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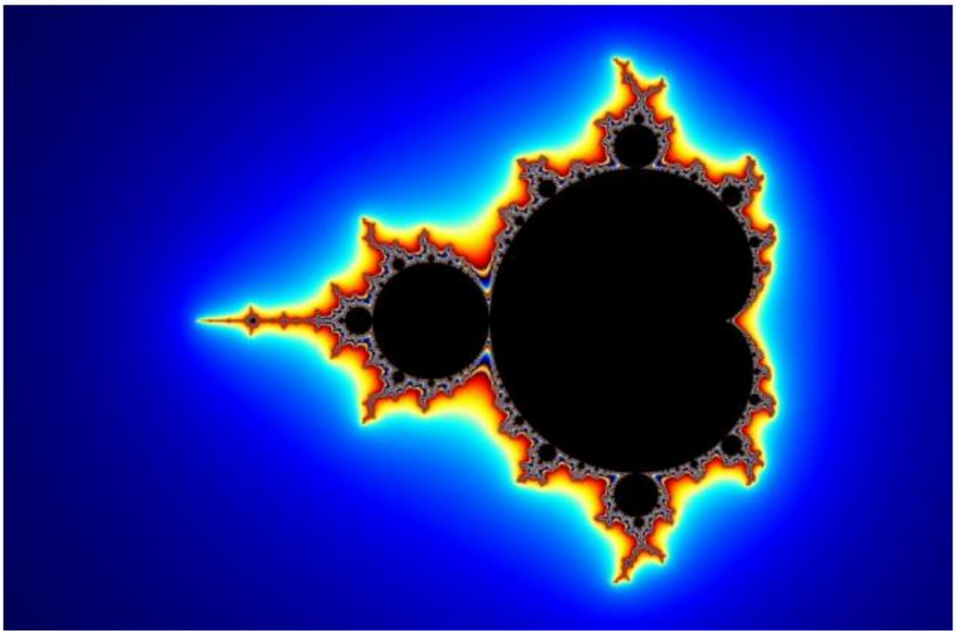
An Introduction to Fractals

with

python™



Professor Stephen Lynch NTF FIMA SFHEA

https://www.lboro.ac.uk/departments/compsci/staff/stephen-lynch/

Definition 1. A *fractal* is an image repeated on an ever-reduced scale.

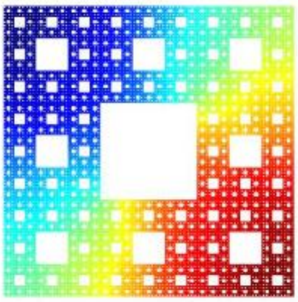
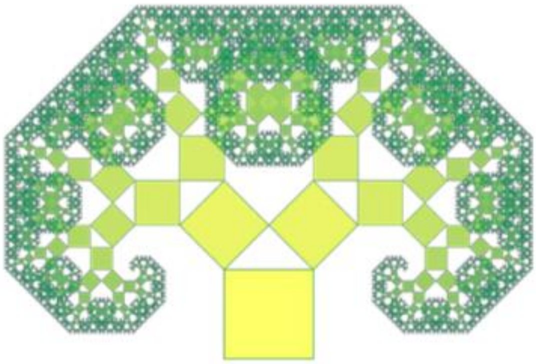
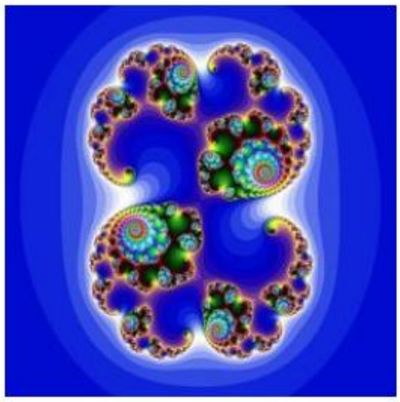
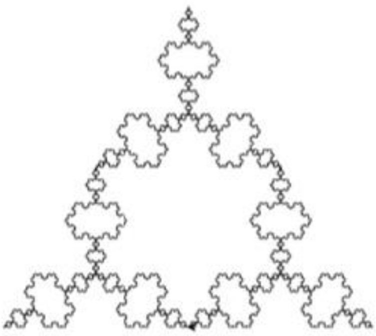
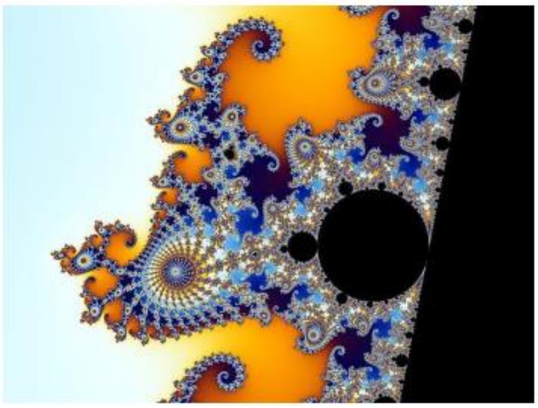
Definition 2. A *fractal* is an object with non-integer dimension.

Fractals in Nature






Mathematical Fractals


Example 1. The *Cantor set* (1870).

Start with a unit line segment and at each stage remove the middle third segment and replace one segment with two segments each of length one third of the previous segment.

IDLE File Edit Format Run Options Window Help

cantor.py - /Users/99900361/Documents/2022 Files/Stephen/MMU

```python
# Program 1: Cantor Fractal.
# Run Module and type >>> cantor(-400, 200 , 400) in Python Shell.
from turtle import *            # Load functions from turtle module.
speed(0)                        # Fastest speed of turtle.
def cantor(x, y, length):
    if length >= 5:             # Stop when length < 5.
        penup()
        pensize(3)
        pencolor('blue')
        setpos(x , y)
        pendown()
        fd(length)
        y -= 60                 # Height difference.
        cantor(x , y, length / 3)
        cantor(x + 2 * length / 3, y, length / 3)
        penup()
        setpos(x , y + 60)
```

Figure 1: The Cantor set to stage 4.

To understand the program, set speed(1) and plot the Cantor set to stage 2. Mimic by-hand what the program is doing.

Problem 1: Edit Program 1 to plot a variation of the Cantor set, where two segments (each one-fifth the length of the previous segment) are removed at each stage.

Example 2. The *Koch curve* (1904).

Start with a unit line segment, remove the middle third segment and replace one segment with four segments each of length one third the previous segment, as illustrated below.

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*KochCurve.py - /Users/99900361/Documents/2022 Files/Stephe...

```python
# Program 2: Koch Curve Fractal
# Run Module and type >>> Koch_Curve(300,4) in Python Shell.
from turtle import *
speed(0)                            # Fastest speed.
def KochCurve(length,level):
    if level==0:
        fd(length)
        return
    KochCurve(length/3,level-1)
    lt(60)                          # Left turn 60 degrees.
    KochCurve(length/3,level-1)
    rt(120)                         # Right turn 120 degrees.
    KochCurve(length/3,level-1)
    lt(60)
    KochCurve(length/3,level-1)
```

Ln: 1 Col: 0

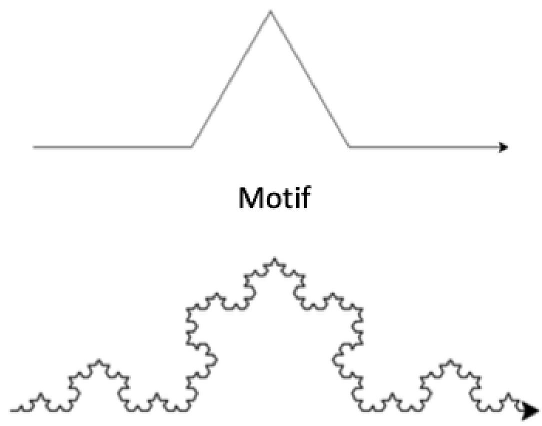

Figure 2: The Koch curve.

Problem 2: Edit Program 2 to plot a Koch square curve, where one segment is replaced with five segments each one-third the length of the segment before.

Example 3. The *Sierpiński triangle* (1915).

Start with a solid equilateral triangle and at each stage remove the middle-inverted triangle. Complete the construction up to stage 4.

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*Sierpinski.py - /Users/99900361/Documents/2022 Files/Stephen/...

```
# Program 3: Sierpinski triangle - save file as Sierpinski.py.
# Run the Module (or type F5).
# In the Python shell, type >>> Sierpinski(400 , 5)
from turtle import *
def Sierpinski(length, level):
    speed(0)       # Fastest speed.
    if level==0:
        return
    begin_fill()  # Fill shape.
    color("red")

    for i in range(3):
        Sierpinski(length/2,level-1)
        fd(length)
        lt(120)    # Left turn 120 degrees.
    end_fill()
```

Ln: 1 Col: 0

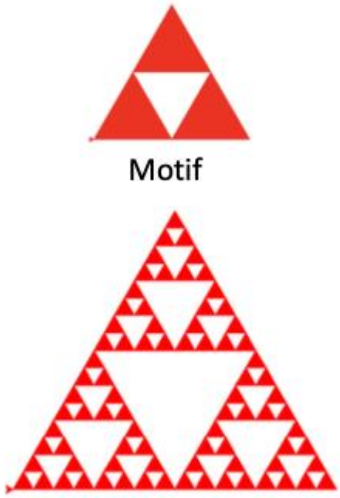


Figure 3: The Sierpiński triangle.

Problem 3: Edit Program 3 to plot a Sierpiński square, where the middle square is removed at each stage. See one face of the Menger Sponge in Figure 4b.

Example 4. Use train tickets to construct stage 1 of the *Menger Sponge (see Figures 4a and 4b).*

Problem 4: Given that you need 6 tickets to make one small block:

How many tickets do you need for the stage 1 construction? (See Figure 4a).

How many tickets would be required for the stage 2 construction?

How many tickets would be needed for the stage 4 construction? (See Figure 4b).

Figure 4a: Menger sponge, stage 1.

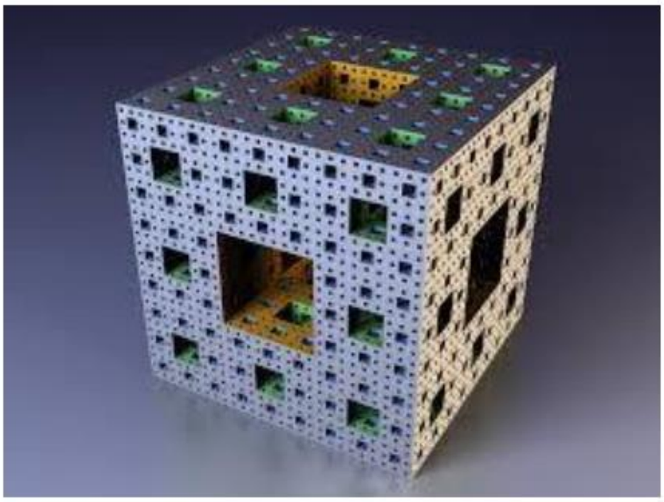


Figure 4b: Menger sponge, stage 4.

Example 5. A Python program for plotting a fractal tree.

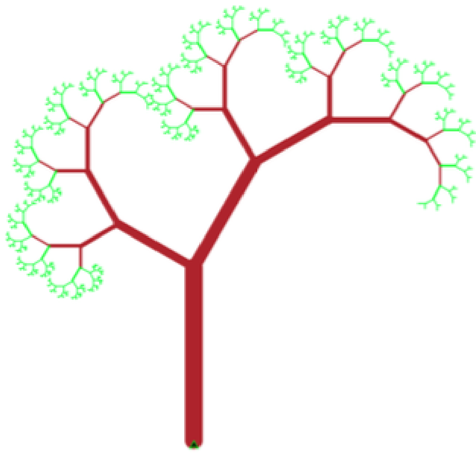
IDLE File Edit Format Run Options Window Help

FractalTreeColor.py - /Users/99900361/Documents/2022 Files/Ste

```
# Program 3: A program to plot a color fractal tree.
# Run Module and type >>> FractalTreeColor(200,10) in Python Shell.
from turtle import *
setheading(90)              # The turtle points straight up.
penup()                     # Lift the pen.
setpos(0,-250)              # Set initial point.
pendown()                   # Pen down.

def FractalTreeColor(length, level):
    """
    Draws a fractal tree
    """
    pensize(length/10)      # Thickness of lines.
    if length < 20:
        pencolor("green")
    else:
        pencolor("brown")

    speed(0)
    if level > 0:
        fd(length)          # Forward.
        rt(30)              # Right turn 30 degrees.
        FractalTreeColor(length*0.7, level-1)
        lt(90)              # Left turn 90 degrees.
        FractalTreeColor(length*0.5, level-1)
        rt(60)              # Right turn 60 degrees.
        penup()
        bk(length)          # Backward.
        pendown()
```



Problem 5: Edit Program 4 to plot a trifurcating tree with three branches at each level.

The formula for working out the fractal dimension D_f, say, is

$$D_f = -\frac{ln(\text{Number of segments})}{ln(\text{Length scale})}.$$

The fractal dimension of the Cantor set is $D_f = \frac{ln2}{ln3} \sim 0.6309$.

The fractal dimension of the Koch curve is $D_f = \frac{ln4}{ln3} \sim 1.2619$.

The fractal dimension of the Sierpiński triangle is $D_f = \frac{ln3}{ln2} \sim 1.5850$.

The fractal dimension of the Menger sponge is $D_f = \frac{ln20}{ln3} \sim 2.7268$.

Problem 6: Work out the fractal dimensions of the other fractals.

Further Information:

URL to download IDLE Python (which is free):

https://www.python.org/

URL for an introduction to the Python Turtle module:

https://docs.python.org/3/library/turtle.html

Python for employability:

https://www.mathscareers.org.uk/python-for-a-level-maths-undergraduate-maths-and-employability/

Python for A-Level Mathematics and Beyond:

https://drstephenlynch.github.io/webpages/Python_for_A_Level_Mathematics_and_Beyond.html

The Mandelbrot Set

URL for the Mandelbrot set (song):

https://www.youtube.com/watch?v=alj30SOoIDM

IMA Workshops and Recent Python Books

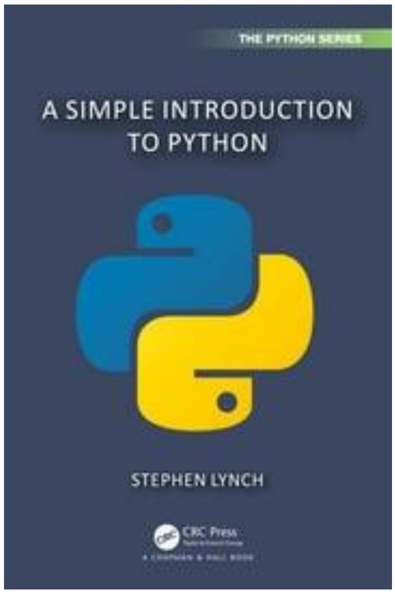
Institute of mathematics & its applications

One-day interactive workshops: Python for A-Level Mathematics and Beyond:

https://ima.org.uk/events/conferences/ (one in winter and one in summer)

Python Books: (click on book)

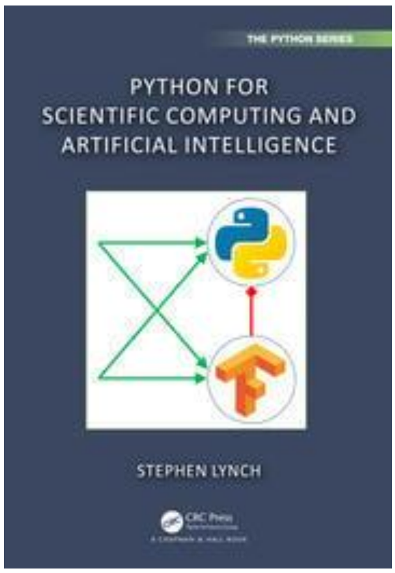
A Simple Introduction to Python (for complete novices)



Features:

- No prior experience in programming is required.
- Demonstrates how to format Jupyter notebooks for publication on the Web.
- Full solutions to exercises are available as a Jupyter notebook on the Web.
- All Jupyter notebook solution files can be downloaded through GitHub.

Python for Scientific Computing and Artificial Intelligence (intermediate level)



Features:

- No prior experience of programming is required.
- Online GitHub repository available with codes for readers to practice.
- Covers applications and examples from biology, chemistry, computer science, data science, electrical and mechanical engineering, economics, mathematics, physics, statistics and binary oscillator computing.
- Full solutions to exercises are available as Jupyter notebooks on the Web.