HOMEWORK N° 11

Chapter 6

A-11

$X_i \sim N(0,1), i = 1 \dots 16$ i.i.d, Solve compound distribution.

Solution:

1. $\bar{X} \sim N(0, \frac{1}{16})$
2. For $X_i^2 \sim \chi^2(1)$, the sum $\sum_{i=1}^{16} X_i^2 \sim \chi^2(16)$.
3. For $3X_1 \sim N(0,9)$, $\sqrt{\sum_{i=2}^{10} X_i^2} \sim \sqrt{\chi^2(9)}$, their fraction yields the student distribution $\frac{3X_1}{\sqrt{\sum_{i=2}^{10} X_i^2}} \sim t(9)$.
4. The same goes on $\frac{X_1+X_2}{\sqrt{X_3^2+X_4^2}} \sim t(2)$.
5. For $\bar{X} - X_1 = -\frac{15}{16}X_1 + \sum_{i=2}^{16} X_i$, its distribution is the linear compound of some normal distributions: $\bar{X} - X_1 \sim N(0, {\frac{15}{16}}^2 + \frac{15}{16^2}) = N(0, \frac{15}{16})$.

B-2

$X \sim N(\mu, \sigma^2)$. Extracts 9 samples. Solve the sampling distribution.

Solution:

1.
$$\frac{3(\bar{X}-\mu)}{\sigma} \sim N(0,1)$$

2.
$$\frac{3(\bar{X}-\mu)}{S} = \frac{\bar{X}-\mu}{\frac{S}{\sqrt{9}}} \sim t(8)$$

3.
$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \Rightarrow \frac{\sum(X_i-\overline{X})^2}{\sigma^2} \sim \chi^2(8)$$

4.
$$\sum_{i=1}^{9}(X_i-\mu)^2 \sim \sigma^2 \cdot \chi^2(9) \Rightarrow \frac{\sum(X_i-\mu)^2}{\sigma^2} \sim \chi^2(9)$$

5.
$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{9}\right) \Rightarrow \frac{\overline{X}-\mu}{\sigma/\sqrt{9}} \sim N(0,1) \Rightarrow \left(\frac{3(\overline{X}-\mu)}{\sigma}\right)^2 \sim \chi^2(1) \Rightarrow \frac{9(\overline{X}-\mu)^2}{\sigma^2} \sim \chi^2(1)$$

6.
$$\frac{\left(\frac{(\overline{X}-\mu)^2}{\sigma^2/9}\right)}{\frac{S^2}{\sigma^2}} \sim \frac{\chi^2(1)/1}{\chi^2(8)/8} \sim F(1,8) \Rightarrow \frac{9(\overline{X}-\mu)^2}{S^2} \sim F(1,8)$$

7.
$$\Rightarrow \frac{2(X_1-X_2)^2}{(X_3-X_4)^2+(X_5-X_6)^2} \sim \frac{\chi^2(1)}{\chi^2(2)/2} \sim F(1,2)$$

8.
$$\Rightarrow \frac{\sum_{i=1}^{3}(X_i-Y_1)^2}{\sum_{i=4}^{6}(X_i-Y_2)^2} \sim \frac{\chi^2(2)}{\chi^2(2)} \sim F(2,2)$$

B-3

Suppose a bivariate population $(X,Y) \sim N(0,0;1,1;\rho)$, and let (X_i,Y_i), $i=1,2,\ldots,10$, be a simple random sample drawn from this population. Define the statistic:

$$Z = a\sum_{i=1}^{10}(X_i+Y_i)^2$$

If $Z \sim \chi^2(n)$, find the values of a and n.

Solution:

We are given $(X,Y) \sim N(0,0;1,1;\rho)$, and define:

$$Z = a\sum_{i=1}^{10}(X_i + Y_i)^2$$

Let $W_i = X_i + Y_i$. Since X_i, Y_i are jointly normal,

$$W_i \sim N(0, \mathrm{Var}(X_i + Y_i)) = N(0, 2(1+\rho))$$

So,

$$W_i^2 = 2(1+\rho)Z_i^2 \quad \text{with } Z_i \sim N(0,1)$$

Then:

$$Z = a\sum_{i=1}^{10} W_i^2 = a \cdot 2(1+\rho)\sum_{i=1}^{10} Z_i^2 = a \cdot 2(1+\rho) \cdot \chi^2(10)$$

To make $Z \sim \chi^2(n)$, we must have:

$$a \cdot 2(1+\rho) = 1 \Rightarrow a = \frac{1}{2(1+\rho)}, \quad n = 10$$

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B-5

Suppose X satisfies a standard Laplacian distribution. Randomly draws 10 examples. Solve the expectation of statistics $\bar{X}, S^2$ and the variance of $\bar{X}$.

Solution:

For $\bar{X}$, there exists some pre-deduced conclusion.

$$\mathbb{E}[\bar{X}] = \mu, \mathbb{E}[S^2] = \sigma^2$$

The variance of $\bar{X}$ is also clear via elementary calculation.

$$Var[\bar{X}] = \frac{\sigma^2}{n}$$

And since the expectation and the variance of a standard Laplacian distribution are:

$$\mu = 0, \sigma^2 = 2$$

Hence:

$$\mathbb{E}[\bar{X}] = 0, Var[\bar{X}] = 0.2, \mathbb{E}[S^2] = 2$$

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B-7

Suppose a population $X \sim Exp(\lambda)$. Exact 10 samples. Solve the expectation and the variance of $\bar{X}$ and $X_{(1)} = \min\{X_1 \ldots X_{10}\}$.

Solution:

1.
$$\mathbb{E}[\bar{X}] = \mu = \frac{1}{\lambda}, Var[\bar{X}] = \frac{1}{10\lambda^2}$$

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2.
$$F_{X_{(1)}}(x) = 1 - [1 - F_{X_i}(x)]^{10} = 1 - e^{-10\lambda x}, x > 0$$
$$\rightarrow f_{X_{(1)}}(x) = 10\lambda e^{-10\lambda x}, x > 0$$

Which is to say:
$$X_{(1)} \sim Exp(10\lambda)$$

Therefore:
$$\mathbb{E}[X_{(1)}] = \frac{1}{10\lambda}, Var[X_{(1)}] = \frac{1}{100\lambda^2}$$

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B-9

Let $X_1, X_2, \ldots, X_8$ be a random sample from a standard normal distribution. An additional observation X_9 is taken independently from the same distribution. Determine the distribution of:

$$Y = \frac{2\sqrt{2}}{3} \cdot \frac{X_9 - \overline{X}}{S}$$

Solution:

Since $X_1, \ldots, X_8 \sim N(0,1)$, the sample mean is:
$$\overline{X} \sim N\left(0, \frac{1}{8}\right).$$

Thus,
$$X_9 - \overline{X} \sim N\left(0, 1 + \frac{1}{8}\right) = N\left(0, \frac{9}{8}\right).$$

Standardizing:
$$\frac{X_9 - \overline{X}}{\sqrt{\frac{9}{8}}} = \frac{2\sqrt{2}}{3}(X_9 - \overline{X}) \sim N(0,1).$$

The sample variance (with $n = 8$) follows:
$$S^2 \sim \frac{\chi^2(7)}{7}.$$

Therefore, $S = \sqrt{S^2}$ is the sample standard deviation.

Rewrite Y as:

$$Y = \frac{\frac{2\sqrt{2}}{3}(X_9 - \overline{X})}{S} = \frac{Z}{S},$$

where $Z \sim N(0, 1)$ and is independent of S (since X_9 is independent of $X_1, \ldots, X_8$). The ratio $\frac{Z}{S}$ follows a Student's t-distribution with 7 degrees of freedom:

$$Y \sim t(7)$$

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B-10

Suppose that two separate i.i.d. series $X_1 \ldots X_5$ and $Y_1 \ldots Y_9$ are risen from the same population $X \sim N(\mu, \sigma^2)$. Solve a in

$$\frac{a(\bar{X} - \bar{Y})}{\sigma} \sim N(0, 1)$$

and b in

$$\frac{b(\bar{X} - \bar{Y})}{\sqrt{S_1^2 + 2S_2^2}} \sim t(12)$$

Solution:

1. Since

$$\frac{\bar{X} - \bar{Y}}{\sigma} \sim N(0, \frac{1}{5} + \frac{1}{9})$$

   It is confident to say that:

$$a = \pm \frac{1}{\sqrt{\frac{1}{5} + \frac{1}{9}}} = \pm \frac{3}{14}\sqrt{70}$$

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2. Likewise:

$$\frac{4S_1^2 + 8S_2^2}{\sigma^2} \sim \chi^2(12)$$

   Therefore:

$$b = \pm \frac{\sqrt{3 \times 45}}{\sqrt{14}} = \pm \frac{3}{14}\sqrt{210}$$

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B-12

Let the population $X \sim \chi^2(n)$. Extract 16 samples. Solve the probability of certain statistics.

Solution:

1. $$\frac{\sum_{i=1}^{8} X_i}{\sum_{i=9}^{16} X_i} = \frac{8n\sum_{i=1}^{8} X_i}{8n\sum_{i=9}^{16} X_i} \sim F(8n, 8n)$$

   And since:

   $$F_{1-\alpha}(n_1, n_2) = \frac{1}{F_\alpha(n_2, n_1)}$$

   It is evident that:

   $$P\{\frac{\sum_{i=1}^{8} X_i}{\sum_{i=9}^{16} X_i} \leq 1\} = \alpha = 0.5$$

2. For any continuous distribution, the probability of taking an exact value is zero:

   $$P\{\frac{\sum_{i=1}^{8} X_i}{\sum_{i=9}^{16} X_i} = 1\} = 0$$