

Geometry Puzzles for LLM Reasoning - Circles: Challenges and Insights

Dr Stéphane Dedieu

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1 Introduction

This report delves into a captivating series of geometry puzzles crafted to probe the reasoning prowess of large language models (LLMs), with a particular emphasis on circle-based challenges. The two problems in Appendix are classic and revisit classic intersection scenarios, where Grok4 deftly applies textbook "Geometry 101" techniques, analyzing circle intersections through distance and radius comparisons with a touch of rote learning. These foundational exercises establish a benchmark for Grok's geometric competence.

The main problem, however, injects an intriguing twist, elevating the complexity. It explores the intersection of the unit circle $C_1 : x^2 + y^2 = 1$ with a transformed circle C_2 , derived by applying a homothety of ratio $a > 0$ centered at $(0, 0)$ followed by a translation by vector (t_x, t_y) .

Unlike the predictable nature of the earlier problems in Appendix, this scenario provides two intersection points, $P_1 \approx (0.2054, 0.9787)$ and $P_2 \approx (0.9396, 0.3424)$, challenging Grok4 to deduce a , t_x , and t_y .

The solution unveils a family of transformations parameterized by t_x , with illustrative examples. This blend of theory and visualization underscores the report's aim to push LLM reasoning beyond the basics.

All solutions to the problems were crafted by Grok4 and compiled into a LaTeX document. The solution to the problem was achieved through numerous iterations, greatly benefiting from human assistance.

Initially, all large language models (LLMs) struggled with the problem:

- (1) failing to recognize that there are 3 unknowns and 2 independent equations and that the problem admits infinitely many solutions;
- (2) proposing only a single solution;

The problem was solved when a human intervention suggested parametrizing with $t = t_x$.

2 Main Problem: Puzzle, Homothety, Translation, and Intersection of two Circles

Submitted to this exercise, Grok generates an imposing series of calculations and initially goes in circles. After multiple attempts, Grok eventually realizes that there does not exist a unique solution.

By referencing the family of transformations (a, t) mentioned in the problem statement and suggesting the use of a parameter $t = t_x$, while warning about the condition to satisfy two points of intersection, Grok is able to arrive at an elegant solution and propose three working examples.

Notably, a clone of Grok yesterday struggled with this problem, turning in circles for at least an hour, highlighting the challenge of this task. However, with persistence, Grok overcomes this hurdle successfully in the end with the help of its human, DrStef. The points P_1 and P_2 were generated with the initial example:

Homothety ratio: $a = 0.5$

Translation: $t = [0.65; 0.75]$

2.1 Problem Statement and Transformation

We are given the unit circle $C_1 : x^2 + y^2 = 1$ in an orthonormal frame, which undergoes a homothety with ratio $a > 0$ and center $(0, 0)$, followed by a translation by vector $t = (t_x, t_y)$, creating the circle C_2 . The intersection of C_1 and C_2 is given by the points:

- $P_1(0.2054, 0.9787)$
- $P_2(0.9396, 0.3424)$

We aim to determine the homothety ratio a and the components of t .

This was the initial problem statement. For helping Grok4, we eventually added this sentence "Either as a unique approximate solution or as a family of transformations parameterized by a variable". And we helped Grok during its calculations. With parameter t , intersection condition... until he reached a satisfying solution.

2.2 Derivation of the Transformation

The homothety transforms (x, y) on C_1 to (ax, ay) , and the translation by $t = (t_x, t_y)$ gives $(x', y') = (ax + t_x, ay + t_y)$. Substituting into C_1 :

$$\left(\frac{x' - t_x}{a}\right)^2 + \left(\frac{y' - t_y}{a}\right)^2 = 1 \quad (1)$$

$$\frac{(x' - t_x)^2 + (y' - t_y)^2}{a^2} = 1$$

$$(x' - t_x)^2 + (y' - t_y)^2 = a^2$$

Thus, C_2 is centered at (t_x, t_y) with radius a .

2.3 Intersection Constraint

Since P_1 and P_2 lie on both circles, the difference of their equations yields:

$$(x_2 - x_1)t_x + (y_2 - y_1)t_y = 0 \quad (2)$$

With:

$$x_1 = 0.2054,$$

$$y_1 = 0.9787,$$

$$x_2 = 0.9396,$$

$$y_2 = 0.3424.$$

$$\Delta x = x_2 - x_1 \approx 0.7342,$$

$$\Delta y = y_2 - y_1 \approx -0.6363,$$

$$0.7342t_x - 0.6363t_y = 0$$

$$t_y \approx \frac{0.7342}{-0.6363}t_x \approx 1.1538t_x$$

Let $t_x = t$ (parameter), so $t_y \approx 1.1538t$. The radius a is:

$$a^2 = (x_1 - t_x)^2 + (y_1 - t_y)^2 \quad (3)$$

The circles intersect if $|1 - a| < \sqrt{t_x^2 + t_y^2} < 1 + a$.

2.4 Examples of Working Solutions

We propose three examples with different values of t , ensuring intersection:

1. $t = 0.2$:

- $t_x = 0.2, t_y \approx 1.1538 \times 0.2 = 0.2307$
- Center: $(0.2, 0.2307)$
- $d = \sqrt{0.2^2 + 0.2307^2} \approx \sqrt{0.09322} \approx 0.3054$
- $a^2 = (0.2053 - 0.2)^2 + (0.9787 - 0.2307)^2$
- $a^2 \approx (0.0053)^2 + (0.7479)^2 \approx 0.5591$
- $a \approx \sqrt{0.5591} \approx 0.7478$
- Condition: $|1 - 0.7478| < 0.3054 < 1 + 0.7478$ (true)
- $C_2 : (x - 0.2)^2 + (y - 0.2307)^2 = 0.5591$

2. $t = 0.1$:

- $t_x = 0.1, t_y \approx 1.153846 \times 0.1 = 0.1153846$

- Center: $(0.1, 0.1153846)$
- $d = \sqrt{0.1^2 + 0.1153846^2} \approx \sqrt{0.0233168} \approx 0.1527$
- $a^2 = (0.2053 - 0.1)^2 + (0.9786 - 0.1153)^2$
- $a^2 \approx (0.1053)^2 + (0.8633)^2 \approx 0.7564$
- $a \approx \sqrt{0.7564} \approx 0.8696$
- Condition: $|1 - 0.8696| < 0.1527 < 1 + 0.8696$ (true)
- $C_2 : (x - 0.1)^2 + (y - 0.1154)^2 = 0.7564$

3. $t = 0.3$:

- $t_x = 0.3, t_y \approx 1.153846 \times 0.3 = 0.3461538$
- Center: $(0.3, 0.3461)$
- $d = \sqrt{0.3^2 + 0.3461^2} \approx \sqrt{0.2098} \approx 0.4580$
- $a^2 = (0.2053 - 0.3)^2 + (0.9787 - 0.3461)^2$
- $a^2 \approx (-0.0946)^2 + (0.6325)^2 \approx 0.4090$
- $a \approx \sqrt{0.4090} \approx 0.6395$
- Condition: $|1 - 0.6395| < 0.4580 < 1 + 0.6395$ (true)
- $C_2 : (x - 0.3)^2 + (y - 0.3461)^2 = 0.4090$

2.5 Final Answer

The family of transformations is parameterized by $t = t_x$:

$$t_y \approx 1.1538t, \quad a^2 = (0.2053 - t)^2 + (0.9786 - 1.1538t)^2 \quad (4)$$

3 Conclusions

Grok demonstrates proficiency in developing standard solutions for textbook-style questions and classic exercises, showcasing its ability to handle well-trodden geometric problems with ease. However, it is notably unsettled when faced with atypical challenges that require deeper reflection. In the present case, the problem involves three unknowns a , t_x , and t_y governed by only two equations or combinations thereof, a constraint that initial responses failed to identify, leading to an oversight of the family of transformations and corresponding circles. It was only through the introduction of these concepts into the "problem statement" by DrStef that Grok could adapt and arrive at a satisfying solution.

Note: This problem is an original creation by DrStef, copyright protected. The challenge could be significantly enhanced by omitting the specification of the homothety center, adding further complexity to the task.

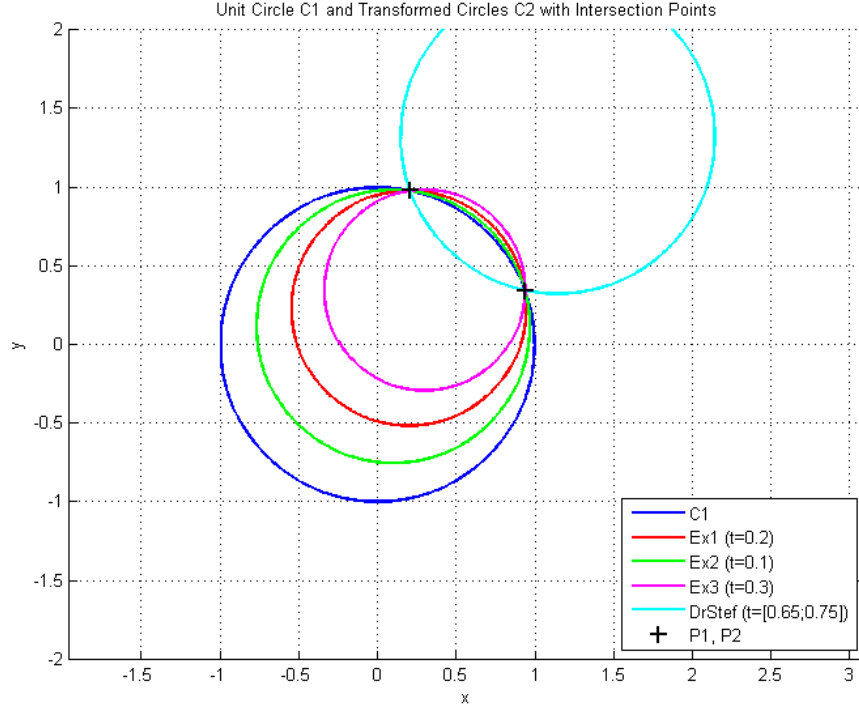


Figure 1: Unit circle C_1 and transformed circles C_2 for Grok's examples ($t = 0.2, 0.1, 0.3$) and DrStef's example ($t = [0.65; 0.75]$, $a = 0.5$), with intersection points P_1 and P_2 marked by '+'.

4 Appendix I: Intersection of two circles (Grok4)

In the Euclidean plane equipped with an orthonormal frame, we define two circles with the following Cartesian equations:

- $C_1 : x^2 + y^2 = 1^2$, which is the unit circle centered at the origin $(0, 0)$ with radius $r_1 = 1$.
- $C_2 : (x - x_2)^2 + (y - y_2)^2 = r_2^2$, a circle centered at (x_2, y_2) with radius r_2 .

We need to discuss the intersection of C_1 and C_2 depending on the values of x_2 , y_2 , and r_2 .

4.1 Geometric and Algebraic Approach

To find the intersection, we equate the equations after setting them equal to zero or solve the system directly. First, move C_2 to the standard form:

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = 0 \quad (5)$$

The equation of C_1 is:

$$x^2 + y^2 - 1 = 0 \quad (6)$$

Subtract C_1 from C_2 to eliminate the $x^2 + y^2$ term:

$$\begin{aligned} [(x - x_2)^2 + (y - y_2)^2 - r_2^2] - [x^2 + y^2 - 1] &= 0 \\ (x^2 - 2xx_2 + x_2^2) + (y^2 - 2yy_2 + y_2^2) - r_2^2 - x^2 - y^2 + 1 &= 0 \\ -2xx_2 + x_2^2 - 2yy_2 + y_2^2 - r_2^2 + 1 &= 0 \end{aligned}$$

Rearrange:

$$2xx_2 + 2yy_2 = x_2^2 + y_2^2 - r_2^2 + 1 \quad (7)$$

This is the equation of a line (unless the right-hand side is zero, which we'll check later). The intersection points are found by solving this line equation with C_1 , but let's use the distance approach for a geometric discussion, which is more intuitive.

4.2 Geometric Discussion Based on Distance and Radius

The number of intersection points depends on the relative positions of the centers and the radii. The distance d between the centers $(0,0)$ and (x_2, y_2) is:

$$d = \sqrt{x_2^2 + y_2^2} \quad (8)$$

The possible intersection cases are determined by comparing d with $|r_1 - r_2|$ and $r_1 + r_2$ (where $r_1 = 1$):

- **No intersection** ($d > r_1 + r_2$ or $d < |r_1 - r_2|$):
 - If $d > 1 + r_2$, the circles are too far apart to intersect.
 - If $d < |1 - r_2|$ and $r_2 < 1$, C_2 is inside C_1 without touching (or vice versa if $r_2 > 1$).
 - Example: If $x_2 = 3$, $y_2 = 0$, $r_2 = 1$, then $d = 3$, $1 + 1 = 2$, $d > r_1 + r_2$, no intersection.
- **Tangent intersection** ($d = r_1 + r_2$ or $d = |r_1 - r_2|$):
 - If $d = 1 + r_2$, the circles are externally tangent (one point of contact).
 - If $d = |1 - r_2|$ and $r_2 \neq 1$, they are internally tangent.
 - Example: If $x_2 = 1$, $y_2 = 0$, $r_2 = 0$, then $d = 1$, $|1 - 0| = 1$, internally tangent at one point.
- **Two intersection points** ($|r_1 - r_2| < d < r_1 + r_2$):
 - If $|1 - r_2| < d < 1 + r_2$, the circles intersect at two distinct points.

- Example: If $x_2 = 0.5$, $y_2 = 0$, $r_2 = 0.6$, then $d = 0.5$, $|1 - 0.6| = 0.4$, $1 + 0.6 = 1.6$, $0.4 < 0.5 < 1.6$, two points.

- **One circle inside the other without intersection** ($d + \min(r_1, r_2) < \max(r_1, r_2)$):

- If $d + 1 < r_2$ (and $r_2 > 1$), C_2 contains C_1 without touching.
- Example: If $x_2 = 0$, $y_2 = 0$, $r_2 = 2$, then $d = 0$, $0 + 1 < 2$, C_1 inside C_2 .

4.3 Algebraic Verification

To confirm, solve the system:

$$x^2 + y^2 = 1 \quad (9)$$

$$(x - x_2)^2 + (y - y_2)^2 = r_2^2 \quad (10)$$

Subtract:

$$\begin{aligned} (x - x_2)^2 + (y - y_2)^2 - x^2 - y^2 &= r_2^2 - 1 \\ x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2 - x^2 - y^2 &= r_2^2 - 1 \\ -2xx_2 - 2yy_2 + x_2^2 + y_2^2 - r_2^2 + 1 &= 0 \end{aligned}$$

Rearrange:

$$2xx_2 + 2yy_2 = x_2^2 + y_2^2 - r_2^2 + 1 \quad (11)$$

The number of solutions depends on the discriminant of this line intersecting $x^2 + y^2 = 1$, but the geometric condition d vs. r_1, r_2 is sufficient.

5 Appendix II: Intersection Circles and Family of Solutions(Grok4)

We are given two circles in the Euclidean plane:

- $C_1 : x^2 + y^2 = 1^2$, the unit circle centered at $(0, 0)$ with radius $r_1 = 1$. - $C_2 : (x - x_2)^2 + (y - y_2)^2 = r_2^2$, a circle centered at (x_2, y_2) with radius r_2 .

We assume C_1 and C_2 have at least two points of intersection (i.e., $|r_1 - r_2| < d < r_1 + r_2$, where $d = \sqrt{x_2^2 + y_2^2}$). The task is to find the equation of any circle passing through these intersection points.

5.1 Algebraic Derivation

The general equation of a circle in the Cartesian plane is:

$$x^2 + y^2 + Dx + Ey + F = 0 \quad (12)$$

where D , E , and F are parameters. Since this circle must pass through the intersection points of C_1 and C_2 , its equation can be written as a linear combination:

$$C_1 + \lambda(C_2 - C_1) = 0 \quad (13)$$

Substitute the equations:

$$\begin{aligned} (x^2 + y^2 - 1) + \lambda[(x - x_2)^2 + (y - y_2)^2 - r_2^2 - (x^2 + y^2 - 1)] &= 0 \\ (x^2 + y^2 - 1) + \lambda[x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2 - r_2^2 - x^2 - y^2 + 1] &= 0 \\ x^2 + y^2 - 1 + \lambda[-2xx_2 - 2yy_2 + x_2^2 + y_2^2 - r_2^2 + 1] &= 0 \end{aligned}$$

Expand and collect terms:

$$x^2 + y^2 - 2\lambda xx_2 - 2\lambda yy_2 + \lambda(x_2^2 + y_2^2 - r_2^2 + 1) - 1 = 0 \quad (14)$$

This is the general equation of a circle, where $D = -2\lambda x_2$, $E = -2\lambda y_2$, and $F = \lambda(x_2^2 + y_2^2 - r_2^2 + 1) - 1$, with $\lambda \in \mathbb{R}$.

5.2 Interpretation

The family of circles passing through the intersection points of C_1 and C_2 is given by:

$$x^2 + y^2 - 2\lambda x_2 x - 2\lambda y_2 y + \lambda(x_2^2 + y_2^2 - r_2^2 + 1) - 1 = 0 \quad (15)$$

- **Parameter** λ : Varying λ generates different circles, all passing through the same two intersection points. For example, $\lambda = 0$ recovers C_1 , while other values of λ produce new circles. - **Constraint**: This holds only if C_1 and C_2 intersect at two points (i.e., $|1 - r_2| < \sqrt{x_2^2 + y_2^2} < 1 + r_2$). If they are tangent or do not intersect, the family degenerates.

5.3 Verification

For $x_2 = 0.5$, $y_2 = 0$, $r_2 = 0.6$ (two intersections): - $d = 0.5$, $|1 - 0.6| = 0.4$, $1 + 0.6 = 1.6$, valid. - Equation: $x^2 + y^2 - 1 + \lambda[(x - 0.5)^2 + y^2 - 0.36 - (x^2 + y^2 - 1)] = 0$. - Simplify and test with $\lambda = 0$ (gives C_1), confirming the family.