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Integral Equations and Operator Theory

The Isometric Representation Theory of Numerical Semigroups

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Abstract. We study representations of numerical semigroups Σ by isometries on Hilbert space with commuting range projections. Our main theorem says that each such representation is unitarily equivalent to the direct sum of a representation by unitaries and a finite number of multiples of particular concrete representations by isometries. We use our main theorem to identify the faithful representations of the C^*-algebra $C^*(\Sigma)$ generated by a universal isometric representation with commuting range projections, and also prove a structure theorem for $C^*(\Sigma)$.

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1. Introduction

Coburn proved that the C^*-algebras generated by non-unitary isometries are all canonically isomorphic [1]. Coburn's result can be viewed as saying that the C^*-algebras generated by representations of the additive semigroup $\mathbb{N}$ by non-unitary isometries are all canonically isomorphic. This result has since been generalised to other semigroups, in particular the positive cones of ordered subgroups of $\mathbb{R}$ by Douglas [2], the positive cones of totally ordered abelian groups by Murphy [6], and amenable quasi-lattice ordered groups by Nica [7] and Laca-Raeburn [5].

Murphy [6] and Jang [3, 4] have observed that such a result does not hold for the additive semigroup $\mathbb{N} \setminus \{1\}$, however, by finding two representations by non-unitary isometries whose C^*-algebras are not canonically isomorphic. This observation motivated an investigation of the isometric representations of $\mathbb{N} \setminus \{1\}$ and the C^*-algebras they generate [8]. The main theorem in [8] says that each isometric representation of $\mathbb{N} \setminus \{1\}$ with commuting range projections is unitarily equivalent to the direct sum of a unitary representation, a multiple of the Toeplitz

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representation on $\ell^2(\mathbb{N} \setminus \{1\})$, and a multiple of a representation by shifts on $\ell^2(\mathbb{N})$. This theorem was used to study the C^*-algebra $C^*(\mathbb{N} \setminus \{1\})$ generated by a universal isometric representation with commuting range projections. The faithful representations of $C^*(\mathbb{N} \setminus \{1\})$ were identified, and the structure of $C^*(\mathbb{N} \setminus \{1\})$ described in terms of the usual Toeplitz algebra.

The purpose of this article is to extend the study in [8] to the whole class of numerical semigroups, of which the semigroup $\mathbb{N} \setminus \{1\}$ is a particular example. As in [8] we study representations by isometries with commuting range projections. We analyse the isometric representations of numerical semigroups with commuting range projections, and investigate the structure of the C^*-algebras they generate.

The organisation of this article is as follows. We discuss the necessary background material on numerical semigroups in Section 2. In Section 3 we describe the class of isometric representations that we investigate, and in particular our main class of concrete examples. In Section 4 we associate to every numerical semigroup Σ a certain collection $\mathcal{A}(\Sigma)$ of subsets of $\mathbb{N}$. To each set in $\mathcal{A}(\Sigma)$ we then associate a particular concrete isometric representation of Σ, and the resulting collection of representations is used in our description of a general representation which constitutes our main theorem. We prove our main theorem in Section 5 by extending a special case of this theorem. The proof of the special case is the subject of Section 6, and the basic strategy is analogous to that for [8, Theorem 3.1]. In Section 7 we use our main theorem to investigate the C^*-algebra $C^*(\Sigma)$ generated by a universal isometric representation of a numerical semigroup Σ with commuting range projections. We obtain a condition describing precisely when a given representation of $C^*(\Sigma)$ is faithful, and describe the structure of $C^*(\Sigma)$ in terms of the usual Toeplitz algebra.

2. Numerical semigroups

Throughout this article, $\mathbb{N}$ denotes the additive semigroup of non-negative integers (including 0). A *numerical semigroup* is a subsemigroup of $\mathbb{N}$ containing 0 such that the greatest common divisor of its elements is equal to one.

The following proposition is proved in [9, page 105, Proposition 10.2] and provides some equivalent characterisations of numerical semigroups.

Proposition 2.1. *Let Σ be a subsemigroup of $\mathbb{N}$ containing 0. Then the following are equivalent:*

1. *Σ is a numerical semigroup;*
2. *Σ generates the additive group $\mathbb{Z}$ of integers;*
3. *Σ has finite complement in $\mathbb{N}$.*

For a numerical semigroup Σ, the complement $\mathbb{N} \setminus \Sigma$ is finite by Proposition 2.1, so $\mathbb{Z} \setminus \Sigma$ contains a largest element (with respect to the usual order on $\mathbb{Z}$), which we call the *conductor* of Σ and denote by $C(\Sigma)$.

Notation 2.2. For a non-empty finite subset $M = \{n_1, \dots, n_r\}$ of $\mathbb{N}$ we shall denote by $\langle M \rangle$ or $\langle n_1, \dots, n_r \rangle$ the subsemigroup of $\mathbb{N}$ generated by M and containing 0.

Remark 2.3. For a non-empty finite subset M of $\mathbb{N}$, $\langle M \rangle$ is a numerical semigroup if and only if the greatest common divisor of the elements of M is equal to one.

The following straightforward proposition shows that every numerical semigroup is finitely generated and has a unique minimal (with respect to inclusion) system of generators. These properties are also discussed in [9, pages 106 and 107].

Proposition 2.4. *Let Σ be a numerical semigroup and define $\Sigma^* := \Sigma \setminus \{0\}$. Then $\Sigma^* \setminus (\Sigma^* + \Sigma^*)$ is finite, generates Σ, and is contained in every system of generators of Σ.*

3. Isometric representations of numerical semigroups

An *isometric representation of a numerical semigroup Σ on a Hilbert space $\mathcal{H}$* is a map $V : \Sigma \to B(\mathcal{H})$ such that each V_n is an isometry, and such that $V_{m+n} = V_m V_n$. We say that V has *commuting range projections* if the range projections $V_n V_n^*$ pairwise commute.

Our main examples are of a particular class.

Example 3.1. Let Σ be a numerical semigroup and let A be a non-empty subset of $\mathbb{N}$. Denote by $\{\, e_{A,a} \mid a \in A \,\}$ the standard orthonormal basis for the Hilbert space $\ell^2(A)$. If $\Sigma + A \subset A$, or equivalently $\Sigma + A = A$, then for each $n \in \Sigma$ the set $\{\, e_{A,n+a} \mid a \in A \,\}$ is orthonormal and hence there is an isometry T_n^A on $\ell^2(A)$ such that $T_n^A e_{A,a} = e_{A,n+a}$. Straightforward calculations show that $T_{m+n}^A = T_m^A T_n^A$, and that the range projections pairwise commute. The map $n \mapsto T_n^A$ is then an isometric representation of Σ on $\ell^2(A)$ with commuting range projections, which we denote by T^A.

In particular, $T^{\mathbb{N}}$ and T^{Σ} are isometric representations. Note that if S is the unilateral shift on $\ell^2(\mathbb{N})$ then $T_n^{\mathbb{N}} = S^n$.

Remark 3.2. If Σ and Γ are numerical semigroups, and A is a non-empty subset of $\mathbb{N}$ such that $\Sigma + A = A$ and $\Gamma + A = A$, then we shall denote both of the associated isometric representations of Σ and Γ from Example 3.1 by T^A without reference to the particular numerical semigroup. The context will make clear which representation is intended.

Remark 3.3. Let Σ be a numerical semigroup. The subsemigroup Σ of the additive group $\mathbb{Z}$ is the positive cone for the partial order $\leq$ on $\mathbb{Z}$ defined by $n \leq m$ if and only if $m - n \in \Sigma$. If $\Sigma \neq \mathbb{N}$ then the pair $(\mathbb{Z}, \Sigma)$ is not quasi-lattice ordered in the sense of Nica [7]. Indeed, the subset $\{C(\Sigma)+1, C(\Sigma)+2\}$ of $\mathbb{Z}$ has an upper bound in Σ, however does not have a least upper bound in Σ. So the general theory of [7] and [5] does not apply in our situation.

Consider an isometric representation $V\colon \Sigma \to B(\mathcal{H})$ of a numerical semigroup Σ with commuting range projections. If $s, m \in \Sigma$ and $m - s \in \Sigma$ then $V_s^* V_m = V_{m-s}$ and $V_m^* V_s = V_{m-s}^*$. If $m - s \notin \Sigma$ and $m - s > 0$, however, then we cannot expect to cancel words such as $V_s^* V_m$ in this manner. Therefore words in terms of V may appear quite complicated, however with the assumption of commuting range projections there are certain words that are also pairwise commuting projections.

Lemma 3.4. *Let Σ be a numerical semigroup and $V\colon \Sigma \to B(\mathcal{H})$ be an isometric representation of Σ on a Hilbert space $\mathcal{H}$ with commuting range projections. Then $V_s^* V_m V_m^* V_s$ is a projection for $s, m \in \Sigma$, and commutes with every other such projection.*

Proof. Since $V_s^* V_m V_m^* V_s$ is self-adjoint and

$$\begin{aligned}(V_s^* V_m V_m^* V_s)^2 &= V_s^*(V_m V_m^*)(V_s V_s^*) V_m V_m^* V_s = V_s^* V_s V_s^* V_m V_m^* V_m V_m^* V_s \\ &= V_s^* V_m V_m^* V_s,\end{aligned}$$

$V_s^* V_m V_m^* V_s$ is a projection. Further,

$$\begin{aligned}&(V_s^* V_m V_m^* V_s)(V_t^* V_n V_n^* V_t) \\ &= V_s^* V_m V_m^* V_s V_t^* (V_s^* V_s) V_n V_n^* (V_s^* V_s) V_t = V_s^* (V_m V_m^*)(V_s V_s^*) V_t^* V_{s+n} V_{s+n}^* V_{s+t} \\ &= V_s^* (V_t^* V_t) V_m V_m^* V_t^* V_{s+n} V_{s+n}^* V_{s+t} = V_{t+s}^* (V_{t+m} V_{t+m}^*)(V_{s+n} V_{s+n}^*) V_{s+t} \\ &= V_{t+s}^* V_{s+n} V_{s+n}^* V_{t+m} V_{t+m}^* V_{s+t} (V_s^* V_s) = V_t^* V_n V_n^* V_s^* V_t (V_m V_m^*)(V_s V_s^*) V_s \\ &= V_t^* V_n V_n^* V_s^* V_t V_s V_s^* V_m V_m^* V_s = (V_t^* V_n V_n^* V_t)(V_s^* V_m V_m^* V_s),\end{aligned}$$

so any two such projections commute. □

4. The collection $\mathcal{A}(\Sigma)$

Throughout this section we let Σ be a numerical semigroup with minimal system of generators $\{m_1, \ldots, m_r\}$, where $m_i < m_j$ if $i < j$. In this section we shall associate to Σ a certain collection $\mathcal{A}(\Sigma)$ of subsets of $\mathbb{N}$, such that each $A \in \mathcal{A}(\Sigma)$ gives an isometric representation T^A as in Example 3.1.

Definition 4.1. We define $\mathcal{A}(\Sigma)$ to be the collection of all subsets A of $\mathbb{N}$ such that $\Sigma + A = A$ and $\Sigma \subset A$.

Remark 4.2. If A is a subset of $\mathbb{N}$ such that $\Sigma + A = A$ then the condition $\Sigma \subset A$ is equivalent to $0 \in A$.

Remark 4.3. We trivially have that Σ and $\mathbb{N}$ are in $\mathcal{A}(\Sigma)$. For $A \in \mathcal{A}(\Sigma)$ we can write $A = \Sigma \cup (A \setminus \Sigma)$, so A is determined by the subset $A \setminus \Sigma$ of $\mathbb{N} \setminus \Sigma$. It must be noted, however, that not every subset of $\mathbb{N} \setminus \Sigma$, when adjoined to Σ, will give a set A such that $\Sigma + A = A$. As an example, consider the numerical semigroup $\langle 3, 4\rangle = \mathbb{N} \setminus \{1, 2, 5\}$, for which $\mathbb{N} \setminus \langle 3, 4\rangle = \{1, 2, 5\}$, and define $A := \langle 3, 4\rangle \cup \{2\}$. Then $\{2\}$ is a subset of $\mathbb{N} \setminus \langle 3, 4\rangle$, however $\langle 3, 4\rangle + A = \mathbb{N} \setminus \{1\}$ is not a subset of A since $5 \notin A$.

Theorem 4.4. *$\mathcal{A}(\Sigma)$ is a finite collection such that:*

1. *For each $A \in \mathcal{A}(\Sigma)$ we have an isometric representation T^A;*
2. *Whenever C is a non-empty subset of $\mathbb{N}$ such that $\Sigma + C = C$ there exists $A \in \mathcal{A}(\Sigma)$ such that T^C is unitarily equivalent to T^A;*
3. *If $A, B \in \mathcal{A}(\Sigma)$ and $A \neq B$ then T^A is not unitarily equivalent to T^B.*

Remark 4.5. Consider the equivalence relation on the collection of all non-empty subsets A of $\mathbb{N}$ satisfying $\Sigma + A = A$ such that A is related to B if and only if the isometric representations T^A and T^B are unitarily equivalent. Then, by Theorem 4.4, $\mathcal{A}(\Sigma)$ consists of precisely one element A from each equivalence class, namely the unique element with $\Sigma \subset A$.

Before proving Theorem 4.4 we shall establish some notation and technical results. As motivation, for $A \in \mathcal{A}(\Sigma)$ we decompose $\ell^2(A)$ as a direct sum

$$\bigoplus_{n=0}^{\infty} \big((T_{m_1}^A)^n (T_{m_1}^A)^{*\,n} - (T_{m_1}^A)^{n+1} (T_{m_1}^A)^{*\,n+1}\big)(\ell^2(A)), \tag{4.1}$$

where $(T_{m_1}^A)^n (T_{m_1}^A)^{*\,n} - (T_{m_1}^A)^{n+1} (T_{m_1}^A)^{*\,n+1}$ is the projection of $\ell^2(A)$ onto the orthogonal complement of $(T_{m_1}^A)^{n+1}(\ell^2(A))$ in $(T_{m_1}^A)^n(\ell^2(A))$. For $m \in A$ and $n \in \mathbb{N}$, $\big((T_{m_1}^A)^n (T_{m_1}^A)^{*\,n} - (T_{m_1}^A)^{n+1} (T_{m_1}^A)^{*\,n+1}\big) e_{A,m}$ is equal to $e_{A,m}$ if n is the largest integer such that $m - nm_1 \in A$ (equivalently, $m - nm_1$ is the smallest integer in A which is congruent to m modulo m_1), and is zero otherwise. Denoting by α the smallest positive integer in Σ such that m_1 and α are relatively prime, we apply this analysis to $(T_\alpha^A)^i e_{A,0} = e_{A,i\alpha}$, for $0 \leq i \leq m_1 - 1$, thereby identifying the subspace of the decomposition (4.1) to which the image of $e_{A,0}$ under $(T_\alpha^A)^i$ belongs. These properties will allow us to characterise the representation T^A, and distinguish between distinct representations of this form. This leads us to the following definition.

Definition 4.6. Let α be the smallest positive integer in Σ such that m_1 and α are relatively prime. For each $A \in \mathcal{A}(\Sigma)$ and $0 \leq i \leq m_1 - 1$ let $b_{A,i}$ be the smallest element of A such that $b_{A,i} \equiv i\alpha \pmod{m_1}$, and let $q_{A,i} \in \mathbb{N}$ be the unique integer such that $b_{A,i} = i\alpha - q_{A,i} m_1$.

Remark 4.7. If Σ has a minimal system of generators $\{m_1, m_2\}$ with $m_1 < m_2$ then $\alpha = m_2$, a fact we use in Section 6.

Remark 4.8. For the set $\{\, b_{A,i} \mid 0 \leq i \leq m_1 - 1 \,\}$, the residue classes modulo m_1 of distinct elements are distinct, and the collection of such classes is a partition of $\mathbb{Z}$.

Lemma 4.9. *Let $A \in \mathcal{A}(\Sigma)$. Each $a \in A$ can be written uniquely in the form $a = nm_1 + b_{A,j}$ for some $n \in \mathbb{N}$ and $0 \leq j \leq m_1 - 1$.*

Proof. As $a \equiv j\alpha \pmod{m_1}$ for some $0 \leq j \leq m_1 - 1$ it follows by the definition of $b_{A,j}$ that $a \equiv b_{A,j} \pmod{m_1}$ and $a \geq b_{A,j}$, hence there exists $n \in \mathbb{N}$ such that $a = nm_1 + b_{A,j}$. For uniqueness, if we can also write $a = n' m_1 + b_{A,i}$ with $n' \in \mathbb{N}$ and $0 \leq i \leq m_1 - 1$ then $b_{A,i} \equiv b_{A,j} \pmod{m_1}$, so $i = j$ and the result follows. $\square$

The following proposition gives an indication of the significance of the $b_{A,i}$.

Proposition 4.10. *If* $A \in \mathcal{A}(\Sigma)$ *then* $\ker((T^A_{m_1})^*) = \operatorname{span}\{\, e_{A,b_{A,i}} \mid 0 \le i \le m_1 - 1 \,\}$.

Proof. The inclusion $\operatorname{span}\{\, e_{A,b_{A,i}} \mid 0 \le i \le m_1 - 1 \,\} \subset \ker((T^A_{m_1})^*)$ follows from the observation that if $0 \le i \le m_1 - 1$ then the definition of $b_{A,i}$ as the smallest element of A such that $b_{A,i} \equiv i\alpha \pmod{m_1}$ gives $(T^A_{m_1})^* e_{A,b_{A,i}} = 0$. For the reverse inclusion consider $a \in A$. By Lemma 4.9 there exist $n \in \mathbb{N}$ and $0 \le j \le m_1 - 1$ such that $a = nm_1 + b_{A,j}$. If $n > 0$ then $(T^A_{m_1})^* e_{A,a} = e_{A,(n-1)m_1 + b_{A,j}}$, and if $n = 0$ then $(T^A_{m_1})^* e_{A,a} = (T^A_{m_1})^* e_{A,b_{A,j}} = 0$. Therefore $\ker((T^A_{m_1})^*)$ is contained in $\operatorname{span}\{\, e_{A,b_{A,i}} \mid 0 \le i \le m_1 - 1 \,\}$. ☐

Lemma 4.11. *If* $A, B \in \mathcal{A}(\Sigma)$ *with* $A \ne B$ *then there exists* $1 \le k \le m_1 - 1$ *such that* $b_{A,k} \ne b_{B,j}$ *for all* $0 \le j \le m_1 - 1$.

Proof. Suppose $b_{A,i} \in \{\, b_{B,j} \mid 0 \le j \le m_1 - 1 \,\}$ for $1 \le i \le m_1 - 1$, and note that we then have $\{\, b_{A,i} \mid 0 \le i \le m_1 - 1 \,\} = \{\, b_{B,j} \mid 0 \le j \le m_1 - 1 \,\}$. If $a \in A$ then, by Lemma 4.9, $a = nm_1 + b_{A,i}$ for some $n \in \mathbb{N}$ and $0 \le i \le m_1 - 1$, and since $b_{A,i} = b_{B,j}$ for some $0 \le j \le m_1 - 1$ we have $a = nm_1 + b_{B,j} \in B$. So A is a subset of B. A similar argument shows that B is a subset of A. ☐

Proposition 4.12. *Let* $A \in \mathcal{A}(\Sigma)$. *Define the polynomial* p_A *in* 4 *non-commuting indeterminates over* $\mathbb{C}$ *as*

$$p_A(v, w, x, y) := \textstyle\prod_{i=0}^{m_1-1} y^i v^{q_{A,i}} (1 - vw) w^{q_{A,i}} x^i.$$

Then $p_A(T^A_{m_1}, (T^A_{m_1})^*, T^A_\alpha, (T^A_\alpha)^*)$ *is the projection of* $\ell^2(A)$ *onto* $\operatorname{span}\{e_{A,0}\}$, *and for* $B \in \mathcal{A}(\Sigma)$ *with* $B \ne A$, $p_A(T^B_{m_1}, (T^B_{m_1})^*, T^B_\alpha, (T^B_\alpha)^*)$ *is zero on* $\ell^2(B)$.

Proof. Let $C \in \mathcal{A}(\Sigma)$. For simplicity of notation we shall denote by Q^C_i the factor

$$(T^C_{i\alpha})^* T^C_{q_{A,i} m_1} \big(1 - T^C_{m_1} (T^C_{m_1})^*\big) (T^C_{q_{A,i} m_1})^* T^C_{i\alpha}$$

of $p_A(T^C_{m_1}, (T^C_{m_1})^*, T^C_\alpha, (T^C_\alpha)^*)$, for each $0 \le i \le m_1 - 1$. Recalling that the range projections of T^C pairwise commute, we observe that each Q^C_i is a projection. Further, it follows from Lemma 3.4 that the factors Q^C_i pairwise commute. So $p_A(T^C_{m_1}, (T^C_{m_1})^*, T^C_\alpha, (T^C_\alpha)^*)$ is a product of commuting projections, hence is itself a projection.

For each $0 \le i \le m_1 - 1$ and $c \in C$, recalling that $b_{A,i} = i\alpha - q_{A,i} m_1$, we have

$$Q^C_i e_{C,c} = \begin{cases} e_{C,c} & \text{if } b_{A,i} + c \in C \text{ and } b_{A,i} + c - m_1 \notin C, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

If $c \in C$ and $c > 0$ then, choosing $0 \le k \le m_1 - 1$ with $-c \equiv b_{A,k} \pmod{m_1}$, we have $b_{A,k} + c = qm_1$ for some integer $q \ge 1$, hence both $b_{A,k} + c$ and $b_{A,k} + c - m_1$ are in C, so $Q^C_k e_{C,c} = 0$ by (4.2). It follows that the range of the projection $p_A(T^C_{m_1}, (T^C_{m_1})^*, T^C_\alpha, (T^C_\alpha)^*)$ is contained in $\operatorname{span}\{e_{C,0}\}$.

The definition of $b_{A,i}$ and (4.2) give $p_A(T^A_{m_1}, (T^A_{m_1})^*, T^A_\alpha, (T^A_\alpha)^*) e_{A,0} = e_{A,0}$, so the range of this projection is as claimed.

It remains to show that $p_A(T^B_{m_1}, (T^B_{m_1})^*, T^B_\alpha, (T^B_\alpha)^*)e_{B,0} = 0$ for $B \neq A$. By Lemma 4.11 there exists $1 \leq k \leq m_1-1$ such that $b_{A,k} \neq b_{B,j}$ for all $0 \leq j \leq m_1-1$. If $b_{A,k} \in B$ then by Lemma 4.9 there exist $n \in \mathbb{N}$ and $0 \leq j \leq m_1 - 1$ such that $b_{A,k} = nm_1 + b_{B,j}$. Since $b_{A,k} \neq b_{B,j}$, $n \geq 1$, so $b_{A,k} - m_1 \in B$. In any case, it follows from (4.2) that $Q^B_k e_{B,0} = 0$, hence $p_A(T^B_{m_1}, (T^B_{m_1})^*, T^B_\alpha, (T^B_\alpha)^*) = 0$. □

Proof of Theorem 4.4. For $A \in \mathcal{A}(\Sigma)$ we can write $A = \Sigma \cup (A \setminus \Sigma)$, so A is determined by the subset $A \setminus \Sigma$ of $\mathbb{N} \setminus \Sigma$. Since $\mathbb{N} \setminus \Sigma$ is finite by Proposition 2.1, there can only be finitely many distinct such sets A, hence $\mathcal{A}(\Sigma)$ is a finite collection.

If $A \in \mathcal{A}(\Sigma)$ then, since $\Sigma + A = A$, we have an isometric representation T^A as in Example 3.1.

Let C be a non-empty subset of $\mathbb{N}$ such that $\Sigma + C = C$, let m be the smallest element of C, and define $A := \{ c - m \mid c \in C \}$. Since $\Sigma + C = C$, $\Sigma + A = A$. Further, $0 \in A$, so $\Sigma \subset A$. Hence $A \in \mathcal{A}(\Sigma)$. The representations T^C and T^A are unitarily equivalent since the map $\phi\colon \ell^2(C) \to \ell^2(A)$ determined by $\phi(e_{C,c}) = e_{A,c-m}$ for $c \in C$ is a unitary isomorphism satisfying $\phi T^C_n = T^A_n \phi$ for $n \in \Sigma$.

Finally, let $A, B \in \mathcal{A}(\Sigma)$ with $A \neq B$, and take p_A as in Proposition 4.12. Then $p_A(T^A_{m_1}, (T^A_{m_1})^*, T^A_\alpha, (T^A_\alpha)^*)$ is non-zero, whereas $p_A(T^B_{m_1}, (T^B_{m_1})^*, T^B_\alpha, (T^B_\alpha)^*)$ is zero, and it follows that T^A and T^B are not unitarily equivalent. □

From a practical point of view, one may wish to determine the collection $\mathcal{A}(\Sigma)$, and Proposition 4.13 shows that the sets in $\mathcal{A}(\Sigma)$ can be obtained by considering the subsets of the finite set $\mathbb{N} \setminus \Sigma$.

Proposition 4.13. *Denote by* $\Lambda(\Sigma)$ *the collection of all subsets* Λ *of* $\mathbb{N} \setminus \Sigma$ *such that* $\{m_1, \ldots, m_r\} + \Lambda \subset \Sigma \cup \Lambda$. *The mapping* $\Lambda \mapsto \Sigma \cup \Lambda$ *is a bijection from* $\Lambda(\Sigma)$ *onto* $\mathcal{A}(\Sigma)$, *with inverse mapping* $A \mapsto A \setminus \Sigma$.

Proof. It is enough to show the mappings are well-defined. Let $\Lambda \in \Lambda(\Sigma)$. Since $\{m_1, \ldots, m_r\} + \Lambda \subset \Sigma \cup \Lambda$, $\Sigma + \Lambda \subset \Sigma \cup \Lambda$, so $\Sigma + (\Sigma \cup \Lambda) \subset \Sigma \cup \Lambda$. Further, $\Sigma \subset \Sigma \cup \Lambda$, so we have $\Sigma \cup \Lambda \in \mathcal{A}(\Sigma)$.

For the inverse, if $A \in \mathcal{A}(\Sigma)$ then, since $\Sigma + A \subset A$, $\{m_1, \ldots, m_r\} + (A \setminus \Sigma)$ is a subset of $\Sigma \cup (A \setminus \Sigma)$, so $A \setminus \Sigma \in \Lambda(\Sigma)$. □

5. The decomposition theorem

In this section we state and prove our main theorem. The proof follows easily from the corresponding result for the special case of numerical semigroups with a minimal system of generators consisting of two elements. We also state the special case in this section, however due to the length of its proof we defer the proof until Section 6.

Given isometric representations V and W of a semigroup P on Hilbert spaces $\mathcal{H}_V$ and $\mathcal{H}_W$, we say that V *is a multiple of* W if there are a Hilbert space $\mathcal{K}$ and a unitary isomorphism $U\colon \mathcal{H}_V \to \mathcal{H}_W \otimes \mathcal{K}$ such that $UV_pU^* = W_p \otimes 1$ for $p \in P$.

For the representations T^A of Example 3.1 we can identify the tensor product $\ell^2(A) \otimes \mathcal{K}$ with $\ell^2(A, \mathcal{K})$, and we move freely between these realisations.

We can now state our main theorem.

Theorem 5.1. *Let $V \colon \Sigma \to B(\mathcal{H})$ be an isometric representation of a numerical semigroup Σ on a Hilbert space $\mathcal{H}$ with commuting range projections. Then there is a unique direct-sum decomposition $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A \in \mathcal{A}(\Sigma)} \mathcal{H}_A)$ such that $\mathcal{H}_U$ and $\mathcal{H}_A$ for $A \in \mathcal{A}(\Sigma)$ are reducing for V, such that $V|_{\mathcal{H}_U}$ consists of unitary operators, and such that $V|_{\mathcal{H}_A}$ is a multiple of T^A for $A \in \mathcal{A}(\Sigma)$.*

Proposition 5.2 is the special case of Theorem 5.1 for numerical semigroups with a minimal system of generators consisting of two elements.

Proposition 5.2. *Suppose Σ is a numerical semigroup with a minimal system of generators consisting of two elements, and let $V \colon \Sigma \to B(\mathcal{H})$ be an isometric representation of Σ on a Hilbert space $\mathcal{H}$ with commuting range projections. Then there is a unique direct-sum decomposition $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A \in \mathcal{A}(\Sigma)} \mathcal{H}_A)$ such that $\mathcal{H}_U$ and $\mathcal{H}_A$ for $A \in \mathcal{A}(\Sigma)$ are reducing for V, such that $V|_{\mathcal{H}_U}$ consists of unitary operators, and such that $V|_{\mathcal{H}_A}$ is a multiple of T^A for $A \in \mathcal{A}(\Sigma)$.*

The key to extending Proposition 5.2 to Theorem 5.1 is the next lemma. We first consider a few preliminary remarks for two numerical semigroups Γ and Σ with $\Gamma \subset \Sigma$.

We have $\mathcal{A}(\Sigma) \subset \mathcal{A}(\Gamma)$. Indeed, if $A \in \mathcal{A}(\Sigma)$ then $\Gamma + A \subset \Sigma + A = A$ and $\Gamma \subset \Sigma \subset A$, so $A \in \mathcal{A}(\Gamma)$.

We say that an isometric representation $V \colon \Sigma \to B(\mathcal{H})$ *extends* an isometric representation $W \colon \Gamma \to B(\mathcal{H})$ if $W = V|_\Gamma$.

Recalling our convention from Remark 3.2, for $A \in \mathcal{A}(\Sigma) \subset \mathcal{A}(\Gamma)$ we shall denote both of the associated representations of Σ and Γ from Example 3.1 by T^A.

Lemma 5.3. *Suppose that Γ and Σ are numerical semigroups with $\Gamma \subset \Sigma$, suppose that $W \colon \Gamma \to B(\mathcal{H})$ is an isometric representation on a Hilbert space $\mathcal{H}$ which extends to an isometric representation $V \colon \Sigma \to B(\mathcal{H})$, and suppose that there is a direct-sum decomposition $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A \in \mathcal{A}(\Gamma)} \mathcal{H}_A)$ such that $\mathcal{H}_U$ and $\mathcal{H}_A$ for $A \in \mathcal{A}(\Gamma)$ are reducing for W, such that $W|_{\mathcal{H}_U}$ consists of unitary operators, and such that $W|_{\mathcal{H}_A}$ is a multiple of T^A for $A \in \mathcal{A}(\Gamma)$. Then $\mathcal{H}_A = \{0\}$ for $A \in \mathcal{A}(\Gamma) \setminus \mathcal{A}(\Sigma)$, $\mathcal{H}_U$ and $\mathcal{H}_A$ for $A \in \mathcal{A}(\Sigma)$ are reducing for V, $V|_{\mathcal{H}_U}$ consists of unitary operators, and $V|_{\mathcal{H}_A}$ is a multiple of T^A for $A \in \mathcal{A}(\Sigma)$.*

Proof. Throughout the proof we let $c := C(\Gamma) + 1$, so that $t \in \Gamma \subset \Sigma$ whenever $t \geq c$.

Fix $A \in \mathcal{A}(\Gamma) \setminus \mathcal{A}(\Sigma)$. We show that $\mathcal{H}_A = \{0\}$. Note that $\Sigma + A$ is not a subset of A, for if $\Sigma + A \subset A$ then, since $0 \in A$, $\Sigma \subset A$, and hence $A \in \mathcal{A}(\Sigma)$. Choose $s \in \Sigma$ and $a \in A$ such that $s + a \notin A$. Suppose $\mathcal{H}_A \neq \{0\}$. Then there exist a non-zero Hilbert space $\mathcal{K}$ and a unitary isomorphism $U \colon \mathcal{H}_A \to \ell^2(A) \otimes \mathcal{K}$

such that $W_t|_{\mathcal{H}_A} = U^*(T_t^A \otimes 1)U$ for $t \in \Gamma$. Choose a non-zero vector k in $\mathcal{K}$. Since $s + a \notin A$, $(T_c^A)^* T_{c+s}^A e_{A,a} = (T_c^A)^* e_{A,c+s+a} = 0$, so

$$\begin{aligned} V_s(U^*(e_{A,a} \otimes k)) = V_c^* V_{c+s}(U^*(e_{A,a} \otimes k)) &= W_c^* W_{c+s}(U^*(e_{A,a} \otimes k)) \\ &= U^*((T_c^A)^* T_{c+s}^A \otimes 1)U(U^*(e_{A,a} \otimes k)) \\ &= U^*((T_c^A)^* T_{c+s}^A e_{A,a} \otimes k) = 0. \end{aligned}$$

Since $e_{A,a} \otimes k$ is a non-zero vector in $\ell^2(A) \otimes \mathcal{K}$, this implies that V_s is not an isometry. Therefore we must have $\mathcal{H}_A = \{0\}$.

We now show that the subspaces $\mathcal{H}_U$ and $\mathcal{H}_A$ for $A \in \mathcal{A}(\Sigma)$ reduce V. Indeed, if $m \in \Sigma$ then

$$V_m = V_c^* V_{c+m} = W_c^* W_{c+m}$$

and

$$V_m^* = V_{c+m}^* V_c = W_{c+m}^* W_c,$$

so any subspace of $\mathcal{H}$ that is reducing for W is also reducing for V, and the claim follows.

To see that $V|_{\mathcal{H}_U}$ consists of unitaries, let $m \in \Sigma$ and $h \in \mathcal{H}_U$. Then, as $W|_{\mathcal{H}_U}$ consists of unitaries,

$$V_m V_m^* h = V_c^* V_{c+m} V_{c+m}^* V_c h = W_c^* W_{c+m} W_{c+m}^* W_c h = W_c^* 1_{\mathcal{H}_U} W_c h = h,$$

hence $V_m|_{\mathcal{H}_U}$ is unitary.

It remains to show that, for $A \in \mathcal{A}(\Sigma)$, $V|_{\mathcal{H}_A}$ is a multiple of T^A. Since $A \in \mathcal{A}(\Gamma)$, $W|_{\mathcal{H}_A}$ is a multiple of T^A, so there exist a Hilbert space $\mathcal{K}$ and a unitary isomorphism $U\colon \mathcal{H}_A \to \ell^2(A) \otimes \mathcal{K}$ such that $UW_tU^* = T_t^A \otimes 1$ for $t \in \Gamma$. If $m \in \Sigma$ then

$$\begin{aligned} UV_mU^* = UV_c^*V_{c+m}U^* &= UW_c^*W_{c+m}U^* = (UW_c^*U^*)(UW_{c+m}U^*) \\ &= ((T_c^A)^* \otimes 1)(T_{c+m}^A \otimes 1) = T_m^A \otimes 1. \end{aligned}$$

Therefore $V|_{\mathcal{H}_A}$ is a multiple of T^A. □

Proof of Theorem 5.1. We begin with the case $\Sigma = \mathbb{N}$. Applying the version of the Wold decomposition stated in [8, Proposition 3.2] to V_1, there is a reducing subspace $\mathcal{H}_U$ for V_1 with $V_1|_{\mathcal{H}_U}$ unitary, and, denoting by S the unilateral shift on $\ell^2(\mathbb{N})$, there is a subspace $\mathcal{H}_0$ and a unitary isomorphism $\phi\colon \mathcal{H}_U^\perp \to \ell^2(\mathbb{N}) \otimes \mathcal{H}_0$ such that $\phi V_1 \phi^* = S \otimes 1$. Then, with $\mathcal{H}_\mathbb{N} := \mathcal{H}_U^\perp$ and noting that $\mathcal{A}(\Sigma) = \{\mathbb{N}\}$, we have $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A\in\mathcal{A}(\Sigma)} \mathcal{H}_A)$. Since $V_n = V_1^n$, for $n \in \mathbb{N}$, and $\mathcal{H}_U$ and $\mathcal{H}_\mathbb{N}$ reduce V_1, $\mathcal{H}_U$ and $\mathcal{H}_\mathbb{N}$ reduce V. The representation $V|_{\mathcal{H}_U}$ consists of unitary operators since each $V_n|_{\mathcal{H}_U} = (V_1|_{\mathcal{H}_U})^n$ is unitary. For $n \in \mathbb{N}$, $T_n^\mathbb{N} = S^n$, so $\phi V_n \phi^* = \phi V_1^n \phi^* = S^n \otimes 1 = T_n^\mathbb{N} \otimes 1$, hence $V|_{\mathcal{H}_\mathbb{N}}$ is a multiple of $T^\mathbb{N}$. The decomposition is unique, for if there is another $\mathcal{H} = \mathcal{H}_U^{'} \oplus \mathcal{H}_\mathbb{N}^{'}$ as in the theorem then applying the Wold decomposition to V_1 gives $\mathcal{H}_U^{'} = \mathcal{H}_U$ and $\mathcal{H}_\mathbb{N}^{'} = \mathcal{H}_\mathbb{N}$.

The only numerical semigroup with a minimal system of generators consisting of one element is $\mathbb{N}$, so we now assume for the remainder of the proof that Σ has a minimal system of generators consisting of at least two elements. Let Γ be the numerical semigroup $\langle C(\Sigma)+1, C(\Sigma)+2\rangle$, which is a subsemigroup of Σ, and let $W\colon \Gamma \to B(\mathcal{H})$ be the restriction of V to Γ. Note that the minimal system of generators of Γ is $\{C(\Sigma)+1, C(\Sigma)+2\}$. By Proposition 5.2 there is a unique direct-sum decomposition $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A\in\mathcal{A}(\Gamma)} \mathcal{H}_A)$ such that $\mathcal{H}_U$ and $\mathcal{H}_A$ for $A \in \mathcal{A}(\Gamma)$ are reducing for W, such that $W|_{\mathcal{H}_U}$ consists of unitary operators, and such that $W|_{\mathcal{H}_A}$ is a multiple of T^A for $A \in \mathcal{A}(\Gamma)$. Since W extends to the representation $V\colon \Sigma \to B(\mathcal{H})$, by Lemma 5.3 we have that $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A\in\mathcal{A}(\Sigma)} \mathcal{H}_A)$, $\mathcal{H}_U$ and $\mathcal{H}_A$ for $A \in \mathcal{A}(\Sigma)$ are reducing for V, $V|_{\mathcal{H}_U}$ consists of unitary operators, and $V|_{\mathcal{H}_A}$ is a multiple of T^A for $A \in \mathcal{A}(\Sigma)$.

For uniqueness, suppose $\mathcal{H} = \mathcal{H}_U^{'} \oplus (\bigoplus_{A\in\mathcal{A}(\Sigma)} \mathcal{H}_A^{'})$ is another decomposition such that $\mathcal{H}_U^{'}$ and $\mathcal{H}_A^{'}$ for $A \in \mathcal{A}(\Sigma)$ are reducing for V, such that $V|_{\mathcal{H}_U^{'}}$ consists of unitary operators, and such that $V|_{\mathcal{H}_A^{'}}$ is a multiple of T^A for $A \in \mathcal{A}(\Sigma)$. If we define $\mathcal{H}_A^{'} := \{0\}$ for $A \in \mathcal{A}(\Gamma) \setminus \mathcal{A}(\Sigma)$ then we have another decomposition $\mathcal{H} = \mathcal{H}_U^{'} \oplus (\bigoplus_{A\in\mathcal{A}(\Gamma)} \mathcal{H}_A^{'})$ such that $\mathcal{H}_U^{'}$ and $\mathcal{H}_A^{'}$ for $A \in \mathcal{A}(\Gamma)$ are reducing for W, such that $W|_{\mathcal{H}_U^{'}}$ consists of unitary operators, and such that $W|_{\mathcal{H}_A^{'}}$ is a multiple of T^A for $A \in \mathcal{A}(\Gamma)$, so by the uniqueness in Proposition 5.2 we have $\mathcal{H}_U^{'} = \mathcal{H}_U$ and $\mathcal{H}_A^{'} = \mathcal{H}_A$ for $A \in \mathcal{A}(\Gamma)$, and the required uniqueness follows. $\square$

6. Proof of Proposition 5.2

We now turn to the proof of Proposition 5.2. In this section we shall assume that our numerical semigroup Σ has a minimal system of generators consisting of two elements $\{m_1, m_2\}$ where $m_1 < m_2$ (note that we must have $m_1 \geq 2$). Our basic strategy in proving Proposition 5.2 is analogous to that in proving Theorem 3.1 of [8].

The representation V is determined by the two isometries V_{m_1} and V_{m_2}. We shall apply the version of the Wold decomposition stated in [8, Proposition 3.2] to the isometry V_{m_1}, and analyse the interaction of V_{m_2} with this decomposition. As motivation for our argument, for $A \in \mathcal{A}(\Sigma)$ we apply the Wold decomposition to $T^A_{m_1}$. For such an isometry we have $\mathcal{H}_U = \{0\}$, and it follows from Proposition 4.10 that we have a decomposition of $\mathcal{H}_0 = T^A_{m_1}(\ell^2(A))^{\perp} = \ker((T^A_{m_1})^*)$ as the direct sum, recalling from Definition 4.6 that $b_{A,i} = im_2 - q_{A,i}m_1$,

$$\mathcal{H}_0 = \bigoplus_{i=0}^{m_1-1} \operatorname{span}\{e_{A,b_{A,i}}\} = \bigoplus_{i=0}^{m_1-1} (T^A_{m_1})^{*q_{A,i}} (T^A_{m_2})^i (\operatorname{span}\{e_{A,0}\}). \tag{6.1}$$

By Lemma 4.9 each $a \in A$ can be expressed uniquely in the form $a = nm_1 + b_{A,j}$ for some $n \in \mathbb{N}$ and $0 \leq j \leq m_1 - 1$, so sending $e_{A,a} \mapsto e_{nj}$ gives a unitary isomorphism of $\ell^2(A)$ onto $\ell^2(\mathbb{N} \times \{0, \ldots, m_1 - 1\})$ which carries T^A into the isometric representation determined on $f \in \ell^2(\mathbb{N} \times \{0, \ldots, m_1 - 1\})$ by

$$(T^A_{m_1} f)_{nj} = \begin{cases} 0 & \text{if } n = 0, \\ f_{(n-1)j} & \text{if } n \geq 1 \end{cases} \tag{6.2}$$

and

$$(T^A_{m_2} f)_{nj} = \begin{cases} f_{(n-q_{A,j}+q_{A,j-1})(j-1)} & \text{if } n \geq q_{A,j} - q_{A,j-1} \text{ and } j \geq 1, \\ 0 & \text{if } n < q_{A,j} - q_{A,j-1} \text{ and } j \geq 1, \\ f_{(n-m_2+q_{A,m_1-1})(m_1-1)} & \text{if } n \geq m_2 - q_{A,m_1-1} \text{ and } j = 0, \\ 0 & \text{if } n < m_2 - q_{A,m_1-1} \text{ and } j = 0, \end{cases} \tag{6.3}$$

where we have used that $q_{A,m_1-1} \leq m_2$, and $q_{A,j-1} \leq q_{A,j}$ for $1 \leq j \leq m_1 - 1$ (we do not prove these two results here, but note that they follow from the more general results in Lemmas 6.7 and 6.8). Note that, with respect to the unitary isomorphism sending $e_{A,a} \mapsto e_{nj}$,

$$\operatorname{span}\{\, e_{nj} \mid 0 \leq j \leq m_1 - 1 \,\}$$

is the subspace of $\ell^2(\mathbb{N} \times \{0, \ldots, m_1 - 1\})$ corresponding to the subspace

$$\big((T^A_{m_1})^n (T^A_{m_1})^{*\,n} - (T^A_{m_1})^{n+1} (T^A_{m_1})^{*\,n+1}\big)(\ell^2(A))$$

of $\ell^2(A)$ in the decomposition (4.1). Viewing the representation T^A in the manner described by (6.2) and (6.3) is rather instructive.

Example 6.1. Consider the numerical semigroup $\langle 3, 4\rangle = \mathbb{N} \backslash \{1, 2, 5\}$. The collection $\mathcal{A}(\langle 3, 4\rangle)$ consists of the five sets $A_1 := \langle 3, 4\rangle$, $A_2 := \mathbb{N}$, $A_3 := \langle 3, 4\rangle \cup \{1, 5\}$, $A_4 := \langle 3, 4\rangle \cup \{2, 5\}$ and $A_5 := \langle 3, 4\rangle \cup \{5\}$. For

$$f = \begin{pmatrix} f_{02} & f_{12} & f_{22} & f_{32} & \cdots \\ f_{01} & f_{11} & f_{21} & f_{31} & \cdots \\ f_{00} & f_{10} & f_{20} & f_{30} & \cdots \end{pmatrix} \in \ell^2(\mathbb{N} \times \{0, 1, 2\}),$$

we have from (6.2) that

$$T^{A_i}_3 f = \begin{pmatrix} 0 & f_{02} & f_{12} & f_{22} & \cdots \\ 0 & f_{01} & f_{11} & f_{21} & \cdots \\ 0 & f_{00} & f_{10} & f_{20} & \cdots \end{pmatrix}$$

for $1 \leq i \leq 5$, and from (6.3) that

$$T^{A_1}_4 f = \begin{pmatrix} f_{01} & f_{11} & f_{21} & f_{31} & f_{41} & f_{51} & \cdots \\ f_{00} & f_{10} & f_{20} & f_{30} & f_{40} & f_{50} & \cdots \\ 0 & 0 & 0 & 0 & f_{02} & f_{12} & \cdots \end{pmatrix},$$

$$T^{A_2}_4 f = \begin{pmatrix} 0 & f_{01} & f_{11} & f_{21} & \cdots \\ 0 & f_{00} & f_{10} & f_{20} & \cdots \\ 0 & 0 & f_{02} & f_{12} & \cdots \end{pmatrix}, \quad T^{A_3}_4 f = \begin{pmatrix} f_{01} & f_{11} & f_{21} & f_{31} & f_{41} & \cdots \\ 0 & f_{00} & f_{10} & f_{20} & f_{30} & \cdots \\ 0 & 0 & 0 & f_{02} & f_{12} & \cdots \end{pmatrix},$$

$$T^{A_4}_4 f = \begin{pmatrix} 0 & 0 & f_{01} & f_{11} & \cdots \\ f_{00} & f_{10} & f_{20} & f_{30} & \cdots \\ 0 & 0 & f_{02} & f_{12} & \cdots \end{pmatrix}$$

and

$$T^{A_5}_4 f = \begin{pmatrix} 0 & f_{01} & f_{11} & f_{21} & f_{31} & \cdots \\ f_{00} & f_{10} & f_{20} & f_{30} & f_{40} & \cdots \\ 0 & 0 & 0 & f_{02} & f_{12} & \cdots \end{pmatrix}.$$

We now begin the proof of Proposition 5.2. Applying the Wold decomposition to the isometry V_{m_1} gives a reducing subspace

$$\mathcal{H}_U := \bigcap_{n=0}^{\infty} V_{m_1}^n(\mathcal{H})$$

such that $V_{m_1}|_{\mathcal{H}_U}$ is unitary, and, since $V_{m_2}^{m_1} = V_{m_1}^{m_2}$, the isometry V_{m_2} and every other $V_{nm_1+pm_2} = V_{m_1}^n V_{m_2}^p$ are also unitary on $\mathcal{H}_U$. The Wold decomposition also says that the complement $\mathcal{H}_U^\perp$ of $\mathcal{H}_U$ in $\mathcal{H}$ can be identified with $\ell^2(\mathbb{N}, \mathcal{H}_0)$ for

$$\mathcal{H}_0 := V_{m_1}(\mathcal{H})^\perp = \ker V_{m_1} V_{m_1}^*.$$

Our goal is to decompose $\mathcal{H}_0$ into subspaces where V behaves like each T^A, and we begin by identifying the subspaces $\mathcal{K}_A$, for $A \in \mathcal{A}(\Sigma)$, of $\mathcal{H}_0$ consisting of vectors which behave under $V_{m_2}^i$ as the vector $e_{00} \in \ell^2(\mathbb{N} \times \{0, \ldots, m_1 - 1\})$ does under $(T_{m_2}^A)^i$ for $1 \le i \le m_1 - 1$. The crucial observation that we make is that $(T_{m_2}^A)^i e_{00} = e_{q_{A,i} i} = (T_{m_1}^A)^{q_{A,i}} e_{0i}$ belongs to the range of $(T_{m_1}^A)^{q_{A,i}}$ and is orthogonal to the range of $(T_{m_1}^A)^{q_{A,i}+1}$.

With these observations in mind, we define for $A \in \mathcal{A}(\Sigma)$

$$\mathcal{K}_A := \{\, h \in \mathcal{H}_0 \mid V_{m_2}^i h \in V_{m_1}^{q_{A,i}}(\mathcal{H}) \ominus V_{m_1}^{q_{A,i}+1}(\mathcal{H}) \text{ for } 1 \le i \le m_1 - 1 \,\}.$$

With the notation $P_n := V_{m_1}^n V_{m_1}^{*n} - V_{m_1}^{n+1} V_{m_1}^{*n+1}$, which is the projection of $\mathcal{H}$ onto the orthogonal complement of $V_{m_1}^{n+1}(\mathcal{H})$ in $V_{m_1}^n(\mathcal{H})$, we may write

$$\mathcal{K}_A = \{\, h \in \mathcal{H}_0 \mid V_{m_2}^i h \in P_{q_{A,i}}(\mathcal{H}) \text{ for } 1 \le i \le m_1 - 1 \,\}$$

and $\mathcal{H}_0 = P_0(\mathcal{H})$.

In order to decompose $\mathcal{H}_0$ into subspaces where the representation V behaves like each T^A we shall determine the projections of $\mathcal{H}$ onto the subspaces of $\mathcal{H}_0$ corresponding to the subspaces $\operatorname{span}\{e_{A,b_{A,i}}\} = (T_{m_1}^A)^{*q_{A,i}} (T_{m_2}^A)^i (\operatorname{span}\{e_{A,0}\})$ of the direct-sum decompositions in (6.1), observing that the subspace $\mathcal{K}_A$ of $\mathcal{H}_0$ corresponds to the subspace $\operatorname{span}\{e_{A,0}\}$ of (6.1). Recall that, for $m \in A$ and $n \in \mathbb{N}$, $\big((T_{m_1}^A)^n (T_{m_1}^A)^{*n} - (T_{m_1}^A)^{n+1} (T_{m_1}^A)^{*n+1}\big) e_{A,m}$ is equal to $e_{A,m}$ if n is the largest integer such that $m - nm_1 \in A$, and is zero otherwise. Applying this analysis to $(T_{m_2}^A)^j e_{A,b_{A,i}} = e_{A, jm_2 + b_{A,i}}$, for $0 \le j \le m_1 - 1$, identifies the subspace of the decomposition (4.1) to which the image of $e_{A,b_{A,i}}$ under $(T_{m_2}^A)^j$ belongs. This leads us to the following definition.

Definition 6.2. For each $A \in \mathcal{A}(\Sigma)$ and $0 \le i, j \le m_1 - 1$ define $r_{A,i,j} \in \mathbb{N}$ to be the unique integer such that $jm_2 + b_{A,i} - r_{A,i,j} m_1$ is the smallest integer in the set $\{\, jm_2 + b_{A,i} - rm_1 \mid r \in \mathbb{N} \,\} \cap A$.

Remark 6.3. For $0 \le j \le m_1 - 1$ we have $r_{A,0,j} = q_{A,j}$. Indeed, if $i = 0$ then, since $b_{A,0} = 0$, we have that $jm_2 + b_{A,i} - r_{A,i,j} m_1 = jm_2 - r_{A,0,j} m_1$ is the smallest integer in the set $\{\, jm_2 - rm_1 \mid r \in \mathbb{N} \,\} \cap A$, hence by the definition of $b_{A,j}$ it follows that $b_{A,j} = jm_2 - r_{A,0,j} m_1$, so $r_{A,0,j} = q_{A,j}$.

The following lemma provides a useful expression for $r_{A,i,j}$.

Lemma 6.4. *Fix $A \in \mathcal{A}(\Sigma)$ and $0 \le i, j \le m_1 - 1$. Choose $0 \le k \le m_1 - 1$ such that $i + j \equiv k \pmod{m_1}$. Then $r_{A,i,j} = q_{A,k} - q_{A,i} + (m_2/m_1)(i + j - k)$.*

Proof. For simplicity of notation define $s := q_{A,k} - q_{A,i} + (m_2/m_1)(i+j-k)$. Since the definition of $r_{A,i,j}$ implies $jm_2 + b_{A,i} - r_{A,i,j}m_1 \in A$, we have

$$b_{A,k} + (s - r_{A,i,j})m_1 = jm_2 + b_{A,i} - r_{A,i,j}m_1 \in A.$$

So, as $b_{A,k}$ is the smallest element of A such that $b_{A,k} \equiv km_2 \pmod{m_1}$, we must have $r_{A,i,j} \le s$. In particular, $s \ge 0$. Further, $jm_2 + b_{A,i} - sm_1 = b_{A,k} \in A$, so by the definition of $r_{A,i,j}$ we must have $r_{A,i,j} = s$. □

We are now in a position to describe the projections of $\mathcal{H}$ onto the relevant subspaces of $\mathcal{H}_0$. Note that, for $s \in \Sigma$ and $m \in \mathbb{N}$, $V_s^* P_m V_s$ is self-adjoint and

$$(V_s^* P_m V_s)^2 = V_s^* P_m (V_s V_s^*) P_m V_s = V_s^* (V_s V_s^*) P_m P_m V_s = V_s^* P_m V_s,$$

so $V_s^* P_m V_s$ is a projection. Furthermore, it follows from Lemma 3.4 that such projections pairwise commute.

Proposition 6.5. *Let $A \in \mathcal{A}(\Sigma)$ and $0 \le i \le m_1 - 1$. Then $\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{r_{A,i,j}} V_{m_2}^j)$ is the projection of $\mathcal{H}$ onto $V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A)$.*

Proof. Each $\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{r_{A,i,j}} V_{m_2}^j)$ is a product of commuting projections and hence is a projection, so it suffices to show that the ranges of these projections are as claimed.

We begin with the case $i = 0$. For $0 \le j \le m_1 - 1$, $V_{m_2}^j(\mathcal{K}_A) \subset P_{q_{A,j}}(\mathcal{H})$ by the definition of $\mathcal{K}_A$, hence $V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j$ is the identity on $\mathcal{K}_A$. It follows that

$$\mathcal{K}_A = \prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)(\mathcal{K}_A) \subset \prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)(\mathcal{H}).$$

For the reverse inclusion we show that the elements of $\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)(\mathcal{H})$ satisfy the defining conditions of $\mathcal{K}_A$. Indeed,

$$\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)(\mathcal{H}) = P_0 \prod_{j=1}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)(\mathcal{H}) \subset \mathcal{H}_0,$$

and, for $1 \le k \le m_1 - 1$,

$$\begin{aligned} P_{q_{A,k}}\Big(V_{m_2}^k\big(\textstyle\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)\big)\Big) &= P_{q_{A,k}} V_{m_2}^k (V_{m_2}^{*k} V_{m_2}^k) \textstyle\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j) \\ &= P_{q_{A,k}} (V_{m_2}^k V_{m_2}^{*k}) V_{m_2}^k \textstyle\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j) \\ &= V_{m_2}^k \big(V_{m_2}^{*k} P_{q_{A,k}} V_{m_2}^k \textstyle\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)\big) \\ &= V_{m_2}^k \textstyle\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j), \end{aligned}$$

so the range of $\prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)$ is contained in $\mathcal{K}_A$.

Suppose now that $i \ge 1$, and consider the partial isometry $V_{m_1}^{*q_{A,i}} V_{m_2}^i$. Since

$$\mathcal{K}_A = \prod_{j=0}^{m_1-1}(V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^j)(\mathcal{H})$$

and

$$
\begin{aligned}
&(V_{m_1}^{*q_{A,i}} V_{m_2}^{i})^* V_{m_1}^{*q_{A,i}} V_{m_2}^{i} \big(\textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j})\big) \\
&= V_{m_2}^{*i} V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}} V_{m_2}^{i} \big(V_{m_2}^{*i} P_{q_{A,i}} V_{m_2}^{i} \textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j})\big) \\
&= (V_{m_2}^{*i} V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}} (V_{m_2}^{i} V_{m_2}^{*i}) P_{q_{A,i}} V_{m_2}^{i}) \textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j}) \\
&= V_{m_2}^{*i} (V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}} P_{q_{A,i}}) V_{m_2}^{i} \textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j}) \\
&= V_{m_2}^{*i} P_{q_{A,i}} V_{m_2}^{i} \textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j}) \\
&= \textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j}),
\end{aligned}
$$

$(V_{m_1}^{*q_{A,i}} V_{m_2}^{i})^* V_{m_1}^{*q_{A,i}} V_{m_2}^{i}$ is the identity on $\mathcal{K}_A$. So $\mathcal{K}_A$ is a closed subspace of the range of $(V_{m_1}^{*q_{A,i}} V_{m_2}^{i})^* V_{m_1}^{*q_{A,i}} V_{m_2}^{i}$, and hence by [8, Lemma 3.4] the projection of $\mathcal{H}$ onto $V_{m_1}^{*q_{A,i}} V_{m_2}^{i}(\mathcal{K}_A)$ is

$$
V_{m_1}^{*q_{A,i}} V_{m_2}^{i} \big(\textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j})\big) V_{m_2}^{*i} V_{m_1}^{q_{A,i}}. \tag{6.4}
$$

Now, for a fixed $0 \leq j \leq m_1 - 1$ choose $0 \leq k \leq m_1 - 1$ with $k \equiv j - i \pmod{m_1}$, then $i + k \equiv j \pmod{m_1}$ and $r_{A,i,k} = q_{A,j} - q_{A,i} + (m_2/m_1)(i + k - j)$ by Lemma 6.4, so

$$
\begin{aligned}
&V_{m_1}^{*q_{A,i}} V_{m_2}^{i} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j}) \\
&= V_{m_1}^{*q_{A,i}} (V_{m_2}^{*j} V_{m_2}^{j}) V_{m_2}^{i} V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j} = V_{m_1}^{*q_{A,i}} V_{m_2}^{*j} V_{m_2}^{i} (V_{m_2}^{j} V_{m_2}^{*j}) P_{q_{A,j}} V_{m_2}^{j} \\
&= V_{m_1}^{*q_{A,i}} V_{m_2}^{*j} V_{m_2}^{i} P_{q_{A,j}} V_{m_2}^{j} = (V_{m_1}^{*q_{A,i}} V_{m_1}^{q_{A,i}}) V_{m_1}^{*q_{A,i}} V_{m_2}^{*j} V_{m_2}^{i} P_{q_{A,j}} (V_{m_2}^{*i} V_{m_2}^{i}) V_{m_2}^{j} \\
&= V_{m_1}^{*q_{A,i}} (V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}}) (V_{m_2}^{*j} V_{m_2}^{i} P_{q_{A,j}} V_{m_2}^{*i} V_{m_2}^{j}) V_{m_2}^{i} \\
&= V_{m_1}^{*q_{A,i}} (V_{m_2}^{*j} V_{m_2}^{i} P_{q_{A,j}} V_{m_2}^{*i} V_{m_2}^{j}) (V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}}) V_{m_2}^{i} \\
&= V_{m_1}^{*q_{A,i}} V_{m_2}^{*j} (V_{m_2}^{*k} V_{m_2}^{k}) V_{m_2}^{i} P_{q_{A,j}} V_{m_2}^{*i} (V_{m_2}^{*k} V_{m_2}^{k}) V_{m_2}^{j} V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}} V_{m_2}^{i} \\
&= V_{m_2}^{*k} V_{m_1}^{*q_{A,i}} V_{m_2}^{*j} V_{m_2}^{k} V_{m_2}^{i} P_{q_{A,j}} V_{m_2}^{*i} V_{m_2}^{*k} V_{m_2}^{j} V_{m_1}^{q_{A,i}} V_{m_2}^{k} V_{m_1}^{*q_{A,i}} V_{m_2}^{i} \\
&= V_{m_2}^{*k} (V_{m_1}^{*q_{A,i}} V_{m_2}^{*j} V_{m_2}^{k} V_{m_2}^{i} V_{m_1}^{q_{A,j}}) P_0 (V_{m_1}^{*q_{A,j}} V_{m_2}^{*i} V_{m_2}^{*k} V_{m_2}^{j} V_{m_1}^{q_{A,i}}) V_{m_2}^{k} V_{m_1}^{*q_{A,i}} V_{m_2}^{i} \\
&= V_{m_2}^{*k} V_{m_1}^{r_{A,i,k}} P_0 V_{m_1}^{*r_{A,i,k}} V_{m_2}^{k} V_{m_1}^{*q_{A,i}} V_{m_2}^{i} \\
&= (V_{m_2}^{*k} P_{r_{A,i,k}} V_{m_2}^{k}) V_{m_1}^{*q_{A,i}} V_{m_2}^{i}.
\end{aligned}
$$

Applying this result, for each $0 \leq j \leq m_1 - 1$, the projection (6.4) becomes

$$
\big(\textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{r_{A,i,j}} V_{m_2}^{j})\big) V_{m_1}^{*q_{A,i}} V_{m_2}^{i} V_{m_2}^{*i} V_{m_1}^{q_{A,i}}. \tag{6.5}
$$

By Lemma 6.4, $r_{A,i,m_1-i} = m_2 - q_{A,i}$, so

$$
\begin{aligned}
&(V_{m_2}^{*m_1-i} P_{r_{A,i,m_1-i}} V_{m_2}^{m_1-i}) V_{m_1}^{*q_{A,i}} V_{m_2}^{i} V_{m_2}^{*i} V_{m_1}^{q_{A,i}} \\
&= V_{m_2}^{*m_1-i} P_{r_{A,i,m_1-i}} V_{m_2}^{m_1-i} V_{m_1}^{*q_{A,i}} (V_{m_2}^{*m_1-i} V_{m_2}^{m_1-i}) V_{m_2}^{i} V_{m_2}^{*i} (V_{m_2}^{*m_1-i} V_{m_2}^{m_1-i}) V_{m_1}^{q_{A,i}} \\
&= V_{m_2}^{*m_1-i} P_{r_{A,i,m_1-i}} V_{m_2}^{m_1-i} V_{m_2}^{*m_1-i} (V_{m_1}^{*q_{A,i}} V_{m_2}^{m_1} V_{m_2}^{*m_1} V_{m_1}^{q_{A,i}}) V_{m_2}^{m_1-i} \\
&= V_{m_2}^{*m_1-i} P_{r_{A,i,m_1-i}} (V_{m_2}^{m_1-i} V_{m_2}^{*m_1-i}) V_{m_1}^{r_{A,i,m_1-i}} V_{m_1}^{*r_{A,i,m_1-i}} V_{m_2}^{m_1-i} \\
&= V_{m_2}^{*m_1-i} (P_{r_{A,i,m_1-i}} V_{m_1}^{r_{A,i,m_1-i}} V_{m_1}^{*r_{A,i,m_1-i}}) V_{m_2}^{m_1-i} \\
&= V_{m_2}^{*m_1-i} P_{r_{A,i,m_1-i}} V_{m_2}^{m_1-i},
\end{aligned}
$$

and then (6.5) becomes

$$
\begin{aligned}
&\Big(\big(\textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{r_{A,i,j}} V_{m_2}^{j})\big) (V_{m_2}^{*m_1-i} P_{r_{A,i,m_1-i}} V_{m_2}^{m_1-i}) \Big) V_{m_1}^{*q_{A,i}} V_{m_2}^{i} V_{m_2}^{*i} V_{m_1}^{q_{A,i}} \\
&= \big(\textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{r_{A,i,j}} V_{m_2}^{j})\big) (V_{m_2}^{*m_1-i} P_{r_{A,i,m_1-i}} V_{m_2}^{m_1-i}) \\
&= \textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{r_{A,i,j}} V_{m_2}^{j}),
\end{aligned}
$$

which is the projection of $\mathcal{H}$ onto $V_{m_1}^{*q_{A,i}} V_{m_2}^{i}(\mathcal{K}_A)$ as claimed. □

Remark 6.6. Let $A \in \mathcal{A}(\Sigma)$, and recall that p_A is the polynomial as defined in Proposition 4.12. Then, setting $i = 0$ in Proposition 6.5, we see that

$$p_A(V_{m_1}, V_{m_1}^*, V_{m_2}, V_{m_2}^*) = \textstyle\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{q_{A,j}} V_{m_2}^{j})$$

is the projection of $\mathcal{H}$ onto $\mathcal{K}_A$.

Our next step is the decomposition of $\mathcal{H}_0$. For this we need some technical lemmas.

Lemma 6.7. *Let $A \in \mathcal{A}(\Sigma)$ and $0 \le i, j \le m_1 - 1$. Then $r_{A,i,j} \le m_2$.*

Proof. Since $j < m_1$, and the definition of $r_{A,i,j}$ implies $jm_2 + b_{A,i} - r_{A,i,j} m_1 \in A$, we have

$$b_{A,i} + (m_2 - r_{A,i,j}) m_1 = (m_1 - j) m_2 + (j m_2 + b_{A,i} - r_{A,i,j} m_1) \in A.$$

Thus, since $b_{A,i}$ is the smallest element of A such that $b_{A,i} \equiv i m_2 \pmod{m_1}$, we must have $r_{A,i,j} \le m_2$. □

Lemma 6.8. *Let $A \in \mathcal{A}(\Sigma)$ and $0 \le i \le m_1 - 1$. If $0 \le j < k \le m_1 - 1$ then $r_{A,i,j} \le r_{A,i,k}$.*

Proof. Since the definition of $r_{A,i,j}$ implies $jm_2 + b_{A,i} - r_{A,i,j} m_1 \in A$, we have

$$k m_2 + b_{A,i} - r_{A,i,j} m_1 = (k - j) m_2 + (j m_2 + b_{A,i} - r_{A,i,j} m_1) \in A,$$

so the definition of $r_{A,i,k}$ implies $r_{A,i,j} \le r_{A,i,k}$. □

Lemma 6.9. *Let $A \in \mathcal{A}(\Sigma)$ and $0 \le i, j \le m_1 - 1$. Choose $0 \le k \le m_1 - 1$ such that $i + j \equiv k \pmod{m_1}$. Then $j m_2 + b_{A,i} - r_{A,i,j} m_1 = b_{A,k}$.*

Proof. Using Lemma 6.4 we have

$$\begin{aligned} jm_2 + b_{A,i} - r_{A,i,j}m_1 &= jm_2 + b_{A,i} - (q_{A,k} - q_{A,i} + (m_2/m_1)(i + j - k))m_1 \\ &= km_2 - q_{A,k}m_1 = b_{A,k}. \end{aligned}$$ □

Lemma 6.10. *For integers $0 \leq p_j \leq m_2$, $1 \leq j \leq m_1 - 1$, where $p_j \leq p_k$ whenever $j < k$, there exist unique $A \in \mathcal{A}(\Sigma)$ and $0 \leq i \leq m_1 - 1$ such that $p_j = r_{A,i,j}$ for $1 \leq j \leq m_1 - 1$.*

Proof. Our aim is to find $A \in \mathcal{A}(\Sigma)$ and $0 \leq i \leq m_1 - 1$ so that, for $1 \leq j \leq m_1 - 1$, $jm_2 + b_{A,i} - p_j m_1$ is the smallest integer in the set $\{ jm_2 + b_{A,i} - rm_1 \mid r \in \mathbb{N} \} \cap A$, and then the definition of $r_{A,i,j}$ will imply that $p_j = r_{A,i,j}$ as required. Our strategy is to use the integers p_j to find the integer $b_{A,i}$, then construct A, and then find i.

To motivate the proof, fix $A \in \mathcal{A}(\Sigma)$ and $0 \leq i \leq m_1 - 1$. If $0 \leq j \leq m_1 - 1$ then, choosing $0 \leq k \leq m_1 - 1$ with $i + j \equiv k \pmod{m_1}$, $jm_2 + b_{A,i} - r_{A,i,j}m_1 = b_{A,k}$ by Lemma 6.9. In particular, $jm_2 + b_{A,i} - r_{A,i,j}m_1 \geq 0$ for $0 \leq j \leq m_1 - 1$. Moreover, if $1 \leq i \leq m_1 - 1$ then $(m_1 - i)m_2 + b_{A,i} - r_{A,i,m_1-i}m_1 = b_{A,0} = 0$, and if $i = 0$ then trivially $b_{A,i} = 0$. It follows that $b_{A,i} = -\min\{ jm_2 - r_{A,i,j}m_1 \mid 0 \leq j \leq m_1 - 1 \}$.

With this motivation we begin by constructing the required $A \in \mathcal{A}(\Sigma)$. Define $p_0 := 0$ and define $c_0 := -\min\{ jm_2 - p_j m_1 \mid 0 \leq j \leq m_1 - 1 \}$. For $1 \leq j \leq m_1 - 1$ define $c_j := jm_2 + c_0 - p_j m_1$. Now, for $0 \leq j \leq m_1 - 1$ we have by the definition of c_0 that $-c_0 \leq jm_2 - p_j m_1$, so $c_j = jm_2 + c_0 - p_j m_1 \geq 0$. Further, if $c_0 \neq 0$ then $c_0 = -(km_2 - p_k m_1)$ for some $1 \leq k \leq m_1 - 1$, so $c_k = km_2 + c_0 - p_k m_1 = 0$, thus in any case some element of $\{ c_j \mid 0 \leq j \leq m_1 - 1 \}$ is equal to zero. We now define $A := \{ c_j + nm_1 \mid 0 \leq j \leq m_1 - 1 \text{ and } n \in \mathbb{N} \}$. We will show that $A \in \mathcal{A}(\Sigma)$. Since $c_j \geq 0$ for $0 \leq j \leq m_1 - 1$ we have $A \subset \mathbb{N}$. Note that if we have $\Sigma + A = A$ then, since $0 \in A$, $\Sigma \subset A$. So it suffices to show $\Sigma + A = A$, or equivalently $\Sigma + A \subset A$, and, since $\{m_1, m_2\}$ generates Σ, the definition of A shows that it is enough to prove that $m_2 + c_j \in A$ for $0 \leq j \leq m_1 - 1$. We consider two cases. If $0 \leq j \leq m_1 - 2$, then

$$\begin{aligned} m_2 + c_j &= m_2 + (jm_2 + c_0 - p_j m_1) = (j + 1)m_2 + c_0 - p_j m_1 \\ &= c_{j+1} + (p_{j+1} - p_j)m_1, \end{aligned}$$

which is in A as $p_{j+1} \geq p_j$. If $j = m_1 - 1$ then

$$m_2 + c_j = m_2 + (jm_2 + c_0 - p_j m_1) = c_0 + (m_2 - p_{m_1-1})m_1,$$

which is in A as $m_2 \geq p_{m_1-1}$.

We now want to show that there exists $0 \leq i \leq m_1 - 1$ such that $p_j = r_{A,i,j}$ for $1 \leq j \leq m_1 - 1$. Note that if $c_j \equiv c_k \pmod{m_1}$ for some $0 \leq j, k \leq m_1 - 1$ then, since $c_j - c_k = (j - k)m_2 - (p_j - p_k)m_1$, $m_1 \mid (j - k)m_2$, so as m_1 and m_2 are relatively prime we must have $j = k$. It follows that the smallest element of A which is congruent to c_j modulo m_1 is c_j itself, for each $0 \leq j \leq m_1 - 1$. So, choosing $0 \leq i \leq m_1 - 1$ such that $c_0 \equiv im_2 \pmod{m_1}$, c_0 is the smallest element of A which is congruent to im_2 modulo m_1, hence by the definition of $b_{A,i}$ we have $c_0 = b_{A,i}$. Moreover, for $1 \leq j \leq m_1 - 1$ we have that $c_j = jm_2 + b_{A,i} - p_j m_1$

is the smallest integer in the set $\{ jm_2 + b_{A,i} - rm_1 \mid r \in \mathbb{N} \} \cap A$, hence, by the definition of $r_{A,i,j}$, we have $p_j = r_{A,i,j}$.

For uniqueness, suppose there also exist $B \in \mathcal{A}(\Sigma)$ and $0 \le k \le m_1 - 1$ such that $p_j = r_{B,k,j}$ for $1 \le j \le m_1 - 1$. We may assume without loss of generality that $b_{B,k} \ge b_{A,i}$. We will show that we must have $b_{B,k} = b_{A,i}$. If $k = 0$ then $b_{A,i} \le b_{B,k} = 0$, so $b_{A,i} = 0 = b_{B,k}$. Suppose $k \ge 1$. Then, by Lemma 6.9,

$$(m_1 - k)m_2 + b_{B,k} - r_{B,k,m_1-k}m_1 = b_{B,0} = 0,$$

and, choosing $0 \le t \le m_1 - 1$ such that $i + (m_1 - k) \equiv t \pmod{m_1}$, Lemma 6.9 also gives

$$(m_1 - k)m_2 + b_{A,i} - r_{A,i,m_1-k}m_1 = b_{A,t},$$

hence

$$(m_1 - k)m_2 + b_{B,k} - r_{B,k,m_1-k}m_1 = 0 \le b_{A,t} = (m_1 - k)m_2 + b_{A,i} - r_{A,i,m_1-k}m_1,$$

which implies, since $r_{B,k,m_1-k} = r_{A,i,m_1-k}$, that $b_{B,k} \le b_{A,i}$. Hence $b_{B,k} = b_{A,i}$.

We now show that $A = B$. Let $0 \le j \le m_1 - 1$, and choose $0 \le s \le m_1 - 1$ such that $s \equiv j - k \pmod{m_1}$. Then $s + k \equiv j \pmod{m_1}$, and by Lemma 6.9 we have $sm_2 + b_{B,k} - r_{B,k,s}m_1 = b_{B,j}$. Further, choosing $0 \le r \le m_1 - 1$ such that $i + s \equiv r \pmod{m_1}$ we have $sm_2 + b_{A,i} - r_{A,i,s}m_1 = b_{A,r}$ by Lemma 6.9. Therefore,

$$b_{B,j} = sm_2 + b_{B,k} - r_{B,k,s}m_1 = sm_2 + b_{A,i} - r_{A,i,s}m_1 = b_{A,r}.$$

It follows that $\{ b_{B,j} \mid 0 \le j \le m_1 - 1 \} \subset \{ b_{A,r} \mid 0 \le r \le m_1 - 1 \}$. So Lemma 4.11 implies that $A = B$.

Finally, since $A = B$, we have $b_{A,i} = b_{A,k}$, which implies that $m_1 \mid (i - k)m_2$, so as m_1 and m_2 are relatively prime we must have $i = k$. □

Proposition 6.11. *There is a direct-sum decomposition*

$$\mathcal{H}_0 = \bigoplus_{A \in \mathcal{A}(\Sigma)} \big(\bigoplus_{i=0}^{m_1-1} V_{m_1}^{*q_{A,i}} V_{m_2}^{i}(\mathcal{K}_A) \big). \tag{6.6}$$

Remark 6.12. The projection of $\mathcal{H}$ onto the summand $V_{m_1}^{*q_{A,i}} V_{m_2}^{i}(\mathcal{K}_A)$ of (6.6) is $\prod_{j=0}^{m_1-1} (V_{m_2}^{*j} P_{r_{A,i,j}} V_{m_2}^{j})$ by Proposition 6.5.

Proof of Proposition 6.11. For $1 \le i \le m_1 - 1$,

$$V_{m_1}^{*m_2+1} V_{m_2}^{i} P_0 = V_{m_1}^{*} V_{m_2}^{*m_1} V_{m_2}^{i} P_0 = V_{m_1}^{*} V_{m_2}^{*m_1-i} P_0 = V_{m_2}^{*m_1-i} V_{m_1}^{*} P_0 = 0,$$

so $\sum_{p_i=0}^{m_2} P_{p_i} V_{m_2}^{i} P_0 = V_{m_2}^{i} P_0$, hence $P_0 = \sum_{p_i=0}^{m_2} V_{m_2}^{*i} P_{p_i} V_{m_2}^{i} P_0$. Therefore,

$$\begin{aligned} P_0 &= \big(\textstyle\prod_{i=1}^{m_1-1} (\sum_{p_i=0}^{m_2} V_{m_2}^{*i} P_{p_i} V_{m_2}^{i}) \big) P_0 \\ &= \big(\textstyle\sum_{p_{m_1-1}=0}^{m_2} \cdots \sum_{p_1=0}^{m_2} \prod_{j=1}^{m_1-1} (V_{m_2}^{*j} P_{p_j} V_{m_2}^{j}) \big) P_0. \end{aligned} \tag{6.7}$$

If $i, j, m, n \in \mathbb{N}$, $i < j$, and $m > n$ then, since $V_{m_2}^{i} V_{m_2}^{*i}$ commutes with P_m, and since $V_{m_1}^{*m} P_n = 0$,

$$\begin{aligned} (V_{m_2}^{*i} P_m V_{m_2}^{i})(V_{m_2}^{*j} P_n V_{m_2}^{j}) &= V_{m_2}^{*i} P_m (V_{m_2}^{i} V_{m_2}^{*i}) V_{m_2}^{*j-i} P_n V_{m_2}^{j} = V_{m_2}^{*i} P_m V_{m_2}^{*j-i} P_n V_{m_2}^{j} \\ &= V_{m_2}^{*i} V_{m_1}^{m} P_0 V_{m_2}^{*j-i} V_{m_1}^{*m} P_n V_{m_2}^{j} = 0, \end{aligned}$$

so if a summand $\prod_{j=1}^{m_1-1}(V_{m_2}^{*j}P_{p_j}V_{m_2}^{j})$ of (6.7) is a non-zero projection then $p_j \leq p_k$ whenever $j < k$.

Further, for integers $0 \leq p_j \leq m_2$, $1 \leq j \leq m_1 - 1$, where $p_j \leq p_k$ whenever $j < k$, Lemma 6.10 ensures the existence of unique $A \in \mathcal{A}(\Sigma)$ and $0 \leq i \leq m_1 - 1$ such that $p_j = r_{A,i,j}$ for $1 \leq j \leq m_1 - 1$. Moreover, for any $A \in \mathcal{A}(\Sigma)$ and $0 \leq i \leq m_1 - 1$, it follows from Lemmas 6.7 and 6.8 that $0 \leq r_{A,i,j} \leq m_2$ for $1 \leq j \leq m_1 - 1$, where $r_{A,i,j} \leq r_{A,i,k}$ whenever $j < k$.

So (6.7) becomes

$$\begin{aligned} P_0 &= \big(\textstyle\sum_{A\in\mathcal{A}(\Sigma)}\sum_{i=0}^{m_1-1}\prod_{j=1}^{m_1-1}(V_{m_2}^{*j}P_{r_{A,i,j}}V_{m_2}^{j})\big)P_0 \\ &= \textstyle\sum_{A\in\mathcal{A}(\Sigma)}\sum_{i=0}^{m_1-1}\Big(\big(\prod_{j=1}^{m_1-1}(V_{m_2}^{*j}P_{r_{A,i,j}}V_{m_2}^{j})\big)P_0\Big) \\ &= \textstyle\sum_{A\in\mathcal{A}(\Sigma)}\sum_{i=0}^{m_1-1}\prod_{j=0}^{m_1-1}(V_{m_2}^{*j}P_{r_{A,i,j}}V_{m_2}^{j}). \end{aligned} \tag{6.8}$$

By (6.8), the sum of the projections $\prod_{j=0}^{m_1-1}(V_{m_2}^{*j}P_{r_{A,i,j}}V_{m_2}^{j})$ is the projection P_0, so the projections $\prod_{j=0}^{m_1-1}(V_{m_2}^{*j}P_{r_{A,i,j}}V_{m_2}^{j})$ have mutually orthogonal ranges, and we have a direct-sum decomposition

$$\mathcal{H}_0 = P_0(\mathcal{H}) = \textstyle\bigoplus_{A\in\mathcal{A}(\Sigma)}\big(\bigoplus_{i=0}^{m_1-1}\prod_{j=0}^{m_1-1}(V_{m_2}^{*j}P_{r_{A,i,j}}V_{m_2}^{j})(\mathcal{H})\big).$$

Further, Proposition 6.5 gives $\prod_{j=0}^{m_1-1}(V_{m_2}^{*j}P_{r_{A,i,j}}V_{m_2}^{j})(\mathcal{H}) = V_{m_1}^{*q_{A,i}}V_{m_2}^{i}(\mathcal{K}_A)$. □

Applying each isometry $V_{m_1}^{m}$ to the decomposition (6.6) of $\mathcal{H}_0 = P_0(\mathcal{H})$ gives decompositions

$$P_m(\mathcal{H}) = V_{m_1}^{m}(\mathcal{H}_0) = \textstyle\bigoplus_{A\in\mathcal{A}(\Sigma)}\big(\bigoplus_{i=0}^{m_1-1}V_{m_1}^{m}V_{m_1}^{*q_{A,i}}V_{m_2}^{i}(\mathcal{K}_A)\big),$$

and, since the subspaces $P_m(\mathcal{H})$ themselves give a direct-sum decomposition of $\mathcal{H}_U^{\perp}$, we have

$$\mathcal{H} = \mathcal{H}_U \oplus \big(\textstyle\bigoplus_{m=0}^{\infty}\big(\bigoplus_{A\in\mathcal{A}(\Sigma)}\big(\bigoplus_{i=0}^{m_1-1}V_{m_1}^{m}V_{m_1}^{*q_{A,i}}V_{m_2}^{i}(\mathcal{K}_A)\big)\big)\big).$$

So defining, for $A \in \mathcal{A}(\Sigma)$,

$$\mathcal{H}_A := \textstyle\bigoplus_{m=0}^{\infty}\big(\bigoplus_{i=0}^{m_1-1}V_{m_1}^{m}V_{m_1}^{*q_{A,i}}V_{m_2}^{i}(\mathcal{K}_A)\big), \tag{6.9}$$

we have the decomposition

$$\mathcal{H} = \mathcal{H}_U \oplus (\textstyle\bigoplus_{A\in\mathcal{A}(\Sigma)}\mathcal{H}_A).$$

This decomposition is unique, for given a decomposition $\mathcal{H} = \mathcal{H}_U' \oplus (\bigoplus_{A\in\mathcal{A}(\Sigma)}\mathcal{H}_A')$ as in Proposition 5.2 this process will yield $\mathcal{H}_U = \mathcal{H}_U'$ and $\mathcal{H}_A = \mathcal{H}_A'$ for $A \in \mathcal{A}(\Sigma)$.

We now show that the subspaces $\mathcal{H}_U$ and $\mathcal{H}_A$, for $A \in \mathcal{A}(\Sigma)$, of $\mathcal{H}$ have the required properties.

Proposition 6.13. *The subspaces $\mathcal{H}_U$ and $\mathcal{H}_A$, for $A \in \mathcal{A}(\Sigma)$, of $\mathcal{H}$ are reducing for V.*

Proof. Since $\{m_1, m_2\}$ generates Σ and $V_{m_2}^* = V_{m_2}^{*m_1} V_{m_2}^{m_1-1} = V_{m_1}^{*m_2} V_{m_2}^{m_1-1}$, to prove that a subspace of $\mathcal{H}$ is reducing for V it is enough to show that the subspace is invariant under V_{m_1}, $V_{m_1}^*$, and V_{m_2}. Each of our subspaces is clearly invariant under V_{m_1}. Since $\mathcal{H}_U = \bigcap_{n=0}^{\infty} V_{m_1}^n(\mathcal{H}) = \bigcap_{n=1}^{\infty} V_{m_1}^n(\mathcal{H})$, it is invariant under $V_{m_1}^*$, and, since $V_{m_2}(V_{m_1}^n(\mathcal{H})) = V_{m_1}^n V_{m_2}(\mathcal{H}) \subset V_{m_1}^n(\mathcal{H})$, it is also invariant under V_{m_2}.

Fix $A \in \mathcal{A}(\Sigma)$ and consider $\mathcal{H}_A$. We have

$$V_{m_1}^*\big(\bigcup_{m=1}^{\infty} \bigcup_{i=0}^{m_1-1} V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A)\big) = \bigcup_{m=0}^{\infty} \bigcup_{i=0}^{m_1-1} V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A) \\ \subset \mathcal{H}_A.$$

Further, $V_{m_2}^i(\mathcal{K}_A)$ is contained in $P_{q_{A,i}}(\mathcal{H}) = V_{m_1}^{q_{A,i}} P_0 V_{m_1}^{*q_{A,i}}(\mathcal{H})$ by the definition of $\mathcal{K}_A$, so $V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A)$ is contained in $P_0(\mathcal{H}) = \mathcal{H}_0$, hence

$$V_{m_1}^*\big(\bigcup_{i=0}^{m_1-1} V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A)\big) = \{0\}.$$

So $\mathcal{H}_A$ is invariant under $V_{m_1}^*$.

In showing that $\mathcal{H}_A$ is invariant under V_{m_2} we will freely use the fact that $V_{m_2}^i(\mathcal{K}_A)$ is contained in the range of the projection $V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}}$ for $1 \le i \le m_1 - 1$. Now, we first calculate

$$\begin{aligned} V_{m_2}\big(V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A)\big) &= V_{m_1}^m V_{m_2} V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A) \\ &= V_{m_1}^m (V_{m_1}^{*q_{A,i}} V_{m_1}^{q_{A,i}}) V_{m_2} V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A) \\ &= V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}(V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A)) \\ &= V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2} V_{m_2}^i(\mathcal{K}_A) \\ &= V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^{i+1}(\mathcal{K}_A), \end{aligned} \tag{6.10}$$

and then consider the two cases $i < m_1 - 1$ and $i = m_1 - 1$. If $i < m_1 - 1$ then, since $q_{A,i} \le q_{A,i+1}$ by Lemma 6.8, we have from (6.10)

$$\begin{aligned} V_{m_2}\big(V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A)\big) &= V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^{i+1}(\mathcal{K}_A) \\ &= V_{m_1}^m V_{m_1}^{*q_{A,i}} (V_{m_1}^{q_{A,i+1}} V_{m_1}^{*q_{A,i+1}} V_{m_2}^{i+1}(\mathcal{K}_A)) \\ &= V_{m_1}^m V_{m_1}^{q_{A,i+1}-q_{A,i}} V_{m_1}^{*q_{A,i+1}} V_{m_2}^{i+1}(\mathcal{K}_A) \\ &= V_{m_1}^{m+q_{A,i+1}-q_{A,i}} V_{m_1}^{*q_{A,i+1}} V_{m_2}^{i+1}(\mathcal{K}_A) \subset \mathcal{H}_A. \end{aligned}$$

If $i = m_1 - 1$ then, since $q_{A,i} \le m_2$ by Lemma 6.7, we have from (6.10)

$$\begin{aligned} V_{m_2}\big(V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^i(\mathcal{K}_A)\big) &= V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^{i+1}(\mathcal{K}_A) = V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_2}^{m_1}(\mathcal{K}_A) \\ &= V_{m_1}^m V_{m_1}^{*q_{A,i}} V_{m_1}^{m_2}(\mathcal{K}_A) = V_{m_1}^m V_{m_1}^{m_2-q_{A,i}}(\mathcal{K}_A) \\ &= V_{m_1}^{m+m_2-q_{A,i}}(\mathcal{K}_A) \subset \mathcal{H}_A. \end{aligned}$$ □

We next show that $V|_{\mathcal{H}_A}$ is equivalent to $T^A \otimes 1_{\mathcal{K}_A}$ for each $A \in \mathcal{A}(\Sigma)$. We identify $\ell^2(A) \otimes \mathcal{K}_A$ with $\ell^2(\mathbb{N} \times \{0, \dots, m_1 - 1\}, \mathcal{K}_A)$ so that $T_{m_1}^A \otimes 1_{\mathcal{K}_A}$ and $T_{m_2}^A \otimes 1_{\mathcal{K}_A}$ on $\ell^2(\mathbb{N} \times \{0, \dots, m_1 - 1\}, \mathcal{K}_A)$ are given by the same formulas (6.2) and (6.3) as

$T^A_{m_1}$ and $T^A_{m_2}$. Denote by Q^A_{00} the projection $\prod_{j=0}^{m_1-1}(V^{*j}_{m_2}P_{q_{A,j}}V^j_{m_2})$ of $\mathcal{H}$ onto $\mathcal{K}_A$ as in Proposition 6.5, and define $W_A\colon \mathcal{H}_A \to \ell^2(\mathbb{N}\times\{0,\ldots,m_1-1\},\mathcal{K}_A)$ by

$$(W_Ah)_{nj} = Q^A_{00}V^{*j}_{m_2}V^{q_{A,j}}_{m_1}V^{*n}_{m_1}h.$$

It follows from the direct-sum decomposition (6.9) of $\mathcal{H}_A$ that W_A is a unitary isomorphism of $\mathcal{H}_A$ onto $\ell^2(\mathbb{N}\times\{0,\ldots,m_1-1\},\mathcal{K}_A)$.

Proposition 6.14. *We have $W_A(V|_{\mathcal{H}_A})W_A^* = T^A\otimes 1_{\mathcal{K}_A}$ for $A\in\mathcal{A}(\Sigma)$.*

Proof. It suffices to prove that $W_A(V_s|_{\mathcal{H}_A}) = (T^A_s\otimes 1)W_A$ for $s=m_1$ and $s=m_2$. Let $h\in\mathcal{H}_A$, $n\in\mathbb{N}$, and $0\le j\le m_1-1$. Since $V^j_{m_2}Q^A_{00}(\mathcal{H}) = V^j_{m_2}(\mathcal{K}_A)$ is orthogonal to $V^{q_{A,j}+1}_{m_1}(\mathcal{H})$,

$$(W_AV_{m_1}h)_{0j} = Q^A_{00}V^{*j}_{m_2}V^{q_{A,j}+1}_{m_1}h = 0 = ((T^A_{m_1}\otimes 1)W_Ah)_{0j}.$$

For $n\ge 1$,

$$\begin{aligned}(W_AV_{m_1}h)_{nj} &= Q^A_{00}V^{*j}_{m_2}V^{q_{A,j}}_{m_1}V^{*n}_{m_1}V_{m_1}h = Q^A_{00}V^{*j}_{m_2}V^{q_{A,j}}_{m_1}V^{*n-1}_{m_1}h = (W_Ah)_{(n-1)j}\\ &= ((T^A_{m_1}\otimes 1)W_Ah)_{nj}.\end{aligned}$$

Before proving the equality $W_A(V_{m_2}|_{\mathcal{H}_A}) = (T^A_{m_2}\otimes 1)W_A$ we will establish that $(W_Ak)_{mi} = Q^A_{00}V^{*i}_{m_2}V^{*m}_{m_1}V^{q_{A,i}}_{m_1}k$ for $k\in\mathcal{H}_A$, $m\in\mathbb{N}$, and $0\le i\le m_1-1$, a result we will use freely throughout the remainder of this proof. Indeed,

$$\begin{aligned}(W_Ak)_{mi} &= Q^A_{00}V^{*i}_{m_2}V^{q_{A,i}}_{m_1}V^{*m}_{m_1}k\\ &= (Q^A_{00}(V^{*i}_{m_2}P_{q_{A,i}}V^i_{m_2}))V^{*i}_{m_2}V^{q_{A,i}}_{m_1}V^{*m}_{m_1}(V^{*q_{A,i}}_{m_1}V^{q_{A,i}}_{m_1})k\\ &= Q^A_{00}V^{*i}_{m_2}P_{q_{A,i}}(V^i_{m_2}V^{*i}_{m_2})(V^{q_{A,i}}_{m_1}V^{*q_{A,i}}_{m_1})V^{*m}_{m_1}V^{q_{A,i}}_{m_1}k\\ &= Q^A_{00}V^{*i}_{m_2}(P_{q_{A,i}}V^{q_{A,i}}_{m_1}V^{*q_{A,i}}_{m_1})V^i_{m_2}V^{*i}_{m_2}V^{*m}_{m_1}V^{q_{A,i}}_{m_1}k\\ &= (Q^A_{00}(V^{*i}_{m_2}P_{q_{A,i}}V^i_{m_2}))V^{*i}_{m_2}V^{*m}_{m_1}V^{q_{A,i}}_{m_1}k = Q^A_{00}V^{*i}_{m_2}V^{*m}_{m_1}V^{q_{A,i}}_{m_1}k.\end{aligned}$$

We now prove the equality $W_AV_{m_2}h = (T^A_{m_2}\otimes 1)W_Ah$ by considering four cases. If $n\ge q_{A,j}-q_{A,j-1}$ and $j\ge 1$ then, noting that $q_{A,j}-q_{A,j-1}\ge 0$ by Lemma 6.8,

$$\begin{aligned}(W_AV_{m_2}h)_{nj} &= Q^A_{00}V^{*j}_{m_2}V^{*n}_{m_1}V^{q_{A,j}}_{m_1}V_{m_2}h = Q^A_{00}V^{*j-1}_{m_2}V^{*n}_{m_1}(V^*_{m_2}V_{m_2})V^{q_{A,j}}_{m_1}h\\ &= Q^A_{00}V^{*j-1}_{m_2}V^{*n}_{m_1}(V^{q_{A,j}-q_{A,j-1}}_{m_1}V^{q_{A,j-1}}_{m_1})h\\ &= Q^A_{00}V^{*j-1}_{m_2}V^{*n-q_{A,j}+q_{A,j-1}}_{m_1}V^{q_{A,j-1}}_{m_1}h = (W_Ah)_{(n-q_{A,j}+q_{A,j-1})(j-1)}\\ &= ((T^A_{m_2}\otimes 1)W_Ah)_{nj}.\end{aligned}$$

If $n<q_{A,j}-q_{A,j-1}$ and $j\ge 1$ then, since $V^{j-1}_{m_2}Q^A_{00}(\mathcal{H}) = V^{j-1}_{m_2}(\mathcal{K}_A)$ is orthogonal to $V^{q_{A,j-1}+1}_{m_1}(\mathcal{H})$ and since $q_{A,j}-n\ge q_{A,j-1}+1$, $V^{j-1}_{m_2}Q^A_{00}(\mathcal{H})$ is orthogonal to $V^{q_{A,j}-n}_{m_1}(\mathcal{H})$, so

$$\begin{aligned}(W_AV_{m_2}h)_{nj} &= Q^A_{00}V^{*j}_{m_2}V^{*n}_{m_1}V^{q_{A,j}}_{m_1}V_{m_2}h = Q^A_{00}V^{*j-1}_{m_2}V^{*n}_{m_1}(V^*_{m_2}V_{m_2})V^{q_{A,j}}_{m_1}h\\ &= Q^A_{00}V^{*j-1}_{m_2}V^{q_{A,j}-n}_{m_1}h = 0 = ((T^A_{m_2}\otimes 1)W_Ah)_{nj}.\end{aligned}$$

If $n \geq m_2 - q_{A,m_1-1}$ and $j = 0$ then, noting that $m_2 - q_{A,m_1-1} \geq 0$ by Lemma 6.7,

$$\begin{aligned}(W_A V_{m_2} h)_{nj} &= Q^A_{00} V^{*n}_{m_1} V_{m_2} h = Q^A_{00} V^{*n}_{m_1} (V^{*m_1-1}_{m_2} V^{m_1-1}_{m_2}) V_{m_2} h \\ &= Q^A_{00} V^{*m_1-1}_{m_2} V^{*n}_{m_1} V^{m_2}_{m_1} h = Q^A_{00} V^{*m_1-1}_{m_2} V^{*n}_{m_1} (V^{m_2-q_{A,m_1-1}}_{m_1} V^{q_{A,m_1-1}}_{m_1}) h \\ &= Q^A_{00} V^{*m_1-1}_{m_2} V^{*n-m_2+q_{A,m_1-1}}_{m_1} V^{q_{A,m_1-1}}_{m_1} h \\ &= (W_A h)_{(n-m_2+q_{A,m_1-1})(m_1-1)} = ((T^A_{m_2} \otimes 1) W_A h)_{nj}.\end{aligned}$$

If $n < m_2 - q_{A,m_1-1}$ and $j = 0$ then, since $V^{m_1-1}_{m_2} Q^A_{00}(\mathcal{H}) = V^{m_1-1}_{m_2}(\mathcal{K}_A)$ is orthogonal to $V^{q_{A,m_1-1}+1}_{m_1}(\mathcal{H})$ and since $m_2 - n \geq q_{A,m_1-1} + 1$, $V^{m_1-1}_{m_2} Q^A_{00}(\mathcal{H})$ is orthogonal to $V^{m_2-n}_{m_1}(\mathcal{H})$, so

$$\begin{aligned}(W_A V_{m_2} h)_{nj} &= Q^A_{00} V^{*n}_{m_1} V_{m_2} h = Q^A_{00} V^{*n}_{m_1} (V^{*m_1-1}_{m_2} V^{m_1-1}_{m_2}) V_{m_2} h \\ &= Q^A_{00} V^{*m_1-1}_{m_2} V^{*n}_{m_1} V^{m_2}_{m_1} h = Q^A_{00} V^{*m_1-1}_{m_2} V^{m_2-n}_{m_1} h = 0 \\ &= ((T^A_{m_2} \otimes 1) W_A h)_{nj}.\end{aligned}$$

□

7. The C^*-algebra of Σ

Throughout this section we let Σ be a numerical semigroup with minimal system of generators $\{m_1, \dots, m_r\}$, where $m_i < m_j$ if $i < j$. We denote by $C^*(\Sigma)$ the unital C^*-algebra generated by an isometric representation $v\colon \Sigma \to C^*(\Sigma)$ with commuting range projections which is universal for such representations: for every isometric representation $V\colon \Sigma \to B$ in a unital C^*-algebra B with commuting range projections there is a unique homomorphism $\pi_V\colon C^*(\Sigma) \to B$ such that $V = \pi_V \circ v$.

The two main results of this section are generalisations of the corresponding results in [8, Section 4] for the numerical semigroup $\mathbb{N} \setminus \{1\}$. Namely, we describe a condition on an isometric representation $V\colon \Sigma \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ with commuting range projections which ensures that π_V is faithful, and give a concrete description of $C^*(\Sigma)$ in terms of the usual Toeplitz algebra.

We recall from Definition 4.6 that α is the smallest positive integer in Σ such that m_1 and α are relatively prime.

Theorem 7.1. *Let $V : \Sigma \to B(\mathcal{H})$ be an isometric representation on a Hilbert space $\mathcal{H}$ with commuting range projections. Then the representation π_V of $C^*(\Sigma)$ is faithful if and only if*

$$\prod_{i=0}^{m_1-1} V^{*i}_\alpha (V^{q_{A,i}}_{m_1} V^{*q_{A,i}}_{m_1} - V^{q_{A,i}+1}_{m_1} V^{*q_{A,i}+1}_{m_1}) V^i_\alpha \neq 0 \quad \textit{for every } A \in \mathcal{A}(\Sigma). \tag{7.1}$$

Remark 7.2. Condition (7.1) is equivalent to requiring that each subspace $\mathcal{H}_A$ in the decomposition of Theorem 5.1 is non-zero.

To see this, first consider the case $\Sigma = \mathbb{N}$. Then (7.1) says that $1 - V_1 V_1^* \neq 0$ (equivalently each V_n, $n \geq 1$, is non-unitary). Theorem 5.1 in conjunction with the observation that the range of the projection $1 - V_1 V_1^*$ is contained in $\mathcal{H}_\mathbb{N} = \mathcal{H}_U^\perp$ shows that $1 - V_1 V_1^* \neq 0$ if and only if $\mathcal{H}_\mathbb{N} \neq \{0\}$.

Now suppose $\Sigma \neq \mathbb{N}$, so that $2 \leq m_1 < \alpha$, and consider the numerical semigroup $\langle m_1, \alpha\rangle$, which has the minimal system of generators $\{m_1, \alpha\}$. Since $\langle m_1, \alpha\rangle \subset \Sigma$ we may consider the restriction $V|_{\langle m_1,\alpha\rangle} \colon \langle m_1, \alpha\rangle \to B(\mathcal{H})$ of V to $\langle m_1, \alpha\rangle$. Applying Theorem 5.1 (or Proposition 5.2) to the representation $V|_{\langle m_1,\alpha\rangle}$ gives a decomposition $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A\in\mathcal{A}(\langle m_1,\alpha\rangle)} \mathcal{H}_A)$. By the uniqueness of this decomposition, each $\mathcal{H}_A$ is the same subspace as in (6.9). So

$$\prod_{i=0}^{m_1-1} V_\alpha^{*i}(V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}} - V_{m_1}^{q_{A,i}+1} V_{m_1}^{*q_{A,i}+1})V_\alpha^i \tag{7.2}$$

is the projection onto the subspace $\mathcal{K}_A$ of $\mathcal{H}_A$ by Proposition 6.5. In particular, (7.2) is non-zero if and only if $\mathcal{H}_A$ is non-zero. Since V extends $V|_{\langle m_1,\alpha\rangle}$, we have, by Lemma 5.3, a decomposition $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A\in\mathcal{A}(\Sigma)} \mathcal{H}_A)$ for V, which is unique by Theorem 5.1, and the claim follows.

Remark 7.3. Theorem 7.1 implies in particular that $\pi_{\bigoplus_{A\in\mathcal{A}(\Sigma)} T^A}$ is faithful.

Proof of Theorem 7.1. Recall that p_A is the polynomial in Proposition 4.12, and note that

$$\prod_{i=0}^{m_1-1} V_\alpha^{*i}(V_{m_1}^{q_{A,i}} V_{m_1}^{*q_{A,i}} - V_{m_1}^{q_{A,i}+1} V_{m_1}^{*q_{A,i}+1})V_\alpha^i = p_A(V_{m_1}, V_{m_1}^*, V_\alpha, V_\alpha^*).$$

Since

$$\pi_{T^A}\big(p_A(v_{m_1}, v_{m_1}^*, v_\alpha, v_\alpha^*)\big) = p_A(T_{m_1}^A, (T_{m_1}^A)^*, T_\alpha^A, (T_\alpha^A)^*)$$

is the projection onto $\operatorname{span}\{e_{A,0}\}$ by Proposition 4.12, $p_A(v_{m_1}, v_{m_1}^*, v_\alpha, v_\alpha^*)$ is non-zero in $C^*(\Sigma)$. So, if π_V is faithful then

$$p_A(V_{m_1}, V_{m_1}^*, V_\alpha, V_\alpha^*) = \pi_V\big(p_A(v_{m_1}, v_{m_1}^*, v_\alpha, v_\alpha^*)\big) \neq 0$$

for $A \in \mathcal{A}(\Sigma)$, hence (7.1) holds.

Now suppose that V satisfies condition (7.1). Then each $\mathcal{H}_A$ is non-zero in the decomposition $\mathcal{H} = \mathcal{H}_U \oplus (\bigoplus_{A\in\mathcal{A}(\Sigma)} \mathcal{H}_A)$ of Theorem 5.1. Write $V_U := V|_{\mathcal{H}_U}$, $V_A := V|_{\mathcal{H}_A}$ for $A \in \mathcal{A}(\Sigma)$, and fix $a \in C^*(\Sigma)$. We can check on generators that $\pi_V = \pi_{V_U} \oplus (\bigoplus_{A\in\mathcal{A}(\Sigma)} \pi_{V_A})$, hence

$$\|\pi_V(a)\| = \max\big(\{\|\pi_{V_U}(a)\|\} \cup \{\,\|\pi_{V_A}(a)\| \mid A \in \mathcal{A}(\Sigma)\,\}\big). \tag{7.3}$$

Since V_A is equivalent to $T^A \otimes 1$, and since we can check on generators that $\pi_{T^A\otimes 1} = \pi_{T^A} \otimes 1$, we have that π_{V_A} is equivalent to $\pi_{T^A} \otimes 1$. Since each $\mathcal{H}_A$ is non-zero, we then have $\|\pi_{V_A}(a)\| = \|\pi_{T^A}(a)\|$, so (7.3) implies

$$\|\pi_V(a)\| = \max\big(\{\|\pi_{V_U}(a)\|\} \cup \{\,\|\pi_{T^A}(a)\| \mid A \in \mathcal{A}(\Sigma)\,\}\big). \tag{7.4}$$

Define $U_1 := ((V_U)_{C(\Sigma)+1})^*(V_U)_{C(\Sigma)+2}$, which is unitary, and denote by S the unilateral shift on $\ell^2(\mathbb{N})$. For $n \in \Sigma$ we have

$$(U_1)^n = ((V_U)_{C(\Sigma)+1})^{*n}((V_U)_{C(\Sigma)+2})^n = (V_U)_n$$

and $S^n = T_n^{\mathbb{N}}$, so the operator $\pi_{V_U}(a) \oplus \pi_{T^{\mathbb{N}}}(a)$ belongs to the C^*-algebra generated by $U_1 \oplus S$. Hence the Lemma on page 724 of [1] implies that $\|\pi_{V_U}(a)\| \leq \|\pi_{T^{\mathbb{N}}}(a)\|$. Thus (7.4) implies

$$\|\pi_V(a)\| = \max\{\,\|\pi_{T^A}(a)\| \mid A \in \mathcal{A}(\Sigma)\,\}.$$

Since every C^*-algebra has a faithful representation and every representation of $C^*(\Sigma)$ has the form π_W for some isometric representation W, there is a faithful representation of the form π_W. Applying Theorem 5.1 to W, we then have that each $\mathcal{H}_A$ is non-zero by the first part of the proof. We can then deduce from the argument of the previous paragraph that

$$\|a\| = \|\pi_W(a)\| = \max\{\, \|\pi_{T^A}(a)\| \mid A \in \mathcal{A}(\Sigma)\,\} = \|\pi_V(a)\|,$$

which, since a is an arbitrary element of $C^*(\Sigma)$, implies that π_V is faithful. □

We can view the Toeplitz algebra $\mathcal{T}$ either as the C^*-subalgebra of $B(\ell^2(\mathbb{N}))$ generated by the unilateral shift, or as the C^*-subalgebra of $B(H^2(\mathbb{T}))$ generated by the Toeplitz operators T_ϕ with symbol $\phi \in C(\mathbb{T})$. In either realisation, the algebra $\mathcal{K}$ of compact operators is an ideal of $\mathcal{T}$, and the quotient $\mathcal{T}/\mathcal{K}$ is naturally isomorphic to $C(\mathbb{T})$. In the proof of Theorem 7.4 we realise $\mathcal{T}$ as a subalgebra of $B(\ell^2(\mathbb{N}))$. We denote by $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{K}$ the quotient map.

Theorem 7.4. *$C^*(\Sigma)$ is isomorphic to*

$$C := \{\, \bigoplus_{A\in\mathcal{A}(\Sigma)} R_A \in \bigoplus_{A\in\mathcal{A}(\Sigma)} \mathcal{T} \mid q(R_{\mathbb{N}}) = q(R_A) \text{ for } A \in \mathcal{A}(\Sigma)\,\}.$$

We use the following lemma to prove this theorem.

Lemma 7.5. *Let $A \in \mathcal{A}(\Sigma)$ and denote $A = \{\, a_n \mid n \in \mathbb{N}\,\}$ where $a_n < a_{n+1}$ for $n \in \mathbb{N}$. Let $U_A\colon \ell^2(\mathbb{N}) \to \ell^2(A)$ be the unitary isomorphism determined by $U_A e_{\mathbb{N},n} = e_{A,a_n}$ for $n \in \mathbb{N}$. Then $U_A^* T_p^A U_A - T_p^{\mathbb{N}}$ is a finite-rank operator on $\ell^2(\mathbb{N})$ for every $p \in \Sigma$.*

Proof. If $A = \mathbb{N}$ then the result is trivial, so suppose $A \neq \mathbb{N}$. Then we must have $\Sigma \neq \mathbb{N}$, so $C(\Sigma) \geq 1$. If $a_{C(\Sigma)} = C(\Sigma)$ then $A = \mathbb{N}$, so we must have $a_{C(\Sigma)} \geq C(\Sigma) + 1$. Thus, for $n \in \mathbb{N}$, $n + a_{C(\Sigma)} = a_{n+C(\Sigma)}$. On the other hand, if $0 \leq q < C(\Sigma)$ then, for $n \in \mathbb{N} \setminus \{0\}$, $n + a_q$ is not necessarily equal to a_{n+q}. Therefore, using the notation $h \otimes \overline{k}$ for the rank-one operator $g \mapsto (\,g \mid k\,)h$ on $\ell^2(\mathbb{N})$,

$$U_A^* T_p^A U_A - T_p^{\mathbb{N}} = \textstyle\sum_{i=0}^{C(\Sigma)-1} (U_A^* e_{A,p+a_i} - e_{\mathbb{N},p+i}) \otimes \overline{e}_{\mathbb{N},i}. \qquad \square$$

Proof of Theorem 7.4. For each $A \in \mathcal{A}(\Sigma)$ take U_A as in Lemma 7.5, and define the mapping $\psi\colon C^*(\Sigma) \to \bigoplus_{A\in\mathcal{A}(\Sigma)} B(\ell^2(\mathbb{N}))$ by $\psi(a) := \bigoplus_{A\in\mathcal{A}(\Sigma)} U_A^* \pi_{T^A}(a) U_A$. We claim that ψ is an isomorphism of $C^*(\Sigma)$ onto C. It is injective because $\bigoplus_{A\in\mathcal{A}(\Sigma)} \pi_{T^A} = \pi_{\bigoplus_{A\in\mathcal{A}(\Sigma)} T^A}$ is faithful by Theorem 7.1. Since the operators $\pi_{T^{\mathbb{N}}}(v_p) = T_p^{\mathbb{N}}$ are all powers of the unilateral shift, and Lemma 7.5 implies that $U_A^* \pi_{T^A}(v_p) U_A = U_A^* T_p^A U_A$ differs from $T_p^{\mathbb{N}}$ by a finite-rank operator, the range of ψ is contained in C. So it remains to prove that every element of C is in the range of ψ.

Let $\bigoplus_{A\in\mathcal{A}(\Sigma)} (R_{\mathbb{N}} + K_A) \in C$, where the K_A are compact operators with $K_{\mathbb{N}} = 0$. Since $(T_{C(\Sigma)+1}^{\mathbb{N}})^* T_{C(\Sigma)+2}^{\mathbb{N}} = \pi_{T^{\mathbb{N}}}(v_{C(\Sigma)+1}^* v_{C(\Sigma)+2})$ is the unilateral shift,

$\pi_{T^{\mathbb{N}}}$ maps $C^*(\Sigma)$ onto $\mathcal{T}$. Thus there exists $a \in C^*(\Sigma)$ such that $\pi_{T^{\mathbb{N}}}(a) = R_{\mathbb{N}}$, and then

$$R_{\mathbb{N}} + K_A = U_A^* \pi_{T^A}(a) U_A + (\pi_{T^{\mathbb{N}}}(a) - U_A^* \pi_{T^A}(a) U_A) + K_A,$$

which we may write as $U_A^* \pi_{T^A}(a) U_A + L_A$, where L_A is compact, since it follows from Lemma 7.5 that $\pi_{T^{\mathbb{N}}}(a) - U_A^* \pi_{T^A}(a) U_A$ is compact. So it suffices to show that, for each $B \in \mathcal{A}(\Sigma)$, $\bigoplus_{A \in \mathcal{A}(\Sigma)} R_A$ is in the range of ψ where $R_A = 0$ for $A \neq B$ and R_B is compact, and for this it is enough to show that this is true when R_B is a rank-one operator $e_{\mathbb{N},i} \otimes \overline{e}_{\mathbb{N},j}$.

Therefore fix $B \in \mathcal{A}(\Sigma)$ and $i, j \in \mathbb{N}$. Denote $B = \{\, b_n \mid n \in \mathbb{N} \,\}$ where $b_n < b_{n+1}$ for $n \in \mathbb{N}$. Recall that p_B is the polynomial in Proposition 4.12. To simplify notation, for an isometric representation V of Σ in a C^*-algebra we denote $p_B(V_{m_1}, V_{m_1}^*, V_\alpha, V_\alpha^*)$ by $p_B(V)$. Now, consider

$$\begin{aligned} &\psi(v^*_{C(\Sigma)+1} v_{C(\Sigma)+1+b_i} p_B(v) v^*_{C(\Sigma)+1+b_j} v_{C(\Sigma)+1}) \\ &\quad = \textstyle\bigoplus_{A \in \mathcal{A}(\Sigma)} U_A^* (T^A_{C(\Sigma)+1})^* T^A_{C(\Sigma)+1+b_i} p_B(T^A) (T^A_{C(\Sigma)+1+b_j})^* T^A_{C(\Sigma)+1} U_A. \end{aligned} \tag{7.5}$$

If $A \neq B$ then, since $p_B(T^A) = 0$ by Proposition 4.12, the summand of (7.5) corresponding to A is zero. Further, since $p_B(T^B)$ is the projection onto $\operatorname{span}\{e_{B,0}\}$ by Proposition 4.12, the summand of (7.5) corresponding to B satisfies

$$\begin{aligned} &\big(U_B^* (T^B_{C(\Sigma)+1})^* T^B_{C(\Sigma)+1+b_i} p_B(T^B) (T^B_{C(\Sigma)+1+b_j})^* T^B_{C(\Sigma)+1} U_B\big) e_{\mathbb{N},n} \\ &\quad = \begin{cases} e_{\mathbb{N},i} & \text{if } n = j, \\ 0 & \text{if } n \neq j \end{cases} = (e_{\mathbb{N},i} \otimes \overline{e}_{\mathbb{N},j}) e_{\mathbb{N},n} \end{aligned}$$

and the result follows. □

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