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HOMEWORK N° 12

Chapter 7

A-4

Suppose a population $X \sim G(p)$, extract n samples with their values $x_1 \dots x_n$, estimating the parameter p using maximum likelihood estimation.

Solution:

The likelihood function of this distribution is:

$$L(p) = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} (1-p)^{x_i - 1} p = p^n (1-p)^{\sum_{i=1}^{n}(x_i - 1)} = p^n (1-p)^{\sum x_i - n}$$

Make it logarithmic:

$$\ln L(p) = n \ln p + \left(\sum x_i - n\right) \ln(1-p)$$

The minimum point is given by such a equation:

$$\frac{d}{dp} \ln L(p) = \frac{n}{p} - \frac{\sum x_i - n}{1-p} = 0$$

$$\frac{n}{p} = \frac{\sum x_i - n}{1-p} \Rightarrow n(1-p) = p(\sum x_i - n) \Rightarrow n - np = p \sum x_i - pn \Rightarrow n = p \sum x_i \Rightarrow \hat{p} = \frac{n}{\sum x_i}$$

Therefore the result:

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} \quad \checkmark$$

A-5

Let the random variable X follow a known discrete distribution, where $0 < \theta < 1$. From this population, a random sample of size 9 is drawn, yielding the observations:

$$2,\ 0,\ 2,\ 1,\ 0,\ 0,\ 0,\ 1,\ 2,\ 1$$

Find:
1. The method of moments estimator of θ
2. The MLE of θ

Solution:

1. We use the first-order moment estimator(population mean) to estimate the desired value.:

$$\mathbb{E}[X] = 0 \cdot \theta^2 + 1 \cdot 2\theta(1-\theta) + 2 \cdot (1-\theta)^2 = 2\theta(1-\theta) + 2(1-\theta)^2$$

$$= 2(1-\theta)\left[\theta + (1-\theta)\right] = 2(1-\theta)(1) = 2(1-\theta)$$

Therefore:

$$\mathbb{E}[X] = 2(1-\theta) \Rightarrow \theta = 1 - \frac{1}{2}\mathbb{E}[X]$$

Substituting $\bar{x}$ would we obtain the desired $\hat{\theta}_{\text{MM}}$

$$\bar{x} = \frac{9}{9} = 1 \Rightarrow \hat{\theta}_{\text{MM}} = 1 - \frac{1}{2} = \frac{1}{2} \checkmark$$

2. Let's denote:

$$n_0 = Card\{X_i = 0\} = 3$$
$$n_1 = Card\{X_i = 1\} = 3$$
$$n_2 = Card\{X_i = 2\} = 3$$

The likelihood function of such distribution is:

$$L(\theta) = (\theta^2)^{n_0} \cdot [2\theta(1-\theta)]^{n_1} \cdot [(1-\theta)^2]^{n_2}$$

$$= \theta^{2n_0} \cdot 2^{n_1} \cdot \theta^{n_1} \cdot (1-\theta)^{n_1} \cdot (1-\theta)^{2n_2} = 2^{n_1} \cdot \theta^{2n_0+n_1} \cdot (1-\theta)^{n_1+2n_2}$$

Transfer it into log-likelihood form:

$$\ln L(\theta) = \text{const} + (2n_0 + n_1)\ln\theta + (n_1 + 2n_2)\ln(1-\theta)$$

Let:

$$a = 2n_0 + n_1 = 2 \times 3 + 3 = 9$$

$$b = n_1 + 2n_2 = 3 + 6 = 9$$

Then:

$$\ln L(\theta) = \text{const} + 9\ln\theta + 9\ln(1-\theta)$$

Differentiate and solve:

$$\frac{d}{d\theta}\ln L(\theta) = \frac{9}{\theta} - \frac{9}{1-\theta} = 0 \Rightarrow \frac{9}{\theta} = \frac{9}{1-\theta}$$

Therefore:

$$\hat{\theta}_{\text{MLE}} = \frac{1}{2} \quad \checkmark$$

A-6

Estimate two parameters with the probability distribution given:

$$P(X=0) = \theta, \quad P(X=1) = \lambda, \quad P(X=2) = 1 - \theta - \lambda,$$

with constraints $0 < \theta, \lambda < 1$ and $\theta + \lambda < 1$, and the observed sample:

$$2,\ 0,\ 2,\ 1,\ 0,\ 0,\ 1,\ 2,\ 1$$

Solution:

Sample counts:

$$n_0 = 3, \quad n_1 = 3, \quad n_2 = 3, \quad n = 9$$

Method of Moments Estimators(Using empirical frequencies to estimate probabilities):

$$\underset{\checkmark}{\hat{\theta}_{\text{MM}} = \frac{n_0}{n} = \frac{1}{3}}, \quad \hat{\lambda}_{\text{MM}} = \frac{n_1}{n} = \frac{1}{3} \quad \checkmark$$

Likelihood:

$$L(\theta, \lambda) = \theta^4 \lambda^2 (1 - \theta - \lambda)^3$$

Log-likelihood:

$$\ell(\theta, \lambda) = 3\ln\theta + 3\ln\lambda + 3\ln(1-\theta-\lambda)$$

Solve:

$$\frac{\partial \ell}{\partial \theta} = \frac{3}{\theta} - \frac{3}{1-\theta-\lambda} = 0, \quad \frac{\partial \ell}{\partial \lambda} = \frac{3}{\lambda} - \frac{3}{1-\theta-\lambda} = 0$$

Result:

$$\hat{\theta}_{\text{MLE}} = \frac{1}{3}, \quad \hat{\lambda}_{\text{MLE}} = \frac{1}{3}$$

$$\checkmark \qquad \checkmark$$

A-7

Let the density function of the population X be

$$f(x;\theta) = \begin{cases} \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta > 0$ is unknown. Let $\mu_2 = E(X^2)$ and $p = P\{X > 1\}$. Given $X_1, X_2, \cdots, X_n$ as a simple random sample from this population, find the maximum likelihood estimators for the parameters θ, μ_2, and p.

Solution:

1. The likelihood function for the sample is:

$$L(\theta) = \prod_{i=1}^{n} f(X_i;\theta) = \prod_{i=1}^{n} \left(\frac{X_i}{\theta} e^{-\frac{X_i^2}{2\theta}} \right) = \left(\prod_{i=1}^{n} X_i \right) \theta^{-n} e^{-\frac{1}{2\theta} \sum_{i=1}^{n} X_i^2}.$$

The log-likelihood function is:

$$\ln L(\theta) = \sum_{i=1}^{n} \ln X_i - n \ln \theta - \frac{1}{2\theta} \sum_{i=1}^{n} X_i^2.$$

To find the maximum likelihood estimator (MLE), take the derivative with respect to θ and set it to zero:

$$\frac{d}{d\theta} \ln L(\theta) = -\frac{n}{\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} X_i^2 = 0.$$

Solving for θ:

$$-\frac{n}{\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} X_i^2 = 0 \implies \frac{1}{2\theta^2} \sum_{i=1}^{n} X_i^2 = \frac{n}{\theta} \implies \sum_{i=1}^{n} X_i^2 = 2n\theta.$$

Thus, the MLE for θ is:

$$\hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} X_i^2. \quad \checkmark$$

2. First, compute $E(X^2)$:

$$E(X^2) = \int_0^\infty x^2 f(x;\theta)\, dx = \int_0^\infty x^2 \cdot \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}\, dx.$$

Let $u = \frac{x^2}{2\theta}$, then $du = \frac{x}{\theta} dx$ and $x^2 = 2\theta u$:

$$E(X^2) = \int_0^\infty 2\theta u \cdot e^{-u}\, du = 2\theta \int_0^\infty u e^{-u}\, du = 2\theta \cdot 1 = 2\theta.$$

Thus, $\mu_2 = 2\theta$. Using the MLE of θ, the MLE for μ_2 is:

$$\hat{\mu}_2 = 2\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i^2. \quad \checkmark$$

3. Compute $P(X > 1)$:

$$P(X > 1) = \int_1^\infty f(x;\theta)\,dx = \int_1^\infty \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}\,dx.$$

Let $u = \frac{x^2}{2\theta}$, then $du = \frac{x}{\theta}dx$:

$$P(X > 1) = \int_{\frac{1}{2\theta}}^\infty e^{-u}\,du = e^{-\frac{1}{2\theta}}.$$

Using the MLE of θ, the MLE for p is:

$$\hat{p} = e^{-\frac{1}{2\hat{\theta}}} = e^{-\frac{n}{\sum_{i=1}^n X_i^2}}.$$

A-9

Let the population $X \sim N(\mu, \sigma^2)$, where μ and σ^2 are unknown. X_1, X_2, X_3 is a simple random sample from the population X. Consider the following estimators for the parameter μ:

$$\hat{\mu}_1 = \frac{1}{2}X_1 + \frac{1}{4}X_2 + \frac{1}{4}X_3,$$

$$\hat{\mu}_2 = 2X_1 - 2X_2 + X_3,$$

$$\hat{\mu}_3 = \frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{3}X_3.$$

Are these estimators unbiased? If so, which one is more efficient?

Solution:

All three estimators $\hat{\mu}_1, \hat{\mu}_2$, and $\hat{\mu}_3$ are unbiased. The most efficient estimator is $\boxed{\hat{\mu}_3}$. Here's why.

An estimator $\hat{\mu}$ is unbiased if $E(\hat{\mu}) = \mu$.

For $\hat{\mu}_1$:

$$E(\hat{\mu}_1) = \frac{1}{2}E(X_1) + \frac{1}{4}E(X_2) + \frac{1}{4}E(X_3) = \frac{1}{2}\mu + \frac{1}{4}\mu + \frac{1}{4}\mu = \mu.$$

Thus, $\hat{\mu}_1$ is unbiased.

For $\hat{\mu}_2$:

$$E(\hat{\mu}_2) = 2E(X_1) - 2E(X_2) + E(X_3) = 2\mu - 2\mu + \mu = \mu.$$

Thus, $\hat{\mu}_2$ is unbiased.

For $\hat{\mu}_3$:

$$E(\hat{\mu}_3) = \frac{1}{3}E(X_1) + \frac{1}{3}E(X_2) + \frac{1}{3}E(X_3) = \frac{1}{3}\mu + \frac{1}{3}\mu + \frac{1}{3}\mu = \mu.$$

Thus, $\hat{\mu}_3$ is unbiased.

The efficiency of an unbiased estimator is determined by its variance. The estimator with the smallest variance is the most efficient.

Variance of $\hat{\mu}_1$:

$$\text{Var}(\hat{\mu}_1) = \left(\frac{1}{2}\right)^2 \sigma^2 + \left(\frac{1}{4}\right)^2 \sigma^2 + \left(\frac{1}{4}\right)^2 \sigma^2 = \frac{1}{4}\sigma^2 + \frac{1}{16}\sigma^2 + \frac{1}{16}\sigma^2 = \frac{3}{8}\sigma^2.$$

Variance of $\hat{\mu}_2$:

$$\text{Var}(\hat{\mu}_2) = 2^2\sigma^2 + (-2)^2\sigma^2 + 1^2\sigma^2 = 4\sigma^2 + 4\sigma^2 + \sigma^2 = 9\sigma^2.$$

Variance of $\hat{\mu}_3$:

$$\text{Var}(\hat{\mu}_3) = \left(\frac{1}{3}\right)^2 \sigma^2 + \left(\frac{1}{3}\right)^2 \sigma^2 + \left(\frac{1}{3}\right)^2 \sigma^2 = \frac{1}{9}\sigma^2 + \frac{1}{9}\sigma^2 + \frac{1}{9}\sigma^2 = \frac{1}{3}\sigma^2.$$

The variances are ordered as:

$$\text{Var}(\hat{\mu}_3) < \text{Var}(\hat{\mu}_1) < \text{Var}(\hat{\mu}_2).$$

Thus, $\hat{\mu}_3$ is the most efficient estimator among the three.

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A-10

Let $\hat{\theta}_1$ and $\hat{\theta}_2$ both be unbiased estimators of θ, and assume that $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent. Given:

$$D(\hat{\theta}_1) = \sigma_1^2, \quad D(\hat{\theta}_2) = \sigma_2^2,$$

a new unbiased estimator for θ is introduced:

$$\hat{\theta}_3 = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2.$$

Determine the constant α such that the variance $D(\hat{\theta}_3)$ is minimized.

Solution:

Since $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ, we have:

$$E(\hat{\theta}_1) = \theta, \quad E(\hat{\theta}_2) = \theta.$$

For $\hat{\theta}_3$ to be unbiased:

$$E(\hat{\theta}_3) = \alpha E(\hat{\theta}_1) + (1-\alpha)E(\hat{\theta}_2) = \alpha\theta + (1-\alpha)\theta = \theta.$$

Thus, $\hat{\theta}_3$ is unbiased for any α.
Given that $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, the variance of $\hat{\theta}_3$ is:

$$D(\hat{\theta}_3) = \alpha^2 D(\hat{\theta}_1) + (1-\alpha)^2 D(\hat{\theta}_2) = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2.$$

To minimize $D(\hat{\theta}_3)$ with respect to α, take the derivative and set it to zero:

$$\frac{d}{d\alpha} D(\hat{\theta}_3) = 2\alpha\sigma_1^2 - 2(1-\alpha)\sigma_2^2 = 0.$$

Solving for α:

$$2\alpha\sigma_1^2 - 2\sigma_2^2 + 2\alpha\sigma_2^2 = 0 \implies \alpha(\sigma_1^2 + \sigma_2^2) = \sigma_2^2 \implies \alpha = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

The second derivative of $D(\hat{\theta}_3)$ with respect to α is:

$$\frac{d^2}{d\alpha^2} D(\hat{\theta}_3) = 2\sigma_1^2 + 2\sigma_2^2 > 0,$$

which confirms that the critical point corresponds to a minimum.
Therefore, the constant α that minimizes the variance $D(\hat{\theta}_3)$ is:

$$\alpha = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad \checkmark$$

B-3

Let the population X have a known probability distribution:

A sample of size n is drawn from the population X, denoted as $X_1, X_2, \cdots, X_n$. Let n_1, n_2, n_3 be the counts of a_1, a_2, a_3 in the sample, respectively, where $n_1 + n_2 + n_3 = n$. Find the method of moments estimator and the maximum likelihood estimator for the parameter θ.

Solution:

The first moment (population mean) of X is:

$$E(X) = a_1 \cdot \theta + a_2 \cdot \frac{1-\theta}{2} + a_3 \cdot \frac{1-\theta}{2}.$$

Simplify:

$$E(X) = a_1\theta + \frac{a_2 + a_3}{2}(1-\theta).$$

The sample mean is:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{a_1 n_1 + a_2 n_2 + a_3 n_3}{n}.$$

Set $E(X) = \bar{X}$ to solve for θ:

$$a_1\theta + \frac{a_2 + a_3}{2}(1-\theta) = \frac{a_1 n_1 + a_2 n_2 + a_3 n_3}{n}.$$

Solve for θ:

$$\theta\left(a_1 - \frac{a_2 + a_3}{2}\right) = \frac{a_1 n_1 + a_2 n_2 + a_3 n_3}{n} - \frac{a_2 + a_3}{2}.$$

$$\hat{\theta}_{\text{MM}} = \frac{\frac{a_1 n_1 + a_2 n_2 + a_3 n_3}{n} - \frac{a_2 + a_3}{2}}{a_1 - \frac{a_2 + a_3}{2}}. \quad \checkmark$$

The likelihood function is:

$$L(\theta) = \theta^{n_1} \left(\frac{1-\theta}{2}\right)^{n_2} \left(\frac{1-\theta}{2}\right)^{n_3} = \theta^{n_1} \left(\frac{1-\theta}{2}\right)^{n_2+n_3}.$$

The log-likelihood function is:

$$\ln L(\theta) = n_1 \ln \theta + (n_2 + n_3) \ln \left(\frac{1-\theta}{2}\right).$$

Take the derivative with respect to θ and set it to zero:

$$\frac{d}{d\theta} \ln L(\theta) = \frac{n_1}{\theta} - \frac{n_2 + n_3}{1-\theta} = 0.$$

Solve for θ:

$$\frac{n_1}{\theta} = \frac{n_2 + n_3}{1-\theta} \implies n_1(1-\theta) = \theta(n_2 + n_3).$$

$$n_1 - n_1\theta = \theta n_2 + \theta n_3 \implies n_1 = \theta(n_1 + n_2 + n_3).$$

$$\hat{\theta}_{\text{MLE}} = \frac{n_1}{n}. \quad \checkmark$$

B-4

Let the population $X \sim N(\mu, \sigma^2)$, where μ and σ^2 are unknown. Three independent samples are drawn: (X_1, X_2), (Y_1, Y_2, Y_3), and (Z_1, Z_2, Z_3, Z_4). Let S_1^2, S_2^2, S_3^2 be the corresponding sample variances. Define $T = aS_1^2 + bS_2^2 + cS_3^2$, where a, b, c are real numbers in the interval $[0, 1]$.

(1) State the necessary and sufficient condition for T to be an unbiased estimator of σ^2;

(2) Determine the values of a, b, c that make T the most efficient estimator.

Solution:

1. The sample variance S^2 for a sample of size n is an unbiased estimator of σ^2, i.e., $E(S^2) = \sigma^2$.

   The expectation of T is:

   $$E(T) = aE(S_1^2) + bE(S_2^2) + cE(S_3^2) = a\sigma^2 + b\sigma^2 + c\sigma^2 = (a+b+c)\sigma^2.$$

   For T to be unbiased for σ^2, we require:

   $$E(T) = \sigma^2 \implies a + b + c = 1.$$

2. The most efficient estimator is the one with the smallest variance. The variance of each sample variance S_i^2 is given by:

   $$\text{Var}(S_i^2) = \frac{2\sigma^4}{n_i - 1}.$$

   Thus: $\text{Var}(S_1^2) = \frac{2\sigma^4}{1}$ (since $n_1 - 1 = 1$), $\text{Var}(S_2^2) = \frac{2\sigma^4}{2} = \sigma^4$, $\text{Var}(S_3^2) = \frac{2\sigma^4}{3}$.

Assuming independence between the samples, the variance of T is:

$$\mathrm{Var}(T) = a^2\mathrm{Var}(S_1^2) + b^2\mathrm{Var}(S_2^2) + c^2\mathrm{Var}(S_3^2) = 2a^2\sigma^4 + b^2\sigma^4 + \frac{2}{3}c^2\sigma^4.$$

To minimize $\mathrm{Var}(T)$ under the constraint $a + b + c = 1$, we use the method of Lagrange multipliers. The Lagrangian is:

$$\mathcal{L} = 2a^2 + b^2 + \frac{2}{3}c^2 - \lambda(a + b + c - 1).$$

Taking partial derivatives and setting them to zero:

$$\frac{\partial \mathcal{L}}{\partial a} = 4a - \lambda = 0 \implies a = \frac{\lambda}{4},$$

$$\frac{\partial \mathcal{L}}{\partial b} = 2b - \lambda = 0 \implies b = \frac{\lambda}{2},$$

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{4}{3}c - \lambda = 0 \implies c = \frac{3\lambda}{4}.$$

Substitute into the constraint $a + b + c = 1$:

$$\frac{\lambda}{4} + \frac{\lambda}{2} + \frac{3\lambda}{4} = 1 \implies \lambda\left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right) = 1 \implies \lambda = \frac{2}{3}.$$

Thus:

$$a = \frac{1}{6}, \quad b = \frac{1}{3}, \quad c = \frac{1}{2}.$$

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