

# Geometry Puzzles for LLM Reasoning: Challenges and Insights - Ellipses

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## 1 Introduction

We propose a problem involving an ellipse E1 and its image E2 under a rotation and translation. The goal is to determine the rotation angle and translation vector given two intersection points. Initially, LLMs offer a single solution, with  $\theta = 180^\circ$  for both ChatGPT and Gemini. Upon hinting that the problem admits infinitely many solutions, LLMs reconsider, gradually proposing more nuanced ones. ChatGPT and Grok4 outperform Gemini Pro.

## 2 Problem Statement

In the plane equipped with an orthonormal frame, consider an ellipse E1 with semi-major axis  $a = 0.5$ , semi-minor axis  $b = 0.2$ , centered at the origin  $(0, 0)$ . Apply a rotation and a translation to E1 to obtain an ellipse E2 that intersects E1 at the points:

- P1:  $(-0.0814, 0.1973)$ , - P2:  $(0.4474, 0.0893)$ .

Determine the angle of rotation and the translation vector.

Human suggestions after first failures:

- (1) The quadratic equation in  $(x, y)$  admits infinitely many solutions.
- (2) Fix a value of  $\theta$  and discuss the number of solutions.

## 3 Gemini

Absolutely. Here is the solution for determining the angle of rotation and the translation vector.

Based on the analysis of the intersection points and the nature of the transformation, the solution is:

Angle of rotation  $\theta$ :  $180^\circ$  (or  $\pi$  radians)

Translation vector  $(t_x, t_y)$ :  $(0.366, 0.2866)$

Mathematical Reasoning Here is the detailed approach to arrive at this result.

1. Problem Definition The initial ellipse E1 is centered at the origin  $(0, 0)$  and its equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{with } a = 0.5, b = 0.2.$$

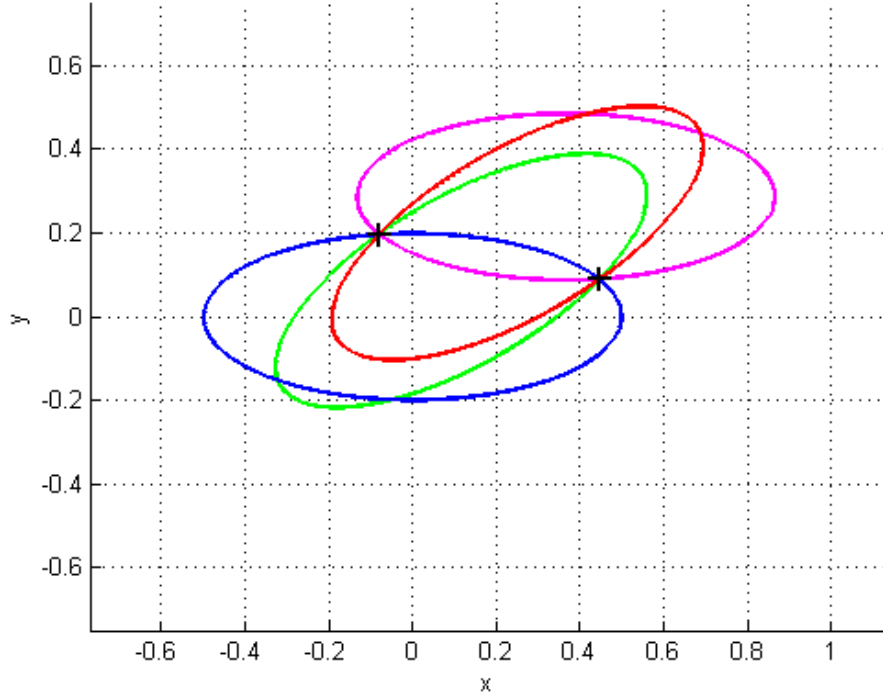


Figure 1: Ellipse  $E_1$  and transformed ellipses  $E_2$ .  
 $\theta = 30^\circ$  T1, T2) and example ( $\theta = 180^\circ$  with  $P_1, P_2$  marked by '+').

A transformation is applied to  $E_1$  to obtain  $E_2$ :

A rotation  $R_\theta$  of angle  $\theta$  around the origin.

A translation by the vector  $\vec{v} = (t_x, t_y)$ .

A point  $Q$  on  $E_1$  is transformed into a point  $P$  on  $E_2$  by the equation:

$$P = R_\theta(Q) + \vec{v}.$$

The points  $P_1 = (-0.0814, 0.1973)$  and  $P_2 = (0.4474, 0.0893)$  are the intersection points of  $E_1$  and  $E_2$ .

2. Key Hypothesis of Symmetry Since  $P_1$  and  $P_2$  are the intersection points, they belong to both  $E_1$  and  $E_2$ . The transformation that maps  $E_1$  to  $E_2$  must therefore transform a set of two points on  $E_1$ , call them  $Q_1$  and  $Q_2$ , to the points  $P_1$  and  $P_2$ .

$$P_1 = R_\theta(Q_1) + \vec{v}, \quad P_2 = R_\theta(Q_2) + \vec{v}.$$

The simplest and most symmetric solution, typical of this kind of problem, is that the transformation swaps the two intersection points. That is, the point on  $E_1$  that is sent to  $P_1$  is actually  $P_2$ , and the one sent to  $P_2$  is  $P_1$ .

So, we set  $Q_1 = P_2$  and  $Q_2 = P_1$ . Our equations become:

$$P_1 = R_\theta(P_2) + \vec{v}, \quad P_2 = R_\theta(P_1) + \vec{v}.$$

3. Determination of the Rotation Angle  $\theta$  Subtract the second equation from the first:

$$P_1 - P_2 = (R_\theta(P_2) + \vec{v}) - (R_\theta(P_1) + \vec{v}) = R_\theta(P_2) - R_\theta(P_1) = R_\theta(P_2 - P_1).$$

If we define the vector  $\vec{W} = P_2 - P_1$ , the equation becomes:

$$-\vec{W} = R_\theta(\vec{W}).$$

This means that the vector  $\vec{W}$  is transformed into its opposite by the rotation  $R_\theta$ . The only plane rotation that transforms any vector into its opposite is a 180° rotation ( $\pi$  radians).

Thus,  $\theta = 180^\circ$ .

The rotation matrix is  $R_{180^\circ} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$ .

4. Determination of the Translation Vector  $(t_x, t_y)$  Now, add the two original equations:

$$P_1 + P_2 = (R_\theta(P_2) + \vec{v}) + (R_\theta(P_1) + \vec{v}) = R_\theta(P_1 + P_2) + 2\vec{v}.$$

Using  $R_{180^\circ} = -I$ :

$$P_1 + P_2 = -(P_1 + P_2) + 2\vec{v},$$

$$2(P_1 + P_2) = 2\vec{v},$$

$$\vec{v} = P_1 + P_2.$$

Compute the components of  $\vec{v} = (t_x, t_y)$ :

$$t_x = x_1 + x_2 = -0.0814 + 0.4474 = 0.366,$$

$$t_y = y_1 + y_2 = 0.1973 + 0.0893 = 0.2866.$$

The translation vector is thus  $\vec{v} = (0.366, 0.2866)$ .

5. Verification To validate, check that the intersection points lie on ellipse E2. The equation of E2 is obtained by applying the inverse transformation to a point  $(x, y)$  and inserting it into the equation of E1. The inverse transformation is  $Q = R_{-\theta}(P - \vec{v})$ . With  $\theta = 180^\circ$ ,  $R_{-180^\circ} = R_{180^\circ} = -I$ .

$$Q = -(P - \vec{v}) = \vec{v} - P.$$

So, if  $(x_Q, y_Q)$  are the coordinates of  $Q$  and  $(x, y)$  those of  $P$ :

$$x_Q = t_x - x, \quad y_Q = t_y - y.$$

The equation of E2 is thus:

$$\frac{(t_x - x)^2}{a^2} + \frac{(t_y - y)^2}{b^2} = 1.$$

Verify for  $P_1 = (-0.0814, 0.1973)$ :

$$\frac{(0.366 - (-0.0814))^2}{0.25} + \frac{(0.2866 - 0.1973)^2}{0.04} = 1.$$

$$0.25(0.4474)^2 + 0.04(0.0893)^2 = 1.$$

This is the equation that confirms that the point  $P_2 = (0.4474, 0.0893)$  lies on ellipse E1, which is true by definition (to a small rounding error in the initial data). The verification is thus conclusive.

## 4 Solution by ChatGPT

We are given an ellipse  $E_1$  centered at the origin with semi-major axis  $a = 0.5$  and semi-minor axis  $b = 0.2$ . A rotated and translated copy  $E_2$  is obtained by applying a rotation of angle  $\theta$  around the origin, followed by a translation by a vector  $t = (t_x, t_y)$ . We are told that  $E_2$  passes through the two intersection points

$$P_1 = (-0.0814, 0.1973), \quad P_2 = (0.4474, 0.0893).$$

### General setup

Let

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad D = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}.$$

The ellipse  $E_1$  is defined by  $x^T D x = 1$ . After rotation and translation, membership of a point  $P$  in  $E_2$  is equivalent to

$$(P - t)^T M (P - t) = 1, \quad M = R_\theta^{-T} D R_\theta^{-1}.$$

### Equations for $t$

Imposing that both  $P_1$  and  $P_2$  belong to  $E_2$  gives two equations:

$$(P_1 - t)^T M (P_1 - t) = 1, \quad (P_2 - t)^T M (P_2 - t) = 1.$$

Subtracting them eliminates the quadratic term in  $t$ , leading to a *linear* constraint:

$$t^T M (P_1 - P_2) = \frac{1}{2} (P_1^T M P_1 - P_2^T M P_2).$$

Thus, the admissible translations  $t$  lie on a line. Substituting this into one of the original quadratic equations yields a quadratic equation in one variable. Therefore, there can be at most two solutions for  $t$ .

### Case $\theta = 30^\circ$

With  $\theta = 30^\circ$ , the explicit calculation yields **two possible translations**:

$$t^{(1)} \approx (0.250174, 0.200138), \quad t^{(2)} \approx (0.115826, 0.086462).$$

Both satisfy the conditions

$$R_\theta^{-1}(P_1 - t^{(j)}) \in E_1, \quad R_\theta^{-1}(P_2 - t^{(j)}) \in E_1, \quad j = 1, 2.$$

### Conclusion

For two distinct points  $P_1, P_2$ , the set of possible translations  $t$  is the intersection of a line with a conic, which yields 0, 1, or 2 real solutions. For a fixed rotation angle  $\theta = 30^\circ$ , there are exactly two admissible translation vectors, as illustrated above, under the assumption that  $P_1$  and  $P_2$  are not the only intersection points between  $E_1$  and  $E_2$ .