Symmetric Three-body Problem Notes

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1 Equations of Motion

From Ekeland page 59 (paperback edition):

> Imagine two stars of equal mass, rotating around their common center of gravity. Newton's law asserts that this is possible, and that the orbit of the two stars will be a circle, along which they travel with equal speed, being exactly opposite to each other at all times. [...] A third body, with very small mass–a comet, for instance–moves along [the axis].

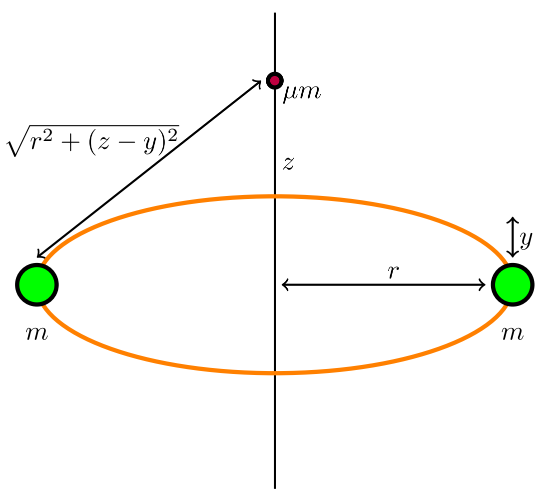


Figure 1: Two stars of the same mass m travel in a circular orbit along with a small comet on the centerline with an initial velocity along the z-axis. The system has cylindrical symmetry so that the plain containing the stars moves as a single unit and the comet does not leave the axis.

Gravitational potentials from the interactions of the 3 masses. The first term is the interaction of the stars while the second term is the sum of the interactions of the comet with each of the stars.

$$\begin{aligned} U_{mm} &= \frac{-Gmm}{2r} && (1) \\ U_{\mu m} &= \frac{-2G\mu mm}{\sqrt{r^2 + (z-y)^2}} && (2) \\ & && (3) \end{aligned}$$

Kinetic energy terms represent radial, circular and motion along the axis.

$$T = \frac{1}{2}\mu m\dot{z}^2 + m\dot{y}^2 + m\dot{r}^2 + m(r\dot{\theta})^2 \tag{4}$$

Lagrangian.

$$\begin{aligned} \mathcal{L} &= T - V && (5) \\ &= \frac{1}{2}\mu m\dot{z}^2 + m\dot{y}^2 + m\dot{r}^2 + m(r\dot{\theta})^2 + \frac{-Gmm}{2r} + \frac{-2G\mu mm}{\sqrt{r^2 + (z-y)^2}} && (6) \end{aligned}$$

Partial derivatives with respect to velocities and postitions.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{z}} &= \mu m\dot{z} && (7) \\ \frac{\partial \mathcal{L}}{\partial \dot{r}} &= 2m\dot{r} && (8) \\ \frac{\partial \mathcal{L}}{\partial \dot{y}} &= 2m\dot{y} && (9) \\ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= 2mr^2\dot{\theta} && (10) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial z} &= \frac{2G\mu m^2(z-y)}{(r^2 + (z-y)^2)^{3/2}} && (11) \\ \frac{\partial \mathcal{L}}{\partial r} &= 2mr\theta^2 - \frac{Gm^2}{r^2} - \frac{2G\mu m^2 r}{(r^2 + (z-y)^2)^{3/2}} && (12) \\ \frac{\partial \mathcal{L}}{\partial y} &= -\frac{2G\mu m^2(z-y)}{(r^2 + (z-y)^2)^{3/2}} && (13) \\ \frac{\partial \mathcal{L}}{\partial \theta} &= 0 && (14) \end{aligned}$$

2 Equations of motion

$$\ddot{z} = \frac{2Gm(z-y)}{(r^2+(z-y)^2)^{3/2}} \tag{15}$$

$$\ddot{y} = \frac{G\mu m(z-y)}{(r^2+(z-y)^2)^{3/2}} \tag{16}$$

$$\ddot{r} = r\dot{\theta}^2 - Gm(\frac{1}{2r^2} + \frac{\mu r}{(r^2+(z-y)^2)^{3/2}}) \tag{17}$$

$$\ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r} \tag{18}$$

Rewrite as a system of 1^{st} differential equations and integrated with a "Leap frog" integrator. This preserves phase space topology.

3 Escape Velocity

Energy at $z=0$ is the energy at $z=\infty$,

$$\frac{1}{2}\mu m v_0^2 - \frac{2G\mu m^2}{r} = \frac{1}{2}\mu m(0)^2 - \frac{2G\mu m^2}{(\infty)} \tag{19}$$

$$v_0 = \sqrt{\frac{4Gm}{r}} \tag{20}$$

$$v_0 = \sqrt{\frac{4(4.302e^{-}3\text{pc sun}^{-1}(\frac{km}{s})^2)(1\text{ sun})}{0.5\text{ pc}}} \tag{21}$$

$$\approx 0.185515\frac{km}{s} \tag{22}$$

4 Small z

For small z, $\tan z/r \approx z/r$ and r is constant,

$$\mu m\ddot{z} = -\frac{2G\mu m^2}{r^2}\frac{z}{r} \tag{23}$$

$$T = 2\pi\sqrt{\frac{2G\mu m^2}{r^3}} \tag{24}$$

$$\approx 23.94885\text{ time units} \tag{25}$$

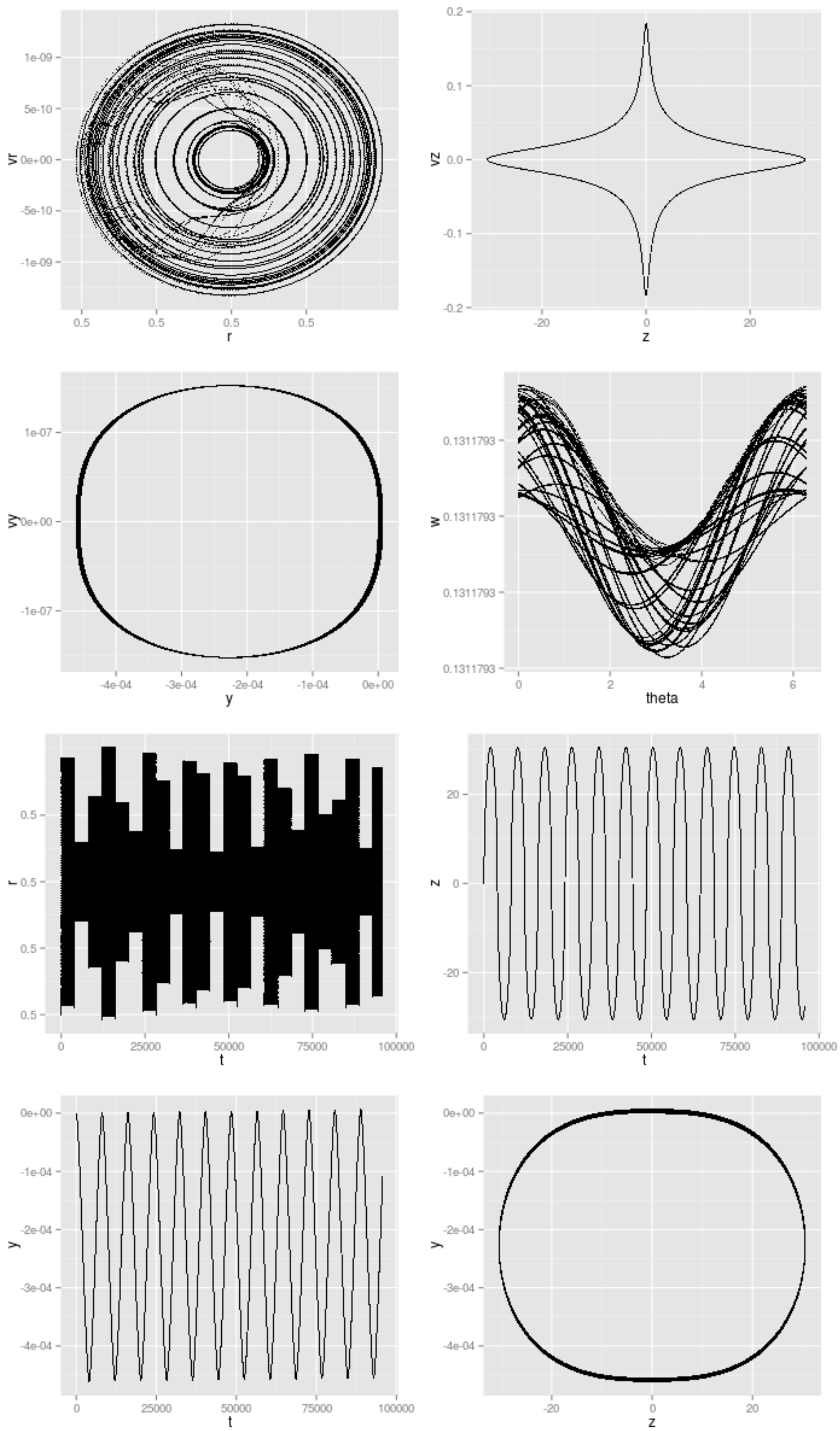
5 Sources and Licensing

For code and example output, please visit and fork or clone `https://github.com/DrSkippy/Gravitational-Three-Body-Symmetric`. If you find errorsor have comments, please email scott@drskippy.net.

References

[Ekeland1990] Ivar Ekeland. Mathematics and the Unexpected. http://www.amazon.com/Mathematics-Unexpected-Ivar-Ekeland/dp/0226199908 1990.

[Drexel] The Leapfrog Integrator, http://einstein.drexel.edu/courses/Comp_Phys/Integrators/leapfrog/.



vr
1e-09
5e-10
0e+00
-5e-10
-1e-09
0.5
0.5
0.5
0.5
r
vz
0.2
0.1
0.0
-0.1
-0.2
-20
0
20
z
vy
1e-07
0e+00
-1e-07
-4e-04
-3e-04
-2e-04
-1e-04
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