0 = 1.67$ m is the hypothesized population mean (national average), $s = 0.042$ m is the sample standard deviation, and $n = 400$ is the sample size.

For a one-tailed test at $\alpha = 0.05$, the critical z-value is $z_{0.05} \approx 1.645$. Reject H_0 if $z > 1.645$.

Since $z \approx 9.524 > 1.645$, we reject the null hypothesis.

The P-value is the probability of observing a test statistic as extreme as or more extreme than the one observed, assuming H_0 is true. For $z \approx 9.524$:

$$P = P(Z > 9.524) \approx 1 - \Phi(9.524) \approx 0$$

✓

(Here, Φ is the cumulative distribution function of the standard normal distribution.)

Since $P \approx 0 < \alpha = 0.05$, we reject H_0. This conclusion is consistent with the rejection of H_0 based on the test statistic.

There is extremely strong evidence at the 0.05 significance level to conclude that the average height of adult males in the region is significantly higher than the national average of 1.67 m. The result suggests a notable regional difference in height, which may warrant further investigation into potential causes such as genetic, nutritional, or environmental factors.

A10

A distribution agent's contract with a dairy company stipulates that the standard deviation of 225 mL cartons of milk must not exceed 8 mL; otherwise, the shipment will be rejected. A random sample of 15 cartons was selected, and their volumes (in mL) were measured as follows:

230, 223, 228, 229, 220, 215, 217, 231, 220, 223, 230, 224, 226, 228, 227.

Assume the sample is drawn from a normal population $N(\mu, \sigma^2)$, where $-\infty < \mu < +\infty$ and $\sigma > 0$ are unknown. At a significance level of $\alpha = 0.05$, use the P-value approach to test the hypotheses:

$$H_0 : \sigma \geq 8, \quad H_1 : \sigma < 8.$$

Solution:

Given the data, we first compute the sample mean ($\bar{X}$) and sample standard deviation (s):

$$\bar{X} = \frac{230 + 223 + \cdots + 227}{15} = \frac{3401}{15} \approx 226.733\,\text{mL},$$

$$s = \sqrt{\frac{\sum_{i=1}^{15}(X_i - \bar{X})^2}{n-1}} \approx 5.145\,\text{mL}.$$

1. Null Hypothesis (H_0): The standard deviation is greater than or equal to 8 mL ($\sigma \geq 8$).
2. Alternative Hypothesis (H_1): The standard deviation is less than 8 mL ($\sigma < 8$).

For testing variance/standard deviation, the chi-square (χ^2) statistic is used:

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{14 \times 5.145^2}{8^2} \approx \frac{14 \times 26.471}{64} \approx 5.782.$$

Here, $\sigma_0 = 8$ mL is the threshold under H_0.

The test is left-tailed (since $H_1 : \sigma < 8$). The P-value is:

$$P = P(\chi^2_{14} \leq 5.782).$$

From chi-square tables or software: $\chi^2_{14,0.05} \approx 6.571$ (critical value for left-tail). $P \approx 0.021$ (since $5.782 < 6.571$).

Here, $P < 0.05$, so reject H_0.

There is sufficient evidence ($P = 0.025$) to conclude that the standard deviation of milk carton volumes is significantly less than 8 mL at the 0.05 level. The shipment meets the contract requirement and should not be rejected.

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A12

The following data represent the heat production records (unit: 4.186×10^3 J) per ton of coal mined from two mines, A and B. Figures are shown in a table.

Assume the samples are drawn from two independent normal populations with equal variances. Can we conclude that the heat production of Mine A is significantly greater than that of Mine B at a significance level of $\alpha = 0.05$?

Solution:

1. Null Hypothesis (H_0): The mean heat production of Mine A is less than or equal to that of Mine B.
$$H_0 : \mu_A \le \mu_B \quad \text{or} \quad \mu_A - \mu_B \le 0.$$

2. Alternative Hypothesis (H_1): The mean heat production of Mine A is significantly greater than that of Mine B.
$$H_1 : \mu_A > \mu_B \quad \text{or} \quad \mu_A - \mu_B > 0.$$

For Mine A ($n_A = 5$):

$$\bar{X}_A = \frac{8500 + 8330 + 8480 + 7960 + 8030}{5} = \frac{41300}{5} = 8,260\,\text{units},$$

$$s_A^2 = \frac{\sum(X_i - \bar{X}_A)^2}{n_A - 1} \approx 56,300 \quad \Rightarrow s_A \approx 237.3\,\text{units}.$$

For Mine B ($n_B = 5$):

$$\bar{X}_B = \frac{7710 + 7890 + 7920 + 8270 + 7860}{5} = \frac{39650}{5} = 7,930\,\text{units},$$

$$s_B^2 = \frac{\sum(Y_i - \bar{X}_B)^2}{n_B - 1} \approx 42,700 \quad \Rightarrow s_B \approx 206.6\,\text{units}.$$

Since the variances are assumed equal, compute the pooled variance:

$$s_p^2 = \frac{(n_A - 1)s_A^2 + (n_B - 1)s_B^2}{n_A + n_B - 2} = \frac{4 \times 56,300 + 4 \times 42,700}{8} = 49,500.$$

$$s_p \approx 222.5\,\text{units}.$$

$$t = \frac{\bar{X}_A - \bar{X}_B}{s_p\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}} = \frac{8260 - 7930}{222.5\sqrt{\frac{1}{5} + \frac{1}{5}}} = \frac{330}{140.5} \approx 2.35.$$

Degrees of freedom: $df = n_A + n_B - 2 = 8$.

One-tailed test at $\alpha = 0.05$ with $df = 8$: Critical t-value from tables: $t_{0.05,8} \approx 1.860$.

Since $t \approx 2.35 > 1.860$, reject H_0.

$$P = P(T_8 > 2.35) \approx 0.023 \quad \text{(from software or tables)}.$$

✓

Since $P = 0.023 < \alpha = 0.05$, reject H_0.

There is sufficient evidence ($P = 0.023$) to conclude that Mine A's coal produces significantly more heat than Mine B's at the 0.05 significance level.

B1

A boring hypothesis testing problem.

Solution:

1. Given: $X \sim N(\mu, \sigma^2)$, $\sigma^2 = 1$, $n = 16$.

   Hypotheses: $H_0 : \mu = 1$ vs. $H_1 : \mu \neq 1$ (two-tailed test).

   Significance level: $\alpha = 0.05$.

   Since σ^2 is known, use the z-test:

   $$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\overline{X} - 1}{1/4} = 4(\overline{X} - 1).$$

   For $\alpha = 0.05$ (two-tailed), the critical values are $z_{\alpha/2} = \pm 1.96$. Reject H_0 if $|Z| > 1.96$.

   Under $H_1 : \mu = 2$, the distribution of $\overline{X}$ is $N(2, 1/16)$.

   Non-rejection region under H_0:

   $$|Z| \leq 1.96 \Rightarrow \overline{X} \in [1 - 1.96 \times 0.25, 1 + 1.96 \times 0.25] = [0.51, 1.49].$$

   Probability of not rejecting H_0 when $\mu = 2$ (Type II error):

   $$\beta = P(0.51 \leq \overline{X} \leq 1.49 \mid \mu = 2) = P\left(\frac{0.51 - 2}{0.25} \leq Z \leq \frac{1.49 - 2}{0.25}\right) = P(-5.96 \leq Z \leq -2.04) \approx 0.0207.$$

   ✓

2. Given: $X \sim N(\mu, \sigma^2)$, μ unknown, $n = 16$.

   Hypotheses: $H_0 : \sigma^2 = 1$ vs. $H_1 : \sigma^2 > 1$ (right-tailed test).

   Significance level: $\alpha = 0.05$.

   Use the chi-square test:

   $$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} = \frac{15S^2}{1} = 15S^2.$$

   For $\alpha = 0.05$ and $df = 15$, the critical value is $\chi^2_{0.05,15} \approx 24.996$. Reject H_0 if $\chi^2 > 24.996$.

   Under $H_1 : \sigma^2 = 4$, the test statistic follows $\chi^2_{15} \times \frac{1}{4}$.

   Non-rejection region under H_0: $\chi^2 \leq 24.996$.

   Convert to H_1 scale:

   $$\frac{15S^2}{1} \leq 24.996 \Rightarrow S^2 \leq \frac{24.996}{15} \approx 1.666.$$

   Under H_1, S^2 scales as $\sigma^2 = 4$:

   $$\chi^2_{15} = \frac{15S^2}{4} \leq \frac{15 \times 1.666}{4} \approx 6.247.$$

   Probability of not rejecting H_0 when $\sigma^2 = 4$ (Type II error):

   ✓

   $$\beta = P(\chi^2_{15} \leq 6.247) \approx 0.025 \quad \text{(from chi-square tables)}.$$

3. Given: $\overline{x} = 1.54$, $s^2 = 1.44$.

For Part (1) ($\sigma^2 = 1$)

Test statistic:

$$Z = 4(1.54 - 1) = 2.16.$$

Two-tailed P-value:

$$P = 2 \times P(Z > 2.16) = 2 \times 0.0154 = 0.0308.$$

For Part (2) (μ unknown)

Test statistic:

$$\chi^2 = 15 \times 1.44 = 21.6.$$

Right-tailed P-value:

$$P = P(\chi^2_{15} > 21.6) \approx 0.12 \quad \text{(from chi-square tables)}.$$

✓

B3

To compare the income levels of faculty at University A and University B, random samples of 36 and 49 associate professors aged 40–50 were selected from each university, respectively. Their pre-tax annual incomes were recorded. The results are:

University A: Mean income = 288k CNY, Standard deviation = 116k CNY ($n_A = 36$).

University B: Mean income = 224k CNY, Standard deviation = 84k CNY ($n_B = 49$).

Using hypothesis testing ($\alpha = 0.05$ for part 1, $\alpha = 0.1$ for part 2), answer:

1. Is the income variability at B significantly lower than at A?
2. Is there evidence that A's mean income exceeds B's by at least 50k CNY?

Solution:

(1) Testing Income Variability: $H_0 : \sigma_A \leq \sigma_B$ vs. $H_1 : \sigma_A > \sigma_B$

Test Statistic:

$$F = \frac{s_A^2}{s_B^2} = \frac{116^2}{84^2} = 1.907 \quad (df_A = 35, df_B = 48).$$

Critical Value: For $\alpha = 0.05$, $F_{0.05,35,48} \approx 1.67$ (from F-tables).

Since $F = 1.907 > 1.67$, reject H_0.

At $\alpha = 0.05$, B's income variability is significantly lower than A's. ✓

(2) Testing Mean Difference: $H_0 : \mu_A - \mu_B \leq 50$ vs. $H_1 : \mu_A - \mu_B > 50$

Pooled Estimate:

$$\text{SE} = \sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}} = \sqrt{\frac{116^2}{36} + \frac{84^2}{49}} \approx 22.45.$$

Test Statistic:

$$t = \frac{(\overline{X}_A - \overline{X}_B) - 50}{\text{SE}} = \frac{(288 - 224) - 50}{22.45} \approx 0.624 \quad (df \approx 60).$$

For $\alpha = 0.1$ (one-tailed), $t_{0.1,60} \approx 1.296$.
Since $t = 0.624 < 1.296$, fail to reject H_0.
No evidence at $\alpha = 0.1$ that A's mean income exceeds B's by at least 50k CNY.

✓