

# Mystery Function Family: Bounding Inequalities and Convergence Twists

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09-25-2025, reviewed October 2, 2025

## 1 Introduction

In the context of studying exponential integrals and special functions, we draw inspiration from the classical formulas in the *Handbook of Mathematical Functions* by Abramowitz and Stegun (1964). Specifically, the definition of the generalized exponential integral  $E_n(x)$  is given in (5.1.4), page 228:

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt, \quad \Re(x) > 0, \quad n = 1, 2, 3, \dots,$$

which represents a truncated form of the Laplace transform of  $1/t^n$ . Formula (5.1.19), page 229, provides a useful recurrence relation for these functions:

$$E_{n+1}(x) = \frac{1}{n} [e^{-x} - xE_n(x)], \quad n \geq 1.$$

Considering the family of functions  $f(x, n) = e^x E_n(x)$ , we obtain a normalized form that facilitates asymptotic analyses and bounds.

The central problem of this document is to find a family of positive functions  $f(x, n)$ , defined for  $x > 0$  and  $n = 1, 2, 3, \dots$  (fixed integer), satisfying the following inequality:

$$\frac{1}{x+n} < f(x, n) \leq \frac{1}{x+n-1},$$

where, for  $n = 1$ , the upper bound is interpreted as  $+\infty$  (or  $1/x$  near  $x \rightarrow 0^+$ ). This family must be strictly decreasing in  $x$  for fixed  $n$  and related to infinite series or integral representations, such as a tail sum of the form  $\sum_{k=0}^\infty \frac{1}{(x+n+k)^p}$  for an appropriate  $p$ .

There are several possible solutions, ranging from elementary forms to more sophisticated special functions. To enrich the exploration, we examine proposals from contemporary artificial intelligences. For instance, ChatGPT suggests a simple rational family  $f(x, n) = 1/(x+n-\theta)$  with  $0 < \theta \leq 1$ , which elegantly saturates the upper bound and remains easy to handle. Gemini, on the other hand, proposes a logarithmic approach, such as  $f(x, n) = \ln\left(1 + \frac{1}{x+n-1}\right)$ , which captures the decay but requires careful verification of the bounds.

These initial suggestions guide toward the intended result: the trigamma family  $\psi^{(1)}(x+n)$ , a polygamma special function that naturally emerges from integral bounds on series, thereby linking exponential integrals to polygammas via tail approximations. We will develop this step by step, incorporating convergence twists for a more nuanced analysis.

## 2 Problem Statement

Consider a family of positive functions  $f(x, n)$  defined for  $x > 0$  and  $n = 1, 2, 3, \dots$  (fixed integer). Seek an analytic expression for  $f(x, n)$  such that, for all  $n \geq 1$ ,

$$\frac{1}{x+n} < f(x, n) \leq \frac{1}{x+n-1},$$

where for  $n = 1$ , the upper bound is interpreted as  $+\infty$  (or  $1/x$  near  $x \rightarrow 0^+$ ). This family must be decreasing in  $x$  for fixed  $n$ , and related to series or integral representations (e.g., a tail sum  $\sum_{k=0}^{\infty} \frac{1}{(x+n+k)^p}$  for appropriate  $p$ ).

## 3 Hints for Solution

1. **Compute the integrals:** For real  $z > 0$ , calculate the incomplete exponential integrals:

- $E_1(z) = \int_1^{\infty} \frac{e^{-zt}}{t} dt,$
- $E_2(z) = \int_1^{\infty} \frac{e^{-zt}}{t^2} dt.$

Use the substitution  $u = zt$  and integration by parts for the second to express  $E_2(z)$  in terms of  $E_1(z)$ . These truncated forms (from 1 to  $\infty$ ) help approximate series tails via integral bounds (integral test for decreasing functions).

2. **Find a candidate for  $f(x, n)$ :** The family should be a known special function, such as a polygamma (derivative of the digamma  $\psi(z)$ ), with an infinite series representation  $\sum_{k=0}^{\infty} \frac{1}{(x+n+k)^2}$ . Verify that classical integral bounds (lower by  $\int_0^{\infty}$ , upper by shifted  $\int_{-1}^{\infty}$ ) reproduce the given inequality. Test numerically for  $n = 2$ ,  $x = 0.5$  (where  $f \approx 0.490 > 1/2.5 = 0.4$  and  $\leq 1/1.5 \approx 0.667$ ).
3. **Is the family  $g(x, n) = e^x E_n(x)$  a candidate?** Check if  $g(x, n) = e^x E_n(x)$  (where  $E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n} dt$ ) satisfies the inequality for  $n = 1, 2, 3, 4$  and various  $x > 0$ . Use the recurrence  $E_{n+1}(x) = \frac{1}{n}(e^{-x} - xE_n(x))$  to compute explicit  $g_n(x)$  (e.g.,  $g_2(x) = 1 - xg_1(x)$ ). Confirm with a numerical table (e.g., for  $n = 2$ ,  $x = 1$ ,  $g_2(1) \approx 0.404 > 1/3 \approx 0.333$  and  $\leq 1/2 = 0.5$ ).
4. **Bonus twist: Convergence toward the trigamma.** For extra flair, explore if  $g(x, n)$  converges asymptotically to  $\psi'(x+n)$  when adjusting tails (e.g., the trigamma remainder  $r_m(x, n) = \sum_{k=m}^{\infty} \frac{1}{(x+n+k)^2} \approx e^{x+n+m} E_2(x+n+m)$  for large  $m$ ). Check uniform convergence on compacts  $[a, b] \subset (0, \infty)$  via the Weierstrass M-test (yes, as the series is uniformly convergent). Globally? Who knows poles near 0 make it wild. For large  $n$ ,  $g(x, n) \sim 1/x^{n-1}$ , a "pseudo-trigamma" teasing uniformity... or not!

## 4 Solution by Grok4 Expert

The primary candidate for the family is the trigamma function:

$$f(x, n) = \psi^{(1)}(x+n) = \sum_{k=0}^{\infty} \frac{1}{(x+n+k)^2},$$

which satisfies the inequality strictly for the lower bound and non-strictly for the upper (derived from integral estimates on the decreasing convex summand  $1/(z+t)^2$  with  $z = x+n > n \geq 1$ ).

### 4.1 Step 1: Computed Integrals

-  $E_1(z) = \int_z^{\infty} \frac{e^{-u}}{u} du$  (via  $u = zt$ ). -  $E_2(z) = e^{-z} - zE_1(z)$  (by parts:  $\frac{e^{-u}}{u^2} = -\frac{d}{du} \left( \frac{e^{-u}}{u} \right) - \frac{e^{-u}}{u}$ ).

These provide tail approximations for series, e.g.,  $\sum_{k=0}^{\infty} e^{-(x+n+k)} \approx E_1(x+n)$ , but for  $p = 2$ , they bound the trigamma tails.

## 4.2 Step 2: Candidate Verification

The trigamma fits perfectly: Lower bound from  $\psi^{(1)}(z) > \int_0^\infty \frac{1}{(z+t)^2} dt = \frac{1}{z}$ ; upper from shifted  $\psi^{(1)}(z) \leq \int_{-1}^\infty \frac{1}{(z+t)^2} dt = \frac{1}{z-1}$  (for  $z > 1$ ). Numerical test for  $n = 2, x = 0.5$ :  $\psi^{(1)}(2.5) \approx 0.490 > 0.4 \leq 0.667$ .

## 4.3 Step 3: Family $g(x, n)$ as Candidate

Yes,  $g(x, n) = e^x E_n(x)$  satisfies the bounds (see table below for  $n = 1$  to 4, various  $x$ ):

$n$	$x$	$g(x, n)$	Lower $1/(x+n)$	Upper $1/(x+n-1)$
1	0.5	0.923	0.667	$\infty$
1	1.0	0.596	0.500	$\infty$
2	0.5	0.539	0.400	0.667
2	1.0	0.404	0.333	0.500
3	0.5	0.365	0.286	0.400
3	1.0	0.298	0.250	0.333
4	0.5	0.272	0.222	0.286
4	1.0	0.234	0.200	0.250

Table 1: Numerical verification: All hold strictly.

Recurrence:  $g_{n+1}(x) = \frac{1}{n}(1 - xg_n(x))$ .

## 4.4 Step 4: Convergence Twist

For fixed  $n$ , as  $m \rightarrow \infty$ , the trigamma tail  $r_m(x, n) \approx e^{x+n+m} E_2(x+n+m)$  (Laplace approx of series by integral). Pointwise convergence to  $\psi'(x+n)$  holds; uniformly on compacts  $[a, b] \subset (0, \infty)$  by Weierstrass (majorized by geometric  $\sum 1/k^2$ ). Globally? No near  $x = 0$ , poles disrupt uniformity. For large  $n$ ,  $g(x, n) \sim x^{1-n}$ , mimicking trigamma asymptotics  $\psi'(x+n) \sim 1/(x+n) + 1/(2(x+n)^2)$ , but "teasing" uniformity... va savoir!

This family bridges exponential integrals to polygammas elegant yet absurdly tight.

## 4.5 Conclusion

Such inequalities are common in asymptotic analysis and special functions literature (e.g., *Concrete Mathematics* by Graham, Knuth, and Patashnik). For precise  $f(x)$ , provide more details!

# 5 Solution by ChatGPT5

We are asked to construct a family of positive functions  $f(x, n)$  defined for  $x > 0$  and integer  $n \geq 1$  such that

$$\frac{1}{x+n} < f(x, n) \leq \frac{1}{x+n-1},$$

with the upper bound interpreted as  $+\infty$  (or  $1/x$ ) for  $n = 1$ . The family must also be decreasing in  $x$  for each fixed  $n$ .

## 5.1 A Simple Rational Family

A first convenient candidate is

$$f(x, n) = \frac{1}{x+n-\theta}, \quad 0 < \theta \leq 1.$$

- For  $0 < \theta \leq 1$ , one has  $n-1 \leq n-\theta < n$ , hence

$$\frac{1}{x+n} < f(x, n) \leq \frac{1}{x+n-1}.$$

- Positivity is obvious, and

$$\frac{\partial}{\partial x} f(x, n) = -\frac{1}{(x + n - \theta)^2} < 0,$$

so the function is strictly decreasing in  $x$ .

- Choosing  $\theta = 1$  gives  $f(x, n) = 1/(x + n - 1)$ , which saturates the upper bound (allowed). Any  $0 < \theta < 1$  provides a strict interior value.

## 5.2 The Trigamma Function

A more elegant solution is provided by the trigamma function (the derivative of the digamma):

$$f(x, n) = \psi^{(1)}(x + n) = \sum_{k=0}^{\infty} \frac{1}{(x + n + k)^2}, \quad x > 0, \quad n \geq 1.$$

- Because the summand is decreasing in  $k$ , integral comparisons give

$$\int_0^{\infty} \frac{dt}{(x + n + t)^2} < \sum_{k=0}^{\infty} \frac{1}{(x + n + k)^2} \leq \int_{-1}^{\infty} \frac{dt}{(x + n + t)^2}.$$

This yields the desired bounds:

$$\frac{1}{x + n} < \psi^{(1)}(x + n) \leq \frac{1}{x + n - 1}.$$

- Differentiating gives

$$\frac{\partial}{\partial x} \psi^{(1)}(x + n) = \psi^{(2)}(x + n) = -2 \sum_{k=0}^{\infty} \frac{1}{(x + n + k)^3} < 0,$$

so  $f(\cdot, n)$  is strictly decreasing in  $x$ .

- Example: for  $n = 2$ ,  $x = 0.5$ ,

$$\psi^{(1)}(2.5) \approx 0.49035,$$

which indeed satisfies  $1/2.5 = 0.4 < 0.49035 \leq 1/1.5 \approx 0.667$ .

## 5.3 The Exponential Integral Family

Another interesting candidate is defined via the exponential integral:

$$g(x, n) = e^x E_n(x), \quad E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n} dt.$$

These satisfy the recurrence

$$g_{n+1}(x) = \frac{1 - xg_n(x)}{n}.$$

Numerical checks show that  $g(x, n)$  also satisfies

$$\frac{1}{x + n} < g(x, n) \leq \frac{1}{x + n - 1}, \quad (n \geq 1, x > 0).$$

For example, for  $n = 2$ ,  $x = 1$ ,

$$g_2(1) \approx 0.404, \quad \text{with} \quad \frac{1}{3} \approx 0.333 < 0.404 \leq 0.5.$$

## 5.4 Asymptotic Behaviour and Tails

The trigamma representation

$$\psi^{(1)}(x+n) = \sum_{k=0}^{\infty} \frac{1}{(x+n+k)^2}$$

has a remainder (tail) after  $m$  terms,

$$r_m(x, n) = \sum_{k=m}^{\infty} \frac{1}{(x+n+k)^2}.$$

Using the Laplace transform representation

$$\frac{1}{s^2} = \int_0^{\infty} t e^{-st} dt,$$

one finds

$$r_m(z) = \int_0^{\infty} t \frac{e^{-(z+m)t}}{1-e^{-t}} dt, \quad z = x+n.$$

Expanding  $1/(1-e^{-t})$  around  $t=0$  leads to the asymptotic expansion

$$r_m(z) \sim \frac{1}{z+m} + \frac{1}{2(z+m)^2} + \frac{1}{6(z+m)^3} + \cdots, \quad m \rightarrow \infty.$$

## 5.5 Uniform Convergence

For any compact interval  $[a, b] \subset (0, \infty)$ ,

$$\sup_{x \in [a, b]} \frac{1}{(x+n+k)^2} \leq \frac{1}{(a+n+k)^2},$$

and the bounding series converges. By the Weierstrass M-test, the defining series for  $\psi^{(1)}(x+n)$  converges uniformly on compacts of  $(0, \infty)$ . Globally uniform convergence on  $(0, \infty)$  fails due to the singular behaviour near  $x=0$ .

## 5.6 Conclusion

Two elegant families solve the problem:

- The rational form  $f(x, n) = 1/(x+n-\theta)$ ,  $0 < \theta \leq 1$ .
- The trigamma function  $f(x, n) = \psi^{(1)}(x+n)$ .

Both satisfy the inequalities and monotonicity. The exponential-integral family  $g(x, n) = e^x E_n(x)$  also works and provides an alternative perspective through classical special functions.

## 6 Solution by Gemini 2.5Pro

The problem is to find one or more families of positive functions  $f(x, n)$  defined for  $x > 0$  and integers  $n \geq 1$  that satisfy two primary conditions:

1. The function is bounded such that  $\frac{1}{x+n} < f(x, n) \leq \frac{1}{x+n-1}$ .
2. The function is decreasing in  $x$  for any fixed  $n$ .

We have identified three distinct families of functions that satisfy these criteria.

## 6.1 Solution 1: A Logarithmic Function

A simple and elegant solution is derived from the integral representation of the logarithm.

$$f(x, n) = \ln \left( \frac{x+n}{x+n-1} \right)$$

### Verification

- **Inequality:** The function  $g(t) = 1/t$  is strictly decreasing. Therefore, for  $x > 0$  and  $n \geq 1$ , we can bound its integral:

$$\int_{x+n-1}^{x+n} \frac{1}{x+n} dt < \int_{x+n-1}^{x+n} \frac{1}{t} dt < \int_{x+n-1}^{x+n} \frac{1}{x+n-1} dt$$

Evaluating the integrals gives the desired bounds:

$$\frac{1}{x+n} < \ln(x+n) - \ln(x+n-1) < \frac{1}{x+n-1}$$

- **Monotonicity:** The partial derivative with respect to  $x$  is:

$$\frac{\partial f}{\partial x} = \frac{1}{x+n} - \frac{1}{x+n-1} < 0$$

since  $x+n > x+n-1$ . The function is therefore decreasing in  $x$ .

- **Positivity:** Since  $x+n > x+n-1$ , the argument of the logarithm is greater than 1, so the function is always positive.
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## 6.2 Solution 2: The Polygamma Function

A second solution can be found in the family of special functions, specifically the **trigamma function**,  $\psi_1(z)$ .

$$f(x, n) = \psi_1(x+n) = \sum_{k=0}^{\infty} \frac{1}{(x+n+k)^2}$$

### Verification

- **Inequality:** The bounds can be established by comparing the infinite sum to an integral. Let  $z = x+n$ .
  - **Lower Bound:** The sum is greater than the integral of the corresponding term from 0 to  $\infty$ :

$$\psi_1(z) > \int_0^{\infty} \frac{1}{(z+t)^2} dt = \left[ -\frac{1}{z+t} \right]_0^{\infty} = \frac{1}{z} = \frac{1}{x+n}$$

- **Upper Bound:** The sum is less than or equal to the integral with a shifted lower limit of  $-1$ :

$$\psi_1(z) \leq \int_{-1}^{\infty} \frac{1}{(z+t)^2} dt = \left[ -\frac{1}{z+t} \right]_{-1}^{\infty} = \frac{1}{z-1} = \frac{1}{x+n-1}$$

- **Monotonicity:** The derivative with respect to  $x$  is  $f'(x) = \psi_2(x+n)$ , where  $\psi_2(z)$  is the pentagamma function. Since  $\psi_2(z) = -2 \sum_{k=0}^{\infty} (z+k)^{-3}$ , it is always negative for  $z > 0$ , so the function is decreasing.
  - **Numerical Test:** For  $n = 2, x = 0.5$ , we test  $f(0.5, 2) = \psi_1(2.5)$ . The bounds are  $1/2.5 = 0.4$  and  $1/1.5 \approx 0.667$ . The calculated value is  $\psi_1(2.5) \approx 0.4904$ , which correctly lies within the bounds.
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### 6.3 Solution 3: The Exponential Integral Function

A third candidate is the family constructed from the generalized exponential integral,  $E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt$ .

$$g(x, n) = e^x E_n(x)$$

#### Recurrence Relation

The functions  $g_n(x)$  obey a simple recurrence relation. Starting with the known recurrence for  $E_n(x)$ :

$$E_{n+1}(x) = \frac{1}{n}(e^{-x} - xE_n(x))$$

Multiplying by  $e^x$  gives the relation for  $g_n(x)$ :

$$e^x E_{n+1}(x) = \frac{1}{n}(1 - xe^x E_n(x)) \implies g_{n+1}(x) = \frac{1}{n}(1 - xg_n(x))$$

For example,  $g_2(x) = 1 - xg_1(x)$ .

#### Verification

We test the inequality for  $n = 1, 2, 3, 4$ .

- **Case n=1:**  $g_1(x) = e^x E_1(x)$ . Well-known bounds for  $E_1(x)$  are  $\frac{e^{-x}}{x+1} < E_1(x) < \frac{e^{-x}}{x}$ . Multiplying by  $e^x$  directly yields the required inequality:  $\frac{1}{x+1} < g_1(x) < \frac{1}{x}$ .
- **Case n=2:**  $g_2(x) = 1 - xg_1(x)$ . The inequality to check is  $\frac{1}{x+2} < g_2(x) \leq \frac{1}{x+1}$ .
  - At  $x = 1$ , the bounds are  $1/3 \approx 0.333$  and  $1/2 = 0.5$ .
  - $g_2(1) = 1 - g_1(1) = 1 - e^1 E_1(1) \approx 1 - 2.718(0.219) \approx 1 - 0.596 = 0.404$ .
  - The result holds:  $0.333 < 0.404 \leq 0.5$ .
- **Case n=3:**  $g_3(x) = \frac{1}{2}(1 - xg_2(x))$ . The inequality is  $\frac{1}{x+3} < g_3(x) \leq \frac{1}{x+2}$ .
  - At  $x = 1$ , the bounds are  $1/4 = 0.25$  and  $1/3 \approx 0.333$ .
  - $g_3(1) = \frac{1}{2}(1 - g_2(1)) \approx \frac{1}{2}(1 - 0.404) = 0.298$ .
  - The result holds:  $0.25 < 0.298 \leq 0.333$ .
- **Case n=4:**  $g_4(x) = \frac{1}{3}(1 - xg_3(x))$ . The inequality is  $\frac{1}{x+4} < g_4(x) \leq \frac{1}{x+3}$ .
  - At  $x = 1$ , the bounds are  $1/5 = 0.2$  and  $1/4 = 0.25$ .
  - $g_4(1) = \frac{1}{3}(1 - g_3(1)) \approx \frac{1}{3}(1 - 0.298) = 0.234$ .
  - The result holds:  $0.2 < 0.234 \leq 0.25$ .

The family  $g(x, n) = e^x E_n(x)$  appears to be a valid candidate, satisfying the required properties of being positive, decreasing, and bounded as specified.

## 7 Sequel: Problem II – Rational Candidates and Convergence Twists

In this sequel, we extend the mystery family by introducing a simple rational candidate from an alternative perspective, while delving deeper into convergence properties of the exponential integral family  $g(x, n) = e^x E_n(x)$  toward the trigamma  $\psi^{(1)}(x + n)$ . The goal remains to satisfy

$$\frac{1}{x+n} < f(x, n) \leq \frac{1}{x+n-1}, \quad x > 0, n \geq 1,$$

with decreasing monotonicity in  $x$ .

## 7.1 A Simple Rational Family Candidate

A straightforward and elegant candidate arises from a parameterized rational form:

$$f(x, n) = \frac{1}{x + n - \theta}, \quad 0 < \theta \leq 1.$$

- For  $0 < \theta \leq 1$ , we have  $n - 1 \leq n - \theta < n$ , so

$$x + n - 1 \leq x + n - \theta < x + n,$$

implying the desired inequality (strict lower, non-strict upper).

- Positivity holds for  $x > 0$ , and

$$\frac{\partial}{\partial x} f(x, n) = -\frac{1}{(x + n - \theta)^2} < 0,$$

ensuring strict decrease in  $x$ .

- For  $\theta = 1$ ,  $f(x, n) = 1/(x + n - 1)$  saturates the upper bound. For  $0 < \theta < 1$ , it lies strictly interior, offering flexibility (e.g.,  $\theta = 1/2$  for midpoint approximation).
- Numerical check: For  $n = 2$ ,  $x = 0.5$ ,  $\theta = 0.5$ :  $f \approx 0.545 > 0.4 \leq 0.667$ .

This rational form is computationally trivial and highlights how the inequality captures a continuum of decreasing hyperbolas, bridging elementary functions to special ones like the trigamma.

## 7.2 Advanced Twists on Convergence

To spice things up, consider the interplay between the rational family, the trigamma  $\psi^{(1)}(x + n)$ , and the exponential approximant  $g(x, n) = e^x E_n(x)$ . We explore convergence in various topologies, with a focus on how tails "tease" uniformity.

### 7.2.1 Pointwise Asymptotic Convergence

For fixed  $n$  and large  $x \rightarrow \infty$ ,

$$\psi^{(1)}(x + n) \sim \frac{1}{x + n} + \frac{1}{2(x + n)^2} + O\left(\frac{1}{(x + n)^3}\right),$$

while the rational  $f(x, n) \sim \frac{1}{x + n - \theta} = \frac{1}{x + n} + \frac{\theta}{(x + n)^2} + O\left(\frac{1}{x^3}\right)$ , matching the leading terms (choose  $\theta = 1/2$  for exact second-order alignment). For  $g(x, n)$ ,

$$g(x, n) \sim \frac{1}{x^{n-1}} - \frac{n-1}{x^n} + \dots,$$

converging pointwise to the trigamma tail for shifted arguments (e.g.,  $g(x + n, 2) \approx \psi^{(1)}(x + n)$  for moderate  $x$ , via Laplace's method on the integral rep).

### 7.2.2 Uniform Convergence on Compacts

The trigamma series  $\sum_{k=0}^{\infty} \frac{1}{(x + n + k)^2}$  converges uniformly on any compact  $[a, b] \subset (0, \infty)$ :

$$\left| \frac{1}{(x + n + k)^2} \right| \leq \frac{1}{(a + n + k)^2},$$

and  $\sum_{k=0}^{\infty} \frac{1}{(a + n + k)^2} < \infty$ , so by Weierstrass M-test, uniform convergence holds. The rational approximant converges uniformly to the trigamma on compacts too (as a continuous shift). However, for  $g(x, n)$ , the exponential tail  $r_m(x, n) \approx e^{x+n+m} E_2(x + n + m)$  converges pointwise to the full trigamma as  $m \rightarrow 0$ , but uniformly only on compacts away from singularities ear  $x = 0$ , the exponential blow-up disrupts it.



### 7.2.3 Global Non-Uniformity and "Teasing" Twists

Globally on  $(0, \infty)$ , uniformity fails dramatically: Near  $x \rightarrow 0^+$ ,  $\psi^{(1)}(n+x) \sim 1/x^2$  (pole-like), while the rational stays bounded for  $\theta > 0$ , and  $g(x, n)$  oscillates in approximation quality (e.g., for small  $x$ ,  $E_n(x)$  involves Ei branches, leading to non-uniform error  $O(1/x)$ ). A con twist: For large  $n$ ,  $g(x, n) \sim x^{1-n}$  acts as a "pseudo-trigamma" that \*teases\* uniformity near infinity but explodes near  $0$  a fractal edge where convergence "dances" on the boundary. Va savoir: On compactified domains (e.g., via stereographic proj to sphere), it might uniformize... but that's for another sequel!

This sequel bridges the rational simplicity to convergence chaos, enriching the mystery family's lore.