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CO-UNIVERSAL C^*-ALGEBRAS ASSOCIATED TO GENERALISED GRAPHS*

BY

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ABSTRACT

We introduce P-graphs, which are generalisations of directed graphs in which paths have a degree in a semigroup P rather than a length in $\mathbb{N}$. We focus on semigroups P arising as part of a quasi-lattice ordered group (G, P) in the sense of Nica, and on P-graphs which are finitely aligned in the sense of Raeburn and Sims. We show that each finitely aligned P-graph admits a C^*-algebra $C^*_{\min}(\Lambda)$ which is co-universal for partial-isometric representations of Λ which admit a coaction of G compatible with the P-valued length function. We also characterise when a homomorphism induced by the co-universal property is injective. Our results combined with those of Spielberg show that every Kirchberg algebra is Morita equivalent to $C^*_{\min}(\Lambda)$ for some $(\mathbb{N}^2 * \mathbb{N})$-graph Λ.

1. Introduction

The Cuntz–Krieger algebras $\mathcal{O}_A$ introduced in [3] provide an extensive array of purely infinite simple C^*-algebras. The study of these algebras has led in particular to the celebrated Kirchberg–Phillips classification theorem which

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says, roughly, that every purely infinite simple C^*-algebra (these are now called Kirchberg algebras) is determined up to isomorphism by its K-theory [18], and that for every pair of abelian groups G, H, there exists a purely infinite simple C^*-algebra A with $K_*(A) = (G, H)$.

However, not every purely infinite simple C^*-algebra is a Cuntz–Krieger algebra: the results of [4] imply that the K-groups of a Cuntz–Krieger algebra are finitely generated and have equal rank, and that the K_1-group is free abelian. Graph C^*-algebras [15, 16] and their higher-rank analogues [13] were developed in part to seek Cuntz–Krieger-like models for the remaining purely infinite simple C^*-algebras.

This program has met with mixed success. On the one hand, graph algebras themselves do not suffice to describe all Kirchberg algebras: the results of [24] and [29] imply that a purely infinite simple C^*-algebra can be realised up to Morita equivalence as a graph algebra if and only if its K_1-group is free abelian. And the question of whether every purely infinite simple C^*-algebra can be realised as a k-graph C^*-algebra remains open. On the other hand, since higher-rank graph C^*-algebras include, in particular, all finite tensor products of graph C^*-algebras [13, Corollary 3.5(iv)], for every pair of abelian groups G, H, there exist 2-graphs Λ_G and Λ_H such that each of $C^*(\Lambda_G)$ and $C^*(\Lambda_H)$ is simple and purely infinite, and $K_*(\Lambda_G) = (G, \{0\})$ while $K_*(\Lambda_H) = (\{0\}, H)$. In [27, 28] Spielberg developed a construction which incorporates Λ_G and Λ_H in a kind of hybrid graph Λ in such a way that the C^*-algebra associated to Λ is itself simple and purely infinite and has K-theory (G, H). So every purely infinite simple C^*-algebra can be realised up to stable isomorphism as the C^*-algebra of one of Spielberg's hybrid graphs, and so can, in a sense, be built from k-graph algebras.

A particularly powerful source of intuition when dealing with graph C^*-algebras and k-graph C^*-algebras is that each k-graph C^*-algebra can be realised up to Morita equivalence as a crossed product of an AF algebra by an action of $\mathbb{Z}^k$ [14]. It is therefore natural to seek an analogous description of Spielberg's models. While Spielberg's construction does not lend itself immediately to such a description, the discussion of [11, Examples 1.5] suggests that one may be able to think of Spielberg's hybrid graphs as generalised k-graphs in which the degree functor from Λ to $\mathbb{N}^k$ has been replaced by a functor taking values in the free product $\mathbb{N}^2 * \mathbb{N}$. The results of [2] then suggest that the purely infinite simple C^*-algebra associated to a hybrid graph can be regarded as a crossed product of an AF core by $\mathbb{N}^2 * \mathbb{N}$.

In this paper, we introduce the notion of a P-graph (Definition 2.1) for a quasi-lattice ordered group (G, P) in the sense of Nica, and associate to each P-graph Λ a C^*-algebra $C^*_{\min}(\Lambda)$. We show that Spielberg's hybrid graphs can be regarded as $(\mathbb{N}^2 * \mathbb{N})$-graphs, and that the associated $(\mathbb{N}^2 * \mathbb{N})$-graph C^*-algebra as constructed in this paper coincides with the purely infinite simple C^*-algebra associated to the hybrid graph by Spielberg (Theorem 6.2). In particular, the class of P-graph algebras contains, up to Morita equivalence, every Kirchberg algebra.

Our approach to the construction of the P-graph algebra associated to a P-graph Λ does not follow the traditional lines used for graphs and k-graphs in the literature (see, for example, [1, 13, 16]). Instead we proceed using the notion of a co-universal C^*-algebra. This approach was inspired by Katsura's description of the C^*-algebras he associates to Hilbert bimodules [12, Proposition 7.14], and was applied in [2] to product systems. Our main result, Theorem 5.3, says that every finitely aligned P-graph Λ admits a C^*-algebra which is co-universal for representations of Λ which are nonzero on generators and carry a natural coaction of G.

Co-universal properties have been explored previously as a means of specifying C^*-algebras associated to directed graphs [25]. However, this approach is relatively new, and one of our motivations for tackling P-graphs in this way is to develop techniques for establishing the existence of a co-universal algebra for a given system of generators and relations. In particular, we address in Examples 6.4 and 6.6 the problems arising in previous approaches to co-universal algebras detailed in [2, Example 3.9] and [26, Example 3.16]. Our other motivation for using co-universal properties is that we deal here with groups which need not be amenable. Since unitary representations of the groups themselves are, in some instances, examples of our construction, one cannot expect to obtain a C^*-algebra which satisfies a version of the gauge-invariant uniqueness theorem as a universal C^*-algebra (see [2, Remark 5.4]).

The notion of a representation of a P-graph Λ and the associated universal C^*-algebra $\mathcal{T}C^*(\Lambda)$ were introduced in [21]. The algebra $\mathcal{T}C^*(\Lambda)$ is generated by partial isometries $\{s_\mu : \mu \in \Lambda\}$ and is spanned by the elements of the form $s_\mu s_\nu^*$ such that $s(\mu) = s(\nu)$. The fixed-point algebra for the canonical coaction of G on $\mathcal{T}C^*(\Lambda)$ is the subalgebra spanned by the elements $s_\mu s_\nu^*$ such that $d(\mu) = d(\nu)$, where $d : \Lambda \to P$ denotes the generalised length function. Analysing this fixed-point subalgebra of $\mathcal{T}C^*(\Lambda)$ is the traditional first step in establishing a uniqueness theorem for $\mathcal{T}C^*(\Lambda)$ itself.

Our innovation in this paper is to begin by developing an analysis of the universal C^*-algebra $\mathcal{B}_\Lambda$ generated by partial isometries

$$\{\omega_{\mu,\nu} : d(\mu) = d(\nu), s(\mu) = s(\nu)\}$$

satisfying the same relations as the $s_\mu s_\nu^*$. In particular, we characterise in Theorem 4.9 the ideals of $\mathcal{B}_\Lambda$ which contain none of the $\omega_{\mu,\nu}$. We then use this analysis to construct a C^*-algebra $\mathcal{B}_\Lambda^{\min}$ which is co-universal for representations of $\mathcal{B}_\Lambda$ by nonzero partial isometries (Theorem 4.12). We prove that $\omega_{\mu,\nu} \mapsto s_\mu s_\nu^*$ determines an isomorphism of $\mathcal{B}_\Lambda$ with the fixed-point algebra in $\mathcal{T}C^*(\Lambda)$. We are then able to use the categorical approach to coactions studied in [7] to construct a C^*-algebra $C^*_{\min}(\Lambda)$ which is generated by a representation of Λ by nonzero partial isometries $\{S_\lambda : \lambda \in \Lambda\}$, and carries a normal coaction of G whose fixed-point algebra coincides with $\mathcal{B}_\Lambda^{\min}$. We present a bootstrapping argument employing the canonical conditional expectations associated to coactions and the universal property of $\mathcal{T}C^*(\Lambda)$ to deduce from the co-universal property of $\mathcal{B}_\Lambda^{\min}$ that $C^*_{\min}(\Lambda)$ is co-universal for representations of Λ by nonzero partial isometries which preserve the canonical coaction of G. A key tool in our analysis of $\mathcal{B}_\Lambda^{\min}$ is Exel's use of filters and ultrafilters as a tool for studying representations of inverse semigroups. Example 6.4 and Remark 6.5 highlight the advantage of this approach.

2. Preliminaries

Following Nica [17], we say that (G, P) is a **quasi-lattice ordered group** if G is a discrete group, P is a subsemigroup of G such that $P \cap P^{-1} = \{e\}$, and, under the partial order $p \leq q \Leftrightarrow p^{-1}q \in P$ on G, every pair of elements $p, q \in G$ with a common upper bound in P has a least common upper bound $p \vee q$ in P. We write $p \vee q = \infty$ to indicate that $p, q \in G$ have no common upper bound in P, and we write $p \vee q < \infty$ otherwise.

Definition 2.1: Let (G,P) be a quasi-lattice ordered group. A P**-graph** (Λ,d) consists of a countable category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Hom}(\Lambda), \mathrm{cod}, \mathrm{dom})$ together with a functor $d\colon \Lambda \to P$, called the **degree map**, which satisfies the **factorisation property**: for every $\lambda \in \Lambda$ and $p,q \in P$ with $d(\lambda) = pq$ there exist unique elements $\mu,\nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = p$, and $d(\nu) = q$.

Notation 2.2: Let (G,P) be a quasi-lattice ordered group, and let Λ be a P-graph. For $p \in P$ we define

$$\Lambda^p := \{\, \lambda \in \Lambda\colon d(\lambda) = p \,\}.$$

The factorisation property implies that $\Lambda^0 = \{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\}$. We define surjections $r, s : \Lambda \to \Lambda^0$ by $r(\lambda) := \mathrm{id}_{\mathrm{cod}(\lambda)}$ and $s(\lambda) := \mathrm{id}_{\mathrm{dom}(\lambda)}$, and we regard Λ^0 as the vertex set of Λ.

For $E \subset \Lambda$ and $\lambda \in \Lambda$ we define

$$\lambda E := \{\, \lambda\mu\colon \mu \in E \text{ and } r(\mu) = s(\lambda) \,\}$$

and

$$E\lambda := \{\, \mu\lambda\colon \mu \in E \text{ and } s(\mu) = r(\lambda) \,\}.$$

Hence, for $E \subset \Lambda$ and $v \in \Lambda^0$,

$$vE := \{\, \mu \in E\colon r(\mu) = v \,\}$$

and

$$Ev := \{\, \mu \in E\colon s(\mu) = v \,\}.$$

We write $\Lambda *_s \Lambda$ for the set $\{(\mu,\nu) \in \Lambda \times \Lambda : s(\mu) = s(\nu)\}$, and write $\Lambda *_{d,s} \Lambda$ for the set $\{(\mu,\nu) \in \Lambda *_s \Lambda : d(\mu) = d(\nu)\}$ consisting of pairs which are balanced with respect to the degree functor. More generally, for any pair U, V of subsets of Λ, we will write $U *_s V$ for $(U \times V) \cap (\Lambda *_s \Lambda)$, and $U *_{d,s} V$ for $(U \times V) \cap (\Lambda *_{d,s} \Lambda)$.

Definition 2.3: Let (G,P) be a quasi-lattice ordered group and let Λ be a P-graph. For $\mu,\nu \in \Lambda$ we say that $\lambda \in \Lambda$ is a **minimal common extension** of μ and ν if $d(\mu) \vee d(\nu) < \infty$, $d(\lambda) = d(\mu) \vee d(\nu)$ and there exist $\alpha \in \Lambda^{d(\mu)^{-1}(d(\mu)\vee d(\nu))}$ and $\beta \in \Lambda^{d(\nu)^{-1}(d(\mu)\vee d(\nu))}$ such that $\lambda = \mu\alpha = \nu\beta$. We write $\mathrm{MCE}(\mu,\nu)$ for the set of minimal common extensions of μ and ν. We say that Λ is **finitely aligned** if $\mathrm{MCE}(\mu,\nu)$ is finite (possibly empty) for all $\mu,\nu \in \Lambda$. Given $v \in \Lambda^0$ we say that $E \subset v\Lambda$ is **exhaustive** if for every $\mu \in v\Lambda$ there exists $\lambda \in E$ such that $\mathrm{MCE}(\mu,\lambda) \neq \emptyset$.

Note that, in particular, if $d(\mu) \vee d(\nu) = \infty$, then $\mathrm{MCE}(\mu, \nu) = \emptyset$.

Notation 2.4: We make frequent use of the abstract C^*-algebras generated by matrix units indexed by countable sets. Fix a countable set X. By [20, Corollary A.9 and Remark A.10], there is a unique (up to canonical isomorphism) C^*-algebra $\mathcal{K}_X$ generated by nonzero elements $\{\Theta_{x,y} : x, y \in X\}$ satisfying

$$\Theta_{x,y}^* = \Theta_{y,x} \quad \text{and} \quad \Theta_{x,y}\Theta_{w,z} = \begin{cases} \Theta_{x,z} & \text{if } y = w, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We call a family satisfying (2.1) a **family of matrix units over** X. In particular, given two such families $\{\alpha_{x,y} : x, y \in X\}$ and $\{\beta_{x,y} : x, y \in X\}$, there is a unique isomorphism $C^*(\{\alpha_{x,y} : x, y \in X\}) \to C^*(\{\beta_{x,y} : x, y \in X\})$ which carries each $\alpha_{x,y}$ to $\beta_{x,y}$. Since the set $\{\Theta_{x,y} : x, y \in X\}$ is closed under adjoints and multiplication, $\mathcal{K}_X = \overline{\operatorname{span}}\,\{\Theta_{x,y} : x, y \in X\}$.

For a finite subset F of X, we denote by P_F the projection $P_F := \sum_{x \in F} \Theta_{x,x}$ in $\mathcal{K}_X$. An $\varepsilon/3$ argument shows that the net $\{P_F : F \subset X \text{ is finite}\}$ is an approximate identity for $\mathcal{K}_X$.

Notation: In this paper, given a finitely aligned P-graph Λ, we deal both with representations of $\Lambda *_{d,s} \Lambda$, and also with representations of Λ itself. In addition, in each case there are two distinguished representations — the universal representation and the co-universal representation — which we frequently wish to talk about.

Our convention will be that Greek letters are used to denote representations of $\Lambda *_{d,s} \Lambda$, and Roman letters are used to denote representations of Λ; and the universal and co-universal representations will be denoted by the same letter in lower case and upper case, respectively.

3. Filters and ultrafilters in P-graphs

In the theories of graph C^*-algebras and of k-graph C^*-algebras, spaces of infinite paths — or an appropriate analogue — are often used to construct a representation by nonzero partial isometries. Precisely what should constitute an infinite path in a P-graph is not immediately clear; in fact, the question is already complicated enough for k-graphs. In this section we show how the roles played by paths and infinite paths in the representation theory of k-graphs can be played by filters and ultrafilters in the setting of P-graphs. We show how initial segments can be appended to or removed from filters and ultrafilters, and use this construction to associate to each P-graph a specific family of partial isometries on Hilbert space which will later be a key ingredient in our construction of the co-universal algebra of the P-graph. We got the idea of using ultrafilters to obtain a minimal representation from Exel, who introduced it in the context of partial-isometric representations of inverse semigroups [8].

Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. We define a relation $\preceq$ on Λ by $\lambda \preceq \mu$ if and only if $\mu = \lambda\mu'$ for some $\mu' \in \Lambda$.

Definition 3.1: A **filter** of Λ is a nonempty subset U of Λ such that

(F1) if $\mu \in U$ and $\lambda \preceq \mu$, then $\lambda \in U$, and
(F2) if $\mu, \nu \in U$, then there exists $\lambda \in U$ such that $\mu, \nu \preceq \lambda$.

Fix a filter U of Λ. The factorisation property and (F2) imply that if $\mu, \nu \in U$, then there is a unique element λ of $\mathrm{MCE}(\mu, \nu)$ such that $\lambda \in U$. This combined with (F1) and that U is nonempty implies that there is a unique $v \in \Lambda^0$ such that $v \in U$, and then we have $r(\lambda) = v$ for all $\lambda \in U$. We write $r(U) = v$.

We write $\widehat{\Lambda}$ for the collection of all filters of Λ, and we regard $\widehat{\Lambda}$ as a partially ordered set under inclusion. An **ultrafilter** of Λ is a filter $U \in \widehat{\Lambda}$ which is maximal; that is, U is not properly contained in any other filter V of Λ. We write $\widehat{\Lambda}_\infty$ for the collection of all ultrafilters of Λ.

Lemma 3.2: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. For each $\lambda \in \Lambda$ there exists an ultrafilter U of Λ such that $\lambda \in U$.*

Proof. We aim to apply Zorn's Lemma. Let $\mathcal{X}_\lambda$ denote the collection of all filters U of Λ such that $\lambda \in U$. Observe that $\mathcal{X}_\lambda$ is nonempty because $\{\mu \in \Lambda : \lambda \in \mu\Lambda\}$ is a filter of Λ which contains λ.

Fix a totally ordered subset $\mathcal{Y}$ of $\mathcal{X}_\lambda$. We claim that $\bigcup \mathcal{Y}$ is an upper bound for $\mathcal{Y}$ in $\mathcal{X}_\lambda$. To see this, it suffices to show that $\bigcup \mathcal{Y}$ is a filter of Λ; that is, we must verify (F1) and (F2). For (F1), suppose that $\mu \in \bigcup \mathcal{Y}$ and $\lambda \preceq \mu$. By definition of $\bigcup \mathcal{Y}$, we have $\mu \in V$ for some filter $V \in \mathcal{Y}$; and then since V is a filter we have $\lambda \in V \subset \bigcup \mathcal{Y}$ also. For (F2), suppose that $\mu, \nu \in \bigcup \mathcal{Y}$. Then there exist $V, W \in \mathcal{Y}$ such that $\mu \in V$ and $\nu \in W$. Since $\mathcal{Y}$ is totally ordered, we may suppose without loss of generality that $V \subset W$. So $\mu, \nu \in W$, and since W is a filter, it follows that there exists $\lambda \in W \subset \bigcup \mathcal{Y}$ such that $\mu, \nu \preceq W$. Hence $\bigcup \mathcal{Y}$ is an upper bound for $\mathcal{Y}$ as claimed.

Zorn's Lemma now implies that $\mathcal{X}_\lambda$ has a maximal element U. We have $\lambda \in U$ by definition of $\mathcal{X}_\lambda$. To see that U is an ultrafilter, observe that if V is a filter with $U \subset V$, then $\lambda \in U \subset V$ forces $V \in \mathcal{X}_\lambda$, and since U is maximal in $\mathcal{X}_\lambda$, it follows that $U = V$. ∎

Lemma 3.3: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Let $\mu \in \Lambda$, and let E be a finite exhaustive subset of $s(\mu)\Lambda$. Let U be an ultrafilter of Λ such that $\mu \in U$. Then there exists $\alpha \in E$ such that $\mu\alpha \in U$.*

Proof. We first claim that there exists $\alpha \in E$ such that $\mathrm{MCE}(\mu\alpha, \lambda) \neq \emptyset$ for all $\lambda \in U$. To see this, suppose for contradiction that for each $\alpha \in E$ there exists $\lambda_\alpha \in U$ such that $\mathrm{MCE}(\mu\alpha, \lambda_\alpha) = \emptyset$. Since U is a filter, there exists $\lambda \in U$ such that $\mu \preceq \lambda$ and $\lambda_\alpha \preceq \lambda$ for all $\alpha \in E$. Fix $\alpha \in E$. Since $\mathrm{MCE}(\mu\alpha, \lambda_\alpha) = \emptyset$ and $\lambda \in \lambda_\alpha\Lambda$, we have $\mathrm{MCE}(\mu\alpha, \lambda) = \emptyset$. Since $\mu \preceq \lambda$ we may factorise $\lambda = \mu\lambda'$ and then since

$$\mu\,\mathrm{MCE}(\alpha, \lambda') = \mathrm{MCE}(\mu\alpha, \mu\lambda') = \mathrm{MCE}(\mu\alpha, \lambda) = \emptyset,$$

we have $\mathrm{MCE}(\alpha, \lambda') = \emptyset$. Since $\alpha \in E$ was arbitrary, this contradicts that E is exhaustive.

Fix $\alpha \in E$ such that $\mathrm{MCE}(\mu\alpha, \lambda) \neq \emptyset$ for all $\lambda \in U$. We will show that $\mu\alpha \in U$. Since Λ, and hence U, is countable there is a cofinal sequence $(\lambda_n)_{n=1}^\infty$ in U with $\lambda_1 = \mu$ and $\lambda_n \preceq \lambda_{n+1}$ for all n. We have $\mathrm{MCE}(\mu\alpha, \lambda_n) \neq \emptyset$ for all n.

We claim that there exists a sequence $(\xi_n)_{n=1}^\infty$ in Λ such that for each n,

(1) $\xi_n \in \mathrm{MCE}(\mu\alpha, \lambda_n)$,
(2) if $n \geq 2$, then $\xi_{n-1} \preceq \xi_n$, and
(3) $\mathrm{MCE}(\xi_n, \lambda_m) \neq \emptyset$ for all m.

We prove the claim by induction on n. When $n = 1$, the path $\xi_1 := \mu\alpha$ satisfies (1) and (3) by choice of α, and (2) is trivial.

Now suppose that there are paths $\xi_1, \ldots, \xi_k$ satisfying (1)–(3). For each $m > k$, the set $\mathrm{MCE}(\xi_k, \lambda_m)$ is nonempty by the inductive hypothesis, so we may fix $\eta_m \in \mathrm{MCE}(\xi_k, \lambda_m)$. Fix $m > k$. Since $\lambda_{k+1} \preceq \lambda_m$, there is a unique $\xi \in \mathrm{MCE}(\xi_k, \lambda_{k+1})$ such that $\xi \preceq \eta_m$. Since Λ is finitely aligned, $\mathrm{MCE}(\xi_k, \lambda_{k+1})$ is finite, so there exists $\xi_{k+1} \in \mathrm{MCE}(\xi_k, \lambda_{k+1})$ such that $\xi_{k+1} \preceq \eta_m$ for infinitely many $m > k$. We claim that this ξ_{k+1} satisfies (1)–(3). It is straightforward to see that ξ_{k+1} satisfies (1) using that ξ_k satisfies (1), and that $\lambda_k \preceq \lambda_{k+1}$. It satisfies (2) by definition. To see that it satisfies (3), fix $m \in \mathbb{N}$. By choice of ξ_{k+1} there exists $m' \geq m$ such that $\xi_{k+1} \preceq \eta_{m'} \in \mathrm{MCE}(\mu\alpha, \lambda_{m'})$ and then that $\lambda_m \preceq \lambda_{m'}$ forces $\mathrm{MCE}(\xi_{k+1}, \lambda_m) \neq \emptyset$ also. This completes the proof of the claim.

Now let $V := \bigcup_{n\in\mathbb{N}}\{\zeta \in \Lambda : \zeta \preceq \xi_n\}$. That the ξ_n are increasing with respect to $\preceq$ implies that V is a filter. Since the λ_n are cofinal in U and since each $\lambda_n \preceq \xi_n$, we have $U \subset V$. Since U is an ultrafilter, it follows that $V = U$, and since $\mu\alpha \preceq \xi_n \in V$ for all n, it follows that $\mu\alpha \in U$ as claimed. ∎

Fix $\lambda \in \Lambda$. For $U \in \widehat{\Lambda}$ with $r(U) = s(\lambda)$, we define

$$\lambda \cdot U := \bigcup_{\mu\in U}\{\alpha \in \Lambda : \alpha \preceq \lambda\mu\}.$$

For $V \in \widehat{\Lambda}$ such that $\lambda \in V$, we define

$$\lambda^* \cdot V := \{\mu \in \Lambda : \lambda\mu \in V\}.$$

Lemma 3.4: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Fix $\lambda \in \Lambda$, and $U, V \in \widehat{\Lambda}$ with $r(U) = s(\lambda)$ and $\lambda \in V$. Then*

1. *$\lambda \cdot U$ and $\lambda^* \cdot V$ belong to $\widehat{\Lambda}$;*
2. *$\lambda^* \cdot (\lambda \cdot U) = U$ and $\lambda \cdot (\lambda^* \cdot V) = V$; and*
3. *$U \in \widehat{\Lambda}_\infty \implies \lambda \cdot U \in \widehat{\Lambda}_\infty$, and $V \in \widehat{\Lambda}_\infty \implies \lambda^* \cdot V \in \widehat{\Lambda}_\infty$.*

Proof. (1) Since $\lambda \in \lambda \cdot U$ and $s(\lambda) \in \lambda^* \cdot V$, both $\lambda \cdot U$ and $\lambda^* \cdot V$ are nonempty.

It is routine to use the factorisation property to check that both $\lambda \cdot U$ and $\lambda^* \cdot V$ satisfy (F1). Suppose $\mu, \nu \in \lambda \cdot U$. Then there exist $\alpha, \beta \in U$ such that $\mu \preceq \lambda\alpha$ and $\nu \preceq \lambda\beta$. Since $U \in \widehat{\Lambda}$, there exists $\eta \in U$ such that $\alpha, \beta \preceq \eta$, and the factorisation property then forces $\mu, \nu \preceq \lambda\eta \in \lambda \cdot U$. So $\lambda \cdot U$ satisfies (F2). Now suppose that $\mu, \nu \in \lambda^* \cdot V$. Then $\lambda\mu, \lambda\nu \in V$. Since $V \in \widehat{\Lambda}$, there exists $\eta \in V$ such that$\lambda\mu, \lambda\nu \preceq \eta$; it then follows from the factorisation property that $\eta = \lambda\eta'$ for some η' with $\mu, \nu \preceq \eta'$, and we have $\eta' \in \lambda^* \cdot V$ by definition. So $\lambda^* \cdot V$ satisfies (F2). This completes the proof of (1).

(2) We have

$$\mu \in \lambda^* \cdot (\lambda \cdot U) \iff \lambda\mu \in \lambda \cdot U \iff \mu \in U,$$

so $\lambda^* \cdot (\lambda \cdot U) = U$. To see that $\lambda \cdot (\lambda^* \cdot V) = V$, we first calculate

$$\begin{aligned} \mu \in \lambda \cdot (\lambda^* \cdot V) &\iff \mu \preceq \lambda\alpha \text{ for some } \alpha \in \lambda^* \cdot V \\ &\iff \mu \preceq \lambda\alpha \text{ for some } \alpha \text{ with } \lambda\alpha \in V. \end{aligned}$$

Since V satisfies (F2) and $\lambda \in V$, every $\nu \in V$ satisfies $\nu \preceq \lambda\alpha$ for some $\lambda\alpha \in V$. That is $\nu \in V$ if and only if $\nu \preceq \lambda\alpha$ for some $\lambda\alpha \in V$, and it follows that $\mu \in \lambda \cdot (\lambda^* \cdot V)$ if and only if $\mu \in V$ as required.

(3) Suppose that $U \in \widehat{\Lambda}_\infty$, and suppose that $U' \in \widehat{\Lambda}$ with $\lambda \cdot U \subset U'$. We must show that $U' = \lambda \cdot U$. We have $\lambda \in \lambda \cdot U \subset U'$, so $\lambda^* \cdot U'$ makes sense. We then have

$$\lambda^* \cdot U' \supset \lambda^* \cdot (\lambda \cdot U) = U$$

by part (2). Since U is an ultrafilter, it follows that $\lambda^* \cdot U' = U$ and then another application of (2) gives

$$U' = \lambda \cdot (\lambda^* \cdot U') = \lambda \cdot U.$$

A similar argument shows that $\lambda^* \cdot V$ is an ultrafilter. ∎

Definition 3.5: Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Define $T : \Lambda \to B(\ell^2(\widehat{\Lambda}))$ by $T_\lambda e_U := \delta_{s(\lambda),r(U)} e_{\lambda \cdot U}$.

Routine calculations using the inner-product on $\ell^2(\widehat{\Lambda})$ (see, for example, [23, Proposition 2.12]) show that the T_λ are partial isometries with adjoints characterised by

$$T_\lambda^* e_U = \begin{cases} e_{\lambda^* \cdot U} & \text{if } \lambda \in U, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

In particular, for each $\lambda \in \Lambda$, the operator $T_\lambda T_\lambda^*$ is the orthogonal projection onto the subspace $\overline{\operatorname{span}}\,\{e_U : U \in \widehat{\Lambda}, \lambda \in U\} \subset \ell^2(\widehat{\Lambda})$.

4. The balanced algebras of a P-graph

In this section we introduce and analyse what we call the **balanced algebras** of a P-graph. We associate to each finitely aligned P-graph two balanced algebras — a universal balanced algebra, and a quotient thereof, which we call the co-universal balanced algebra. For a k-graph, the universal balanced algebra would correspond to the fixed-point algebra for the gauge-action on the Toeplitz algebra of the k-graph, and the co-universal balanced algebra to the fixed-point algebra for the gauge action on the Cuntz–Krieger algebra of the k-graph.

We show in the next section that the universal balanced algebra of a P-graph is isomorphic to the fixed-point algebra for the canonical coaction of G on the Toeplitz algebra of the P-graph. We then use this and a bootstrapping argument to construct the co-universal algebra of the P-graph. It turns out that the analysis of [2] is greatly simplified by first demonstrating that the fixed-point subalgebra of the Toeplitz algebra has a universal property and admits a co-universal quotient in its own right.

4.1. The universal balanced algebra. In this subsection we define the universal balanced algebra of a P-graph Λ and characterise the representations of this balanced algebra.

Definition 4.1: Let (G,P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. A **representation** of $\Lambda *_{d,s} \Lambda$ in a C^*-algebra B is a map $\tau : \Lambda *_{d,s} \Lambda \to B$, $(\mu,\nu) \mapsto \tau_{\mu,\nu}$ such that for all $(\mu,\nu),(\xi,\eta) \in \Lambda *_{d,s} \Lambda$,

(B1) $\tau^*_{\mu,\nu} = \tau_{\nu,\mu}$, and

(B2) $\tau_{\mu,\nu}\tau_{\xi,\eta} = \sum_{\nu\alpha=\xi\beta\in \mathrm{MCE}(\nu,\xi)} \tau_{\mu\alpha,\eta\beta}$.

We denote by $C^*(\tau)$ the C^*-subalgebra of B generated by the $\tau_{\mu,\nu}$.

Observe that $\mathrm{MCE}(\mu,\mu) = \{\mu\}$ for all μ. Hence (B1) and (B2) imply that $\tau_{\mu,\mu} = \tau^2_{\mu,\mu} = \tau^*_{\mu,\mu}$, so each $\tau_{\mu,\mu}$ is a projection. Moreover, $\tau^*_{\mu,\nu}\tau_{\mu,\nu} = \tau_{\nu,\nu}$ for all $(\mu,\nu) \in \Lambda *_{d,s} \Lambda$. So the range of a representation of $\Lambda *_{d,s} \Lambda$ consists of partial isometries. Finally, since $\mathrm{MCE}(\mu,\nu) = \mathrm{MCE}(\nu,\mu)$ for all μ and ν, condition (B2) implies that the projections $\{\tau_{\mu,\mu} : \mu \in \Lambda\}$ pairwise commute.

Lemma 4.2: *Let (G,P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Let $T : \Lambda \to B(\ell^2(\widehat{\Lambda}))$ be as in Definition 3.5. Then the map $(\mu,\nu) \mapsto T_\mu T^*_\nu$ is a representation of $\Lambda *_{d,s} \Lambda$.*

Proof. Condition (B1) is obvious. For (B2), fix $(\mu,\nu),(\xi,\eta) \in \Lambda *_{d,s} \Lambda$. Then for $U \in \widehat{\Lambda}$, we have

$$T_\nu T^*_\nu T_\xi T^*_\xi e_U = \begin{cases} e_U & \text{if } \nu,\xi \in U, \\ 0 & \text{otherwise.} \end{cases}$$

As discussed after Definition 3.1, $\mathrm{MCE}(\nu,\xi)\cap U$ has at most one element, so

$$\sum_{\nu\alpha=\xi\beta\in\mathrm{MCE}(\nu,\xi)} T_{\nu\alpha}T_{\nu\alpha}^* e_U = \begin{cases} e_U & \text{if } \mathrm{MCE}(\nu,\xi)\cap U\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

Conditions (F1) and (F2) imply that $\mathrm{MCE}(\nu,\xi)\cap U\neq\emptyset$ if and only if $\nu,\xi\in U$. So $T_\nu T_\nu^* T_\xi T_\xi^* = \sum_{\lambda\in\mathrm{MCE}(\nu,\xi)} T_\lambda T_\lambda^*$.

Since the T_λ are all partial isometries, it follows that

$$T_\mu T_\nu^* T_\xi T_\eta^* = T_\mu T_\nu^* T_\nu T_\nu^* T_\xi T_\xi^* T_\xi T_\eta^* = \sum_{\nu\alpha=\xi\beta\in\mathrm{MCE}(\nu,\xi)} T_\mu T_\nu^* T_{\nu\alpha} T_{\xi\beta}^* T_\xi T_\eta^*.$$

So it is enough to fix $U\in\widehat{\Lambda}$ and show that for $\nu\alpha=\xi\beta\in\mathrm{MCE}(\nu,\xi)$, we have $T_\mu T_\nu^* T_{\nu\alpha} e_U = T_{\mu\alpha} e_U$ (it will follow from symmetry that $T_{\xi\beta}^* T_\xi T_\eta^* e_U = T_{\eta\beta}^* e_U$). If $r(U)\neq s(\alpha)$ then both sides are equal to zero, so suppose that $r(U)=s(\alpha)$. Then

$$\begin{aligned}\mu\cdot(\nu^*\cdot(\nu\alpha\cdot U)) &= \{\kappa\in\Lambda : \kappa\preceq\mu\zeta \text{ for some } \zeta\in\nu^*\cdot(\nu\alpha\cdot U)\}\\ &= \{\kappa\in\Lambda : \kappa\preceq\mu\zeta \text{ for some } \zeta \text{ such that } \nu\zeta\in\nu\alpha\cdot U\}\\ &= \{\kappa\in\Lambda : \kappa\preceq\mu\zeta \text{ for some } \zeta\in\alpha\cdot U\}\\ &= \{\kappa\in\Lambda : \kappa\preceq\mu\alpha\zeta' \text{ for some } \zeta'\in U\}\\ &= \mu\alpha\cdot U.\end{aligned}$$

Hence

$$T_\mu T_\nu^* T_{\nu\alpha} e_U = e_{\mu\cdot(\nu^*\cdot(\nu\alpha\cdot U))} = e_{\mu\alpha\cdot U} = T_{\mu\alpha} e_U.$$

So $(\mu,\nu)\mapsto T_\mu T_\nu^*$ satisfies (B1) and (B2) as required. ∎

Proposition 4.3: *Let (G,P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. There exists a C^*-algebra $\mathcal{B}_\Lambda$ generated by a representation ω of $\Lambda *_{d,s} \Lambda$ which is universal in the following sense: for every representation τ of $\Lambda *_{d,s}\Lambda$, there is a C^*-homomorphism $\rho_\tau : \mathcal{B}_\Lambda \to C^*(\tau)$ satisfying $\rho_\tau\circ\omega=\tau$. Moreover, the partial isometries $\omega_{\mu,\nu}$ are all nonzero, and for each $\mu\in\Lambda$ and each finite exhaustive set $E\subset s(\mu)\Lambda\setminus\{s(\mu)\}$,*

$$\textstyle\prod_{\alpha\in E}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})\neq 0.$$

Proof. An argument along the lines of [20, pages 12 and 13] shows that there is a C^*-algebra $\mathcal{B}_\Lambda$ generated by a representation ω of $\Lambda *_{d,s}\Lambda$ which is universal for representations of $\Lambda *_{d,s}\Lambda$.

To see that each $\omega_{\mu,\nu}$ is nonzero, observe that for each $\lambda \in \Lambda$, there exists $U \in \widehat{\Lambda}$ such that $\lambda \in U$, and hence the partial isometries T_λ of Definition 3.5 are all nonzero. It follows that

$$T_\mu T_\nu^* \neq 0 \quad \text{for all } (\mu,\nu) \in \Lambda *_{d,s} \Lambda. \tag{4.1}$$

By Lemma 4.2 and the universal property of $\mathcal{B}_\Lambda$, there is a homomorphism which takes each $\omega_{\mu,\nu}$ to $T_\mu T_\nu^*$, and it follows that the $\omega_{\mu,\nu}$ are nonzero as well.

Fix $\mu \in \Lambda$ and a finite exhaustive set $E \subset s(\mu)\Lambda \setminus \{s(\mu)\}$. Let $U_\mu := \{\lambda \in \Lambda : \mu \in \lambda\Lambda\}$. Then $U_\mu \in \widehat{\Lambda}$ and we have $\mu \in U_\mu$, but $\mu\alpha \notin U_\mu$ for all $\alpha \in E$. So $T_\mu T_\mu^* e_{U_\mu} = e_{U_\mu}$, but $T_{\mu\alpha} T_{\mu\alpha}^* e_{U_\mu} = 0$ for all $\alpha \in E$. Thus

$$\prod_{\alpha\in E} (T_\mu T_\mu^* - T_{\mu\alpha} T_{\mu\alpha}^*) e_{U_\mu} = e_{U_\mu} \neq 0. \tag{4.2}$$

Lemma 4.2 and the first statement of this proposition imply that there is a homomorphism taking $\prod_{\alpha\in E}(\omega_{\mu,\mu} - \omega_{\mu\alpha,\mu\alpha})$ to $\prod_{\alpha\in E}(T_\mu T_\mu^* - T_{\mu\alpha}T_{\mu\alpha}^*)$, so $\prod_{\alpha\in E}(\omega_{\mu,\mu} - \omega_{\mu\alpha,\mu\alpha})$ is nonzero also. ∎

THEOREM 4.4: *Let (G,P) be a quasi-lattice ordered group, let Λ be a finitely aligned P-graph, and let τ be a representation of $\Lambda *_{d,s} \Lambda$. Then the homomorphism $\rho_\tau : \mathcal{B}_\Lambda \to C^*(\tau)$ induced by the universal property of $\mathcal{B}_\Lambda$ is injective if and only if*

1. *$\tau_{\mu,\mu} \neq 0$ for all $\mu \in \Lambda$, and*
2. *$\prod_{\alpha\in E}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha}) \neq 0$ for each $\mu \in \Lambda$ and each finite exhaustive $E \subset s(\mu)\Lambda \setminus \{s(\mu)\}$.*

Moreover, $\mathcal{B}_\Lambda$ is an AF algebra.

To prove the theorem, we first analyse the structure of $\mathcal{B}_\Lambda$. We require the notion of a $\vee$-closed subset of P.

A subset F of P is called $\vee$-closed if, whenever $p, q \in F$ satisfy $p \vee q < \infty$, we have $p \vee q \in F$. Since the $\vee$ operation is both commutative and associative, given $G \subset P$ the formula

$$\vee G := p_1 \vee (p_2 \vee (p_3 \vee \cdots \vee p_{|G|})) \in P \cup \{\infty\}$$

is well-defined. So if $F \subset P$ is finite, then

$$\overline{F} := \{\vee G : G \subset F, \vee G \neq \infty\}$$

is a finite $\vee$-closed subset of P, which contains F since $\vee\{p\} = \{p\}$ for all $p \in P$. It follows that the collection of finite $\vee$-closed subsets of P is directed under $\subseteq$

and the union of all finite $\vee$-closed subsets of P is P itself. A minimal element of a finite $\vee$-closed subset F of P is an element $p \in F$ such that $q \in F$ implies $q \not\leq p$.

Definition 4.5: For each $\vee$-closed subset F of P, we define

$$B_F := \Big\{ \sum_{p \in F} a_p : a_p \in \overline{\operatorname{span}}\,\{\omega_{\mu,\nu} : (\mu,\nu) \in \Lambda^p *_s \Lambda^p\} \text{ for each } p \in F \Big\},$$

and for each $p \in P$, we write B_p for $B_{\{p\}}$. So B_F consists of finite linear combinations of elements of the B_p where p ranges over F.

For the following lemma, recall from Section 2 our notation for the abstract algebra $\mathcal{K}_X$ generated by matrix units indexed by a countable set X.

LEMMA 4.6: *Let (G,P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Then*

1. *for each $p \in P$ there is an isomorphism $B_p \cong \bigoplus_{v \in \Lambda^0} \mathcal{K}_{\Lambda^p v}$ satisfying $\omega_{\mu,\nu} \mapsto \Theta_{\mu,\nu}$;*
2. *for each finite $\vee$-closed subset F of P, the set B_F is an AF C^*-subalgebra of $\mathcal{B}_\Lambda$; and*
3. *$\mathcal{B}_\Lambda = \varinjlim B_F$ where the collection of finite $\vee$-closed subsets of P is directed by inclusion.*

Proof. For (1), one uses (B1) and (B2) to see that $\{\omega_{\mu,\nu} : (\mu,\nu) \in \Lambda^p v \times \Lambda^p v\}$ is a family of matrix units for each $v \in \Lambda^0$. The $\omega_{\mu,\nu}$ are all nonzero by Proposition 4.3. The uniqueness of $\mathcal{K}_{\Lambda^p v}$ implies that

$$B_p(v) := \overline{\operatorname{span}}\,\{\omega_{\mu,\nu} : (\mu,\nu) \in \Lambda^p v \times \Lambda^p v\}$$

is isomorphic to $\mathcal{K}_{\Lambda^p v}$ via $\omega_{\mu,\nu} \mapsto \Theta_{\mu,\nu}$. Moreover, (B2) implies that if $\mu,\nu \in \Lambda^p v$ and $\xi,\eta \in \Lambda^p w$ for distinct v,w, then $\omega_{\mu,\nu}\omega_{\xi,\eta} = 0$. Hence $B_p = \bigoplus_{v \in \Lambda^0} B_p(v) \cong \bigoplus_{v\in\Lambda^0} \mathcal{K}_{\Lambda^p v}$.

For (2), we proceed by induction on $|F|$ as in [2, Lemma 3.6]. When $|F| = 1$, statement (2) follows from (1). Now suppose that B_F is an AF C^*-subalgebra of $\mathcal{B}_\Lambda$ whenever $|F| \leq k$, and fix a $\vee$-closed subset F of P with $|F| = k+1$. Fix a minimal element $m \in F$. Then $G := F \setminus \{m\}$ is also $\vee$-closed. By the inductive hypothesis, B_G is an AF C^*-algebra. Moreover, one can check on spanning elements using (B2) that $B_m B_G, B_G B_m \subset B_G$. Hence [6, Corollary 1.8.4] implies that B_F is a C^*-algebra. To see that B_F is AF, observe that B_G is an ideal of B_F with quotient $B_F/B_G \cong B_m/(B_m \cap B_G)$. Both B_G and B_m are AF by the inductive hypothesis, and since quotients of AF algebras are AF, it follows that B_F is an extension of an AF algebra by an AF algebra, and hence itself AF (see, for example, [5, Theorem III.6.3]).

For (3), observe that if $G \subset F$ are both $\vee$-closed, then $B_G \subset B_F$, and that $\bigcup_F B_F$ contains all the generators of $\mathcal{B}_\Lambda$. ∎

We now establish two technical results which we shall use to prove Theorem 4.4.

Lemma 4.7: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Let τ be a representation of $\Lambda *_{d,s} \Lambda$. Fix $v \in \Lambda^0$ and a finite subset $H \subset v\Lambda \setminus \{v\}$. For $\mu, \nu \in \Lambda^p v$, let*

$$\theta_{\mu,\nu} := \tau_{\mu,\nu} \prod_{\lambda \in H} (\tau_{\nu,\nu} - \tau_{\nu\lambda,\nu\lambda}).$$

Then for $\mu, \nu, \rho, \sigma \in \Lambda^p v$,

$$\theta^*_{\mu,\nu} = \theta_{\nu,\mu} \quad \text{and} \quad \theta_{\mu,\nu}\theta_{\rho,\sigma} = \delta_{\nu,\rho}\theta_{\mu,\sigma}.$$

In particular, if τ satisfies conditions (1) *and* (2) *of Theorem* 4.4*, then $\{\theta_{\mu,\nu} : \mu, \nu \in \Lambda^p v\}$ is a family of nonzero matrix units.*

Proof. For each $\mu, \nu \in \Lambda^p v$ we have $(\tau_{\nu,\nu} - \tau_{\nu\lambda,\nu\lambda})\tau_{\nu,\mu} = \tau_{\nu,\mu}(\tau_{\mu,\mu} - \tau_{\mu\lambda,\mu\lambda})$, and hence

$$\Big(\prod_{\lambda \in H} (\tau_{\nu,\nu} - \tau_{\nu\lambda,\nu\lambda})\Big)\tau_{\nu,\mu} = \tau_{\nu,\mu} \prod_{\lambda \in H} (\tau_{\mu,\mu} - \tau_{\mu\lambda,\mu\lambda}). \tag{4.3}$$

It now follows from (B1) and (4.3) that $\theta^*_{\mu,\nu} = \theta_{\nu,\mu}$. Condition (B2) and (4.3) imply that for $\rho, \sigma \in \Lambda^p v$ we have

$$\theta_{\mu,\nu}\theta_{\rho,\sigma} = \begin{cases} \big(\prod_{\lambda \in H}(\tau_{\mu,\mu} - \tau_{\mu\lambda,\mu\lambda})\big)\tau_{\mu,\sigma}\big(\prod_{\eta \in H}(\tau_{\sigma,\sigma} - \tau_{\sigma\eta,\sigma\eta})\big) & \text{if } \nu = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Thus (4.3) for $\mu, \sigma \in \Lambda^p v$ and that $\prod_{\eta \in H}(\tau_{\sigma,\sigma} - \tau_{\sigma\eta,\sigma\eta})$ is a projection imply that $\theta_{\mu,\nu}\theta_{\rho,\sigma} = \delta_{\nu,\rho}\theta_{\mu,\sigma}$. Hence $\{\theta_{\mu,\nu} : \mu, \nu \in \Lambda^p v\}$ is a family of matrix units.

Now suppose that τ satisfies conditions (1) and (2) of Theorem 4.4. It suffices to show $\theta_{\mu,\mu} \neq 0$ for $\mu \in \Lambda^p v$. If H is exhaustive, then in particular $H \neq \emptyset$, and $\theta_{\mu,\mu} = \prod_{\lambda \in H}(\tau_{\mu,\mu} - \tau_{\mu\lambda,\mu\lambda}) \neq 0$ by assumption. If H is not exhaustive, then there exists $\eta \in v\Lambda$ with $\mathrm{MCE}(\lambda, \eta) = \emptyset$ for all $\lambda \in H$. It follows that $\tau_{\mu\eta,\mu\eta}\theta_{\mu,\mu} = \tau_{\mu\eta,\mu\eta} \neq 0$. Hence $\theta_{\mu,\mu} \neq 0$. ∎

In the proof of the next lemma we need some notation from [9]. Given subsets U and V of a finitely aligned P-graph Λ, we write $\operatorname{Ext}(U;V)$ for the set

$$\{\alpha \in \Lambda : \text{ there exist } \mu \in U \text{ and } \nu \in V \text{ such that } \mu\alpha \in \operatorname{MCE}(\mu,\nu)\}.$$

Roughly speaking, $\operatorname{Ext}(U;V)$ is the set of tails which extend paths in U to minimal common extensions with paths in V. Since Λ is finitely aligned, if U and V are finite, then so is $\operatorname{Ext}(U;V)$.

Lemma 4.8: *Let (G,P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Let τ be a representation of $\Lambda *_{d,s} \Lambda$ which satisfies conditions* (1) *and* (2) *of Theorem* 4.4. *Let F be a finite $\vee$-closed subset of P and let m be a minimal element of F. For each $p \in F$, let X_p be a finite subset of Λ^p, and let $X = \bigcup_{p \in F} X_p$. Fix scalars $\{a_{\mu,\nu} : (\mu,\nu) \in X *_{d,s} X\}$. Then*

$$\Big\| \sum_{(\mu,\nu)\in X *_{d,s} X} a_{\mu,\nu}\tau_{\mu,\nu} \Big\| \geq \Big\| \sum_{(\mu,\nu)\in X_m *_s X_m} a_{\mu,\nu}\tau_{\mu,\nu} \Big\|.$$

Proof. For each $p \in F$, let $a_p := \sum_{(\mu,\nu)\in X_p *_s X_p} a_{\mu,\nu}\tau_{\mu,\nu}$, and let $a := \sum_{p\in F} a_p$. We must prove that $\|a\| \geq \|a_m\|$. If $\|a_m\| = 0$, then the result is trivial. So assume $\|a_m\| > 0$, and in particular $X_m \neq \emptyset$. Define

$$H := \operatorname{Ext}(X_m; X \setminus X_m) \quad \text{and} \quad Q := \sum_{\mu \in X_m} \Big(\prod_{\lambda \in s(\mu)H} (\tau_{\mu,\mu} - \tau_{\mu\lambda,\mu\lambda}) \Big).$$

Then Q is a projection, and with $\theta_{\mu,\nu}$ defined as in Lemma 4.7,

$$\begin{aligned} a_m Q = \sum_{(\mu,\nu)\in X_m *_s X_m} a_{\mu,\nu}\tau_{\mu,\nu} Q &= \sum_{(\mu,\nu)\in X_m *_s X_m} a_{\mu,\nu}\tau_{\mu,\nu}\theta_{\nu,\nu} \\ &= \sum_{(\mu,\nu)\in X_m *_s X_m} a_{\mu,\nu}\theta_{\mu,\nu}. \end{aligned}$$

We claim that $a_p Q = 0$ for all $p \in F \setminus \{m\}$. To see this, fix $p \in F \setminus \{m\}$ and $\rho,\sigma \in X_p$. If $\operatorname{MCE}(\mu,\sigma) = \emptyset$ for all $\mu \in X_m$, then $\tau_{\rho,\sigma}Q = 0$. If there exists $\mu \in X_m$ with $\operatorname{MCE}(\mu,\sigma) \neq \emptyset$, then there exists $\lambda \in H$ with $\mu\lambda = \sigma$. So $\tau_{\rho,\sigma}(\tau_{\mu,\mu} - \tau_{\mu\lambda,\mu\lambda}) = 0$, and it follows that $\tau_{\rho,\sigma}Q = 0$. This proves the claim.

It follows from (B1) and (B2) that $\{\tau_{\mu,\nu} : (\mu,\nu) \in X_m *_s X_m\}$ is a family of matrix units. They are nonzero because each $\tau_{\mu,\mu} \neq 0$. By Lemma 4.7, the set $\{\theta_{\mu,\nu} : (\mu,\nu) \in X_m *_s X_m\}$ is also a family of nonzero matrix units, so as in Notation 2.4, $\theta_{\mu,\nu} \mapsto \tau_{\mu,\nu}$ determines an isomorphism

$$\overline{\operatorname{span}}\,\{\theta_{\mu,\nu} : (\mu,\nu) \in X_m *_s X_m\} \to \overline{\operatorname{span}}\,\{\tau_{\mu,\nu} : (\mu,\nu) \in X_m *_s X_m\}.$$

It follows that

$$\|a\| \geq \|aQ\| = \|a_m Q\| = \Big\| \sum_{(\mu,\nu)\in X_m * X_m} a_{\mu,\nu}\theta_{\mu,\nu} \Big\| = \Big\| \sum_{(\mu,\nu)\in X_m *_s X_m} a_{\mu,\nu}\tau_{\mu,\nu} \Big\|,$$

completing the proof. ∎

Proof of Theorem 4.4. The "only if" statement follows from Proposition 4.3. For the "if" statement, suppose that τ satisfies (1) and (2).

Fix a nonempty finite $\vee$-closed subset F of P, and let m be a minimal element of F. For each $p \in F$, fix $b_p \in B_p$, and let $b = \sum_{p\in F} b_p$. Suppose $b \neq 0$; we will show that $\rho_\tau(b) \neq 0$.

For each $p \in F$ and each finite subset $X \subseteq \Lambda^p$, let Q_X denote the projection $\sum_{\lambda\in X} \omega_{\lambda,\lambda}$. By Lemma 4.6(1) and Notation 2.4, we have

$$b_p = \lim_{X\subseteq\Lambda^p, |X|<\infty} Q_X b_p Q_X.$$

For each $p \in F$ fix a finite set $X_p \subset \Lambda^p$ such that

$$\|b_p - Q_{X_p} b_p Q_{X_p}\| \leq \frac{\|b_m\|}{4|F|}.$$

In particular, $\|Q_{X_m} b_m Q_{X_m}\| \geq 3\|b_m\|/4$.

For each $p \in F$ we have $Q_{X_p} b_p Q_{X_p} \in \operatorname{span}\{\omega_{\mu,\nu} : \mu,\nu \in \Lambda^p\}$. Lemma 4.8 therefore implies that $\|\rho_\tau(\sum_{p\in F} Q_{X_p} b_p Q_{X_p})\| \geq \|Q_{X_m} b_m Q_{X_m}\|$. We now have

$$\begin{aligned}
\|\rho_\tau(b)\| &= \Big\|\rho_\tau(b) - \rho_\tau\Big(\sum_{p\in F} Q_{X_p} b_p Q_{X_p}\Big) + \rho_\tau\Big(\sum_{p\in F} Q_{X_p} b_p Q_{X_p}\Big)\Big\| \\
&\geq \Big\|\rho_\tau\Big(\sum_{p\in F} Q_{X_p} b_p Q_{X_p}\Big)\Big\| - \Big\|\rho_\tau(b) - \rho_\tau\Big(\sum_{p\in F} Q_{X_p} b_p Q_{X_p}\Big)\Big\| \\
&\geq \Big\|\rho_\tau\Big(\sum_{p\in F} Q_{X_p} b_p Q_{X_p}\Big)\Big\| - \sum_{p\in F}\Big\|\rho_\tau(b_p) - \rho_\tau(Q_{X_p} b_p Q_{X_p})\Big\| \\
&\geq \|Q_{X_m} b_m Q_{X_m}\| - \sum_{p\in F} \frac{\|b_m\|}{4|F|} \\
&\geq \frac{3\|b_m\|}{4} - \frac{\|b_m\|}{4} \\
&> 0.
\end{aligned}$$

Hence $\rho_\tau(b) \neq 0$, and it follows that ρ_τ is injective on B_F. It now follows from Lemma 4.6(3) that ρ_τ is injective on $\Lambda *_{d,s} \Lambda$; and Lemma 4.6(3) and (2) show that $\Lambda *_{d,s} \Lambda$ is AF because direct limits of AF algebras are also AF (see, for example, [5, Theorem III.3.4]). ∎

4.2. Ideals of the balanced algebra. In this subsection we prove our key technical result, Theorem 4.9. This theorem identifies generating elements for any ideal $I \lhd \mathcal{B}_\Lambda$ which contains none of the generators $\omega_{\mu,\nu}$. We use Theorem 4.9 in the next subsection to see that there is a unique largest ideal containing no $\omega_{\mu,\nu}$. The corresponding quotient of $\mathcal{B}_\Lambda$ is the desired co-universal balanced algebra of Λ.

Theorem 4.9: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Let τ be a representation of $\Lambda *_{d,s} \Lambda$ such that each $\tau_{\mu,\mu}$ is nonzero, and let $\rho_\tau : \mathcal{B}_\Lambda \to C^*(\tau)$ be the homomorphism induced by the universal property of $\mathcal{B}_\Lambda$. Then $\ker(\rho_\tau)$ is generated by the set*

$$\bigcup_{\mu\in\Lambda} \Big\{ \prod_{\alpha\in E} (\omega_{\mu,\mu} - \omega_{\mu\alpha,\mu\alpha}) : E \subset s(\mu)\Lambda \text{ is finite exhaustive and } \prod_{\alpha\in E} (\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha}) = 0 \Big\}.$$

By Lemma 4.6(3), it suffices to show that the ideals of the B_F are all generated by the appropriate elements. We require the following technical lemma.

Lemma 4.10: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Let F be a finite $\vee$-closed subset of P, let m be a minimal element of F, and let H be a finite subset of Λ^m. Then*

$$A_H := \operatorname{span}\{\omega_{\mu,\nu} : (\mu, \nu) \in H *_s H\} + B_{F\setminus\{m\}}$$

is a C^*-subalgebra of B_F and $B_F = \varinjlim A_H$.

Proof. Calculations using (B1) and (B2) show that $\operatorname{span}\{\omega_{\mu,\nu} : (\mu, \nu) \in H *_s H\}$ is a finite-dimensional C^*-algebra. Using (B2) and that m is minimal, one checks that $\operatorname{span}\{\omega_{\mu,\nu} : (\mu, \nu) \in H *_s H\}$ is absorbed by $B_{F\setminus\{m\}}$ under multiplication. The result then follows from [6, Corollary 1.8.4] since each spanning element of B_F belongs to some A_H. ∎

Proposition 4.11: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Let τ be a representation of $\Lambda *_{d,s} \Lambda$ such that each $\tau_{\mu,\mu}$ is nonzero, and let $\rho_\tau : \mathcal{B}_\Lambda \to C^*(\tau)$ be the homomorphism induced by the universal property of $\mathcal{B}_\Lambda$. Then for each finite $\vee$-closed subset F of P, $\ker(\rho_\tau) \cap B_F$ is generated by the set*

$$\bigcup_{q\in F} \Big\{ \prod_{\alpha\in E} (\omega_{\mu,\mu} - \omega_{\mu\alpha,\mu\alpha}) : \mu \in \Lambda^q, E \subset \bigcup_{q\le p\in F} s(\mu)\Lambda^{q^{-1}p} \text{ is}$$

$$\text{finite exhaustive, and } \prod_{\alpha\in E} (\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha}) = 0 \Big\}. \tag{4.4}$$

We will need the following notation in the proof of the proposition. Given a finite subset G of Λ, we define $\mathrm{MCE}(G) := \big\{\lambda \in \bigcap_{\mu\in G} \mu\Lambda : d(\lambda) = \bigvee_{\mu\in G} d(\mu)\big\}$; in particular, if $\bigvee_{\mu\in G} d(\mu) = \infty$, then $\mathrm{MCE}(G) = \emptyset$. Given a subset F of Λ, we define

$$\vee F := \bigcup_{G\subset F} \mathrm{MCE}(G).$$

Since Λ is finitely aligned, $\vee F$ is finite. It is not hard to see that $\vee F$ is the smallest subset of Λ which contains F and is closed under taking minimal common extensions.

Proof of Proposition 4.11. For each finite $\vee$-closed subset F of P denote by I^F_τ the ideal of B_F generated by the set (4.4). Fix a finite $\vee$-closed set F. Since each element of the set (4.4) belongs to $\ker(\rho_\tau) \cap B_F$ by definition, we have $I^F_\tau \subset \ker(\rho_\tau) \cap B_F$, so it suffices to establish the reverse inclusion.

We proceed by induction on $|F|$. Since ρ_τ is injective on each B_p, we have $\{0\} = \ker(\rho_\tau) \cap B_F \subset I^F_\tau$ if $|F| = 1$. Suppose now that $\ker(\rho_\tau) \cap B_F \subset I^F_\tau$ whenever $|F| < k$, and fix F with $|F| = k$. Let m be a minimal element of F, and define $G := F \setminus \{m\}$. Recall from Lemma 4.10 that for finite $H \subseteq \Lambda^m$, A_H denotes $\mathrm{span}\{\omega_{\mu,\nu} : (\mu,\nu) \in H *_s H\} + B_{F\setminus\{m\}}$. By Lemma 4.10, it suffices to show that $\ker(\rho_\tau) \cap A_H \subset I^F_\tau \cap A_H$ for all H.

Fix scalars $\{a_{\mu,\nu} : (\mu,\nu) \in H *_s H\}$. Without loss of generality, we may assume that for each $v \in s(H)$ there exists $\mu,\nu \in Hv$ with $a_{\mu,\nu} \ne 0$. Fix $T_p \in B_p$ for each $p \in G$. Then

$$a := \sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\omega_{\mu,\nu} + \sum_{p\in G} T_p$$

is a typical element of A_H. Suppose $\rho_\tau(a) = 0$. We must show that $a \in I_\tau^F$. We proceed in three steps:

1. Decompose a as $a = a_{\{m\}} + a_G$, where $a_G \in B_G$, and $a_{\{m\}}$ is a linear combination of elements of the form $\prod_{\alpha\in E}(\omega_{\mu,\mu} - \omega_{\mu\alpha,\mu\alpha})\omega_{\mu,\nu}$ where E is a finite exhaustive $\vee$-closed subset of $\bigcup_{p\in G, m\leq p}\Lambda^{m^{-1}p}$ (see (4.9) below).
2. Show that $a_{\{m\}} \in \ker(\rho_\tau)$.
3. Deduce that $a_G \in \ker(\rho_\tau)$ and then apply the inductive hypothesis.

Step 1. (Decompose a as $a = a_{\{m\}} + a_G$.) Let

$$M := \min_{v\in s(H)}\Big\|\sum_{\mu,\nu\in Hv} a_{\mu,\nu}\omega_{\mu,\nu}\Big\|.$$

Then $M \neq 0$ by our assumption on the scalars $a_{\mu,\nu}$. For each $p \in G$ the isomorphism $B_p \cong \bigoplus_{v\in\Lambda^0}\mathcal{K}_{\Lambda^p v}$ of Lemma 4.6(1) and the approximate identities of the $\mathcal{K}_{\Lambda^p v}$ obtained from Notation 2.4 imply that there is a finite subset $F_p \subseteq \Lambda^p$ such that $\|T_p - (\sum_{\lambda\in F_p}\omega_{\lambda,\lambda})T_p\| < M/(2|G||H|^2)$. Without loss of generality, we may assume that if $(\mu,\nu) \in H *_s H$ and $\mu\alpha \in F_p$, then $\nu\alpha \in F_p$. Let

$$E_p := \{\alpha \in \Lambda^{m^{-1}p} : \text{ there exists } \mu \in \Lambda^m \text{ such that } \mu\alpha \in F_p\}.$$

Let $E_G := \bigcup_{p\in G} E_p$. For each $v \in \Lambda^0$, the set $v(\vee E_G)$ is equal to $\vee(vE_G)$ and so is closed under minimal common extensions. Each $v(\vee E_G)$ is finite because Λ is finitely aligned.

For each $\mu \in H$ we have

$$\Big\|\Big(\omega_{\mu,\mu} - \sum_{\alpha\in s(\mu)E_p}\omega_{\mu\alpha,\mu\alpha}\Big)T_p\Big\| = \Big\|\omega_{\mu,\mu}\Big(T_p - \Big(\sum_{\lambda\in F_p}\omega_{\lambda,\lambda}\Big)T_p\Big)\Big\| < \frac{M}{2|G||H|^2}.$$

Hence

$$\Big\|\sum_{\mu\in H}\Big(\omega_{\mu,\mu} - \sum_{\alpha\in s(\mu)E_p}\omega_{\mu\alpha,\mu\alpha}\Big)T_p\Big\| < \sum_{\mu\in H}\frac{M}{2|G||H|^2} = \frac{M}{2|G||H|}. \tag{4.5}$$

Claim: *Each $v(\vee E_G)$ is exhaustive.*

To prove this claim, we suppose that $v \in s(H)$ has the property that $v(\vee E_G)$ is not exhaustive and seek a contradiction. Since $v(\vee E_G)$ is not exhaustive, there exists $\lambda \in v\Lambda$ with $\mathrm{MCE}(\lambda, \alpha) = \emptyset$ for all $\alpha \in v(\vee E_G)$. We have

$$\Big(\sum_{\mu \in Hv} \omega_{\mu\lambda,\mu\lambda}\Big) a = \sum_{\mu,\nu \in Hv} a_{\mu,\nu}\omega_{\mu\lambda,\nu\lambda} + \sum_{\substack{p \in G \\ \mu \in Hv}} \omega_{\mu\lambda,\mu\lambda} T_p.$$

Since $\rho_\tau(a) = 0$, applying ρ_τ to both sides of the above equation and rearranging yields

$$\sum_{\mu,\nu \in Hv} a_{\mu,\nu}\tau_{\mu\lambda,\nu\lambda} = -\rho_\tau\Big(\sum_{\substack{p \in G \\ \mu \in Hv}} \omega_{\mu\lambda,\mu\lambda} T_p\Big).$$

It follows that

$$\begin{aligned}\Big\| \sum_{\mu,\nu \in Hv} a_{\mu,\nu}\tau_{\mu\lambda,\nu\lambda}\Big\| &\leq \sum_{p \in G} \Big\| \sum_{\mu \in Hv} \omega_{\mu\lambda,\mu\lambda} T_p \Big\| \\ &= \sum_{p \in G} \Big\| \sum_{\mu \in Hv} \omega_{\mu\lambda,\mu\lambda}\omega_{\mu,\mu} T_p \Big\| \\ &\leq \sum_{p \in G} \Big(\Big\| \sum_{\mu \in Hv} \omega_{\mu\lambda,\mu\lambda}\Big(\sum_{\alpha \in s(\mu)E_p} \omega_{\mu\alpha,\mu\alpha}\Big) T_p \Big\|\Big) \\ &\qquad + \sum_{p \in G} \Big(\Big\| \sum_{\mu \in Hv} \omega_{\mu\lambda,\mu\lambda}\Big(\omega_{\mu,\mu} - \sum_{\alpha \in s(\mu)E_p} \omega_{\mu\alpha,\mu\alpha}\Big) T_p \Big\|\Big),\end{aligned} \tag{4.6}$$

where the inequality on the final line follows from the triangle inequality. Since $\mathrm{MCE}(\lambda, \alpha) = \emptyset$ for all $\alpha \in v(\vee E_G)$, it follows from (B2) that

$$\sum_{p \in G} \Big(\Big\| \sum_{\mu \in Hv} \omega_{\mu\lambda,\mu\lambda}\Big(\sum_{\alpha \in s(\mu)E_p} \omega_{\mu\alpha,\mu\alpha}\Big) T_p \Big\|\Big) = 0.$$

Using that $H \subset \Lambda^m$ in the second line, we calculate

$$\begin{aligned}
&\Big\| \sum_{\mu,\nu\in Hv} a_{\mu,\nu}\tau_{\mu\lambda,\nu\lambda}\Big\| \\
&\qquad \le \sum_{p\in G}\bigg(\Big\|\sum_{\mu\in Hv}\omega_{\mu\lambda,\mu\lambda}\Big(\omega_{\mu,\mu}-\sum_{\alpha\in s(\mu)E_p}\omega_{\mu\alpha,\mu\alpha}\Big)T_p\Big\|\bigg)\\
&\qquad = \sum_{p\in G}\bigg(\Big\|\sum_{\mu\in Hv}\omega_{\mu\lambda,\mu\lambda}\Big(\sum_{\mu'\in H}\Big(\omega_{\mu',\mu'}-\sum_{\alpha\in s(\mu')E_p}\omega_{\mu'\alpha,\mu'\alpha}\Big)\Big)T_p\Big\|\bigg)\\
&\qquad \le \sum_{p\in G}\bigg(\sum_{\mu\in Hv}\|\omega_{\mu\lambda,\mu\lambda}\|\Big\|\sum_{\mu'\in H}\Big(\omega_{\mu',\mu'}-\sum_{\alpha\in s(\mu')E_p}\omega_{\mu'\alpha,\mu'\alpha}\Big)T_p\Big\|\bigg)\\
&\qquad = \sum_{\substack{p\in G\\ \mu\in Hv}}\Big\|\sum_{\mu'\in H}\Big(\omega_{\mu',\mu'}-\sum_{\alpha\in s(\mu')E_p}\omega_{\mu'\alpha,\mu'\alpha}\Big)T_p\Big\|.
\end{aligned}$$

Equation (4.5) therefore implies that

$$\Big\|\sum_{\mu,\nu\in Hv} a_{\mu,\nu}\tau_{\mu\lambda,\nu\lambda}\Big\| < \sum_{\substack{p\in G\\ \mu\in Hv}}\frac{M}{2|G||H|} = \frac{|Hv|M}{2|H|}\le\frac{M}{2}. \tag{4.7}$$

Since $\tau_{\mu,\mu}\neq 0$ for all $\mu\in\Lambda$, we have $\tau_{\mu\lambda,\nu\lambda}\neq 0$ for all $\mu,\nu\in Hv$. In particular, both $\{\omega_{\mu,\nu} : \mu,\nu\in Hv\}$ and $\{\tau_{\mu\lambda,\nu\lambda} : \mu,\nu\in Hv\}$ are families of nonzero matrix units. Hence the uniqueness of $\mathcal{K}_{Hv}$ implies that $\omega_{\mu,\nu}\mapsto\tau_{\mu\lambda,\nu\lambda}$ extends to an isomorphism $C^*(\{\omega_{\mu,\nu} : \mu,\nu\in Hv\})\cong C^*(\{\tau_{\mu\lambda,\nu\lambda} : \mu,\nu\in Hv\})$. Thus

$$\Big\|\sum_{\mu,\nu\in Hv} a_{\mu,\nu}\tau_{\mu\lambda,\nu\lambda}\Big\| = \Big\|\sum_{\mu,\nu\in Hv} a_{\mu,\nu}\omega_{\mu,\nu}\Big\|\ge M.$$

This contradicts (4.7), completing the proof of the claim.

To finish off Step 1, for each $v\in s(H)$, each $\mu\in Hv$, and each $\alpha\in v(\vee E_G)$, let

$$Q^{v(\vee E_G)}_{\mu\alpha} := \omega_{\mu\alpha,\mu\alpha}\prod_{\alpha\zeta\in v(\vee E_G)\setminus\{\alpha\}}(\omega_{\mu\alpha,\mu\alpha}-\omega_{\mu\alpha\zeta,\mu\alpha\zeta}).$$

Each $Q^{v(\vee E_G)}_{\mu\alpha}\in B_G$ by definition of the $v(\vee E_G)$. Let $K := \{v,\mu\}\cup\mu(\vee E_G)$. Then $\vee K = K$. Reversing the edges in Λ yields a finitely aligned product system over P of graphs as in [21, Example 3.1], and then the arguments of [21, Proposition 8.6] applied to the set K show that

$$\omega_{\mu,\mu} = \prod_{\alpha\in s(\mu)(\vee E_G)}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha}) + \sum_{\alpha\in s(\mu)(\vee E_G)} Q^{s(\mu)(\vee E_G)}_{\mu\alpha} \tag{4.8}$$

(see the displayed equation immediately below equation (8.5) on page 421 of [21]).

Let

$$
\begin{aligned}
a_{\{m\}} &:= \sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)} (\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})\omega_{\mu,\nu}\Big) \quad \text{and} \\
a_G &:= \sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\sum_{\alpha\in s(\mu)(\vee E_G)} Q^{s(\mu)(\vee E_G)}_{\mu\alpha}\omega_{\mu,\nu}\Big) + \sum_{p\in G} T_p.
\end{aligned} \tag{4.9}
$$

It follows from (4.8) that a decomposes as $a = a_{\{m\}} + a_G$, which is the desired decomposition of a. This completes Step 1.

Step 2. (Show that $a_{\{m\}} \in \ker(\rho_\tau)$.)

To begin Step 2, fix $\mu' \in H$ and $\alpha' \in E_{s(\mu)}$. We claim that

$$
\Big(\sum_{\mu\in H}\prod_{\alpha\in s(\mu)(\vee E_G)} (\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})\Big) Q^{s(\mu')(\vee E_G)}_{\mu'\alpha'} = 0. \tag{4.10}
$$

First suppose that $\mu \neq \mu'$. Then $H \subset \Lambda^m$ implies $\omega_{\mu,\mu}\omega_{\mu',\mu'} = 0$, giving (4.10) when $\mu \neq \mu'$. Now suppose that $\mu = \mu'$. Then the product $\prod_{\alpha\in s(\mu)(\vee E_G)}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})$ contains a factor of $\omega_{\mu',\mu'} - \omega_{\mu'\alpha',\mu'\alpha'}$. Since $\omega_{\mu',\mu'} \geq \omega_{\mu'\alpha',\mu'\alpha'} \geq Q^{s(\mu)(\vee E_G)}_{\mu'\alpha'}$, we have $(\omega_{\mu',\mu'} - \omega_{\mu'\alpha',\mu'\alpha'})Q^{s(\mu')(\vee E_G)}_{\mu'\alpha'} = 0$, which establishes (4.10) when $\mu = \mu'$.

Using (4.9) and then (4.10), we calculate

$$
\begin{aligned}
&\Big(\sum_{\mu\in H}\prod_{\alpha\in s(\mu)(\vee E_G)} (\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})\Big) a \\
&\qquad = \Big(\sum_{\mu\in H}\prod_{\alpha\in s(\mu)(\vee E_G)} (\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})\Big)(a_{\{m\}} + a_G) \\
&\qquad = \sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)} (\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})\omega_{\mu,\nu}\Big) \\
&\qquad\qquad + \sum_{\substack{p\in G\\ \mu\in H}}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)} (\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})\Big) T_p.
\end{aligned}
$$

Since $\rho_\tau(a) = 0$, applying ρ_τ to both sides of the above equation and rearranging yields

$$\begin{aligned}\sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)}(\tau_{\mu,\mu}-\tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}\Big)\\ = -\rho_\tau\Big(\sum_{p\in G,\,\mu\in H}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})\Big)T_p\Big).\end{aligned}\tag{4.11}$$

Using (4.11),

$$\begin{aligned}&\Big\|\sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)}(\tau_{\mu,\mu}-\tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}\Big)\Big\|\\ &\le \sum_{p\in G}\Big\|\sum_{\mu\in H}\prod_{\alpha\in s(\mu)(\vee E_G)}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})T_p\Big\|\\ &=\sum_{p\in G}\Big\|\sum_{\mu\in H}\prod_{\alpha'\in s(\mu)(\vee E_G)}(\omega_{\mu,\mu}-\omega_{\mu\alpha',\mu\alpha'})\prod_{\alpha\in s(\mu)E_p}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})T_p\Big\|,\end{aligned}$$

since the $\omega_{\mu,\mu} - \omega_{\mu\alpha,\mu\alpha}$ are projections and $s(\mu)E_p \subset s(\mu)(\vee E_G)$. Since $H \subset \Lambda^m$, condition (B2) yields

$$\begin{aligned}&\Big\|\sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)}(\tau_{\mu,\mu}-\tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}\Big)\Big\|\\ &\le \sum_{p\in G}\Big\|\Big(\sum_{\mu\in H}\prod_{\alpha'\in s(\mu)(\vee E_G)}(\omega_{\mu,\mu}-\omega_{\mu\alpha',\mu\alpha'})\Big)\\ &\qquad\Big(\sum_{\mu\in H}\prod_{\alpha\in s(\mu)E_p}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})T_p\Big)\Big\| \qquad (4.12)\\ &\le \sum_{p\in G}\Big\|\sum_{\mu\in H}\prod_{\alpha\in s(\mu)E_p}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})T_p\Big\|, \qquad (4.13)\end{aligned}$$

because the $\prod_{\alpha\in s(\mu)(\vee E_G)}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha})$ are mutually orthogonal so that their sum is a projection. The $\omega_{\mu\alpha,\mu\alpha}$, where $\alpha \in s(\mu)E_p$, are mutually orthogonal, so each $\prod_{\alpha\in s(\mu)E_p}(\omega_{\mu,\mu}-\omega_{\mu\alpha,\mu\alpha}) = \omega_{\mu,\mu} - \sum_{\alpha\in s(\mu)E_p}\omega_{\mu\alpha,\mu\alpha}$. Hence combining

(4.13) with (4.5), we obtain

$$\begin{aligned}\Big\| \sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)} (\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}\Big)\Big\| \\ \le \sum_{p\in G}\Big\| \sum_{\mu\in H}\Big(\omega_{\mu,\mu} - \sum_{\alpha\in s(\mu)E_p} \omega_{\mu\alpha,\mu\alpha}\Big)T_p\Big\| < \frac{M}{2}.\end{aligned} \tag{4.14}$$

Now, we claim that $\prod_{\alpha\in v(\vee E_G)}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu} = 0$ for all $v \in s(H)$ and $\mu,\nu \in Hv$. Suppose for contradiction that $v \in s(H)$ and $\mu,\nu \in Hv$ with $\prod_{\alpha\in v(\vee E_G)}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu} \neq 0$. Then Lemma 4.7 and (4.3) imply that the set $\{\prod_{\alpha\in v(\vee E_G)}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu} : \mu,\nu \in Hv\}$ is a family of matrix units. Since $\prod_{\alpha\in v(\vee E_G)}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu} \neq 0$, all the matrix units are nonzero. It follows that $\omega_{\mu,\nu} \mapsto \prod_{\alpha\in v(\vee E_G)}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}$ determines an isomorphism $C^*(\{\omega_{\mu,\nu} : \mu,\nu \in Hv\}) \cong \mathcal{K}_{Hv}$. We then have

$$\begin{aligned}\Big\| \sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)} (\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}\Big)\Big\| \\ = \max_{w\in s(H)}\Big\| \sum_{\mu,\nu\in Hw} a_{\mu,\nu}\Big(\prod_{\alpha\in w(\vee E_G)} (\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}\Big)\Big\| \\ \ge \Big\| \sum_{\mu,\nu\in Hv} a_{\mu,\nu}\Big(\prod_{\alpha\in v(\vee E_G)} (\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}\Big)\Big\| \\ = \Big\| \sum_{\mu,\nu\in Hv} a_{\mu,\nu}\omega_{\mu,\nu}\Big\| \\ \ge M,\end{aligned}$$

which contradicts (4.14). This establishes the claim, and hence

$$\begin{aligned}\rho_\tau\Big(\sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)} (\omega_{\mu,\mu} - \omega_{\mu\alpha,\mu\alpha})\omega_{\mu,\nu}\Big)\Big) \\ = \sum_{(\mu,\nu)\in H*_s H} a_{\mu,\nu}\Big(\prod_{\alpha\in s(\mu)(\vee E_G)} (\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha})\tau_{\mu,\nu}\Big) = 0.\end{aligned}$$

This completes Step 2.

Step 3. (Deduce that $a_G \in \ker(\rho_\tau)$ and then apply the inductive hypothesis.)

To complete the proof, observe that $a \in \ker(\rho_\tau)$ by hypothesis. Step 1 gives $a_G = a - a_{\{m\}}$, and Step 2 gives $a_{\{m\}} \in \ker(\rho_\tau)$, and it follows that $a_G \in \ker(\rho_\tau)$

as well. The inductive hypothesis now forces $a_G \in I_\tau^G \subset I_\tau^F$. It therefore suffices to show that $a_{\{m\}} \in I_\tau^F$. By definition of $a_{\{m\}}$, it suffices to show that

$$\prod_{\alpha \in s(\lambda) \vee E_G} \tau_{\lambda,\lambda} - \tau_{\lambda\alpha,\lambda\alpha} = 0 \qquad \text{for every } \lambda \in H. \tag{4.15}$$

So fix $\lambda \in H$. Recall that by choice of H there exist $\mu, \nu \in Hs(\lambda)$ such that $a_{\mu,\nu} \neq 0$. Hence, with $P_\mu := \prod_{\alpha \in s(\lambda)(\vee E_G)} \tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha}$ and $P_\nu := \prod_{\alpha \in s(\lambda)(\vee E_G)} \tau_{\nu,\nu} - \tau_{\nu\alpha,\nu\alpha}$, we have

$$\prod_{\alpha \in s(\lambda) \vee E_G} \tau_{\lambda,\lambda} - \tau_{\lambda\alpha,\lambda\alpha} = \frac{1}{a_{\mu,\nu}} \tau_{\lambda,\mu} P_\mu \rho_\tau(a_{\{m\}}) P_\nu \tau_{\nu,\lambda}.$$

Since Step 2 forces $\rho_\tau(a_{\{m\}}) = 0$, this establishes (4.15) as required. ∎

4.3. The co-universal balanced algebra. In this subsection we use the analysis of Subsection 4.2 to establish that there is a representation of $\Lambda *_{d,s} \Lambda$ by nonzero partial isometries which is co-universal in the sense that every other representation of $\Lambda *_{d,s} \Lambda$ by nonzero partial isometries factors through it.

Theorem 4.12: *Let (G, P) be a quasi-lattice ordered group, and let (Λ, d) be a finitely aligned P-graph. There is a C^*-algebra $\mathcal{B}_\Lambda^{\min}$ generated by a representation Ω of $\Lambda *_{d,s} \Lambda$ such that:*

1. *each $\Omega_{\mu,\nu}$ is nonzero, and*
2. *given any other representation τ of $\Lambda *_{d,s} \Lambda$ with each $\tau_{\mu,\nu}$ nonzero, there is a homomorphism $\phi_\tau : C^*(\tau) \to \mathcal{B}_\Lambda^{\min}$ satisfying $\phi_\tau(\tau_{\mu,\nu}) = \Omega_{\mu,\nu}$ for each $(\mu, \nu) \in \Lambda *_{d,s} \Lambda$.*

Moreover, $\mathcal{B}_\Lambda^{\min}$ is unique up to canonical isomorphism, and given a representation τ of $\Lambda *_{d,s} \Lambda$ as in (2), ϕ_τ is injective if and only if

$$\prod_{\alpha \in E} (\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha}) = 0 \quad \text{for all } \mu \in \Lambda \text{ and finite exhaustive } E \subset s(\mu)\Lambda.$$

Proof of Theorem 4.12. Recall from Section 3 that $\widehat{\Lambda}$ denotes the space of filters of Λ, and $\widehat{\Lambda}_\infty$ denotes the space of ultrafilters. Lemma 3.4(3) implies that the subspace $\ell^2(\widehat{\Lambda}_\infty) \subset \ell^2(\widehat{\Lambda})$ is invariant for the partial isometries $\{T_\lambda : \lambda \in \Lambda\}$. Define $\Omega : \Lambda *_{d,s} \Lambda \to B(\ell^2(\widehat{\Lambda}_\infty))$ by

$$\Omega_{\mu,\nu} := (T_\mu T_\nu^*)|_{\ell^2(\widehat{\Lambda}_\infty)} \quad \text{for all } (\mu, \nu) \in \Lambda *_{d,s} \Lambda.$$

Since the restriction map from $B(\ell^2(\widehat{\Lambda}))$ to $B(\ell^2(\widehat{\Lambda}_\infty))$ is a homomorphism on $C^*(\{T_\lambda : \lambda \in \Lambda\})$, Lemma 4.2 implies that Ω is a representation of $\Lambda *_{d,s} \Lambda$.

Fix $(\mu,\nu) \in \Lambda *_{d,s} \Lambda$. By Lemma 3.2, there is an ultrafilter U of Λ with $\nu \in U$. Hence $\Omega_{\mu,\nu} e_U = e_{\mu\cdot(\nu^*\cdot U)} \neq 0$. Thus Ω satisfies (1).

Fix a representation τ of $\Lambda *_{d,s} \Lambda$ as in (2). The universal property of $\mathcal{B}_\Lambda$ yields homomorphisms $\rho_\tau : \mathcal{B}_\Lambda \to C^*(\tau)$ and $\rho_\Omega : \mathcal{B}_\Lambda \to \mathcal{B}_\Lambda^{\min}$ such that $\rho_\tau(\omega_{\mu,\nu}) = \tau_{\mu,\nu}$ and $\rho_\Omega(\omega_{\mu,\nu}) = \Omega_{\mu,\nu}$. We will show that $\ker(\rho_\tau) \subset \ker(\rho_\Omega)$; condition (2) will follow because there is then a well defined homomorphism $\phi_\tau : C^*(\tau) \to \mathcal{B}_\Lambda^{\min}$ satisfying $\phi_\tau \circ \rho_\tau = \phi_\Omega$.

By Theorem 4.9, it suffices to show that whenever $\mu\in\Lambda$ and E is a finite exhaustive subset of $s(\mu)\Lambda$ such that $\prod_{\alpha\in E}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha}) = 0$, we have $\prod_{\alpha\in E}(\Omega_{\mu,\mu} - \Omega_{\mu\alpha,\mu\alpha}) = 0$. In particular, it is enough to establish that

$$\prod_{\alpha\in E}(\Omega_{\mu,\mu} - \Omega_{\mu\alpha,\mu\alpha}) = 0 \text{ for all } \mu \in \Lambda \text{ and finite exhaustive } E \subset s(\mu)\Lambda; \tag{4.16}$$

and this follows from Lemma 3.3.

We have now proved that Ω satisfies (1) and (2). To see that $\mathcal{B}_\Lambda^{\min}$ is unique up to canonical isomorphism, suppose that τ is another representation of $\Lambda *_{d,s} \Lambda$ such that $C^*(\tau)$ satisfies (1) and (2). Then property (2) of $C^*(\tau)$ gives a homomorphism from $\mathcal{B}_\Lambda^{\min}$ to $C^*(\tau)$ which is an inverse for ϕ_τ.

Finally, to see that ϕ_τ is injective if and only if $\prod_{\alpha\in E}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha}) = 0$ for every $\mu \in \Lambda$ and every finite exhaustive subset $E \subset s(\mu)\Lambda$, observe first that the "only if" implication follows from (4.16). For the "if" implication, suppose that τ is a representation of $\Lambda *_{d,s} \Lambda$ with each $\tau_{\mu,\nu}$ nonzero, and with $\prod_{\alpha\in E}(\tau_{\mu,\mu} - \tau_{\mu\alpha,\mu\alpha}) = 0$ for every $\mu \in \Lambda$ and finite exhaustive subset $E \subset s(\mu)\Lambda$. Then in particular, $\ker(\rho_\tau)$ contains $\prod_{\alpha\in E}(\omega_{\mu,\mu} - \omega_{\mu\alpha,\mu\alpha})$ for every $\mu \in \Lambda$ and finite exhaustive subset $E \subset s(\mu)\Lambda$, and it follows from Theorem 4.9 that $\ker(\rho_\Omega) \subset \ker(\rho_\tau)$. Since $\rho_\Omega = \phi_\tau \circ \rho_\tau$, it follows that ϕ_τ is injective. ∎

5. The C^*-algebra of a P-graph

We are now ready to state and prove our main theorem. We define what we mean by a representation of a P-graph, and we show that the C^*-algebra $\mathcal{T}C^*(\Lambda)$ which is universal for such representations admits a co-universal quotient. That is, there is a smallest quotient of $\mathcal{T}C^*(\Lambda)$ in which the canonical coaction of G is preserved and the images of all the generators are nonzero. The point is that the hard work is largely already done in the results of Section 4:

we use the uniqueness theorem for $\mathcal{B}_\Lambda$ established in Subsection 4.1 to identify $\mathcal{B}_\Lambda$ with the fixed-point algebra of $\mathcal{T}C^*(\Lambda)$, and we then present a fairly generic argument, based on coaction theory, to bootstrap the co-universal property of $\mathcal{B}_\Lambda^{\min}$ up to the desired co-universal quotient of $\mathcal{T}C^*(\Lambda)$.

Definition 5.1: Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. A **representation** of Λ in a C^*-algebra B is a map $t : \Lambda \to B$, $\lambda \mapsto t_\lambda$ such that:

(T1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(T2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(T3) $t_\mu^* t_\mu = t_{s(\mu)}$ for all $\mu \in \Lambda$; and
(T4) $t_\mu t_\mu^* t_\nu t_\nu^* = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} t_\lambda t_\lambda^*$ for all $\mu, \nu \in \Lambda$.

As in [10, Theorem 6.3], there exists a C^*-algebra $\mathcal{T}C^*(\Lambda)$ generated by a representation s of Λ which is universal in the sense that given any other representation t of Λ, there is a homomorphism $\pi_t : \mathcal{T}C^*(\Lambda) \to C^*(t) := C^*(\{t_\lambda : \lambda \in \Lambda\})$ satisfying $\pi_t \circ s = t$.

We need to dip a little into the theory of coactions; but not too far because the coactions we deal with are all coactions of discrete groups. For more detail on coactions, see [7, Section A.3]. The following summary is adapted from [2, Section 3]. All tensor products of C^*-algebras (here and later in the section) are minimal tensor products.

Let G be a discrete group, and let $g \mapsto U_g$ be the universal unitary representation of G in $C^*(G)$. There is a homomorphism $\delta_G\colon C^*(G) \to C^*(G) \otimes C^*(G)$ determined by $\delta_G(U_g) = U_g \otimes U_g$. A full coaction of G on a C^*-algebra A is an injective nondegenerate homomorphism $\delta\colon A \to A \otimes C^*(G)$ such that $(\delta \otimes \mathrm{id}_{C^*(G)}) \circ \delta = (\mathrm{id}_A \otimes \delta_G) \circ \delta$. The fixed-point algebra for G is the subalgebra $A_e^\delta := \{a \in A : \delta(a) = a \otimes 1_{C^*(G)}\}$. By [19, Lemma 1.3(a)] there is a conditional expectation Φ^δ from A to A_e^δ determined by $\Phi^\delta(a) = a$ if $a \in A_e^\delta$, and $\Phi^\delta(a) = 0$ if $\delta(a) = a \otimes U_g$ for some other $g \in G$. A **normal** coaction is one for which Φ^δ is faithful on positive elements (there are a number of equivalent characterisations of normality for coactions, but this is the one most useful from our point of view). Given a coaction $\delta : A \to A \otimes C^*(G)$, there is a quotient A^r of A, and a normal coaction $\delta^n : A^r \to A^r \otimes C^*(G)$ such that the coaction crossed-products $A \otimes_\delta G$ and $A^r \otimes_{\delta^n} G$ are identical. The cosystem (A^r, δ^n, G) is called the **normalisation** of (A, G, δ) (see [7, Section A.7]). We write q_δ for the quotient map from A to A^r. We have $\delta^n \circ q_\delta = (q_\delta \otimes 1) \circ \delta$, and q_δ restricts to an isomorphism $A_e^\delta \cong (A^r)_e^{\delta^n}$ of fixed-point algebras. Indeed q_δ is isometric on $\overline{\operatorname{span}}\{a \in A : \delta(a) = a \otimes U_g\}$ for each fixed $g \in G$.

Given a quasi-lattice ordered group (G, P) and a finitely aligned P-graph Λ, a standard argument using the universal property of $\mathcal{T}C^*(\Lambda)$ shows that there is a coaction δ of G on $\mathcal{T}C^*(\Lambda)$ which satisfies $\delta(s_\lambda) = s_\lambda \otimes U_{d(\lambda)}$ for all $\lambda \in \Lambda$.

Remark 5.2: Let (G, P) be a quasi-lattice ordered group, let Λ be a finitely aligned P-graph, and let $t : \Lambda \to B$ be a representation of Λ. Relation (T4) and the factorisation property imply that $t_\mu^* t_\nu = \sum_{\mu\alpha=\nu\beta\in\mathrm{MCE}(\mu,\nu)} t_\alpha t_\beta^*$, and it follows that $C^*(t) = \overline{\operatorname{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}$. Thus if $C^*(t)$ admits a coaction α of G satisfying $\alpha(t_\lambda) = t_\lambda \otimes U_{d(\lambda)}$ for all $\lambda \in \Lambda$, then that $C^*(t)^\alpha = \Phi^\alpha(C^*(t))$ forces $C^*(t)^\alpha = \overline{\operatorname{span}}\{t_\mu t_\nu^* : d(\mu) = d(\nu)\}$.

Theorem 5.3: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. There exists a C^*-algebra $C^*_{\min}(\Lambda)$ generated by a representation S of Λ such that*

1. *each S_λ is nonzero, and there is a coaction β of G on $C^*_{\min}(\Lambda)$ satisfying $\beta(S_\lambda) = S_\lambda \otimes U_{d(\lambda)}$ for all λ; and*
2. *given any other representation t of Λ with each t_λ nonzero such that $C^*(t)$ carries a coaction α of G satisfying $\alpha(t_\lambda) = t_\lambda \otimes U_{d(\lambda)}$ for all λ, there is a homomorphism $\psi_t : C^*(t) \to C^*_{\min}(\Lambda)$ satisfying $\psi_t(t_\lambda) = S_\lambda$ for all λ.*

Moreover, $C^*_{\min}(\Lambda)$ is unique up to canonical isomorphism, and the homomorphism ψ_t of (2) is injective if and only if α is normal and

$$\prod_{\alpha\in E}(t_\mu t_\mu^* - t_{\mu\alpha} t_{\mu\alpha}^*) = 0 \quad \textit{for all } \mu \in \Lambda \textit{ and finite exhaustive } E \subset s(\mu)\Lambda.$$

To prove the theorem we require a preliminary result:

Lemma 5.4: *Let (G, P) be a quasi-lattice ordered group, and let Λ be a finitely aligned P-graph. Then there is an isomorphism $\mathcal{B}_\Lambda \cong \mathcal{T}C^*(\Lambda)^\delta$ which takes $\omega_{\mu,\nu}$ to $s_\mu s_\nu^*$ for all $(\mu, \nu) \in \Lambda *_{d,s} \Lambda$.*

Proof. It is routine that $\tau_{\mu,\nu} := s_\mu s_\nu^*$ determines a representation τ of $\Lambda *_{d,s} \Lambda$. So there is a homomorphism ρ_τ from $\mathcal{B}_\Lambda$ to $\mathcal{T}C^*(\Lambda)$ which takes each $\omega_{\mu,\nu}$ to $s_\mu s_\nu^*$. The partial isometries T_λ of Definition 3.5 clearly satisfy $T_\mu T_\nu = T_{\mu\nu}$ when $s(\mu) = r(\nu)$, and combined with Lemma 4.2, this shows that T is a representation of Λ. Hence there is a homomorphism $\pi_T : \mathcal{T}C^*(\Lambda) \to C^*(T)$ which takes each s_μ to T_μ. Thus (4.1) and (4.2) imply that each $s_\mu s_\nu^* \neq 0$ and that for every $\mu \in \Lambda$ and finite exhaustive set $E \subset s(\mu)\Lambda$,

$$\prod_{\alpha \in E}(s_\mu s_\mu^* - s_{\mu\alpha} s_{\mu\alpha}^*) \neq 0.$$

Hence Theorem 4.4 implies that $\rho_\mathcal{T}$ is injective.

It remains to show that the range of $\rho_\mathcal{T}$ is $\mathcal{T}C^*(\Lambda)^\delta$. We have $s_\mu s_\nu^* = 0$ unless $s(\mu) = s(\nu)$ by (T3). Hence

$$\operatorname{range}(\rho_\mathcal{T}) = \overline{\operatorname{span}}\,\{s_\mu s_\nu^* : d(\mu) = d(\nu)\},$$

and this is equal to $\mathcal{T}C^*(\Lambda)^\delta$ by Remark 5.2. ∎

Proof of Theorem 5.3. Lemma 3.4(3) implies that $\ell^2(\widehat{\Lambda}_\infty) \subset \ell^2(\widehat{\Lambda})$ is invariant for the partial isometries T_λ of Definition 3.5. Hence the partial isometries $T_\lambda|_{\ell^2(\widehat{\Lambda}_\infty)}$ form a representation of Λ on $B(\ell^2(\widehat{\Lambda}_\infty))$.

Define a map $\widetilde{T} : \Lambda \to B(\ell^2(\widehat{\Lambda}_\infty)) \otimes C^*(G)$ by

$$\widetilde{T}_\lambda := T_\lambda|_{\ell^2(\widehat{\Lambda}_\infty)} \otimes U_{d(\lambda)}.$$

It is straightforward to see that $\widetilde{T}$ is a representation of Λ.

Let β_0 be the canonical coaction on $B(\ell^2(\widehat{E}_\infty)) \otimes C^*(G)$ given by $\beta_0(a \otimes U_g) := (a \otimes U_g) \otimes U_g$. Since sums of the form $\sum_{v \in F} \widetilde{T}_v$, where F increases over finite subsets of Λ^0, form an approximate identity for $B(\ell^2(\widehat{E}_\infty))$, and since G is discrete, β_0 restricts to a coaction, also denoted β_0, on

$$C^*(\widetilde{T}) = \overline{\operatorname{span}}\,\{\widetilde{T}_\mu \widetilde{T}_\nu^* : \mu, \nu \in \Lambda\}$$

(see [7, Remark A.22(3)]). Let $(C^*_{\min}(\Lambda), \beta)$ be the normalisation of the cosystem $(C^*(\widetilde{T}), \beta_0)$ as in [7, Definition A.56]: so $C^*_{\min}(\Lambda) = C^*(\widetilde{T})^r$ and $\beta = \beta_0^n$. Recall that q_{β_0} denotes the canonical quotient map from $C^*(\widetilde{T})$ to $C^*_{\min}(\Lambda)$. For each $\lambda \in \Lambda$, let $S_\lambda := q_{\beta_0}(\widetilde{T}_\lambda)$, so S is a representation of Λ. Since q_{β_0} is isometric on $\overline{\operatorname{span}}\,\{a \in C^*(\widetilde{T}) : \beta_0(a) = a \otimes U_{d(\lambda)}\}$, it follows from Lemma 3.2 that each S_λ is nonzero. Hence the triple $(C^*_{\min}(\Lambda), S, \beta)$ satisfies (1).

To prove (2), fix a representation t of Λ and a coaction α of G on $C^*(t)$ as in (2). Let $\pi_t : \mathcal{T}C^*(\Lambda) \to C^*(t)$ be the homomorphism obtained from the universal property of $\mathcal{T}C^*(\Lambda)$. It suffices to show that $\ker(\pi_t) \subset \ker(\pi_S)$, for then π_S descends to the desired homomorphism from $C^*(t)$ to $C^*_{\min}(\Lambda)$.

Let

$$\Phi^\delta : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda)^\delta, \quad \Phi^\alpha : C^*(t) \to C^*(t)^\alpha \text{ and}$$
$$\Phi^\beta : C^*_{\min}(\Lambda) \to C^*_{\min}(\Lambda)^\beta$$

be the conditional expectations obtained from the coactions δ, α, β.

Fix $a \in \mathcal{T}C^*(\Lambda)$ with $\pi_t(a) = 0$. Then $\Phi^\alpha(\pi_t(a^*a)) = 0$. Since π_t intertwines δ and α on spanning elements, we have $\Phi^\alpha \circ \pi_t = \pi_t \circ \Phi^\delta$, and hence

$$\pi_t(\Phi^\delta(a^*a)) = 0. \tag{5.1}$$

Claim: *There is an isomorphism $C^*_{\min}(\Lambda)^\beta \cong \mathcal{B}^{\min}_\Lambda$ satisfying $S_\mu S_\nu^* \mapsto \Omega_{\mu,\nu}$ for all $(\mu,\nu) \in \Lambda *_{d,s} \Lambda$.*

To prove the claim, first note that Remark 5.2 implies that $C^*_{\min}(\Lambda)^\beta$ is generated by the representation of $\Lambda *_{d,s} \Lambda$ defined by $(\mu,\nu) \mapsto S_\mu S_\nu^*$. Since q_{β_0} is injective on $C^*(\widetilde{T})^{\beta_0}$, we have $S_\mu S_\nu^* \neq 0$ for all $(\mu,\nu) \in \Lambda *_{d,s} \Lambda$. The co-universal property of $\mathcal{B}^{\min}_\Lambda$ therefore induces a homomorphism $\phi_S : C^*_{\min}(\Lambda)^\beta \to \mathcal{B}^{\min}_\Lambda$ satisfying $\phi_S(S_\mu S_\nu^*) = \Omega_{\mu,\nu}$. Lemma 3.3 implies that for each $\mu \in \Lambda$ and each finite exhaustive subset E of $s(\mu)\Lambda$,

$$\prod_{\alpha \in E}(S_\mu S_\mu^* - S_{\mu\alpha} S_{\mu\alpha}^*) = 0.$$

Hence the final assertion of Theorem 4.12 implies that ϕ_S is an isomorphism. This proves the claim.

The claim combined with the co-universal property of $\mathcal{B}^{\min}_\Lambda$ implies that there is a homomorphism $\phi_t : C^*(t)^\alpha \to C^*_{\min}(\Lambda)^\beta$ which takes $t_\mu t_\nu^*$ to $S_\mu S_\nu^*$ for all $(\mu,\nu) \in \Lambda *_{d,s} \Lambda$. In particular, $\phi_t \circ \pi_t|_{\mathcal{T}C^*(\Lambda)^\delta} = \pi_S|_{\mathcal{T}C^*(\Lambda)^\delta}$. Hence (5.1) implies that $\pi_S(\Phi^\delta(a^*a)) = 0$.

Since π_S intertwines δ and β, it follows that $\Phi^\beta(\pi_S(a^*a)) = 0$. As β is a normal coaction, we deduce that $\pi_S(a^*a) = 0$, and hence that $\pi_S(a) = 0$ as required.

The uniqueness assertion follows from an argument identical to the one establishing uniqueness of $\mathcal{B}^{\min}_\Lambda$. It remains to show that ψ_t is injective if and only if α is normal and

$$\prod_{\alpha \in E}(t_\mu t_\mu^* - t_{\mu\alpha} t_{\mu\alpha}^*) = 0 \quad \text{for all } \mu \in \Lambda \text{ and finite exhaustive } E \subset s(\mu)\Lambda.$$

The "only if" direction is clear because β is normal and each $\prod_{\alpha \in E}(S_\mu S_\mu^* - S_{\mu\alpha} S_{\mu\alpha}^*) = 0$. For the "if" direction, suppose that the above two conditions are satisfied. Then the final assertion of Theorem 4.12 and the claim above imply that ψ_t restricts to an isomorphism of $C^*(t)^\alpha$. That α is normal implies that $\Phi^\alpha : C^*(t) \to C^*(t)^\alpha$ is faithful on positive elements; hence

$$\begin{aligned}\psi_t(a) = 0 \implies \Phi^\beta(\psi_t(a^*a)) = 0 &\implies \psi_t(\Phi^\alpha(a^*a)) = 0 \\ &\implies \Phi^\alpha(a^*a) = 0 \implies a = 0.\end{aligned}$$

Thus ψ_t is injective. ∎

The following corollary shows that if $P = \mathbb{N}^k$, then our P-graph C^*-algebra coincides with the k-graph C^*-algebra of [23].

Corollary 5.5: *Let Λ be a finitely aligned k-graph. Then the co-universal C^*-algebra $C^*_{\min}(\Lambda)$ obtained from Theorem 5.3 by regarding Λ as an $\mathbb{N}^k$-graph is canonically isomorphic to the k-graph C^*-algebra of [23].*

Proof. By definition, $C^*(\Lambda)$ is generated by a representation t of Λ. Corollary 4.3 of [23] shows that $t_v \neq 0$ for all $v \in \Lambda^0$, and that there is an action γ of $\mathbb{T}^k$ on $C^*(\Lambda)$ satisfying $\gamma_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for all λ. Under action-coaction duality, γ determines a coaction α of $\mathbb{Z}^k$ satisfying $\alpha(t_\lambda) = t_\lambda \otimes U_{d(\lambda)}$ for all $\lambda \in \Lambda$. The coaction α is normal because $\mathbb{Z}^k$ is abelian and hence amenable. Moreover, the t_λ satisfy

$$\prod_{\lambda \in E} (t_v - t_\lambda t_\lambda^*) = 0 \quad \text{for all } v \in \Lambda^0 \text{ and finite exhaustive } E \subset v\Lambda$$

by definition of $C^*(\Lambda)$ (see [23, Definition 2.5]). The result therefore follows from Theorem 5.3. ∎

The following corollary shows that our construction is compatible with inclusions of quasi-lattice ordered groups. We use it in Example 6.6 below.

Corollary 5.6: *Let (G, P) be a quasi-lattice ordered group, and let (H, Q) be a subgroup; that is, $H \leq G$, $Q \leq P$, the order on H agrees with that on G, and Q is closed under taking least upper bounds in P. Suppose that Q is hereditary in the sense that if $p, q \in P$ with $pq \in Q$, then $p, q \in Q$. Let $\iota : H \to G$ be the inclusion map. Let Λ be a finitely aligned Q-graph. Define Λ_P to be a copy of Λ endowed with the degree map $d_P : \Lambda^P \to P$ given by $d_P = \iota \circ d$. Then Λ_P is a finitely aligned P-graph, and $C^*_{\min}(\Lambda_P) \cong C^*_{\min}(\Lambda)$.*

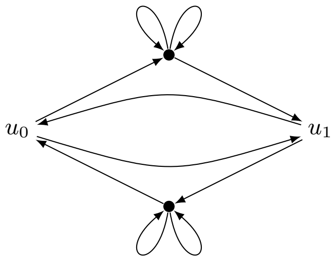
Proof. To see that Λ_P is a finitely aligned P-graph, the only difficulty is checking the factorisation property, and this follows from the assumption that $pq \in Q$ forces $p, q \in Q$.

It is routine to verify that every representation of Λ_P is a representation of Λ and vice versa. Both $C^*_{\min}(\Lambda)$ and $C^*_{\min}(\Lambda_P)$ carry coactions of G satisfying $S_\lambda \mapsto S_\lambda \otimes U_{d(\lambda)}$ for all λ (the coaction of G on $C^*_{\min}(\Lambda)$ is the inflation of the canonical coaction of H). Theorem 5.3 implies that the co-universal representations of both algebras consist of nonzero partial isometries, so the co-universal properties of the two algebras yield mutually inverse homomorphisms between them. ∎

6. Examples

6.1. Spielberg's C^*-algebras of hybrid graphs. For the following discussion, we need to recall the cartesian-product graph introduced by Kumjian and Pask [13]. Given a k-graph Λ and an l-graph Γ, the cartesian-product graph is the $(k+l)$-graph which is equal to the cartesian product $\Lambda \times \Gamma$, with pointwise operations and structure maps. By [13, Corollary 3.5(iv)], $C^*(\Lambda \times \Gamma) \cong C^*(\Lambda) \otimes C^*(\Gamma)$.

We recall the construction of a hybrid graph [27, Definition 2.1]. A warning: as frequently happens when treating constructions involving directed graphs using ideas based on k-graphs, it is easiest to reverse the directions of the edges from [27]. Let D be the following directed graph:



Fix, for the rest of the section, irreducible directed graphs E_0, E_1, F_0 and F_1 each containing at least one infinite receiver. We fix infinite receivers $v_i \in E_i^0$ and $w_i \in F_i^0$, and we attach the 2-graphs $E_i \times F_i$ to D by identifying $u_i \in D^0$

with $(v_i, w_i) \in E_i^0 \times F_i^0$. We call the resulting object the **hybrid graph**. The range and source maps coming from D and each $E_i \times F_i$ extend to range and source maps r and s on the hybrid graph.

A **finite path** in the hybrid graph is a finite string $\mu_1 \cdots \mu_k$, where

(1) $\mu_j \in D^* \sqcup \left(\bigsqcup_{i=0,1} E_i^* \times F_i^*\right)$ for each $1 \le j \le k$;
(2) $s(\mu_j) = r(\mu_{j+1})$ for each $1 \le j < k$; and
(3) $\mu_j \in D^* \iff \mu_{j+1} \in \bigsqcup_{i=0,1} E_i^* \times F_i^*$ for each $1 \le j < k$.

We say that paths $\mu, \nu \in D^* \sqcup \left(\bigsqcup_{i=0,1} E_i^* \times F_i^*\right)$ are of **different type** if one of them belongs to D^* and the other to $E_i^* \times F_i^*$.

The range and source maps extend naturally to finite paths: $r(\mu_1 \cdots \mu_k) := r(\mu_1)$ and $s(\mu_1 \cdots \mu_k) := s(\mu_k)$. Let $l : D^* \to \mathbb{N}$ be the length function, and let $l : E_i^* \times F_i^*$ be the standard degree function on the cartesian-product graph; that is, $l(\alpha, \beta) := (l(\alpha), l(\beta))$. Denote by Λ the set of all finite paths in the hybrid graph. Then Λ is a category under concatenation. Define $d : \Lambda \to \mathbb{N}^2 * \mathbb{N}$ by defining $d(\mu_i \cdots \mu_k)$ to be the word $l(\mu_1) \cdots l(\mu_k)$.

Lemma 6.1: *The pair (Λ, d) described above is an $(\mathbb{N}^2 * \mathbb{N})$-graph. Moreover, given finite paths $\mu = \mu_1 \cdots \mu_m$ and $\nu = \nu_1 \cdots \nu_n$ with $m \le n$, we have*

$$\operatorname{MCE}(\mu,\nu) = \begin{cases} \{\nu\} & \text{if } n > m,\ \mu_i = \nu_i \text{ for } i < m, \\ & \quad \text{and } \nu_m = \mu_m \nu'_m; \\ (\mu_1 \cdots \mu_{m-1}) \operatorname{MCE}(\mu_m, \nu_m) & \text{if } n = m \text{ and } \mu_i = \nu_i \text{ for } i < m; \\ \emptyset & \text{otherwise.} \end{cases} \tag{6.1}$$

In particular, Λ is finitely aligned.

Proof. We must first show that (Λ, d) satisfies the factorisation property. Indeed, suppose that $d(\lambda) = wx$ where $w, x \in \mathbb{N}^2 * \mathbb{N}$. Write $w = w_1 \cdots w_m$ and $x = x_1 \cdots x_n$ where each $w_i, x_j \in \mathbb{N}^2 \sqcup \mathbb{N}$ and $w_i \in \mathbb{N}^2 \implies w_{i+1} \in \mathbb{N}$ and similarly for x. By definition of d, we have $\lambda = \mu_1 \cdots \mu_{m-1} \gamma \nu_2 \cdots \nu_n$ where $d(\mu_i) = w_i$ and $d(\nu_i) = x_i$ for all i, and $d(\gamma) = w_m x_1$. If $w_m \in \mathbb{N}^2$ and $x_1 \in \mathbb{N}$ or vice versa, then the definition of d again forces $\gamma = \mu_m \nu_1$ where $d(\mu_m) = w_m$ and $d(\nu_1) = x_1$. If, instead, we have both w_m and x_1 in $\mathbb{N}^2$ or both w_m and x_1 in $\mathbb{N}$, then the factorisation property in either $\bigcup_{i=0,1} E_i^* \times F_i^*$ or D^* implies that we can factorise γ uniquely as $\gamma = \mu_m \nu_1$ where $d(\mu_m) = w_m$ and $d(\nu_1) = x_1$. In particular, $\mu := \mu_1 \cdots \mu_m$ and $\nu := \nu_1 \cdots \nu_n$ are the unique paths such that $d(\mu) = w$, $d(\nu) = x$, and $\lambda = \mu\nu$. Hence (Λ, d) satisfies the factorisation property as claimed. In particular, (Λ, d) is a $(\mathbb{N}^2 * \mathbb{N})$-graph.

We must establish (6.1). To see this, fix $\mu, \nu \in \Lambda$. First observe that if $d(\mu) \vee d(\nu) = \infty$, then $\operatorname{MCE}(\mu, \nu)$ is empty, so we may assume $d(\mu) \vee d(\nu) < \infty$. Let $w := d(\mu)$ and $x = d(\nu)$, and write $w = w_1 \cdots w_m$ and $x = x_1 \cdots x_n$ so that each $w_i, x_j \in \mathbb{N}^2 \sqcup \mathbb{N}$, and $w_i \in \mathbb{N}^2 \implies w_{i+1} \in \mathbb{N}$ and likewise for x. Write $\mu = \mu_1 \cdots \mu_m$ and $\nu = \nu_1 \cdots \nu_n$ with $d(\mu_i) = w_i$ and $d(\nu_i) = x_i$. Without loss of generality, assume that $m \le n$. By definition of the free product, since $w \vee x \neq \infty$, we must have $w_i = x_i$ for $i < n$. By the factorisation property, we have

(6.2)
$$\operatorname{MCE}(\mu, \nu) = \begin{cases} (\mu_1 \cdots \mu_{m-1}) \operatorname{MCE}(\mu_m, \nu_m \cdots \nu_n) & \text{if } \mu_i = \nu_i \text{ for } i < m, \\ \emptyset & \text{otherwise.} \end{cases}$$

So suppose $\mu_i = \nu_i$ for $i < m$. We must consider two cases: either $m = n$ or $m < n$. First suppose that $m = n$. Then (6.2) gives $|\operatorname{MCE}(\mu, \nu)| = |\operatorname{MCE}(\mu_m, \nu_m)| < \infty$ because each of D^* and $\bigsqcup_{i=0,1} E_i^* \times F_i^*$ is finitely aligned. Now suppose that $m < n$. For any $\alpha \in \Lambda$, the element $d(\nu_m \cdots \nu_n \alpha)$ has the form $x_1 \cdots x_n y_1 \cdots y_l$ for some y_i, and it follows, in particular, that $d(\mu_m \beta) = d(\nu_m \cdots \nu_n \alpha)$ forces $w_m = d(\mu_m) \le d(\nu_m) = x_m$. The factorisation property in either D^* or $\bigsqcup_{i=0,1} E_i^* \times F_i^*$ then implies that $\nu_m = \mu_m \nu_m'$. Hence ν is the unique element of $\operatorname{MCE}(\mu, \nu)$. This shows that Λ is finitely aligned as claimed. ∎

Recall from [27, Definition 3.3] that the C^*-algebra Θ associated to the hybrid graph is the universal C^*-algebra generated by mutually-orthogonal projections $\{S_v : v \in \Lambda^0\}$ and partial isometries

$$\Big\{ S_\nu : \nu \in D^1 \sqcup \Big(\bigsqcup_{i=0,1} (E_i^1 \times F_i^0) \sqcup (E_i^0 \times F_i^1) \Big) \Big\}$$

satisfying conditions (i)–(v) of [27, Definition 3.3]:

- (i) Each S_v is a projection and each S_ν is a partial isometry.
- (ii) For each $v \in E_i^0$ the projections $\{S_{(v,w)} : w \in F_i^0\}$ and partial isometries $\{S_{(v,f)} : f \in F_i^1\}$ satisfy the Cuntz–Krieger relations for the graph F_i.
- (ii') For each $w \in F_i^0$ the projections $\{S_{(v,w)} : v \in E_i^0\}$ and partial isometries $\{S_{(e,w)} : e \in E_i^1\}$ satisfy the Cuntz–Krieger relations for the graph E_i.

(iii) The projections $\{S_v : v \in D^0\}$ are mutually orthogonal, and the partial isometries $\{S_e : e \in D^1\}$ satisfy $S_e^* S_e = S_{s(e)}$ for all $e \in D^1$, and

$$\sum_{r(e)=v} S_e S_e^* \leq S_v \text{ for all } v \in D^0, \text{ with equality if } v \notin \{u_0, u_1\}.$$

(iv) If $e \in D^1$ and $f \in \left(\bigsqcup_{i=0,1}((E_i^1 \times F_i^0) \sqcup (E_i^0 \times F_i^1))\right)$, then $S_e^* S_f = 0$.

(v) For $e \in E_i^1$ and $f \in F_i^1$, we have

$$S_{(e,r(f))} S_{(f,s(e))} = S_{(f,r(e))} S_{(e,s(f))}, \text{ and}$$
$$S_{(e,r(f))}^* S_{(r(e),f)} = S_{(s(e),f)} S_{(e,s(f))}^*.$$

In what follows, we extend the generating family in Spielberg's C^*-algebra Θ to a representation S of the associated $(\mathbb{N}^2 * \mathbb{N})$-graph Λ. This should not be confused with the co-universal representation of Λ in $C^*_{min}(\Lambda)$.

Proposition 6.2: *The C^*-algebra Θ defined above is isomorphic to $C^*_{\min}(\Lambda)$.*

Proof. A vertex in Λ is an element $v \in \Lambda^{(0,0)} \cup \Lambda^0$, and an edge in Λ is an element $\nu \in \Lambda^{(1,0)} \cup \Lambda^{(0,1)} \cup \Lambda^1$. Given $\mu \in D^*$, say $\mu = d_1 d_2 \cdots d_n$ with each $d_i \in D^1$, we define $S_\mu = S_{d_1} \cdots S_{d_n}$. For $(\mu, \nu) \in E_i^* \times F_i^*$, with $\mu = e_1 \cdots e_m$ and $\nu = f_1 \cdots f_n$ with each $e_j \in E_i^1$ and each $f_j \in F_i^1$, define

$$S_{(\mu,\nu)} := S_{(e_1,r(\nu))} \cdots S_{(e_m,r(\nu))} S_{(s(\mu),f_1)} \cdots S_{(s(\mu),f_n)}.$$

Now for a finite path $\mu = \mu_1 \cdots \mu_m$ in Λ, define $S_\mu := S_{\mu_1} \cdots S_{\mu_m}$.

We claim that $\mu \mapsto S_\mu$ determines a representation of Λ in Θ. Condition (T1) of Definition 5.1 follows immediately from (i)–(iii) of [27, Definition 3.3]. For the other conditions, let $\mu, \nu \in \Lambda$ with $\mu = \mu_1 \cdots \mu_m$ and $\nu = \nu_1 \cdots \nu_n$. Suppose $s(\mu) = r(\nu)$. So $s(\mu_m) = r(\nu_1)$. If μ_m and ν_1 are of different type, then $S_\mu S_\nu = S_{\mu\nu}$ follows immediately. If μ_m and ν_1 are the same type, then the Cuntz–Krieger relations of $E_i \times F_i$ and Toeplitz–Cuntz–Krieger relations of D imply that $S_{\mu_j} S_{\nu_1} = S_{\mu_j \nu_1}$. We then have

$$S_\mu S_\nu = S_{\mu_1} \cdots S_{\mu_j} S_{\nu_1} \cdots S_{\nu_k} = S_{\mu_1} \cdots S_{\mu_j \nu_1} \cdots S_{\nu_k} = S_{\mu_1 \cdots (\mu_j \nu_1) \cdots \nu_k} = S_{\mu\nu},$$

and so condition (T2) is satisfied.

Fix a finite path $\mu = \mu_1 \cdots \mu_m$ in Λ. The Cuntz–Krieger relations of $E_i \times F_i$, and Toeplitz–Cuntz–Krieger relations of D imply that $S_{\mu_j}^* S_{\mu_j} = S_{s(\mu_j)}$ for all $1 \leq j \leq m$. An inductive argument then gives $S_\mu^* S_\mu = S_{s(\mu)}$, so (T3) is satisfied.

The proof of (T4) is tedious, so we set it aside as a claim.

Claim: *The map $\mu \mapsto S_\mu$ satisfies (T4).*

To prove this claim, fix $\mu, \nu \in \Lambda$. We must show that

$$S_\mu S_\mu^* S_\nu S_\nu^* = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} S_\lambda S_\lambda^*. \tag{6.3}$$

Write $\mu = \mu_1 \cdots \mu_m$ and $\nu = \nu_1 \cdots \nu_n$. We may assume without loss of generality that $m \le n$. First suppose that μ_1 and ν_1 are of different type. Then (iv) ensures that

$$S_\mu S_\mu^* S_\nu S_\nu^* = S_\mu S_{\mu_2\cdots\mu_m}^* S_{\mu_1}^* S_{\nu_1} S_{\nu_2\cdots\nu_n} S_\nu^* = 0.$$

Lemma 6.1 implies that $\mathrm{MCE}(\mu,\nu) = \emptyset$ also, so (T4) is satisfied.

Now suppose that μ_1 and ν_1 have the same type. Condition (3) then implies that μ_i and ν_i have the same type for $i \le m$. If $l \le m$ satisfies $\mu_i = \nu_i$ for all $i < l$, then repeated applications of (T3) show that

$$S_\mu S_\mu^* S_\nu S_\nu^* = S_\mu S_{\mu_l\cdots\mu_m}^* S_{\nu_l\cdots\nu_n} S_\nu^*,$$

and then conditions (ii), (ii′) and (iii) imply that

$$S_\mu S_\mu^* S_\nu S_\nu^* = \sum_{\mu_l\alpha = \nu_l\beta \in \mathrm{MCE}(\mu_l,\nu_l)} S_\mu S_{\mu_m}^* \cdots S_{\mu_{l+1}}^* S_\alpha S_\beta^* S_{\nu_l} \cdots S_{\nu_n} S_\nu^*. \tag{6.4}$$

If $\mu_l \ne \nu_l$ for some $l < m$, then $\mu_l\alpha = \nu_l\beta \in \mathrm{MCE}(\mu,\nu)$ forces at least one of $d(\alpha), d(\beta) > 0$ and then (iv) forces one of $S_{\mu_{l+1}}^* S_\alpha$ and $S_\beta^* S_{\nu_{l+1}}$ to be equal to zero. Hence $\mu_l \ne \nu_l$ for some $l < m$ forces $S_\mu S_\mu^* S_\nu S_\nu^* = 0$. Since Lemma 6.1 implies that $\mathrm{MCE}(\mu,\nu) = \emptyset$, unless $\mu_i = \nu_i$ for all $i < m$, we have now established (6.3) whenever $\mu_l \ne \nu_l$ for some $l < m$.

So we suppose that $\mu_l = \nu_l$ for all $l < m$, and consider two cases: $m = n$ or $m < n$. Suppose first that $m = n$. Then (6.4) reduces to

$$\begin{aligned} S_\mu S_\mu^* S_\nu S_\nu^* &= \sum_{\mu_m\alpha = \nu_m\beta \in \mathrm{MCE}(\mu_m,\nu_m)} S_\mu S_\alpha S_\beta^* S_\nu^* \\ &= \sum_{\mu_m\alpha = \nu_m\beta \in \mathrm{MCE}(\mu_m,\nu_m)} S_{\mu_1\cdots\mu_{m-1}} S_{\mu_m\alpha} S_{\nu_m\beta}^* S_{\nu_1\cdots\nu_{m-1}}^*, \end{aligned}$$

and since Lemma 6.1 gives $\mathrm{MCE}(\mu,\nu) = \mu_1\cdots\mu_{m-1}\,\mathrm{MCE}(\mu_m,\nu_m)$, this establishes (6.3) in the case $m = n$. Now suppose that $m < n$. Suppose that $\nu_m \ne \mu_m\nu_m'$. Then $\mathrm{MCE}(\mu,\nu) = \emptyset$ by Lemma 6.1. Also, $d(\beta) > 0$, and since β and ν_{m+1} are of different type, condition (iv) again gives $S_\beta^* S_{\nu_{m+1}} = 0$. Hence $S_\mu S_\mu^* S_\nu S_\nu^* = 0 = \sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} S_\lambda S_\lambda^*$, establishing (6.3) in the case $m < n$ and

$\nu_m \neq \mu_m\nu'_m$. Finally, suppose that $\nu_m = \mu_m\nu'_m$. Then $\mathrm{MCE}(\mu,\nu) = \{\nu\}$ by Lemma 6.1, and $\mathrm{MCE}(\mu_m,\nu_m) = \{\nu_m\}$. Hence (6.4) reduces to

$$S_\mu S_\mu^* S_\nu S_\nu^* = S_\mu S_{\nu'_m} S_{s(\nu_m)}^* S_{\nu_{m+1}\cdots\nu_n} S_\nu^* = S_\nu S_\nu^*.$$

We have now established (6.3) in all possible cases. This proves the claim.

The claim completes the proof that $\mu \mapsto S_\mu$ is a representation of Λ in Θ.

We denote by e the identity of $\mathbb{Z}^2 * \mathbb{Z}$. Straightforward calculations show that $\{S_v \otimes 1_{C^*(G)} : v \in \Lambda^{(0,0)} \cup \Lambda^0\} \cup \{S_\nu \otimes U_{d(\nu)} : \nu \in \Lambda^{(0,1)} \cup \Lambda^{(1,0)} \cup \Lambda^1\}$ is a set of projections and partial isometries in $\Theta \otimes C^*(\mathbb{Z}^2 * \mathbb{Z})$ satisfying conditions (i)–(v) of [27, Definition 3.3]. The universal property of Θ then gives a $*$-homomorphism $\alpha : \Theta \to \Theta \otimes C^*(\mathbb{Z}^2 * \mathbb{Z})$ such that $\alpha(S_\mu) = S_\mu \otimes U_{d(\mu)}$.

We show that α is a coaction. It is straightforward to check that α satisfies the coaction identity on generators of Θ. Since increasing finite sums $P_F := \sum_{v \in F} S_v$ where $F \subset \Lambda^{(0,0)} \cup \Lambda^0$ form an approximate identity for Θ such that $\alpha(P_F) = P_F \otimes 1$ for all F, the homomorphism α is nondegenerate, and it follows (see [7, Remark A.22(3)]) that it is a coaction.

Theorem 5.3 now yields a surjective homomorphism $\psi_S : \Theta \to C^*_{\min}(\Lambda)$. Corollary 3.19 of [27] implies that Θ is simple, and it follows that ψ_S is an isomorphism. ∎

Corollary 6.3: *Let A be a simple purely infinite nuclear C^*-algebra belonging to the UCT class. Then there exists a finitely aligned $(\mathbb{N}^2 * \mathbb{N})$-graph Λ such that A is stably isomorphic to $C^*_{\min}(\Lambda)$.*

Proof. Fix K-groups K_0 and K_1. The argument of [28, Theorem 2.2] (see also [29]) shows that there are graphs E_i, F_i, $i = 0, 1$, each with a unique infinite receiver, such that the C^*-algebra Θ of the associated hybrid graph is a simple, purely infinite nuclear C^*-algebra in the UCT class with $K_*(\Theta) = (K_0, K_1)$. The result therefore follows from Lemma 6.1 and Proposition 6.2. ∎

6.2. Other examples.

We now consider two examples discussed in [26] and [2].

Example 6.4: Consider the product system of [26, Example 3.16]. That is, let $G = (\mathbb{Z}^2, +)$, let $S = \{0\} \times \mathbb{N} \subset \mathbb{Z}^2$, and let $P := S \sqcup ((\mathbb{N} \setminus \{0\}) \times \mathbb{Z})$. Then (G, P) is a quasi-lattice ordered group, and the order on G induced by P is lexicographic order.

We define a P-graph Λ as follows. As a set, $\Lambda = \{f_s, g_s : s \in P\}$, and the degree map is given by $d(f_s) = d(g_s) = s$. Define range and source maps by $r(f_s) = s(f_s) = f_0$ for all $s \in P$; $r(g_s) = s(g_s) = g_0$ for $s \in S$; and $r(g_s) = f_0$, $s(g_s) = g_0$ for $s \in P \setminus S$. So for $s \in P$, the directed graph $(\Lambda^0, \Lambda^s, r, s)$ has one of two forms:

Define composition by

$$f_s f_t = f_{s+t}, \quad g_s g_t = g_{s+t} \text{ for } t \in S, \quad \text{and} \quad f_s g_t = g_{s+t} \text{ for } t \in P \setminus S.$$

It is routine to check that this determines a composition map which is defined on all composable pairs, that composition is associative, and that it satisfies the factorisation property. Hence Λ is a P-graph. Since $v\Lambda^s$ is a singleton for all $v \in \Lambda^0$ and $s \in P$, Λ is finitely aligned.

If X is the product system over P whose fibre over $s \in P$ is the usual graph C^*-correspondence (see [21, Proposition 3.2]), then X is isomorphic to the product system described in [26, Example 3.16].

Identifying representations of the product system X with representations of Λ as in [21, Theorem 4.2], the discussion of [26, Example 3.16] shows that every representation t of Λ corresponding to a CNP-covariant representation of X satisfies $t_{g_0} = 0$. In particular, the universal generating representation of Λ in the algebra $\mathcal{NO}_X$ defined in [26] does not satisfy $t_v \neq 0$ for all $v \in \Lambda^0$.

However, since the set $\{g_0, g_1, g_2, \dots\}$ is an ultrafilter of Λ, the representation T of Definition 3.5 satisfies $T_{g_0} \neq 0$ and hence the generating representation of Λ in $C^*_{\min}(\Lambda)$ consists of nonzero partial isometries. That is, while the algebra $\mathcal{NO}_X$ does not satisfy Criterion (A) of [26, Section 1.2], our $C^*_{\min}(\Lambda)$ does.

Remark 6.5: A brief explanation is in order here. The bimodules $\widetilde{X}_s$ of [26] were intended to model the sets $\Lambda^{\leq s}$ of finite paths in a k-graph whose degree is smaller than s but which cannot be extended nontrivially in direction s (see [22]). The Cuntz–Pimsner covariance condition of [26] was then intended to model the Cuntz–Krieger relation of [22].

In a P-graph Λ (as opposed to a k-graph), the analogue of $\Lambda^{\leq s}$ would be

$$\Lambda^{\leq s} = \{\mu \in \Lambda : d(\mu) \leq s, d(\mu) < p \leq s \implies s(\mu)\Lambda^{d(\mu)^{-1}p} = \emptyset\}.$$

In the example above, we have $g_0\Lambda^s = \{g_s\}$ if $s \in S$, and $g_0\Lambda^s = \emptyset$ if $s \in P \setminus S$. Since $s \in S$ and $t \in P \setminus S$ implies $s \leq t$, it follows that $g_0\Lambda^{\leq s} = \emptyset$ for $s \in P \setminus S$. In particular, the Cuntz–Krieger relations of [22], adapted to P-graphs, would force $t_{g_0} = 0$.

The point is that since for $t \in P \setminus S$, the set $\{s \in P : s \leq t\}$ is infinite, the above definition of $\Lambda^{\leq s}$ is inappropriate. Instead, Exel's insight, when applied to this example, is that the set of ultrafilters whose elements all have degree smaller than s is the appropriate analogue of $\Lambda^{\leq s}$.

Example 6.6: Let $\mathbb{F}_2$ be the free group on two generators, $\mathbb{F}_2 = \langle a, b\rangle$, and let $\mathbb{F}_2^+$ be subsemigroup generated by a and b. Let Λ be the $\mathbb{N}$-graph with $\Lambda^n = \{e_n\}$ for all $n \in \mathbb{N}$, and let $\Lambda_{\mathbb{F}_2^+}$ be the $\mathbb{F}_2^+$-graph obtained from the first assertion of Corollary 5.6 applied to the embedding of $\mathbb{N}$ in $\mathbb{F}_2^+ = \langle a, b\rangle$ given by $1 \mapsto a$.

The second assertion of Corollary 5.6 implies that $C^*_{\min}(\Lambda_{\mathbb{F}_2^+}) \cong C^*_{\min}(\Lambda)$. Lemma 5.5 implies that $C^*_{\min}(\Lambda) \cong C^*(\Lambda)$. It is well-known (see, for example, [20, Example 2.14]) that $C^*(\Lambda) \cong C(\mathbb{T})$. By contrast, [2, Example 3.9] shows that if X is the product system over $\mathbb{F}_2^+$ corresponding to $\Lambda_{\mathbb{F}_2^+}$, then $\mathcal{NO}_X \cong \mathcal{T}$ where $\mathcal{T}$ denotes the Toeplitz algebra. In particular, $\mathcal{NO}_X$ is not co-universal for gauge-compatible representations of X. Moreover, passing to the normalisation $\mathcal{NO}^r_X$ as in [2, Section 4] doesn't help because the coaction of $\mathbb{F}_2^+$ on $\mathcal{NO}_X$ is already normal. In particular, our construction avoids the pathology arising in this example for product systems (see [2, Remark 4.2]).

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