HOMEWORK N° 9

Chapter 4

A-12

$X \sim Pois(2), Y \sim B(2, 0.4)$, i.i.d, Solve $\mathbb{E}[2X - Y], Var[2X - Y], \mathbb{E}[(2X - Y)^2]$.

Solution:

$$\mathbb{E}[X] = 2, Var[X] = 2; \mathbb{E}[Y] = 0.8, Var[Y] = 0.48$$

For i.i.d X and Y:

$$\mathbb{E}[2X - Y] = 2\mathbb{E}[X] - \mathbb{E}[Y] = 3.2 \checkmark$$

$$Var[2X - Y] = 4Var[X] + Var[Y] = 8.48 \quad \checkmark$$

$$\mathbb{E}[(2X - Y)^2] = Var(2X - Y) + \mathbb{E}(2X - Y)^2 = 18.72 \checkmark$$

A-16

$Cov(X, Y)$ and ρ_{xy} of a joint distribution $f(x, y)$.

Solution:

$$\mathbb{E}[X] = \int_0^x x\left(\int_x^2 \frac{3x}{4} dy\right)dx = \int_0^2 \frac{3x^2(2-x)}{4} dx = 1$$

$$\mathbb{E}[Y] = \int_0^2 y\left(\int_0^y \frac{3x}{4} dx\right)dy = 1.5$$

Therefore:

$$Cov(X,Y)=\int_0^2\int_x^2(x-1)(y-\frac{3}{2})(\frac{3x}{4})dydx=\frac{1}{10}$$

And because of the variance the two variables bear:

$$Var[X]=\mathbb{E}[X^2]-\mathbb{E}[X]^2=\int_0^2\frac{3}{4}x^3(2-x)dx-1=\frac{1}{5}$$

$$Var[Y]=\int_0^2\frac{3}{8}y^4dy-(\frac{3}{2})^2=\frac{3}{20}$$

The correlation coefficient of which is:

$$\rho_{xy}=\frac{Cov(X,Y)}{\sqrt{Var[X]}\sqrt{Var[Y]}}=\frac{\sqrt{3}}{3}$$

✓

A-17

Set $(X,Y)\sim N(-1,1:4,4:0.6)$, solve $\mathbb{E}[XY]$.

Solution:

$$\mathbb{E}[XY]=Cov(X,Y)+\mathbb{E}[X]\mathbb{E}[Y]=\rho_{XY}\sqrt{Var[X]}\sqrt{Var[Y]}+\mathbb{E}[X]\mathbb{E}[Y]=0.6\times\sqrt{4}^2+(-1)\times1=1.4$$

✓

A-18

Articulate whether X,Y are correlated and whether they are individually independent.

Solution:

For whether they are correlated:

$$\mathbb{E}[X]=0.6\times1+0.4\times2=1.4$$

$$\mathbb{E}[Y]=0.4\times1+0.3\times2=1$$

$$\mathbb{E}[XY]=0.4\times1+0.1\times2+0.2\times4=1.4$$

$$Cov(X,Y)=\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]=0,\ \text{Not correlated}$$

✓

For whether they are individually independent:

$$\text{No. For } P(X=0,Y=1)=0.1\neq P(X=0)P(Y=1)=0.18$$

✓

A-19

$X, Y \sim N(-1, 1; 4, 4; 0.6)$. Derive a that makes $X + Y$ and $X - aY$ independent.

Solution:

Set,

$$\mathbf{x} = \begin{pmatrix} X \\ Y \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 1 & -a \end{pmatrix}$$

The distribution could be formulated in matrix form:

$$\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Where:

$$\boldsymbol{\mu} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 4 & 2.4 \\ 2.4 & 4 \end{pmatrix}$$

Therefore, the transferred two variables satisfy:

$$\mathbf{y} = \mathbf{C}\mathbf{x} \sim N(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{C})$$

Since the correlation and the independence are strictly equivalent in normal distribution, one would only be required to pay attention to the newly-formed variance matrix.

$$\boldsymbol{\Sigma}_{\boldsymbol{y}} = \mathbf{C}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{C} = \begin{pmatrix} 12.8 & 6.4(1-a) \\ 6.4(1-a) & 4a^2 - 4.8a + 4 \end{pmatrix}$$

Therefore, to make them irrelevant:

$$Cov(X, Y) = \boldsymbol{\Sigma}_{\boldsymbol{y}}(1, 2) = 6.4(1 - a) = 0$$

Which give us an unsurprising result:

$$a = 1 \quad \checkmark$$

B-20

Solve the mean and the variance of a Laplace distribution.

Solution:

$$X \sim Laplace(0, 1)$$

Therefore:

$$\mathbb{E}[X] = 0 \quad \checkmark$$

With:

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2(\frac{1}{2}e^{-|x|})dx = \int_{0}^{+\infty} x^2e^{-x}dx = \mathcal{F}(s)|_{s=1} = \frac{2!}{s^3}|_{s=1} = 2$$

The variance of which is:

$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2$$

In fact, for Laplace distribution, there always exist such a rule:

$$\text{For } X \sim Laplace(\mu, b), \mathbb{E}[X] = \mu, Var[X] = 2b^2$$

For $Y = |X|$:

$$f_Y(y) = f_X(y) + f_X(-y) = e^{-y}, y \geq 0$$

$$Y \sim Exp(1), \mathbb{E}[Y] = 1, Var[Y] = 1$$

B-22

$X \sim B(1, 0.5)$,$Y \sim B(1, 0.5)$, i.i.d.

Solution:

Therefore, according to the rule of Bernoulli distribution:

$$X + Y \sim B(2, 0.5)$$

Hence:

$$P(X + Y \geq 1) = 1 - P(X + Y = 0) = 1 - 0.5^2 = 0.75$$

Their joint distribution $P(X = i, Y = j)$ is:

$$P(X = i, Y = j) = 0.25, \forall(i, j)$$

Hence:

$$\mathbb{E}[(-1)^Y X] = 0.25 \times [1 + (-1)] = 0$$

$$Var[(-1)^Y X] = \mathbb{E}[X^2] = 0.5$$

B-24

Is the X and $Y = |X|$ correlated in the B-20?Independence?

Solution:

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

With:

$$\mathbb{E}[XY] = \mathbb{E}[X|X|] = 0, \text{Since the distribution is a even function}$$

Therefore:

$$\rho_{xy} = Cov(X,Y) = 0$$

They aren't correlated. Yet they aren't independent, either, for:

$$f(x,y) = f_X(x)\delta(y-|x|) \neq f_X(x)f_Y(y)$$

B-25

Judge whether X, Y and X^2, Y^2 is i.i.d.

Solution:

1. For X, Y:

$$f_X(x) = \int_{-1}^{1} \frac{1}{4}(1+xy)dy = \frac{1}{2}, x \in (-1,1)$$
$$f_Y(y) = \frac{1}{2}, y \in (-1,1)$$
$$\mathbb{E}[X] = 0, \mathbb{E}[Y] = 0$$
$$Var[X] = \mathbb{E}[X^2] = \frac{1}{3} = Var[Y]$$
$$\mathbb{E}[XY] = \int_{-1}^{1}\int_{-1}^{1} xy\frac{1}{4}(1+xy)dydx = \frac{1}{9}$$
$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{9}$$
$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var[X]}\sqrt{Var[Y]}} = \frac{1}{3}$$

   Thus, they are not independent and positively correlated.

2. For X^2, Y^2:

$$\mathbb{E}[X^2] = \frac{1}{3} = \mathbb{E}[Y^2]$$
$$\mathbb{E}[X^2Y^2] = \int_{-1}^{1}\int_{-1}^{1} x^2y^2\frac{1}{4}(1+xy)dydx = \frac{1}{9}$$
$$Cov(X,Y) = \mathbb{E}[X^2Y^2] - \mathbb{E}[X^2]\mathbb{E}[Y^2] = 0 = \rho_{X^2Y^2}$$

   And since:

$$P(X^2 \leq u, Y^2 \leq v) = \int_{-\sqrt{u}}^{-\sqrt{u}}\int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{4}(1+xy)dydx = \sqrt{uv} = P(X^2 \leq u)P(Y^2 \leq v)$$

   It is confident that they are i.i.d.

B-31

Suppose there are two boxes, Box A and Box B, each containing 2 white balls and 3 black balls. First, randomly take one ball from Box A and put it into Box B. Then, randomly take out one ball from Box B. Let X and Y denote the number of white balls taken from Box A and Box B, respectively. Judge whether they are i.i.d.

Solution:

		Y		$p_{i\cdot}$
		0	1	
X	0	0.4	0.2	0.6
	1	0.2	0.2	0.4
$p_{\cdot j}$		0.6	0.4	

Extremely evident that X, Y are not independent. $\checkmark$

$$\mathbb{E}[X] = 0.4, \mathbb{E}[Y] = 0.4, \mathbb{E}[XY] = 0.2$$

$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 0.24 = Var[Y]$$

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.04$$

$$\rho_{xy} = \frac{Cov(X, Y)}{\sqrt{Var[X]}\sqrt{Var[Y]}} = \frac{1}{6}$$

Therefore, they are positively correlated.

$\checkmark$