HOMEWORK N° #

The 1st Assignment of Robot Modeling and Control

2-1

Prove that $|RP| = |P|$.

Solution:

By definition, the special orthogonal group $SO(3)$ consists of all 3×3 real matrices R satisfying:

$$R^T R = I, \quad \det(R) = 1$$

Such matrices R are called rotation matrices. The length (norm) of a vector $P \in \mathbb{R}^3$ is given by:

$$|P| = \sqrt{P^T P}$$

We consider the length of the transformed vector RP:

$$|RP| = \sqrt{(RP)^T (RP)}$$

Expand the expression above:

$$|RP| = \sqrt{(RP)^T (RP)} = \sqrt{P^T R^T R P}$$

Since $R \in SO(3)$, we have $R^T R = I$. Thus:

$$P^T (R^T R) P = P^T I P = P^T P = |P|^2$$

Taking the square root (and noting lengths are non-negative):

$$|RP| = |P|$$

This proves that any rotation $R \in SO(3)$ preserves lengths of vectors, as expected from the geometric meaning of rotations.

2-2

Elaborate the $(1,4)$ element of $\mathbf{T}_B^A$.

Solution:

5.The x coordinate of point B in the coordinate system A.

2-3

Solve $\mathbf{O}_A^B$.

Solution:

Since,

$$\mathbf{O}_A^B = -\mathbf{R}_A^B \mathbf{O}_B^A$$

And

$$\mathbf{R}_A^B = (\mathbf{R}_B^A)^T = \begin{pmatrix} 0.25 & 0.87 & 0.43 \\ 0.43 & -0.50 & 0.75 \\ 0.86 & 0.00 & -0.50 \end{pmatrix}$$

One could easily get the answer.

$$\mathbf{O}_A^B = -\begin{pmatrix} 0.25 & 0.87 & 0.43 \\ 0.43 & -0.50 & 0.75 \\ 0.86 & 0.00 & -0.50 \end{pmatrix} \begin{pmatrix} 5.0 \\ -4.0 \\ 3.0 \end{pmatrix} = \begin{pmatrix} -0.94 \\ 6.40 \\ 2.80 \end{pmatrix}$$

2-4

Solve $\mathbf{R}_{y'x'z'}(\alpha,\beta,\gamma)$ and $\mathbf{R}_{x'z'x'}(\alpha,\beta,\gamma)$.

Solution:

$$\begin{aligned}
\mathbf{R}_{y'x'z'}(\alpha,\beta,\gamma) &= \begin{pmatrix} c_\alpha & 0 & s_\alpha \\ 0 & 1 & 0 \\ -s_\alpha & 0 & c_\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\beta & -s_\beta \\ 0 & s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.5 & 0 & 0.87 \\ 0 & 1 & 0 \\ -0.87 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.87 & 0.5 \\ 0 & -0.5 & 0.87 \end{pmatrix} \begin{pmatrix} -0.7 & -0.7 & 0 \\ 0.7 & -0.7 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -0.66 & -0.05 & 0.75 \\ 0.61 & -0.61 & 0.50 \\ 0.44 & 0.79 & 0.43 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{x'z'x'}(\alpha,\beta,\gamma) &= \begin{pmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\beta & -s_\beta \\ 0 & s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.5 & 0.87 & 0 \\ -0.87 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.87 & 0.5 \\ 0 & -0.5 & 0.87 \end{pmatrix} \begin{pmatrix} -0.7 & -0.7 & 0 \\ 0.7 & -0.7 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -0.88 & 0.18 & -0.43 \\ -0.31 & -0.92 & 0.25 \\ -0.35 & 0.35 & 0.87 \end{pmatrix}
\end{aligned}$$

2-5

Solve $\mathbf{R}_B^A$.

Solution:

$$\begin{aligned}
\mathbf{R}_B^A &= \mathbf{R}_{z,\theta_1}\mathbf{R}_{x,\theta_2} \\
&= \begin{pmatrix} c_{\theta_1} & -s_{\theta_1} & 0 \\ s_{\theta_1} & c_{\theta_1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\theta_2} & -s_{\theta_2} \\ 0 & s_{\theta_2} & c_{\theta_2} \end{pmatrix} \\
&= \begin{pmatrix} c_{\theta_1} & -s_{\theta_1}c_{\theta_2} & s_{\theta_1}s_{\theta_2} \\ s_{\theta_1} & c_{\theta_1}c_{\theta_2} & -c_{\theta_1}s_{\theta_2} \\ 0 & s_{\theta_2} & c_{\theta_2} \end{pmatrix}
\end{aligned}$$

2-6

Prove the equation given by $z-y-z$, $\beta \in [0, \pi]$.

Solution:

The equation:

$$\mathbf{R}_z(\pm\pi+\alpha)\mathbf{R}_y(-\beta)\mathbf{R}_y(\pm\pi+\gamma) = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma)$$

Proof

$$\begin{aligned}\mathbf{R}_z(\pm\pi+\alpha)\mathbf{R}_y(-\beta)\mathbf{R}_y(\pm\pi+\gamma) &= \begin{pmatrix}-c_\alpha & s_\alpha & 0\\ -s_\alpha & -c_\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}c_\beta & 0 & -s_\beta\\ 0 & 1 & 0\\ s_\beta & 0 & c_\beta\end{pmatrix}\begin{pmatrix}-c_\gamma & s_\gamma & 0\\ -s_\gamma & -c_\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \mathbf{R}_z(\alpha)(\mathbf{diag}_{(-1,-1,1)}\mathbf{R}_y(-\beta)\mathbf{diag}_{(-1,-1,1)})\mathbf{R}_z(\gamma)\\ &= \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma)\end{aligned}$$

2-7

Calculate $\mathbf{R}_K(\theta)$.

Solution:

Under tedious calculation would one obtain such a equation,

$$\begin{aligned}\mathbf{R}_K(\theta) &= \begin{pmatrix}k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta\\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta\\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_y^2 v_\theta + c_\theta\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{3} & \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3}\\ \frac{1+\sqrt{3}}{3} & \frac{1}{3} & \frac{1-\sqrt{3}}{3}\\ \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1}{3}\end{pmatrix}\end{aligned}$$

2-8

Calculate a series of matrices.

Solution:

The transition matrix is disassembled as follows,

$$\mathbf{T}_1\mathbf{T}_B^A\mathbf{T}_2 = \begin{pmatrix} \mathbf{I} & \mathbf{Q}^A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_K(\theta_1) & 0 \\ 0 & 1 \end{pmatrix} \mathbf{T}_B^A \begin{pmatrix} \mathbf{I} & \mathbf{P}^B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_L(\theta_2) & 0 \\ 0 & 1 \end{pmatrix}$$

Hence,

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{R}_K(\theta_1) & \mathbf{Q}^A \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{T}_2 = \begin{pmatrix} \mathbf{R}_L(\theta_2) & \mathbf{P}^B \\ 0 & 1 \end{pmatrix}$$

Thus,

$$\mathbf{Q}^A = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}, \mathbf{P}^B = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

The rotation matrix could be calculated through the expression below,

$$\theta_1 = \arccos\left(\frac{tr(\mathbf{R}_K(\theta_1)) - 1}{2}\right)$$

$$K_A = \frac{1}{2\sin(\theta_1)} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

Thus,

$$\theta_1 = 0.74, \theta_2 = 0.52$$

$$\mathbf{K}^A = \begin{pmatrix} 0.69 \\ -0.19 \\ 0.69 \end{pmatrix}, \mathbf{L}^B = \begin{pmatrix} 0.58 \\ 0.58 \\ 0.58 \end{pmatrix}$$

2-9,2-10

Deduce equivalent rotation matrix for small θ and prove that two infinite small rotation matrix is swap-able when multiply.

Solution:

Directly substituting the would one get,

$$\mathbf{R}_K(\theta) = \begin{pmatrix} 1 & -\theta k_z & \theta k_y \\ \theta k_z & 1 & -\theta k_x \\ -\theta k_y & \theta k_x & 1 \end{pmatrix}$$

and since the absolute element of $\mathbf{R}_K(\theta)$ is symmetrical, it is relatively easy to deduce that,

$$\mathbf{R}_K(\theta_1)\mathbf{R}_K(\theta_2) = \mathbf{R}_K(\theta_2)\mathbf{R}_K(\theta_1)$$

Q.E.D

2-11

Prove that the product of two unit quaternions are still unit quaternion.

Solution:

A quaternion $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ has **conjugate**

$$\overline{q} \;=\; a \;-\; b\,\mathbf{i} \;-\; c\mathbf{j} \;-\; d\,\mathbf{k},$$

and its **norm** satisfies

$$\|q\|^2 \;=\; q\,\overline{q}.$$

For two quaternions q_1, q_2, we use $\overline{q_1\,q_2} = \overline{q_2}\,\overline{q_1}$. Then

$$\|q_1q_2\|^2 \;=\; (q_1q_2)\,\overline{(q_1q_2)} \;=\; (q_1q_2)\,(\overline{q_2}\,\overline{q_1}) \;=\; q_1\,(q_2\,\overline{q_2})\,\overline{q_1} \;=\; \|q_2\|^2\,(q_1\,\overline{q_1}) \;=\; \|q_1\|^2\,\|q_2\|^2.$$

Taking square roots yields

$$\|\,q_1q_2\| \;=\; \|q_1\|\;\|q_2\|.$$

Hence if both q_1 and q_2 are **unit quaternions** (i.e. $\|q_1\| = \|q_2\| = 1$), their product $q_1\,q_2$ also has norm 1 and remains a **unit quaternion**.

2-12

Prove that the reciprocal of a unit quaternion is its conjugate.

Solution:

$$\overline{q}q = q\overline{q} = \|q\|^2 = 1$$

Append. 1

Solve $\mathbf{T}_C^B$.

Solution:

$$\begin{aligned}\mathbf{T}_C^B &= \mathbf{T}_A^B\mathbf{T}_U^A\mathbf{T}_C^U \\ &= \mathbf{T}_A^B(\mathbf{T}_A^U)^{-1}(\mathbf{T}_U^C)^{-1} \\ &= \begin{pmatrix} 0.500 & 0.750 & 0.433 & -6.575 \\ -0.750 & 0.625 & -0.216 & 19.788 \\ -0.433 & -0.216 & 0.875 & -28.318 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

Append. 2

Prove that for any $R \in SO(3)$,$|R| = 1$.

Solution:

Since,

$$\mathbf{R} = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$$
$$c_3 = c_1 \times c_2$$

The determinant of R is,

$$|\mathbf{R}| = (c_1 \times c_2) \cdot c_3 = \|c_3\|^2 = 1$$

Append. 3

Solve several problems relating to Euler's parameter.

Solution:

Its Graßmann product is determined by following expression,

$$\begin{pmatrix} at - bx - cy - dz & ax + bt + cz - dy & ay - bz + ct + dx & az + bt - cx + dt \end{pmatrix}^T$$

hence its value is,

$$\begin{pmatrix} \tau \\ \rho \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{8} - \frac{\sqrt{6}}{24} & \frac{\sqrt{2}-\sqrt{6}}{8} & -\frac{\sqrt{2}}{4} - \frac{5\sqrt{6}}{24} & \frac{\sqrt{2}}{8} - \frac{\sqrt{6}}{6} \end{pmatrix}^T$$

From those quaternions could one conclude,

$$\eta = 2\arccos 0 = \pi, \varepsilon = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})^T, \mathbf{R}_\varepsilon(\eta) = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

The same goes for $\mathbf{R}_\sigma(\xi)$ and $\mathbf{R}_\rho(\tau)$,

$$\mathbf{R}_\sigma(\xi) = \begin{pmatrix} -\frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ \frac{1}{8} & \frac{3}{16} & -\frac{9\sqrt{3}}{16} \\ -\frac{3\sqrt{3}}{8} & \frac{7\sqrt{3}}{16} & \frac{1}{16} \end{pmatrix}, \mathbf{R}_\rho(\tau) = \begin{pmatrix} -\frac{2+3\sqrt{3}}{12} & \frac{7\sqrt{3}-2}{24} & \frac{1-10\sqrt{3}}{24} \\ \frac{11-6\sqrt{3}}{24} & \frac{17+14\sqrt{3}}{48} & \frac{2+13\sqrt{3}}{48} \\ \frac{14+3\sqrt{3}}{24} & \frac{26-7\sqrt{3}}{48} & -\frac{1+14\sqrt{3}}{48} \end{pmatrix}$$

And it is easy to verify that $\mathbf{R}_\rho(\tau) = \mathbf{R}_\varepsilon(\eta)\mathbf{R}_\sigma(\xi)$.