HOMEWORK N° 6

Chapter 4

A2

$f(x)$ known, solve $E(X), P(X > E(X))$.

Solution:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} xf(x)dx = \int_1^3 \frac{x^2}{4}dx = \frac{13}{6} \quad \checkmark$$

$$P(X > \mathbb{E}[X]) = P(X > \frac{13}{6}) = \int_{\frac{13}{6}}^3 \frac{x}{4}dx = \frac{155}{288}$$

$\checkmark$

A4

Solve $E(Z)$ in $Z = XY; Z = min(X,Y); Z = max(X,Y)$.

Solution:

1. $Z = XY$
$$\mathbb{E}[Z] = \sum xyp_{ij} = 0.1 + 2 \times 0.3 + 4 \times 0.2 = 1.5 \quad \checkmark$$
2. $Z = min(X,Y)$
$$\mathbb{E}[Z] = \sum min(x,y)p_{ij} = (0.1 + 0.3) \times 1 + 0.2 \times 2 = 0.8$$
$\checkmark$
3. $Z = max(X,Y)$
$$\mathbb{E}[Z] = \sum max(x,y)p_{ij} = 0.3 \times 1 + 0.7 \times 2 = 1.7$$
$\checkmark$

A6

Solve $E(3X^2 - \frac{1}{3X^2})$

Solution:

For

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

The result would be

$$\mathbb{E}[3X^2 - \frac{1}{3X^2}] = \int_{-\infty}^{\infty} (3x^2 - \frac{1}{3x^2})f(x)dx = \int_0^1 (9x^4 - 1)dx = \frac{4}{5}$$

✓

B2

A soon-to-be university graduate is considering signing an employment contract with a company. The company offers two annual salary options:

1. A fixed annual salary of 100,000 yuan.
2. A base salary of 60,000 yuan, plus a 100,000 yuan performance bonus if performance meets company requirements. If not, there's no bonus. The probability of meeting the performance requirement is 80%.

Which option should he choose? Explain your reasoning.

Solution:

In option 2, the expected salary would be

$$\mathbb{E}[X_2] = 6000 + (0 \times 0.2 + 10000 \times 0.8) = 14000 > \mathbb{E}[X_1] = 10000$$

Therefore, he shall choose option two for greater salary expectations.

✓

B8

Two people, A and B, agree to meet between 8:00 and 9:00 AM. They each arrive at a random time during this hour, independently and uniformly distributed. Find the expected waiting time for the person who arrives first.

Solution:

Let X and Y be the arrival times of A and B, respectively, measured in minutes after 8:00. So $X, Y \sim U[0, 60]$, and independent. The waiting time is $|X - Y|$.

We want to find the expected value of this:

$$\mathbb{E}[|X - Y|]$$

This is a well-known result:

$$\mathbb{E}[|X - Y|] = \frac{1}{60^2} \int_0^{60} \int_0^{60} |x - y| \, dx \, dy = 20$$

✓

B15

From the open interval $(0, 1)$, select n points uniformly and independently at random (where $n \geq 2$). Find the expected distance between the two farthest points, i.e., the expected range of the sample.

Solution:

Let $X_1, X_2, \ldots, X_n \sim \text{Uniform}(0, 1)$ be i.i.d. random variables.
Let:

$$M = \max(X_1, \ldots, X_n)$$

$$m = \min(X_1, \ldots, X_n)$$

We want to compute:

$$\mathbb{E}[M - m] = \mathbb{E}[M] - \mathbb{E}[m]$$

The CDF of $M = \max(X_1, \ldots, X_n)$ is:

$$F_M(x) = P(M \leq x) = P(X_1 \leq x, \ldots, X_n \leq x) = P(X_1 \leq x)^n = x^n$$

So the PDF is:

$$f_M(x) = \frac{d}{dx} F_M(x) = n x^{n-1}, \quad x \in (0, 1)$$

Then:

$$\mathbb{E}[M] = \int_0^1 x \cdot f_M(x) \, dx = \int_0^1 x \cdot n x^{n-1} dx = n \int_0^1 x^n dx = n \cdot \frac{1}{n+1} = \frac{n}{n+1}$$

Similarly, the CDF of $m = \min(X_1, \ldots, X_n)$ is:

$$F_m(x) = P(m \leq x) = 1 - P(X_1 > x)^n = 1 - (1 - x)^n$$

So the PDF is:

$$f_m(x) = \frac{d}{dx} F_m(x) = n(1 - x)^{n-1}, \quad x \in (0, 1)$$

Then:

$$\mathbb{E}[m] = \int_0^1 x \cdot f_m(x) \, dx = n \int_0^1 x(1 - x)^{n-1} dx$$

Use substitution or known beta integral:

$$n \int_0^1 x(1-x)^{n-1} dx = n \cdot \frac{1}{n(n+1)} = \frac{1}{n+1}$$

Therefore:

$$\mathbb{E}[m] = \frac{1}{n+1}$$

$$\mathbb{E}[R] = \mathbb{E}[M-m] = \mathbb{E}[M] - \mathbb{E}[m] = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}$$

Which gives the desired result:

$$\boxed{\mathbb{E}[R] = \frac{n-1}{n+1}}$$

✓