HOMEWORK N° 5

The 5th Assignment of Robot Modeling and Control

5-5

Solve the transformation matrix that turns the $z-y-x$ Euler angle speed vector to the actual rigid body's angular speed vector.

Solution:

The overall w could be represented as three angular velocity directly caused by the presence of Euler angles.

$$\omega = \omega_\alpha + \omega_\beta + \omega_\gamma$$

With each have its transformation expression (the coordinate mark was neglected due to good reason, or it would be a chaotic symbolic confusion),

$$\omega_\alpha = \mathbf{R}_z^{zyx} \begin{pmatrix} 0 \\ 0 \\ \dot{\alpha} \end{pmatrix}, \omega_\beta = \mathbf{R}_{zy}^{zyx} \begin{pmatrix} 0 \\ \dot{\beta} \\ 0 \end{pmatrix}, \omega_\gamma = \begin{pmatrix} \dot{\gamma} \\ 0 \\ 0 \end{pmatrix}$$

Implying the result,

$$\begin{aligned}
\omega &= (\mathbf{R}_{y,\beta}\mathbf{R}_{x,\gamma})^{-1}\begin{pmatrix}0\\0\\\dot{\alpha}\end{pmatrix} + \mathbf{R}_{x,\gamma}^{-1}\begin{pmatrix}0\\\dot{\beta}\\0\end{pmatrix} + \begin{pmatrix}\dot{\gamma}\\0\\0\end{pmatrix}\\
&= \begin{pmatrix}c_\beta & 0 & -s_\beta\\ s_\beta s_\gamma & c_\gamma & c_\beta s_\gamma\\ s_\beta c_\gamma & -s_\gamma & c_\beta c_\gamma\end{pmatrix}\begin{pmatrix}0\\0\\\dot{\alpha}\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & c_\gamma & s_\gamma\\ 0 & -s_\gamma & c_\gamma\end{pmatrix}\begin{pmatrix}0\\\dot{\beta}\\0\end{pmatrix} + \begin{pmatrix}\dot{\gamma}\\0\\0\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & -s_\beta\\ 0 & c_\gamma & c_\beta s_\gamma\\ 0 & -s_\gamma & c_\beta c_\gamma\end{pmatrix}\begin{pmatrix}\dot{\gamma}\\\dot{\beta}\\\dot{\alpha}\end{pmatrix}
\end{aligned}$$

Flipping the matrix horizontally would one obtain the final result.

$$\mathbf{B}(\Psi) = \begin{pmatrix}-s_\beta & 0 & 1\\ c_\beta s_\gamma & c_\gamma & 0\\ c_\beta c_\gamma & -s_\gamma & 0\end{pmatrix}$$

5-7

Four deduction method on the Jacobian matrix of a 3-DOF robot arm.

Solution:

1. By velocity transmission. Other than,

$$\mathbf{R}_1^2 = \begin{pmatrix}c_2 & 0 & s_2\\ -s_2 & 0 & c_2\\ 0 & -1 & 0\end{pmatrix}, \mathbf{R}_3^4 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

The revolution matrices are,

$$\mathbf{R}_i^{i+1} = \begin{pmatrix}c_{i+1} & s_{i+1} & 0\\ -s_{i+1} & c_{i+1} & 0\\ 0 & 0 & 1\end{pmatrix}, \omega_{i+1}^{i+1} = \mathbf{R}_i^{i+1}\omega_i^i + \dot{\theta}_{i+1}\hat{Z}_{i+1}^{i+1}, v_{i+1}^{i+1} = \mathbf{R}_i^{i+1}(v_i^i + \omega_i^i \times P_{i+1}^i)$$

Thus recursively solving the equations,

$$v_0^0 = 0, \omega_0^0 = 0; v_1^1 = 0, \omega_1^1 = \begin{pmatrix}0\\0\\\dot{\theta}_1\end{pmatrix}; v_2^2 = \begin{pmatrix}0\\0\\-\dot{\theta}_1\end{pmatrix}, \omega_2^2 = \begin{pmatrix}s_2\dot{\theta}_1\\c_2\dot{\theta}_1\\\dot{\theta}_2\end{pmatrix}$$

$$v_3^3 = \begin{pmatrix}0.6s_3\dot{\theta}_2\\0.6c_3\dot{\theta}_2\\-(1+0.6c_2)\dot{\theta}_1\end{pmatrix}, \omega_3^3 = \begin{pmatrix}s_{23}\dot{\theta}_1\\c_{23}\dot{\theta}_1\\\dot{\theta}_2+\dot{\theta}_3\end{pmatrix}$$

$$v_4^4 = \begin{pmatrix} 0.6s_3\dot{\theta}_2 \\ (0.2+0.6c_3)\dot{\theta}_2 + 0.2\dot{\theta}_3 \\ -(1+0.6c_2+0.2c_{23})\dot{\theta}_1 \end{pmatrix}, \omega_4^4 = \omega_3^3$$

And applying the rotation matrix $\mathbf{R}_4^0$ in the front to get v_4^0, ω_4^0,

$$\mathbf{R}_4^0 = \begin{pmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{pmatrix}, \omega_0^4 = \begin{pmatrix} s_1(\dot{\theta}_2+\dot{\theta}_3) \\ -c_1(\dot{\theta}_2+\dot{\theta}_3) \\ \dot{\theta}_1 \end{pmatrix}$$

$$v_4^0 = \mathbf{R}_4^0 v_4^4 = \begin{pmatrix} -(s_1+0.6s_1c_2+0.2s_1c_{23})\dot{\theta}_1 - (0.6c_1s_2+0.2c_1s_{23})\dot{\theta}_2 - 0.2c_1s_{23}\dot{\theta}_3 \\ (c_1+0.6c_1c_2+0.2c_1c_{23})\dot{\theta}_1 - (0.6s_1s_2+0.2s_1s_{23})\dot{\theta}_2 - 0.2s_1s_{23}\dot{\theta}_3 \\ (0.2c_{23}+0.6c_2)\dot{\theta}_2 + 0.2c_{23}\dot{\theta}_3 \end{pmatrix}$$

Therefore, the final Jacobian matrix is,

$$\mathcal{J}(\theta) = \begin{pmatrix} -s_1-0.6s_1c_2-0.2s_1c_{23} & -0.6c_1s_2-0.2c_1s_{23} & -0.2c_1s_{23} & 0 \\ c_1+0.6c_1c_2+0.2c_1c_{23} & -0.6s_1s_2-0.2s_1s_{23} & -0.2s_1s_{23} & 0 \\ 0 & 0.2c_{23}+0.6c_2 & 0.2c_{23} & 0 \\ 0 & s_1 & s_1 & s_1 \\ 0 & -c_1 & -c_1 & -c_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Why the $\mathcal{J}_{w_4} = (s_1, -c_1, 0)^{\mathrm{T}}$ is the direct result of coordinating with solution 4 discussed later.

2. By static force transmission.

$$f_i^i = \mathbf{R}_{i+1}^i f_{i+1}^{i+1}; n_i^i = \mathbf{R}_{i+1}^i n_{i+1}^{i+1} + \mathbf{P}_{i+1}^i \times f_i^i$$

Different from velocity transmission method, the direction of recursion is from tip to bottom.

$$f_4^4 = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, n_4^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; f_3^3 = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, n_3^3 = \begin{pmatrix} 0 \\ -0.2f_z \\ 0.2f_y \end{pmatrix}$$

$$f_2^2 = \begin{pmatrix} c_3f_x - s_3f_y \\ s_3f_x + c_3f_y \\ f_z \end{pmatrix}, n_2^2 = \begin{pmatrix} 0.2s_3f_z \\ -(0.2c_3+0.6)f_z \\ 0.6s_3f_x + (0.6c_3+0.2)f_y \end{pmatrix};$$

$$f_1^1 = \begin{pmatrix} c_{23}f_x - s_{23}f_y \\ -f_z \\ s_{23}f_x + c_{23}f_y \end{pmatrix}, n_1^1 = \begin{pmatrix} (0.2s_{23}+0.6s_2)f_z \\ (0.6s_3 - s_{23})f_x + (0.6c_3 + 0.2 - c_{23})f_y \\ -(1+0.6c_2+0.2c_{23})f_z \end{pmatrix}$$

Thus, the torque provided by joint mechanism itself must be,

$$\tau_i = {n_i^i}^{\mathrm{T}} \cdot \hat{Z}_i^i$$

With its exact value being,

$$\tau_1 = -(1+0.6c_2+0.2c_{23})f_z; \tau_2 = 0.6s_3f_x + (0.6c_3+0.2)f_y; \tau_3 = 0.2f_y$$

Thus,

$$\mathcal{J}(\theta)^4 = \begin{pmatrix} 0 & 0.6s_3 & 0 \\ 0 & 0.6c_3+0.2 & 0.2 \\ -(1+0.6c_2+0.2c_{23}) & 0 & 0 \end{pmatrix}$$

Hence the actual result would be,

$$\mathcal{J}(\theta) = \mathbf{R}_4^0\mathcal{J}(\theta)^4 = \begin{pmatrix} -s_1-0.6s_1c_2-0.2s_1c_{23} & -0.6c_1s_2-0.2c_1s_{23} & -0.2c_1s_{23} \\ c_1+0.6c_1c_2+0.2c_1c_{23} & -0.6s_1s_2-0.2s_1s_{23} & -0.2s_1s_{23} \\ 0 & 0.2c_{23}+0.6c_2 & 0.2c_{23} \end{pmatrix}$$

Worth noting that the matrix is 3×3 wide. It is the same regarding to solution 3 as well.

3. Directly differentiate the kinematic equations.

   Likewise,

$$\mathbf{T}_4^0 = \begin{pmatrix} c_1c_{23} & -c_1s_{23} & s_1 & c_1(1+0.6c_2+0.2c_{23}) \\ s_1c_{23} & -s_1s_{23} & -c_1 & s_1(1+0.6c_2+0.2c_{23}) \\ s_{23} & c_{23} & 0 & 0.6s_2+0.2s_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   Differentiating the $\mathbf{P}$ vector would one obtain,

$$\mathcal{J}(\theta) = \frac{d}{d\theta}\mathbf{P} = \begin{pmatrix} -s_1-0.6s_1c_2-0.2s_1c_{23} & -0.6c_1s_2-0.2c_1s_{23} & -0.2c_1s_{23} \\ c_1+0.6c_1c_2+0.2c_1c_{23} & -0.6s_1s_2-0.2s_1s_{23} & -0.2s_1s_{23} \\ 0 & 0.2c_{23}+0.6c_2 & 0.2c_{23} \end{pmatrix}$$

4. Adopt the vector cross-product construction method.

   Accordingly, for revolute joints,

$$\mathcal{J}_{v_i} = \hat{Z}_i \times (o_n - o_i), \mathcal{J}_{\omega_i} = \hat{Z}_i$$

   Their exact value being,

$$\hat{Z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \hat{Z}_2 = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}, \hat{Z}_3 = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}, \hat{Z}_4 = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}$$

$$o_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, o_2 = \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}, o_3 = \begin{pmatrix} c_1(1+0.6c_2) \\ s_1(1+0.6c_2) \\ 0.6s_2 \end{pmatrix}, o_4 = \begin{pmatrix} c_1(1+0.6c_2+0.2c_{23}) \\ s_1(1+0.6c_2+0.2c_{23}) \\ 0.6s_2+0.2s_{23} \end{pmatrix}$$

   Hence,

$$\mathcal{J}(\theta) = \begin{pmatrix} -s_1-0.6s_1c_2-0.2s_1c_{23} & -0.6c_1s_2-0.2c_1s_{23} & -0.2c_1s_{23} & 0 \\ c_1+0.6c_1c_2+0.2c_1c_{23} & -0.6s_1s_2-0.2s_1s_{23} & -0.2s_1s_{23} & 0 \\ 0 & 0.2c_{23}+0.6c_2 & 0.2c_{23} & 0 \\ 0 & s_1 & s_1 & s_1 \\ 0 & -c_1 & -c_1 & -c_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

From the $|\mathcal{J}(\theta)|$ would one acquire the kinematic singularities of the robot arm,

$$|\mathcal{J}(\theta)| = -\frac{3}{125}s_3(3c_2 + c_{23} + 5) \rightarrow \theta_3 = 0|\pi$$

5-8

Deduction method on the Jacobian matrix of the 2-DOF RP robot arm mentioned before.

Solution:

The end effector position vector $\mathbf{P}$ is,

$$\mathbf{P} = \begin{pmatrix} -s_1 d_2 \\ c_1 d_2 \\ 0 \end{pmatrix}$$

Hence,

$$\mathcal{J}_v = \frac{d}{d\theta}\mathbf{P} = \begin{pmatrix} -c_1 d_2 & -s_1 \\ -s_1 d_2 & c_1 \\ 0 & 0 \end{pmatrix}$$

And the $\mathcal{J}_{w_i}$ yields,

$$\mathcal{J}_{w_1} = \hat{Z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathcal{J}_{w_2} = \mathbf{0}$$

Thus the overall $\mathcal{J}(\theta)$ is,

$$\mathcal{J}(\theta) = \begin{pmatrix} -c_1 d_2 & -s_1 \\ -s_1 d_2 & c_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

No doubt that the linear velocity of end effector could be solved via relations bonded by Jacobian matrix,

$$v = \mathcal{J}_{\square}\dot{\theta} = \begin{pmatrix} -c_1 d_2 \dot{\theta}_1 - s_1 \dot{d}_2 \\ -s_1 d_2 \dot{\theta}_1 + c_1 \dot{d}_2 \\ 0 \end{pmatrix}$$

Its kinematic singularities could be solved hence,

$$(-s_1^2 - c_1^2)d_2 = 0 \rightarrow d_2 = 0$$

5-9

Rinse and repeat on a 3-DOF PRR robot arm.

Solution:

1. By velocity transmission.

$$\mathbf{R}_3^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_2^3 = \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_1^2 = \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_0^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With v_i^i, ω_i^i solved recursively,

$$v_0^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \omega_0^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; v_1^1 = \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix}, \omega_1^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; v_2^2 = \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix}, \omega_2^2 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}$$

$$v_3^3 = \begin{pmatrix} l_2 s_3 \dot{\theta}_2 \\ l_2 c_3 \dot{\theta}_2 \\ \dot{d}_1 \end{pmatrix}, \omega_3^3 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix}; v_4^4 = \begin{pmatrix} l_2 s_3 \dot{\theta}_2 \\ l_2 c_3 \dot{\theta}_2 + l_3(\dot{\theta}_2 + \dot{\theta}_3) \\ \dot{d}_1 \end{pmatrix}, \omega_4^4 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix}$$

Rotate the end one back into coordinate 0,

$$\mathbf{R}_4^0 = \begin{pmatrix} c_{23} & -s_{23} & 0 \\ s_{23} & c_{23} & 0 \\ 0 & 0 & 1 \end{pmatrix}, v_4^0 = \begin{pmatrix} -(l_2 s_2 + l_3 s_{23})\dot{\theta}_2 - l_3 s_{23}\dot{\theta}_3 \\ (l_2 c_2 + l_3 c_{23})\dot{\theta}_2 + l_3 c_{23}\dot{\theta}_3 \\ \dot{d}_1 \end{pmatrix}, \omega_4^0 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix}$$

Therefore,

$$\mathcal{J}(\theta) = \begin{pmatrix} 0 & -(l_2 s_2 + l_3 s_{23}) & -l_3 s_{23} & 0 \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

2. By static force transmission.

Vise versa,

$$f_4^4 = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, n_4^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; f_3^3 = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, n_3^3 = \begin{pmatrix} 0 \\ -l_3 f_z \\ l_3 f_y \end{pmatrix}$$

$$f_2^2 = \begin{pmatrix} f_x c_3 - f_y s_3 \\ f_x s_3 + f_y c_3 \\ f_z \end{pmatrix}, n_2^2 = \begin{pmatrix} l_3 s_3 f_z \\ -(l_2 + l_3 c_3) f_z \\ l_2 s_3 f_x + (l_2 c_3 + l_3) f_y \end{pmatrix}$$

$$f_1^1 = \begin{pmatrix} f_x c_{23} - f_y s_{23} \\ f_x s_{23} + f_y c_{23} \\ f_z \end{pmatrix}, n_1^1 = \begin{pmatrix} (l_3 s_{23} + l_2 s_2) f_z \\ -(l_1 + l_2 c_2 + l_3 c_{23}) f_z \\ (l_1 s_{23} + l_2 s_3) f_x + (l_1 c_{23} + l_2 c_3 + l_3) f_y \end{pmatrix}$$

Thus one obtains the torque and the Jacobian matrix under coordinate 3,

$$\begin{cases} \tau_1 = f_z \\ \tau_2 = l_2 s_3 f_x + (l_2 c_3 + l_3) f_y \\ \tau_3 = l_3 f_y \end{cases}, \mathcal{J}(\theta)^4 = \begin{pmatrix} 0 & l_2 s_3 & 0 \\ 0 & l_2 c_3 + l_3 & l_3 \\ 1 & 0 & 0 \end{pmatrix}$$

And transfer it,

$$\mathcal{J}(\theta) = \mathbf{R}_4^0 \mathcal{J}(\theta)^4 = \begin{pmatrix} 0 & -(l_2 s_2 + l_3 s_{23}) & -l_3 s_{23} \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} \\ 1 & 0 & 0 \end{pmatrix}$$

3. Directly differentiate the kinematic equation.

   For the transfer matrix being,

$$\mathbf{T}_4^0 = \begin{pmatrix} c_{23} & -s_{23} & 0 & l_1 + l_2 c_2 + l_3 c_{23} \\ s_{23} & c_{23} & 0 & l_2 s_2 + l_3 s_{23} \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{J}(\theta) = \frac{d}{d\theta}\mathbf{P} = \begin{pmatrix} 0 & -(l_2 s_2 + l_3 s_{23}) & -l_3 s_{23} \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} \\ 1 & 0 & 0 \end{pmatrix}$$

4. Adopt the vector cross-product construction method. Likewise,

$$\hat{Z}_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, o_1 = \begin{pmatrix} 0 \\ 0 \\ d_1 \end{pmatrix}, o_2 = \begin{pmatrix} l_1 \\ 0 \\ d_1 \end{pmatrix}, o_3 = \begin{pmatrix} l_1 + l_2 c_2 \\ l_2 s_2 \\ d_1 \end{pmatrix}, o_4 = \begin{pmatrix} l_1 + l_2 c_2 + l_3 c_{23} \\ l_2 s_2 + l_3 s_{23} \\ d_1 \end{pmatrix}$$

   Therefore,

$$\mathcal{J}(\theta) = \begin{pmatrix} 0 & -(l_2 s_2 + l_3 s_{23}) & -l_3 s_{23} & 0 \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Its kinematic sigularities are hence solved,

$$|\mathcal{J}(\theta)| = l_2 l_3 s_3 \to \theta_3 = 0|\pi$$