On the Turán number of 1-subdivision of $K_{3,t}$

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Abstract

For a graph H, the 1-subdivision of H, denoted by H', is the graph obtained by replacing the edges of H by internally disjoint paths of length 2. Recently, Conlon, Janzer and Lee (arXiv: 1903.10631) asked the following question: For any integer $s \geq 2$, estimate the smallest t such that $\mathrm{ex}(n, K'_{s,t}) = \Omega(n^{\frac{3}{2}-\frac{1}{2s}})$. In this paper, we consider the case $s = 3$. More precisely, we provide an explicit construction giving

$$\mathrm{ex}(n, K'_{3,30}) = \Omega(n^{\frac{4}{3}}),$$

which reduces the estimation for the smallest value of t from a magnitude of 10^{56} to the number 30. The construction is algebraic, which is based on some equations over finite fields.

Key words and phrases: Turán number, algebraic construction, subdivision.

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1 Introduction

Given a graph H, the Turán number $\mathrm{ex}(n, H)$ is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph. The estimation of $\mathrm{ex}(n, H)$ for various graphs

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H is one of the most important problems in extremal graph theory. For general graph H, the famous Erdős-Stone-Simonovits Theorem [10] gives that

$$\mathrm{ex}(n, H) = (1 - \frac{1}{\chi(H) - 1} + o(1))\binom{n}{2},$$

where $\chi(H)$ is the chromatic number of H. This theorem asymptotically solves the problem when $\chi(H) \geq 3$. However, for bipartite graphs H, it only gives $\mathrm{ex}(n, H) = o(n^2)$. So far, there are only a few bipartite graphs for which the asymptotics of whose Turán numbers are known. A well-known theorem of Kövari, Sós and Turán [18] showed that $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s})$ for any integers $t \geq s$. When $s = 2, 3$, matched lower bounds were found in [3, 9]. For general values of s, we have $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-\frac{1}{s}})$ when $t \geqslant (s-1)! + 1$ [1, 17]. Recently, using the random algebraic method, Bukh [4] gave a new construction of $K_{s,t}$-free graphs which also yields $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-\frac{1}{s}})$, where t is sufficiently large.

The 1-subdivision of a graph H, denoted by H', is the graph obtained by replacing the edges of H by internally disjoint paths of length 2. Recently, the 1-subdivision of graphs received a lot of attention due to a paper of Kang, Kim and Liu [16], where they made the following conjecture about the 1-subdivision of a general bipartite graph.

Conjecture 1.1 ([16]). *Let H be a bipartite graph. If* $\mathrm{ex}(n, H) = O(n^{1+\alpha})$ *for some $\alpha > 0$, then* $\mathrm{ex}(n, H') = O(n^{1+\frac{\alpha}{2}})$.

Apart from being interesting on its own, somewhat surprisingly, Kang, Kim and Liu [16] showed that this seemingly unrelated conjecture implies the rational exponent conjecture as follows.

Conjecture 1.2 ([8]). *For every rational number $r \in [1, 2]$, there exists a graph F with* $\mathrm{ex}(n, F) = \Theta(n^r)$.

For more information on the recent active study of the Turán problem for subdivisions, we refer the readers to [6, 7, 12, 13, 14, 15] and the references therein.

In this paper, we focus on the 1-subdivision of complete bipartite graphs. In [6], Conlon, Janzer and Lee showed that $\mathrm{ex}(n, K'_{s,t}) = O(n^{\frac{3}{2}-\frac{1}{2s}})$ for $2 \leq s \leq t$, which proved Conjecture 1.1 for complete bipartite graphs. Combining the random algebraic construction in [5], they also showed the upper bound is tight when t is sufficiently large compared to s. Since the random algebraic method requires the parameter t to be very large, in the same paper, they asked the following problem:

Problem 1.3. *For any integer $s \geq 2$, estimate the smallest t such that* $\mathrm{ex}(n, K'_{s,t}) = \Omega(n^{\frac{3}{2}-\frac{1}{2s}})$.

The previously known smallest value of t from random algebraic method in [5] is $s^{O(s^2)}$. In particular, when $s = 3$, their construction showed that $t \approx 10^{56}$. Moreover, the random algebraic method falls well short of this due to Lang-Weil bound [19]. The case $s = 2$ amounts to estimating the extremal number of the theta graph $\theta_{4,t}$. Very recently, Verstraëte and Williford [20] gave an algebraic construction which yields $\mathrm{ex}(n, \theta_{4,3}) = \Omega(n^{\frac{5}{4}})$. However deriving a similar bound for $\mathrm{ex}(n, \theta_{4,2})$ is likely to be very difficult, as it would solve the famous open problem of estimating $\mathrm{ex}(n, C_8)$. In this paper, we consider the next case $s = 3$, and prove the following result.

Theorem 1.4. $\mathrm{ex}(n, K'_{3,30}) = \Omega(n^{\frac{4}{3}})$.

Combining with the above upper bound, we have

Corollary 1.5. $\mathrm{ex}(n, K'_{3,30}) = \Theta(n^{\frac{4}{3}})$.

The rest of this paper is organized as follows. In Section 2, we will give some basics about the resultant of polynomials. In Section 3, we prove our main result. Section 4 concludes our paper. All computations have been done by MAGMA [2].

2 Preliminaries

Our main technique is the resultant of polynomials, which has been used in [21]. For the convenience of readers, we recall some basics about the resultant of polynomials, which will be used in the following section. Let $\mathbb{F}$ be a field, and $\mathbb{F}[x]$ be the polynomial ring with coefficients in $\mathbb{F}$.

Definition 2.1. *Let $f(x), g(x) \in \mathbb{F}[x]$ with $f(x) = a_m x^m + \cdots + a_1 x + a_0$ and $g(x) = b_n x^n + \cdots + b_1 x + b_0$, then the resultant of f and g is defined by the determinant of the following $(m+n+2) \times (m+n+2)$ matrix,*

$$\begin{pmatrix}
a_0 & a_1 & \cdots & a_m & & & & \\
 & a_0 & \cdots & a_{m-1} & a_m & & & \\
 & & \cdots & \cdots & \cdots & & & \\
 & & & & & a_0 & \cdots & a_m \\
b_0 & b_1 & \cdots & \cdots & b_n & & & \\
 & b_0 & \cdots & \cdots & b_{n-1} & b_n & & \\
 & & \cdots & \cdots & \cdots & \cdots & & \\
 & & & & b_0 & \cdots & \cdots & b_n
\end{pmatrix},$$

which is denoted by $R(f, g)$.

The resultant of two polynomials has the following property.

Lemma 2.2 ([11]). *If $gcd(f(x), g(x)) = h(x)$, where $deg(h(x)) \geq 1$, then $R(f, g) = 0$. In particular, if f and g have a common root in $\mathbb{F}$, then $R(f, g) = 0$.*

When we consider multivariable polynomials, we can define the resultant similarly, and the above lemma still holds when we fix one variable. For any $f, g \in \mathbb{F}[x_1, \dots, x_n]$, let $R(f, g, x_i)$ denote the resultant of f and g with respect to x_i, then we have $R(f, g, x_i) \in \mathbb{F}[x_1, \dots, x_{i-1}, x_{i+1}, x_n]$.

3 Construction of $K'_{3,30}$-free graphs

In this section, we construct a $K'_{3,30}$-free graph with n vertices and $\Omega(n^{\frac{4}{3}})$ edges. Let $\mathbb{F}_p$ be a finite field, where p is an odd prime with $p \equiv 5 \pmod 6$ and $p > 11$. Let $S = \{x : x \in \mathbb{F}_p, x \in [1, \frac{p-5}{6}]\}$. Then we have the following lemma.

Lemma 3.1. *For any $x, y, z, t \in S$, we have $x + y \neq 0$, $x + y + z + t \neq 0$, $x + 5y \neq 0$, and $x^2 + xy + y^2 \neq 0$.*

Proof. Since $x, y, z, t \in \mathbb{F}_p$ and $x, y, z, t \in [1, \frac{p-5}{6}]$, then we have $x + y \neq 0$, $x + y + z + t \neq 0$, $x + 5y \neq 0$.

Note that $p \equiv 5 \pmod 6$, then -3 is a non-quadratic residue module p. Hence, $x^2+xy+y^2 = (x + \frac{y}{2})^2 + 3(\frac{y}{2})^2 \neq 0$. □

Now we can give our construction.

Construction 3.2. *The graph G_p is defined with vertex set $V := S \times \mathbb{F}_p \times \mathbb{F}_p$, where $x = (x_1, x_2, x_3) \in V$ is joined to $y = (y_1, y_2, y_3) \in V$ if $x \neq y$ and*

$$x_2 + y_3 = x_1 y_1^2,$$
$$x_3 + y_2 = x_1^2 y_1.$$

By fixing a vertex x, it is easy to see that the choice of y_1 determines a unique neighbor y of x except $x = y$. Therefore each vertex has degree at least $\frac{p-11}{6}$. Hence G_p has $n := \frac{(p-5)p^2}{6}$ vertices and at least $\frac{1}{72}(p-5)(p-11)p^2 = \Omega(n^{\frac{4}{3}})$ edges. In the following of this section, we will prove that G_p is $K'_{3,30}$-free.

We begin with the following simple lemma.

Lemma 3.3. *If $x, y \in V$ are distinct and have a common neighbor, then $x_1 \neq y_1$, $x_2 \neq y_2$ and $x_3 \neq y_3$.*

Proof. We will only prove $x_3 \neq y_3$, the others are similar. Suppose x, y have a common neighbor u, then

$$\begin{aligned} x_3 + u_2 &= x_1^2 u_1, \\ y_3 + u_2 &= y_1^2 u_1. \end{aligned}$$

If $x_3 = y_3$, then $u_1(x_1^2 - y_1^2) = 0$. Hence $x_1 = y_1$ or $x_1 + y_1 = 0$. If $x_1 = y_1$, then it is easy to get that $x = y$, which is a contradiction. If $x_1 + y_1 = 0$, this contradicts to the definition of S. Hence $x_3 \neq y_3$. $\square$

For any given $a, b, c \in V$ with a, b, c pairwise distinct, we estimate the number of sequences $(x, y, z, w) \in V^4$ with x, y, z, w pairwise distinct, such that ax, xw, by, yw, cz, zw are edges in G_p. We will prove that there are at most 29 different such sequences. By the definition of graph G_p, we have

$$\begin{aligned} a_2 + x_3 &= a_1 x_1^2, & a_3 + x_2 &= a_1^2 x_1, \\ w_2 + x_3 &= w_1 x_1^2, & w_3 + x_2 &= w_1^2 x_1, \\ b_2 + y_3 &= b_1 y_1^2, & b_3 + y_2 &= b_1^2 y_1, \\ w_2 + y_3 &= w_1 y_1^2, & w_3 + y_2 &= w_1^2 y_1, \\ c_2 + z_3 &= c_1 z_1^2, & c_3 + z_2 &= c_1^2 z_1, \\ w_2 + z_3 &= w_1 z_1^2, & w_3 + z_2 &= w_1^2 z_1. \end{aligned}$$

Cancelling $x_2, x_3, y_2, y_3, z_2, z_3$ from the above equations, we can get the following equations

$$f_1 := a_2 - w_2 - x_1^2(a_1 - w_1) = 0, \tag{1}$$
$$f_2 := a_3 - w_3 - x_1(a_1^2 - w_1^2) = 0, \tag{2}$$
$$f_3 := b_2 - w_2 - y_1^2(b_1 - w_1) = 0, \tag{3}$$
$$f_4 := b_3 - w_3 - y_1(b_1^2 - w_1^2) = 0, \tag{4}$$
$$f_5 := c_2 - w_2 - z_1^2(c_1 - w_1) = 0, \tag{5}$$
$$f_6 := c_3 - w_3 - z_1(c_1^2 - w_1^2) = 0. \tag{6}$$

In the following of this section, we divide our discussions into three subsections.

3.1 $a_1 = b_1$ or $a_1 = c_1$ or $b_1 = c_1$

Without loss of generality, we assume that $a_1 = b_1$. Then Equations (1)-(6) become

$$
\begin{aligned}
f_1 &:= a_2 - w_2 - x_1^2(a_1 - w_1) = 0,\\
f_2 &:= a_3 - w_3 - x_1(a_1^2 - w_1^2) = 0,\\
f_3 &:= b_2 - w_2 - y_1^2(a_1 - w_1) = 0,\\
f_4 &:= b_3 - w_3 - y_1(a_1^2 - w_1^2) = 0,\\
f_5 &:= c_2 - w_2 - z_1^2(c_1 - w_1) = 0,\\
f_6 &:= c_3 - w_3 - z_1(c_1^2 - w_1^2) = 0.
\end{aligned}
$$

We begin with the following lemma.

Lemma 3.4. *If $a_1 = b_1$, then $a_2 \neq b_2$ and $a_3 \neq b_3$.*

Proof. If $a_2 = b_2$, then by the above equations, we have $x_1^2 = y_1^2$, hence $x_1 = y_1$ or $x_1 = -y_1$, which contradicts to Lemmas 3.1 and 3.3.

If $a_3 = b_3$, then by the above equations again, we have $x_1 = y_1$, which contradicts to Lemma 3.1. □

Now we regard f_i $(i = 1, 2, \ldots, 6)$ as polynomials with variables $x_1, y_1, z_1, w_1, w_2, w_3$. By a MAGMA program, we can get that

$$
\begin{aligned}
R(f_1, f_2, x_1) &= g_1 \cdot (a_1 - w_1),\\
R(f_3, f_4, y_1) &= g_2 \cdot (a_1 - w_1),\\
R(f_5, f_6, z_1) &= g_3 \cdot (c_1 - w_1).
\end{aligned}
$$

By Lemma 3.3, we have $a_1 \neq w_1$ and $c_1 \neq w_1$. Then we can compute to get that

$$
\begin{aligned}
R(g_1, g_2, w_2) &= g_4 \cdot (a_1 - w_1)(a_1 + w_1)^2,\\
R(g_1, g_3, w_2) &= g_5.
\end{aligned}
$$

By Lemmas 3.1 and 3.3, $a_1 + w_1 \neq 0$ and $a_1 - w_1 \neq 0$. Let $h = R(g_4, g_5, w_3)$, then h is a polynomial of w_1 with degree 8. We can write h as $h = \sum_{i=0}^{8} h_i w_1^i$. Then we can compute to get that

$$h_8 = (a_2 - b_2)^2(a_1 - c_1).$$

By Lemma 3.4, $a_2 \neq b_2$ and $a_3 \neq b_3$. If $a_1 \neq c_1$, then there are at most 8 solutions for w_1. For any fixed w_1, g_4 is a polynomial of w_3 with degree 1. We write g_4 as $g_4 = s_1 w_3 + s_0$, then $s_1 = a_3 - b_3 \neq 0$. Hence there is at most 1 solution for w_3. If w_1 and w_3 are given, then all the remaining variables are uniquely determined. Hence there are at most 8 different sequences of (x, y, z, w) for this case.

If $a_1 = c_1$, then we can compute to get that $g_5 = g_5' \cdot (a_1 - w_1)(a_1 + w_1)^2$. Let $h' = R(g_4, g_5', w_3)$, then h' is a polynomial of w_1 with degree 3. We can write h' as $h' = \sum_{i=0}^{3} k_i w_1^i$. Regarding k_i $(i = 0, 1, 2, 3)$ as polynomials with variable a_2, then by a MAGMA program, we have

$$R(k_0, k_3, a_2) = (a_3 - b_3)^2(a_3 - c_3)(a_3 - b_3).$$

By Lemma 3.4, $R(k_0, k_3, a_2) \neq 0$. Hence there are at most 3 solutions for w_1. Similarly as above, for any fixed w_1, there is at most 1 solution for w_3. If w_1 and w_3 are given, then all the remaining variables are uniquely determined. Hence there are at most 3 different sequences of (x, y, z, w) for this case.

Therefore, if $a_1 = b_1$ or $a_1 = c_1$ or $b_1 = c_1$, then there are at most 8 different sequences of (x, y, z, w).

3.2 $a_1 \neq b_1$, $a_1 \neq c_1$, $b_1 \neq c_1$ and $a_2 = b_2$ or $a_2 = c_2$ or $b_2 = c_2$

Without loss of generality, we assume that $a_2 = b_2$. Then Equations (1)-(6) become

$$\begin{aligned}
f_1 &:= a_2 - w_2 - x_1^2(a_1 - w_1) = 0,\\
f_2 &:= a_3 - w_3 - x_1(a_1^2 - w_1^2) = 0,\\
f_3 &:= a_2 - w_2 - y_1^2(b_1 - w_1) = 0,\\
f_4 &:= b_3 - w_3 - y_1(b_1^2 - w_1^2) = 0,\\
f_5 &:= c_2 - w_2 - z_1^2(c_1 - w_1) = 0,\\
f_6 &:= c_3 - w_3 - z_1(c_1^2 - w_1^2) = 0.
\end{aligned}$$

Now we regard f_i $(i = 1, 2, \ldots, 6)$ as polynomials with variables $x_1, y_1, z_1, w_1, w_2, w_3$. By a MAGMA program, we can get that

$$\begin{aligned}
R(f_1, f_2, x_1) &= g_1 \cdot (a_1 - w_1),\\
R(f_3, f_4, y_1) &= g_2 \cdot (b_1 - w_1),\\
R(f_5, f_6, z_1) &= g_3 \cdot (c_1 - w_1).
\end{aligned}$$

By Lemma 3.3, we have $a_1 \neq w_1, b_1 \neq w_1$ and $c_1 \neq w_1$. Then we can compute to get that

$$\begin{aligned}
R(g_1, g_2, w_2) &= g_4,\\
R(g_1, g_3, w_2) &= g_5,\\
R(g_4, g_5, w_3) &= h \cdot (a_1 - w_1)^2(a_1 + w_1)^4,
\end{aligned}$$

where h is a polynomial of w_1 with degree 10. By Lemmas 3.1 and 3.3, $a_1 \neq w_1$ and $a_1 + w_1 \neq 0$. We can write h as $h = \sum_{i=0}^{10} h_i w_1^i$. Then we can compute to get that

$$h_{10} = (a_2 - c_2)^2 (a_1 - b_1)^2.$$

If $a_2 \neq c_2$, then there are at most 10 solutions for w_1. For any fixed w_1, g_4 and g_5 are polynomials of w_3 with degree 2. We write g_4 and g_5 as $g_4 = \sum_{i=0}^{2} s_i w_3^i$ and $g_5 = \sum_{i=0}^{2} t_i w_3^i$, then $s_2 = s_2' \cdot (a_1 - b_1)$ and $t_2 = t_2' \cdot (a_1 - c_1)$. We can compute to get that $s_2' - t_2' = (b_1 - c_1)(a_1 + b_1 + c_1 + w_1)$. By Lemmas 3.1 and 3.3, we have $s_2' - t_2' \neq 0$. Hence there is at least one of s_2, t_2 not being 0, then there are at most 2 solutions for w_3. If w_1 and w_3 are given, then all the remaining variables are uniquely determined. Hence there are at most 20 different sequences of (x, y, z, w) for this case.

If $a_2 = c_2$, then $h_i = 0$ for $i = 6, 7, 8, 9, 10$. We can regard h_4 and h_5 as polynomials with variable a_1. Then by a MAGMA program, we have

$$R(h_4, h_5, a_1) = (b_1 - c_1)^2 (a_3 - b_3)^4 (a_3 - c_3)^4 (b_3 - c_3)^4.$$

If $R(h_4, h_5, a_1) \neq 0$, then at least one of h_4, h_5 is not 0. Hence there are at most 5 solutions for w_1. For any fixed w_1, through a similar discussion as above, there are at most 2 solutions for w_3. If w_1 and w_3 are given, then all the remaining variables are uniquely determined. Hence there are at most 10 different sequences of (x, y, z, w) for this case.

If $R(h_4, h_5, a_1) = 0$, without loss of generality, we assume that $a_3 = b_3$. Then we can compute to get that

$$g_4 = (a_3 - w_3)^2 (a_1 - b_1) \cdot g_4',$$

where $g_4' = -w_1^2 + (a_1 + b_1) w_1 + a_1^2 + a_1 b_1 + b_1^2$. It is easy to see that $a_3 \neq w_3$ and $a_1 \neq b_1$, then there are at most 2 solutions for w_1. For any fixed w_1, g_5 is a polynomial of w_3 with degree 2. We write g_5 as $g_5 = \sum_{i=0}^{2} t_i w_3^i$, then $t_2 = t_2' \cdot (a_1 - c_1)$. We can compute to get that $g_4' - t_2' = (b_1 - c_1)(a_1 + b_1 + c_1 + w_1)$. By Lemmas 3.1 and 3.3, we have $g_4' - t_2' \neq 0$. Hence there are at most 2 solutions for w_3. If w_1 and w_3 are given, then all the remaining variables are uniquely determined. Hence there are at most 4 different sequences of (x, y, z, w) for this case.

Therefore, if $a_1 \neq b_1$, $a_1 \neq c_1$, $b_1 \neq c_1$ and $a_2 = b_2$ or $a_2 = c_2$ or $b_2 = c_2$, then there are at most 20 different sequences of (x, y, z, w).

3.3 $a_1 \neq b_1$, $a_1 \neq c_1$, $b_1 \neq c_1$, $a_2 \neq b_2$, $a_2 \neq c_2$ and $b_2 \neq c_2$

For this case, we begin with the following lemmas. A theta graph $\theta_{k,t}$ is a graph made of t internally disjoint paths of length k connecting two endpoints.

Lemma 3.5. *If G_p contains a $\theta_{3,3}$, and the set of edges $\{da, db, dc, ax, by, cz, wx, wy, wz\}$ form a $\theta_{3,3}$, then $w_1 = a_1 + b_1 + c_1$ or $a_1y_1 - a_1z_1 - b_1x_1 + b_1z_1 + c_1x_1 - c_1y_1 = 0$.*

Proof. We first consider the edges da, ax, xw, wy, yb, bd, which form a hexagon. By the definition of G_p, we have

$$d_3 + a_2 = d_1^2a_1, \qquad a_3 + d_2 = a_1^2d_1, \tag{7}$$
$$a_2 + x_3 = a_1x_1^2, \qquad x_2 + a_3 = x_1a_1^2, \tag{8}$$
$$x_3 + w_2 = x_1^2w_1, \qquad w_3 + x_2 = w_1^2x_1, \tag{9}$$
$$w_2 + y_3 = w_1y_1^2, \qquad y_2 + w_3 = y_1w_1^2, \tag{10}$$
$$y_3 + b_2 = y_1^2b_1, \qquad b_3 + y_2 = b_1^2y_1, \tag{11}$$
$$b_2 + d_3 = b_1d_1^2, \qquad d_2 + b_3 = d_1b_1^2. \tag{12}$$

Then we can compute to get that

$$f_1 := d_1^2a_1 - a_1x_1^2 + x_1^2w_1 - w_1y_1^2 + y_1^2b_1 - b_1d_1^2 = 0,$$
$$f_2 := d_1a_1^2 - a_1^2x_1 + x_1w_1^2 - w_1^2y_1 + y_1b_1^2 - b_1^2d_1 = 0,$$

where f_1 is from the left six equations of (7)-(12) and f_2 is from the right six equations of (7)-(12). Regarding f_1, f_2 as polynomials with variables $a_1, u_1, v_1, b_1, x_1, w_1$, we can compute to get that

$$\begin{aligned}R(f_1, f_2, b_1) =&(a_1 - w_1)(x_1 - y_1)(d_1 - y_1)(d_1 - x_1)(d_1^2a_1 + d_1^2w_1 - d_1a_1x_1 - d_1a_1y_1+\\ &d_1x_1w_1 + d_1w_1y_1 - a_1x_1^2 - a_1x_1y_1 - a_1y_1^2 + x_1^2w_1 + x_1w_1y_1 - w_1y_1^2).\end{aligned}$$

By Lemma 3.3 and $f_1 = f_2 = 0$, we have

$$\begin{aligned}&d_1^2a_1 + d_1^2w_1 - d_1a_1x_1 - d_1a_1y_1 + d_1x_1w_1 + d_1w_1y_1 - a_1x_1^2 - a_1x_1y_1-\\ &a_1y_1^2 + x_1^2w_1 + x_1w_1y_1 - w_1y_1^2 = 0.\end{aligned}$$

Similarly, the edges da, ax, xw, wz, zc, cd form a hexagon, we have

$$\begin{aligned}&d_1^2a_1 + d_1^2w_1 - d_1a_1x_1 - d_1a_1z_1 + d_1x_1w_1 + d_1w_1z_1 - a_1x_1^2 - a_1x_1z_1-\\ &a_1z_1^2 + x_1^2w_1 + x_1w_1z_1 - w_1z_1^2 = 0.\end{aligned}$$

From the above two equations, we have

$$(d_1 + x_1)(w_1 - a_1)y_1 - (w_1 + a_1)y_1^2 = (d_1 + x_1)(w_1 - a_1)z_1 - (w_1 + a_1)z_1^2.$$

Then we have

$$\frac{d_1 + x_1}{w_1 + a_1} = \frac{y_1 + z_1}{w_1 - a_1}.$$

By the symmetry of $\theta_{3,3}$, we have

$$\frac{w_1 + a_1}{d_1 + x_1} = \frac{b_1 + c_1}{d_1 - x_1}.$$

From the above two equations, we can get

$$g_1 = (y_1 + z_1)(b_1 + c_1) - (w_1 - a_1)(d_1 - x_1) = 0.$$

By the symmetry of $\theta_{3,3}$ again, we also have

$$g_2 = (x_1 + z_1)(a_1 + c_1) - (w_1 - b_1)(d_1 - y_1) = 0,$$
$$g_3 = (x_1 + y_1)(a_1 + b_1) - (w_1 - c_1)(d_1 - z_1) = 0.$$

Now we regard g_i $(i = 1, 2, 3)$ as polynomials with variables d_1, w_1. By a MAGMA program, we can get that

$$R(g_1, g_2, d_1) = h_1 \cdot (a_1 + b_1 + c_1 - w_1),$$
$$R(g_1, g_3, d_1) = h_2 \cdot (a_1 + b_1 + c_1 - w_1).$$

If $a_1 + b_1 + c_1 - w_1 \neq 0$, then we can compute to get that $R(h_1, h_2, w_1) = (y_1 + z_1)(a_1 y_1 - a_1 z_1 - b_1 x_1 + b_1 z_1 + c_1 x_1 - c_1 y_1) = 0$. Hence $a_1 y_1 - a_1 z_1 - b_1 x_1 + b_1 z_1 + c_1 x_1 - c_1 y_1 = 0$. ☐

Remark 3.6. It is easy to see that if $d_1 \in (\mathbb{F}_p^* \backslash S)$, then Lemma 3.5 still holds.

Lemma 3.7. *If $w_1 = a_1 + b_1 + c_1$, then there are at most 2 different sequences of (x, y, z, w).*

Proof. Substituting $w_1 = a_1 + b_1 + c_1$ into Equations (1)-(6), we have

$$f_1 := a_2 - w_2 + x_1^2(b_1 + c_1) = 0,$$
$$f_2 := a_3 - w_3 - x_1(a_1^2 - (a_1 + b_1 + c_1)^2) = 0,$$
$$f_3 := b_2 - w_2 + y_1^2(a_1 + c_1) = 0,$$
$$f_4 := b_3 - w_3 - y_1(b_1^2 - (a_1 + b_1 + c_1)^2) = 0,$$
$$f_5 := c_2 - w_2 + z_1^2(a_1 + b_1) = 0,$$
$$f_6 := c_3 - w_3 - z_1(c_1^2 - (a_1 + b_1 + c_1)^2) = 0.$$

Now we regard f_i $(i = 1, 2, \ldots, 6)$ as polynomials with variables x_1, y_1, z_1, w_2, w_3. By a MAGMA program, we can get that

$$R(f_1, f_2, x_1) = g_1 \cdot (b_1 + c_1),$$
$$R(f_3, f_4, y_1) = g_2 \cdot (a_1 + c_1),$$
$$R(f_5, f_6, z_1) = g_3 \cdot (a_1 + b_1),$$
$$R(g_1, g_2, w_2) = g_4,$$
$$R(g_1, g_3, w_2) = g_5,$$

where g_4 and g_5 are polynomials of w_3 with degree 2. We write g_4 and g_5 as $g_4 = \sum_{i=0}^{2} s_i w_3^i$ and $g_5 = \sum_{i=0}^{2} t_i w_3^i$, then $s_2 = s_2' \cdot (a_1 - b_1)$ and $t_2 = t_2' \cdot (a_1 - c_1)$. Now we regard s_2', t_2' as polynomials with variable c_1, then $R(s_2', t_2', c_1) = (a_1 - b_1)(a_1 + b_1)(a_1^2 + a_1 b_1 + b_1^2) \neq 0$. Hence there is at least one of s_2, t_2 not being 0, then there are at most 2 solutions for w_3. If w_1 and w_3 are given, then all the remaining variables are uniquely determined. Hence there are at most 2 different sequences of (x, y, z, w) for this case. $\square$

By Lemma 3.3, if d, a, b, c, x, y, z, w form a $\theta_{3,3}$ with edge set $\{da, db, dc, ax, by, cz, wx, wy, wz\}$, then $a_3 \neq b_3$.

Lemma 3.8. *If $a_1 y_1 - a_1 z_1 - b_1 x_1 + b_1 z_1 + c_1 x_1 - c_1 y_1 = 0$ and $a_3 \neq b_3$, then there are at most 27 different sequences of (x, y, z, w).*

Proof. Let $f_7 = a_1 y_1 - a_1 z_1 - b_1 x_1 + b_1 z_1 + c_1 x_1 - c_1 y_1 = 0$, then we regard f_7 and f_i $(i = 1, 2, \ldots, 6)$ in Equations (1)-(6) as polynomials with variables $x_1, y_1, z_1, w_1, w_2, w_3$. Let $g_1 = f_1 - f_3$, $g_2 = f_1 - f_5$ and $g_3 = f_2 - f_4$. By a MAGMA program, we can get that

$$
\begin{aligned}
&R(g_1, g_2, w_1) = g_5,\\
&R(g_1, g_3, w_1) = g_6 \cdot (x_1 - y_1),\\
&R(f_7, g_5, x_1) = h_1 \cdot (y_1 - z_1),\\
&R(f_7, g_6, x_1) = h_2,\\
&R(h_1, h_2, z) = (b_1 - c_1)^3 (a_1 - b_1)^6 (a_1 y_1^2 - b_1 y_1^2 - a_2 + b_2)^2 \cdot s,
\end{aligned}
$$

where s is a polynomial of y_1 with degree 8. Since at most one of $u, -u$ belongs to S, then $a_1 y_1^2 - b_1 y_1^2 - a_2 + b_2 = 0$ has at most 1 solution for y_1. We write $s = \sum_{i=0}^{8} s_i y_1^i$, then we can compute to get that $s_8 = s_8' \cdot (b_1 - c_1)^3 (a_1 - c_1)^4 (a_1 - b_1)^4$ and $s_7 = s_7' \cdot (b_1 - c_1)^3 (a_3 - b_3)(a_1 - c_1)^3 (a_1 - b_1)^3$. We regard s_8' and s_7' as polynomials of a_1, then $R(s_7', s_8', a_1) = (b_1 + c_1)(b_1 + 5c_1)(b_1^2 + b_1 c_1 + c_1^2) \neq 0$. Hence there are at most 9 $(= 8 + 1)$ solutions for y_1 (note that we have proved $a_1 y_1^2 - b_1 y_1^2 - a_2 + b_2 = 0$ has at most 1 solution for y_1 previously). For any given y_1, h_1 is a polynomial of z_1 with degree 3. We write h_1 as $h_1 = \sum_{i=0}^{3} s_i z_1^i$, then $s_3 = (a_1 - c_1)(a_1 - b_1)^2 \neq 0$. Hence, there are at most 3 solutions for z_1. If y_1 and z_1 are given, then all the remaining variables are uniquely determined. Hence there are at most 27 different sequences of (x, y, z, w) for this case. $\square$

Now we regard f_i $(i = 1, 2, \ldots, 6)$ in Equations (1)-(6) as polynomials with variables $x_1, y_1, z_1, w_1, w_2, w_3$. By a MAGMA program, we can get that

$$
\begin{aligned}
&R(f_1, f_2, x_1) = g_1 \cdot (a_1 - w_1),\\
&R(f_3, f_4, y_1) = g_2 \cdot (b_1 - w_1),\\
&R(f_5, f_6, z_1) = g_3 \cdot (c_1 - w_1).
\end{aligned}
$$

By Lemma 3.3, we have $a_1 \neq w_1$ and $c_1 \neq w_1$. Then we can compute to get that

$$\begin{aligned} R(g_1, g_2, w_2) &= g_4, \\ R(g_1, g_3, w_2) &= g_5, \\ R(g_4, g_5, w_3) &= h \cdot (a_1 - w_1)^2 (a_1 + w_1)^4, \end{aligned}$$

where h is a polynomial of w_1 with degree 10. By Lemma 3.1, $a_1 + w_1 \neq 0$. We can write h as $h = \sum_{i=0}^{10} h_i w_1^i$.

If at least one of h_i is not 0, then there are at most 10 solutions for w_1. For any fixed w_1, g_4 and g_5 are polynomials of w_3 with degree 2. We write g_4 and g_5 as $g_4 = \sum_{i=0}^{2} s_i w_3^i$ and $g_5 = \sum_{i=0}^{2} t_i w_3^i$, then $s_2 = s_2' \cdot (a_1 - b_1)$ and $t_2 = t_2' \cdot (a_1 - c_1)$. We can compute to get that $s_2' - t_2' = (b_1 - c_1)(a_1 + b_1 + c_1 + w_1)$. By Lemmas 3.1 and 3.3, we have $s_2' - t_2' \neq 0$. Hence there is at least one of s_2, t_2 not being 0, then there are at most 2 solutions for w_3. If w_1 and w_3 are given, then all the remaining variables are uniquely determined. Hence there are at most 20 different sequences of (x, y, z, w) for this case.

If $h_i = 0$ for $0 \leq i \leq 10$. Now we regard h_i as polynomials with variables b_2, c_2, c_3. We can compute to get that

$$h_{10} = (a_1 b_2 - a_1 c_2 - a_2 b_1 + a_2 c_1 + b_1 c_2 - b_2 c_1)^2.$$

Let $h_{10}' = a_1 b_2 - a_1 c_2 - a_2 b_1 + a_2 c_1 + b_1 c_2 - b_2 c_1$. By a MAGMA program, we can get that

$$\begin{aligned} R(h_{10}', h_8, c_2) &= (a_2 - b_2)(a_1 - b_1) \cdot k_1, \\ R(h_{10}', h_6, c_2) &= (a_2 - b_2)(a_1 - b_1) \cdot k_2, \\ R(k_1, k_2, c_3) &= (b_1 - c_1)^4 (a_1 - c_1)^4 (a_1 - b_1)^4 \cdot r_1^2, \\ R(k_1, k_2, b_2) &= (b_1 - c_1)^2 (a_1 - c_1)^2 (a_1 - b_1) \cdot r_2^2, \end{aligned}$$

where

$$r_1 = a_1^3 a_2 - a_1^3 b_2 + a_1^2 a_2 b_1 - a_1^2 b_1 b_2 - a_1 a_2 b_1^2 + a_1 b_1^2 b_2 - a_2 b_1^3 - a_3^2 + 2 a_3 b_3 + b_1^3 b_2 - b_3^2, \tag{13}$$

$$r_2 = a_1^2 b_3 - a_1^2 c_3 - a_3 b_1^2 + a_3 c_1^2 + b_1^2 c_3 - b_3 c_1^2. \tag{14}$$

We can also compute to get that

$$\begin{aligned} R(h_{10}', h_8, b_2) &= (a_2 - c_2)(a_1 - c_1) \cdot k_3, \\ R(h_{10}', h_6, b_2) &= (a_2 - c_2)(a_1 - c_1) \cdot k_4, \\ R(k_3, k_4, c_3) &= (b_1 - c_1)^4 (a_1 - c_1)^2 (a_1 - b_1)^4 \cdot r_3^2, \end{aligned}$$

where

$$\begin{aligned} r_3 = {} & a_1^4 a_2 - a_1^4 c_2 - 2 a_1^2 a_2 b_1^2 + 2 a_1^2 b_1^2 c_2 - a_1 a_3^2 + 2 a_1 a_3 b_3 - a_1 b_3^2 + a_2 b_1^4 + a_3^2 c_1 - \\ & 2 a_3 b_3 c_1 - b_1^4 c_2 + b_3^2 c_1. \end{aligned} \tag{15}$$

Now we define $d_1 := \frac{a_3-b_3}{a_1^2-b_1^2}$, $d_2 := a_1^2(\frac{a_3-b_3}{a_1^2-b_1^2}) - a_3$, and $d_3 := a_1(\frac{a_3-b_3}{a_1^2-b_1^2})^2 - a_2$. Then by $r_1 = r_2 = r_3 = 0$ (see Equations (13), (14) and (15)), it is easy to check that

$$\begin{aligned} a_2 + d_3 &= a_1 d_1^2, & a_3 + d_2 &= a_1^2 d_1, \\ b_2 + d_3 &= b_1 d_1^2, & b_3 + d_2 &= b_1^2 d_1, \\ c_2 + d_3 &= c_1 d_1^2, & c_3 + d_2 &= c_1^2 d_1. \end{aligned}$$

Then the vertex $d = (d_1, d_2, d_3)$ is a common neighbor of a, b, c. Hence the vertex d, a, b, c, x, y, z, w form a $\theta_{3,3}$, by Lemmas 3.5, 3.7 and 3.8, there are at most 29 different sequences of (x, y, z, w) for this case.

Remark 3.9. Note that d_1 may not be in the set S, and then $d \notin V$, but by Remark 3.6, we still have the same result. Actually, we do not need the notation $\theta_{3,3}$. If the point $d = (d_1, d_2, d_3)$ satisfies the above equations with a, b, c, then we have the statements of Lemma 3.5.

Therefore, if $a_1 \neq b_1$, $a_1 \neq c_1$, $b_1 \neq c_1$, $a_2 \neq b_2$, $a_2 \neq c_2$ and $b_2 \neq c_2$, then there are at most 29 different sequences of (x, y, z, w).

3.4 Proof of Theorem 1.4

From the previous discussions, for any given $a, b, c \in V$, there are at most 29 different sequences of (x, y, z, w) such that ax, xw, by, yw, cz, zw are edges in G_p. Hence G_p is $K'_{3,30}$-free.

4 Conclusion and remarks

In this paper, we study the Turán number of 1-subdivision of $K_{3,t}$. More precisely, we provide an explicit construction giving

$$\mathrm{ex}(n, K'_{3,30}) = \Omega(n^{\frac{4}{3}}),$$

which makes progress on the known estimation for the smallest value of t concerning a problem posed by Conlon, Janzer and Lee [6]. It would be interesting to consider the Conlon-Janzer-Lee problem for $s \geq 4$.

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