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ERROR PROPAGATION EQUATIONS FOR ESTIMATING THE UNCERTAINTY IN HIGH-SPEED WIND TUNNEL TEST RESULTS*

Edward L. Clark†
Sandia National Laboratories
Albuquerque, New Mexico

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Abstract

Error propagation equations, based on the Taylor series model, are derived for the nondimensional ratios and coefficients most often encountered in high-speed wind tunnel testing. These include pressure ratio and coefficient, static force and moment coefficients, dynamic stability coefficients, and calibration Mach number. The error equations contain partial derivatives, denoted as sensitivity coefficients, which define the influence of free-stream Mach number, M_∞, on various aerodynamic ratios. To facilitate use of the error equations, sensitivity coefficients are derived and evaluated for five fundamental aerodynamic ratios which relate free-stream test conditions to a reference condition.

Nomenclature

Primary symbols

a	speed of sound
A	cross-sectional area of stream tube or channel
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
C_F	force coefficient, Eq. (71)
C_l	rolling moment coefficient, Eq. (85)
C_{l_p}	rolling moment dynamic stability coefficient, Eq. (85)
C_m	pitching moment coefficient, Eq. (74)
C_{m_q}	pitching moment dynamic stability coefficient, Eq. (76)
C_p	pressure coefficient, $(p-p_\infty)/q_\infty$
d	reference diameter
E	error, either bias or precision
E'	error from single source
f	function
F	aerodynamic force
l	reference length
L	aerodynamic rolling moment
L_p	derivative of rolling moment with respect to roll rate, $\partial L/\partial p$
M	Mach number, V/a
M	aerodynamic pitching moment
M_q	derivative of pitching moment with respect to pitch rate, $\partial M/\partial q$
p	pressure
p	vehicle roll rate
P	pressure ratio, p/p_r
q	dynamic pressure, $\rho V^2/2$
q	vehicle pitch rate
R	aerodynamic ratio
R	calculated test result
R_A	stream-tube area ratio, A_∞/A^*
R_g	gas constant, $c_p - c_v$
S	reference area
T	absolute temperature
V	flow velocity
x_i	ith parameter in test result
γ	ratio of specific heats, c_p/c_v
Δx_i	interval for numerical evaluation of sensitivity coefficient
θ	absolute sensitivity coefficient, Eq. (5)
θ'	relative sensitivity coefficient, Eq. (8)
ρ	mass density
$\rho_{x_i x_j}$	coefficient of correlation

Subscripts

a	atmospheric
g	gage pressure measurement
r	reference condition
t	total (stagnation) condition
1	conditions just upstream of a normal shock wave
2	conditions just downstream of a normal shock wave
∞	free-stream condition

Superscripts

*	critical conditions (local speed equal to local speed of sound, i.e., M=1.0)

Introduction

Evaluation of the uncertainty in an experimental result has always been an important, but frequently neglected, part of the test process. With recent emphasis on quality and traceability of results, it has become essential. Some journals, such as the *Journal of Fluids Engineering* published by the American Society of Mechanical Engineers (ASME), will no longer accept experimental results unless

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†Senior Member of Technical Staff, Associate Fellow AIAA

they include uncertainty estimates. The experimentalist who has learned to use uncertainty estimates finds them invaluable both before and after testing. Pre-test estimates are very useful in selecting the instrumentation and test techniques which will best accomplish the test goals. Post-test estimates provide the user with error bounds which can be indispensable if the data are to be used in design studies. Several organizations including the ASME, Instrument Society of America (ISA), and the International Organization for Standardization (ISO) have established standards which define the uncertainty estimation process. Aerospace organizations will soon be joining this group as the Advisory Group for Aerospace Research & Development (AGARD) and the American Institute of Aeronautics and Astronautics (AIAA) develop their standards for uncertainty analysis.

Procedures for estimating the uncertainty in experimental measurements are well documented[1-6]. Although there are some differences in the methodologies recommended by various authors, it is generally agreed that the procedure can be divided into three steps: (1) estimate measurement errors, (2) propagate measurement errors to final results and (3) calculate uncertainty. In the first step, the experimentalist defines all independent measurements involved in the experiment. The elemental errors associated with each measurement are then defined, according to categories such as (a) calibration, (b) data acquisition and (c) data reduction. At this point the errors must be classified as bias (fixed or systematic) or precision (random) errors. Finally, the elemental errors are combined by root-sum-squaring, to provide an estimate of the total bias and precision error for each measurement. In most cases, the desired test result is not the actual measurement, but is a calculated result incorporating one or more measurements, for example, a nondimensional ratio or coefficient. Thus, the second step in the uncertainty analysis is to estimate the error in the test result using an error propagation method based on a Taylor series model. The bias and precision errors are propagated separately. Finally, in the third step, the bias and precision errors obtained in the previous step are combined to provide an uncertainty estimate for the test result. The two factors to be combined are the bias error and the precision error multiplied by t, the 95th percentile point for the two-tailed Student t distribution. The combined uncertainty is estimated by direct addition of the two factors, or by a root-sum-squared addition.

The purpose of the present work is to facilitate, for high-speed aerodynamic testing, the calculations required in step two, that is, propagation of measurement errors to the final result. The Taylor series method requires that the test result functional relationship be differentiated, with respect to each of its parameters, x_i, to define a set of sensitivity coefficients, θ_{x_i}. For most experimental aerodynamic results, the calculus and algebra required to evaluate the sensitivity coefficients are difficult. This is especially true for the sensitivity coefficient, θ_{M_∞}, of the free-stream Mach number, M_∞. In this paper, error propagation equations are derived for many of the nondimensional ratios and coefficients used in high-speed wind tunnel testing. Most of the results are also applicable to low speed wind tunnel tests, but emphasis is on high-speed testing where Mach number is an important parameter. Equations and plotted values of the sensitivity coefficients, θ_{M_∞}, are provided as functions of M_∞. An extended coverage of this subject, including additional aerodynamic ratios and Reynolds number, with tabulated results is presented in Ref. 7.

Error Propagation Equations

Taylor series model

The previously cited references [1-6] all recommend that the Taylor series model be used for error propagation into a result. Detailed derivations of the model are provided in Refs. 1, 2, 4 and 5 and will not be repeated here. Consider a test result, R, which is a function of n parameters, x_i,

$$R = R(x_1, x_2, \ldots, x_n) \qquad (1)$$

Then, the bias or precision error, E_R, in R is approximated by

$$E_R^2 = \left(\frac{\partial R}{\partial x_1} E_{x_1}\right)^2 + \left(\frac{\partial R}{\partial x_2} E_{x_2}\right)^2 + \ldots + \left(\frac{\partial R}{\partial x_n} E_{x_n}\right)^2 + 2\frac{\partial R}{\partial x_1}\frac{\partial R}{\partial x_2} \rho_{x_1 x_2} E_{x_1} E_{x_2} + \ldots \qquad (2)$$

where E_{x_i} is the bias or precision error in x_i. As was mentioned in the previous section, bias and precision errors are propagated separately and are then combined to estimate the uncertainty in the result. The cross product terms provide the error contribution due to correlated errors and $\rho_{x_1 x_2}$ is the coefficient of correlation between x_1 and x_2. Evaluation of ρ is difficult, and Coleman and Steele [8] have suggested an alternate equation,

$$E_R^2 = \left(\frac{\partial R}{\partial x_1}E_{x_1}\right)^2 + \left(\frac{\partial R}{\partial x_2}E_{x_2}\right)^2 + \ldots + \left(\frac{\partial R}{\partial x_n}E_{x_n}\right)^2 + 2\frac{\partial R}{\partial x_1}\frac{\partial R}{\partial x_2}E'_{x_1}E'_{x_2} + \ldots \quad (3)$$

where E'_{x_1} and E'_{x_2} are the portions of the error in x_1 and x_2 that arise from the same source and are presumed to be perfectly correlated (ρ=1.0). This equation is easier to apply than Eq. (2). If the correlation between parameters is a result of the measurements, precision errors are assumed to be independent (ρ=0). However, if the correlation occurs as the result of multiple occurrences of a parameter within the equation for the test result, even precision errors will be correlated and Eq. (2) or (3) must be used. In any case, if the bias errors are not independent, Eq. (2) or (3) must be used. In evaluating the partial derivatives, it is easy to ignore the sign of the derivatives since most of the terms are squared. However, it is essential that the signs be retained for the correlation terms which are not squared.

If the errors are independent, Eq. (2) can be simplified to,

$$E_R^2 = \left(\frac{\partial R}{\partial x_1}E_{x_1}\right)^2 + \left(\frac{\partial R}{\partial x_2}E_{x_2}\right)^2 + \ldots + \left(\frac{\partial R}{\partial x_n}E_{x_n}\right)^2 \quad (4)$$

The partial derivatives are denoted as absolute sensitivity coefficients, θ_i, that is,

$$\theta_i = \frac{\partial R}{\partial x_i} \quad (5)$$

Then,

$$E_R^2 = \left(\theta_1 E_{x_1}\right)^2 + \left(\theta_2 E_{x_2}\right)^2 + \ldots + \left(\theta_n E_{x_n}\right)^2 \quad (6)$$

Dividing each side of Eq. (4) by R^2 gives the "relative error" form of the propagation equation,

$$\left(\frac{E_R}{R}\right)^2 = \left(\frac{x_1}{R}\frac{\partial R}{\partial x_1}\frac{E_{x_1}}{x_1}\right)^2 + \left(\frac{x_2}{R}\frac{\partial R}{\partial x_2}\frac{E_{x_2}}{x_2}\right)^2 + \ldots + \left(\frac{x_n}{R}\frac{\partial R}{\partial x_n}\frac{E_{x_n}}{x_n}\right)^2 \quad (7)$$

This form of the equation is especially useful if the measurement errors are defined as relative errors, E_{x_i}/x_i, and if the test result parameters are related only by multiplication and division, e.g., $R=x_1x_2/x_3$. When parameters are added or subtracted, e.g., $R=(x_1+x_2)/x_3$, the relative error form is less useful. The relative sensitivity coefficient, θ'_i, is defined as

$$\theta'_i = \frac{x_i}{R}\frac{\partial R}{\partial x_i} \quad (8)$$

Then,

$$\left(\frac{E_R}{R}\right)^2 = \left(\theta'_1\frac{E_{x_1}}{x_1}\right)^2 + \left(\theta'_2\frac{E_{x_2}}{x_2}\right)^2 + \ldots + \left(\theta'_n\frac{E_{x_n}}{x_n}\right)^2 \quad (9)$$

In many cases, analytical evaluation of the sensitivity coefficients is difficult or impossible. Examples include results which involve: (1) complex and/or implicit functions; (2) interpolation of tabulated functions and (3) iterative calculations. The procedure in this case is to use a numerical evaluation of the partial derivatives. Using a central differencing scheme, the derivatives can be approximated by,

$$\frac{\partial R}{\partial x_i} = \lim_{\Delta x_i \to 0}\left[\frac{R_{x_i+\Delta x_i/2} - R_{x_i-\Delta x_i/2}}{\Delta x_i}\right]_{x_j = \mathrm{const}\,(j \neq i)} \quad (10)$$

With this method, nominal values are selected for all of the parameters, x_i. Then, R is sequentially evaluated by perturbing each x_i (i=1,n) by $\pm\Delta x_i/2$ while the remaining n-1 x_j values remain constant. The two perturbed values of R are differenced and divided by Δx_i. The approximate derivatives are used with Eq. (4) to estimate the error in the result. This approach is discussed in detail in Refs. 9 and 10. The advantages of this method are that it avoids complex differentiation and algebra, automatically calculates the effect of correlated variables, and may be the only solution possible. The disadvantage is that it provides only a "point" solution and the functional variation of test result error with a given variable cannot be determined. The method is very powerful and was used in the present report to verify the relative sensitivity coefficients for each of the applications. For these checks, the increment was $\Delta x_i = 0.005\ x_i$ and the numerical results were in excellent agreement (4 to 5 significant digits) with the analytical values. A simple code which demonstrates this technique is listed in Ref. 7.

Application of the model

Application of the Taylor series model is straightforward. However, there are two cautions. First, the correct model must be used. If the measurements are independent, Eq. (4) can be used, but if they are not independent, i.e., if two or more of the measurements are correlated, then either Eq. (2) or (3) must be used. This is very important and will be emphasized again in later sections. Second, it is essential that the data reduction equation be solved for the experimental result before performing an uncertainty analysis[11]. This mistake is most likely to occur during a pre-test analysis to estimate the allowable error in a measurement for a given error in the result. For example, let

$$R = xy \tag{11}$$

Estimate the allowable error, E_x, in the measurement, x, for a given error, E_R, in the result, R. From Eq. (4),

$$\begin{aligned} E_R^2 &= \left(\frac{\partial R}{\partial x}E_x\right)^2 + \left(\frac{\partial R}{\partial y}E_y\right)^2 \\ &= (yE_x)^2 + (xE_y)^2 \\ &= R^2\left[\left(\frac{E_x}{x}\right)^2 + \left(\frac{E_y}{y}\right)^2\right] \end{aligned}$$

Then,

$$E_x = x\left[\left(\frac{E_R}{R}\right)^2 - \left(\frac{E_y}{y}\right)^2\right]^{1/2} \tag{12}$$

This is the correct estimate of E_x. However, if the original equation for R is solved for x and E_x is estimated from the resulting equation, the result will be in error. Solving Eq. (11) for x gives,

$$x = \frac{R}{y} \tag{13}$$

The error in x is given by,

$$\begin{aligned} E_x^2 &= \left(\frac{\partial x}{\partial R}E_R\right)^2 + \left(\frac{\partial x}{\partial y}E_y\right)^2 \\ &= \left(\frac{E_R}{y}\right)^2 + \left(\frac{R}{y^2}E_y\right)^2 \end{aligned}$$

Then,

$$E_x = x\left[\left(\frac{E_R}{R}\right)^2 + \left(\frac{E_y}{y}\right)^2\right]^{1/2} \tag{14}$$

Comparison of Eqs. (12) and (14) shows that the second result is incorrect.

<u>Example - pressure ratio</u>

<u>Independent pressure measurements</u>

As an example of the use of error propagation equations, consider the simple pressure ratio, P,

$$P = \frac{p}{p_r} \tag{15}$$

where p and p_r are two independent absolute pressure measurements. Typically, in a high-speed wind tunnel test, the reference pressure, p_r, could be p_∞ (measured on the sidewall or with a probe), p_t or p_{t_2}. The measurements are assumed to be independent - this may not be the case if the same standard was used to calibrate both transducers and/or if they share common signal conditioning. From Eq. (4),

$$E_P = \left[\left(\frac{\partial P}{\partial p}E_p\right)^2 + \left(\frac{\partial P}{\partial p_r}E_{p_r}\right)^2\right]^{1/2} \tag{16}$$

Differentiating Eq. (15) and substituting the results in Eq. (16) gives

$$E_P = \left[\left(\frac{1}{p_r}E_p\right)^2 + \left(-\frac{p}{p_r^2}E_{p_r}\right)^2\right]^{1/2}$$

and simplifying,

$$E_P = \frac{1}{p_r}\left[E_p^2 + \left(PE_{p_r}\right)^2\right]^{1/2} \tag{17}$$

In the relative error form,

$$\frac{E_P}{P} = \left[\left(\frac{E_p}{p}\right)^2 + \left(\frac{E_{p_r}}{p_r}\right)^2\right]^{1/2} \tag{18}$$

This is the simplest form of error propagation and shows that the relative error in the result is the root-sum-square of the relative errors in the two measurements.

<u>Correlated pressures</u>

Now, assume that the two pressure measurements are gage, not absolute. Then, the two absolute pressures required for the ratio are given by

$$\begin{aligned} p &= p_g + p_a \\ p_r &= p_{r,g} + p_a \end{aligned} \tag{19}$$

where p_g and $p_{r,g}$ are the two independent gage pressure measurements and p_a is the measured atmospheric pressure. In this case, the calculated pressures p and p_r are correlated by p_a. The pressure ratio, P, is given by,

$$P = \frac{p_g + p_a}{p_{r,g} + p_a} \tag{20}$$

Calculation of the error in P is much more complicated than in the previous case, where p and p_r were independent. The error will be evaluated analytically, using two techniques, and numerically.

Analytical evaluation – independent variables

If the required calculus and algebra associated with determining the sensitivity coefficients permits, it is *always* best to express the result in terms of the individual measurements, that is, in its most elementary form. This avoids the problems which can occur with correlated variables as is demonstrated in the next section. Since p_g, $p_{r,g}$, and p_a are independent measurements, Eq. (20) is the desired form and the error estimate is given by Eq. (4),

$$E_P = \left[\left(\frac{\partial P}{\partial p_g}E_{p_g}\right)^2 + \left(\frac{\partial P}{\partial p_{r,g}}E_{p_{r,g}}\right)^2 + \left(\frac{\partial P}{\partial p_a}E_{p_a}\right)^2\right]^{1/2} \tag{21}$$

Performing the indicated partial differentiation and simplifying gives,

$$E_P = P\left[\left(\frac{E_{p_g}}{p}\right)^2 + \left(\frac{E_{p_{r,g}}}{p_r}\right)^2 + \left(\frac{1}{p} - \frac{1}{p_r}\right)^2 E_{p_a}^2\right]^{1/2} \tag{22}$$

or,

$$E_P = P\left[\left(\frac{E_{p_g}}{p}\right)^2 + \left(\frac{E_{p_{r,g}}}{p_r}\right)^2 + \left(\frac{1}{p^2} + \frac{1}{p_r^2} - \frac{2}{pp_r}\right) E_{p_a}^2\right]^{1/2} \tag{23}$$

Notice that in this equation, the pressures in the denominators are the absolute pressures p and p_r, not the gage pressures p_g and $p_{r,g}$. Therefore, this is not the usual relative error form.

Analytical evaluation – dependent variables

Frequently, it is easier to differentiate the unexpanded equation, Eq. (15) in this case. If we were to take this easier course and treat p and p_r as independent variables, we would obtain the error in P as,

$$E_P = \left[\left(\frac{\partial P}{\partial p}E_p\right)^2 + \left(\frac{\partial P}{\partial p_r}E_{p_r}\right)^2\right]^{1/2} \tag{24}$$

Then, differentiating Eq. (15) and simplifying,

$$E_P = P\left[\left(\frac{E_p}{p}\right)^2 + \left(\frac{E_{p_r}}{p_r}\right)^2\right]^{1/2} \tag{25}$$

The effect of errors in p_a is concealed in this equation. Expanding p and p_r, Eq. (19), and estimating their errors gives,

$$E_p^2 = E_{p_g}^2 + E_{p_a}^2 \tag{26}$$

and

$$E_{p_r}^2 = E_{p_{r,g}}^2 + E_{p_a}^2 \tag{27}$$

Substituting these expressions into Eq. (25) gives

$$E_P = P\left[\left(\frac{E_{p_g}}{p}\right)^2 + \left(\frac{E_{p_{r,g}}}{p_r}\right)^2 + \left(\frac{1}{p^2} + \frac{1}{p_r^2}\right) E_{p_a}^2\right]^{1/2} \tag{28}$$

Comparing Eq. (28) to Eq. (23) shows that there is a missing term, $-2E_{p_a}^2/pp_r$. Therefore, Eq. (28) will overpredict the error in E_P. The reason for this is that p and p_r were treated as independent variables, whereas they are actually correlated by p_a.

For a correct error estimate with correlated variables, use Eq. (2) or (3). Using Eq. (3),

$$E_P^2 = \left(\frac{\partial P}{\partial p}E_p\right)^2 + \left(\frac{\partial P}{\partial p_r}E_{p_r}\right)^2 + 2\frac{\partial P \partial P}{\partial p \partial p_r}E'_p E'_{p_r}(p_a) \tag{29}$$

where E'_p and E'_{p_r} are the portions of the error in p and p_r that originate from the correlating factor, p_a. From Eqs. (26) and (27),

$$E'_p = E_{p_a}$$

$$E'_{p_r} = E_{p_a}$$

The sensitivity coefficients are,

$$\frac{\partial P}{\partial p} = \frac{1}{p_r} \qquad \text{and} \qquad \frac{\partial P}{\partial p_r} = -\frac{p}{p_r^2}$$

Substituting these relations into Eq. (29) and simplifying gives,

$$E_P = P\left[\left(\frac{E_{p_g}}{p}\right)^2 + \left(\frac{E_{p_{r,g}}}{p_r}\right)^2 + \left(\frac{1}{p^2}+\frac{1}{p_r^2}-\frac{2}{pp_r}\right)E_{p_a}^2\right]^{1/2} \quad (30)$$

which agrees with Eq. (23).

Numerical evaluation

A program utilizing Eq. (10) was used with Eq. (20) to estimate the three sensitivity coefficients - $\partial P/\partial p_g$, $\partial P/\partial p_{r,g}$ and $\partial P/\partial p_a$. The numerical results agreed with Eq. (23).

Sensitivity Coefficients - Aerodynamic Ratios

The aerodynamic ratios considered in this section are basic to high-speed aerodynamics and, in general, relate calculated free-stream conditions to measured total (stagnation) conditions. There are two reasons for interest in these ratios. First, they provide a means of estimating the uncertainty in the calculated free-stream conditions. Second, they are required for evaluating the uncertainty in the more complex applications which are discussed later. In all cases, these aerodynamic ratios are functions of only the free-stream Mach number, M_∞, and the ratio of specific heats, γ. In the previous section, the uncertainty in the pressure ratio, $R = p/p_r$, was examined. In that case, both pressures were measured values. In the present section, only the total condition is measured–the free-stream value is to be calculated from the aerodynamic ratio, R, and the total condition. For example, consider the ratio of free-stream pressure, p_∞, to total pressure, p_t,

$$R = \frac{p_\infty}{p_t} = f(M_\infty, \gamma) \quad (31)$$

Then, the free-stream static pressure is calculated with

$$p_\infty = R(M_\infty, \gamma)\, p_t \quad (32)$$

Assuming that the variables are independent, the error in p_∞ is given by

$$E_{p_\infty} = \left[\left(\frac{\partial p_\infty}{\partial M_\infty}E_{M_\infty}\right)^2 + \left(\frac{\partial p_\infty}{\partial \gamma}E_\gamma\right)^2 + \left(\frac{\partial p_\infty}{\partial p_t}E_{p_t}\right)^2\right]^{1/2} \quad (33)$$

Letting $E_\gamma = 0$ and expanding the partial derivative in the first term (chain rule),

$$E_{p_\infty} = \left[\left(\frac{\partial p_\infty}{\partial R}\frac{\partial R}{\partial M_\infty}E_{M_\infty}\right)^2 + \left(\frac{\partial p_\infty}{\partial p_t}E_{p_t}\right)^2\right]^{1/2} \quad (34)$$

Differentiating Eq. (32) and substituting in Eq. (34) gives the absolute error

$$E_{p_\infty} = \left[\left(p_t\frac{\partial R}{\partial M_\infty}E_{M_\infty}\right)^2 + \left(RE_{p_t}\right)^2\right]^{1/2} \quad (35)$$

The sensitivity coefficient for Mach number is,

$$\theta_{M_\infty} \equiv \frac{\partial p_\infty}{\partial M_\infty} = p_t\frac{\partial R}{\partial M_\infty} \quad (36)$$

Dividing both sides of Eq. (35) by p_∞ gives the relative error,

$$\frac{E_{p_\infty}}{p_\infty} = \left[\left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2} \quad (37)$$

The relative sensitivity coefficient for M_∞ is

$$\theta'_{M_\infty} = \frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty} \quad (38)$$

In evaluating Eq. (35) or (37), if p_t is calculated from a gage pressure measurement, $p_{t,g}$, then

$$p_t = p_{t,g} + p_a \quad (39)$$

where p_a is measured atmospheric pressure. The error in p_t is

$$E_{p_t} = \left[\left(\frac{\partial p_t}{\partial p_{t,g}}E_{p_{t,g}}\right)^2 + \left(\frac{\partial p_t}{\partial p_a}E_{p_a}\right)^2\right]^{1/2}$$

$$E_{p_t} = \left[E_{p_{t,g}}^2 + E_{p_a}^2\right]^{1/2} \quad (40)$$

This value of E_{p_t} would then be substituted in Eq. (35) or (37).

The analysis in this section makes several assumptions. First, it will be assumed that p_t is an absolute pressure measurement (or equivalently, that E_{p_a} is negligible). If this is not the case, Eq. (40) can be used with Eq. (35) or (37). Second, it will be assumed that there is negligible error in γ, i.e., $E_\gamma = 0$. Finally, it is assumed that M_∞ and p_t are independent. This may not be true in many tunnels where the free-stream Mach number is determined from a calibration relation which is a function of total pressure. In this case there are three options. First, expand the equation of interest to its elemental parameters, which are independent, and differentiate it with respect to each parameter to evaluate the sensitivity coefficients. Equation (4) can then be used to define the error equation. Second, modify the error equation to include the effect of correlated parameters, e.g., use

Eq. (3). Then use the sensitivity coefficients for Mach number, which are given in the following sections, with the appropriate error estimates defined by the M_∞ calibration relation. Third, use a numerical evaluation which defines sensitivity coefficients that include the effects of correlated parameters. These coefficients can be used with an error equation based on Eq. (4).

In the following subsections, the derivative $\partial R/\partial M_\infty$ and the relative sensitivity coefficient, $\theta'_{M_\infty} = (M_\infty/R)(\partial R/\partial M_\infty)$ will be evaluated as functions of M_∞ for five fundamental aerodynamic ratios. Values of R, $\partial R/\partial M_\infty$, and θ'_{M_∞} are plotted for $\gamma = 1.40$. The calculated values of R were checked by comparing them to the values given in Ref. 12. Analytical values of $\partial R/\partial M_\infty$ were checked by numerical differentiation of the tabulated values of R.

Free-stream static pressure ratio, $R = p_\infty / p_t$

The equations for free-stream static pressure, p_∞, and its absolute and relative errors are given above by Eqs. (32), (35) and (37), respectively:

$$p_\infty = R(M_\infty, \gamma)\, p_t \qquad [32]$$

$$E_{p_\infty} = \left[\left(p_t \frac{\partial R}{\partial M_\infty} E_{M_\infty}\right)^2 + \left(R E_{p_t}\right)^2\right]^{1/2} \qquad [35]$$

$$\frac{E_{p_\infty}}{p_\infty} = \left[\left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2} \qquad [37]$$

For an isentropic flow of a perfect gas[1], Eq.(44) of Ref. 12 gives,

$$R = \frac{p_\infty}{p_t} = \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)^{-\frac{\gamma}{\gamma-1}} \qquad (41)$$

Differentiating Eq. (41) with respect to M_∞ gives

[1]Following the notation of Ref. 12, a "thermally perfect" gas is one which obeys the thermal equation of state, $p=\rho R_g T$. A "calorically perfect" gas is one which has constant specific heats, c_p and c_v. A "perfect" gas is both thermally and calorically perfect.

$$\frac{\partial R}{\partial M_\infty} = -\frac{\gamma M_\infty R}{1 + \frac{\gamma-1}{2}M_\infty^2} \qquad (42)$$

The relative sensitivity coefficient is

$$\theta'_{M_\infty} = \frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty} = -\frac{\gamma M_\infty^2}{1 + \frac{\gamma-1}{2}M_\infty^2} \qquad (43)$$

For $\gamma = 1.40$, values of R, $\partial R/\partial M_\infty$ and $(M_\infty/R)(\partial R/\partial M_\infty)$ are plotted in Figure 1.

Free-stream dynamic pressure ratio, $R = q_\infty / p_t$

The free-stream dynamic pressure, q_∞, is calculated from

$$q_\infty = R(M_\infty, \gamma)\, p_t \qquad (44)$$

and the error equations are

$$E_{q_\infty} = \left[\left(p_t \frac{\partial R}{\partial M_\infty} E_{M_\infty}\right)^2 + \left(R E_{p_t}\right)^2\right]^{1/2} \qquad (45)$$

and

$$\frac{E_{q_\infty}}{q_\infty} = \left[\left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2} \qquad (46)$$

For an isentropic flow of a perfect gas, Eq. (48) of Ref. 12 gives,

$$R = \frac{q_\infty}{p_t} = \frac{\gamma}{2}M_\infty^2\left(1 + \frac{\gamma-1}{2}M_\infty^2\right)^{-\frac{\gamma}{\gamma-1}} \qquad (47)$$

Differentiating Eq. (47) with respect to M_∞ gives

$$\frac{\partial R}{\partial M_\infty} = \frac{R}{M_\infty}\left(\frac{2 - M_\infty^2}{1 + \frac{\gamma-1}{2}M_\infty^2}\right) \qquad (48)$$

The relative sensitivity coefficient is

$$\theta'_{M_\infty} = \frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty} = \frac{2 - M_\infty^2}{1 + \frac{\gamma-1}{2}M_\infty^2} \qquad (49)$$

For $\gamma = 1.40$, values of R, $\partial R/\partial M_\infty$ and $(M_\infty/R)(\partial R/\partial M_\infty)$ are plotted in Figure 2.

Pitot pressure ratio, $R = p_{t_2}/p_{t_1}$

The pitot pressure, p_{t_2}, is calculated from

$$p_{t_2} = R(M_\infty, \gamma)\, p_{t_1} \tag{50}$$

The error equations are

$$E_{p_{t_2}} = \left[\left(p_{t_1}\frac{\partial R}{\partial M_\infty}E_{M_\infty}\right)^2 + \left(RE_{p_{t_1}}\right)^2\right]^{1/2} \tag{51}$$

and

$$\frac{E_{p_{t_2}}}{p_{t_2}} = \left[\left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2 + \left(\frac{E_{p_{t_1}}}{p_{t_1}}\right)^2\right]^{1/2} \tag{52}$$

For free-stream Mach numbers less than 1.0, $p_{t_2} = p_{t_1}$ and $R = 1.0$. For $M_\infty \geq 1.0$ with adiabatic flow of a perfect gas, Eq (99) of Ref. 12 gives,

$$R = \frac{p_{t_2}}{p_{t_1}} = \left[\frac{(\gamma+1)M_\infty^2}{(\gamma-1)M_\infty^2+2}\right]^{\frac{\gamma}{\gamma-1}} \times \left[\frac{\gamma+1}{2\gamma M_\infty^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \tag{53}$$

Differentiating Eq. (53) with respect to M_∞ gives

$$\frac{\partial R}{\partial M_\infty} = \frac{4\gamma R}{\gamma-1}\left[\frac{1}{M_\infty[(\gamma-1)M_\infty^2+2]} - \frac{M_\infty}{2\gamma M_\infty^2-(\gamma-1)}\right] \tag{54}$$

The relative sensitivity coefficient is

$$\theta'_{M_\infty} = \frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty} = \frac{4\gamma M_\infty}{\gamma-1}\left[\frac{1}{M_\infty[(\gamma-1)M_\infty^2+2]} - \frac{M_\infty}{2\gamma M_\infty^2-(\gamma-1)}\right] \tag{55}$$

For $\gamma = 1.40$, values of R, $\partial R/\partial M_\infty$ and $(M_\infty/R)(\partial R/\partial M_\infty)$ are plotted in Figure 3.

Rayleigh pitot ratio, $R = p_\infty/p_{t_2}$

The Rayleigh pitot ratio is normally used, not to calculate p_∞ or p_{t_2}, but to calculate M_∞ from the two measured pressures. This application will be discussed in more detail in a later section. At free-stream Mach numbers less than 1.0, $p_{t_2} = p_t$ and $R = p_\infty/p_t$. For $M_\infty \geq 1.0$ with adiabatic flow of a perfect gas, Eq. (100) of Ref. 12 gives,

$$R = \frac{p_\infty}{p_{t_2}} = \frac{p_1}{p_{t_2}} = \left[\frac{2}{(\gamma+1)M_\infty^2}\right]^{\frac{\gamma}{\gamma-1}} \times \left[\frac{2\gamma M_\infty^2-(\gamma-1)}{\gamma+1}\right]^{\frac{1}{\gamma-1}} \tag{56}$$

Differentiating Eq. (56) with respect to M_∞ gives

$$\frac{\partial R}{\partial M_\infty} = \frac{R}{M_\infty}\left[\frac{2\gamma\left(1-2M_\infty^2\right)}{2\gamma M_\infty^2-(\gamma-1)}\right] \tag{57}$$

The relative sensitivity coefficient is

$$\theta'_{M_\infty} = \frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty} = \frac{2\gamma\left(1-2M_\infty^2\right)}{2\gamma M_\infty^2-(\gamma-1)} \tag{58}$$

For $\gamma = 1.40$, values of R, $\partial R/\partial M_\infty$ and $(M_\infty/R)(\partial R/\partial M_\infty)$ are plotted in Figure 4.

Stream-tube area ratio, $R = A_\infty / A^*$

The stream-tube area ratio will be used in the applications to estimate the error in q_∞ / V_∞. For an isentropic flow of a perfect gas, Eq. (80) of Ref. 12 gives

$$R = \frac{A_\infty}{A^*} = \frac{1}{M_\infty}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{59}$$

Differentiating Eq. (59) with respect to M_∞ gives

$$\frac{\partial R}{\partial M_\infty} = R\left[\frac{M_\infty^2-1}{M_\infty\left(1+\frac{\gamma-1}{2}M_\infty^2\right)}\right] \tag{60}$$

The relative sensitivity coefficient is

$$\theta'_{M_\infty} = \frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty} = \frac{M_\infty^2-1}{1+\frac{\gamma-1}{2}M_\infty^2} \tag{61}$$

For $\gamma = 1.40$, values of R, $\partial R/\partial M_\infty$ and $(M_\infty/R)(\partial R/\partial M_\infty)$ are plotted in Figure 5.

The stream-tube area ratio can also be used to estimate the sensitivity of M_∞ to changes in the sonic throat cross-sectional area and to changes in the test section cross-sectional area (e.g., blockage at subsonic speeds). Since

$$R = f(M_\infty, \gamma)$$

$$E_R = \frac{\partial R}{\partial M_\infty} E_{M_\infty}$$

Except for $M_\infty = 1.0$, where $\partial R/\partial M_\infty = 0$,

$$E_{M_\infty} = \left(\frac{\partial R}{\partial M_\infty}\right)^{-1} E_R \tag{62}$$

Then, with $\partial R/\partial M_\infty$ from Eq. (60), the sensitivity of M_∞ to stream-tube area ratio, $R = A_\infty/A^*$, can be estimated with Eq. (62).

Applications

In this section, error estimates will be given for several nondimensional ratios and coefficients which are frequently used in presenting the results of high-speed wind tunnel tests. These estimates will rely heavily on the results of the previous section. It will be assumed that the variables are independent. For those results which involve both an arbitrary measured pressure and a measured tunnel condition pressure, such as total pressure, p_t, this implies that the two pressures are absolute measurements or are gage measurements with a negligible error in the measured atmospheric pressure. If this is not the case, a more complicated analysis will be required. Also it is assumed that M_∞ and p_t are not correlated through the tunnel calibration. If this is not true, the equations may be much more complex and a numerical estimate may be required. Finally, it is assumed that there is no error in the ratio of specific heats, γ.

Pressure ratio, P

For the first application we will again examine that ubiquitous parameter, the pressure ratio, P, where

$$P = \frac{p}{p_\infty} \tag{63}$$

In this equation, p is an arbitrary measured pressure and p_∞ is calculated from M_∞, γ, and the measured total pressure, p_t, using Eq. (41). Assuming that the variables in Eq. (63) are independent, the error in P is given by

$$E_P = P\left[\left(\frac{E_p}{p}\right)^2 + \left(\frac{E_{p_\infty}}{p_\infty}\right)^2\right]^{1/2} \tag{64}$$

where E_p is the error in the measured pressure and E_{p_∞} was given by Eq. (37) as,

$$\frac{E_{p_\infty}}{p_\infty} = \left[\left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2} \qquad [37]$$

where $R = p_\infty/p_t$, E_{M_∞} is the error in the free-stream Mach number, and E_{p_t} is the error in the measured total pressure. The relative sensitivity coefficient, $(M_\infty/R)(\partial R/\partial M_\infty)$, is given by Eq. (43) and is plotted in Figure 1 for $\gamma=1.40$.

Pressure coefficient, C_p

The pressure coefficient, C_p, is defined as

$$C_p = \frac{p - p_\infty}{q_\infty} \tag{65}$$

For high-speed wind tunnels, the dynamic pressure is usually calculated from the measured total pressure and the free-stream Mach number with Eq. (47). The static pressure, p_∞ is calculated with Eq. (41). For a thermally perfect gas, q_∞ and p_∞ are related by Eq. (47) of Ref. 12,

$$q_\infty = \frac{\gamma}{2} p_\infty M_\infty^2 \tag{66}$$

Then, substituting Eq. (66) in Eq. (65),

$$C_p = \frac{p - p_\infty}{\frac{\gamma}{2} p_\infty M_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

and substituting $p_\infty = R(M_\infty, \gamma)\, p_t$ gives

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_t R} - 1\right) \tag{67}$$

Assuming p, p_t and M_∞ are independent, the error in C_p is given by

$$E_{C_p} = \left[\left(\frac{\partial C_p}{\partial p} E_p\right)^2 + \left(\frac{\partial C_p}{\partial p_t} E_{p_t}\right)^2 + \left(\frac{\partial C_p}{\partial M_\infty} E_{M_\infty}\right)^2\right]^{1/2} \tag{68}$$

$$E_{C_p} = C_p\left(\frac{P}{P-1}\right)\left[\left(\frac{E_p}{p}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2 + \left[2\left(\frac{P-1}{P}\right) + \frac{M_\infty \partial R}{R\, \partial M_\infty}\right]^2 \left(\frac{E_{M_\infty}}{M_\infty}\right)^2\right]^{1/2} \tag{69}$$

where $P = p / p_\infty$ and $R = p_\infty / p_t$. If desired, P can be replaced with C_p by using the relation,

$$P = 1 + \frac{\gamma M_\infty^2}{2} C_p \tag{70}$$

The relative sensitivity coefficient, $(M_\infty/R)(\partial R/\partial M_\infty)$, is given by Eq. (43) and is plotted in Figure 1 for γ=1.40.

Force and moment coefficients, C_F and C_m

Nondimensional force coefficients are defined by

$$C_F = \frac{F}{q_\infty S} \tag{71}$$

where F is an aerodynamic force, q_∞ the free-stream dynamic pressure, and S a reference area. The error in the force coefficient is given by

$$E_{C_F} = C_F\left[\left(\frac{E_F}{F}\right)^2 + \left(\frac{E_{q_\infty}}{q_\infty}\right)^2 + \left(\frac{E_S}{S}\right)^2\right]^{1/2} \tag{72}$$

where E_F is the error in the measured force, E_S the error in the reference area and E_{q_∞} is given by Eq. (46) as

$$\frac{E_{q_\infty}}{q_\infty} = \left[\left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2} \qquad [46]$$

where $R = q_\infty / p_t$, E_{M_∞} is the error in the free-stream Mach number and E_{p_t} is the error in the measured total pressure. The relative sensitivity coefficient, $(M_\infty/R)(\partial R/\partial M_\infty)$, is given by Eq. (49) and is plotted in Figure 2 for γ= 1.40. Frequently, $S = \pi d^2/4$, where d is a reference diameter. Then,

$$\frac{E_S}{S} = 2\frac{E_d}{d} \tag{73}$$

Nondimensional moment coefficients are defined by

$$C_m = \frac{M}{q_\infty S l} \tag{74}$$

where M is an aerodynamic moment and l is a reference length. The error in C_m is given by

$$E_{C_m} = C_m\left[\left(\frac{E_M}{M}\right)^2 + \left(\frac{E_{q_\infty}}{q_\infty}\right)^2 + \left(\frac{E_S}{S}\right)^2 + \left(\frac{E_l}{l}\right)^2\right]^{1/2} \tag{75}$$

This equation is similar to Eq. (72) with E_M the error in the measured moment and E_l the error in the reference length. Frequently, $l = d$ and $S = \pi d^2/4$. Then, l and S are correlated and the last two terms of Eq. (75) must be replaced by

$$\left(\frac{3E_d}{d}\right)^2$$

to avoid underestimating the error contribution due to d.

Dynamic stability coefficients, C_{m_q} and C_{l_p}

Nondimensional dynamic stability coefficients are defined as derivatives of the moment coefficient with respect to a normalized angular rate. For example, the pitching moment coefficient derivative, C_{m_q}, is defined by

$$C_{m_q} = \frac{\partial C_m}{\partial (ql/V_\infty)} = \frac{\partial (M/q_\infty S l)}{\partial (ql/V_\infty)}$$

$$= \frac{\partial M/\partial q}{q_\infty S l^2/V_\infty} = \frac{M_q}{q_\infty S l^2 /V_\infty} \tag{76}$$

The derivative, M_q, is determined from the experiment. Note that in these equations, M is the aerodynamic pitching moment, whereas M_∞ is the free-stream Mach number, and q is the angular pitch rate, whereas q_∞ is the free-stream dynamic pressure. The estimated error in C_{m_q} is

$$E_{C_{m_q}} = C_{m_q}\left[\left(\frac{E_{M_q}}{M_q}\right)^2 + \left(\frac{E_{q_\infty/V_\infty}}{q_\infty/V_\infty}\right)^2 + \left(\frac{E_S}{S}\right)^2 + \left(\frac{2E_l}{l}\right)^2\right]^{1/2} \tag{77}$$

As was noted in the previous section, if $l = d$ and $S = \pi d^2/4$, l and S are correlated and the last two terms in Eq. (77) should be replaced with $(4E_d/d)^2$. By definition,

$$\frac{q_\infty}{V_\infty} = \frac{\rho_\infty V_\infty^2}{2V_\infty} = \frac{\rho_\infty V_\infty}{2} \tag{78}$$

For a thermally perfect gas,

$$\rho_\infty = \frac{p_\infty}{R_g T_\infty} \tag{79}$$

and

$$V_\infty = M_\infty \sqrt{\gamma R_g T_\infty} \quad (80)$$

Substituting these equations into Eq. (78) and using Eq. (41) to relate p_∞ to p_t and Eq. (43) of Ref. 12 to relate T_∞ to T_t,

$$\frac{T_\infty}{T_t} = \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{-1}$$

gives

$$\frac{q_\infty}{V_\infty} = \frac{1}{2}\sqrt{\frac{\gamma}{R_g}} \frac{p_t}{\sqrt{T_t}} M_\infty \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{-\frac{(\gamma + 1)}{2(\gamma - 1)}} \quad (81)$$

The relative error in q_∞/V_∞ is given by

$$\frac{E_{q_\infty/V_\infty}}{q_\infty/V_\infty} = \left[\left(\frac{E_{p_t}}{p_t}\right)^2 + \left(\frac{E_{T_t}}{2T_t}\right)^2 + \left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2\right]^{1/2} \quad (82)$$

where $R = q_\infty / V_\infty$. Differentiating Eq. (81) with respect to M_∞ gives

$$\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty} = \frac{1 - M_\infty^2}{1 + \frac{\gamma - 1}{2} M_\infty^2} \quad (83)$$

Comparing Eq. (83) with Eq. (61) shows that

$$\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty} = -\frac{M_\infty}{R_A}\frac{\partial R_A}{\partial M_\infty} \quad (84)$$

where $R_A = A_\infty/A^*$. Then, Eq. (61) and the results plotted in Figure 5 can be used to evaluate $(M_\infty/R)(\partial R/\partial M_\infty)$ for $R = q_\infty/V_\infty$. Equations (82) and (77) are used to estimate the error in C_{m_q}.

The rolling moment coefficient derivative, C_{l_p}, is defined by,

$$C_{l_p} = \frac{\partial C_l}{\partial (pl/V_\infty)} = \frac{\partial (L/q_\infty S l)}{\partial (pl/V_\infty)}$$

$$= \frac{\partial L/\partial p}{q_\infty S l^2/V_\infty} = \frac{L_p}{q_\infty S l^2/V_\infty} \quad (85)$$

where L is the aerodynamic rolling moment and p is the roll rate. The derivative L_p is determined from the experiment. The error in C_{l_p} is given by

$$E_{C_{l_p}} = C_{l_p}\left[\left(\frac{E_{L_p}}{L_p}\right)^2 + \left(\frac{E_{q_\infty/V_\infty}}{q_\infty/V_\infty}\right)^2 + \left(\frac{E_S}{S}\right)^2 + \left(\frac{2E_l}{l}\right)^2\right]^{1/2} \quad (86)$$

which is evaluated with Eqs. (61) and (82) as indicated above for Eq. (77).

Mach number, M_∞

In high-speed wind tunnels, the free-stream Mach number is usually determined during tunnel calibration by one of three methods, each involving a measured pressure ratio, R. First, at subsonic and transonic Mach numbers, a static pipe may be used to measure free-stream pressure, p_∞. The measured tunnel total pressure, p_t, is used in the ratio, $R = p_\infty/p_t$ with the isentropic flow equation to determine Mach number. From Eq. (41),

$$R = \frac{p_\infty}{p_t} = \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{-\frac{\gamma}{\gamma - 1}} \quad [41]$$

Solving for M_∞ gives

$$M_\infty = \left[\frac{2}{\gamma - 1}\left[\left(\frac{p_\infty}{p_t}\right)^{-\frac{\gamma - 1}{\gamma}} - 1\right]\right]^{\frac{1}{2}} \quad (87)$$

Second, at supersonic and hypersonic Mach numbers, the static pressure can be very low and the Mach number is usually determined with a pitot tube which measures the total pressure, p_{t_2}, behind the tube's normal shock. This pressure is divided by the tunnel total pressure, p_t, to give a ratio, R, which is related to Mach number by the normal shock relation, Eq. (53),

$$R = \frac{p_{t_2}}{p_{t_1}} = \left[\frac{(\gamma + 1) M_\infty^2}{(\gamma - 1) M_\infty^2 + 2}\right]^{\frac{\gamma}{\gamma - 1}} \times \left[\frac{\gamma + 1}{2\gamma M_\infty^2 - (\gamma - 1)}\right]^{\frac{1}{\gamma - 1}} \quad [53]$$

This equation is solved numerically for M_∞, by using an iterative root solver. Finally, a pitot-static probe may be used to determine Mach number. In this case, the ratio and Mach number dependency is given by the Rayleigh formula, Eq. (56),

$$R = \frac{p_\infty}{p_{t_2}} = \frac{p_1}{p_{t_2}} = \left[\frac{2}{(\gamma+1)M_\infty^2}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\times\left[\frac{2\gamma M_\infty^2 - (\gamma-1)}{\gamma+1}\right]^{\frac{1}{\gamma-1}} \quad [56]$$

Again, this equation is solved numerically for M_∞.

The error equation for the Mach number calculated from any of these three methods is the same. Let

$$R(M_\infty, \gamma) = \frac{p_x}{p_y} \quad (88)$$

The uncertainty in the measured ratio is given by,

$$E_R = R\left[\left(\frac{E_{p_x}}{p_x}\right)^2 + \left(\frac{E_{p_y}}{p_y}\right)^2\right]^{1/2} \quad (89)$$

Although an explicit relation for M_∞ is not possible for two of the ratios, the uncertainty in M_∞ can be determined implicitly, when $\partial R/\partial M_\infty \neq 0$, by solving the error equation,

$$E_R = \frac{\partial R}{\partial M_\infty} E_{M_\infty} \quad (90)$$

to give

$$E_{M_\infty} = \frac{\partial M_\infty}{\partial R} E_R \quad (91)$$

Substituting Eq. (89) in Eq. (91) gives

$$\frac{E_{M_\infty}}{M_\infty} = \frac{R}{M_\infty}\frac{\partial M_\infty}{\partial R}\left[\left(\frac{E_{p_x}}{p_x}\right)^2 + \left(\frac{E_{p_y}}{p_y}\right)^2\right]^{1/2}$$

For nonzero values of M_∞ and $\partial R/\partial M_\infty$,

$$\frac{R}{M_\infty}\frac{\partial M_\infty}{\partial R} = \left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\right)^{-1} \quad (92)$$

Then,

$$\frac{E_{M_\infty}}{M_\infty} = \left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\right)^{-1}\left[\left(\frac{E_{p_x}}{p_x}\right)^2 + \left(\frac{E_{p_y}}{p_y}\right)^2\right]^{1/2} \quad (93)$$

Substituting the measured pressures for p_x and p_y gives the following estimates of error in calibration Mach number for the three methods:

(1) For $R = p_\infty/p_t$, the relative error in Mach number is

$$\frac{E_{M_\infty}}{M_\infty} = \left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\right)^{-1}\left[\left(\frac{E_{p_\infty}}{p_\infty}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2} \quad (94)$$

where $(M_\infty/R)(\partial R/\partial M_\infty)$ is given by Eq. (43).

(2) For $R = p_{t_2}/p_t$, the relative error in Mach number is

$$\frac{E_{M_\infty}}{M_\infty} = \left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\right)^{-1}\left[\left(\frac{E_{p_{t_2}}}{p_{t_2}}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2} \quad (95)$$

where $(M_\infty/R)(\partial R/\partial M_\infty)$ is given by Eq. (55).

(3) For $R = p_\infty/p_{t_2}$, the relative error in Mach number is

$$\frac{E_{M_\infty}}{M_\infty} = \left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\right)^{-1}\left[\left(\frac{E_{p_\infty}}{p_\infty}\right)^2 + \left(\frac{E_{p_{t_2}}}{p_{t_2}}\right)^2\right]^{1/2} \quad (96)$$

where $(M_\infty/R)(\partial R/\partial M_\infty)$ is given by Eq. (58).

Concluding Remarks

Estimation of the uncertainty in results from high-speed wind tunnel tests is hindered by the need to evaluate the sensitivity coefficients, $\partial R/\partial M_\infty$, for various aerodynamic ratios, R. The ratios can be complex functions of M_∞ and calculating the partial derivative is a demanding task. To simplify the error estimation procedure, sensitivity coefficients were evaluated for five fundamental aerodynamic ratios. In general, these ratios relate a free-stream test condition (static or dynamic) to a reference (total or sonic) condition. Methods of applying these sensitivity coefficients in error estimation were demonstrated for several nondimensional ratios and coefficients which are used in high-speed testing. In addition, the sensitivity coefficients can be applied directly to estimating the error in calculated free-stream conditions. The presence of dependent or correlated variables in a calculated test result is frequently a challenge. Examples were given which compared two analytical methods for estimating the error when variables are dependent. Also, a numerical method which approximates the partial derivatives was discussed.

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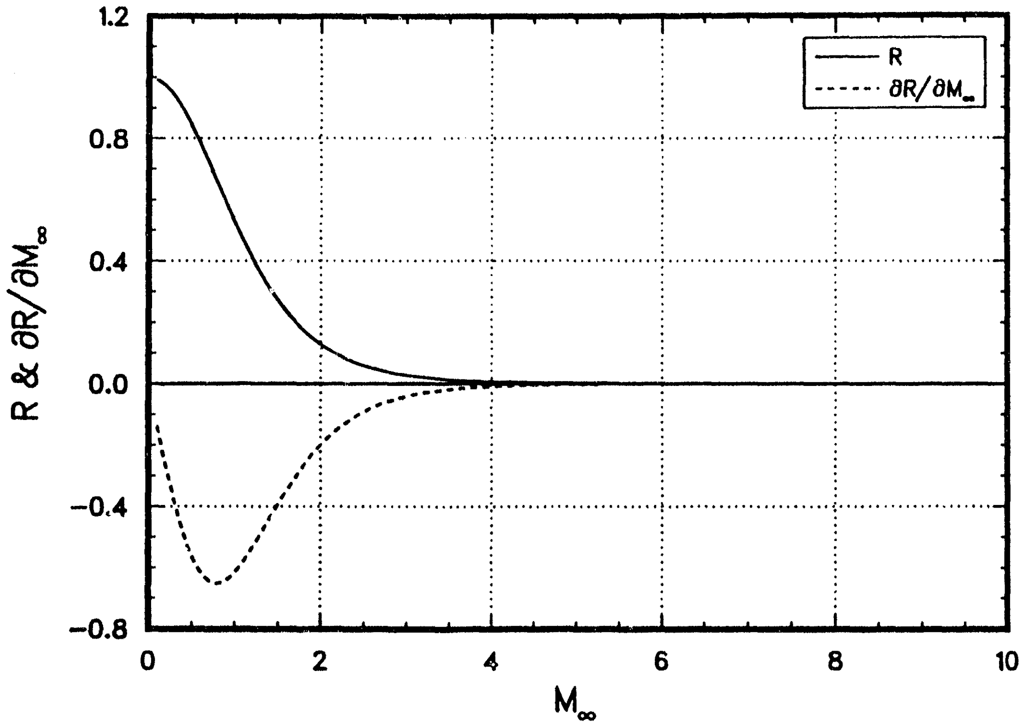
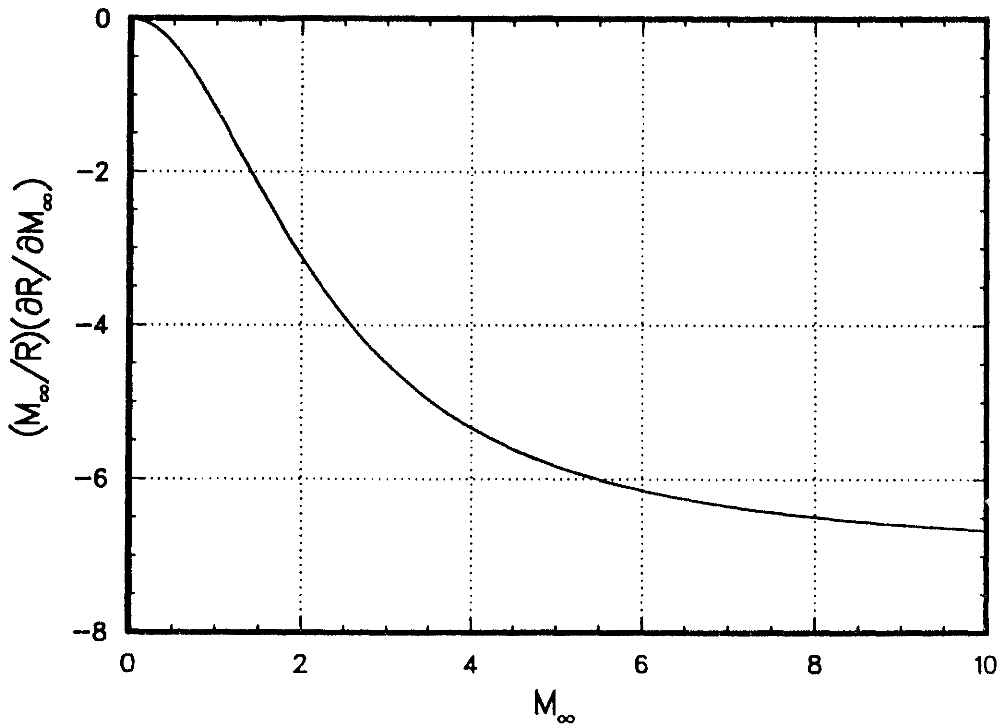
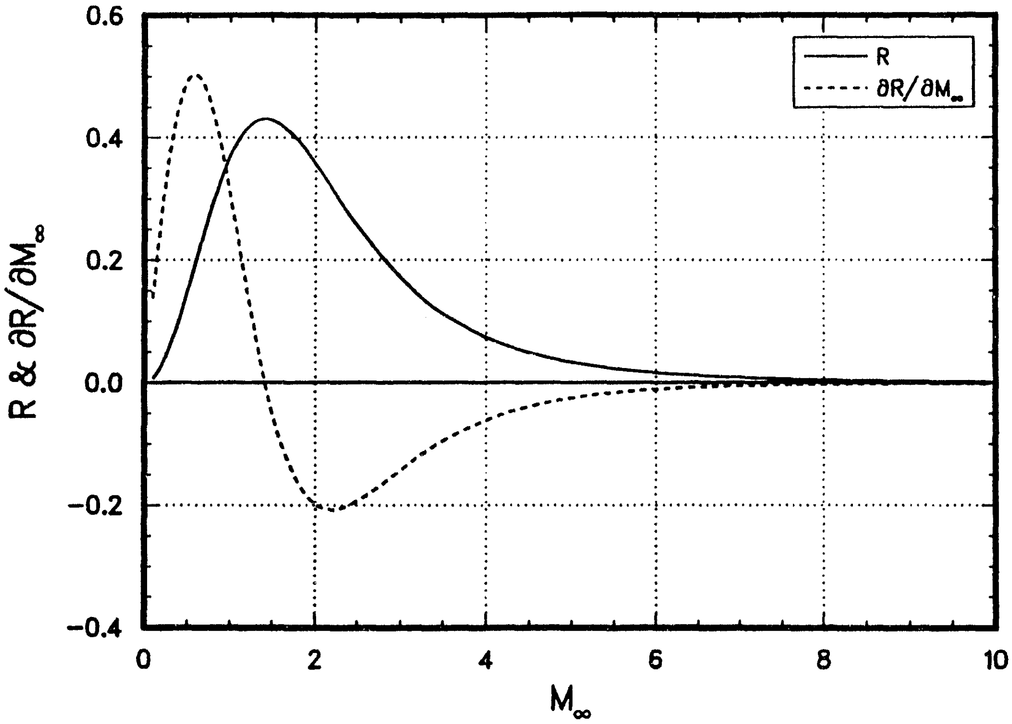


a. R and $\partial R / \partial M_\infty$

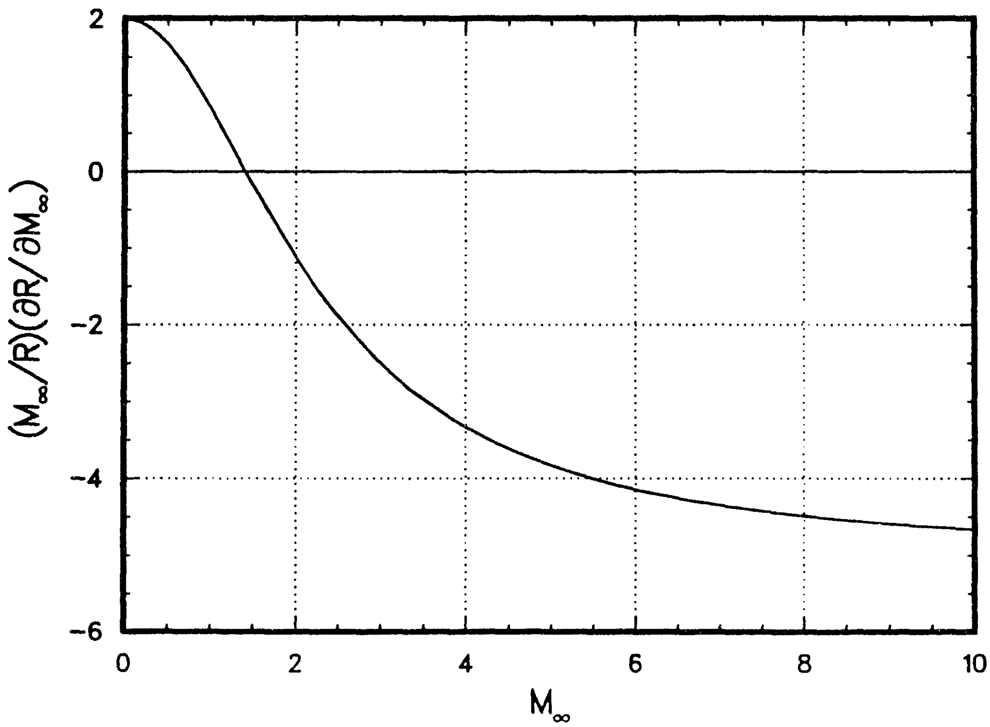


b. $\theta'_{M_\infty} = (M_\infty/R)\,(\partial R/\partial M_\infty)$

Figure 1. Variation of R, $\partial R/\partial M_\infty$, and θ'_{M_∞} with M_∞ for $R = p_\infty / p_t$ and $\gamma = 1.40$

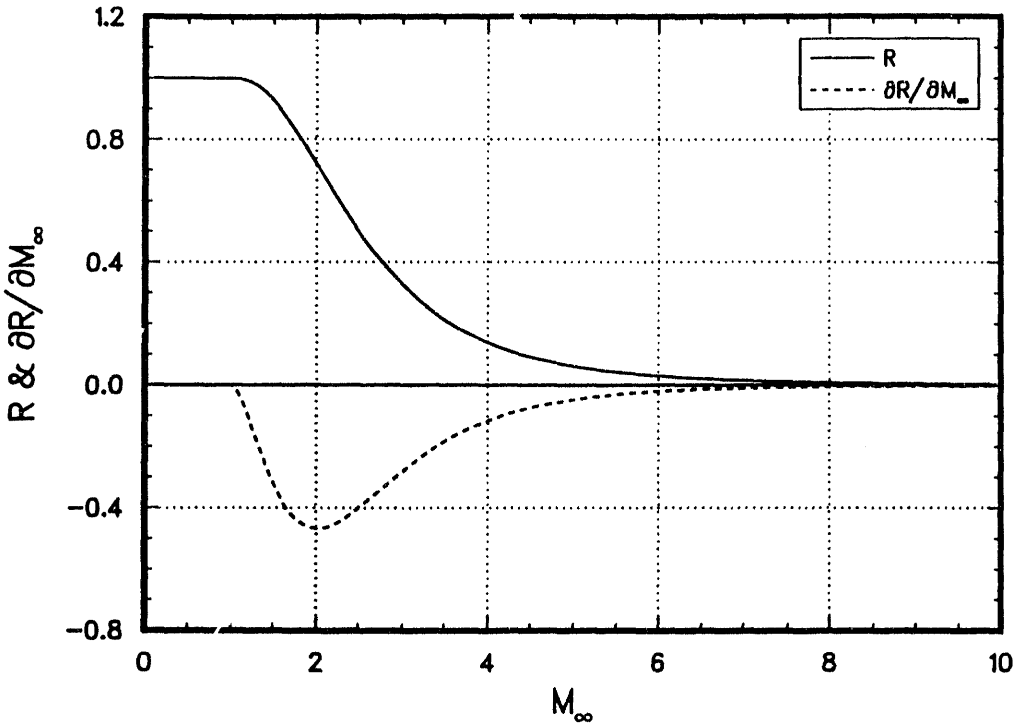


a. R and $\partial R / \partial M_\infty$

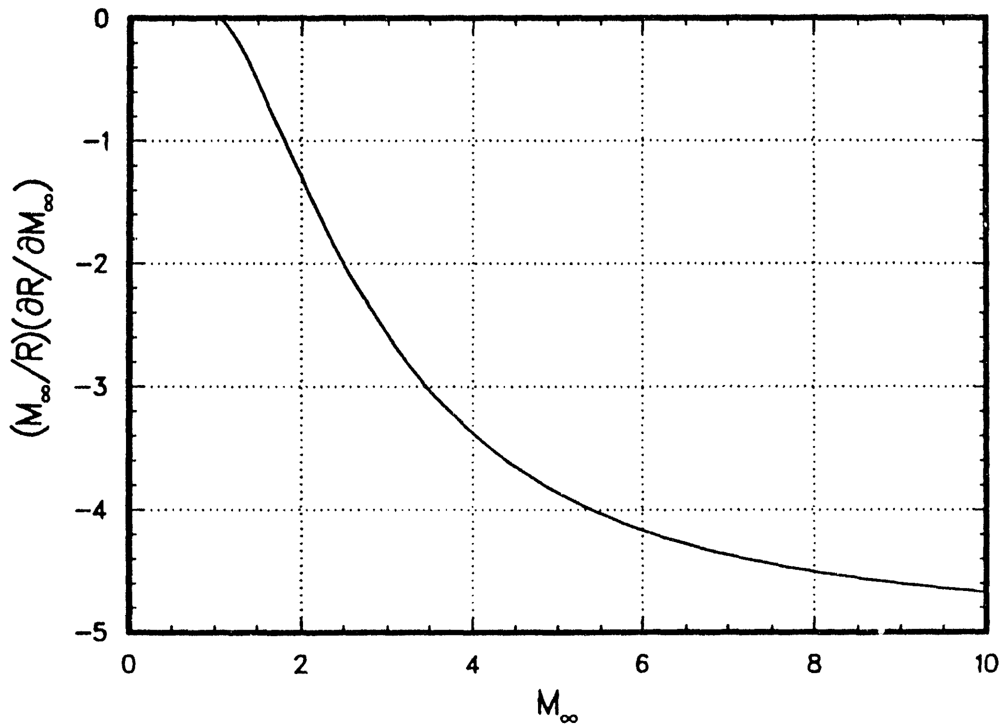


b. $\theta'_{M_\infty} = (M_\infty/R)(\partial R/\partial M_\infty)$

Figure 2. Variation of R, $\partial R/\partial M_\infty$, and θ'_{M_∞} with M_∞ for $R = q_\infty / p_t$ and $\gamma = 1.40$

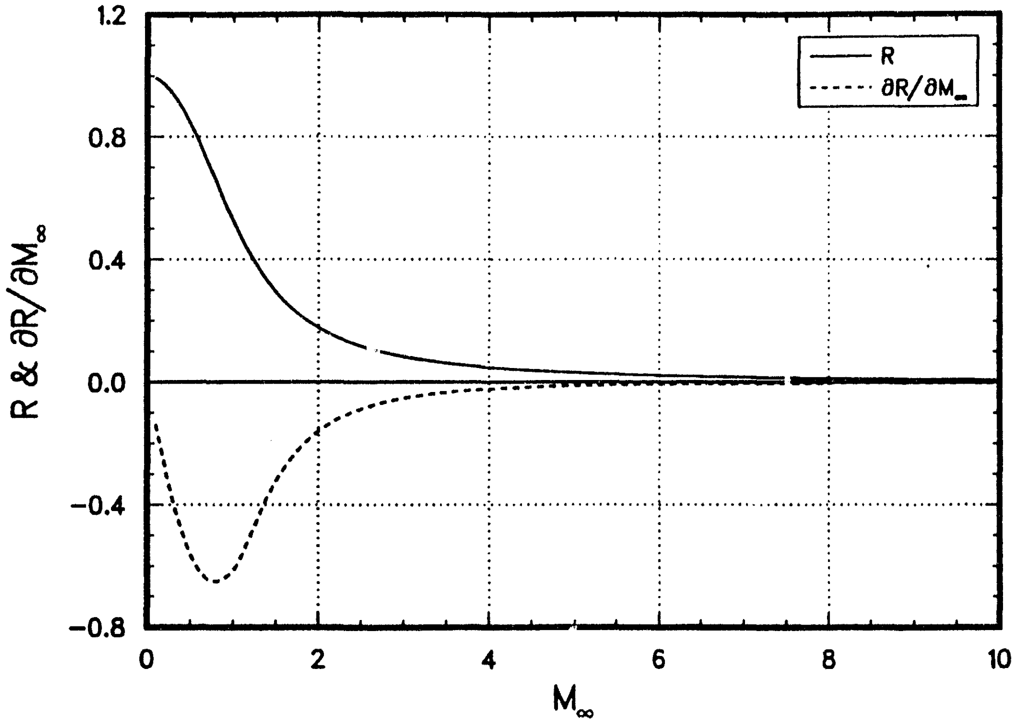


a. R and $\partial R / \partial M_\infty$

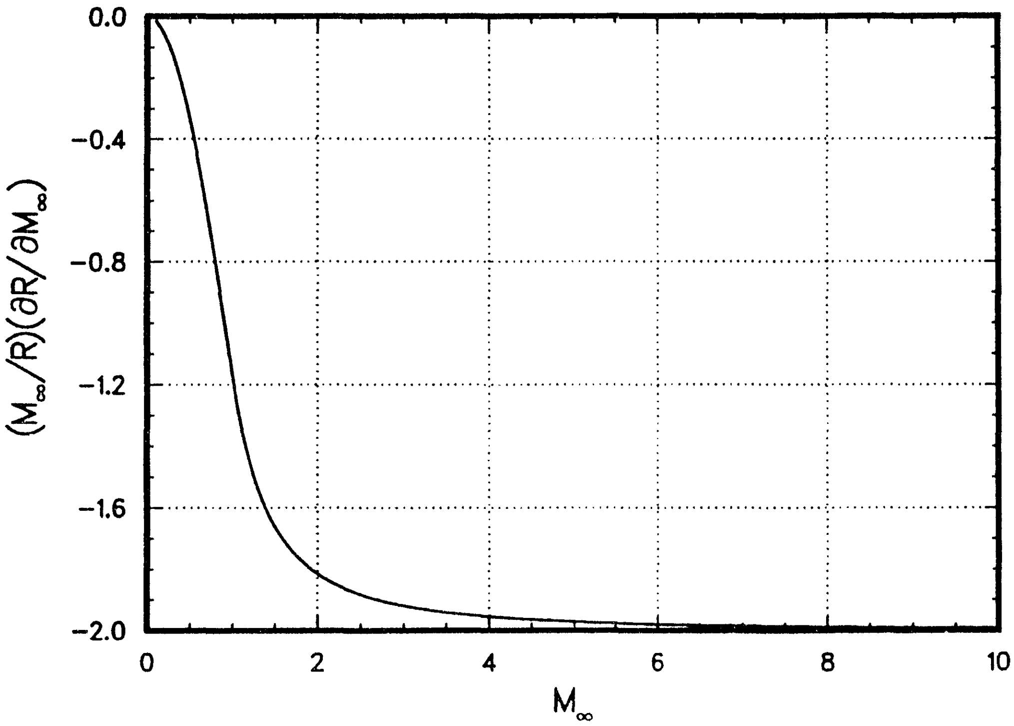


b. $\theta'_{M_\infty} = (M_\infty/R)\,(\partial R/\partial M_\infty)$

Figure 3. Variation of R, $\partial R/\partial M_\infty$, and θ'_{M_∞} with M_∞ for $R = p_{t_2}/p_{t_1}$ and $\gamma = 1.40$

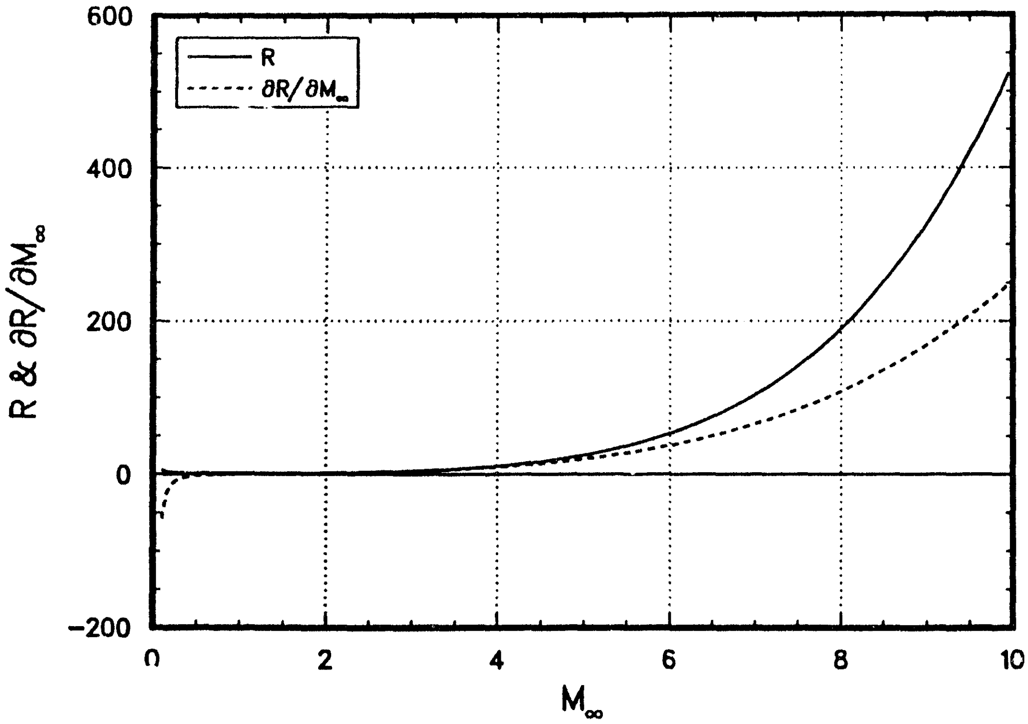


a. R and $\partial R / \partial M_\infty$

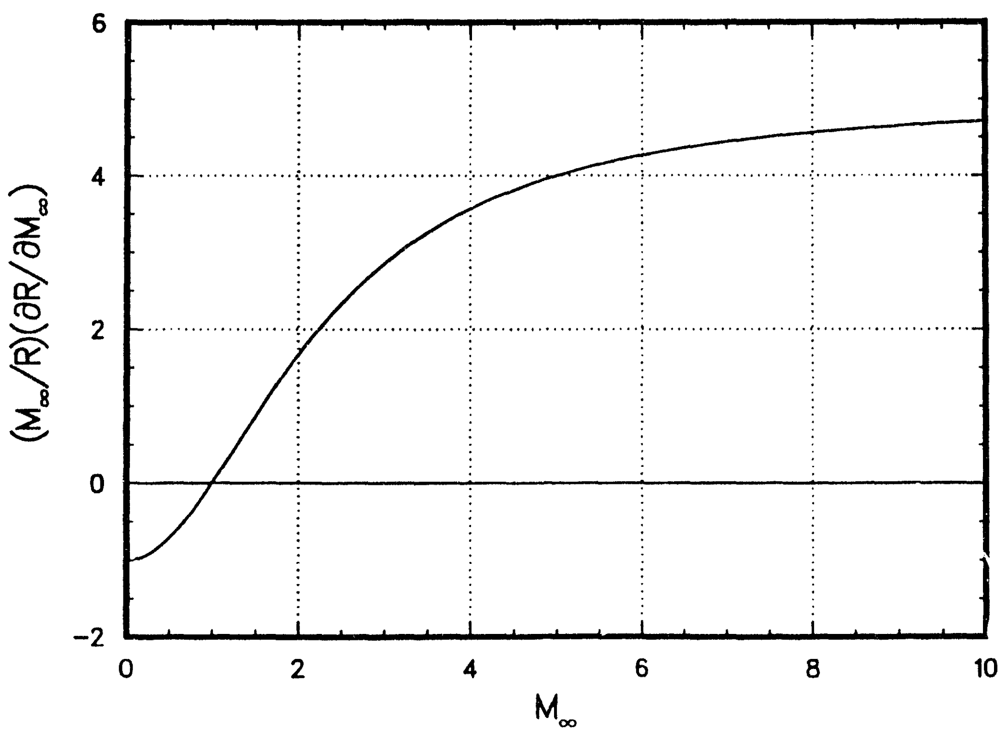


b. $\theta'_{M_\infty} = (M_\infty/R)\,(\partial R/\partial M_\infty)$

Figure 4 Variation of R, $\partial R/\partial M_\infty$, and θ'_{M_∞} with M_∞ for $R = p_\infty/p_{t_2}$ and $\gamma = 1.40$



a. R and $\partial R / \partial M_\infty$



b. $\theta'_{M_\infty} = (M_\infty/R)\,(\partial R/\partial M_\infty)$

Figure 5. Variation of R, $\partial R/\partial M_\infty$, and θ'_{M_∞} with M_∞ for $R = A_\infty/A^*$ and $\gamma = 1.40$

ADDENDUM

Sensitivity Coefficients for Ratio of Specific Heats, γ

A colleague recently suggested that I should examine the sensitivity coefficients for the ratio of specific heats, γ. In the main body of the paper, it is assumed that the error due to γ is negligible, i.e., $E_\gamma = 0$. At hypersonic speeds, where real gas effects are significant and errors in γ are probable, the sensitivity coefficients for γ are large. Therefore, any errors in γ will be magnified.

The error equation for free-stream static pressure, p_∞, is, from Eq. (33),

$$E_{p_\infty} = \left[\left(\frac{\partial p_\infty}{\partial M_\infty}E_{M_\infty}\right)^2 + \left(\frac{\partial p_\infty}{\partial \gamma}E_\gamma\right)^2 + \left(\frac{\partial p_\infty}{\partial p_t}E_{p_t}\right)^2\right]^{1/2}$$

Expanding to the relative error form gives

$$\frac{E_{p_\infty}}{p_\infty} = \left[\left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2 + \left(\frac{\gamma}{R}\frac{\partial R}{\partial \gamma}\frac{E_\gamma}{\gamma}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2}$$

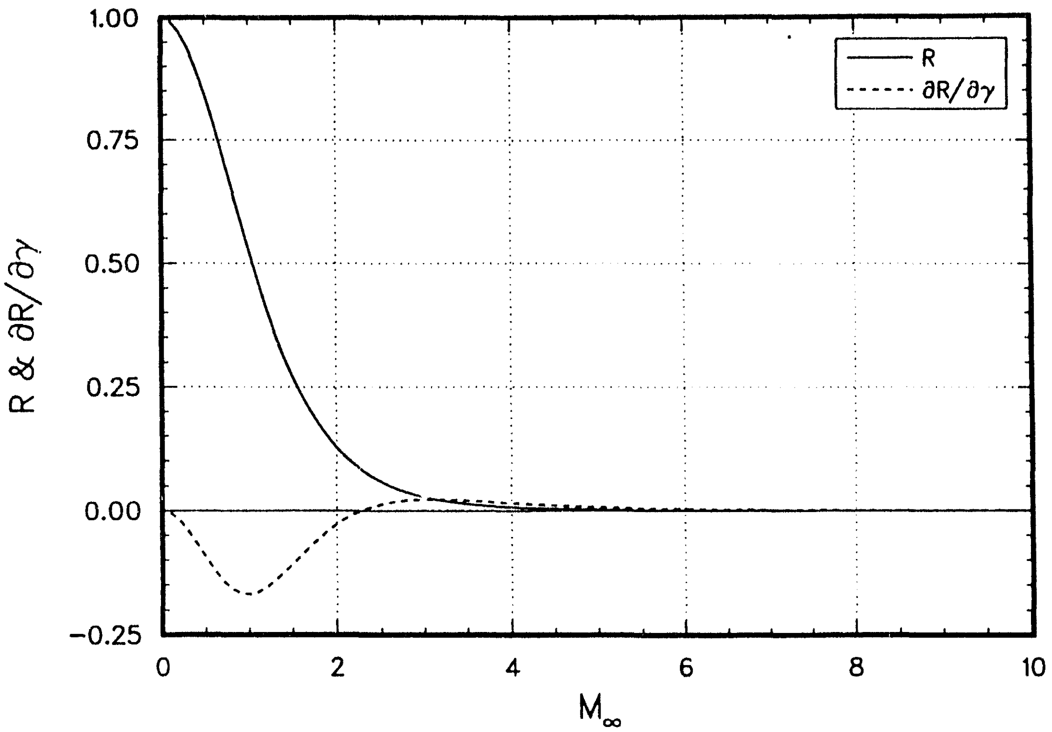
where $R = p_\infty/p_t$. The absolute and relative sensitivity coefficients, $\theta_\gamma = \frac{\partial R}{\partial \gamma}$ and $\theta_\gamma' = \frac{\gamma \partial R}{R \partial \gamma}$, respectively, were evaluated using numerical differentiation (Eq. (10) with $\Delta\gamma/\gamma = 0.004$) and are presented in Figure A-1.

The relative error equation for free-stream dynamic pressure is the same as that given above for static pressure, except that $R = q_\infty/p_t$,

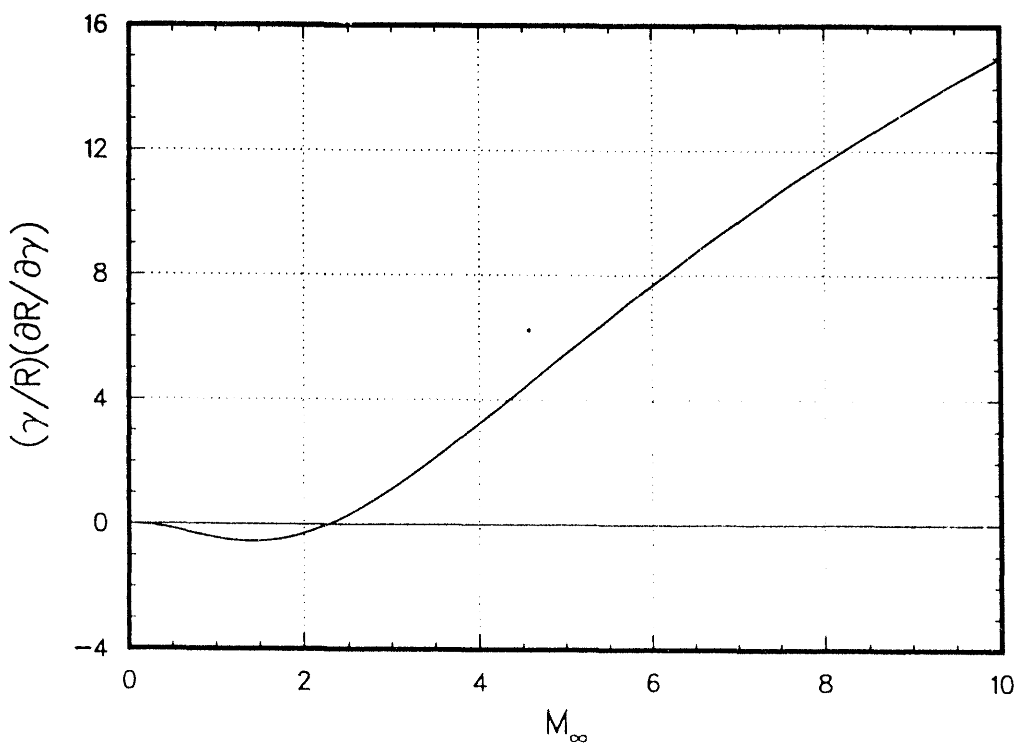
$$\frac{E_{q_\infty}}{q_\infty} = \left[\left(\frac{M_\infty}{R}\frac{\partial R}{\partial M_\infty}\frac{E_{M_\infty}}{M_\infty}\right)^2 + \left(\frac{\gamma}{R}\frac{\partial R}{\partial \gamma}\frac{E_\gamma}{\gamma}\right)^2 + \left(\frac{E_{p_t}}{p_t}\right)^2\right]^{1/2}$$

The sensitivity coefficients for q_∞ are given in Figure A-2.

The significant influence of errors in γ at hypersonic Mach numbers is evident from the plotted coefficients. For example, at $M_\infty = 8$, an error in γ of 0.001 (relative error of 0.07%) will result in relative errors of approximately 0.8% in p_∞ and 0.9% in q_∞.

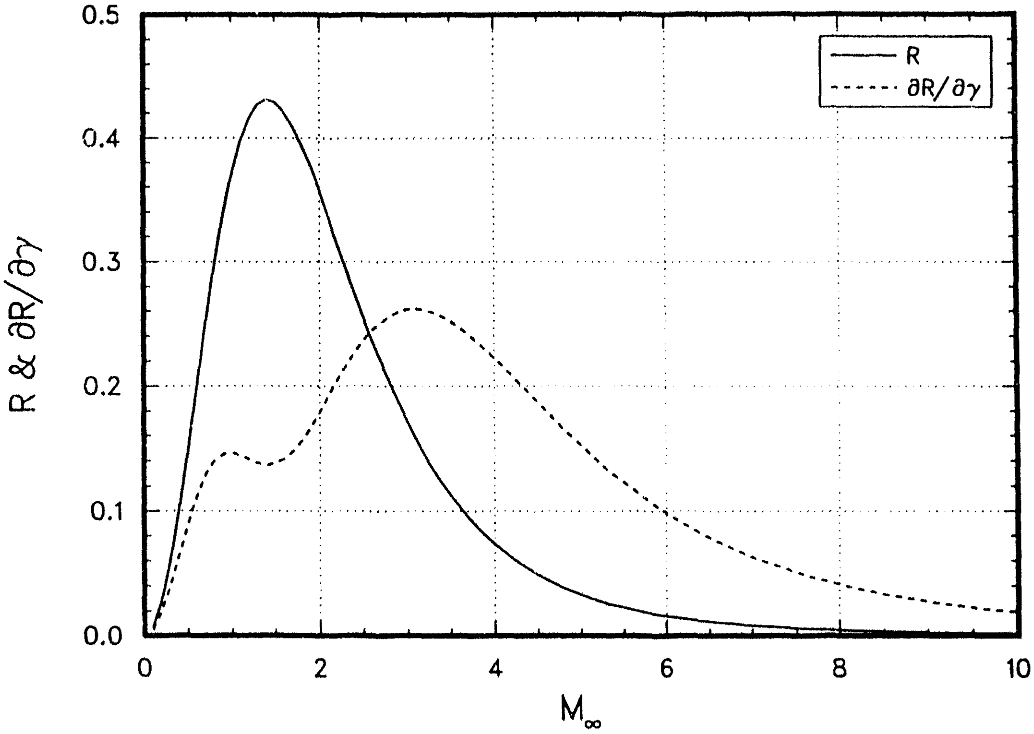


a. R and $\partial R / \partial \gamma$

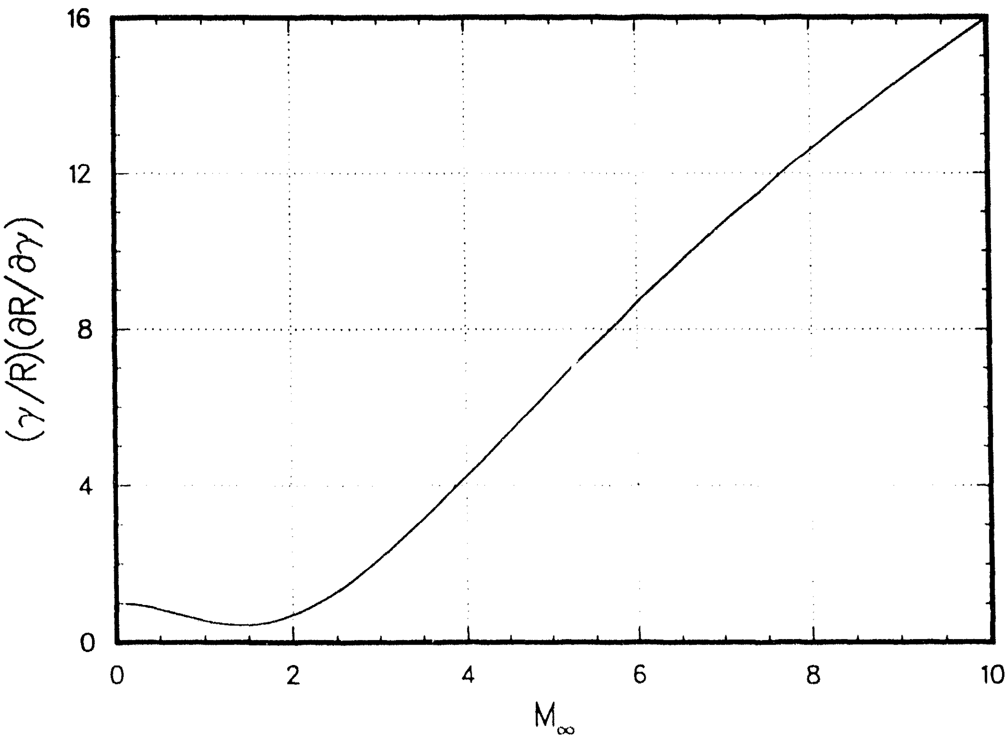


b. $\theta'_\gamma = (\gamma/R)\ (\partial R/\partial \gamma)$

Figure A-1. Variation of R, $\partial R/\partial\gamma$, and θ'_γ with M_∞ for $R = p_\infty / p_t$ and $\gamma = 1.40$



a. R and $\partial R / \partial \gamma$



b. $\theta'_\gamma = (\gamma/R)\ (\partial R/\partial \gamma)$

Figure A-2. Variation of R, $\partial R/\partial \gamma$, and θ'_γ with M_∞ for $R = q_\infty / p_t$ and $\gamma = 1.40$

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