Introduction to Financial Mathematics

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Preface

The most important concept in this course is the concept of arbitrage. Informally, arbitrage is profit without risk.

These notes are designed around the following learning objectives:

1. Learn how to price assets so that no arbitrage opportunities appear for competitors.

2. Learn how to recognize and exploit arbitrage opportunities.

The course is divided into two parts.

In Part 1, we study the basic theory of mathematical finance.

Part 1 consists of Chapters 1 to 8. In Chapter 1, we present the basic probability theory we will need. In Chapter 2, we introduce the fundamental concepts of portfolio, replication, and arbitrage. In Chapter 3, we discuss compound interest, zero coupon bonds, and the time value of money. In Chapter 4, we introduce derivative contracts and the study the simplest type: forward contracts. In Chapter 5, we study forward interest rates and forward rate agreements, including on Libor. In Chapter 6, we study swap contracts. In Chapter 7, we give a brief introduction to futures contracts. In Chapter 8, we introduce options and study some basic properties of option prices; however, we leave the non-trivial problem of actually calculating the price of options to Part 2 of these notes.

In Part 2, we study the problem of option pricing.

Part 2 consists of Chapters 9 to 13. In Chapter 9, we present some more advanced probability theory needed to tackle the option pricing problem. In Chapter 10, we introduce risk-neutral probability measures and the fundamental theorem of asset pricing, which will be our main tools for option pricing in the market models we consider. In Chapter 11, we introduce the discrete-time binomial tree model and use the fundamental theorem and risk-neutral probability to price options in this model. In Chapter 12, we prove the fundamental theorem in the one-step binomial tree model. In Chapter 13, we present the probability theory needed to introduce the Black-Scholes model, namely the normal distribution and the central limit theorem. In Chapter 14, we introduce the Black-Scholes model as the continuous-time limit of the binomial tree model and derive the famous Black-Scholes formula of option pricing.

Chapter 1

Probability Theory: Basics

The future values of financial assets are uncertain. Financial mathematics is built on probability theory, the mathematical theory of modeling uncertainty. We will give a brief introduction to probability theory (without measure-theoretic subtleties and with minimal set theory). The purpose is not to be completely rigorous, but to build the correct intuition.

1.1 Sample Space, Events, Random Variables

Consider an uncertain outcome that we wish to model, such as a die roll, the result of an experiment, or the state of the world an hour from now.

Definition 1.1.1. The set of all possible outcomes is called the sample space. It is typically denoted by Ω. Individual outcomes, i.e. elements of Ω, are typically denoted by ω. $\triangle$

Definition 1.1.2. A subset of possible outcomes is called an **event**. $\triangle$

Example 1.1.1. Flip a fair coin three times.
Sample space: $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. The set $A = \{HHH, HHT, HTH, HTT\}$ is the event that the first flip is heads. The set $B = \{HTH, HTT, TTH, TTT\}$ is the event that the second flip is tails.

$\triangle$

Definition 1.1.3. A **random variable** X is a function from the sample space Ω to the set of real numbers $\mathbb{R}$. In other words, X assigns to each outcome $\omega \in \Omega$ a real number $X(\omega)$. (The symbol "$\in$" means "in"). $\triangle$

Example 1.1.2. Flip a fair coin twice. Sample space: $\Omega = \{HH, TT, HT, TH\}$. A random variable: $X =$ number of heads. If the outcome is $\omega = HT$, then $X(\omega) = 1$. If the outcome is $\omega = TT$, then $X(\omega) = 0$. Etc.

If the outcome is $\omega = HH$, what is $X(\omega)$? $\triangle$

Events can be written in terms of random variables.

Example 1.1.3. Flip a fair coin twice. Sample space: $\Omega = \{HH, TT, HT, TH\}$. $X =$ number of heads. The event that number of heads is at least one is

$$\begin{aligned}\{X \geq 1\} &= \{\omega \in \Omega : X(\omega) \geq 1\} \\ &= \text{ set of all outcomes } \omega \text{ in the sample space } \Omega \text{ such that } X(\omega) \geq 1 \\ &= \{HH, HT, TH\}\,.\end{aligned}$$

△

1.2 Probability Measure

Definition 1.2.1. A **probability measure** P on a sample space Ω is a function that assigns to each event A a real number $P(A)$ such that $0 \leq P(A) \leq 1$, $P(\Omega) = 1$ and P is countably additive (as defined below). The number $P(A)$ is called the **probability** of the event A. △

Interpretation. The probability $P(A)$ encodes our knowledge or belief about how likely event A is. $P(A) = 0$ means the event cannot occur. $P(A) = 1$ means the event is certain to occur.

To define countably additive, we need some other definitions first.

Definition 1.2.2.

- The event $\emptyset = \{\ \}$ is the called the **empty event**. It is the event that nothing happens.
- The **intersection** of events A and B is the event $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$. It is the event that both A and B occur.
- The events A and B are called **disjoint** if $A \cap B = \emptyset$. This means that there is no outcome ω where both A and B occur.
- The **union** of events A and B is the event $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$. It is the event that A or B (or both) occur. =
- The **union** of an infinite sequence of events $A_1, A_2, A_3, \ldots$ is the event $\bigcup_{i=1}^{\infty} A_i = \{\omega : \omega \in A_i \text{ for at least one } i = 1, 2, \ldots\}$. It is the event that at least one of $A_1, A_2, A_3, \ldots$ occurs.

△

Definition 1.2.3. P is **countably additive** means that for every infinite sequence of events $A_1, A_2, A_3, \ldots$ such that A_i and A_j are disjoint for all $i \neq j$, we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

$\triangle$

We won't need to work with the countable additive property of probability measures in this course. The follow intuitive properties will be enough

Theorem 1.2.1. If P is a probability measure, then

- $P(\{\omega_1, \omega_2, \ldots, \omega_n\}) = \sum_{i=1}^{n} P(\{\omega_i\})$ for any set of outcomes $\{\omega_1, \ldots, \omega_n\}$.
- $P(X \leq a) = 1 - P(X > a)$ for all random variables X and all real numbers a.

$\triangle$

Example 1.2.2. Roll two fair six-sided dice. The possible outcomes ω are pairs (i, j), where i is the number shown on the first die and j is the number shown on the second die. The sample space is $\Omega = \{(1,1), (1,2), (1,3), \ldots, (6,6)\}$. The dice are fair, so all outcomes are equally likely, i.e., the probability measure P satisfies

$$P(\{\omega\}) = \frac{1}{36} \text{ for all } \omega \in \Omega.$$

Consider random variables $X_1 =$ number on first die, $X_2 =$ number on second die, $Y =$ sum of the dice.

For the outcome $\omega = (2, 5)$,

$$\begin{aligned} X_1(\omega) &= X_1((2,5)) = 2 \\ X_2(\omega) &= X_2((2,5)) = 5 \\ Y(\omega) &= Y((2,5)) = 2 + 5 = 7 \end{aligned}$$

Sum of the dice is at least 11
Event: $\{Y \geq 11\} = \{(5,6), (6,5), (6,6)\}$
Probability: $P(Y \geq 11) = \frac{3}{36} = \frac{1}{12}$

Sum of the dice is less than 11
Event: $\{Y < 11\}$
Probability: $P(Y < 11) = 1 - P(Y \geq 11) = 1 - \frac{1}{12} = \frac{11}{12}$

Both dice show same number
Event: $\{X_1 = X_2\} = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$
Probability: $P(X_1 = X_2) = \frac{6}{36} = \frac{1}{6}$ $\triangle$

Remark 1.2.3. If the sample space is finite, the probability measure P can be defined by defining $P(\{\omega\})$ for every outcome ω. If the sample space is infinite, it may not be possible to define the probability measure P just by defining $P(\{\omega\})$ for every outcome ω. The next example illustrates this. Except for Chapters 13 and 14, we can assume the sample spaces we work with are finite and $P(\{\omega\}) > 0$ for every outcome ω in the sample space. △

Example 1.2.4. Pick a point uniformly at random from the interval $[0, 1]$. Sample space: $\Omega = [0, 1]$. The word "uniformly" here means that the probability that the point belongs to a given subinterval $[a, b]$ in $[0, 1]$ is proportional to the length of the interval:
$P([a, b]) = b - a$ for $0 \leq a \leq b \leq 1$
In particular, $P(\{c\}) = P([c, c]) = 0$ for any $0 \leq c \leq 1$. △

Exercise 1.2.1. Flip a fair coin 5 times. Let X = totals number of heads.
(a) Write down three possible outcomes ω from the sample space Ω.
(b) How many outcomes are in the sample space?
(c) Compute $P(\{\omega\})$ for each outcome ω you wrote down in part (a).
(d) Compute $X(\omega)$ for each outcome ω you wrote down in part (a).
(e) Find $P(X \leq 3)$. Hint: Find $P(X > 3)$ first.

Exercise 1.2.2. Consider a coin where the probability of heads is $0 < p < 1$. Do **not** assume $p = 1/2$. Flip it until the first tails occurs. Let X = number of flips needed to see the first tails.
(a) How many outcomes are in the sample space?
(b) Find $P(X = 1)$, $P(X = 2)$, and $P(X = 3)$.
(c) Write down a formula for $P(X = k)$, where k is a positive integer.

Exercise 1.2.3. Prove Theorem 1.2.1.

1.3 Discrete Random Variables

Let X be a random variable on a sample space Ω.

Definition 1.3.1. A **countable** set is a set that can be listed as either a finite sequence $a_1, a_2, \ldots, a_n$ or an infinite sequence $a_1, a_2, a_3, \ldots$. △

Example 1.3.1. The sets $\{-1, 0, 1\}$, $\{1, 1/2, \ldots, 1/10\}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are countable sets. $\mathbb{R}$ is an uncountable set. △

Definition 1.3.2. Let X be a random variable on a sample space Ω. The **range** of X is

$$R(X) = \{X(\omega) : \omega \in \Omega\} = \text{ the set of all possible outputs } X(\omega)$$

X is called a **discrete random variable** if $R(X)$ is a countable set. △

Example 1.3.2. Flip a fair coin until the first tails occurs. X = number of heads in the first two flips and Y = total number of heads. $R(X) = \{0, 1, 2\}$ and $R(Y) = \{0, 1, 2, \ldots\}$. X and Y are discrete. △

Remark. The sample space in Example 1.3.2 can be taken to be the set of all possible infinite sequences of heads and tails, like

$$HTHTTTTTTHHTTTHTHTHTHHHTHHHHTTTHTT\ldots.$$

In particular, the sample space is an infinite set.

Remark. Except in Chapter 14, all the random variables we work with are discrete.

Exercise 1.3.1. Show that $\mathbb{R}$ (the set of all real numbers) is uncountable. Show that $\mathbb{Q}$ (the set of all rational numbers) is countable.

1.4 Expectation

Definition 1.4.1. Let Ω be a sample space, let P be probability measure on Ω, and let X be a discrete random variable on Ω. The expectation of X (with respect to P) is

$$\mathbb{E}(X) = \sum_{k \in R(X)} kP(X = k).$$

△

The expectation of X is a weighted average of the possible outputs of X, with the weights being the probability of each output.

Terminology: expectation = expected value = average = mean = first moment

Example 1.4.1. Roll a fair six-sided die. The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Since the die is fair, the probability measure P is $P(\{\omega\}) = \frac{1}{6}$ for all $\omega \in \Omega$. $X =$ the number shown. $R(X) = \{1, 2, 3, 4, 5, 6\}$. The expectation of X is

$$\mathbb{E}(X) = \sum_{k \in R(X)} kP(X = k) = (1)(1/6)+(2)(1/6)+...+(5)(1/6)+(6)(1/6) = 21/6 = 3.5$$

△

Theorem 1.4.2 (Linearity of Expectation)**.** Let X and Y be discrete random variables, and let a, b, c be real numbers (constants). Then

$$\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c.$$

△

Example 1.4.3. Roll three fair six-sided dice. Let Z be the sum of the dice. Find $\mathbb{E}(Z)$.

The easiest way is to use linearity of expectation and something we already know. Define X_i = number on i-th die. Then $Z = X_1 + X_2 + X_3$, $\mathbb{E}X_i = 3.5$, and

$$\mathbb{E}(Z) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 3.5 + 3.5 + 3.5 = 10.5.$$

We could instead compute $\mathbb{E}(Z)$ from the definition. First note that $P(\{\omega\}) = \frac{1}{6^3} = \frac{1}{216}$ for each $\omega = (i, j, k)$ in the sample space $\Omega = \{(i, j, k) : 1 \le i, j, k \le 6\} = \{(1, 1, 1), (1, 1, 2), \ldots, (6, 6, 6)\}$. Then calculate $P(Z = n) = P(\{(i, j, k) : i + j + k = n\})$ for $n = 3, \ldots, 18$. Finally compute

$$\mathbb{E}(Z) = \sum_{n \in R(Z)} nP(Z = n) = 3 \cdot P(Z = 3) + 4 \cdot P(Z = 4) + \cdots + 18 \cdot P(Z = 18) = \cdots$$

We leave it as a challenge for the reader to check that we get the same result. △

Theorem 1.4.4 (Change of Variable or Law of the Unconscious Statistician)**.** Let X be a discrete random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a function. The expectation of the random variable $g(X)$ is

$$\mathbb{E}(g(X)) = \sum_{k \in R(g(X))} kP(g(X) = k) = \sum_{k \in R(X)} g(k)P(X = k). \tag{1.4.1}$$

△

Remark 1.4.5. The first sum in (1.4.1) is just the definition of the expectation of $g(X)$. The two sums are equal, but the second is often easier to compute. △

Example 1.4.6. Let X be a random variable with $P(X = k) = \frac{1}{3}$ for $k = -1, 0, 1$. Find $\mathbb{E}(X^2)$.

We take $g(x) = x^2$.

Using the second sum in (1.4.1),

$$\mathbb{E}(X^2) = \sum_{k \in R(X)} k^2 P(X = k) = (-1)^2 \left(\frac{1}{3}\right) + (0)^2 \left(\frac{1}{3}\right) + (1)^2 \left(\frac{1}{3}\right) = \frac{2}{3}$$

To use the first sum in (1.4.1), we first note that $R(X^2) = \{0, 1\}$, $P(X^2 = 0) = P(X = 0) = 0$, and $P(X^2 = 1) = P(X = -1 \text{ or } X = 1) = P(X = -1) + P(X = 1) = \frac{2}{3}$. Then

$$\mathbb{E}(X^2) = \sum_{k \in R(X^2)} kP(X^2 = k) = (0) \left(\frac{1}{3}\right) + (1) \left(\frac{2}{3}\right).$$

It was slightly easier to use the second sum in (1.4.1) because we didn't need to work out $R(X^2)$ and $P(X^2 = k)$. We leave it as a challenge for the reader to come up with more complicated examples that increase or reverse the difference in difficulty. △

Exercise 1.4.1. Use Definition 1.4.1 (not Theorem 1.4.2 or Theorem 1.4.4) to prove that $\mathbb{E}(cX) = c\mathbb{E}(X)$ for every discrete random variable X and real constant c.

Exercise 1.4.2. Consider a class of 50 students. For each student, a fair six-sided die will be rolled to determine the student's final grade. If the die shows 6, the grade is 90. If the

die shows any other number, the grade is 40. Let X_i be the grade of the i-th student.
(a) Let X_i be the grade of the i-th student. Find $\mathbb{E}(X_i)$.
(b) Let Z be the class average. Write down the formula for Z in terms of $X_1, \ldots, X_{50}$.
(c) If only 7 students roll a 6, what is the class average?
(d) What is the expectation of the class average?

Exercise 1.4.3. Consider a coin where the probability of heads is p. Flip the coin n times. $X_i = 1$ if i-th flip is heads, 0 if i-th flip is tails. Define $Y = \sum_{i=1}^{n} X_i$.
(a) Express the event that there are exactly k heads in terms of Y.
(b) Find $\mathbb{E}(Y)$.

Exercise 1.4.4. The **variance** of a random variable X is

$$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2).$$

It is the average squared-distance between X and its average $\mathbb{E}(X)$.
(a) Use the properties of expectation to prove $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.
(b) Let X be the number shown after rolling of a fair six-sided die. Find $\mathbb{E}(X^2)$ and $\text{Var}(X)$.

Chapter 2

Assets, Portfolios, and Arbitrage

In this chapter, we explain our mathematical model of the financial market, introduce basic definitions, and (most) importantly introduce the concept of arbitrage.

2.1 Assets

Definition 2.1.1. An **asset** (or **security** or **instrument**) is a valuable thing that can be owned and traded. △

Examples of assets are stocks (shares) , bonds, cash (domestic or foreign currency), real estate, and resource rights. (Don't worry if you don't know what these are yet.)

Definition 2.1.2. The **value** or **price** of an asset is the amount of cash it can be traded for. We measure the amount of cash in units of a fixed but typically unspecified currency. Unless we are dealing with multiple currencies, we omit writing words like "dollar" or currency symbols like $\$$. △

We model the prices of assets over time. The times we consider are real numbers $T \geq 0$. We will always measure time in units of years.

Let S_T denote the price at time T of a certain asset. Let t denote the present time. If $T < t$ (the past) or $T = t$ (the present), then we assume S_T is a constant. If $T > t$ (the future), then we assume S_T is a random variable. This reflects the idea that asset prices in the past and present are known, while future asset prices are unknown. We sometimes use "current" instead of "present."

The prices at futures times of all assets are assumed to be random variables defined on some fixed sample space Ω. We can think of the sample space as all possible states of the world.

We assume there is a probability measure P defined on Ω. If S_T is the price of an asset at some future time T, $P(S_T = 10)$ is the probability that the price of the asset will be 10 at time T. P represents the objective or real-world probabilities, which in practice are determined a priori from observations on the market or on the basis of historical stock data. In other words, P is estimated by looking at the real world.

2.2 Portfolios

Definition 2.2.1. A **portfolio** (or **trading strategy**) is a collection of assets along with a sequence of trades of those assets at specified times. Only certain types are trades are allowed: Trades cannot spontaneously create or destroy value and trades cannot be based on future information. $\triangle$

Example 2.2.1. Let $S_T^{(F)}$ denote the price of FB stock at time T. Let $S_T^{(G)}$ denote the price of GOOGL stock at time T.

Here is an example portfolio: At the present time $T = 0$, the portfolio consists of 3 shares of FB, 5 shares of GOOGL, and $10,000$ cash. At time $T = 1$, sell 5 shares of GOOGL for $5S_1^{(G)}$ cash. At time $T = 2$, buy 10 shares of FB stock for $10S_2^{(F)}$ cash.

We often consider portfolios that just hold assets and makes no trades. For example: At present time $T = t$, the portfolio is 8 shares of FB stock.

Trades cannot spontaneously create or destroy value. For example, trades like the following are not allowed.

- At time $T = 1$, your mom and dad give you $1,000,000$ dollars.
- At time $T = 2$, you throw all your AAPL stocks into the fires of Mount Doom.

You may think of this as a law of conservation of value for portfolios or as portfolios as being closed systems. Note that specifying what the portfolio contains at the present time does not count as a trade, so it doesn't violate this rule.

Trades cannot be based on future information. For example, a trade like the following is not allowed: At the time AAPL stock reaches its maximum value for the time period between now and 10 years from now, sell all shares of AAPL stock. However, a trade like the following is allowed: If at anytime during the next 10 years AAPL stock price is more than 500, sell all shares of AAPL stock. $\triangle$

Remark 2.2.2. A portfolio can hold any amount of an asset, including fractional and negative amounts. A holding of -1 asset is a debt of 1 asset. For example, if you have no apples and you owe Johnny 3 apples, you have -3 apples. $\triangle$

Definition 2.2.2. The **value** of a portfolio at time T is the sum of the values of the assets in the portfolio at time T. $V^A(T)$ denotes the value of a portfolio A at time T. If t is the current time and $T > t$, then $V^A(t)$ is a constant and $V^A(T)$ is a random variable. $\triangle$

Example 2.2.3. Let A be the first portfolio from Example 2.2.1: At the present time $T = 0$, A consists of 3 shares of FB, 5 shares of GOOGL, and $10,000$ cash. At time $T = 1$, sell 1 share of GOOGL for $S_1^{(G)}$ cash. At time $T = 2$, buy 10 shares of FB stock for $S_2^{(F)}$ cash.

Remember $S_T^{(F)}$ is the price of FB stock at time T, and $S_T^{(G)}$ is the price of GOOGL stock at time T. We assume (for now) that the cash does not accrue interest. Then

$$
\begin{aligned}
V^A(0) &= 3S_0^{(F)} + 5S_0^{(G)} + 10,000 \\
V^A(1) &= 3S_1^{(F)} + (10,000 + 5S_1^{(G)}) = 3S_1^{(F)} + 5S_1^{(G)} + 10,000 \\
V^A(2) &= 13S_2^{(F)} + (10,000 + 5S_1^{(G)} - 10S_2^{(F)}) = 3S_2^{(F)} + 5S_1^{(G)} + 10,000.s
\end{aligned}
$$

$\triangle$

2.3 Arbitrage

Definition 2.3.1. A portfolio A is called an **arbitrage** portfolio if the following conditions hold:

(i) At current time t,
$$V^A(t) \leq 0.$$

(ii) At some future time $T > t$,
$$V^A(T) \geq 0 \text{ with probability one} \quad \text{and} \quad V^A(T) > 0 \text{ with positive probability.}$$

In symbols, the second condition is: At some future time $T > t$, $P(V^A(T) \geq 0) = 1$ and $P(V^A(T) > 0) > 0$. $\triangle$

An arbitrage portfolio represents "free lunch" or "getting something for nothing." (It is more accurate (but less snappy) to say an arbitrage portfolio represents "free lunch without risk" or "getting something for nothing without risk.")

We have two basic goals in these notes. We will learn how to:

1. Price assets so that no arbitrage opportunities appear for competitors.

2. Recognize and exploit arbitrage opportunities.

For the first goal, we will use the

No-Arbitrage Principle. There are no arbitrage portfolios. $\triangle$

The idea is that if, under the assumption of the no-arbitrage principle, we can deduce what the price of an asset must be, then we are guaranteed that no arbitrage portfolios can be constructed with the asset at that price.

Unless otherwise indicated, we will always assume the no-arbitrage principle.

You may have heard the phrase ”there is no such thing as a free lunch.” This phrase is an informal expression of the no-arbitrage principle. (Again, it is more accurate to say ”there is no such thing as a free lunch without risk”.)

For the second goal, there is a general rule: If the price of an asset does not match the no-arbitrage price (i.e., the price implied by the no-arbitrage assumption), then we can construct an arbitrage portfolio. We will get lots of practice constructing arbitrage portfolios in specific examples and in proofs. In fact, the construction of an arbitrage portfolio is contained in a proof in the next section.

Remark 2.3.1. It may help to consider a finite sample $\Omega = \{\omega_1, \ldots, \omega_n\}$ and a probability measure P with $P(\{\omega_i\}) > 0$ for all i. Then A is an arbitrage portfolio if

- At current time t, $V^A(t) \leq 0$.
- At some future time $T > t$, $V^A(T)(\omega_i) \geq 0$ for all i and $V^A(T)(\omega_j) > 0$ for some j.

$\triangle$

2.4 Monotonicity and Replication

The replication principle and monotonicity principle are important consequences of the no-arbitrage principle. We will use them frequently to find the price of assets under the assumption of no-arbitrage.

Monotonicity Principle Let A and B be portfolios and let $T > t$, where t is the current time. If $V^A(T) \geq V^B(T)$ with probability one, then $V^A(t) \geq V^B(t)$. $\triangle$

We will prove the the no-arbitrage principle implies the monotonicity principle. Remember that the no-arbitrage principle is always assumed, unless indicated otherwise.

Proof. Assume $V^A(T) \geq V^B(T)$ with probability one.

We want to conclude $V^A(t) \geq V^B(t)$. We will use an argument called proof by contradiction where we assume the desired conclusion is false and show that this leads to a contradiction.

Assume $V^A(t) < V^B(t)$. Define $\epsilon = V^B(t) - V^A(t)$. Consider the portfolio C consisting of A minus B plus ϵ of cash. (For example, if A consists of 5 shares of AAPL and B consists of 3 shares of GOOGL, then C consists of 5 shares of AAPL, -3 shares of GOOGL, and ϵ of cash.) Then

- $V^C(t) = V^A(t) - V^B(t) + \epsilon = 0,$

- $V^C(T) = V^A(T) - V^B(T) + \epsilon \geq \epsilon > 0$ with probability one.

Therefore C is an arbitrage portfolio. This contradicts the no-arbitrage principle. □

Remark 2.4.1. Note that we may view the portfolio C in the previous proof as starting empty. At time t, we borrow the portfolio B, sell it for $V^B(T)$ cash, use $V^A(T)$ of the cash to buy portfolio A, leaving $\epsilon = V^B(t) - V^A(t)$ cash left over. Thus at time t we have portfolio A, ϵ cash, and a debt of portfolio B. At time T, the debt of portfolio B is worth $V^B(T)$. △

Replication Principle. Let A and B be portfolios and let $T > t$, where t is the current time. If $V^A(T) = V^B(T)$ with probability one, then $V^A(t) = V^B(t)$. △

We will show that the monotoncity principle, hence also the no-arbitrage principle, implies the replication principle.

Proof. $V^A(T) = V^B(T)$ means $V^A(T) \geq V^B(T)$ and $V^A(T) \leq V^B(T)$. The monotonicity theorem implies $V^A(t) \geq V^B(t)$ and $V^A(t) \leq V^B(t)$, which means $V^A(t) = V^B(t)$. □

Definition 2.4.1. Let A and B be portfolios and let $T > t$, where t is the current time. If $V^A(T) = V^B(T)$ with probability one, we say that A **replicates** B (and B **replicates** A). △

Exercise 2.4.1. This exercise is to practice "proof by contradiction." Prove:
(a) Let $a \geq 0$. If $a \leq \epsilon$ for every $\epsilon > 0$, then $a = 0$.
(b) $\sqrt{2}$ is irrational.

Exercise 2.4.2. We showed above that the no-arbitrage implies the monotonicity theorem. Consider the **Strong Monotonicity Principle.** Let A and B be portfolios and let $T > t$, where t is the current time. If $V^A(T) \geq V^B(T)$ with probability one, and $V^A(T) > V^B(T)$ with positive probability, then $V^A(t) > V^B(t)$. △
(a) Show that the no-arbitrage principle implies the strong monotonicity theorem.
(b) Show that the strong monotonicity theorem implies the monotonicity theorem.
(c) Show that the strong monotonicity theorem implies the no-arbitrage principle. Hint: If A is an arbitrage portfolio, apply the monotonicity theorem to A and an empty portfolio B to deduce a contradiction.

Chapter 3

Compound Interest, Discounting, and Basic Assets

3.1 Interest Rates and Compounding

Definition 3.1.1. If we invest (lend, deposit) N dollars at interest rate r **compounded annually**, then:

- After one year the value of the investment is $N(1+r)$
- After two years: $N(1+r)^2$
- After T years: $N(1+r)^T$

The time T is measured in years and can be any non-negative real number. N is called the **notional** or **principle**.

If we owe a debt of N at interest rate r compounded annually, then after T years we owe $N(1+r)^T$. In other words, after T years we have $-N(1+r)^T$. A debt of N is like investing $-N$. $\triangle$

Definition 3.1.2. If we invest N at interest rate r **compounded** m **times per year**, then:

- After $1/m$ years the value of the investment is $N(1+r/m)$.
- After $2/m$ years: $N(1+r/m)^2$.
- After T years (i.e., mT/m years): $N(1+r/m)^{mT}$.

The time T is measured in years and can be any non-negative real number. m is called the **compounding frequency**. $\triangle$

Remember from calculus that $\lim_{m\to\infty}(1+r/m)^{mT} = e^{rT}$.

Definition 3.1.3. If we invest N at interest rate r **compounded continuously**, then:

- After T years we have Ne^{rT}.

The time T is measured in years and can be any non-negative real number. $\triangle$

Example 3.1.1. If we invest 500 at rate $4\% = 0.04$ with compounding frequency 4 (quarterly compounding), the value after 3 years is

$$500(1+0.04/4)^{4\cdot 3} = 563.4125\ldots$$

$\triangle$

Example 3.1.2. If we borrow 500 at two-month compounded interest rate $4\% = 0.04$ (this means compounding 6 times per year), the value after 3 years is

$$500(1+0.04/6)^{6\cdot 3} = 563.5239\ldots$$

$\triangle$

Example 3.1.3. If we invest 500 at rate $4\% = 0.04$ with daily compounding (assuming 365 days per year), the value after 3 years is

$$500(1+0.04/365)^{365\cdot 3} = 563.7447\ldots$$

$\triangle$

Example 3.1.4. If we invest 500 at rate $4\% = 0.04$ with continuous compounding, the value after 3 years is

$$500e^{0.04\cdot 3} = 563.7484\ldots$$

$\triangle$

We will always deal with so-called per-year interest rates. This means time is measured in units of years. The only exception is the next example, where we consider per-month interest rates.

Example 3.1.5. If we borrow 500 at rate $1.2\% = 0.012$ per month with monthly compounding, then after T months we owe

$$500(1+0.012)^T.$$

Notice T is measured in months.

If we borrow 500 at rate $1.2\% = 0.012$ per month with compounding twice a month, then after T months we owe

$$500(1+0.012/2)^{2T}.$$

If we borrow 500 at rate $1.2\% = 0.012$ per month with daily compounding (assuming each month has 30 days), then after T months we owe

$$500(1 + 0.012/30)^{30T}.$$

$\triangle$

Result 3.1.6. If the interest rate with compounding frequency m is r_m and the interest rate with continuous compounding is r_∞, then

(a) $\left(1 + \frac{r_m}{m}\right)^{mT} = e^{r_\infty T}$ for all $T > 0$

(b) $r_\infty = m \ln\left(1 + \frac{r_m}{m}\right)$

(c) $r_m = m\left(e^{r_\infty/m} - 1\right)$

$\triangle$

Proof. First, not that if we have any one of (a),(b),(c), then we get the other two by rearranging. So we only prove (a). The idea is proof by contradiction: If (a) does not hold, then we can build an arbitrage portfolio.

If $\left(1 + \frac{r_m}{m}\right)^{mT} > e^{r_\infty T}$ for some $T > 0$, we consider the portfolio

A: At time 0, invest 1 at rate r_m with compounding frequency m and borrow 1 at rate r with continuous compounding.

Then $V^A(0) = 0$ and $V^A(T) = \left(1 + \frac{r_m}{m}\right)^{mT} - e^{r_\infty T} > 0$ with probability one. So A is an arbitrage portfolio. This contradicts the no-arbitrage principle.

If $\left(1 + \frac{r_m}{m}\right)^{mT} < e^{r_\infty T}$ for some $T > 0$, then a similar arguments also leads to a contradiction.

$\square$

Note that we may view the portfolio in the previous proof as starting empty.

A proof based on the replication principle is also possible:

Proof. Consider portfolios

A: At time 0, invest amount $M = \left(1 + \frac{r_m}{m}\right)^{-mT}$ at rate r_m with compounding frequency m.
B: At time 0, invest amount $N = e^{-r_\infty T}$ at rate r_∞ with continuous compounding.

Note $V^A(T) = M\left(1+\frac{r_m}{m}\right)^{mT} = \left(1+\frac{r_m}{m}\right)^{-mT}\left(1+\frac{r_m}{m}\right)^{mT} = 1$. Likewise $V^B(T) = Ne^{r_\infty T} = e^{-r_\infty T}e^{r_\infty T} = 1$. Thus $V^A(T) = V^B(T) = 1$ with probability 1. By the replication principle, $V^A(0) = V^B(0)$. But $V^A(0) = \left(1+\frac{r_m}{m}\right)^{-mT}$ and $V^B(0) = e^{-r_\infty T}$. Thus $\left(1+\frac{r_m}{m}\right)^{mT} = e^{r_\infty T}$. □

Note that (by Taylor expansion or L'Hopital's rule)

$$\lim_{m\to 0}(1+r/m)^{mT} = (1+Tr)$$

Definition 3.1.4. If we invest N at **simple interest rate** r, then:

- After T years we have $N(1+Tr)$.

△

Example 3.1.7. If we borrow 500 at simple interest rate $4\% = 0.04$, the value after 3 years is

$$500(1+(3)(0.04)) = 560.$$

△

Result 3.1.8. If the simple interest rate is r_0 and the interest rate with continuous compounding is r_∞, then

$$(1+Tr_0) = e^{r_\infty T} \quad \text{for all} T > 0.$$

△

The proof is an exercise.

Remark. We will always assume we can lend and borrow at non-negative interest rates. It is possible to consider negative interest rates, but we will not do so for simplicity. As we move through the course, the reader should think about how results would change if interest rates were negative.

Remark. We have implicitly assumed above that interest rates are constant in time. This is typically not the case. They depend on the period over which we lend/borrow. We will consider this generalization later.

Exercise 3.1.1. Show that if the interest rate with compounding frequency m_1 is r_1 and the interest rate with compounding frequency m_2 is r_2, then

(a) $\left(1+\frac{r_1}{m_1}\right)^{m_1 T} = \left(1+\frac{r_2}{m_2}\right)^{m_2 T}$ for all $T > 0$

(b) $r_1 = m_1\left[\left(1+\frac{r_2}{m_2}\right)^{m_2/m_1} - 1\right]$

Exercise 3.1.2. Show that if the simple interest rate is r_0 and the interest rate with continuous compounding is r_∞, then

$$(1 + Tr_0) = e^{r_\infty T} \quad \text{for all} T > 0.$$

Exercise 3.1.3. If the six-month compounded interest rate is $4.3\% = 0.043$, find the annually compounded rate r_A and the continuously compounded rate r_∞. Hint: The six-month compounded interest rate is the interest rate with compounding twice per year.

Exercise 3.1.4. (a) Show that if the interest rate with compounding frequency m is r_m and the simple interest rate is r_0, then $\left(1 + \dfrac{r}{m}\right)^{mT} = (1 + Tr_0)$ for all $T > 0$. (b) If the simple interest rate is 2.1%, find the annually compounded rate r_A and the continuously compounded rate r_∞.

Exercise 3.1.5. Suppose the definition of annual compounding was slightly different in that interest is only accrued annually. That is, if you invest N at annual rate r, then

- After 1 years the value of the investment is $N(1 + r)$
- After 1.1 years: $N(1 + r)$
- After 1.9 years: $N(1 + r)$
- After 2 years: $N(1 + r)^2$
- After T years: $N(1 + r)^{\lfloor T \rfloor}$

Here $\lfloor T \rfloor$ is the largest integer less than or equal to T.

Construct an arbitrage portfolio.

3.2 Time Value of Money, Zero Coupon Bonds, and Discounting

Would you rather have a dollar today or a dollar tomorrow? The dollar today, because you can invest it to receive interest, so you'll have more than a dollar tomorrow. This idea is called the time value of money. We will see more quantitative versions below.

Definition 3.2.1. A **zero coupon bond (ZCB)** with maturity T is an asset that pays 1 at time T (and nothing else). Its value at time $t \leq T$ is denoted $Z(t, T)$. By definition, $Z(T, T) = 1$. $\triangle$

What is the value of a ZCB at time $t \leq T$? In other words, what is the value today of the promise of a dollar tomorrow?

Result 3.2.1. If the continuously compounded interest rate from time t to time T has constant value r, then

$$Z(t,T) = e^{-r(T-t)}.$$

$\triangle$

Proof. Consider two portfolios.
A: At time t, a ZCB with maturity T
B: At time t, investment of $N = e^{-r(T-t)}$ with continuously compounded interest rate r.
Then $V^A(T) = 1$ and $V^B(T) = Ne^{r(T-t)} = e^{-r(T-t)}e^{r(T-t)} = 1$.
Therefore $V^A(T) = V^B(T)$ with probability one.
By the replication principle, $V^A(t) = V^B(t)$.
In other words, $Z(t,T) = e^{-r(T-t)}$. $\square$

Remark 3.2.2. This is the first of many proofs where we use the replication principle. Make sure you understand it. $\triangle$

Remark 3.2.3. In principal, anybody can write and sell a ZCB. Just write on a piece of paper "I promise to pay the holder of this paper of paper 1 dollar at time T." Then sell that piece of paper. That peice of paper is a ZCB. Of course, the buyer must trust that the promise of the ZCB will be honored. $\triangle$

$Z(t,T)$ is also called a **discount factor**. It depends on the interest rate and compounding frequency over the period from t to T.

Determining the value of an asset at time t based on its value at some future time $T > t$ is called **discounting** or **present valuing**. The value at time t is called the **discounted value** or **present value**.

Example 3.2.4. Consider an asset that pays 500 and matures 3 years from now. If the continuously compounded interest rate is 2.1%, what is its present value?

If the present time is t, the asset is is equivalent to 500 ZCBs with maturity $T = 3 + t$, and the present value is

$$500Z(t,T) = 500e^{-r(T-t)} = 500e^{-0.021(3)}.$$

$\triangle$

Result 3.2.5. If the interest rate with compounding frequency m from time t to time T has constant value r, then

$$Z(t,T) = (1 + r/m)^{-m(T-t)}.$$

$\triangle$

The proof is an exercise.

Result 3.2.6. If the simple interest rate from time t to time T has constant value r, then

$$Z(t,T) = (1 + rT)^{-1}.$$

$\triangle$

The proof is an exercise.

Definition 3.2.2. Because of the Results 3.2.1, 3.2.5, and 3.2.6, we call an interest rate which is constant for a period t to T a **zero rate**. For example, if the continuous interest rate for the period t to T is has constant value $r = 3\% = 0.03$, we would say the continuous zero rate for t to T is $r = 3\% = 0.03$. $\triangle$

Exercise 3.2.1. Consider an asset that pays N at maturity 3 years from now. Suppose the annually compounded interest rate is 3% and the present value is 300. Find N.

Exercise 3.2.2. Show that if the interest rate with compounding frequency m from time t to time T has constant value r, then

$$Z(t,T) = (1 + r/m)^{-m(T-t)}.$$

(a) Do this by combining Result 3.1.6 and Result 3.2.1.
(b) Do this by a no-arbitrage argument as in the proof of Result 3.2.1.

Exercise 3.2.3. Show that if the simple interest rate from time t to time T has constant value r, then

$$Z(t,T) = (1 + rT)^{-1}.$$

(a) Do this by combining Result 3.1.8 and Result 3.2.1.
(b) Do this by a no-arbitrage argument as in the proof of Result 3.2.1.

3.3 Annuities

Definition 3.3.1. An **annuity** is a series of fixed payments C at times $T_1, \ldots, T_n$. It is equivalent to the following collection of ZCBs:

- C ZCBs with maturity T_1
- C ZCBs with maturity T_2
- $\vdots$
- C ZCBs with maturity T_n

Its value at time $t \leq T_1$ is

$$V_t = C \sum_{i=1}^{n} Z(t, T_i).$$

Its value at time $T_1 < t \leq T_2$ is

$$V_t = C \sum_{i=2}^{n} Z(t, T_i).$$

because the 1st payment has already been made. △

Result 3.3.1. Consider an annuity starting at time t that pays C each year for M years. Assume the annually compounded zero rate is r_A for all maturities $T = t + 1, \ldots, t + M$. The value at its starting time t is

$$V_t = C \cdot \frac{1 - (1 + r_A)^{-M}}{r_A}.$$

△

Proof. For simplicity, we assume $C = 1$ and $t = 0$ As an exercise, adjust the proof for general C and t. Consider an annuity starting at time 0 that pays 1 each year for M years. Assume the annually compounded zero rate is r_A for all maturities $T = 1, \ldots, M$. This means that

$$Z(0, T) = (1 + r_A)^{-T} \quad \text{for } T \in \{1, \ldots, M\}$$

The value of the annuity at time $t = 0$ is

$$V_0 = \sum_{T=1}^{M} Z(0, T) = \sum_{T=1}^{M} \frac{1}{(1 + r_A)^T}.$$

We simplify this geometric sum by a standard trick. Observe that

$$\frac{1}{1 + r_A} V_0 - V_0 = \sum_{T=2}^{M+1} \frac{1}{(1 + r_A)^T} - \sum_{T=1}^{M} \frac{1}{(1 + r_A)^T} = \frac{1}{(1 + r_A)^{M+1}} - \frac{1}{1 + r_A}$$

and solve for V to obtain

$$V_0 = \frac{1 - (1 + r_A)^{-M}}{r_A}.$$

□

Example 3.3.2. In the US, a \$100 million Powerball lottery jackpot is typically structured as an annuity paying $\$4$ million per year for 25 years. With an annually compounded interest rate of 3%, the value of the jackpot at time $t = 0$ is

$$4 \cdot 10^6 \sum_{T=1}^{25} Z(0, T) = 4 \cdot 10^6 \sum_{T=1}^{25} \frac{1}{(1 + 0.03)^T} = \frac{1 - (1 + 0.03)^{-25}}{0.03} \approx 69.65 \text{ million}$$

△

Example 3.3.3. A loan of 1000 is to be paid back in 5 equal installments due yearly. Interest of 15% of the balance is applied each year, before the installment is paid. This type of loan is called an **amoritized loan**.

Find the amount C of each installment.

For the lender, the loan is equivalent to an annuity.
Assume it starts at $t = 0$. So it pays C at times $T = 1, 2, 3, 4, 5$.
The 15% yearly interest on the balance is equivalent to a 15% annually compounded interest rate.
By Result 3.3.1, the value at time 0 is

$$V_0 = C\frac{1-(1+(0.15))^{-5}}{0.15}$$

On the other hand, we know $V_0 = 1000$.
Therefore

$$C = 1000 \cdot \frac{0.15}{1-(1+(0.15))^{-5}} \approx 298.32.$$

△

Example 3.3.4. Consider an amortized loan with initial value V due in M years with equal annual installments C and annually compounded interest rate r. As in Example 3.3.3, each installment is

$$C = V\frac{r}{1-(1+r)^{-M}}.$$

The balance after the 1st installment is

$$B_1 = V(1+r) - C.$$

The balance after the 2nd installment is

$$B_2 = (V(1+r) - C)(1+r) - C = V(1+r)^2 - C(1+r) - C.$$

The balance after the k-th installment is

$$B_k = V(1+r)^k - C\sum_{i=0}^{k-1}(1+r)^i.$$

We can rewrite this expression by substituting

$$C = V\frac{r}{1-(1+r)^{-M}} \quad \text{and} \quad \sum_{i=0}^{k-1}(1+r)^i = \frac{1-(1+r)^k}{1-(1+r)}$$

and doing some algebra. We find the balance after the k-th installment is

$$B_k = V\frac{(1+r)^M - (1+r)^k}{(1+r)^M - 1}.$$

$B_0 = V$ is the initial balance.

The interest at the 1st installment is

$$B_0 r = Vr.$$

The interest at the 2nd installment is

$$B_1 r = (V(1+r) - C)r$$

The interest at the k-th installment is

$$B_{k-1} r.$$

The amount of the initial loan repaid in the k-th installment (that is, the amount of the k-th installment that does not go towards interest) is

$$C - B_{k-1} r$$

△

The geometric sum trick we used the proof of Result 3.3.1 can be used to prove the following result. You may prefer to remember this result, rather than the trick.

Result 3.3.5. 1. $\displaystyle\sum_{k=0}^{N} R^k = 1 + R + R^2 + \ldots + R^k = \frac{1 - R^{N+1}}{1 - R}$

2. $\displaystyle\sum_{k=1}^{N} R^k = R(1 + R + R^2 + \ldots + R^{k-1}) = \frac{R(1 - R^N)}{1 - R}$

3. $\displaystyle\sum_{k=1}^{N} \frac{1}{(1+R)^k} = \frac{1 - (1+R)^{-N}}{R}$

△

Exercise 3.3.1. Prove Result 3.3.1 for general C.

Exercise 3.3.2. Consider the loan in Example 3.3.3.
(a) What is the amount of interest included in each installment?
(b) How much of the initial loan is repaid in each installment?
(c) What is the outstanding balance after each installment is paid?

Exercise 3.3.3. Consider an annuity starting at time 0 that pays 1 each year for M years. Assume the annually compounded zero rate is r_A for all maturities $T = 1, \ldots, M$. By Result 3.3.1, the value of this annuity at its starting time 0 is

$$V_0 = \sum_{i=1}^{M} Z(0, i) = \frac{1 - (1 + r_A)^{-M}}{r_A}.$$

Find the value of the annuity at time t', where $0 < t' < 1$.

Exercise 3.3.4. Consider an annuity starting at time t that pays 1 each year for M years. Assume the annually compounded zero rate is r_A for all maturities $T = t+1, \dots, t+M$. According to Result 3.3.1, the value of this annuity at its starting time t is

$$V_t = \sum_{i=1}^{M} Z(t, t+i) = \frac{1 - (1+r_A)^{-M}}{r_A}.$$

Find the value of the annuity at time t', where $t < t' < t+1$.

Exercise 3.3.5. Consider an annuity that pays 1 every quarter for M years. In other words, the payment times are $T = t + \frac{1}{4}, t + \frac{2}{4}, \dots, t + \frac{4M}{4}$. Show that the value at present time t is

$$V_t = \frac{1 - (1 + r_4/4)^{-4M}}{r_4/4},$$

assuming the quarterly compounded interest rate has constant value r_4.

Exercise 3.3.6. Consider an annuity that pays 1 every quarter for M years. In other words, the payment times are $T = t + \frac{1}{4}, t + \frac{2}{4}, \dots, t + \frac{4M}{4}$. Show that the value at present time t is

$$V_t = \frac{1 - (1 + r_8/8)^{-8M}}{(1 + r_8/8)^2 - 1},$$

assuming the interest rate with compounding 8 times per year has constant value r_8.

3.4 Bonds

Definition 3.4.1. A **fixed rate bond** with notional N, coupon c, start date T_0, maturity T_n, and term length α is an asset that pays N at time T_n and coupon payments $\alpha N c$ at times T_i for $i = 1, \dots, n$, where $T_{i+1} = T_i + \alpha$. △

Result 3.4.1. Consider a fixed rate bond with coupon c, notional N, maturity M years from now, and annual coupon payments. Assume the annually compounded interest rate has constant value r_A. The value of the bond at present time t is

$$V_t = cN \frac{1 - (1+r_A)^{-M}}{r_A} + N(1+r_A)^{-M}.$$

△

Proof. The bond is equivalent to an annuity paying cN each year for M years plus N ZCBs with maturity M. Use Result 3.3.1 and Result 3.2.5. □

Exercise 3.4.1. (a) Consider an annuity that pays 1 every quarter for M years. In other words, the payment times are $T = t + \frac{1}{4}, t + \frac{2}{4}, \dots, t + \frac{4M}{4}$. Show that the value at present time t is

$$V_t = \frac{1 - (1 + r_4/4)^{-4M}}{r_4/4},$$

assuming the quarterly compounded interest rate has constant value r_4.
(b) Consider a fixed rate bond with notional N and coupon c that starts now, matures M years from now, and has quarterly coupon payments. Show that the value at present time t is

$$V_t = \frac{cN}{4} \cdot \frac{1-(1+r_4/4)^{-4M}}{r_4/4} + N(1+r_4/4)^{-4M},$$

assuming the quarterly compounded interest rate has constant value r_4.

Exercise 3.4.2. (a) Consider an annuity that pays 1 every quarter for M years. In other words, the payment times are $T = t+\frac{1}{4}, t+\frac{2}{4}, \ldots, t+\frac{4M}{4}$. Show that the value at present time t is

$$V_t = \frac{1-(1+r_8/8)^{-8M}}{(1+r_8/8)^2-1},$$

assuming the interest rate with compounding 8 times per year has constant value r_8.
(b) Consider a fixed rate bond with notional N and coupon c that starts now, matures M years from now, and has quarterly coupon payments. Show that the value at present time t is

$$V_t = \frac{cN}{4} \cdot \frac{1-(1+r_8/8)^{-8M}}{(1+r_8/8)^2-1} + N(1+r_8/8)^{-8M},$$

assuming the interest rate with compounding 8 times per year has constant value r_8.

3.5 Stocks

Definition 3.5.1. A **stock** or **share** is an asset giving ownership of a fraction of a company. The price of a stock at time T is denoted by S_T. If t is the current time, then the known price S_t is called the **spot** price, and S_T is a random variable for $T > t$. A stock may sometimes pay a dividend, which is a cash payment usually expressed as a percentage of the stock price. $\triangle$

3.6 Foreign Exchange Rates

Example 3.6.1. The current euro (EUR) to US dollar (USD) exchange rate is

$$0.89\,\text{EUR}/\text{USD}.$$

Therefore the USD to EUR exchange rate is

$$\frac{1}{0.89\,\text{EUR}/\text{USD}} \approx 1.12\,\text{USD}/\text{EUR}.$$

Then

$$150\,\text{USD} = (150\,\text{USD})\left(0.89\,\frac{\text{EUR}}{\text{USD}}\right) = 150(0.89)\,\text{EUR} = 133.50\,\text{EUR}.$$

$\triangle$

Exercise 3.6.1. The current US Dollar (USD) to Japense Yen (JPY) exchange rate is $0.0098\text{USD}/\text{JPY}$.
(a) Find the JPY to USD exchange rate.
(b) Find the value in USD of 300,000 JPY

Chapter 4

Forward Contracts

4.1 Derivative Contracts

Definition 4.1.1. A **derivative contract** or **derivative** is a financial contract between two entities whose value is a function of (derives from) the value of another variable. The two entities in the contract are called **counterparties**. The variable could be the price of a stock, a foreign exchange rate, an interest rate, or even the weather. △

Example 4.1.1 (A Weather Derivative)**.** A contract where one counterparty pays either 100 or 0 to the other counterparty one year from now depending on whether the total snowfall in Boston over the year is greater than 50 inches. △

We will only consider derivatives of financial variables.

4.2 Forward Contract Definition

Our first derivative is the forward contract.

Definition 4.2.1. In a **forward contract** or **forward**, two counterparties agree to trade a specific asset (like a stock) at a certain future time T and a certain price K.

One counterparty agrees to buy the asset at time T and price K, and the other counterparty agrees to sell the asset at time T and price K.

We say the buyer is **long** the forward contract, and the seller is **short** the forward contract.

K is the called the **delivery price**. T is called the **maturity** or **delivery date**. △

4.3 Value of Forward

Fix an asset. Consider a forward on the asset with delivery price K and maturity T.

Definition 4.3.1. $V_K(t, T)$ denotes the value (price) of the forward to the long counterparty at time $t \leq T$.

Then $-V_K(t, T)$ is the value (price) of the forward to the short counterparty at time $t \leq T$. The value at maturity $\triangle$

Note that K and T are fixed at the time the forward contract is agreed to, but the value of the forward contract may change over time.

To take at time t the long position in a forward contract with maturity T and delivery price K, we must pay $V_K(t, T)$ at time t to the counterparty party taking the short position. Note that we must also pay K at time T to buy the asset.

You can think of $V_K(t, T)$ as the amount the long counterparty must pay upfront (time t) to convince the short counterparty to agree to the forward contract. The long counterparty must still pay K at time T to buy the asset.

Note that if $V_K(t, T)$ is negative, it is actually the short counterparty that pays upfront. Paying a negative amount means receiving.

Here is one more way to understand the value of a forward contract. Suppose two counterparties have agreed (at some time in the past) to a forward contract with maturity T and delivery price K. At current time $t \leq T$, we want to buy we buy the long position from the long counterparty, so that we become the long counterparty. To do so, we need to pay the current long counterparty $V_K(t, T)$ at time t. Note that we will still need to pay pay K at time T to buy the asset itself.

4.4 Payoff

Definition 4.4.1. Fix an asset. Let S_t be its price at time t. Consider a forward on the asset with delivery price K and maturity T.

At time T, we know the counterparty long the forward must pay K to buy the asset whose value is S_T. Therefore the value at maturity (i.e. at time T) long the forward (i.e., for the long counterparty) is

$$V_K(T, T) = S_T - K.$$

We call

$$g(S_T) = S_T - K$$

the **payoff** or **payout** long the forward. Here the function $g(x) = x - K$ is called the **long forward payoff function**.

The value at maturity (i.e. at time T) short the forward (i.e., for the short counterparty) is

$$-V_K(T,T) = K - S_T.$$

The payoff **payoff** or **payout** short the forward is

$$h(S_T) = K - S_T$$

where the function $h(x) = x - K$ is the **short forward payoff function**. △

Example 4.4.1. Consider a forward with delivery price 100 and maturity $T = 2$ years. If the underlying asset has price 95.10 at maturity, find value of the forward to the long party at maturity.

We have $K = 100$, $T = 2$, $S_T = 95.10$. Therefore $V_K(T,T) = S_T - K = 95.10 - 100 = -4.90$. △

Example 4.4.2. Consider a forward with delivery price 100 and maturity T. Suppose underlying asset has price

$$S_T = \begin{cases} 110 & \text{with probability } 0.6 \\ 90 & \text{with probability } 0.4. \end{cases}$$

Find the expected payoff long the forward.

The payoff at maturity long the forward is

$$g(S_T) = S_T - 100 = \begin{cases} 110 - 100 & \text{with probability } 0.6 \\ 90 - 100 & \text{with probability } 0.4 \end{cases} = \begin{cases} 10 & \text{with probability } 0.6 \\ -10 & \text{with probability } 0.4 \end{cases}$$

The expected payoff at maturity long the forward is

$$\mathbb{E}(g(S_T)) = \sum_{k \in R(g(S_T))} kP(g(S_T) = k) = (10)(0.6) + (-10)(0.4) = 2$$

△

Exercise 4.4.1. Consider a forward with delivery price 200 and maturity T. Suppose the underlying asset has price

$$S_T = \begin{cases} 150 & \text{with probability } 0.3 \\ 200 & \text{with probability } 0.5 \\ 250 & \text{with probability } 0.2 \end{cases}.$$

(a) Find the payoff long the forward.
(b) Find the expected value at maturity long the forward.
(c) Find the expected value of the forward to the short counterparty at maturity.

4.5 Forward Price

Pre-emptive clarification:

$$\text{forward price} \neq \text{price of forward} = \text{value of forward} = V_K(t,T)$$

The terminology is unfortunate, but we are stuck with it.

Definition 4.5.1. Fix an asset. The **forward price** of the asset at time $t \leq T$ is the number $F(t,T)$ such that the forward contract with maturity T and delivery price $K = F(t,T)$ has value $V_K(t,T) = 0$ at time t. Thus

$$V_K(t,T) = 0 \text{ if and only if } K = F(t,T).$$

There is no cost to enter a forward contract (as either the long or short counterparty) at t if the delivery price K equals the forward price $F(t,T)$. △

Remark 4.5.1. Here is another way to describe the forward price that may be help understand it. Consider a forward contract on the asset with maturity T. Suppose the two counterparties are negotiating the delivery price K before they agree to the contract. Remember that at future time T the long counterparty will pay K to the short counterparty in exchange for the asset. If K is too small, the long counterparty must pay $V_K(t,T)$ at current time t to convince the short counterparty to agree to the contract. If K is too large, the short counterparty must pay $-V_K(t,T)$ ($V_K(t,T) < 0$ in this situation) at current time t to convince the long counterparty to agree to the contract. The delivery price K where each counterparty pays zero ($V_K(t,T) = 0$) at t to convince the other is called the **forward price** and is denoted $F(t,T)$. A better name for the forward price may be the "fair delivery price" or the "neutral delivery price." △

Remark 4.5.2. Let S_T be the price of the asset at time T. Since $V_K(T,T) = S_T - K$ and since $F(T,T)$ is the value of K that makes $V_K(T,T) = 0$, we have

$$F(T,T) = S_T.$$

△

4.6 Value of Forward and Forward Price for Asset Paying No Income

Result 4.6.1. Suppose the continuous zero rate for period t to T is r. Suppose an asset pays no income (no dividends, no coupons, no payment at maturity, no rent, etc). Then

$$V_K(t,T) = S_t - KZ(t,T) = S_t - Ke^{-r(T-t)}$$

and

$$F(t,T) = \frac{S_t}{Z(t,T)}$$

where S_t is the price of asset at time t. △

Proof. Consider two portfolios.
A: At time t, one unit of the asset.
B: At time t, one long forward contract with maturity T and delivery price K plus K ZCBs.

Then $V^A(T) = S_T$ and $V^B(T) = (S_T - K) + K = S_T$.
So $V^A(T) = V^B(T)$.
By replication, $V^A(t) = V^B(t)$.
This means

$$S_t = V_K(t,T) + KZ(t,T). \tag{4.6.1}$$

Solving $V_K(t,T)$ gives

$$V_K(t,T) = S_t - KZ(t,T).$$

The forward price $F(t,T)$ is the value of K such that $V_K(t,T) = 0$.
Setting $K = F(t,T)$ and $V_K(t,T) = 0$ in (4.6.1) leads to

$$F(t,T) = \frac{S_t}{Z(t,T)}.$$

By the definition of continuous zero rate, $Z(t,T) = e^{-r(T-t)}$. □

Example 4.6.2. The current price of a certain stock paying no income is 20. Assume the continuous zero rate will be 2.1% for the next 6 months. Find the forward price with maturity in 6 months.

$T - t = 6$ months = 0.5 years.

$$F(t,T) = \frac{S_t}{Z(t,T)} = \frac{20}{e^{-(0.021)(0.5)}}$$

△

Example 4.6.3. The current price of a certain stock paying no income is 20. Assume the continuous zero rate will be 2.1% for the next 6 months. Find the value of a forward contract on the stock if the delivery price is 25 and maturity is in 6 months.

$T - t = 6$ months $= 0.5$ years , $S_t = 20$, $K = 25$, $r = 0.021$. Therefore

$$V_K(t,T) = 20 - 25e^{-(0.021)(0.5)}$$

△

If the price of an asset does not match the no-arbitrage price (i.e., the price implied by the no-arbitrage assumption), then we can construct an arbitrage portfolio.

Example 4.6.4. At current time t, a certain stock paying no income has price 45, the forward price with maturity T on the stock is 48, and the price of a zero coupon bond with maturity T is 0.95. Determine whether there is an arbitrage opportunity. If so, construct an arbitrage portfolio.

Given $S_t = 45$, $F(t,T) = 48$, $Z(t,T) = 0.95$. Notice

$$\frac{S_t}{Z(t,T)} = \frac{45}{0.95} \approx 47.37.$$

So $F(t,T) > \dfrac{S_t}{Z(t,T)}$. This violates the result above deduced from the no-arbitrage assumption. So there must be arbitrage portfolio.

At current time t, portfolio A is

- $-S_t$ cash (i.e., S_t cash debt)
- 1 stock
- 1 short forward contract with maturity T and delivery price equal to the forward price $F(t,T)$

$$V^A(t) = -S_t + S_t + V_{F(t,T)}(t,T) = 0.$$

At time T, we must sell 1 stock at price $F(t,T)$, and our debt has grown to $S_t e^{r(T-t)} = S_t/Z(t,T)$, where r is the continuous zero rate for period t to T.

At time T, portfolio A is:

- $-S_t e^{r(T-t)}$ cash
- $F(t,T)$ cash

The value of A is

$$V^A(T) = F(t,T) - S_t e^{r(T-t)} = F(t,T) - \frac{S_t}{Z(t,T)} > 0.$$

with probability one.

Therefore A is an arbitrage portfolio. $\triangle$

Exercise 4.6.1. At current time t, a certain stock has price 45, the forward price with maturity T on the stock is 40, and the price of a zero coupon bond with maturity T is 0.95. Construct an arbitrage portfolio if possible. Verify the portfolio you construct is an arbitrage portfolio.

Exercise 4.6.2. The current price of a certain stock paying no income is 30. Assume the annually compounded zero rate will be 3% for the next 2 years.
(a) Find the current value of a forward contract on the stock if the delivery price is 25 and maturity is in 2 years.
(b) If the stock has price 35 at maturity, find the value of the forward to the short counterparty at maturity.

Exercise 4.6.3. Let S_t be the current price of a stock that pays no income. Let r_{BID} be the interest rate at which one can lend/invest money, and r_{OFF} be the interest rate at which one can borrow money. Both rates are continuously compounded. Assume $r_{BID} \leq r_{OFF}$, except in (a).
(a) Assume $r_{BID} > r_{OFF}$. Find an arbitrage portfolio. Verify it is an arbitrage portfolio.
(b) Use a no-arbitrage argument to prove the forward price with maturity T for the stock satisfies the upper bound

$$F(t,T) \leq S_t e^{r_{OFF}(T-t)}.$$

(c) Use a no-arbitrage argument to prove a similar lower bound for the forward price.
(d) Assume the stock has bid price $S_{t,BID}$ and offer (or ask) price $S_{t,OFF}$. The bid price is the price for which you can sell the stock. The offer price is the price for which you can buy the stock. How do the upper and lower bounds in (a) and (b) change? Prove these bounds using no-arbitrage.

4.7 Value of Forward and Forward Price for Asset Paying Known Income

Result 4.7.1. Suppose the continuous zero rate for period t to T is r. Suppose an asset an pays a known income stream (for example, dividends, coupons, payment at maturity, rent) during the life of the forward contract. Then

$$V_K(t,T) = S_t - I_t - KZ(t,T) = S_t - I_t - Ke^{r(T-t)}$$

and

$$F(t,T) = \frac{S_t - I_t}{Z(t,T)} = (S_t - I_t)e^{r(T-t)},$$

where S_t is the price of asset at time t, and I_t is the present value of the income stream at time t. △

Notice the forward price for an asset paying known income is lower than it would be if the asset paid no income. Until maturity of the forward, the short counterparty holds the asset and collects any income it pays, while the long counterparty gets no income. Thus, for an asset paying income, there is an advantage to buying it spot (i.e., buying it immediately at time t) rather than buying it forward. So the forward price is lower to compensate.

Proof. A: At time t, one unit of the asset and $-I_t$ cash
B: At time t, one long forward contract with maturity T and delivery price K plus K ZCBs.

We have
$V^A(T) = S_T + I_t e^{r(T-t)} - I_t e^{r(T-t)} = S_T,$

$V^B(T) = S_T - K + K = S_T$.
So $V^A(T) = V^B(T)$ with probability 1
By replication, $V^A(t) = V^B(t)$.
Therefore

$$S_t - I_t = V_K(t,T) + KZ(t,T). \tag{4.7.1}$$

Solving for $V_K(t,T)$ gives

$$V_K(t,T) = S_t - I_t - KZ(t,T)$$

The forward price $F(t,T)$ is the value of K such that $V_K(t,T) = 0$.
Setting $K = F(t,T)$ and $V_K(t,T) = 0$ in (4.7.1) leads to

$$F(t,T) = \frac{S_t - I_t}{Z(t,T)}.$$

By the definition of continuous zero rate, $Z(t,T) = e^{-r(T-t)}$. □

Example 4.7.2. Assume the continuously compounded interest rate is a constant r. Consider an asset that pays d at times $T_1 \leq T_2 \leq \cdots \leq T_n$. Find the forward price $F(t,T)$ when $t \leq T_1$ and $T_n \leq T$.
I_t = present value of the income = present value of the stream of payments.

The payment of d at T_i is equivalent to d ZCBs with maturity T_i. Therefore

$$I_t = d\sum_{i=1}^{n} Z(t,T_i) = d\sum_{i=1}^{n} e^{-r(T_i-t)}$$

Hence

$$F(t,T) = \frac{S_t - I_t}{Z(t,T)} = \frac{S_t - d\sum_{i=1}^{n} e^{-r(T_i-t)}}{e^{-r(T-t)}}.$$

△

Exercise 4.7.1. The income may be negative if the asset has carrying costs, such as insurance or storage costs. Suppose you have a single gold bar (400 troy ounces) in a storage unit in Mountain View, California. Rent for the storage unit is 100 per month (with payments starting immediately, not at the end of the month). The current price of gold is 1325 per troy ounce. Find the current forward price for the gold bar if maturity is 1 year from now, assuming the continuously compounded interest rate has constant value 3%. Hint: From the point of view of the rental company, the rent is a sequence of ZCBs with maturities after 0 months, 1 month, 2 months,. . ., 11 months.

4.8 Value of Forward and Forward Price for Stock Paying Known Dividend Yield

Result 4.8.1. Suppose the continuous zero rate for period t to T is r. Suppose a stock pays dividends as a percentage q of the stock price on a continuously compounded per-year basis. q is called the **dividend yield**. Suppose the dividends are automatically reinvested in the stock. Then

$$V_K(t,T) = e^{-q(T-t)}S_t - KZ(t,T).$$

and

$$F(t,T) = \frac{S_t e^{-q(T-t)}}{Z(t,T)} = S_t e^{(r-q)(T-t)}$$

△

Dividends are often assumed to be paid continuously, rather than at discrete times. This assumption is appropriate for a portfolio of many stocks paying dividends at many different times throughout the year. If you wish, replace the word "stock" in Result 4.8.1 by "portfolio of many stocks."

Proof. Consider two portfolios at time t.
A: $N = e^{-q(T-t)}$ units of stock
B: 1 forward with delivery price K and maturity T, K ZCBs with maturity T
At time T, the number of units of stock in A is $Ne^{q(T-t)} = e^{-q(T-t)}e^{q(T-t)} = 1$. Therefore
$V^A(T) = S_T$
$V^B(T) = (S_T - K) + K = S_T$
Since $V^A(T) = V^B(T)$ with probability one, the replication theorem gives $V^A(t) = V^B(t)$.
We have $V^A(t) = e^{-q(T-t)}S_t$
$V^B(t) = V_K(t,T) + KZ(t,T)$
Therefore

$$e^{-q(T-t)}S_t = V_K(t,T) + KZ(t,T). \tag{4.8.1}$$

Solving for $V_K(t,T)$ gives

$$V_K(t,T) = e^{-q(T-t)}S_t - KZ(t,T).$$

The forward price $F(t,T)$ is the value of K such that $V_K(t,T) = 0$. Setting $K = F(t,T)$ gives $V_K(t,T) = 0$, and so (4.8.1) leads to

$$F(t,T) = \frac{S_t e^{-q(T-t)}}{Z(t,T)}.$$

By the definition of continuous zero rate, $Z(t,T) = e^{-r(T-t)}$. □

Exercise 4.8.1. Suppose a stock pays dividends m times per year at equally spaced times with annual yield q. So each dividend payment is equal to q/m of the stock price and the payments are made at times $t + 1/m$, $t + 2/m$, $\ldots$, where t is the current time. Suppose the dividends are automatically reinvested in the stock.
(a) If you have 1 unit of stock at time t, how many will you have $1/m$ years later when the

first dividend is paid?
(b) If $T - t$ is an integer multiple of $1/m$, show that the forward price for the stock is

$$F(t,T) = \frac{S_t(1+q/m)^{-m(T-t)}}{Z(t,T)}. \tag{4.8.2}$$

(c) Compute the limit as $m \to \infty$.
(d) Suppose $m = 1$ and $T - t = 0.5$ (so $T - t$ is not an integer multiple of m). Show that if (4.8.2) holds, then you can build an arbitrage portfolio. Verify the portfolio is an arbitrage portfolio.

Exercise 4.8.2. Fix times $t_0 < t < T$. Consider an asset that pays no income. Suppose that at time t_0 you go short a forward contract with maturity T and delivery price equal to the forward price $F(t_0, T)$. At time t suppose both the price of the asset and interest rates are unchanged.
(a) How much money have you made or lost? This is called the **carry** of the trade at time t. Hint: Compare the value of the short forward at time t_0 to its value at time t.
(b) How does your answer change if the asset pays dividends at constant rate q? As usual, assume the dividends are paid continuously and automatically reinvested in the stock.

4.9 Forward Price for Currency

Result 4.9.1. Suppose X_t is the price at time t in USD of one unit of foreign currency. (For example, 1 CAD = 0.77 USD.) Let $r_\$$ be the zero rate for USD. Let r_f be the zero rate for foreign currency, both constant and compounded continuously. Then the forward price for one unit of foreign currency is

$$F(t,T) = X_t e^{(r_\$ - r_f)(T-t)}.$$

$\triangle$

Exercise 4.9.1. Prove Result 4.9.1. Hint: In the proof of Result 4.8.1, replace the stock by foreign currency. The foreign currency is analogous to a stock paying known dividend yield. The foreign interest rate corresponds to the dividend yield.

4.10 Relationship Between Value of Forward and Forward Price

Result 4.10.1. Suppose the continuous zero rate for period t to T is r. For any asset,

$$V_K(t,T) = (F(t,T) - K)Z(t,T) = (F(t,T) - K)e^{-r(T-t)}.$$

$\triangle$

Proof. Consider two portfolios at time t.
A: one long forward with maturity T and delivery price K, one short forward with maturity T and delivery price $F(t,T)$
B: $(F(t,T)-K)$ ZCBs with maturity T
We have
$V^A(T)=(S_T-K)+(F(t,T)-S_T)=F(t,T)-K$
$V^B(T)=F(t,T)-K$
So $V^A(T)=V^B(T)$ with probability one. By replication, $V^A(t)=V^B(t)$. But $V^A(t)=V_K(t,T)+0=V_K(t,T)$
$V^B(t)=(F(t,T)-K)Z(t,T)$
Therefore

$$V_K(t,T)=(F(t,T)-K)Z(t,T).$$

□

Example 4.10.2. Assume $V_K(t,T)<(F(t,T)-K)Z(t,T)$. Construct an arbitrage portfolio.
A is "A from the proof" minus $V_K(t,T)$ cash (or $V_K(t,T)/Z(t,T)$ ZCBs with maturity T)
$V^A(t)=0$.
$V^A(T)=(S_T-K)+(F(t,T)-S_T)-\dfrac{V_K(t,T)}{Z(t,T)}>0$ with probability one. △

Exercise 4.10.1. Assume $V_K(t,T)>(F(t,T)-K)Z(t,T)$. Construct an arbitrage portfolio.

Exercise 4.10.2. The current time is $t=0$. Suppose the present value of a forward contract on a certain asset is 10. The delivery price is $K=100$ and the maturity is $T=5$. Suppose the forward price on the asset is 110. Suppose the continuous interest rate is 2% for time 0 to T. Determine whether there is an arbitrage opportunity. If there is, find an arbitrage portfolio. Verify the portfolio you construct is an arbitrage portfolio.

Chapter 5

Forward Rates and Libor

5.1 Forward Interest Rates

Definition 5.1.1. If $t < T_1 < T_2$, an interest rate agreed at t for lending/borrowing from T_1 to T_2 is called a **forward rate.** $\triangle$

In contrast, an interest rate agreed at t for lending/borrowing from t to T is a zero rate.

Example 5.1.1. If the continuously compounded forward rate at current time $t = 0$ for period $T_1 = 5$ to $T_2 = 10$ is 7%, then we can agree now to invest $1,000,000$ dollars 5 years from now and receive

$$(1,000,000)e^{(0.07)(10-5)} = 1419067.54\ldots$$

10 years from now. $\triangle$

For $t < T_1 < T_2$, let

- $r_1 =$ zero rate for period t to T_1
- $r_2 =$ zero rate for period t to T_2
- $f_{12} =$ forward rate at t for period T_1 to T_2

Figure 5.1 indicates two possible strategies:

- Lend (borrow) from t to T_2 at rate r_2.
- Lend (borrow) from t to T_1 at rate r_1, then at T_1 lend (borrow) the amount gained (owed) from T_1 to T_2 at rate f_{12}.

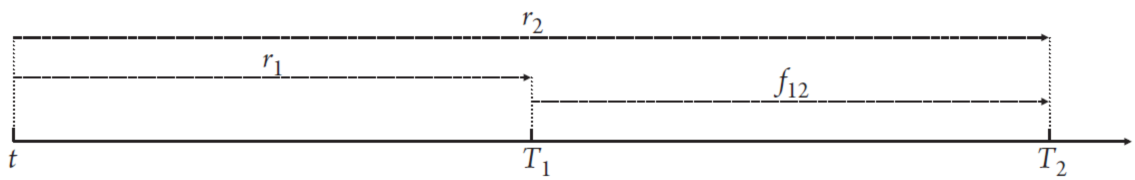


Figure 5.1: Forward and zero rates

The next result says they are equivalent.

Result 5.1.2. In the notation above, if the rates are for continuous compounding, then

$$e^{r_2(T_2-t)} = e^{r_1(T_1-t)}e^{f_{12}(T_2-T_1)}. \tag{5.1.1}$$

△

We give two proofs.

Proof 1. The idea is that if the strategies return different amounts (i.e., if (5.1.1) does not hold), then we can borrow with one and lend with the other to create an arbitrage portfolio.

Assume (5.1.1) does not hold.

If $e^{r_2(T_2-t)} > e^{r_1(T_1-t)}e^{f_{12}(T_2-T_1)}$, we consider portfolio C:

- At time t: Start with nothing. Borrow 1 from t to T_1 at rate r_1; Lend 1 from t to T_2 at rate r_2; We know that at time T_1 we will owe a debt of $e^{r_1(T_1-t)}$; To pay the debt, we arrange at t to borrow $e^{r_1\cdot(T_1-t)}$ from T_1 to T_2 at rate f_{12}.
- At time T_1: We owe $e^{r_1\cdot(T_1-t)}$. As we arranged at t, to pay off the debt we borrow $e^{r_1\cdot(T_1-t)}$ from T_1 to T_2 at the rate f_{12}.
- At time T_2: We receive $e^{r_2\cdot(T_2-t)}$ and must pay $e^{r_1\cdot(T_1-t)}e^{f_{12}\cdot(T_2-T_1)}$.

Then $V^C(t) = 0$ and $V^C(T) = e^{r_2\cdot(T_2-t)} - e^{r_1\cdot(T_1-t)}e^{f_{12}\cdot(T_2-T_1)} > 0$. Thus C is an arbitrage portfolio, which contradicts no-arbitrage.

If $e^{r_2(T_2-t)} < e^{r_1(T_1-t)}e^{f_{12}(T_2-T_1)}$, a similar construction also gives an arbitrage portfolio. □

Proof 2. Consider

A: Time t: Lend amount $M = e^{-r_2(T_2)}$ from t until T_2 at rate r_2.

B: Time t: Lend $N = e^{-r_1(T_1-t)}e^{-f_{12}(T_2-T_1)}$ from t until T_1 at rate r_1. Also arrange at t to lend the amount that will be gained from the first loan, namely $Ne^{r_1(T_1-t)} = e^{-f_{12}(T_2-T_1)}$, from T_1 until T_2 at rate f_{12}.

Then $V^A(T_2) = Me^{r_2(T_2)} = 1$ and $V^B(T_2) = Ne^{r_1(T_1-t)}e^{f_{12}(T_2-T_1)} = 1$. Thus $V^A(T_2) = V^B(T_2)$ with probability one. By replication,

$$V^A(t) = V^B(t)$$

or equivalently

$$e^{-r_2(T_2)} = e^{-r_1(T_1-t)}e^{-f_{12}(T_2-T_1)}.$$

Take reciprocals. □

Remark 5.1.3. The portfolio C in the Proof 1 above can be viewed as $C = A' - B'$ where

A: Time t: Lend amount 1 from t until T_2 at rate r_2.

B: Time t: Lend 1 from t until T_1 at rate r_1. Also arrange at t to lend the amount gained from the first loan, namely $e^{r_1(T_1-t)}$, from T_1 until T_2 at rate f_{12}. △

For compounding m times per year and for simple interest, results analogous to Result 5.1.2 can be obtained by analogous arguments.

For example,

Result 5.1.4. In the notation above, if the rates are for compounding m times per year, then

$$\left(1 + \frac{r_2}{m}\right)^{m(T_2-t)} = \left(1 + \frac{r_1}{m}\right)^{m(T_1-t)} \left(1 + \frac{f_{12}}{m}\right)^{m(T_2-T_1)}$$

△

Example 5.1.5. One year from now, your business plan requires a loan of $100,000$ to purchase new equipment. You plan to repay the loan one year after that. You want to arrange the interest rate of the loan today, rather than gamble on the interest rate one year from now.

Assume all rates are for continuous compounding. The current time is $t = 0$. The interest rate for period 0 to 1 is 8%. The interest rate for period 0 to 2 is 9%. What must the interest rate on the loan be (assuming no-arbitrage)?

Have

- $t = 0, T_1 = 1, T_2 = 2,$
- $r_1 = 0.08$ (rate agreed at $t = 0$ to borrow from $t = 0$ to $T_1 = 1$)
- $r_2 = 0.09$ (rate agreed at $t = 0$ to borrow from $t = 0$ to $T_2 = 2$)

Want f_{12} = forward rate agreed at $t = 0$ to borrow from $T_1 = 1$ to $T_2 = 2$.

DRAW DIAGRAM LIKE Figure 5.1.

$$e^{r_2(T_2-t)} = e^{r_1(T_1-t)} e^{f_{12}(T_2-T_1)}.$$

Solve for f_{12}:

$$f_{12} = \frac{r_2(T_2-t) - r_1(T_1-t)}{T_2 - T_1} = \frac{0.09(2-0) - 0.08(1-0)}{2-1} = 0.1 = 10\%$$

$\triangle$

Definition 5.1.2. The zero rate for the period starting now and ending M years later is called the M**-year zero rate**. It is the zero rate for period t to $T = t + M$.

The forward rate for the period starting M years from now and and ending N years later is the M**-year forward** N**-year rate**. It is denoted f_{MN}. It is also called the MyNy rate. It is the forward rate at current time t for period $T_1 = t + M$ to $T_2 = T_1 + N = t + M + N$. $\triangle$

Remark. Notice that the notation for the one-year forward two-year rate is f_{12}. This conflicts with the notation f_{12} used earlier for the forward rate for period T_1 to T_2. This is unfortunate, but both notations are used by the textbook. If you are careful, you will be able to keep things straight.

Example 5.1.6. Assume all rates are annually compounded. The one-year and two-year zero rates are 1% and 2%, respectively.

(a) What is the one-year forward one-year rate?

Given:
$r_1 = 0.01$ = zero rate for period starting now and ending 1 year later,
$r_2 = 0.02$ = zero rate for period starting now and ending 2 years later.
Want: f_{11} = 1y1y forward rate = one-year forward one-year rate = forward rate for the period starting one year from now and ending one year later.

We can assume current time $t = 0$. Then $T_1 = 1$, $T_2 = 2$.
Given
$r_1 = 0.01 =$ zero rate for period 0 to 1
$r_2 = 0.02 =$ zero rate for period 0 to 2
Want
f_{11} = forward rate for period 1 to 2

DRAW DIAGRAN LIKE Figure 5.1.

Then

$$(1 + r_1)^{T_1 - t}(1 + f_{11})^{T_2 - T_1} = (1 + r_2)^{T_2 - t}$$

Solving for f_{11} gives

$$f_{11} = \left(\frac{(1+r_2)^{T_2-t}}{(1+r_1)^{T_1-t}}\right)^{1/(T_2-T_1)} - 1 = \left(\frac{(1+0.02)^{2-0}}{(1+0.01)^{1-0}}\right)^{1/(2-1)} - 1$$
$$= 0.0300990099\ldots = 3.00990099\ldots\%$$

(b) If the two-year one-year forward rate is 3%, what is the three-year zero rate?

In class, I will draw a picture like Figure 5.1.

We can assume current time $t = 0$.
$f_{21} = 0.03$ = 2y1y forward rate = two-year forward one-year rate = forward rate for the period starting two years from now and ending one year later = forward rate at $t = 0$ for $T_2 = 2$ to $T_3 = 3$
$r_2 = 0.02$ = zero rate for 0 to 2
r_3 = zero rate for 0 to 3 = ???
Then

$$(1+r_2)^{T_2-t}(1+f_{21})^{T_3-T_2} = (1+r_3)^{T_3-t}$$

i.e.,

$$(1+0.02)^{2-0}(1+0.03)^{3-2} = (1+r_3)^{3-0}.$$

Solving for r_3 gives

$$r_3 = 0.02332249903\ldots.$$

△

5.2 Forward Zero Coupon Bond Prices

Definition 5.2.1. Let $t \le T_1 \le T_2$. Consider a forward contract with delivery price K and maturity T_1 where the underlying asset is a ZCB with maturity T_2. The price of the ZCB at t is $S_t = Z(t, T_2)$.

The value/price of the forward is denoted $V_K(t, T_1, T_2)$.

The forward price of the ZCB (or forward ZCB price) is denoted $F(t, T_1, T_2)$.

△

A forward on a ZCB is a forward on an asset paying no income. So Result 4.6.1 (with $T = T_1$ and $S_t = Z(t, T_2)$) implies

Result 5.2.1. Let $t \le T_1 \le T_2$. Then

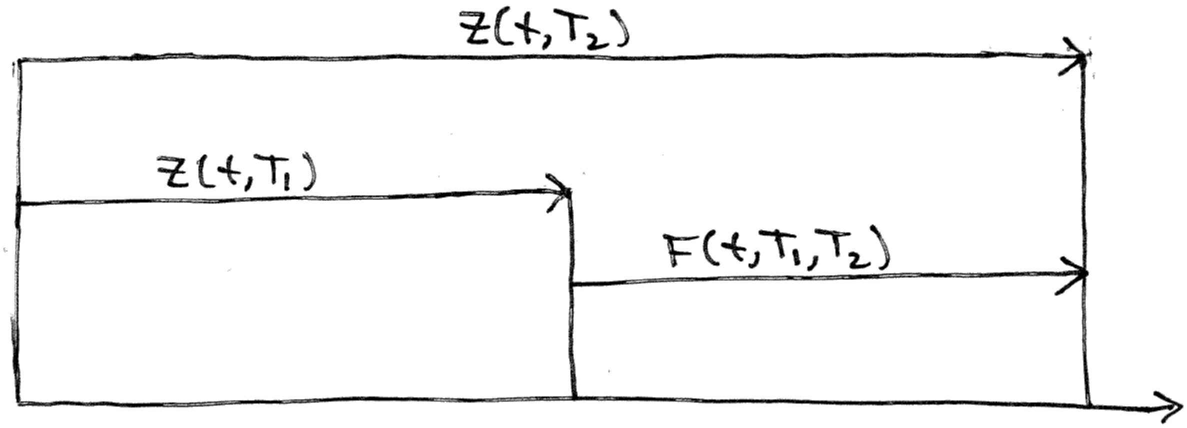
$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}$$

or equivalently

$$Z(t, T_2) = Z(t, T_1)F(t, T_1, T_2).$$

△

Compare this to Result 5.1.2. Compare also Figure 5.1 to Figure 5.2.



Result 5.2.2. Let $t \le T_1 \le T_2$. If f_{12} is the forward rate at t for T_1 to T_2 with continuous compounding

$$F(t, T_1, T_2) = e^{-f_{12}(T_2 - T_1)}$$

△

Similar results hold for compounding m times per year and for simple interest.

We give two proofs.

Proof 1. If r_i is the continuous zero rate for t to T_i, then $Z(t, T_i) = e^{-r_i(T_i - t)}$, so Result 5.1.2 says

$$Z(t, T_2) = Z(t, T_1)e^{-f_{12}(T_2 - T_1)}.$$

On the other hand, Result 5.2.1 says

$$Z(t, T_2) = Z(t, T_1)F(t, T_1, T_2).$$

It follows that

$$F(t, T_1, T_2) = e^{-f_{12}(T_2 - T_1)}.$$

□

Proof 2. Consider

A: Time t: $M = F(t, T_1, T_2)$ ZCBs with maturity T_1. Enter a long forward contract with maturity T_1 and delivery price $K = M = F(t, T_1, T_2)$ on a ZCB with maturity T_2. These contracts have no cost to enter, by the definition of $F(t, T_1, T_2)$. Note that at T_1 the M ZCBs will give us M cash while the long forward will give us 1 ZCB with maturity T_2 and require us to pay M cash. That will leave us with 1 ZCB with maturity T_2.

B: Time t: $N = e^{-f_{12}(T_2-T_1)}$ ZCBs with maturity T_1. Agree at t to lend N (the amount that will be gained from the ZCBs) from T_1 until T_2 at rate f_{12}.

Then $V^A(T_2) = Z(T_2, T_2) = 1$ and $V^B(T_2) = Ne^{f_{12}(T_2-T_1)} = 1$. Thus $V^A(T_2) = V^B(T_2)$ with probability one. By replication,

$$V^A(t) = V^B(t)$$

or equivalently

$$F(t, T_1, T_2)Z(t, T_1) = e^{-f_{12}(T_2-T_1)}Z(t, T_1).$$

Divide by $Z(t, T_1)$. □

Example 5.2.3. Assume $t \leq T_1 \leq T_2$, where t = current time. What can you say about interest rates between T_1 and T_2 if
(a) $Z(t, T_1) = Z(t, T_2)$.
(b) $Z(t, T_1) > 0$ and $Z(t, T_2) = 0$.

For annual compounding:

$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)} = (1 + f_{12})^{-(T_2-T_1)}.$$

(a) We have

$$(1 + f_{12})^{-(T_2-T_1)} = \frac{Z(t, T_2)}{Z(t, T_1)} = 1,$$

so the forward rate f_{12} between T_1 and T_2 must be 0.
(b) We have

$$(1 + f_{12})^{-(T_2-T_1)} = \frac{Z(t, T_2)}{Z(t, T_1)} = 0,$$

so the forward rate f_{12} between T_1 and T_2 must be ∞.

The same is true for any compounding frequency.

For continuous compounding:

$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)} = e^{-f_{12}(T_2-T_1)}.$$

(i) We have

$$e^{-f_{12}(T_2-T_1)} = \frac{Z(t, T_2)}{Z(t, T_1)} = 1,$$

so the forward rate f_{12} between T_1 and T_2 must be 0.
(ii) We have

$$e^{-f_{12}(T_2-T_1)} = \frac{Z(t, T_2)}{Z(t, T_1)} = 0,$$

so the forward rate f_{12} between T_1 and T_2 must be ∞. △

Exercise 5.2.1. Give a third proof of Result 5.2.2 by assuming $F(t,T_1,T_2) \neq e^{-f_{12}(T_2-T_1)}$ and constructing an arbitrage portfolio.

5.3 Libor

Definition 5.3.1. The simple interest rate at which banks borrow or lend money to each other is called the **libor rate** or **libor**. △

Libor stands for London InterBank Offered Rate. Most interest rate derivatives are libor derivatives.

Definition 5.3.2. $L_t[t, t+\alpha]$ is the libor (simple interest) rate agreed at t to lend/borrow from t to $t+\alpha$. Here $\alpha > 0$.

In other words, $L_t[t, t+\alpha]$ is the rate agreed at time t such that we lend/borrow N at time T and receive/return

$$N\left(1 + \alpha L_t[t, t+\alpha]\right)$$

at time $t+\alpha$.

When $\alpha = 1$, the libor $L_t[t, t+\alpha]$ is called the 1-year libor or twelve-month libor or 12mL.

When $\alpha = 0.25$, the libor $L_t[t, t+\alpha]$ is called the quarter-year libor or three-month libor or 3mL. △

Example 5.3.1. The current six-month libor is 3.46%. If a bank lends $2,000,000$ at this rate, how much will they receive at term?

$$N(1 + \alpha L_t[t, t+\alpha]) = 2,000,000(1 + (0.5)(0.0346)) = 2,034,600$$

△

Remark 5.3.2. For $t < T$, you might guess that $L_t[T, T+\alpha]$ is the libor rate agreed at t to lend/borrow from T to $T+\alpha$ and it should be called the forward libor rate.

To match the textbook, we will define $L_t[T, T+\alpha]$ in a different way (Definition 5.6.1), but it will turn out to be equivalent to the above guess (Result 5.6.4).

△

Definition 5.3.3. If t is the current time and $T > t$, then $L_T[T, T+\alpha]$ is the libor that will be available at T to lend/borrow from T to $T+\alpha$. It is unknown at time t, thus it is a random variable and said to be **floating**. △

Remark 5.3.3. In practice, libor (London InterBank Offered Rate) is the interest rate at which a bank offers to lend money and libid (London InterBank Bid Rate) is the interest rate at which a bank is willing to borrow money. For large transactions the two rates are very close together. To keep things simple, we assume libor = libid. △

5.4 Fixed and Floating Payments

Let $t < T < T + \alpha$, where t is the current time. If notional 1 is invested at simple interest rate r from time T until time $T + \alpha$, then it will grow to $1 + \alpha r$. The amount of interest accrued is αr. If r is a fixed (known, non-random) rate K, then the interest accrued is αK. If r is the floating (unknown, random) rate $L_T[T, T + \alpha]$, then the interest accrued is $L_T[T, T + \alpha]$.

Result 5.4.1. Fix a constant $K > 0$. The value at t of an agreement to receive the fixed (i.e., known, non-random) payment αK at time $T + \alpha$ is

$$\alpha K Z(t, T + \alpha)$$

△

Proof. The agreement is equivalent to αK ZCBs. □

Result 5.4.2. The value at t of an agreement to receive the floating (i.e., unknown, random) payment $\alpha L_T[T, T + \alpha]$ at time $T + \alpha$ is

$$Z(t, T) - Z(t, T + \alpha),$$

which is the same value as an agreement to receive 1 at time T and pay back 1 at time $T + \alpha$. △

Remark. Notice the value is non-random. It does not even depend on the distribution of the random variable $L_T[T, T + \alpha]$.

Proof. Consider two portfolios.

A: An agreement to receive $\alpha L_T[T, T + \alpha]$ at time $T + \alpha$.

B:

- Time t: An agreement to receive 1 at time T and pay back 1 at time $T + \alpha$. (Equivalently, $+1$ ZCB with maturity T, -1 ZCB with maturity $T + \alpha$.)
- Time T: Deposit the 1 received from the agreement at random Libor $L_T[T, T + \alpha]$ until time $T + \alpha$.

At time $T + \alpha$:

$$\begin{aligned} V^A(T + \alpha) &= \alpha L_T[T, T + \alpha], \\ V^B(T + \alpha) &= 1 + \alpha L_T[T, T + \alpha] - 1 = \alpha L_T[T, T + \alpha]. \end{aligned}$$

By replication, $V^A(t) = V^B(t) = Z(t, T) - Z(t, T + \alpha)$ □

5.5 Forward Rate Agreements

Definition 5.5.1. A **forward rate agreement (FRA)** a derivative to exchange the floating (unknown, random) amount of cash $\alpha L_T[T, T+\alpha]$ for the fixed (known, non-random) amount of cash αK.

It has three parameters: K = fixed rate or delivery price; T = maturity; α = term length or daycount fraction or accrual factor.

The buyer (long counterparty) of the FRA agrees at time $t \leq T$ to

$$\left.\begin{array}{l}\text{receive } \alpha L_T[T, T+\alpha] \\ \text{pay } \alpha K\end{array}\right\} \text{ at time } T+\alpha.$$

The seller (short counterparty) agrees to do the opposite.

The payout of the FRA for the buyer is

$$g(L_T[T, T+\alpha]) = \alpha L_T[T, T+\alpha] - \alpha K$$

at time $T+\alpha$. Here $g(x) = \alpha x - \alpha K$ is the payout function.

The value of the FRA at time $t \leq T$ is denoted $V_K(t, T)$. △

$\alpha L_T[T, T+\alpha]$ is the amount of simple interest that would be accrued on notional 1 at floating (unknown, random) rate $L_T[T, T+\alpha]$ over period T to $T+\alpha$.

αK is the amount of simple interest that would be accrued on notional 1 at fixed (known, non-random) rate K over period T to $T+\alpha$.

Forwards are derivatives of the price S_T of an asset. FRAs are derivatives of the libor rate $L_T[T, T+\alpha]$.

Result 5.5.1. The value of the FRA with fixed rate K, maturity T, and term length α at time $t \leq T$ is

$$V_K(t, T) = Z(t, T) - Z(t, T+\alpha) - \alpha K Z(t, T+\alpha). \tag{5.5.1}$$

△

Proof. Combine Results 5.4.1 and 5.4.2. □

Remark 5.5.2. Notice the value at present time t does not depend on the distribution of the random variable $L_T[T, T+\alpha]$. This is just like how the value of a forward contract on an asset does not depend on the the random variable S_T (asset price). △

5.6 Forward Libor Rate

Definition 5.6.1. Let T and α be given. The **forward libor rate** at time $t \leq T$ is the number $L_t[T, T+\alpha]$ such that the forward rate agreement with fixed rate $K = L_t[T, T+\alpha]$, maturity T, and term length α is $V_K(t,T) = 0$ at time t. Thus,

$$V_K(t,T) = 0 \text{ if and only if } K = L_t[T, T+\alpha].$$

△

Note the similarity with the forward price $F(t,T)$ for a forward contract on an asset.

Result 5.6.1. Let T and α be given. The forward libor rate at time $t \leq T$ is

$$L_t[T, T+\alpha] = \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)}. \tag{5.6.1}$$

△

Proof. Set $K = L_t[T, T+\alpha]$ and $V_K(t,T) = 0$ in (5.5.1), and solve for $L_t[T, T+\alpha]$. □

Example 5.6.2. Suppose the one-year and two-year continuous zero rates are 1% and 2%, respectively. What is the one-year forward one-year libor rate?

Assume current time $t = 0$.

one-year continuous zero = continuous zero rate for period 0 to 1 = 1%

two-year continuous zero = continuous zero rate for period 0 to 2 = 2%

The m-year forward n-year libor rate is the forward libor rate for the period starting m years from now and ending n years after that.

Want: one-year forward one-year libor rate = $L_t[T, T+\alpha]$
with $t = 0$, $T = 1$, $\alpha = 1$, $T + \alpha = 2$.

$$\begin{aligned} L_t[T, T+\alpha] &= \frac{Z(t,T) - Z(t, T+\alpha)}{\alpha Z(t, T+\alpha)} = \frac{Z(0,1) - Z(0,2)}{Z(0,2)} \\ &= \frac{e^{-(0.01)(1)} - e^{-(0.02)(2)}}{e^{-(0.02)(2)}} = 0.0304545\ldots \end{aligned}$$

△

Result 5.6.3. For $t < T < T + \alpha$,

$$Z(t, T+\alpha) = Z(t,T) \frac{1}{1 + \alpha L_t[T, T+\alpha]}.$$

△

Proof. Rearrange (5.6.1). □

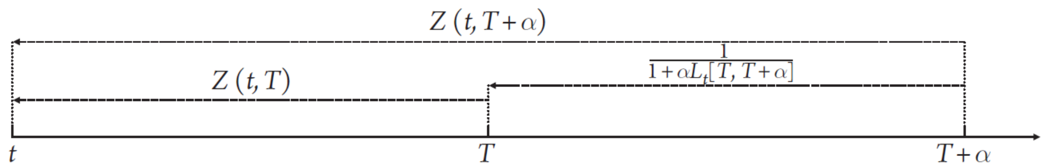


Figure 5.2: Discounting by ZCBs and the forward libor rate

Compare Result 5.6.3 to Results 5.1.2 and 5.1.4. Compare also Figure 5.2 to Figures 5.1 and 5.2.

Result 5.6.4. Let $t < T$ and $\alpha > 0$. The forward libor rate $L_t[T, T+\alpha]$ is the libor (simple interest) rate that we can agree on at t to lend/borrow N at time T and receive/return

$$N(1 + \alpha L_t[T, T + \alpha])$$

at time $T + \alpha$. △

Proof. We consider the case of lending. The borrowing case is analogous.

We agree at time t to do the following:

Time t: We go short on N FRAs with with maturity T, term length α, and fixed rate $K = L_t[T, T + \alpha]$. By the definition of the forward libor rate (Definition 5.6.1), this is free to do.

Time T: Lend N at the random libor rate $L_T[T, T + \alpha]$ until $T + \alpha$.

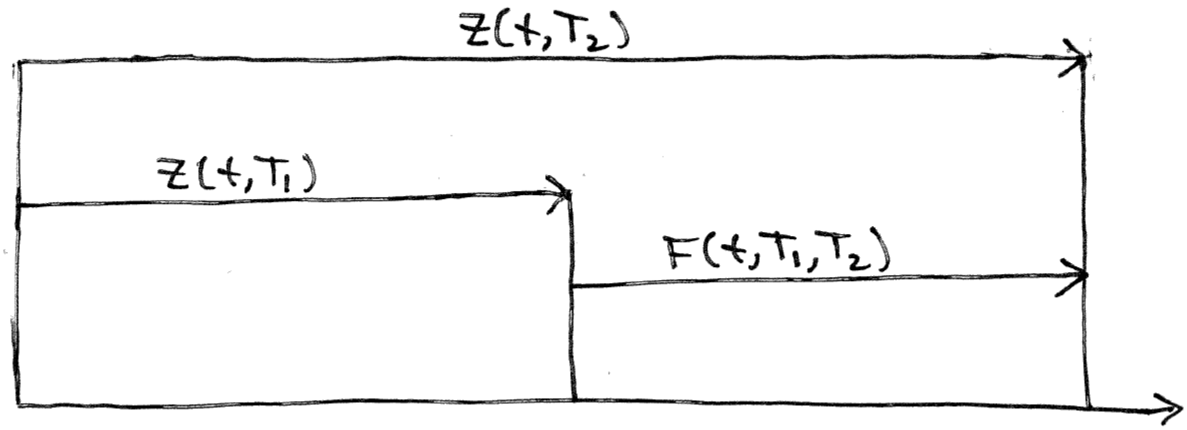
Time $T + \alpha$: Receive $N(1 + \alpha L_T[T, T + \alpha])$ because of the loan we made at T. Receive $N\alpha L_t[T, T + \alpha]$ and pay $N\alpha L_T[T, T + \alpha])$ because of the short FRAs we entered at t. In total, we get

$$N(1+\alpha L_T[T,T+\alpha])+N\alpha L_t[T,T+\alpha]-N\alpha L_T[T,T+\alpha]) = N(1+N\alpha L_t[T,T+\alpha]).$$

Summary: At time t we entered into an agreement at no cost that allowed us to give out N at time T and get back $N(1 + \alpha L_t[T, T + \alpha])$ at time. □

5.7 Forward Rates Unified

Let's write down the pieces.



Result 5.7.1 (Restatement of Result 5.2.1)**.** For $t \le T_1 \le T_2$,

$$Z(t,T_2) = Z(t,T_1)F(t,T_1,T_2)$$

△

Result 5.7.2. For $t \le T$,

$$Z(t,T) = \begin{cases} e^{-r(T-t)} & \text{if } r \text{ is continuous zero rate at } t \text{ for period } t \text{ to } T \\ \left(1+\frac{r}{m}\right)^{-m(T-t)} & \text{if } r \text{ is } m\text{-times-per-year-compounding zero rate at } t \text{ for period } t \text{ to } T \\ (1+r(T-t))^{-1} & \text{if } r = L_t[t,T] \text{ is libor (simple) rate at } t \text{ for period } t \text{ to } T \end{cases} \tag{5.7.1}$$

△

Proof. Combine Results 3.2.1, 3.2.5, 3.2.6. □

Result 5.7.3. For $t \le T_1 \le T_2$,

$$F(t,T_1,T_2) = \begin{cases} e^{-f_{12}(T_2-T_1)} & \text{if } r \text{ is continuous forward rate at } t \text{ for period } T_1 \text{ to } T_2 \\ \left(1+\frac{f_{12}}{m}\right)^{-m(T_2-T_1)} & \text{if } f_{12} \text{ is } m\text{-times-per-year-compounding forward rate} \\ & \text{at } t \text{ for period } T_1 \text{ to } T_2 \\ (1+f_{12}(T_2-T_1))^{-1} & \text{if } r = L_t[T_1,T_2] \text{ is forward libor (simple) rate} \\ & \text{at } t \text{ for period } T_1 \text{ to } T_2 \end{cases} \tag{5.7.2}$$

△

Proof. Mimic the proof of Result 5.2.2. □

The next example shows how the pieces fit together.

Example 5.7.4. The two-year forward one-year libor rate is 3%. The price of a ZCB maturing in 3 years is 0.7. Find the continuous two-year zero rate.

We are given

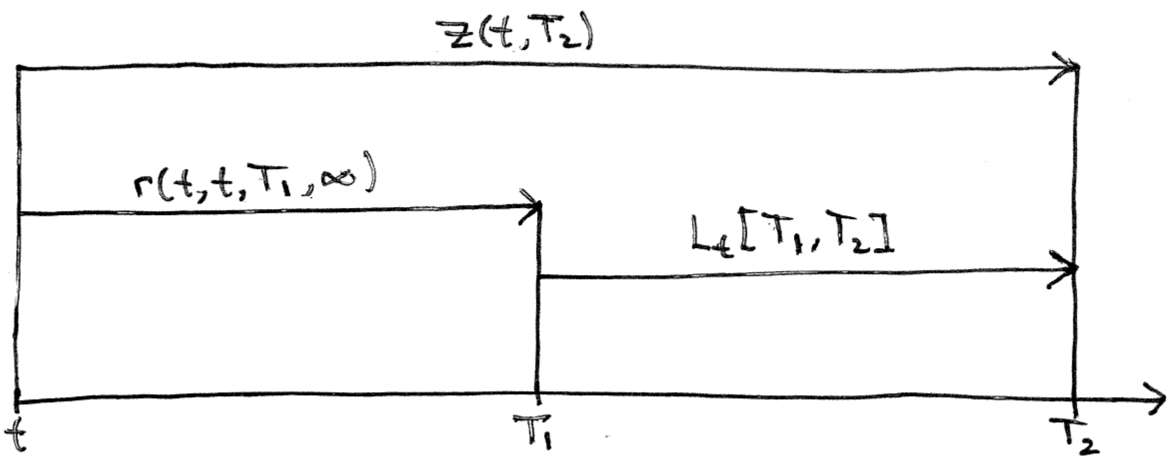
$$L_t[T_1, T_2] = 0.03 \quad \text{and} \quad Z(t, T_2) = 0.7,$$

where t is the current time, $T_1 = t + 2$, and $T_2 = T_1 + 1 = t + 3$.

We want $r =$ continuous zero rate at t for period $T_1 = t + 2$ and $T_2 = t + 3$.

By the results above, the quantities are related by

$$Z(t, T_2) = e^{-r\cdot(T_1 - t)}(1 + (T_2 - T_1)L_t[T_1, T_2])^{-1}.$$



MAKE THIS FIGURE BETTER

Solving for r and substituting gives

$$\begin{aligned} r &= \frac{1}{T_1 - t} \ln\left((Z(t, T_2))^{-1}(1 + (T_2 - T_1)L_t[T_1, T_2])^{-1}\right) \\ &= \frac{1}{2} \ln\left((0.7)^{-1}(1 + (1)(0.03))^{-1}\right) \\ &= 0.163558\ldots \end{aligned}$$

$\triangle$

Chapter 6

Interest Rate Swaps

Interest rate swaps are the most widely traded and most liquid of all over-the-counter derivative contracts.

6.1 Swap Definition

Definition 6.1.1. A **swap** is an agreement between two counterparties to exchange a sequence of floating (unknown, random) cash flows for a sequence of fixed (known, non-random) cash flows.

Parameters: K = fixed rate or delivery price; T_0 = start date; T_n = maturity; $T_1, \ldots, T_n$ = payment dates.

We assume $T_{i+1} = T_i + \alpha$ for every i with fixed $\alpha > 0$.

So knowing n, T_0, and T_n is enough to determine α and every T_i: $T_n = T_0 + n\alpha$ and $T_i = T_0 + i\alpha$.

One counterparty (called "buyer", "payer", or "long") agrees at t to pay αK and receive $\alpha L_{T_i}[T_i, T_i+\alpha]$ at $T_i+\alpha$ for each $i = 0, \ldots, n-1$. The other counterparty (called "seller", "receiver", or "short") does the opposite.

Therefore a swap is a sequence of forward rate agreements (FRAs).

The **floating leg** of the swap consists of the payments $\alpha L_{T_i}[T_i, T_i+\alpha]$ at times $T_{i+1} = T_i+\alpha$ for $i = 0, \ldots, n-1$.

The **fixed leg** of the swap consists of the payments αK at times $T_{i+1} = T_i + \alpha$ for $i = 0, \ldots, n-1$.

The buyer receives the floating leg and pays the fixed leg. The seller does the opposite. $\triangle$

$\alpha L_{T_i}[T_i, T_i + \alpha]$ is the amount of simple interest that would be accrued on notional 1 at floating (unknown, random) rate $L_{T_i}[T_i, T_i + \alpha]$ from T_i to $T_i + \alpha$ and paid at $T_i + \alpha$.

αK is the amount of simple interest that would be accrued on notional 1 at fixed (known, non-random) rate K from T_i to $T_i + \alpha$ and paid at $T_i + \alpha$.

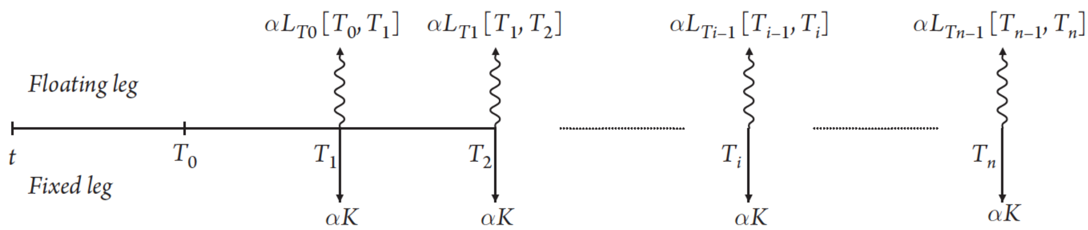


Figure 6.1: A swap

Remark 6.1.1. The definition above is for a swap with notional 1. To treat a swap with notional N, either consider N swaps of notional 1 or make the payments $N\alpha L_{T_i}[T_i, T_i+\alpha]$ and $N\alpha K$. △

Remark 6.1.2. There are more general types of swaps. For example, α can be different for each period, so that $T_{i+1} = T_i + \alpha_i$. As another example, the payment times for the floating and fixed legs may differ. This will be explored in the exercises. △

6.2 Value of Swap

Consider a swap from T_0 to T_n with fixed rate K.

Definition 6.2.1. For $t \leq T_0$,

$$\begin{aligned} V_K^{SW}(t) &= \text{value of the swap at } t \\ V^{FL}(t) &= \text{value of the floating leg at } t \\ V_K^{FXD}(t) &= \text{value of the fixed leg } t \end{aligned}$$

△

Result 6.2.1.

$$V_K^{SW}(t) = V^{FL}(t) - V_K^{FXD}(t).$$

△

Note that $V_K^{SW}(t)$ is the value to the buyer. The value to the seller is the $-V_K^{SW}(t)$.

The fixed leg is a sequence of ZCBs with maturities at $T_1, \ldots, T_n$, so

Result 6.2.2.

$$V_K^{FXD}(t) = \sum_{i=1}^{n} \alpha K Z(t, T_i) = K P_t[T_0, T_n],$$

where we define

$$P_t[T_0, T_n] = \sum_{i=1}^{n} \alpha Z(t, T_i).$$

$\triangle$

The floating leg is a sequence of floating libor payments at $T_1, \ldots, T_n$, so by Result 5.4.2 we have

Result 6.2.3.

$$\begin{aligned} V^{FL}(t) &= \sum_{i=1}^{n} (\text{value at } t \text{ of payment } \alpha L_{T_{i-1}}[T_{i-1}, T_i] \text{ at } T_i = T_{i-1} + \alpha) \\ &= \sum_{i=1}^{n} \alpha L_t[T_{i-1}, T_i] Z(t, T_i) = \sum_{i=1}^{n} [Z(t, T_{i-1}) - Z(t, T_i)] \\ &= Z(t, T_0) - Z(t, T_n) \end{aligned}$$

$\triangle$

This says the value of receiving a stream of libor interest payments on an investment of 1 is equal to the value of receiving one dollar at the beginning of the stream and paying it back at the end. We simply take the dollar and repeatedly invest in a sequence of libor deposits.

By combining Results 6.2.1, 6.2.2, and 6.2.3, we can write $V_K^{SW}(t)$ as a linear combination of ZCB prices.

Result 6.2.4.

$$V_K^{SW}(t) = Z(t, T_0) - Z(t, T_n) - \alpha K \sum_{i=1}^{n} Z(t, T_i).$$

$\triangle$

Example 6.2.5. Consider a swap starting now with fixed rate 3%, quarterly payment frequency, and ending in 2 years. Suppose the quarterly compounded zero rates for all payment times are 2%. Find the present value of
(a) the floating leg
(b) the swap

(a) Given $t = T_0$, $T_n - t = 2$, $\alpha = 0.25$, $r_4 = 0.02$.

$$\begin{aligned} V^{FL}(t) &= Z(t, T_0) - Z(t, T_n) = 1 - (1 + r_4/4)^{-4(T_n - t)} \\ &= 1 - (1 + 0.02/4)^{-4(2)} = 0.039114\ldots \end{aligned}$$

(b) Since $V_K^{SW}(t) = V^{FL}(t) - V_K^{FXD}(t)$, we just need $V_K^{FXD}(t)$.

Have $K = 0.03$. Have $T_i - t = T_i - T_0 = i\alpha = i0.25$. Using $T_n = T_0 + n\alpha$ (or drawing a picture), we get $n = 8$. Then

$$V_K^{FXD}(t) = \alpha K \sum_{i=1}^{n} Z(t, T_i) = \alpha K \sum_{i=1}^{n} (1 + r_4/4)^{-4(T_i - t)} = \alpha K \sum_{i=1}^{n} (1 + r_4/4)^{-i}$$

The sum

$$\sum_{i=1}^{n} (1 + r_4/4)^{-i} = \sum_{i=1}^{8} (1 + 0.02/4)^{-i}$$

only has 8 terms, so we could just compute it directly. Or we can use the formula

$$\sum_{i=1}^{n} (1 + r_4/4)^{-i} = \frac{1 - (1 + r_4/4)^{-n}}{r_4/4}.$$

Then

$$V_K^{FXD}(t) = \alpha K \frac{1 - (1 + r_4/4)^{-n}}{r_4/4} = (0.25)(0.03) \frac{1 - (1 + 0.02/4)^{-8}}{0.02/4} = 0.05867219\ldots$$

Therefore

$$\begin{aligned} V_K^{SW}(t) &= V^{FL}(t) - V_K^{FXD}(t) \\ &= 1 - (1 + 0.02/4)^{-4(2)} - (0.25)(0.03) \frac{1 - (1 + 0.02/4)^{-8}}{0.02/4} \\ &= -0.0195573\ldots \end{aligned}$$

△

6.3 Forward Swap Rate

Definition 6.3.1. The **forward swap rate** $y_t[T_0, T_n]$ is the special number such that the value at t of the swap from T_0 to T_n with fixed rate $K = y_t[T_0, T_n]$ has value $V_K^{SW}(t) = 0$. Thus,

$$V_K^{SW}(t) = 0 \quad \text{if and only if } K = y_t[T_0, T_n].$$

△

Note the similarity to the forward price $F(t, T)$ and forward libor rate $L_t[T, T + \alpha]$

Result 6.3.1. The forward swap rate at $t \leq T_0$ for a swap from T_0 to T_n is

$$y_t[T_0, T_n] = \frac{Z(t, T_0) - Z(t, T_n)}{P_t[T_0, T_n]} = \frac{\sum_{i=1}^{n} \alpha L_t[T_{i-1}, T_i] Z(t, T_i)}{\sum_{i=1}^{n} \alpha Z(t, T_i)}. \tag{6.3.1}$$

△

Proof. We use Results 6.2.1, 6.2.2, 6.2.3. Setting $K = y_t[T_0, T_n]$ gives

$$\begin{aligned} V_K^{SW}(t) = 0 \quad &\Leftrightarrow \quad V_K^{FXD}(t) = V^{FL}(t) && (6.3.2)\\ &\Leftrightarrow \quad KP_t[T_0, T_n] = Z(t, T_0) - Z(t, T_n) && (6.3.3)\\ &\Leftrightarrow \quad K\sum_{i=1}^{n} \alpha Z(t, T_i) = \sum_{i=1}^{n} \alpha L_t[T_{i-1}, T_i]Z(t, T_i) && (6.3.4) \end{aligned}$$

Solving (6.3.3) for K gives the first equality in (6.3.1).

Solving (6.3.4) for K gives the second equality in (6.3.1). □

6.4 Value of Swap in Terms of Forward Swap Rate

Result 6.4.1. The value at $t \leq T_0$ of a swap from T_0 to T_n with fixed rate K is

$$V_K^{SW}(t) = (y_t[T_0, T_n] - K)P_t[T_0, T_n].$$

△

Remark. Compare this to the relationship between the value of a forward on an asset and the forward price of the asset.

$$V_K(t, T) = (F(t, T) - K)Z(t, T).$$

Proof. By Results 6.2.1, 6.2.2, and 6.2.3, we have

$$V_K^{SW}(t) = V^{FL}(t) - V_K^{FXD}(t) = Z(t, T_0) - Z(t, T_n) - KP_t[T_0, T_n].$$

By Result 6.3.1, we have

$$Z(t, T_0) - Z(t, T_n) = y_t[T_0, T_n]P_t[T_0, T_n].$$

□

6.5 Swaps as Difference Between Bonds

Definition 6.5.1. A **fixed rate bond** with notional N, coupon c, start date T_0, maturity T_n, and term length α is an asset that pays N at time T_n and also coupon payments αNc at times T_i for $i = 1, \dots, n$, where $T_{i+1} = T_i + \alpha$.

If $N = 1$, the price at t of the fixed rate bond is denoted $B_c^{FXD}(t)$. △

Definition 6.5.2. A **floating rate bond** with notional N, start date T_0, maturity T_n, and term length α is an asset that pays N at time T_n and also coupon payments $\alpha N L_{T_{i-1}}[T_{i-1}, T_i]$ at times T_i for $i = 1, \ldots, n$, where $T_{i+1} = T_i + \alpha$.

If $N = 1$, the price at t of the floating rate bond is denoted $B^{FL}(t)$. △

Result 6.5.1. Consider a swap from T_0 to T_n with fixed rate K. For $t \leq T_0$,

$$V_K^{SW}(t) = B^{FL}(t) - B_K^{FXD}(t)$$

△

Proof. The fixed rate bond with notional 1 and coupon K equals the fixed leg of the swap plus a payment of 1 at T_n.

The floating rate bond with notional 1 equals the floating leg of the swap plus a payment of 1 at T_n.

Therefore

$$\begin{aligned} B^{FL}(t) - B_K^{FXD}(t) &= (V^{FL}(t) + Z(t, T_n)) - (V_K^{FXD}(t) + Z(t, T_n)) \\ &= V^{FL}(t) - V_K^{FXD}(t) \\ &= V_K^{SW}(t). \end{aligned}$$

□

6.6 Par or Spot-Starting Swaps

Definition 6.6.1. When $t = T_0$, we call $y_{T_0}[T_0, T_n]$ a **par swap rate** or **spot-starting swap rate**. △

Remark 6.6.1. Given par swap rates $y_{T_0}[T_0, T_k]$ for every T_k, we can recover the ZCB price $Z(T_0, T_k)$ for every T_k by using Result 6.3.1:

$$y_t[T_0, T_k] = \frac{Z(t, T_0) - Z(t, T_k)}{P_t[T_0, T_k]}$$

This process is known as bootstrapping and is used frequently in practice. It will be explored in the exercises. △

Reminder:

Definition 6.6.2. A **fixed rate bond** with notional N, coupon c, start date T_0, maturity T_n, and term length α is an asset that pays N at time T_n and also coupon payments $\alpha N c$ at times T_i for $i = 1, \ldots, n$, where $T_{i+1} = T_i + \alpha$.

If $N = 1$, the price at t of the fixed rate bond is denoted $B_c^{FXD}(t)$. △

Result 6.6.2.

$$B_c^{FXD}(T_0) = 1 \text{ if and only if } c = y_{T_0}[T_0, T_n] = \text{ par swap rate}$$

$\triangle$

Remark 6.6.3. Because of this result, the par swap rate $y_{T_0}[T_0, T_n]$ is sometimes called the **coupon rate**. $\triangle$

Remark 6.6.4. To put it other terms, this result says we can invest 1 at time T_0, receive 1 back at time T_n, and receive fixed payments of $\alpha y_{T_0}[T_0, T_n]$ at times $T_1, \ldots, T_n$ in between. And this isn't true if $y_{T_0}[T_0, T_n]$ is replaced by any other coupon c. $\triangle$

Proof. By definition of the par swap rate $y_{T_0}[T_0, T_n]$, we have

$$V_c^{SW}(T_0) = 0 \quad \text{if and only if} \quad c = y_{T_0}[T_0, T_n].$$

But

$$V_c^{SW}(T_0) = B^{FL}(T_0) - B_c^{FXD}(T_0)$$

and

$$B^{FL}(T_0) = V^{FL}(T_0) + Z(T_0, T_n) = Z(T_0, T_0) - Z(T_0, T_n) + Z(T_0, T_n) = Z(T_0, T_0) = 1.$$

Therefore

$$1 - B_c^{FXD}(T_0) = 0 \quad \text{if and only if} \quad c = y_{T_0}[T_0, T_n].$$

$\square$

Chapter 7

Futures Contracts

7.1 Physical and Cash Settlement

We briefly recall the definition of a forward contract.

Fix an asset. Let S_t be its price at time t. Fix times $t \leq T$. A forward contract (or forward) on the asset with maturity T and delivery price K is an agreement to trade the asset. At maturity T, the long counterparty receives (pays if negative) value

$$S_T - K = F(T,T) - K.$$

The short counterparty does the opposite. There are no payments in between. If K equals the forward price $F(t,T)$, the contract has no cost to enter (the value is zero) at time t.

Definition 7.1.1. In a **physically settled** forward, at time T the long counterparty receives the asset (value S_T) and pays K cash. In a **cash settled** forward, the long counterparty receives S_T cash and pays K cash. $\triangle$

Previously, we have always considered physically settled rather than cash settled forwards. Both have the same value at maturity T, hence at any time $t \leq T$. However, a cash settled forward has no exposure to the asset price after T, while a physically settled contract (where one owns the asset at T) continues to have exposure to asset price movements.

Other types of derivative contracts can be either physically-settled or cash-settled. When we define futures contracts below, it may be helpful to cash settlement in mind, rather than physical settlement.

7.2 Futures Definition

Definition 7.2.1. Fix an asset. Let S_t be its price at time t. Fix times $t \leq T$. A **futures contract** (or **future**) on the asset with maturity T and delivery price K is an agreement to trade the asset. Unlike a forward contract, there are payments every day until the maturity date T. We describe the payments below. If K equals the **futures price** $\Phi(t,T)$, the contract has no cost to enter (i.e., the value is zero) at time t.

Let $\Delta = \frac{1}{365}$. Since we measure time in years, day 1 is time $t+\Delta$ and day i is time $t+i\Delta$. We define day n to be maturity T, so that $T = t + n\Delta$.

On day 0, the long counterparty receives (pays if negative) the amount

$$\Phi(t,T) - K.$$

On day 1, the long counterparty receives (pays if negative) the amount

$$\Phi(t+\Delta,T) - \Phi(t,T).$$

On each day $1 < i < n$, the long counterparty receives (pays if negative) the amount

$$\Phi(t+i\Delta,T) - \Phi(t+(i-1)\Delta,T)$$

On day n, the long counterparty receives (pays if negative) the amount

$$\Phi(T,T) - \Phi(t+(n-1)\Delta,T)$$

The payment amount on each day is called the **mark-to-market payment** (or **mark-to-market change** or **variation margin**). The mark-to-market payments are invested (or borrowed) when they are paid, so they accrue interest.

For the short counterparty, the payments are, of course, the negatives of these amounts.

Usually $K = \Phi(t,T)$ (i.e., the delivery price is the futures price), so that there is no mark-to-market change on day zero.

$\Phi(t+i\Delta,T)$ is the future price on day i. It costs nothing to enter a futures contract at $t+i\Delta$ if the delivery price is $\Phi(t+i\Delta,T)$ and maturity is T.

At current time t, $\Phi(t+i\Delta,T)$ is unknown (random) for $i \geq 1$.

The future price at maturity T is defined to be $\Phi(T,T) = S_T$.

Over the life of the futures contract, the long counterparty receives mark-to-market payments that total

$$S_T - K = \Phi(T,T) - K$$

$$= \Phi(t+n\Delta,T) - \Phi(t+(n-1)\Delta,T) + \cdots + \Phi(t+i\Delta,T) - \Phi(t+(i-1)\Delta,T) + \cdots + \Phi(t+\Delta,T) - K$$

However, each payment is made at a different time and accrues interest, so the value of the payments at T will not in general equal $S_T - K$. $\triangle$

Remark. We have described the payments for cash settlement, where the final payment is $\Phi(T,T) = S_T$ cash minus $\Phi(t+(n-1)\Delta, T)$ cash. For physical settlement, the final payment is the asset (value $\Phi(T,T) = S_T$) minus $\Phi(t+(n-1)\Delta, T)$ cash.

Remark. In practice, at t each counterparty in a futures contract makes a deposit called an **initial margin** or **performance bond** in an account at an exchange. Each day, the exchange transfers the appropriate variation margin amount from one counterparties account to the other. The initial margin needs to be large enough to cover several days of likely variation margin transfers. A typical initial margin is 5% to 15% of the value $S_t - K$. If the account balance of a counterparty drops below a certain level, called the **maintenance margin**, the exchange may issue a margin call to that counterparty. A **margin call** is a demand to replenish the account. If the account is not replenished, then the futures contract is closed and the counterparties are paid the balance of their accounts. Note that the accounts accrue interest. In other words, the initial margin and variation margin accrue interest.

7.3 Futures Prices When Rates Are Constant: Result and Examples

Result 7.3.1. If interest rates are constant, then

$$\Phi(t,T) = F(t,T)$$

for all $t \leq T$. $\triangle$

Before we give the proof, here are some examples.

Example 7.3.2. If interest rates are constant, then

$$\Phi(t+i\Delta, T) = F(t+i\Delta, T)$$

for all $t \leq T$, $i = 0, 1, \ldots, n$. $\triangle$

Example 7.3.3. If interest rates are constant and if the underlying asset is a stock paying no income, then

$$\Phi(t,T) = S_t e^{r(T-t)}.$$

$\triangle$

Example 7.3.4. If interest rates are constant and if the underlying asset is a stock that pays dividends at continuous rate q, then

$$\Phi(t,T) = S_t e^{(r-q)(T-t)}.$$

$\triangle$

Example 7.3.5. The table below is for a future contract maturing on day 5 with delivery price equal to the futures price. The underlying asset is a stock paying no income, and the constant continuously compounded interest rate 8%. The MTM column lists market-to-market payment The interest column lists the interest on the mark-to-market payment that will be accumulated over the life of the contract. All values are rounded off.

As an exercise, do the calculations to reproduce the last three columns in this table.

day	S_t	$\Phi(t,T)$	MTM	interest
0	1000	1001.10	0	0
1	1020	1020.89	19.80	0.017
2	980	980.64	-40.25	-0.026
3	1020	1020.45	39.80	0.017
4	1050	1050.23	29.78	0.007
5	1030	1030	-20.23	0
		sum:	28.90	0.015

Note that value of the corresponding forward contract at maturity is

$$S_T - F(t,T) = 1030 - 1001.10 = 28.90.$$

$\triangle$

7.4 Futures Prices When Rates Are Constant: Proof

Proof of Result 7.3.1. For simplicity assume $t = 0$. Then $T = n\Delta$. Let the constant continuously compounded interest rate be r.

Note that we can make trades in a portfolio, as long as they have no cost. For example, we can't just add or subtract cash to a portfolio.

Consider the portfolio having the following strategy.

At time 0, go long $e^{-r(n-1)\Delta}$ futures contracts with maturity T at deliver price equal to the futures price $\Phi(0,T)$. By the definition of the futures prices, we can do this at no cost.

At time Δ, increase position to $e^{-r(n-2)\Delta}$ futures at futures price $\Phi(\Delta,T)$. That is, we go long on $e^{-r(n-2)\Delta} - e^{-r(n-1)\Delta}$ futures with maturity T at delivery price equal to the futures price $\Phi(\Delta,T)$. Again, by the definition of the futures prices, we can do this at no cost.

At time $i\Delta$ (for $i = 2,\dots,n-2$), increase position to $e^{-r(n-i-1)\Delta}$ futures at futures price $\Phi(i\Delta,T)$.

At time $(n-1)\Delta$, increase position to 1 futures contract at futures price $\Phi((n-1)\Delta,T)$.

With this strategy we receive the following amounts.

At time Δ, we receive mark-to-market gain/loss

$$(\Phi(\Delta, T) - \Phi(0, T))e^{-r(n-1)\Delta}.$$

This will be invested at rate r. So by time $T = n\Delta$ (after time $T - \Delta = (n-1)\Delta$ has passed) it becomes

$$(\Phi(\Delta, T) - \Phi(0, T))e^{-r(n-1)\Delta} \cdot e^{r(n-1)\Delta} = \Phi(\Delta, T) - \Phi(0, T).$$

At time $(i+1)\Delta$, we receive mark-to-market gain/loss

$$(\Phi((i+1)\Delta, T) - \Phi(i\Delta, T))e^{-r(n-i-1)\Delta}.$$

This will be invested at rate r. So by time $T = n\Delta$ (after time $T-(i+1)\Delta = (n-i-1)\Delta$ has passed) it becomes

$$(\Phi((i+1)\Delta, T) - \Phi(i\Delta, T))e^{-r(n-i-1)\Delta} \cdot e^{r(n-i-1)\Delta} = \Phi((i+1)\Delta, T) - \Phi(i\Delta, T).$$

Therefore the value at time $T = n\Delta$ of the portfolio is

$$\sum_{i=0}^{n-1} \Phi((i+1)\Delta, T) - \Phi(i\Delta, T) = \Phi(n\Delta, T) - \Phi(0, T) = S_T - \Phi(0, T).$$

Let A be the above portfolio plus $\Phi(0, T)$ ZCBs maturing at T (or e^{-rT} cash). Then

$$V^A(T) = S_T.$$

Let B be the portfolio consisting at time 0 of one long forward contract on the asset with maturity T and delivery price equal to the forward price $F(0, T)$ plus $F(0, T)$ ZCBs maturing at T (or e^{-rT} cash). Then also

$$V^B(T) = S_T.$$

By replication,

$$V^A(t) = V^B(t),$$

which means

$$\Phi(0, T)Z(0, T) = F(0, T)Z(0, T).$$

Divide by $Z(0, T)$ to conclude. □

Result 7.4.1. Suppose that at the present time t we know what the interest rate between any two days will be. IThen

$$\Phi(t, T) = F(t, T)$$

for any given $t \leq T$. △

Exercise 7.4.1. Prove Result 7.4.1. You may assume $t = 0$ for simplicity. Use the notation r_{ij} for the continuous interest rate between day i and day j.

7.5 Futures Convexity Correction

When interest rates are not constant, we may have $\Phi(t,T) \neq F(t,T)$.

Definition 7.5.1. The difference $\Phi(t,T) - F(t,T)$ is called the **futures convexity correction**. $\triangle$

A detailed analysis of the futures convexity correction is beyond our scope. For details, see the textbook and references therein.

Chapter 8

Options

8.1 European Option Definitions

Fix an asset. Let S_t be its price at time t.

Definition 8.1.1. A **European call option** on the asset with **strike** (or **exercise price**) K and **maturity** (or **exercise date**) T is the right—but not the obligation—to buy the asset for K at time T. Using the right to trade the asset is called **exercising** the option.

In contrast, a long forward contract is the obligation to buy the asset for K at time T.

We would only choose to pay K for an asset worth S_T if $S_T \geq K$. Thus the payout of a European call option is s

$$\begin{cases} S_T - K & \text{if } S_T \geq K \\ 0 & \text{if } S_T \leq K \end{cases} = \max\{S_T - K, 0\} = (S_T - K)^+.$$

We can also write the payout as $g(S_T)$, where $g(x) = (x - K)^+ = \max\{x - K, 0\}$ is called the payout function for the European call option. △

Example 8.1.1. Suppose the strike is $K = 100$ for an call option with maturity T. If $S_T = 110$, then we receive value $S_T - K = 110 - 100 = 10$ by exercising the option. On the other hand, if $S_T = 90$, then $S_T - K = -10$, so we let the option expire and receive 0. △

Definition 8.1.2. A **European put option** is the right to sell the asset for K at time T.

The payout of a European put option is

$$\begin{cases} K - S_T & \text{if } S_T \leq K \\ 0 & \text{if } S_T \geq K \end{cases} = \max\{K - S_T, 0\} = (K - S_T)^+.$$

△

Remark 8.1.2. Notice that the payout of a put is *not* the negative of the payout of a call. △

Definition 8.1.3. Suppose we buy a European call option at time t. The price we pay is denoted $C_K(t,T)$. The party who sold us the call is said to be **short** on the call (or the **writer** of the call). Since we are holding the call, we are said to be **long** on the call (or the **owner** of the call). As the long counterparty, we have the *right* to buy the asset from the short counterparty for K at T. If this right is exercised, the short counterparty has the *obligation* to immediately sell the asset to us. Note that

$$\text{short 1 call} \neq \text{long 1 put}.$$

The payout at maturity when short a call is

$$-\max\{S_T - K, 0\}$$

which is the negative of the payout when long the call. However, don't forget that the short counterparty was paid $C_K(t,T)$ at t by the long counterparty. △

Definition 8.1.4. A **European straddle option** is a European call plus a European put. It has payout

$$|S_T - K| = \begin{cases} S_T - K & \text{if } S_T \geq K \\ K - S_T & \text{if } S_T \leq K. \end{cases}$$

△

Remark 8.1.3. Note that an option is automatically exercised at maturity if the payout is positive. △

8.2 American Option Definitions

While a European option allows exercise only at maturity T, an American option allows exercise at any time $t \leq T$.

Definition 8.2.1. An **American call option** on an asset with **strike** (or **exercise price**) K and **maturity** (or **exercise date**) T is the right to buy the asset for K at any time $t \leq T$. For an **American put option**, replace "buy" with "sell." △

There are many other less common types of options (such as Bermudan, Asian, and Lookback options). We won't study them in this course.

Remark. Note that an option is automatically exercised at maturity if the payout is positive.

8.3 More Definitions

Definition 8.3.1. At time $t \leq T$, a call option with strike K is said to be:

in-the-money (ITM) if $S_t > K$ (the call option will have positive payout if the asset price remains unchanged)

out-of-the-money (OTM) if $S_t < K$ (the call option will be worthless if the asset price remains unchanged)

at-the-money (ATM) if $S_t = K$

in-the-money-forward (ITMF) if $F(t, T) > K$

out-of-the-money-forward (OTMF) if $F(t, T) < K$

at-the-money-forward (ATMF) if $F(t, T) = K$

As usual, $F(t, T)$ is the forward price on the asset at t for maturity T.

For a put option, we reverse these inequalities. For example, a put option is ITM if $K > S_t$. △

Definition 8.3.2. The **intrinsic value** at t of a call option is $\max\{S_t - K, 0\}$. The **intrinsic value** at t of a put option is $\max\{K - S_t, 0\}$. Thus an option is in-the-money if its intrinsic value is positive.

The intrinsic value is the payout if we had to immediately exercise the option or let it expire.

The intrinsic value is *not* the value/price of the option. △

8.4 Option Prices

Fix an asset. Let S_t be its price at time t. At this point, we make no assumptions about the nature of the asset.

Definition 8.4.1. For $t \leq T$:

$C_K(t, T)$ = price/value at time t of a European call with strike K and maturity T.

$P_K(t, T)$ = price/value at time t of a European put with strike K and maturity T.

$\tilde{C}_K(t, T)$ = price/value at time t of an American call with strike K and maturity T.

$\tilde{P}_K(t, T)$ = price/value at time t of an American put with strike K and maturity T. △

Because an option must either be exercised or expired at maturity, we have

Result 8.4.1.

$$\tilde{C}_K(T, T) = C_K(T, T) = \max\{S_T - K, 0\}.$$

and

$$\tilde{P}_K(T, T) = P_K(T, T) = \max\{K - S_T, 0\}.$$

△

In words, the next result says: Option prices are never negative, and American options are always worth at least as much as European options (with the same strike and maturity).

Result 8.4.2.

$$\tilde{C}_K(t,T) \geq C_K(t,T) \geq 0 \quad \text{and} \quad \tilde{P}_K(t,T) \geq P_K(t,T) \geq 0.$$

△

Proof. It is intuitively clear that

$$\tilde{C}_K(t,T) \geq C_K(t,T) \quad \text{and} \quad \tilde{P}_K(t,T) \geq P_K(t,T)$$

because an American option gives the same rights as a European option, and more. (As an exercise, prove these inequalities using the no-arbitrage principle or the monotonicity theorem.)

In light of the assertion above, we just need to prove.

$$C_K(t,T) \geq 0 \quad \text{and} \quad P_K(t,T) \geq 0.$$

These inequalities should also be intuitively clear because an option can never lose money for its owner. For completeness, we prove the first inequality in detail. We apply the monotonicity theorem to the portfolios

A = 1 call and B = empty.

We see that

$$V^A(T) = C_K(T,T) = \max\{S_T - K, 0\} \geq 0 = V^B(T).$$

So by monotonicity,

$$V^A(t) \geq V^B(t)$$

that is,

$$C_K(t,T) \geq 0.$$

The proof of the second inequality is similar.

□

8.5 Put-Call Parity

Fix an asset. Let S_t be its price at t. Again, we make no assumptions about the nature of the asset.

Recall that the value of a forward is related to the forward price and delivery price by

$$V_K(t,T) = (F(t,T) - K)Z(t,T).$$

The next result relates the value of a forward to the price of a European call and the price of a European put.

Result 8.5.1 (Put-Call Parity)**.**

$$V_K(t,T) = C_K(t,T) - P_K(t,T)$$

△

Proof. Consider two portfolios.

A: long 1 forward with delivery price K and maturity T.

B: long 1 European call and short 1 European put (i.e., $+1$ call and -1 put), both with strike K and maturity T.

We have

$$V^A(T) = S_T - K$$
$$V^B(T) = \left\{ \begin{array}{ll} (S_T - K) - 0 & \text{if } S_T \geq K \\ 0 - (K - S_T) & \text{if } S_T \leq K \end{array} \right\} = S_T - K.$$

By the replication theorem,

$$V^A(t) = V^B(t)$$

which means

$$V_K(t,T) = C_K(t,T) - P_K(t,T).$$

□

In words, put-call parity says:

$$V_K(t,T) = C_K(t,T) - P_K(t,T)$$

long 1 forward equals long 1 call and short 1 put

$$C_K(t,T) = V_K(t,T) + P_K(t,T)$$

long 1 call equals long 1 forward and long 1 put

$$P_K(t,T) = -V_K(t,T) + C_K(t,T)$$

long 1 put equals short 1 forward and long 1 call

Here "equals" means "has the same value as."

Remember the options are European.

There is a version of put-call parity for American options, but it involves inequalities rather than an equality, and it depends on the income paid by the stock. We will not study it here.

Result 8.5.2.

$$C_K(t,T) = P_K(t,T)$$

if and only if $K = F(t,T)$ if and only if $K = F(t,T)$ if and only if the call and put are both at-the-money-forward (ATMF) at t. △

Proof. Use put-call parity and the definition of forward price. □

Example 8.5.3. Suppose that a stock paying no income is trading at price 30 per share. European puts on the stock with strike 35 and exercise date in six months are trading at price 10. The six-month libor rate is 4%. What is the price of a European call with the same strike and exercise date?

Put-call parity says $V_K(t,T) = C_K(t,T) - P_K(t,T)$, so

$$C_K(t,T) = V_K(t,T) + P_K(t,T).$$

Since the stock pays no income, we have

$$V_K(t,T) = (F(t,T) - K)Z(t,T) = S_t - KZ(t,T) = S_t - K(1 + (T-t)L_t[t,T])^{-1}$$

Therefore

$$\begin{aligned} C_K(t,T) &= S_t - K(1 + (T-t)L_t[t,T])^{-1} + P_K(t,T) \\ &= 30 - 35(1 + (0.5)(0.04))^{-1} + 10 = 5.68627\ldots. \end{aligned}$$

△

8.6 European Call Prices for Assets Paying No Income

Result 8.6.1. For an asset paying no income, the European call price satisfies

$$S_t - KZ(t,T) \leq C_K(t,T) \leq S_t.$$

△

Proof. We have

$$C_K(T,T) = \max\{S_T - K, 0\} \leq S_T$$

To prove the upper bound

$$C_K(t,T) \leq S_t,$$

apply the monotonicity theorem to the following portfolios.
A: 1 call.
B: 1 stock.

We have

$$S_T - K \leq \max\{S_T - K, 0\} = C_K(T,T).$$

Applying the monotonicity theorem to the portfolios
A: 1 stock, $-K$ ZCBs with maturity T
B: 1 call
proves the lower bound

$$S_t - KZ(t,T) \leq C_K(t,T).$$

□

Result 8.6.2. For an asset paying no income, the price of a European call is at least the intrinsic value of the call: $C_K(t,T) \geq \max\{S_t - K, 0\}$. △

Proof. We know $C_K(t,T) \geq 0$. So we just need to observe that

$$C_K(t,T) \geq S_t - KZ(t,T) \geq S_t - K$$

by Result 8.6.1. □

8.7 Equality of American and European Call Prices for Assets Paying No Income

Result 8.7.1. For an asset paying no income, the price of an American call and the price of a European call (with the same strike and maturity) are always equal, i.e.,

$$\tilde{C}_K(t,T) = C_K(t,T)$$

for all $t \leq T$. △

Proof. We already know (Result 8.4.2) that $\tilde{C}_K(t,T) \geq C_K(t,T)$.

To prove $\tilde{C}_K(t,T) \leq C_K(t,T)$, we assume $\tilde{C}_K(t,T) > C_K(t,T)$ and deduce a contradiction. Consider the portfolio C with the following strategy.

At time t, the portfolio is empty. Write and sell 1 American call. The portfolio becomes short 1 American call and $\tilde{C}_K(t,T)$ cash. Spend $C_K(t,T)$ cash to go long 1 European call. The portfolio becomes short 1 American call, long 1 European call, and $\epsilon = \tilde{C}_K(t,T) - C_K(t,T)$ cash. By our assumption, $\epsilon > 0$. It is not necessary to invest the cash, we can just hold it.

Now the portfolio acts depending on what the buyer of the American call does with it.

Case 1. The American call expires without being exercised. Then

$$V^C(T) = \epsilon + C_K(T,T) \geq \epsilon > 0.$$

Case 2. The American call is exercised at some time T_0 with $t \leq T_0 \leq T$. At T_0, we are obligated to give away 1 asset worth S_{T_0} and receive K cash. To meet our obligation, we borrow S_{T_0} cash to buy 1 asset and give it away. We also sell the European call. Then we have $\epsilon + K - S_{T_0} + C_K(T_0, T)$ cash. Nothing else happens until T. Thus, by Result 8.6.1 and the fact $Z(T_0, T) \leq 1$, we have

$$\begin{aligned} V^C(T) &\geq \epsilon + K - S_{T_0} + C_K(T_0, T) \\ &\geq \epsilon + K - S_{T_0} + S_{T_0} - KZ(T_0, T) \\ &\geq \epsilon \\ &> 0 \end{aligned}$$

Combining both cases, we have $V^C(T) > 0$ with probability one and $V^C(t) = 0$. This contradicts the no-arbitrage assumption. □

8.8 No Early Exercise for American Calls for Assets Paying No Income

Result 8.8.1. For an asset paying no income, an American call will never be exercised before maturity if interest rates are positive. △

Proof. Suppose $t < T$. By Result 8.6.1, we have

$$\tilde{C}_K(t, T) \geq \tilde{C}_K(t, T) \geq S_t - KZ(t, T).$$

If the interest rates for period t to T are positive, then $Z(t, T) < 1$, and so

$$\tilde{C}_K(t, T) > S_t - K.$$

We would receive value $S_t - K$ if we exercise an American call at t. But we would receive a strictly larger value $\tilde{C}_K(t, T)$ if we sell the call instead. So we would never choose to exercise at $t < T$. □

8.9 Put Prices for Assets Paying No Income

Result 8.9.1. For an asset paying no income,

$$KZ(t, T) - S_t \leq P_K(t, T) \leq KZ(t, T)$$

△

Result 8.9.2. For an asset paying no income,

$$K - S_t \leq \tilde{P}_K(t, T) \leq K.$$

△

Recall that (with the same strike and maturity) American options are always worth at least as much as European options, and that, for a stock paying no income, American and European calls have the same value.

However, American puts may be worth strictly more than European puts. This is because, by exercising early, we receive the strike price K early and can invest this amount. The next example illustrates this.

Example 8.9.3. The current price of a stock paying no income is $S_t = 10$. The one-year annually compounded interest rate is $r = 16\%$.

Consider an American put expiring in one year with strike $K = 80$. Exercising now, we can gain

$$S_T - K = 80 - 10 = 70,$$

which can be invested to become

$$70(1 + 0.16) = 81.20$$

after one year.

Consider a European put expiring in one year with strike $K = 80$. The payout at T is at most

$$\max\{80 - S_T, 0\} \leq 80 < 81.20$$

after one year.

Since we can potentially make more money with the American put, its current price will be strictly larger than the price of a European put. $\triangle$

Proof of Result 8.9.1. As with the European call bounds of Result 8.6.1, we can prove (8.9.1) directly by replication. Instead, to illustrate another method, we prove (8.9.1) using European call bounds and put-call parity.

We prove the upper bound $P_K(t,T) \leq KZ(t,T)$ in detail. According to Result 8.6.1,

$$C_K(t,T) \leq S_t.$$

By put-call parity,

$$C_K(t,T) = V_K(t,T) + P_K(t,T)$$

Hence

$$P_K(t,T) \leq S_t - V_K(t,T).$$

But, for an asset paying no income,

$$V_K(t,T) = (F(t,T) - K)Z(t,T) = S_t - KZ(t,T).$$

Therefore

$$P_K(t,T) \leq KZ(t,T).$$

To prove the lower bound $KZ(t,T) - S_t \le P_K(t,T)$, start with

$$0 \le C_K(t,T)$$

and argue the same way. □

Proof of Result 8.9.2. For the lower bound: If we exercise the American put at t, we receive value $K - S_t$, so the price $\tilde{P}_K(t,T)$ must be at least this much.

To prove the upper bound $\tilde{P}_K(t,T) \le K$, we assume $\tilde{P}_K(t,T) > K$ and deduce a contradiction. Consider portfolio C with the following strategy.

At time t, the portfolio is empty. Write and sell 1 American put. The portfolio becomes short 1 American put and $\tilde{P}_K(t,T)$ cash. It is not necessary to invest the cash, we can just hold it.

Now the portfolio acts depending what the long party of the American put does.

Case 1. The American put expires without being exercised. Then

$$V^C(T) = \tilde{P}_K(t,T) \ge \tilde{P}_K(t,T) - K > 0.$$

Case 2. The American put is exercised at some time. At that moment, we must pay K to buy 1 asset from the holder of the put. The portfolio is then $\tilde{P}_K(t,T) - K$ non-invested cash and 1 asset. Nothing else happens until T, so

$$V^C(T) = \tilde{P}_K(t,T) - K + S_T > 0.$$

Combining both cases, we have $V^C(T) > 0$ with probability one and $V^C(t) = 0$. This contradicts the no-arbitrage assumption.

□

8.10 Call and Put Prices for Stocks Paying Known Dividend Yield

Result 8.10.1. For a stock paying dividends at continuous yield q with automatic reinvestment,

$$S_t e^{-q(T-t)} - KZ(t,T) \le C_K(t,T) \le S_t e^{-q(T-t)} \tag{8.10.1}$$

$$KZ(t,T) - S_t e^{-q(T-t)} \le P_K(t,T) \le KZ(t,T) \tag{8.10.2}$$

$$S_t - K \le \tilde{C}_K(t,T) \le S_t \tag{8.10.3}$$

$$K - S_t \le \tilde{P}_K(t,T) \le K \tag{8.10.4}$$

△

Recall that (with the same strike and maturity) American options are always worth at least as much as European options, and that, for a stock paying no income, American and European calls have the same value.

However, for a stock paying dividends, an American call may be worth more than a European call. This is because, by exercising early, we may receive dividends from the stock which we would not have received otherwise. The next example illustrates this.

Example 8.10.2. Consider a stock paying dividends at continuous yield $q = 5\%$ with automatic reinvestment. The current price of the stock is $S_t = 10$. The one-year continuously compounded interest rate is $r = 5\%$.

Consider an American call expiring in one year with strike $K = 20$. Exercising now, we get the stock and a debt of $K = 20$. If the stock price in one year is $S_T = 30$, then we will have gained

$$S_T e^{q(T-t)} - K e^{r(T-t)} = 30e^{0.05} - 20e^{0.05} \approx 16.49$$

after one year.

A European call expiring in one year with strike $K = 20$ would payout only

$$S_T - K = 30 - 20 = 10$$

after one year. △

Exercise 8.10.1. Prove Result 8.10.1.

8.11 Call and Put Spreads

In this section, we consider only European options.

Fix an asset. Let S_t be its price at t.

Definition 8.11.1. Let $K_1 < K_2$. A (K_1, K_2) **call spread** is a portfolio consisting of long 1 call option with strike K_1 and short 1 call option with strike K_2, both with maturity T. △

Result 8.11.1. For a (K_1, K_2) call spread:

- The payout at maturity T is

$$\begin{cases} 0 & \text{if } S_T \leq K_1 \text{ (neither exercised)} \\ S_T - K_1 & \text{if } K_1 \leq S_T \leq K_2 \text{ (only } K_1\text{-call exercised)} \\ K_2 - K_1 & \text{if } S_T \geq K_2 \text{ (both exercised)} \end{cases}$$

- The value at $t \leq T$ is

$$C_{K_1}(t, T) - C_{K_2}(t, T).$$

- The value at $t \le T$ satisfies

$$0 \le C_{K_2}(t,T) - C_{K_1}(t,T) \le (K_2 - K_1)Z(t,T).$$

$\triangle$

Proof. Only the last point needs justification.

The payout at T is ≥ 0, so (by the monotonicity theorem) the value at t is also ≥ 0.

The payout at T is always $\le K_2 - K_1$, which is the value at T of a portfolio of $K_2 - K_1$ ZCBs with maturity T. So (by the monotonicity theorem) the value of the spread at t is $\le (K_2 - K_1)Z(t,T)$. $\square$

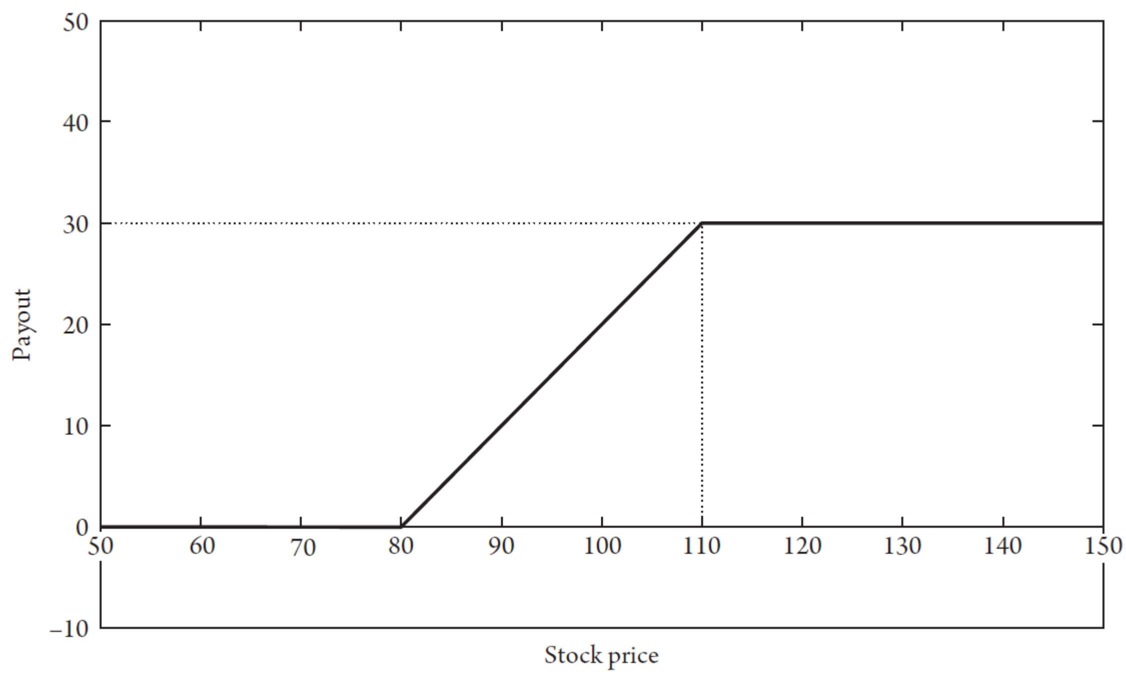


Figure 8.1: Payout of $(80, 110)$ call spread

Definition 8.11.2. Let $K_1 < K_2$. A (K_2, K_1) **put spread** is a portfolio consisting of short 1 put option with strike K_1 and long 1 put option with strike K_2, both with maturity T. $\triangle$

Result 8.11.2. For a (K_2, K_1) put spread:

- The payout at maturity T is

$$\begin{cases} K_2 - K_1 & \text{if } S_T \le K_1 \text{ (both exercised)} \\ K_2 - S_T & \text{if } K_1 \le S_T \le K_2 \text{ (only } K_2\text{-put exercised)} \\ 0 & \text{if } S_T \ge K_2 \text{ (neither exercised)} \end{cases}$$

- The value at $t \leq T$ is
$$P_{K_2}(t,T) - P_{K_1}(t,T).$$
- The value at $t \leq T$ satisfies
$$0 \leq P_{K_2}(t,T) - P_{K_1}(t,T) \leq (K_2 - K_1)Z(t,T)$$

$\triangle$

Remark 8.11.3. The previous two results imply If $K_1 \leq K_2$, then

$$C_{K_1}(t,T) \geq C_{K_2}(t,T) \quad \text{and} \quad P_{K_1}(t,T) \leq P_{K_2}(t,T)$$

In other words, $C_K(t,T)$ is a decreasing function of the strike K, and $P_K(t,T)$ is an increasing function of the strike K. $\triangle$

8.12 Butterflies and Convexity of Option Price

Fix an asset. Let S_t be its price at t.

Definition 8.12.1. Let $K_1 < K^* < K_2$. Then there is a unique $\lambda \in (0,1)$ such that

$$K^* = \lambda K_1 + (1-\lambda)K_2.$$

The portfolio consisting of

- $+2\lambda$ calls with strike K_1,
- -2 calls with strike K^*,
- $+2(1-\lambda)$ calls with strike K_2,

all with the same maturity, is called a (K_1, K^*, K_2) **call butterfly**.

If $\lambda \neq 1/2$, it is called an **asymmetric call butterfly**.

If $\lambda = 1/2$, it is called an **symmetric call butterfly**.

Since symmetric call butterflies are vastly more common in practice, if the shorter term **call butterfly** is used without further qualification, we will always assume it means **symmetric call butterfly**.

For a symmetric call butterfly, the portfolio is

- $+1$ call with strike K_1,

- -2 calls with strike $K^* = \frac{1}{2}(K_1 + K_2)$,
- $+1$ call with strike K_2.

$\triangle$

Result 8.12.1. For a (K_1, K^*, K_2) symmetric call butterfly, at maturity T the payout is

$$\begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 \leq S_T \leq K^* \\ K_2 - S_T & \text{if } K^* \leq S_T \leq K_2 \\ 0 & \text{if } S_T \geq K_2 \end{cases}$$

$\triangle$

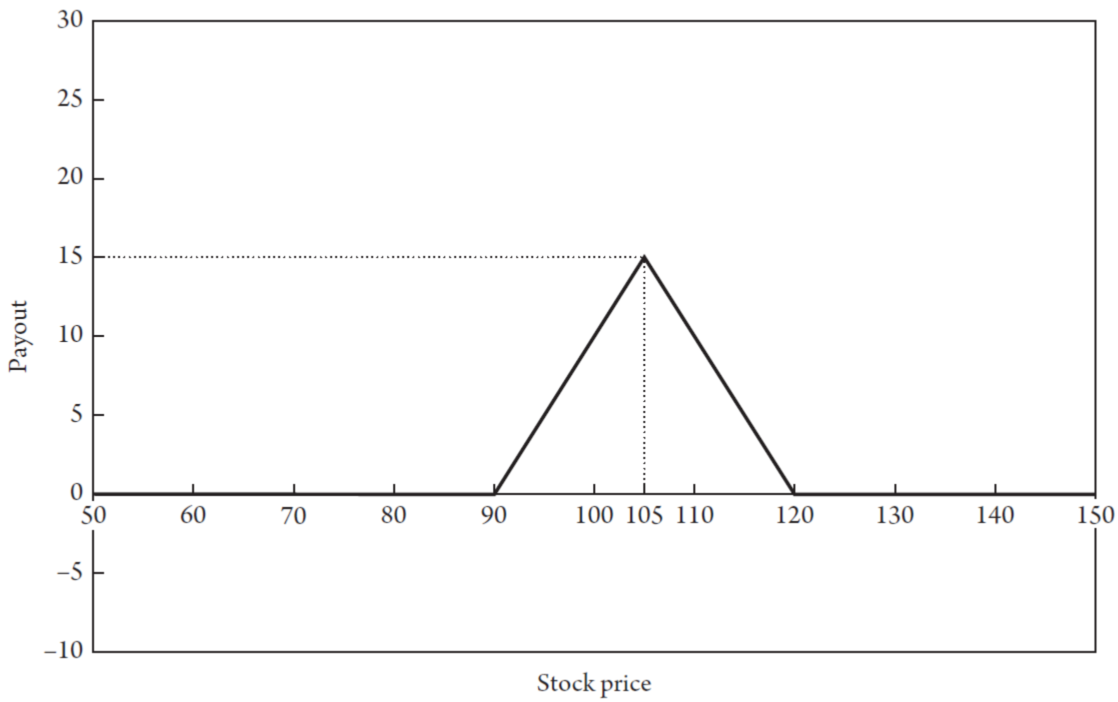


Figure 8.2: Payout of a $(90, 105, 120)$ call butterfly.

Result 8.12.1 is a special case of

Result 8.12.2. For a (K_1, K^*, K_2) (asymmetric or symmetric) call butterfly:

- The payout at maturity T is

$$\begin{cases} 0 & \text{if } S_T \leq K_1 \\ 2\lambda(S_T - K_1) & \text{if } K_1 \leq S_T \leq K^* \\ 2(1-\lambda)(K_2 - S_T) & \text{if } K^* \leq S_T \leq K_2 \\ 0 & \text{if } S_T \geq K_2 \end{cases}$$

- The value at $t \leq T$ is

$$2\lambda C_{K_1}(t,T) + 2(1-\lambda)C_{K_2}(t,T) - 2C_{K^*}(t,T).$$

- The payout at T is ≥ 0, so (by the monotonicity theorem) the value at $t \leq T$ satisfies

$$0 \leq 2\lambda C_{K_1}(t,T) + 2(1-\lambda)C_{K_2}(t,T) - 2C_{K^*}(t,T) \tag{8.12.1}$$

$\triangle$

Proof. Only the first point needs justification. the cases $S_T \geq K_1$ and $K_1 \leq S_T \leq K^*$ are clear. If $K^* \leq S_T \leq K_2$, the payout is

$$2\lambda(S_T - K_1) - 2(S_T - K^*) = 2\lambda(S_T - K_1) - 2(S_T - \lambda K_1 - (1-\lambda)K_2) = 2(1-\lambda)(K_2 - S_T)$$

If $S_T \geq K_2$, the payout is

$$2\lambda(S_T - K_1) - 2(S_T - K^*) + 2(1-\lambda)(S_T - K_1) = 0$$

$\square$

Remark 8.12.3. (8.12.1) can be rearranged to

$$C_{K^*}(t,T) \leq \lambda C_{K_1}(t,T) + (1-\lambda)C_{K_2}(t,T),$$

which means that $C_K(t,T)$ is convex (concave up) function of K. $\triangle$

Exercise 8.12.1. Draw the payout diagram for an asymmetric call butterfly with strikes $(70, 80, 110)$.

Exercise 8.12.2. Consider an arbitrary (K_1, K^*, K_2) asymmetric or symmetric call butterfly. (a) Use put-call parity to express it as a portfolio of only puts.

(b) Show that the value at $t \leq T$ is

$$2\lambda P_{K_1}(t,T) + 2(1-\lambda)P_{K_2}(t,T) - 2P_{K^*}(t,T).$$

(c) Show that $P_K(t,T)$ is a convex (concave up) function of K by showing that

$$P_{K^*}(t,T) \leq \lambda P_{K_1}(t,T) + (1-\lambda)P_{K_2}(t,T).$$

8.13 Digital Options

Fix an asset. Let S_t be its price at t.

Definition 8.13.1. A **digital call option** with strike K, payout β, and maturity T has payout at T

$$\begin{cases} \beta & \text{if } S_T \geq K \\ 0 & \text{if } S_T < K \end{cases}$$

A **digital put option** with strike K, payout β, and maturity T has payout at T

$$\begin{cases} \beta & \text{if } S_T \leq K \\ 0 & \text{if } S_T > K \end{cases}$$

△

Digital options are also called binary options or all-or-nothing options.

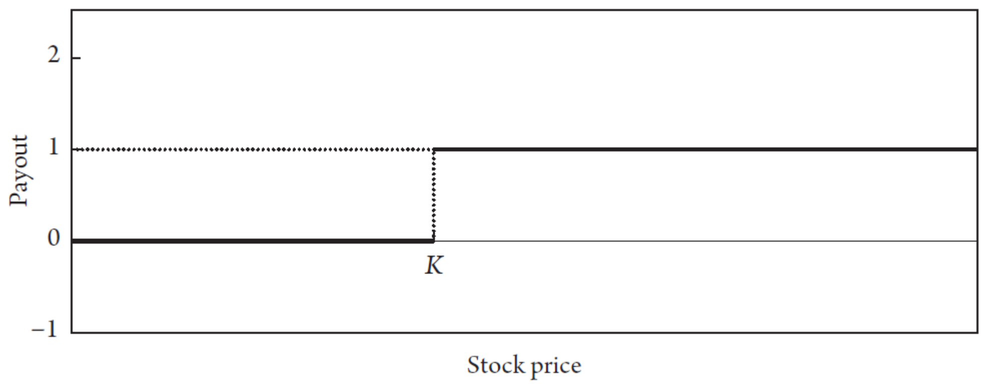


Figure 8.3: Payout of a K-strike 1-payout digital call

Result 8.13.1. The price/value at time $t \leq T$ of a digital call option with strike K, payout 1, and maturity T is

$$-\frac{d}{dK}C_K(t,T),$$

provided $C_K(t,T)$ is differentiable in K. △

The proof is Exercise 8.13.1

Exercise 8.13.1. Fix an asset. Let S_t be its price at $t \leq T$. In this exercise, you will prove Result 8.13.1 (and a bit more). Parts (g) and (i) may be difficult if you haven't studied ϵ-δ analysis. For each positive integer n, define the portfolio

$$A(n) : n \ (K - 1/n, K) \text{ call spreads with maturity } T.$$

Let B be the portfolio consisting of a digital call option with strike K, payout 1, and maturity T. (a) Find $V^{A(n)}(T)$ and $V^{B}(T)$. (b) On the same axes, draw payout diagrams for B and $A(n)$ when $K = 10$ and $n = 1, 5, 10$.
(c) Show that $\lim_{n\to\infty} V^{A(n)}(t) = -\frac{d}{dK}C_K(t,T)$ if $C_K(t,T)$ is differentiable in K. Hint:

$V^{A(n)}(t) = \dfrac{C_{K-1/n}(t,T) - C_K(t,T)}{1/n}$.

(d) Find $\lim_{n\to\infty} V^{A(n)}(T)$ when $S_T \geq K$.
(e) Find $\lim_{n\to\infty} V^{A(n)}(T)$ when $S_T < K$. Justify your answer.
(f) How is $\lim_{n\to\infty} V^{A(n)}(T)$ related to $V^B(T)$?
(g) Use a no-arbitrage argument to prove that $\lim_{n\to\infty} V^{A(n)}(t) = V^B(t)$.
(h) Conclude using parts (c) and (g).
(i) Use Remark 8.12.3 to prove that the limit $\lim\limits_{n\to\infty} V^{A(n)}(t)$ always exists.
(j) Without knowing whether $C_K(t,T)$ is differentiable in K, what formula can you write down for the price/value at time $t \leq T$ of a digital call option with strike K, payout 1, and maturity T?

Exercise 8.13.2. (a) What is the price at time t of a digital put option with strike K, payout 1, and maturity T? (You may assume $P_K(t,T)$ is differentiable in K.) (b) Prove it.

Chapter 9

Probability Theory: Advanced Ideas

In this chapter, we introduce the more advanced concepts in probability that we will need later. As before, the purpose is not to be completely rigorous, but to build the correct intuition.

9.1 Equivalent Probability Measures

Multiple probability measures can be defined on the same sample space.

Example 9.1.1. Roll a six-sided die. Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Consider the probability measure P on Ω defined by $P(\{\omega\}) = \frac{1}{6}$ for all $\omega \in \Omega$. This would be the appropriate measure if the die is fair.

Consider also the probability measure P^* on Ω defined by $P^*(\{\omega\}) = \frac{1}{7}$ for all $\omega \in \{1, 2, 3, 4, 5\}$ and $P^*(\{6\}) = \frac{2}{7}$. This would be an appropriate measure if the die is unfair (weighted) in a certain way.

Consider the random variable $X =$ the number shown on the die. $R(X) = \{1, 2, 3, 4, 5, 6\}$.

The expectation of X with respect to P is

$$\mathbb{E}(X) = \sum_{k \in R(X)} kP(X = k) = (1)(1/6)+(2)(1/6)+...+(5)(1/6)+(6)(1/6) = 21/6 = 3.5$$

The expectation of X with respect to P^* is

$$\mathbb{E}(X) = \sum_{k \in R(X)} kP(X = k) = (1)(1/7)+(2)(1/7)+...+(5)(1/7)+(6)(2/7) = 54/14 = 3.85714285714$$

$\triangle$

Definition 9.1.1. Two probability measures P and P^* on the same sample space are called **equivalent** if

(i) the events that have probability one according to P are exactly the events that have probability one according to P^*,

(ii) the events that have probability zero according to P are exactly the events that have probability zero according to P^*,

(iii) the events that have probability non-zero according to P are exactly the events that have probability non-zero according to P^*.

In symbols, for every event A,

(i) $P(A) = 1$ if and only if $P^*(A) = 1$,

(ii) $P(A) = 0$ if and only if $P^*(A) = 0$,

(iii) $P(A) > 0$ if and only if $P^*(A) > 0$.

$\triangle$

Remark 9.1.2. We don't need to include all the conditions (i),(ii),(iii) in the definition. As soon as we have one condition, we have have the other two. $\triangle$

Remark 9.1.3. Recall the definition of arbitrage portfolio: A portfolio A is an arbitrage portfolio if the following conditions are satisfied:

- At current time t,
$$V^A(t) \leq 0.$$

- At some future time T,
$$P(V^A(T) \geq 0) = 1 \quad \text{and} \quad P(V^A(T) > 0) > 0.$$

Notice the definition only cares about events having probability one and probability non-zero. Therefore, if P and P^* are equivalent probability measures, then they have the same arbitrage portfolios; in other words, the portfolios that are arbitrage portfolios with respect to P are exactly the portfolios that are arbitrage portfolios with respect to P^*. $\triangle$

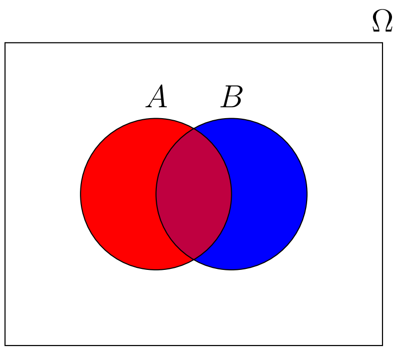
9.2 Conditional Probability

Fix a sample space Ω and probability measure P.

Definition 9.2.1. Let A and B be events (i.e., subsets of Ω). The **conditional probability** of A given B is
$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}.$$
It is undefined if $P(B) = 0$. $\triangle$

Intuition. $P(A|B)$ is the probability that event A occurs assuming event B occurs.

Out of the probability assigned to the outcomes in B, $P(A|B)$ is the fraction assigned to those outcomes that are in both A and B.



Example 9.2.1. Experiment: Flip a fair coin three times.
Sample space: $\Omega = \{$HHH,HHT,HTH,THH,HTT,THT,TTH,TTT$\}$ Probability Measure: $P(\{\omega\}) = \frac{1}{8}$ for all $\omega \in \Omega$
X = number of heads
$Y = 1$ if 1st flip heads, 0 otherwise
$Z = 1$ if 2nd flip heads, 0 otherwise

Find $P(X = 2)$, $P(X = 2|Y = 1)$, and $P(X = 2|Y = 1, Z = 0)$.
Notation: $\{Y = 1, Z = 0\} = \{Y = 1$ **and** $Z = 0\}$.

$P(X = 2) = P(\{\text{HHT,HTH,THH}\}) = \dfrac{3}{8}$

$P(X = 2|Y = 1) = \dfrac{P(X = 2, Y = 1)}{P(Y = 1)}$.
$\{Y = 1\} = \{HHH, HHT, HTH, HTT\}$ and $\{X = 2, Y = 1\} = \{HHT, HTH\}$
$P(X = 2|Y = 1) = \dfrac{P(X = 2, Y = 1)}{P(Y = 1)} = \dfrac{2/8}{4/8} = \dfrac{1}{2}$.

$P(X = 2|Y = 1, Z = 0) = \dfrac{P(X = 2, Y = 1, Z = 0)}{P(Y = 1, Z = 0)}$
$\{Y = 1, Z = 0\} = \{HTH, HTT\}$ and $\{X = 2, Y = 1, Z = 0\} = \{HHT\}$
$P(X = 2|Y = 1, Z = 0) = \dfrac{P(X = 2, Y = 1, Z = 0)}{P(Y = 1, Z = 0)} = \dfrac{1/8}{2/8} = \dfrac{1}{2}$. $\triangle$

9.3 Independence

Fix a sample space Ω and probability measure P.

Definition 9.3.1. Events A and B are called **independent** if

$$P(A \text{ and } B) = P(A)P(B) \tag{9.3.1}$$

$\triangle$

If $P(B) > 0$, we can use the definition $P(A|B) = P(A \text{ and } B)/P(B)$ to rewrite (9.3.1) as

$$P(A|B) = P(A),$$

so, intuitively, independence of A and B means knowledge of B does not affect the probability of A. We get a similar statement with $P(B|A)$ if $P(A) > 0$.

Definition 9.3.2. Discrete random variables X and Y are **independent** if

$$P(X = x, Y = y) = P(X = x)P(Y = y) \quad \text{for all } x, y$$

Remember: The "," here means "and" $\triangle$

Intuitively, X and Y are independent if knowing one of them gives no information about the other.

Example 9.3.1. Roll two fair six-sided dice. X = sum of the dice, Y = number on first die, Z = number on second die.

X and Y are not independent because (for example)

$$P(X = 2|Y = 6) = 0 \neq \frac{1}{36} = P(X = 2).$$

Y and Z are independent. Indeed, we can check that (exercise)

$$P(Y = y|Z = z) = P(Y = y) \quad \text{for all } y, z$$

but this should be intuitively clear. $\triangle$

Definition 9.3.3. Discrete random variables $X_1, \ldots, X_n$ are **independent** if

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdot \cdots \cdot P(X_n = x_n) \quad \text{for all } x_1, \ldots, x_n.$$

An infinite sequence of discrete random variables $X_1, X_2, \ldots$ is **independent** if any finite sub-collection is independent. $\triangle$

Theorem 9.3.2.
(i) If $X_1, X_2, \ldots$ are independent, then $X_1, X_2, X_3, X_5, X_8, X_{13}$ are independent.

(ii) If $X_1, X_2, \ldots$ are independent, then $g(X_1), g(X_2), \ldots$ are independent for any function $g : \mathbb{R} \to \mathbb{R}$.

(iii) If $X_1, X_2, \ldots$ are independent with respect to P and if P^* is an equivalent probability measure to P, then $X_1, X_2, \ldots$ are independent with respect to P^*.

$\triangle$

9.4 Conditional Expectation

Fix a sample space Ω and probability measure P.

Recall that the expectation of a discrete random variable X is

$$\mathbb{E}(X) = \sum_{k \in R(X)} kP(X = k),$$

where $R(X)$ is the range of X.

Definition 9.4.1. Let X and Y be discrete random variables. The **conditional expectation** of X given $Y = y$ is

$$\mathbb{E}(X|Y = y) = \sum_{k \in R(X)} kP(X = k|Y = y).$$

$\triangle$

Interpretation. $\mathbb{E}(X|Y = y)$ is the expectation of X assuming that $Y = y$ occurs.

Example 9.4.1. Roll two fair six-sided dice. X = sum of the dice, Y = number on first die, Z = number on second die.

$$\mathbb{E}(X) = \mathbb{E}(Y + Z) = \mathbb{E}(Y) + \mathbb{E}(Z) = 3.5 + 3.5 = 7$$

$$\mathbb{E}(X|Y = 6) = \mathbb{E}(Y + Z|Y = 6) = \mathbb{E}(Y|Y = 6) + \mathbb{E}(Z|Y = 6) = 6 + 3.5 = 9.5.$$

$\triangle$

Theorem 9.4.2 (Properties of Conditional Expectation $\mathbb{E}(X|Y = y)$)**.** Let X and Y be discrete random variables, let a, b, c, y be real numbers (constants), and let $g : \mathbb{R} \to \mathbb{R}$ be any function. Then

(i) If X and Y are independent, then $\mathbb{E}(X|Y = y) = \mathbb{E}(X)$.

(ii) When conditioning on $Y = y$, replace Y by y: $\mathbb{E}(Xg(Y)|Y) = \mathbb{E}(Xg(y)|Y) = g(y)\mathbb{E}(X|Y)$

(iii) Linearity: $\mathbb{E}(aX + bY + c|Y = y) = \mathbb{E}(aX + by + c|Y = y) = a\mathbb{E}(X|Y) + by + c$.

$\triangle$

Example 9.4.3. Roll two fair six-sided dice. X = sum of the dice, Y = number on first die, Z = number on second die.

$$\mathbb{E}(Z|X = 3) = \sum_{k \in R(Z)} kP(Z = k|X = 3)$$

For $k \geq 3$, $P(Z = k|X = 3) = 0$.

$$P(Z = 1|X = 3) = \frac{P(Z = 1 \text{ and } X = 3)}{P(X = 3)} = \frac{P(\{(2,1)\})}{P(\{(1,2),(2,1)\})} = \frac{1/36}{2/36} = \frac{1}{2}.$$

Similarly, $P(Z = 2|X = 3) = \frac{1}{2}$. Therefore

$$\mathbb{E}(Z|X = 3) = (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right) = 1.5$$

Intuitively, this make sense. If $X = 3$, then Z must be 1 or 2, with equal probability. △

Theorem 9.4.4 (Change of Variable or Law of the Unconscious Statistician)**.** Let X and and Y be a discrete random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a function. The conditional expectation of the random variable $g(X)$ given $Y = y$ is

$$\mathbb{E}(g(X)|Y = y) = \sum_{k \in R(g(X))} kP(g(X) = k|Y = y) = \sum_{k \in R(X)} g(k)P(X = k|Y = y). \tag{9.4.1}$$

△

Definition 9.4.2. Let X and Y be discrete random variables, and let $R(Y) = \{y_1, y_2, y_3, \ldots\}$. The **conditional expectation** of X given Y is

$$E(X|Y) = \begin{cases} E(X|Y = y_1) & \text{if } Y = y_1 \\ E(X|Y = y_2) & \text{if } Y = y_2 \\ \vdots & \vdots \\ E(X|Y = y_i) & \text{if } Y = y_i \\ \vdots & \vdots \end{cases} \tag{9.4.2}$$

Notice that $E(X|Y)$ is a random variable. Remember that a random variable is a function $Z : \Omega \to \mathbb{R}$. For each $\omega \in \Omega$, $Z(\omega)$ is a real number. Indeed, we can define $E(X|Y)$ more concisely as

$$E(X|Y)(\omega) = E(X|Y = Y(\omega)) \quad \text{for all } \omega \in \Omega.$$

△

Example 9.4.5. Roll two fair six-sided dice. X = sum of the dice, Y = number on first die, Z = number on second die.

$$\mathbb{E}(X|Y) = \mathbb{E}(Y + Z|Y) = \mathbb{E}(Y|Y) + \mathbb{E}(Z|Y) = Y + \mathbb{E}(Z|Y) = Y + \mathbb{E}(Z) = Y + 3.5$$

△

Theorem 9.4.6 (Properties of Conditional Expectation $\mathbb{E}(X|Y)$)**.** Let X and Y be discrete random variables, let a, b, c be real numbers (constants), and let $g : \mathbb{R} \to \mathbb{R}$ be any function. Then

(i) If X and Y are independent, then $\mathbb{E}(X|Y) = \mathbb{E}(X)$

(ii) When conditioning on Y, treat Y as constant: $\mathbb{E}(Xg(Y)|Y) = g(Y)\mathbb{E}(X|Y)$

(iii) Linearity: $\mathbb{E}(aX + bY + c|Y) = a\mathbb{E}(X|Y) + bY + c$

$\triangle$

Example 9.4.7. Roll two fair six-sided dice. X = sum of the dice, Y = number on first die, Z = number on second die.

$$E(Z|X) = \begin{cases} E(Z|X=2) & \text{if } X = 2 \\ E(Z|X=3) & \text{if } X = 3 \\ \vdots & \vdots \\ E(Z|X=12) & \text{if } X = 12 \\ \vdots & \vdots \end{cases}$$

In a previous example, we found $E(Z|X = 3) = 1.5$. We leave it as a challenge for the reader to compute $E(Z|X = k)$ for the other values of k, and thus find $E(Z|X)(\omega)$ for each $\omega \in \Omega = \{(1,1), (1,2), \ldots, (6,6)\}$. $\triangle$

Chapter 10

Asset Pricing and the Fundamental Theorem.

In the general setting of Chapter 8, we were only able to derive upper and lower bounds for option prices. In this chapter, we develop tools that will allow us to find exact formulas for the prices of options and other derivatives when we have additional information about the probability measure P.

10.1 The European Pricing Problem

Fix an asset. Let S_t be its price at time t. Consider a derivative on the asset with price D_t at time t. Assume the derivative pays out $g(S_T)$ at time T and has no payout otherwise. Note that $D_T = g(S_T)$. Examples of derivatives of this type include forwards (where $g(S_T) = S_T - K$), European calls (where $g(S_T) = (S_T - K)^+$), and European puts (where $g(S_T) = (K - S_T)^+$). Derivatives of this type are called **European** derivatives.

Problem. Determine the derivative's price D_t at current time t.

10.2 Replication Pricing

Our standard approach to solving problems like the European Pricing Problem has been what is often called replication pricing. We first find a portfolio A which replicates the derivative at T (i.e., $D_T = V^A(T)$ with probability one) and which has known value $V^A(t)$ at time t. Then we apply the replication theorem (to A and the portfolio B consisting of only the derivative) to conclude that $D_t = V^A(t)$.

We used this approach successfully when the derivative was a forward on various types of underlying assets. However, for other types of derivatives, like options, it is not always possible to find a suitable replicating portfolio A. If we have some specific information about the real-world probability measure P, then it may be possible to find a suitable replicating portfolio A. In Section 12.2, we demonstrate this in a simple model called the one-step binomial tree. However, in the present chapter, our main purpose is to introduce a different approach.

10.3 Risk-Neutral Pricing

If we had never heard of arbitrage (or if we don't care about being exploited by arbitrageurs), we might take the following approach to solving the European Pricing Problem. We simply set the current price equal to the future expected payout:

$$D_t = Z(t,T)\mathbb{E}(D_T|S_t). \tag{10.3.1}$$

This is called risk-neutral pricing. It is the pricing that an investor that is neither disposed nor averse to buy assets would perform because he/she believes the best guess for the future price of an asset, after discounting, is the current price.

The risk-neutral pricing formula (10.3.1) is wonderfully simple. However, it has a critical flaw: It can lead to violations the no-arbitrage principle, as the next example illustrates.

Example 10.3.1. Consider a one-year European call option on a stock with strike 40. Suppose:

- Current stock price = $S_t = 35$
- There are three possible states of the world at maturity T: $\omega_1, \omega_2, \omega_3$
- $P(\{\omega_1\}) = 1/2, \quad P(\{\omega_2\}) = 1/3, \quad P(\{\omega_3\}) = 1/6$
- $S_T(\omega_1) = 50, \quad S_T(\omega_2) = 55, \quad S_T(\omega_3) = 30$
- $Z(t,T) = 0.9.$

(a) Find the discounted present value of the expected payout: $Z(t,T)\mathbb{E}(S_T - K)^+$.

Since

$$(S_T - K)^+ = \begin{cases} 50 - 40 = 10, & \text{with probability } 1/2, \\ 55 - 40 = 15, & \text{with probability } 1/3, \\ 0 & \text{with probability } 1/6, \end{cases}$$

we have

$$\mathbb{E}(S_T - K)^+ = 10 \cdot \frac{1}{2} + 15 \cdot \frac{1}{3} + 0 \cdot \frac{1}{6} = 10.$$

Therefore $Z(t,T)\mathbb{E}(S_T - K)^+ = 0.9 \cdot 10 = 9$.

(b) Suppose we take the call price to be discounted present value of the expected payout:

$$C_K(t,T) = Z(t,T)\mathbb{E}(S_T - K, 0)^+ = 9.$$

Does this represent an arbitrage opportunity? If so, build an arbitrage portfolio. Verify it is an arbitrage portfolio.

Yes, there is an arbitrage opportunity.

Start with portfolio C empty at t. Sell one $K = 40$ call for 9 cash, borrow additional 26 cash, buy 1 stock for 35 cash.

C at t: short one $K = 40$ call, one stock, -26 cash.

$$V^C(t) = -9 + 35 - 26 = 0.$$

$$V^C(T) = S_T - (S_T - K)^+ - \frac{26}{Z(t,T)} = \begin{cases} 50 - 10 - 26/0.9 = 11.1111, & \text{with probability } 1/2, \\ 55 - 15 - 26/0.9 = 11.1111, & \text{with probability } 1/3, \\ 30 - 0 - 26/0.9 = 1.1111, & \text{with probability } 1/6. \end{cases}$$

Therefore $V^C(T) > 0$ with probability one. So C is an arbitrage portfolio. △

The flaw is not fatal. The risk-neutral pricing formula (10.3.1) can be made compatible with the no-arbitrage principle if we adjust how we assign probabilities in the world. This might seem bizarre, but it will be explained in the next section where we introduce the fundamental theorem of asset pricing. This might seem bizarre, but it will be explained in the next two sections where we discuss the fundamental theorem of asset pricing. In fact, the fundamental theorem of asset pricing essentially says that the no-arbitrage principle holds if and only if probabilities can be adjusted to make the risk-neutral pricing formula (10.3.1) work.

10.4 The Fundamental Theorem of Asset Pricing and Risk-Neutral Probability Measures

Theorem 10.4.1 (Fundamental Theorem of Asset Pricing)**.** There are no arbitrage portfolios if and only if there exists a probability measure P^* equivalent to P such that

$$V^A(t) = Z(t,T)\mathbb{E}^*(V^A(T)|\mathcal{I}_t) \quad \text{for all times } t < T \text{ and all portfolios } A. \qquad (10.4.1)$$

△

Notation. $\mathcal{I}_t$ is an abbreviation for all random variables relevant to the portfolio at time t. Typically, there will only be one relevant variable. For a portfolio that depends on a stock price, $\mathcal{I}_t = S_t =$ stock price at t. For a derivative that depends on an interest rate, $\mathcal{I}_t =$ interest rate at t.

Definition 10.4.1. A probability measure P^* equivalent to P that satisfies (10.4.1) is called a **risk-neutral** probability measure (or an **equivalent martingale** probability measure). △

Theorem 10.4.2 (Restatement of The Fundamental Theorem of Asset Pricing)**.** No-arbitrage if and only if there exists a risk-neutral probability measure. △

We will not give a general proof of the fundamental theorem. Though we will not get into details here, we note that it can be proved in very general settings. We prove the fundamental theorem in a simple model in Chapter 11.

Corollary 10.4.3 (Applications of the Fundamental Theorem)**.** Let S_t be the price at t of an arbitrary asset. If there are no-arbitrage portfolios, then there exists a risk neutral probability measure P^* and:

(i) If the asset pays no income, then

$$S_t = Z(t,T)\mathbb{E}^*(S_T|S_t) \quad \text{for all times } t \leq T$$

(ii) If D_t is the price at t of a European derivative on the asset (i.e., a derivative that pays out $D_T = g(S_T)$ at T and nothing otherwise), then

$$D_t = Z(t,T)\mathbb{E}^*(D_T|S_t) \quad \text{for all times } t \leq T$$

△

Proof. (i): Let portfolio A consist only of the asset. (ii): Let portfolio A consist only of the derivative. □

Corollary 10.4.3(ii) is the "corrected" version of the risk-neutral pricing formula (10.3.1).

As the name implies, and as we will soon see, the fundamental theorem of asset pricing is a powerful tool for determining the prices of derivatives and other assets. However, to actually use it, we must be able to compute the conditional expectation $\mathbb{E}^*(D_T|S_t)$ (or, more generally, $\mathbb{E}^*(V^A(T)|\mathcal{I}_t)$. Of course, to do this we must have some specific information about the risk-neutral probability measure P^* and (since it is equivalent) the real-world world probability measure P.

In the following chapters, we consider models where we have enough information about P and P^* to use the fundamental theorem to price options and other derivatives.

However, as the next example illustrates, we can already use the fundamental theorem to reproduce some things we already know.

Example 10.4.4. Assume no-arbitrage. Consider a forward with delivery price K and maturity T on a stock paying no income. The prices of the forward and stock at time t are denoted $V_K(t,T)$ and S_t. Let A be the portfolio consisting of only the forward. By the fundamental theorem,

$$\begin{aligned} V_K(t) &= V^A(t) = Z(t,T)\mathbb{E}^*(V^A(T)|S_t) \\ &= Z(t,T)\mathbb{E}^*(S_T - K|S_t) = Z(t,T)\mathbb{E}^*(S_T|S_t) - Z(t,T)K \end{aligned}$$

By Corollary 10.4.3(i), we get

$$V_K(t,T) = S_t - Z(t,T)K,$$

which is the same result we got by replication. △

Remark 10.4.5. The probability measure P represents the real-world probabilities determined from observation and data. A risk-neutral probability measure P^* represents the probabilities in an arbitrage-free world.

When we price assets using the no-arbitrage principle or the replication principle (which is a consequence of the no-arbitrage principle), such prices are guaranteed to not produce arbitrage opportunities because we are working in an arbitrage-free world.

It thus makes perfect sense that if we want to set a price according to the risk-neutral formula that the current price is the discounted expected future price, and if we want that price to not produce arbitrage opportunities, we must compute the expectation in the arbitrage-free world, the world of P^*. △

Exercise 10.4.1. Assume no arbitrage. Consider a forward with delivery price K and maturity T on a stock paying dividends at continuous yield q with automatic reinvestment. The prices of the forward and stock at time t are denoted $V_K(t)$ and S_t. Use the fundamental theorem (not replication) to prove $V_K(t) = S_t e^{-q(T-t)} - KZ(t,T)$. Hint: At some point you will need to show $E(S_T|S_t) = S_t e^{-q(T-t)}/Z(t,T)$. To do so, consider portfolio B consisting of just the stock, find $V^B(T)$, and apply the fundamental theorem to B.

Exercise 10.4.2. Assume no arbitrage. Consider a forward rate agreement with maturity T, fixed rate K, and term length α. Use the fundamental theorem (not replication), to prove that the value at time t is $V_K(t,T) = Z(t,T) - Z(t,T+\alpha) - \alpha K Z(t,T+\alpha)$.

Chapter 11

The Binomial Tree

11.1 Definition of the Binomial Tree

Consider a stock paying no income with price S_T at time T.

A **binomial tree** for the stock is a model of the stock price at discrete times. It has parameters ΔT, r, u, d, and p, which satisfy $\Delta T > 0$, $r > 0$, $0 < 1 + d < 1 + u$, and $0 < p < 1$. The times are $0, \Delta T, 2\Delta T, \ldots, n\Delta T, \ldots$. Often we take time step $\Delta T = 1$. r = constant interest rate with compounding frequency $1/\Delta T$. At time 0 the stock price S_0 is known. At time n the stock price goes from $S_{(n-1)\Delta T}$ up to $S_{n\Delta T} = (1 + u)S_{(n-1)\Delta T}$ with probability p, or down to $S_{n\Delta T} = (1+d)S_{(n-1)\Delta T}$ with probability $(1-p)$. Whether the price goes up or down at time n is independent of whether the price goes up or down at any other time.

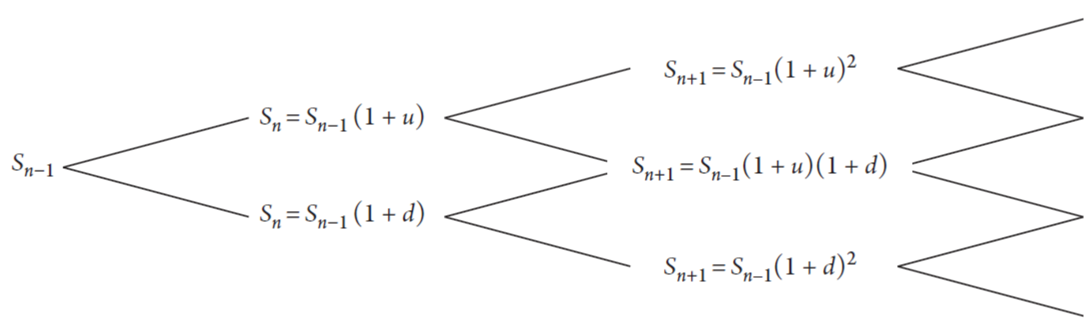


Figure 11.1: Branch of a Binomial Tree

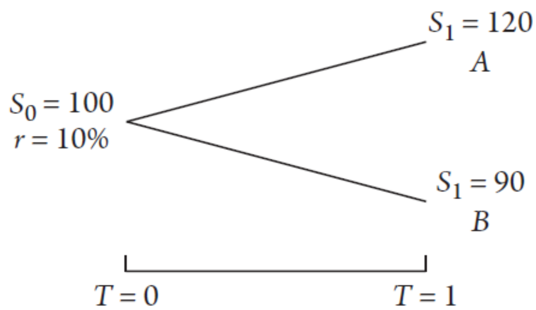


Figure 11.2: One-Step Binomial Tree with $r = 0.1$, $d = -0.1$, and $u = 0.2$

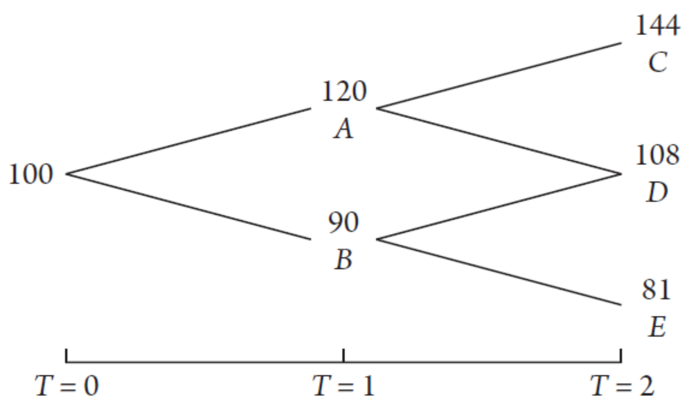


Figure 11.3: Two-Step Binomial Tree with $d = -0.1$ and $u = 0.2$

Here is an equivalent way to define how the price changes at each time:

$$S_{n\Delta T} = \xi_n S_{(n-1)\Delta T} \quad \text{for } n = 1, 2, \ldots, \tag{11.1.1}$$

where $\xi_1, \xi_2, \ldots$ is a sequence of independent random variables with

$$\xi_i = \begin{cases} (1+u) & \text{with probability } p \\ (1+d) & \text{with probability } 1-p \end{cases}$$

By applying (11.1.1) repeatedly, we see

$$S_{n\Delta T} = \xi_n S_{(n-1)\Delta T} = \xi_n \xi_{n-1} S_{(n-2)\Delta T} = \ldots = \xi_n \cdots \xi_1 S_0.$$

The probability p of an "up" movement determines the probability measure P:

Result 11.1.1.

$$P(S_{n\Delta T} = (1+u)^k (1+d)^{n-k} S_0) = P(\text{exactly } k \text{ "up" movements in period } 0 \text{ to } n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

for $0 \le k \le n$. $\triangle$

Proof. From time 0 to time n, the stock price changes n times. If

$$S_{n\Delta T} = (1+u)^k(1+d)^{n-k}S_0,$$

it means the stock price moves "up" exactly k times out of the n possible times. For example, if $n = 5$ and $k = 3$, a typical path with exactly k "up" movements is

up-down-up-up-down.

There are $\binom{n}{k}$ possible paths with exactly k up movements: Out of the n changes, choose k of them to be "up" and the rest "down." Each such path has probability $p^k(1-p)^{n-k}$. Therefore

$$P(S_{n\Delta T} = (1+u)^k(1+d)^{n-k}S_0) = \binom{n}{k}p^k(1-p)^{n-k}$$

for $0 \le k \le n$. □

11.2 Arbitrage-Free Binomial Tree

Consider a binomial tree (parameters ΔT,r,u,d,p) for a stock paying no income. The stock price at T is S_T.

Result 11.2.1. Assume the binomial tree has no arbitrage portfolios. Then

$$p^* = \frac{r\Delta T - d}{u-d},$$

where p^* is the probability of an "up" movement in the stock price with respect to the risk-neutral probability measure P^*. △

Proof. For simplicity, we assume $\Delta T = 1$ and leave the general case as an exercise. By the fundamental theorem, there is a risk-neutral probability measure P^*. Recall p is the probability of an "up" movement in the stock price with respect to P. Let p^* denote the probability of an "up" movement in the stock price with respect to P^*.

PUT A PICTURE OF THE TREE HERE

By the fundamental theorem,

$$\begin{aligned}
S_0 &= Z(0,1)\mathbb{E}^*(S_1|S_0) \\
&= (1+r)^{-1}\mathbb{E}^*(S_1|S_0) \\
&= (1+r)^{-1}\cdot(p^*(1+u)S_0 + (1-p^*)(1+d)S_0) \\
&= (1+r)^{-1}\cdot(p^*(1+u) + (1-p^*)(1+d))\cdot S_0.
\end{aligned}$$

Divide both sides by S_0 to get

$$1 = (1+r)^{-1}(p^*(1+u) + (1-p^*)(1+d)).$$

Solving for p^* gives

$$p^* = \frac{r-d}{u-d}.$$

$\square$

Exercise 11.2.1. Prove Result 11.2.1 in the case of arbitrary ΔT by modifying the proof above.

11.3 Pricing on the Binomial Tree. Part 1.

Consider a binomial tree (parameters ΔT,r,u,d,p) for a stock paying no income. The stock price at T is S_T.

Result 11.3.1 (Cox-Rubenstein Formula for European Calls)**.** Assume there are no arbitrage portfolios. The price at 0 of a European call with maturity $T = n\Delta T$ and strike K is

$$C_K(0,T) = (1+r\Delta T)^{-n} \sum_{k=0}^{n} \max\left\{(1+u)^k(1+d)^{n-k}S_0 - K, 0\right\} \binom{n}{k} (p^*)^k (1-p^*)^{n-k},$$

where $p^* = \dfrac{r\Delta T - d}{u-d}$ and $1 - p^* = \dfrac{u - r\Delta T}{u-d}$. $\triangle$

Remark This formula is a bit complicated, but it's exact. It's something you can program into a computer.

Proof. For simplicity, we assume $\Delta T = 1$, and the leave the general case as an exercise. We have

$$C_K(T,T) = \max\{S_T - K, 0\} = \max\{S_n - K, 0\} = g(S_n).$$

and

$$Z(0,T) = Z(0,n) = (1+r)^{-n}.$$

By the fundamental theorem,

$$C_K(0,T) = Z(0,T)\mathbb{E}^*(C_K(T,T)|S_0) = (1+r)^{-n}\mathbb{E}^*(g(S_n)|S_0).$$

By the Law of the Unconscious Statistician,

$$C_K(0,T) = (1+r)^{-n} \sum_{s\in R(S)} g(s)P^*(S_n = s|S_0).$$

The possible values of S_n are

$$(1+u)^k(1+d)^{n-k}S_0 \quad \text{for } k = 0, 1, \ldots, n.$$

Hence

$$C_K(0,T) = (1+r)^{-n} \sum_{k=0}^{n} g((1+u)^k(1+d)^{n-k}S_0)P^*(S_n = (1+u)^k(1+d)^{n-k}S_0|S_0).$$

By Result 11.2.1,

$$p^* = \frac{r-d}{u-d},$$

is the probability of an "up" movement in the stock price with respect to the risk-neutral probability measure P^*. Therefore, by Result 11.1.1,

$$P^*(S_n = (1+u)^k(1+d)^{n-k}S_0|S_0) = \binom{n}{k} p^{*k}(1-p^*)^{n-k} \quad \text{for } k = 0, 1, \ldots, n.$$

Hence

$$C_K(0,T) = (1+r)^{-n} \sum_{k=0}^{n} g((1+u)^k(1+d)^{n-k}S_0)\binom{n}{k} p^{*k}(1-p^*)^{n-k}.$$

Since $g(x) = \max\{x-K, 0\}$, we are done. $\square$

Adjusting the proof (exercise) gives:

Result 11.3.2 (General Cox-Rubenstein Formula)**.** Assume there are no arbitrage portfolios. Consider a European derivative on the stock with payout $g(S_T)$ at $T = n\Delta T$ (and no payout otherwise). The price at time 0 is

$$D(0,T) = Z(0,T)E^*(g(S_T)|S_0) = (1+r\Delta T)^{-n} \sum_{k=0}^{n} g((1+u)^k(1+d)^{n-k}S_0)\binom{n}{k} p^{*k}(1-p^*)^{n-k},$$

where $p^* = \dfrac{r\Delta T - d}{u-d}$ and $1 - p^* = \dfrac{u - r\Delta T}{u-d}$. $\triangle$

Let's do a computation with this formula.

Example 11.3.3. Assume no-arbitrage. The constant annually compounded interest rate is 10%. The time step is $\Delta T = 1$. At current time 0, a stock paying no income has price 50. Suppose that at each time point, the stock price can go up by 30% or down by 20%. Find the price at 0 of a European put with strike 47 and maturity 2.

$K = 47$, $T = n = 2$, $u = 0.3$, $d = -0.2$, $r = 0.1$.

$$p^* = \frac{r-d}{u-d} = 0.6$$

$$P_K(0,T) = (1+r)^{-n} \sum_{k=0}^{n} \max\{K - (1+u)^k (1+d)^{n-k} S_0, 0\} \cdot \binom{n}{k} (p^*)^k (1-p^*)^{n-k}$$

$$= (1+0.1)^{-2} \cdot [\max\{47 - (1+0.3)^0 (1-0.2)^2 \cdot 50, 0\} \cdot \binom{2}{0} (0.6)^0 (0.4)^2$$
$$+ \max\{47 - (1+0.3)^1 (1-0.2)^1 \cdot 50, 0\} \cdot \binom{2}{1} (0.6)^1 (0.4)^1$$
$$+ \max\{47 - (1+0.3)^2 (1-0.2)^0 \cdot 50, 0\} \cdot \binom{2}{2} (0.6)^2 (0.4)^0]$$

$$= (1+0.1)^{-2} \cdot [(47-32) \cdot \binom{2}{0} (0.6)^0 (0.4)^2 + 0 + 0]$$

$$= 1.9834\ldots$$

△

Exercise 11.3.1. Adjust the proof of Result 11.3.1 above to prove Result 11.3.2 for arbitrary ΔT.

11.4 Pricing on the Binomial Tree. Part 2.

Consider a binomial tree (parameters ΔT,r,u,d,p) for a stock paying no income. The stock price at T is S_T.

Let's do Example 11.3.3 by a different method.

Example 11.4.1. Assume no-arbitrage. The constant annually compounded interest rate is 10%. The time step is $\Delta T = 1$. At current time 0, a stock paying no income has price 50. Suppose that at each time point, the stock price can go up by 30% or down by 20%. Find the price at 0 of a European put with strike 47 and maturity 2. △

$K = 47, \quad T = n = 2, \quad u = 0.3, \quad d = -0.2, \quad r = 0.1$

$$p^* = \frac{r-d}{u-d} = 0.6 = \text{prob. of up move}$$

T	0	1	2
			$(1+u)65 = 84.5$
		$(1+u)50 = 65$	
S_T	50		52
		$(1+d)50 = 40$	
			32

T	0	1	2
			D
		B	
$P_K(T,T)$	A		E
		C	
			F

D: $P_K(2,2) = \max\{K - S_2, 0\} = \max\{47 - 84.5, 0\} = 0$

E: $P_K(2,2) = \max\{47 - 52, 0\} = 0$

F: $P_K(2,2) = \max\{47 - 32, 0\} = 15$

T	0	1	2
			0
		B	
$P_K(T,T)$	A		0
		C	
			15

B: $P_K(1,2) = Z(1,2)\,\mathbb{E}^*(P_K(2,2) \mid B)$
$= (1+r)^{-1} \cdot (p^* \cdot 0 + (1-p^*) \cdot 0)$
$= 0$

C: $P_K(1,2) = Z(1,2)\,\mathbb{E}^*(P_K(2,2) \mid C)$
$= (1+r)^{-1}(p^* \cdot 0 + (1-p^*) \cdot 15)$
$= (1+0.1)^{-1}((0.6)(0) + (0.4)(15))$
$= 5.\overline{45}$

T	0	1	2
			0
		0 (B)	
$P_K(T,T)$	A		0
		$5.\overline{45}$ (C)	
			10

At time 1, the put has the same value as a derivative expiring at 1 with payout

$$\begin{cases} 0 & \text{in state B} \\ 5.\overline{45} & \text{in state C} \end{cases}$$

By replication, the put and the derivative have the same value at time 0.
Let D_T be the value of the derivative at T.
Therefore

$$\begin{aligned} A: \quad P_K(0,2) = D_0 &= Z(0,1)\mathbb{E}^*(D_1 \mid A) \\ &= (1+r)^{-1}\left(p^*(0) + (1-p^*)(5.\overline{45})\right) \\ &= (1+0.1)^{-1}\left((0.6)(0) + (0.4)(5.\overline{45})\right) \\ &= 1.9834\ldots \end{aligned}$$

Let's do the same example again, but with an American put.

Example 11.4.2. Assume no-arbitrage. The constant annually compounded interest rate is 10%. The time step is $\Delta T = 1$. At current time 0, a stock paying no income has price 50. Suppose that at each time point, the stock price can go up by 30% or down by 20%. Find the price at 0 of a American put with strike 47 and maturity 2.

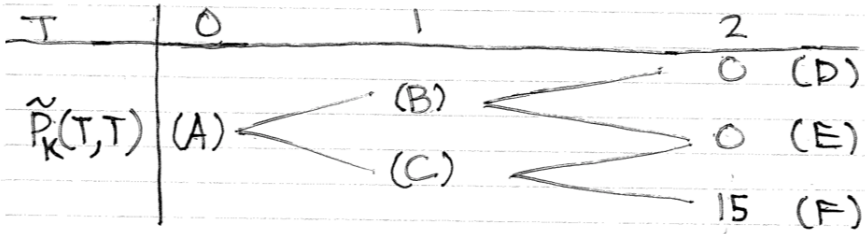
$K = 47, \quad T = n = 2, \quad u = 0.3, \quad d = -0.2, \quad r = 0.1$

$$p^* = \frac{r-d}{u-d} = 0.6 = \text{prob. of up move}$$

T	0	1	2
			84.5 (D)
		$65 = (1+u)50$ (B)	
S_T	50 (A)		52 (E)
		40 (C)	
			32 (F)

T	0	1	2
			(D)
		(B)	
$\tilde{P}_K(T,T)$	(A)		(E)
		(C)	
			(F)

D: $\tilde{P}_K(2,2) = \max\{K - S_2, 0\} = \max\{47 - 84.5\} = 0$
E: $\tilde{P}_K(2,2) = \max\{47 - 52, 0\} = 0$
F: $\tilde{P}_K(2,2) = \max\{47 - 32, 0\} = 15$



At time 1, the owner of the put decides between exercising the put or holding it until time 2. The owner will choose the one with highest value.

C: The payout at 1 for exercising the put is
$\max\{K - S_1, 0\} = \max\{47 - 40, 0\} = 7$

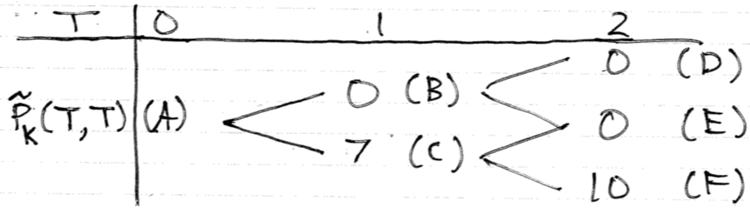
At time 1, the value of holding the put until time 2 is the same as the value of a derivative that expires at time 2 with payout
$\begin{cases} 0 & \text{in state } E \\ 15 & \text{in state } F \end{cases}$
Let D_T be the value of the derivative at T.
Thus the value of holding the put until 2 is

$$\begin{aligned} D_1 &= Z(1,2)\,E^*(D_2 \mid C) \\ &= (1+r)^{-1}(p^* \cdot 0 + (1-p^*) \cdot 15) \\ &= (1+0.1)^{-1}((0.6)(0) + (0.4)(15)) \\ &= 5.\overline{45} \end{aligned}$$

7 vs $5.\overline{45} \longrightarrow 7 = \tilde{P}_K(1,2)$

B: The payout at 1 for exercising the put is
$\max\{47 - 65, 0\} = 0$
The value at 1 of holding the put until 2 is
$(1+r)^{-1}(p^* \cdot 0 + (1-p^*) \cdot 0) = 0$

0 vs. $0 \longrightarrow 0 = \tilde{P}_K(1,2)$



A: Payout at 0 for exercise $= \max\{47-50, 0\} = 0$
Value at 0 for waiting
$$= (1+r)^{-1}(p^* \cdot 0 + (1-p^*) \cdot 7)$$
$$= (1+0.1)^{-1}((0.6)(0) + (0.4)(7))$$
$$= 2.\overline{54}$$

0 vs $2.\overline{54} \longrightarrow 2.\overline{54}$

$\therefore \tilde{P}_k(0,2) = 2.\overline{54}$

△

Remark. The examples above show that an American put can have price strictly greater than a European put with the same strike and maturity.

Remark. The examples in this section demonstrate an algorithm you can program into a computer.

Chapter 12

Replication and Proof of the Fundamental Theorem on the One-Step Binomial Tree

12.1 Setting: One-Step Binomial Tree

Throughout this chapter, we consider a stock paying no income and a one-step binomial tree with times $T = 0$ and $T = 1$. At time $T = 0$, the stock price S_0 is known. At time T=1, there are two possible states of the world: either $S_1 = S_1 = (1 + u)S_0$ (state A) with probability p or $S_1 = (1 + d)S_0$ (state B) with probability $1 - p$. The sample space is $\Omega = \{\omega_A, \omega_B\}$. Of course, $\Delta T = 1$, $0 < 1 + d < 1 + u$, $0 < p < 1$, and $r > 0$ is the constant annually compounded interest rate. An example is given in Figure 12.1.

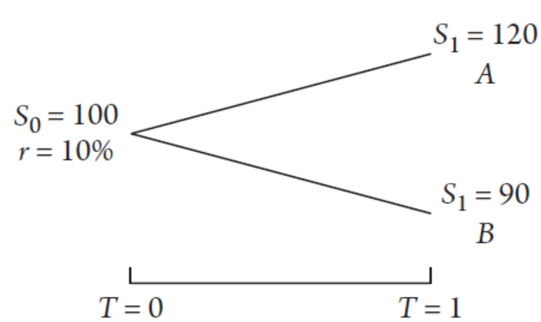


Figure 12.1: One-Step Binomial Tree with $r = 0.1$, $d = -0.1$, and $u = 0.2$

12.2 Replication on the One-Step Binomial Tree

In this section, we fulfill a promise that we made in Section 10.2. Namely, to demonstrate that an arbitrary derivative can be replicated in the one-step binomial tree model.

In other words, the payout is

$$g(S_1) = \begin{cases} \alpha & \text{if } S_1 = S_0(1+u) \\ \beta & \text{if } S_1 = S_0(1+d) \end{cases}$$

Every derivative contract in the one-step tree is of this form. Let D_T be the price of the derivative at T.

Consider the portfolio A of x stocks and y ZCBs (with maturity at $T = 1$). The following statements are logically equivalent.

$$A \text{ replicates the derivative at } T = 1$$
$$V^A(1) = D_1 \text{ with probability one (i.e., in all states of the world)}$$
$$xS_1 + yZ(1,1) = g(S_1) \text{ with probability one (i.e., in all states of the world)}$$
$$\begin{cases} xS_0(1+u) + y = \alpha & \text{if } S_1 = S_0(1+u) \\ xS_0(1+d) + y = \beta & \text{if } S_1 = S_0(1+d) \end{cases}$$

Solving for x and y gives

$$x = \frac{\alpha - \beta}{S_0(u-d)} \tag{12.2.1}$$

$$y = \alpha - \frac{\alpha - \beta}{u-d}(1+u) \tag{12.2.2}$$

Thus the portfolio A of x stocks and y ZCBs (with x,y as above) replicates the derivative at $T = 1$. By the replication theorem, the price of the derivative at $T = 0$ is

$$D_0 = xS_0 + yZ(0,1) = \frac{\alpha - \beta}{u-d} + \left(\alpha - \frac{\alpha - \beta}{u-d}(1+u)\right) Z(0,1). \tag{12.2.3}$$

Remark. By iterating the argument above, we can replicate any European derivative (and some more general derivatives) on a multi-step binomial tree.

Exercise 12.2.1. Show that Result 11.3.2 also gives (12.2.3).

12.3 Proof of the Fundamental Theorem on the One-Step Binomial Tree

On the one-step binomial tree, the fundamental theorem is:

Theorem 12.3.1. There are no arbitrage portfolios if and only if there exists a probability measure P^* equivalent to P such that

$$V^A(0) = Z(0,1)\mathbb{E}^*(V^A(1)|S_0) \quad \text{for all portfolios } A. \tag{12.3.1}$$

$\triangle$

Proof: Part 1. Assume there exists a probability measure P^* equivalent to P such that (12.3.1) holds. We will show there are no arbitrage portfolios.

Let A be any portfolio on the one step-binomial tree. We will assume A is an arbitrage portfolio and deduce a contradiction. We have

- $V^A(0) \leq 0$
- $P(V^A(1) \geq 0) = 1$
- $P(V^A(1) > 0) > 0$

Since P^* is equivalent to P,

(i) $V^A(0) \leq 0$

(ii) $P^*(V^A(1) \geq 0) = 1$

(iii) $P^*(V^A(1) > 0) > 0$

We must have

$$V^A(1) = \begin{cases} \alpha & \text{if } S_1 = S_0(1+u) \\ \beta & \text{if } S_1 = S_0(1+d) \end{cases}$$

for some numbers α, β. Let p^* be the probability of an up movement with respect to P^*. Therefore

$$V^A(1) = \begin{cases} \alpha & \text{with probability } p^* \\ \beta & \text{with probability } 1-p^* \end{cases}$$

Then (ii) implies $\alpha \geq 0$ and $\beta \geq 0$, and (iii) implies at least one of α and β is positive. Without loss of generality, assume $\alpha > \beta \geq 0$. Since (12.3.1) holds,

$$V^A(0) = Z(0,1)\mathbb{E}^*(V^A(1)|S_0) = Z(0,1)(\alpha p^* + \beta(1-p^*)) > 0$$

This contradicts (i). $\square$

Proof: Part 2. Assume there are no-arbitrage portfolios.

We must have $d < r < u$. Indeed, if either $r \geq u > d$ or $d < u \leq r$, then we can construct an arbitrage portfolio. In the first case we proceed as follows. At $T = 0$ borrow 1 stock, sell it for S_0 cash, invest the cash at interest rate r; at $T = 1$ we have $S_0(1+r)$ cash and a debt of one stock, which has positive probability $1-p$ of having price $S_0(1+d) < S_0(1+r)$. The second case is similar.

Define $p^* = \frac{r-d}{u-d}$. Since $d < r < u$, we have $0 < p^* < 1$. Remember the sample space is $\Omega = \{\omega_A, \omega_B\}$. Define the probability measure P^* by $P^*(\{\omega_A\}) = P^*(S_1 = S_0(1+u)) = p^*$ and $P^*(\{\omega_B\}) = P^*(S_1 = S_0(1+d)) = 1-p^*$. Note $P(E) = 1$ if and only if $E = \Omega$ if and only if $P^*(E) = 1$. Therefore P and P^* are equivalent probability measures.

Note that

$$\begin{aligned}
\mathbb{E}^*(S_1|S_0) &= S_0(1+u)P^*(S_1 = S_0(1+u)) + S_0(1+d)P^*(S_1 = S_0(1+d)) \\
&= S_0(1+u)p^* + S_0(1+d)(1-p^*) \\
&= S_0(1+u)\frac{r-d}{u-d} + S_0(1+d)\frac{u-r}{u-d} \\
&= S_0\frac{r-d+ur-ud+u-r+du-dr}{u-d} \\
&= S_0\frac{u-d+r(u-d)}{u-d} \\
&= S_0(1+r) \\
&= \frac{S_0}{Z(0,1)}.
\end{aligned}$$

In summary,

$$Z(0,1)\mathbb{E}^*(S_1|S_0) = S_0. \tag{12.3.2}$$

Let A be any portfolio. We must have

$$V^A(1) = \begin{cases} \alpha & \text{if } S_1 = S_0(1+u) \\ \beta & \text{if } S_1 = S_0(1+d) \end{cases}$$

Since the no-arbitrage principle holds, the replication principle holds. By arguing as in Section 12.2, the portfolio A' of $x = \frac{\alpha-\beta}{S_0(u-d)}$ stocks and $y = \alpha - \frac{\alpha-\beta}{u-d}(1+u)$ ZCBs satisfies $V^A(1) = V^{A'}(1)$ and (by replication) $V^A(0) = V^{A'}(0)$.

Using this and (12.3.2), we have

$$\begin{aligned}
Z(0,1)\mathbb{E}^*(V^A(1)|S_0) &= Z(0,1)\mathbb{E}^*(V^{A'}(1)|S_0) = Z(0,1)\mathbb{E}^*(xS_1 + y|S_0) \\
&= Z(0,1)(x\mathbb{E}^*(S_1|S_0) + y) = xZ(0,1)\mathbb{E}^*(S_1|S_0) + yZ(0,1) \\
&= xS_0 + yZ(0,1) = V^{A'}(0) = V^A(0).
\end{aligned}$$

□

Chapter 13

Probability Theory: Normal Distribution and Central Limit Theorem

13.1 Normal Distribution

Let X be a random variable. If there are constants $\mu \in \mathbb{R}$ and $\sigma > 0$ such that

$$P(a \leq X \leq b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

for all real numbers $a \leq b$, then we write

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

and we say that X has **normal distribution** (with respect to P). Note that X is non-discrete with $R(X) = \mathbb{R}$.

Theorem 13.1.1. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then:

(i) For any function $g(x)$, the expectation of $g(X)$ is

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} g(x) dx$$

(ii) The expectation of X is

$$\mathbb{E}(X) = \mu.$$

(iii) The variance of X is

$$\text{Var}(X) = \mathbb{E}((X-\mu)^2) = \mathbb{E}(X^2) - \mu^2 = \sigma^2.$$

△

13.2 Standard Normal Distribution

Definition 13.2.1. If $W \sim N(0,1)$, we say that W has **standard normal distribution**. △

Definition 13.2.2. The function

$$\Phi(t) = P(X \leq t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

is called the **standard normal cumulative distribution function** or **standard normal cdf**. △

Definition 13.2.3. The function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

is called the **standard normal probability density function** or **standard normal pdf**. △

Theorem 13.2.1. If $W \sim N(0,1)$ and if a and b are real numbers, then

$$X = a + bW \sim N(a, b^2).$$

△

13.3 Central Limit Theorem

The central limit theorem is the reason the normal distribution is so important. Basically, it says that any sum of many independent random effects is approximately normally distributed.

Theorem 13.3.1 (Central Limit Theorem)**.** If $X_1, X_2, \ldots$ are independent identically distributed random variables with $\mathbb{E}(X_i) = 0$ and $\text{Var}(X_i) = 1$, then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \to N(0,1)$$

which is an abbreviation for

$$\lim_{n \to \infty} P\left(a \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \leq b\right) = \int_{a}^{b} \frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2} dx$$

for all real numbers a,b. △

Chapter 14

Continuous-Time Limit and Black-Scholes

14.1 Binomial Tree to Black-Scholes

We motivate the Black-Scholes model as a continuous-time limit of the binomial tree model.

Consider a stock paying no income whose price at time T is S_T. Assume there are no-arbitrage portfolios. Fix a large time T_0. We divide the time interval $[0, \infty)$ at very closely spaced time points 0, ΔT, $2\Delta T$, ..., $(n-1)\Delta T$, $n\Delta T, \ldots$ with step size

$$\Delta T = \frac{1}{N},$$

where N is a very large positive integer. We aim to let $N \to \infty$. We suppose the stock obeys a binomial tree model with ΔT, r, u, d, and $p = 1/2$. Thus the stock price satisfies

$$S_{i\Delta T} = \xi_i^{(N)} S_{(i-1)\Delta T} \quad \text{for } i = 0, 1, 2, \ldots$$

Here $\xi_1^{(N)}, \xi_2^{(N)}, \ldots$ are independent random variables with identical distributions

$$\xi_i^{(N)} = \begin{cases} 1+u & \text{with } P \text{ probability } 1/2 \\ 1+d & \text{with } P \text{ probability } 1/2 \end{cases}$$

Under the risk-neutral probability measure P^*, we have

$$\xi_i = \begin{cases} 1+u & \text{with } P^* \text{ probability } p^* \\ 1+d & \text{with } P^* \text{ probability } 1-p^* \end{cases}$$

where

$$p^* = \frac{r\Delta T - d}{u - d}.$$

The stock price at $T = n\Delta T$ is

$$S_T = S_{N\Delta T} = \xi_N^{(N)} S_{(N-1)\Delta T} = \xi_N^{(N)}\xi_{N-1}^{(N)} S_{(N-2)\Delta T} = \cdots = \xi_N^{(N)}\xi_{N-1}^{(N)}\cdots\xi_1^{(N)} S_0,$$

Hence, for $t = k\delta T < n\delta T = T$,

$$S_T/S_t = \xi_n^{(N)}\xi_{n-1}^{(N)}\cdots\xi_{k+1}^{(N)}. \tag{14.1.1}$$

If $t_1 < T_1 \leq t_2 < T_2$ (with $t_i = k_i\Delta T$ and $T_i = n_i\Delta T$), then the random variables

$$S_{T_1}/S_{t_1} = \xi_{n_1}^{(N)}\xi_{n_1-1}^{(N)}\cdots\xi_{k_1+1}^{(N)} \quad \text{and} \quad S_{T_2}/S_{t_2} = \xi_{n_2}^{(N)}\xi_{n_2-1}^{(N)}\cdots\xi_{k_2+1}^{(N)}$$

are independent.

Fix $t = k\Delta T < n\Delta T = T$. For simplicity, assume $t = 0$.

Then (14.1.1) is

$$S_T/S_0 = \xi_n^{(N)}\xi_{n-1}^{(N)}\cdots\xi_1^{(N)}.$$

We aim to take the limit $N \to \infty$ ($\Delta T \to 0$) and use the central limit theorem. But the central limit theorem applies to sums of independent identically distributed random variables, and we have a product $\xi_n^{(N)}\xi_{N-1}^{(N)}\cdots\xi_1^{(N)}$. To change multiplication into addition, we take logarithms:

$$\ln(S_T/S_0) = \sum_{i=1}^{n} \ln \xi_i^{(N)}.$$

We define μ and $\sigma > 0$ by

$$\mu\Delta T = \mathbb{E}\left(\ln(S_{i\Delta T}/S_{(i-1)\Delta T})\right) = \mathbb{E}\left(\ln(\xi_i^{(N)})\right) \tag{14.1.2}$$

$$\sigma^2\Delta T = \text{Var}\left(\ln(S_{i\Delta T}/S_{(i-1)\Delta T})\right) = \text{Var}\left(\ln(\xi_i^{(N)})\right) \tag{14.1.3}$$

Note that this definition does not depend on i because $\xi_1, \xi_2, \ldots$ are identically distributed.

By some algebra, we can show

$$\ln \xi_i^{(N)} = \mu\Delta T + \sigma\sqrt{\Delta T} X_i$$

where and $X_1, X_2, \ldots,$ are independent with identical distributions

$$X_i = \begin{cases} +1 & \text{with } P \text{ probability } 1/2 \\ -1 & \text{with } P \text{ probability } 1/2 \end{cases} = \begin{cases} +1 & \text{with } P^* \text{ probability } p^* \\ -1 & \text{with } P^* \text{ probability } 1-p^* \end{cases}$$

Note $\mathbb{E}(X_i) = 0$ and $\text{Var}(X_i) = 1$ for all i.

Now

$$\begin{aligned}\ln(S_T/S_0) &= \sum_{i=1}^{n} \ln \xi_i^{(N)} \\ &= \mu \Delta T n + \sigma \sqrt{\Delta T} n \sum_{i=1} X_i \\ &= \mu T + \sigma \sqrt{T} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i.\end{aligned}$$

Now we let $N \to \infty$. Since $T = n\Delta T = n/N$ and since T is fixed, we have $n \to \infty$ also. Thus the central limit theorem implies

$$\ln(S_T/S_0) \to \mu T + \sigma \sqrt{T} W, \quad \text{where } W \sim N(0,1) \text{ w.r.t } P$$

hence

$$\ln(S_T/S_0) \to N(\mu T, \sigma^2 T)$$

with repsect to P. With some adjustments, we can show for any $t < T$ that

$$\ln(S_T/S_t) \to N\left(\mu(T-t), \sigma^2(T-t)\right)$$

with respect to P.

With a more complicated argument, we can show for any $t < T$ that

$$\ln(S_T/S_0) \to N\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$$

with respect to P^*. Here r is the continuously compounded interest rate.

14.2 Black-Scholes Model

We consider a stock paying no income whose price at time T is S_T.

Black-Scholes Model

1. There are no arbitrage portfolios.
2. There is a constant continuous interest rate r.
3. The stock price at S_0 is a known constant.
4. For any times $0 \leq t_1 < T_1 \leq t_2 < T_2$, $\ln(S_{T_1}/S_{t_1})$ and $\ln(S_{T_2}/S_{t_2})$ are independent.
5. There are constants μ and $\sigma > 0$ such that for all times $t < T$,

   $$\ln(S_T/S_t) \sim \mathcal{N}\left(\mu(T-t); \sigma^2(T-t)\right)$$

   with respect to P (the real-world probability measure).

6. There is a risk-neutral probability measure P^* such that for all times $t < T$,

$$\ln(S_T/S_t) \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t); \sigma^2(T-t)\right)$$

with respect to P^*.

Definition 14.2.1. The constant μ is called the **drift** of the stock. The constant σ is called the **volatility** of the stock. △

The following result is extremely very useful.

Result 14.2.1. Assume the Black-Scholes model for a stock paying no income. For any function $g : \mathbb{R} \to \mathbb{R}$ and any times $t < T$,

$$\mathbb{E}^*(g(\ln S_T)|S_t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} e^{-(x-\nu)^2/2\sigma^2(T-t)} g(x)dx$$

△

14.3 Black-Scholes Formula

Assume the Black-Scholes model for a stock paying no income.

Result 14.3.1 (Black-Scholes Formula)**.** The price at t of a European call option with strike K and maturity T is

$$C_K(t,T) = Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\ln K}^{\infty} e^{-(x-\nu)^2/2\sigma^2(T-t)}(e^x - K)dx, \quad (14.3.1)$$

This can also be written as

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2) \quad (14.3.2)$$

or

$$C_K(t,T) = Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2)). \quad (14.3.3)$$

where $F(t,T) = \frac{S_t}{Z(t,T)}$ is the forward price, $\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$. is the standard normal cdf, and △

Proof. Since P^* is a risk-neutral probability measure,

$$\begin{aligned} C_K(t,T) &= Z(t,T)E^*((S_T - K)^+|S_t) \\ &= Z(t,T)E^*((e^{\ln S_T} - K)^+|S_t) \\ &= Z(t,T)E^*(g(\ln S_T)|S_t) \end{aligned}$$

where

$$g(x) = (e^x - K)^+.$$

Under the risk-neutral probability measure P^*,

$$\ln S_T \sim \mathcal{N}(\nu, \sigma^2(T-t)),$$

where

$$\nu = \ln S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t).$$

Therefore

$$\begin{aligned} C_K(t,T) &= Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\int_{-\infty}^{\infty} e^{-(x-\nu)^2/2\sigma^2(T-t)} g(x)dx \\ &= Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\int_{-\infty}^{\infty} e^{-(x-\nu)^2/2\sigma^2(T-t)} (e^x - K)^+ dx. \end{aligned}$$

Notice that

$$(e^x - K)^+ = \begin{cases} e^x - K & \text{if } x \geq \ln K \\ 0 & \text{if } x \leq \ln K \end{cases}$$

Therefore

$$C_K(t,T) = Z(t,T)\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\int_{\ln K}^{\infty} e^{-(x-\nu)^2/2\sigma^2(T-t)} (e^x - K)dx.$$

By making a clever change of variable, we can write the integral above in terms of the standard normal cdf

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Doing so, we obtain

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2)$$

with d_1 and d_2 as above.

Using the forward price formula for a stock paying no income

$$F(t,T) = \frac{S_t}{Z(t,T)},$$

we can rewrite the previous result as

$$C_K(t,T) = Z(t,T)(F(t,T)\Phi(d_1) - K\Phi(d_2)).$$

□

Example 14.3.2. Use the Black-Scholes formula to find the current price of a European call on a stock paying no income with strike 40 and maturity 18 months from now. Assume the current stock price is 50, the stock volatility is 15%, and the constant continuously compounded interest rate is 5%.

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln(50/40) + (0.05 + \frac{1}{2}(0.15)^2)(1.5)}{0.15\sqrt{1.5}} \\ &= 1.7147438\ldots \end{aligned}$$

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{T-t} \\ &= (1.7147438\ldots) - (0.15)\sqrt{1.5} \\ &= 1.531032\ldots \end{aligned}$$

$$\begin{aligned} C_K(t,T) &= S_t\Phi(d_1) - KZ(t,T)\Phi(d_2) \\ &= 50\Phi(1.7147438\ldots) - 40e^{-0.05(1.5)}\Phi(1.531032\ldots) \\ &= 13.06\ldots \end{aligned}$$

△

14.4 Properties of Black-Scholes Formula

Consider a stock paying no income. Assume the Black-Scholes model of the stock. The Black-Scholes formula for a European call is

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2)$$

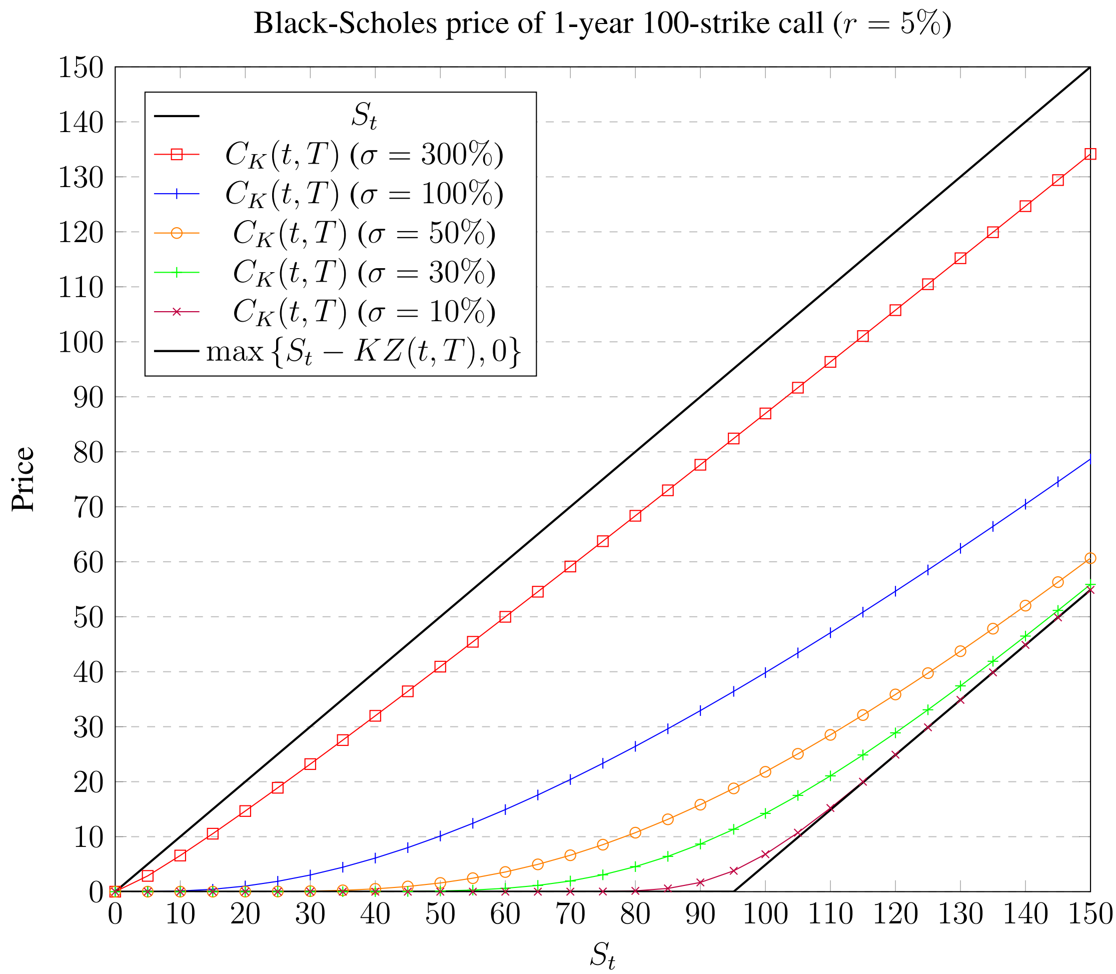
where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Recall the bounds on a European call we obtained in Chapter 8:

$$\max\{S_t - KZ(t,T), 0\} \leq C_K(t,T) \leq S_t$$

These bounds are illustrated when we plot the price of a 1-year 80-strike call under the Black-Scholes model for various values of the volatility σ.



As the plot indicates, the bounds are achieved in limiting cases:

Result 14.4.1. (a)

$$C_K(t,T) \to \max\{S_t - KZ(t,T), 0\} \qquad \text{as } \sigma \to 0.$$

(b)

$$C_K(t,T) \to S_t \qquad \text{as } \sigma \to \infty.$$

$\triangle$

Proof. We prove (a) and leave (b) as an exercise. We have

$$
\begin{aligned}
\lim_{\sigma\to 0} d_1 &= \lim_{\sigma\to 0} \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
&= \lim_{\sigma\to 0} \frac{\ln(S_t/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \\
&= \begin{cases} +\infty & \text{if } \ln(S_t/K) + r(T-t) > 0 \\ 0 & \text{if } \ln(S_t/K) + r(T-t) = 0 \\ -\infty & \text{if } \ln(S_t/K) + r(T-t) < 0 \end{cases} \\
&= \begin{cases} +\infty & \text{if } S_t > Ke^{-r(T-t)} \\ 0 & \text{if } S_t = Ke^{-r(T-t)} \\ -\infty & \text{if } S_t < Ke^{-r(T-t)} \end{cases}
\end{aligned}
$$

Therefore

$$
\lim_{\sigma\to 0} \Phi(d_1) = \lim_{\sigma\to 0} P(X \le d_1) = \begin{cases} 1 & \text{if } S_t > Ke^{-r(T-t)} \\ \frac{1}{2} & \text{if } S_t = Ke^{-r(T-t)} \\ 0 & \text{if } S_t < Ke^{-r(T-t)}. \end{cases}
$$

where $X \sim N(0,1)$. Since

$$
\lim_{\sigma\to 0} d_2 = \lim_{\sigma\to 0}(d_1 - \sigma\sqrt{T-t}) = \lim_{\sigma\to 0} d_1,
$$

we have

$$
\lim_{\sigma\to 0} \Phi(d_2) = \lim_{\sigma\to 0} \Phi(d_1) = \begin{cases} 1 & \text{if } S_t > Ke^{-r(T-t)} \\ \frac{1}{2} & \text{if } S_t = Ke^{-r(T-t)} \\ 0 & \text{if } S_t < Ke^{-r(T-t)}. \end{cases}
$$

Thus

$$
\begin{aligned}
\lim_{\sigma\to 0} C_K(t,T) &= \lim_{\sigma\to 0}(S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)) \\
&= \begin{cases} S_t - Ke^{-r(T-t)} & \text{if } S_t > Ke^{-r(T-t)} \\ 0 & \text{if } S_t = Ke^{-r(T-t)} \\ 0 & \text{if } S_t < Ke^{-r(T-t)}. \end{cases} \\
&= \max\{S_t - KZ(t,T), 0\}
\end{aligned}
$$

□

14.5 The Greeks: Delta and Vega

Definition 14.5.1. The **delta** of an option (or other derivative contract) is the partial derivative of its price with respect to the underlying asset price. △

Result 14.5.1. Assuming the Black-Scholes model for a stock paying no income, the delta of a European call is

$$\frac{\partial C_K(t,T)}{\partial S_t} = \Phi(d_1).$$

△

We outline the proof at the end of this section.

Since $\Phi(d_1) = P(X \leq d_1)$, where $X \sim N(0,1)$, the delta of a European call satisfies

$$0 \leq \frac{\partial C_K(t,T)}{\partial S_t} \leq 1.$$

If the delta of a contract is positive, the price of the contract increases as the stock price increases, and the party long on the contract is said to have a **long delta position**.

If the delta of a contract is negative, the price of the contract decreases as the stock price increases, and the party long on the contract is said to have a **short delta position**.

Example 14.5.2. The owner (long counterparty) of a European call has a long delta position, and so will profit if the stock price increases.

The writer (short counterparty) of a European call has a short delta position. Indeed, the delta is

$$\frac{\partial}{\partial S_t}(-C_K(t,T)) = -\Phi(d_1).$$

△

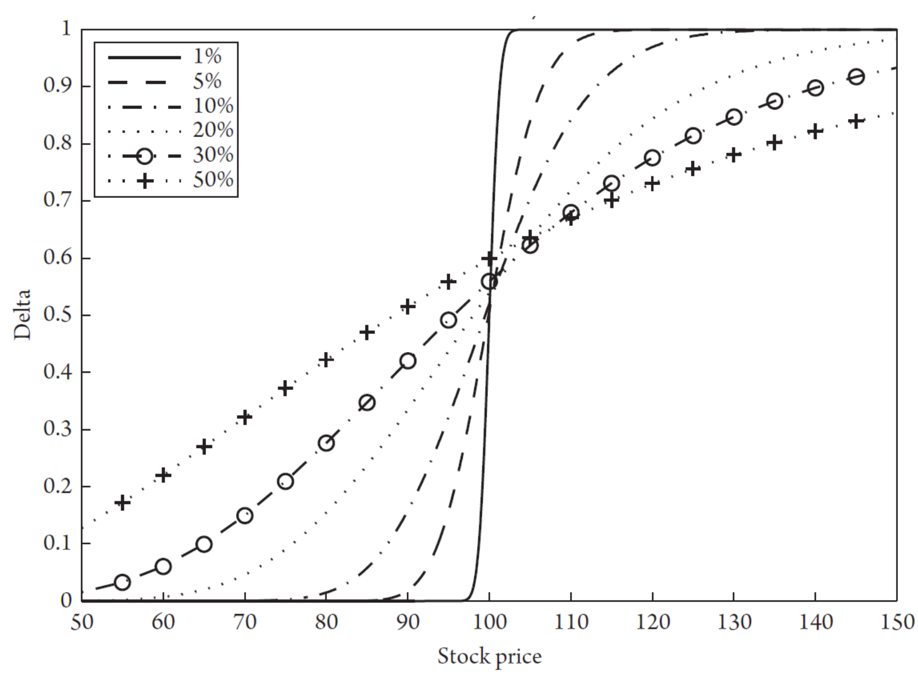


Figure 14.1: Black-Scholes delta of 1-year 100-strike call (for different volatilities σ)

Example 14.5.3 (Delta Hedging)**.** Let $\Delta(s)$ be the delta of a European call when the stock price is $S_t = s$:

$$\Delta(s) = \left.\frac{\partial C_K(t,T)}{\partial S_t}\right|_{S_t=s}.$$

For example, assume $K = 100$, $T - t = 1$, $r = 10\%$, and $\sigma = 20\%$. Then

$$\begin{aligned}
\Delta(90) &= \left.\frac{\partial C_K(t,T)}{\partial S_t}\right|_{S_t=90} = \Phi(d_1)|_{S_t=90} \\
&= \Phi\left(\frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) = \Phi\left(\frac{\ln(90/100) + (0.10 + \frac{1}{2}(0.20)^2)(1)}{(0.20)^2\sqrt{1}}\right) \\
&= \Phi(0.365987\ldots) = 0.642813.
\end{aligned}$$

Similarly,

$$\Delta(80) = 0.00496022.$$

Fix a number s. Consider the portfolio consisting of +1 European call and short $\Delta(s)$ stocks. The delta of the portfolio is

$$\frac{\partial}{\partial S_t}(C_K(t,T) - \Delta(s)S_t) = \frac{\partial C_K(t,T)}{\partial S_t} - \Delta(s)\frac{\partial S_t}{\partial S_t} = \frac{\partial C_K(t,T)}{\partial S_t} - \Delta(s).$$

Therefore when the stock price is $S_t = s$, the delta is

$$\left.\frac{\partial}{\partial S_t}(C_K(t,T) - \Delta(s)S_t)\right|_{S_t=s} = \left.\frac{\partial C_K(t,T)}{\partial S_t}\right|_{S_t=s} - \Delta(s) = 0.$$

So, when $S_t = s$, the value of the portfolio (consisting of +1 European call and short $\Delta(s)$) stocks is unaffected by small changes in the stock price. The portfolio is said to be **delta-hedged** or **delta-neutral** at $S_t = s$ because its delta is zero when $S_t = s$. It has no exposure to changes in the stock price around $S_t = s$.

Similarly, if $D(t,T)$ is the price at t of a derivative contract with maturity T on the stock, then the portfolio consisting of +1 contract and short $\left.\frac{\partial D(t,T)}{\partial S_t}\right|_{S_t=s}$ stocks is **delta-hedged** at $S_t = s$. △

Definition 14.5.2. The **vega** of an option (or other derivative contract) is the partial derivative of its price with respect to volatility σ. △

Result 14.5.4. Assuming the Black-Scholes model for a stock paying no income, the vega of a European call is

$$\frac{\partial C_K(t,T)}{\partial \sigma} = S_t\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}.$$

△

If the vega of a contract is positive, the price of the contract increases as the volatility increases, and the party long on the contract is said to have a **long vega position**.

If the vega of a contract is negative, the price of the contract increases as the volatility decreases, and the party long on the contract is said to have a **short vega position**.

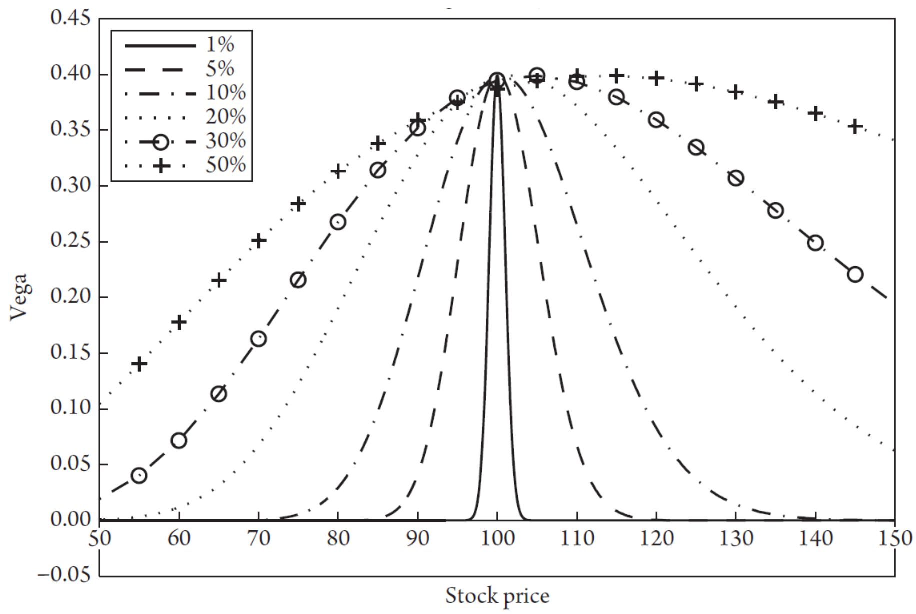


Figure 14.2: Black-Scholes vega of 1-year 100-strike call (for different volatilities)

The next result shows how the delta (and vega) of a call and a put are related.

Result 14.5.5. In the Black-Scholes model for a stock paying no income,

$$(a) \qquad \frac{\partial C_K(t,T)}{\partial S_t} - \frac{\partial P_K(t,T)}{\partial S_t} = 1$$

$$(b) \qquad \frac{\partial C_K(t,T)}{\partial \sigma} - \frac{\partial P_K(t,T)}{\partial \sigma} = 0$$

△

Proof. Recall that put-call parity states

$$C_K(t,T) - P_K(t,T) = V_K(t,T) = S_t - KZ(t,T)$$

Differentiating with respect to S_t gives (a). Differentiating with respect to σ gives (a). □

By combining Results 14.5.1, 14.5.4, and 14.5.5, we obtain

Result 14.5.6. In the Black-Scholes model for a stock paying no income,

$$\frac{\partial P_K(t,T)}{\partial S_t} = \frac{\partial C_K(t,T)}{\partial S_t} - 1 = \Phi(d_1) - 1$$

and

$$\frac{\partial P_K(t,T)}{\partial \sigma} = \frac{\partial C_K(t,T)}{\partial \sigma} = S_t\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$$

△

Example 14.5.7. Suppose we are short a (K_2, K_1) put spread, where $0 < K_1 < K_2$.

Determine whether this position is long vega (vega ≥ 0), short vega (vega ≤ 0), has no volatility exposure (vega $= 0$), or the volatility exposure is indeterminate

(a) Recall that a (K_2, K_1) put spread is $+1$ K_2 put and -1 K_1 put.

Since we are short the spread, our position is:

-1 K_2 put and $+1$ K_1 put

We will need to indicate that d_1 depends on K. So we write

$$d_1(K) = \frac{\ln S_t - \ln K + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Since $K_1 < K_2$, we have

$$d_1(K_1) > d_1(K_2),$$

and so

$$e^{-(d_1(K_1))^2/2} < e^{-(d_1(K_2))^2/2}.$$

Then

$$\begin{aligned}
\text{vega} &= -\frac{\partial P_{K_2}(t,T)}{\partial \sigma} + \frac{\partial P_{K_1}(t,T)}{\partial \sigma} \\
&= -S_t\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-(d_1(K_2))^2/2} + S_t\sqrt{T-t}\frac{1}{\sqrt{2\pi}}e^{-(d_1(K_1))^2/2} \\
&= S_t\sqrt{T-t}\frac{1}{\sqrt{2\pi}}\left(e^{-(d_1(K_1))^2/2} - e^{-(d_1(K_2))^2/2}\right)
\end{aligned}$$

Therefore $vega < 0$. It is a short vega position. △

Outline of Proof of Result 14.5.1. The Black-Scholes formula says

$$C_K(t,T) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Notice d_1 and d_2 depend on S_t. Then

$$\frac{\partial C_K(t,T)}{\partial S_t} = \frac{\partial}{\partial S_t}\left(S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)\right).$$

By the product rule

$$\frac{\partial C_K(t,T)}{\partial S_t} = \Phi(d_1) + S_t\frac{\partial}{\partial S_t}\Phi(d_1) - Ke^{-r(T-t)}\frac{\partial}{\partial S_t}\Phi(d_2)$$

By the fundamental theorem of calculus and the chain rule,

$$\begin{aligned}
\frac{\partial}{\partial S_t}\Phi(d_1) &= \frac{\partial}{\partial S_t}\int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{\partial}{\partial S_t}d_1 \\
&= \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{\partial}{\partial S_t}\left(\frac{\ln S_t - \ln K + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{\partial}{\partial S_t}\frac{\ln S_t}{\sigma\sqrt{T-t}} \\
&= \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{1}{S_t\sigma\sqrt{T-t}}
\end{aligned}$$

By a similar calculation,

$$\frac{\partial}{\partial S_t}\Phi(d_2) = \frac{1}{\sqrt{2\pi}}e^{-d_2^2/2}\frac{1}{S_t\sigma\sqrt{T-t}}$$

Exercise: Complete the proof. □

14.6 Volatility

For a stock paying no income, the Black-Scholes model assumes for any $t \leq T$ that

$$\ln(S_T/S_t) \sim \mathcal{N}((r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)).$$

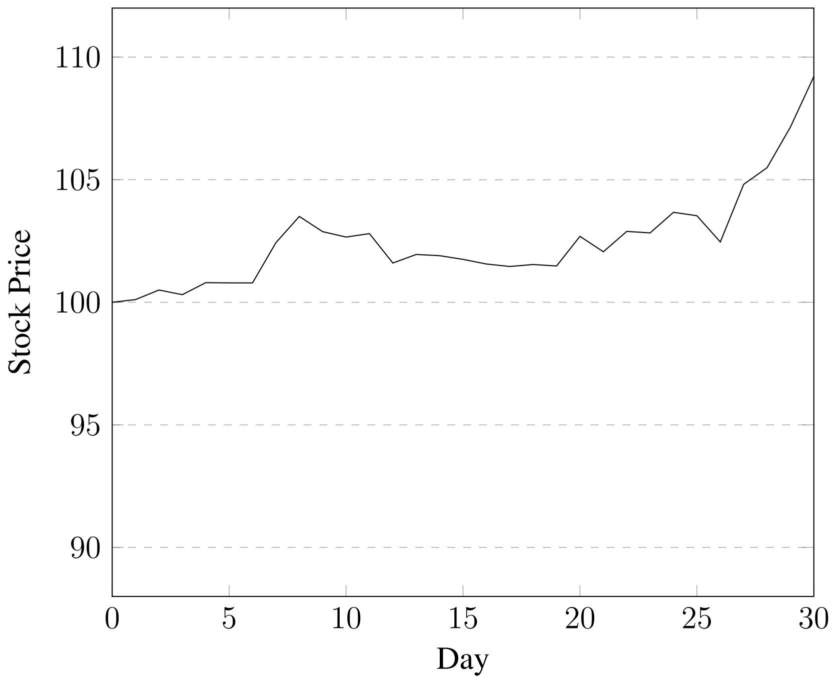
under the risk-neutral measure probability measure P^*.

r is the continuously compounded interest rate, which is assumed to be constant.

σ is a positive constant called the **volatility** of the stock. Roughly, it measures how much the stock price can be expected to change over time. Large volatility means large changes in the stock price are likely. Small volatility means large changes in the stock price are unlikely.

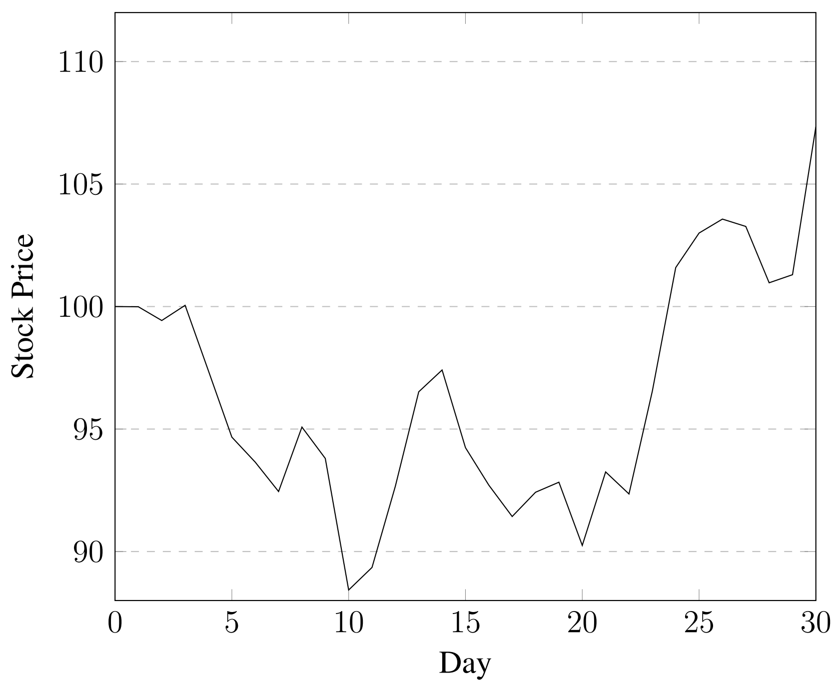
Price History of Stock A

Day	Stock Price
0	100.00
1	100.11
2	100.50
3	100.31
4	100.80
5	100.79
6	100.79
7	102.43
8	103.50
9	102.88
10	102.66
11	102.80
12	101.60
13	101.95
14	101.90
15	101.75
16	101.56
17	101.46
18	101.54
19	101.48
20	102.69
21	102.06
22	102.89
23	102.83
24	103.67
25	103.53
26	102.46
27	104.81
28	105.49
29	107.15
30	109.21



Price History of Stock B

Day	Stock Price
0	100.00
1	99.99
2	99.43
3	100.05
4	97.37
5	94.67
6	93.65
7	92.45
8	95.08
9	93.80
10	88.43
11	89.35
12	92.67
13	96.52
14	97.41
15	94.24
16	92.71
17	91.43
18	92.42
19	92.83
20	90.25
21	93.25
22	92.35
23	96.55
24	101.59
25	103.00
26	103.57
27	103.27
28	100.97
29	101.30
30	107.34



Based on the graphs, it looks like stock B has higher volatility than stock A.

We now explain how to estimate the volatility of stocks A and B based on the data above.

Some Theory.

Let $S_{i\Delta T}$ be the stock price on day i, where ΔT = $1/365$ years = 1 day. Assuming the Black-Scholes model for the stock, we have

$$\ln(S_{i\Delta T}/S_{(i-1)\Delta T}) \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right),$$

Define

$$X_i = \ln(S_{i\Delta T}/S_{(i-1)\Delta T}).$$

Then

$$X_i = \ln(S_{i\Delta T}/S_{(i-1)\Delta T}) = \ln(S_{i\Delta T}) - \ln(S_{(i-1)\Delta T}) \sim \mathcal{N}\left((r - \frac{1}{2}\sigma^2)\Delta T, \sigma^2\Delta T.\right)$$

Therefore

$$\sigma^2\Delta T = \text{Var}^*\left(X_i\right).$$

Procedure to Estimate σ From Data.

We compute $X_i = \ln(S_{i\Delta T}/S_{(i-1)\Delta T})$ for day $i = 1, \ldots, n = 30$.

We compute the sample variance of the data points $X_1, \ldots, X_n$:

$$s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \quad \text{where} \quad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Then

$$s_n^2 \approx \text{Var}^*\left(X_i\right) = \sigma^2\Delta T.$$

Therefore

$$\sigma \approx \sqrt{\frac{1}{\Delta T}s_n^2}$$

Remark. We can use a spreadsheet to do the calculations. In Excel and Google Sheets, VAR() computes the sample variance of a set of data points.

Example 14.6.1 (Estimating the Volatility)**.** We use a spreadsheet to estimate the volatility of stocks A and B according to the procedure described above. You can find a sample spreadsheet on the course website.

Stock A

Day	Stock Price	$\ln\left(\frac{S_{i\Delta T}}{S_{(i-1)\Delta T}}\right)$
0	100.00	
1	100.11	0.001058
2	100.50	0.003958
3	100.31	-0.001968
4	100.80	0.004884
5	100.79	-0.000050
6	100.79	0.000027
7	102.43	0.016101
8	103.50	0.010389
9	102.88	-0.006053
10	102.66	-0.002059
11	102.80	0.001347
12	101.60	-0.011804
13	101.95	0.003454
14	101.90	-0.000480
15	101.75	-0.001444
16	101.56	-0.001891
17	101.46	-0.000995
18	101.54	0.000788
19	101.48	-0.000588
20	102.69	0.011868
21	102.06	-0.006163
22	102.89	0.008077
23	102.83	-0.000548
24	103.67	0.008119
25	103.53	-0.001372
26	102.46	-0.010321
27	104.81	0.022598
28	105.49	0.006494
29	107.15	0.015586
30	109.21	0.019046
	Sample Variance =	0.000066628
	Volatility = $\sigma \approx$	0.1559

Stock B

Day	Stock Price	$\ln\left(\frac{S_{i\Delta T}}{S_{(i-1)\Delta T}}\right)$
0	100.00	
1	99.99	-0.000053
2	99.43	-0.005705
3	100.05	0.006300
4	97.37	-0.027170
5	94.67	-0.028109
6	93.65	-0.010864
7	92.45	-0.012910
8	95.08	0.028077
9	93.80	-0.013594
10	88.43	-0.058895
11	89.35	0.010312
12	92.67	0.036433
13	96.52	0.040808
14	97.41	0.009134
15	94.24	-0.033096
16	92.71	-0.016330
17	91.43	-0.013951
18	92.42	0.010781
19	92.83	0.004450
20	90.25	-0.028251
21	93.25	0.032770
22	92.35	-0.009707
23	96.55	0.044447
24	101.59	0.050907
25	103.00	0.013816
26	103.57	0.005456
27	103.27	-0.002878
28	100.97	-0.022518
29	101.30	0.003265
30	107.34	0.057947
	Sample Variance =	0.00074733
	Volatility = $\sigma \approx$	0.5222

From the data, we estimate the volatility as

Stock A: $\sigma = 15.59\%$

Stock B: $\sigma = 52.22\%$ $\triangle$

Remark. In fact, the price data was generated with
Stock A: $\sigma = 15\%$
Stock B: $\sigma = 50\%$ Our estimates are not bad for only 30 data points.

Example 14.6.2 (Option Pricing With Estimated Volatility)**.** Assume the constant continuously compounded interest rate is 10%. Use the volatility estimates
Stock A: $\sigma = 15.59\%$
Stock B: $\sigma = 52.22\%$
to find call prices for two-year 100-strike European calls on stocks A and B. The current time is Day 30.

The Black-Scholes formula is

$$C_K(t,T) = S_t\Phi(d_1) - KZ(t,T)\Phi(d_2).$$

Stock A:

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln(109.21/100) + (0.10 + \frac{1}{2}(0.1559)^2)(2)}{(0.1559)\sqrt{2}} \\ &= 1.41697\ldots \end{aligned}$$

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{T-t} \\ &= (1.41697\ldots) - (0.1559)\sqrt{2} \\ &= 1.19649\ldots \end{aligned}$$

$$Z(t,T) = e^{-0.10(2)}$$

Therefore

$$\begin{aligned} C_K(t,T) &= S_t\Phi(d_1) - KZ(t,T)\Phi(d_2) \\ &= 109.21\Phi(1.41697\ldots) - 100e^{-0.10(2)}\Phi(1.19649\ldots) \\ &= 28.26\ldots \end{aligned}$$

Stock B:

$$
\begin{aligned}
d_1 &= \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\ln(107.64/100) + (0.10 + \frac{1}{2}(0.5222)^2)(2)}{0.5222\sqrt{2}} \\
&= 0.739761\ldots
\end{aligned}
$$

$$
\begin{aligned}
d_2 &= d_1 - \sigma\sqrt{T-t} \\
&= (0.739761\ldots) - (0.15)\sqrt{1.5} \\
&= 0.00125836\ldots
\end{aligned}
$$

$$
Z(t,T) = e^{-0.10(2)}
$$

Therefore

$$
\begin{aligned}
C_K(t,T) &= S_t\Phi(d_1) - KZ(t,T)\Phi(d_2) \\
&= 107.64\Phi(0.739761\ldots) - 100e^{-0.10(2)}\Phi(0.00125836\ldots) \\
&= 41.93\ldots
\end{aligned}
$$

$\triangle$

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