HOMEWORK N° 5

Chapter 2

A15

$X \sim N(1,4)$.

Solution:

1.
$$P(X \leq 0) = \Phi(\frac{0-1}{2}) = 1 - \Phi(0.5) \approx 0.31$$
$$P(|X-1| \leq 2) = P(-1 \leq X \leq 3) = \Phi(1) - (1 - \Phi(1)) \approx 0.68$$

2. Evident that $a = \mu$,
$$a = 1$$

3.
$$P(|X| \leq 2) = \Phi(0.5) - \Phi(-1.5) \approx 0.62$$

✓

A17

Distribution law of $Y = 2X - 1, Z = X^2$

Solution:

Y	-3	-1	1	3
p	0.3	0.1	0.2	0.4

Z	0	1	4
p	0.1	0.5	0.4

✓

Which is convenient for one to solve,

$$F_z(x) = \begin{cases} 0 & ,x < 0 \\ 0.1 & ,0 \le x < 1 \\ 0.6 & ,1 \le x < 4 \\ 1 & ,x \ge 4 \end{cases}$$

A18

X satisfies,

$$f_X(x) = \begin{cases} \frac{x}{4} & ,1 < x < 3 \\ 0 & ,x_{others} \end{cases}$$

Solution:

$$F_X(x) = \begin{cases} 0 & ,x < 1 \\ \frac{x^2-1}{8} & ,1 \le x < 3 \\ 1 & ,x \ge 3 \end{cases}$$

1. $Y = 2X$.

$$F_Y(x) = P(Y \le x) = P(X \le \frac{x}{2}) = F_X(\frac{x}{2})$$

Therefore,

$$F_X(x) = \begin{cases} 0 & ,x < 2 \\ \frac{x^2-4}{32} & ,2 \le x < 6 \\ 1 & ,x \ge 6 \end{cases}, f_Y(x) = \begin{cases} \frac{x}{16} & ,2 \le x < 6 \\ 0 & ,x_{others} \end{cases}$$

2. $Y = 2 - X$.

$$F_Y(x) = P(Y \le x) = P(X \ge 2 - x) = 1 - F_X(2 - x)$$

Therefore,

$$F_X(x) = \begin{cases} 0 & ,x < -1 \\ \frac{9-(x-2)^2}{8} & ,-1 \le x < 1 \\ 1 & ,x \ge 1 \end{cases}, f_Y(x) = \begin{cases} \frac{2-x}{4} & ,-1 \le x < 1 \\ 0 & ,x_{others} \end{cases}$$

3. $Y = X^2$.

$$F_Y(x) = P(Y \le x) = P(-\sqrt{x} \le X \le \sqrt{x}) = F_X(\sqrt{x})$$

Therefore,

$$F_X(x) = \begin{cases} 0 & ,x < 1 \\ \frac{x-1}{8} & ,1 \le x < 9 \\ 1 & ,x \ge 9 \end{cases}, f_Y(x) = \begin{cases} \frac{1}{8} & ,1 \le x < 9 \\ 0 & ,x_{others} \end{cases}$$

A19

$X \sim N(1,4)$, what does $Y = \frac{X-1}{2}, Z = 2 - X$ satisfy?

Solution:

For Y,

$$\mu_Y = \frac{\mu_X - 1}{2} = 0, \sigma_Y = \frac{\sigma_X}{2} = 1$$

A standard normal distribution.

$$Y \sim N(0,1)$$

Where for Z,

$$\mu_Z = 2 - \mu_X = 1, \sigma_Z = \sigma_X$$

Thus,

$$Z \sim N(1,4)$$

✓

B22

$N \sim N(170, 5.0^2)$.

Solution:

1.
$$P(X > 170) = 1 - \Phi(0) = 0.5$$
✓

2.
$$P(165 \leq X < 175) = 2\Phi(1) - 1 \approx 0.68$$
✓

3.
$$P(X < 172) = \Phi(0.4) \approx 0.66$$
✓

B23

$X \sim N(22.5, 2.5^2)$

Solution:

1.
$$P(X \leq 25) = \Phi(1) = 0.842, P(25 < X \leq 27.5) = 0.135, P(X > 27.5) = 0.023$$

Thus,

$$P(H) = \sum P(H|X_i)P(X_i) = 0.111$$

✓

2.

$$P(X > 25|H) = \frac{P(X > 25, H)}{P(H)} = 0.244$$

3. Set Y to be the number of people affecting hypertension.

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - (1 - P(H))^3 = 0.298$$

✓

B26

Normal distribution reverse prediction with $X \sim N(15, 4)$.

Solution:

$$\Phi(0) = 0.5, \Phi(1) \approx 0.84$$

Therefore,

$$x_1 = \sigma x_{\Phi_1} + \mu = 15, x_2 = \sigma x_{\Phi_2} + \mu = 17$$

✓

B27

Solve the PDF parameter of a normal distribution.

Solution:

1.

$$\mu = 0, \sigma^2 = \frac{1}{2}$$

$X \sim N(0, \frac{1}{2})$. Therefore,

$$a = \frac{1}{\sqrt{\pi}}$$

2.

$$P(X > \frac{1}{2}) = 1 - \Phi(\frac{\sqrt{2}}{2}) \approx 0.24$$

✓

B33

$$f_X(x) = \begin{cases} c(4 - x^2) & , -1 < x < 2 \\ 0 & , x_{others} \end{cases}$$

Solution:

1.
$$\int_{-1}^{2} f_X(x)dx = 1 \to c[(8-\frac{8}{3})-(-4+\frac{1}{3})] = 9$$

Thus,
$$c = \frac{1}{9}$$

2.
$$F_X(x) = \begin{cases} 0 & ,x < -1 \\ \frac{1}{9}(4x - \frac{1}{3}x^3 + \frac{11}{3}) & ,-1 \le x < 2 \\ 1 & ,x \ge 2 \end{cases}$$

Thus, for$Y = 3X$
$$P(Y \le x) = P(X \le \frac{x}{3}) = F_X(\frac{x}{3})$$

$$F_Y(x) = \begin{cases} 0 & ,x < -3 \\ \frac{1}{9}(\frac{4}{3}x - \frac{1}{81}x^3 + \frac{11}{3}) & ,-3 \le x < 6 \\ 1 & ,x \ge 6 \end{cases}$$

$$f_Y(x) = \begin{cases} \frac{1}{9}(\frac{4}{3} - \frac{1}{27}x^2) & ,-3 < x < 6 \\ 0 & ,x_{others} \end{cases}$$

3. For $Z = |X|$
$$P(Z \le x) = P(-x \le X \le x) = \begin{cases} F_X(x) - F_X(-x) & ,x > 0 \\ 0 & ,x \le 0 \end{cases}$$

Thus,
$$F_Z(x) = \begin{cases} 0 & ,x < 0 \\ -\frac{2}{27}x^3 + \frac{8}{9}x & ,0 \le x < 1 \\ \frac{4}{9}x - \frac{1}{27}x^3 + \frac{11}{27} & ,1 \le x < 2 \\ 1 & ,x \ge 2 \end{cases}$$

$$f_Z(x) = \begin{cases} -\frac{2}{9}x^2 + \frac{8}{9} & ,0 < x < 1 \\ -\frac{1}{9}x^2 + \frac{4}{9} & ,1 < x < 2 \\ 0 & ,x_{others} \end{cases}$$

B35

$X \sim U(0,1)$, solve $Y = X^n$.

Solution:

$$F_X(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \le x < 1 \\ 1 & , x \ge 1 \end{cases}$$

Therefore,

$$P(Y \le x) = P(X \le \sqrt[n]{x})$$

$$F_Y(x) = \begin{cases} 0 & , x < 0 \\ \sqrt[n]{x} & , 0 \le x < 1 \\ 1 & , x \ge 1 \end{cases}$$

$$f_Y(x) = \begin{cases} \frac{1}{n} x^{\frac{1}{n}-1} & , 0 < x < 1 \\ 0 & , x_{others} \end{cases}$$

✓

B36

$X \sim U(0, \frac{3}{2}\pi)$, $Y = \cos X$, solve Y's CDF.

Solution:

$$F_X(x) = \begin{cases} 0 & , x < 0 \\ \frac{2}{3\pi} x & , 0 \le x < \frac{3}{2}\pi \\ 1 & , x \ge \frac{3}{2}\pi \end{cases}$$

Therefore,

$$P(Y \le x) = P(\arccos x \le X \le 2\pi - \arccos x) = F_X(2\pi - \arccos x) - F_X(\arccos x)$$

$$F_Y(x) = \begin{cases} 0 & , x < -1 \\ \frac{4}{3} - \frac{4}{3\pi} \arccos x & , -1 \le x < 0 \\ 1 - \frac{2}{3\pi} \arccos x & , 0 \le x < 1 \\ 1 & , x \ge 1 \end{cases}$$

✓

B39

$X \sim N(0, 1)$,$Y = e^X$,$Z = \ln |X|$

Solution:

1.

$$F_X(x) = \Phi(x)$$

And,

$$P(Y \le x) = P(X \le \ln x) = F_X(\ln x), x > 0$$

Thus,

$$F_Y(x) = \Phi(\ln x), f_Y(x) = \Phi'(\ln x)\frac{1}{x} = \frac{1}{x\sqrt{2\pi}}e^{-\frac{(\ln x)^2}{2}}, x > 0$$

$$f_Y(x) = \begin{cases} \frac{1}{x\sqrt{2\pi}}e^{-\frac{(\ln x)^2}{2}} & , x > 0 \\ 0 & , x_{others} \end{cases}$$

2.

$$F_z(x) = P(Z \le x) = P(-e^x \le X \le e^x) = 2\Phi(e^x) - 1$$

Thus,

$$f_z(x) = 2\Phi'(e^x)e^x = \sqrt{\frac{2}{\pi}}e^{x-\frac{e^{2x}}{2}}, x \in \mathbf{R}$$

HOMEWORK N° 5

Chapter 3

A1

(X, Y) co-distribution.

Solution:

1.

$$0.7 + 3c = 1 \to c = 0.1$$

2.

$$P(X \le 1, Y \ge 1) = 0.1 \times 4 + 0.2 = 0.6$$

3.

$$P(X = 0) = 0.5, P(X = 1) = 0.2, P(X = 2) = 0.3$$

$$P(Y = 0) = 0.2, P(Y = 1) = 0.2, P(Y = 2) = 0.3, P(Y = 3) = 0.3$$

A2

Solve certain distribution value in (X, Y).

Solution:

1.

$$0.3 + 3a = 0.6 \to a = 0.1$$

$$0.2 + b = 0.4 \to b = 0.2$$

2.

$$b = 0.1, 3a = 0.4 \to a = \frac{2}{15}$$

3.

$$3a + 0.2 = 0.35 \to a = 0.05, b = 0.35$$

A4

(X, Y) joint and marginal probability distribution.

Solution:

1. No-replaced

X	Y		$P\{X = i\}$
	0	1	
0	0.1	0.3	0.4
1	0.3	0.3	0.6
$P\{Y = j\}$	0.4	0.6	

2. Re-placed

X	Y		$P\{X=i\}$
	0	1	
0	0.16	0.24	0.4
1	0.24	0.36	0.6
$P\{Y=j\}$	0.4	0.6	

✓

A6

Solution:

$$\frac{0.1+a}{0.8-c}=0.5, b+c=0.5, 0.3+a=0.5$$

Thus,

$$a=0.2, b=0.3, c=0.2$$

✓