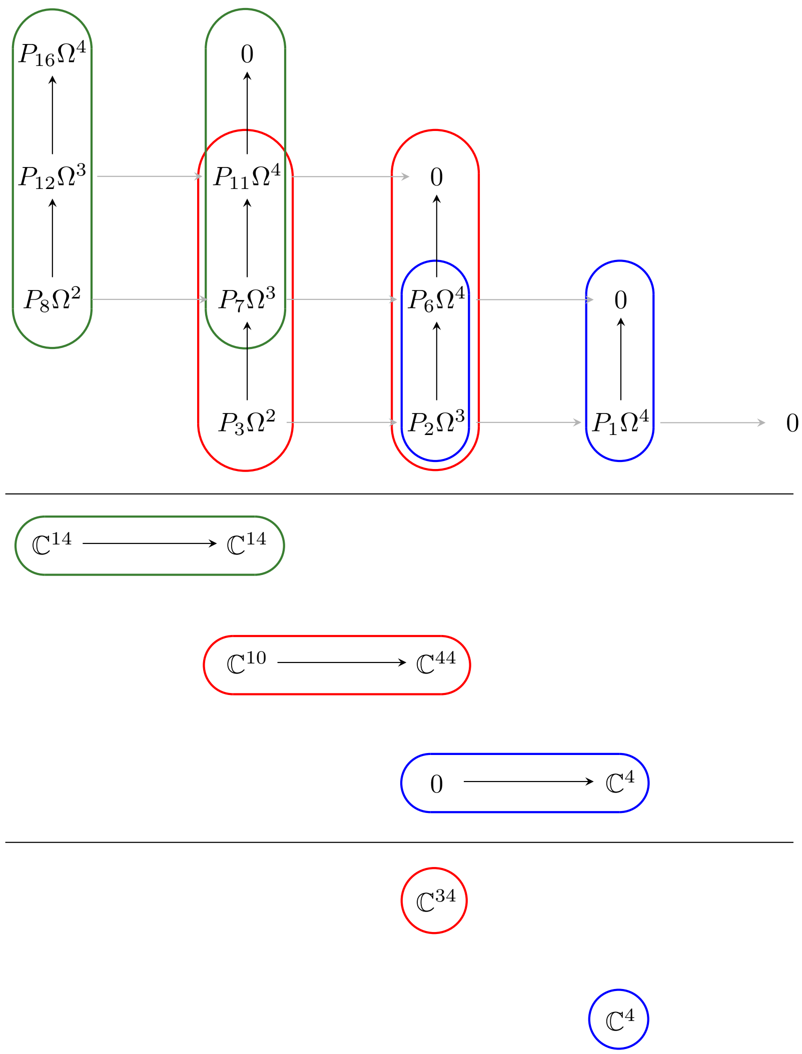
Spectral Sequence of a Quintic with 14 Nodes

Let $f = 3x_0^2x_1^2(x_0+x_1) - x_2x_3(2x_0^3+2x_1^3-x_2^3-x_3^3) \in \mathbb{Z}[x_0,x_1,x_2,x_3]$, a homogeneous quintic, then f has 14 ordinary double points and no other singularities. Let $\zeta = e^{2\pi i/3}$, then the 14 singular points are

$$[1:0:0:0], [1:0:0:\zeta^k\sqrt[3]{2}], [1:0:\zeta^k\sqrt[3]{2}:0],$$
$$[0:1:0:0], [0:1:0:\zeta^k\sqrt[3]{2}], [0:1:\zeta^k\sqrt[3]{2}:0]$$

for $k = 0,1,2$. The spectral sequence of f is below.



The zeta function of f is

$$Z(f,T) = \frac{1}{(1-T)(1-pT)(1-p^2T)P(T)}$$

where $P(T)$ is the "interesting" part of the zeta function and $\deg(P(T)) = 34+4 = 38$. Of course this agrees with the theory which says that

$$\deg(P(T)) = \frac{1}{5}((5-1)^4 + 5 - 1) - 14 = 52 - 14 = 38$$

note that 52 is from the smooth case. I was able to calculate $P(T)$ for $p = 5$ which gives us an example of the algorithm working when the prime divides the degree of f.

p	$P(T)$
5	$(1-5T)^9(1+5T)^5(1+25T^2)^2(1-6T+25T^2)$ $(1+15T^2-100T^3+375T^4+5^6T^6)(1+15T^2+100T^3+375T^4+5^6T^6)$ $(1+6T+45T^2+200T^3+1125T^4+3750T^5+5^6T^6)$
7	$(1-7T)^{10}(1+7T)^6(1-10T+49T^2)(1-7T+49T^2)^4$ $(1+4T+42T^2+196T^3+7^4T^4)(1+9T+91T^2+441T^3+7^4T^4)^2$
11	$(1-11T)^{10}(1+11T)^4(1+121T^2)^2(1-3T+121T^2)$ $(1-T-44T^2+726T^3-5324T^4-11^4T^5+11^6T^6)$ $(1+T-44T^2-726T^3-5324T^4+11^4T^5+11^6T^6)$ $(1+11T+187T^2+2178T^3+22627T^4+11^5T^5+11^6T^6)$

Remarks: On the factorization of $P(T)$.
From the theory of zeta functions we know that

$$P(T) = \prod_{k=1}^{38}(1-\alpha_k T)$$

where $|\alpha_k| = p^{(3-1)/2} = p$, for all $k = 1, 2, \ldots, 38$. Our computations show us what $P(T)$ over $\mathbb{Q}$ is, and curiosity drives us to find $P(T)$ over $\mathbb{C}$, i.e., 38 linear factors. For $p = 5$, it is obvious that

$$\begin{aligned} 1+25T^2 &= (1+5iT)(1-5iT) \\ 1-6T+25T^2 &= (1-(3+4i)T)(1-(3-4i)T) \end{aligned}$$

so we are left with the three sextics above. Let $g_1(T) = 1+15T^2-100T^3+375T^4+5^6T^6$, then finding α_k such that $g_1(T) = \prod(1-\alpha_k T)$ is equivalent to finding the roots of the polynomial $T^6 g_1(1/T)$ since

$$T^6 g_1\left(\frac{1}{T}\right) = T^6 \prod_{k=1}^{6}\left(1-\frac{\alpha_k}{T}\right) = \prod_{k=1}^{6}(T-\alpha_k).$$

Now

$$\begin{aligned} T^6 g_1\left(\frac{1}{T}\right) &= T^6\left(1+\frac{15}{T^2}-\frac{100}{T^3}+\frac{375}{T^4}+\frac{5^6}{T^6}\right) \\ &= T^6+15T^4-100T^3+375T^2+5^6 \end{aligned}$$

the reciprocal polynomial of g_1, call it G_1. Since the absolute value of each α_k is 5, if G_1 had a real root then it would either be 5 or -5. However $G_1(5) = 37500$ and $G_1(-5) = 62500$, hence all roots of G_1 are complex. In addition, the coefficients of G_1 are all real numbers so the complex roots come in conjugate pairs. Lastly, since $|\overline{\alpha_k}| = |\alpha_k| = 5$ we are led to the following factorization of G_1

$$\begin{aligned} G_1(T) &= (T^2+aT+25)(T^2+bT+25)(T^2+cT+25) \\ &= T^6+(a+b+c)T^5+(ab+ac+bc+75)T^4+(50(a+b+c)+abc)T^3 \\ &\quad +25(ab+ac+bc+75)T^2+625(a+b+c)T+5^6. \end{aligned}$$

Equating the coefficients yields the following system of equations

$$\begin{aligned} a+b+c &= 0 \\ ab+ac+bc &= -60 \\ abc &= -100 \end{aligned}$$

which can be solved by elimination and or substitution. However, since the left hand side of the equations above are the elementary symmetric polynomials in three variables we can construct a cubic polynomial whose roots are a, b, and c. Recall that

$$(x-a)(x-b)(x-c) = x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$$

so for our case we are interested in the roots of $x^3-60x+100$. One can use the intervals $[-9,-8]$, $[1,2]$, and $[6,7]$ together with the Intermediate Value Theorem to show that this cubic has three real roots. And since $x^3-60x+100\in\mathbb{Q}[x]$ is irreducible, expressing the real roots in terms of radicals requires complex numbers by Casus Irreducibilis. One can do this by using Cardano's formula for the depressed cubic x^3+px+q

$$x=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}.$$

Our particular cubic is depressed and $p=-60, q=100$

$$\Rightarrow \sqrt[3]{-50+10i\sqrt{55}}+\sqrt[3]{-50-10i\sqrt{55}}\approx 6.71649$$

is a root of $x^3-60x+100$. Let $\zeta=e^{2\pi i/3}$, then the other two roots are

$$\begin{aligned}\zeta\sqrt[3]{-50+10i\sqrt{55}}+\zeta^2\sqrt[3]{-50-10i\sqrt{55}}&\approx -8.47357\\ \zeta^2\sqrt[3]{-50+10i\sqrt{55}}+\zeta\sqrt[3]{-50-10i\sqrt{55}}&\approx 1.75708.\end{aligned}$$

With these numbers we can now factor $G_1(T)$ over the reals

$$\begin{aligned}G_1(T)&=T^6+15T^4-100T^3+375T^2+5^6\\ &=(T^2+\left(\sqrt[3]{-50+10i\sqrt{55}}+\sqrt[3]{-50-10i\sqrt{55}}\right)T+25)\\ &\quad\times(T^2+\left(\zeta\sqrt[3]{-50+10i\sqrt{55}}+\zeta^2\sqrt[3]{-50-10i\sqrt{55}}\right)T+25)\\ &\quad\times(T^2+\left(\zeta^2\sqrt[3]{-50+10i\sqrt{55}}+\zeta\sqrt[3]{-50-10i\sqrt{55}}\right)T+25).\end{aligned}$$

Each of these quadratics can be factored using the quadratic formula and therefore our original sextic $1+15T^2-100T^3+375T^4+5^6T^6$ can be written as $\prod(1-\alpha_kT)$ where α_k runs through the following 6 numbers

$$-\frac{\sqrt[3]{10}}{2}\left(\sqrt[3]{-5+i\sqrt{55}}+\sqrt[3]{-5-i\sqrt{55}}\right)+\frac{\sqrt[3]{10}}{2}\sqrt{\left(\sqrt[3]{-5+i\sqrt{55}}+\sqrt[3]{-5-i\sqrt{55}}\right)^2-10\sqrt[3]{10}}$$

$$-\frac{\sqrt[3]{10}}{2}\left(\sqrt[3]{-5+i\sqrt{55}}+\sqrt[3]{-5-i\sqrt{55}}\right)-\frac{\sqrt[3]{10}}{2}\sqrt{\left(\sqrt[3]{-5+i\sqrt{55}}+\sqrt[3]{-5-i\sqrt{55}}\right)^2-10\sqrt[3]{10}}$$

$$-\frac{\sqrt[3]{10}}{2}\left(\zeta\sqrt[3]{-5+i\sqrt{55}}+\zeta^2\sqrt[3]{-5-i\sqrt{55}}\right)+\frac{\sqrt[3]{10}}{2}\sqrt{\left(\zeta\sqrt[3]{-5+i\sqrt{55}}+\zeta^2\sqrt[3]{-5-i\sqrt{55}}\right)^2-10\sqrt[3]{10}}$$

$$-\frac{\sqrt[3]{10}}{2}\left(\zeta\sqrt[3]{-5+i\sqrt{55}}+\zeta^2\sqrt[3]{-5-i\sqrt{55}}\right)-\frac{\sqrt[3]{10}}{2}\sqrt{\left(\zeta\sqrt[3]{-5+i\sqrt{55}}+\zeta^2\sqrt[3]{-5-i\sqrt{55}}\right)^2-10\sqrt[3]{10}}$$

$$-\frac{\sqrt[3]{10}}{2}\left(\zeta^2\sqrt[3]{-5+i\sqrt{55}}+\zeta\sqrt[3]{-5-i\sqrt{55}}\right)+\frac{\sqrt[3]{10}}{2}\sqrt{\left(\zeta^2\sqrt[3]{-5+i\sqrt{55}}+\zeta\sqrt[3]{-5-i\sqrt{55}}\right)^2-10\sqrt[3]{10}}$$

$$-\frac{\sqrt[3]{10}}{2}\left(\zeta^2\sqrt[3]{-5+i\sqrt{55}}+\zeta\sqrt[3]{-5-i\sqrt{55}}\right)-\frac{\sqrt[3]{10}}{2}\sqrt{\left(\zeta^2\sqrt[3]{-5+i\sqrt{55}}+\zeta\sqrt[3]{-5-i\sqrt{55}}\right)^2-10\sqrt[3]{10}}.$$

As a reminder, the absolute value of each of the six numbers above is 5 (the theory is so beautiful). For the next sextic, $1+15T^2+100T^3+375T^4+5^6T^6$, we have the same story as above but now with the cubic $x^3-60x-100$. Again the α_k's are solvable in terms of radicals so we are left with the last sextic $1+6T+45T^2+200T^3+1125T^4+3750T^5+5^6T^6$. Factoring its reciprocal polynomial

$$T^6+6T^5+45T^4+200T^3+1125T^2+3750T+5^6=(T^2+aT+25)(T^2+bT+25)(T^2+cT+25)$$

gives us the following system of equations

$$\begin{aligned} a+b+c &= 6 \\ ab+ac+bc &= -30 \\ abc &= -100. \end{aligned}$$

Solving for a, b, and c is equivalent to finding the roots of $x^3-6x^2-30x+100$. Notice that this cubic is not depressed so we make the change of variables $x \to y-2$ to yield the cubic $y^3-42y+24$ whose roots can be calculated using Cardano's formula. Hence

$$\begin{aligned} &T^6+6T^5+45T^4+200T^3+1125T^2+3750T+5^6 \\ &= (T^2+\left(2+\sqrt[3]{-12+10i\sqrt{26}}+\sqrt[3]{-12-10i\sqrt{26}}\right)T+25) \\ &\quad\times (T^2+\left(2+\zeta\sqrt[3]{-12+10i\sqrt{26}}+\zeta^2\sqrt[3]{-12-10i\sqrt{26}}\right)T+25) \\ &\quad\times (T^2+\left(2+\zeta^2\sqrt[3]{-12+10i\sqrt{26}}+\zeta\sqrt[3]{-12-10i\sqrt{26}}\right)T+25) \end{aligned}$$

where $\zeta = e^{2\pi i/3}$. The roots of the three quadratics above provide us with the the six α_k such that

$$1+6T+45T^2+200T^3+1125T^4+3750T^5+5^6T^6 = \prod_{k=1}^{6}(1-\alpha_k T).$$

For $p=7$ we have

$$\begin{aligned} 1-10T+49T^2 &= (1-(5+2i\sqrt{6})T)(1-(5-2i\sqrt{6})T) \\ 1-7T+49T^2 &= \left(1-\frac{7(1+i\sqrt{3})}{2}T\right)\left(1-\frac{7(1-i\sqrt{3})}{2}T\right) \end{aligned}$$

and for the quartics they factor as $\prod(1-\alpha_k T)$ where α_k runs through the 4 numbers below each quartic.

$$\begin{array}{ll} 1+4T+42T^2+196T^3+7^4T^4 & 1+9T+91T^2+441T^3+7^4T^4 \\ -1+\sqrt{15}\pm i\sqrt{33+2\sqrt{15}} & \frac{1}{4}\left(-9+\sqrt{109}\pm 3i\sqrt{66+2\sqrt{109}}\right) \\ -1-\sqrt{15}\pm i\sqrt{33-2\sqrt{15}} & \frac{1}{4}\left(-9-\sqrt{109}\pm 3i\sqrt{66-2\sqrt{109}}\right) \end{array}$$

For $p=11$ we have

$$\begin{aligned} 1+121T^2 &= (1-11iT)(1+11iT) \\ 1-3T+121T^2 &= \left(1-\frac{3+5i\sqrt{19}}{2}T\right)\left(1-\frac{3-5i\sqrt{19})}{2}T\right) \end{aligned}$$

and for the three sextics they are all solvable by radicals. Factoring them into the form $\prod(1-\alpha_k T)$ requires the same steps that we went through for the prime 5. As an example, one of the roots of

$$T^6-T^5-44T^4+726T^3-5324T^2-11^4T+11^6$$

is

$$\begin{aligned} &\frac{1}{6\sqrt[3]{2}}\Bigg(\sqrt[3]{2}-\sqrt[3]{22471+33i\sqrt{6238959}}-\sqrt[3]{22471-33i\sqrt{6238959}} \\ &\quad+\sqrt{\left(\sqrt[3]{2}-\sqrt[3]{22471+33i\sqrt{6238959}}-\sqrt[3]{22471-33i\sqrt{6238959}}\right)^2-4356\sqrt[3]{4}}\Bigg). \end{aligned}$$

Here are the bases that I used for the cohomology spaces on the E_1 page. For $H^4(K_f^\bullet)_1$ the trivial basis

$$\{x_0, x_1, x_2, x_3\} dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3.$$

For $H^4(K_f^\bullet)_6$ the following 44 monomials were used.

x_0^6	$x_0^5 x_1$	$x_0^5 x_2$	$x_0^5 x_3$	$x_0^4 x_1^2$	$x_0^4 x_1 x_2$	$x_0^4 x_1 x_3$	$x_0^4 x_2^2$
$x_0^4 x_3^2$	$x_0^3 x_1^3$	$x_0^3 x_1^2 x_2$	$x_0^3 x_1^2 x_3$	$x_0^3 x_1 x_2^2$	$x_0^3 x_1 x_3^2$	$x_0^3 x_2^3$	$x_0^3 x_3^3$
$x_0^2 x_1^4$	$x_0^2 x_1^3 x_2$	$x_0^2 x_1^3 x_3$	$x_0^2 x_1^2 x_2^2$	$x_0^2 x_1^2 x_3^2$	$x_0^2 x_1 x_2^3$	$x_0^2 x_1 x_2^2 x_3$	$x_0^2 x_1 x_2 x_3^2$
$x_0^2 x_1 x_3^3$	$x_0^2 x_2^4$	$x_0^2 x_2^3 x_3$	$x_0 x_1^2 x_2^3$	$x_0 x_1^2 x_3^3$	$x_0 x_1 x_2^4$	$x_0 x_1 x_2^3 x_3$	$x_0 x_1 x_2^2 x_3^2$
$x_0 x_2^5$	$x_0 x_2^3 x_3^2$	x_1^6	$x_1^5 x_2$	$x_1^5 x_3$	$x_1^4 x_2^2$	$x_1^4 x_3^2$	$x_1^3 x_2^3$
$x_1^3 x_3^3$	$x_1 x_2^5$	$x_1 x_2^3 x_3^2$	x_2^6				

As a reminder each monomial is being multiplied by the 4-form $dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$. Lastly for $H^3(K_f^\bullet)_7$ let

$$\begin{aligned}
\omega_1 = {} & 8x_3(2x_0^4 + 3x_0^3 x_1 - 3x_0 x_1^3 - 2x_1^4) dx_0 \wedge dx_1 \wedge dx_2 \\
& + 2x_2(2x_0^4 + 3x_0^3 x_1 - 3x_0 x_1^3 - 2x_1^4) dx_0 \wedge dx_1 \wedge dx_3 \\
& + x_1(3x_0 + 2x_1)(2x_0^3 + 2x_1^3 - 5x_3^3) dx_0 \wedge dx_2 \wedge dx_3 \\
& + x_0(2x_0 + 3x_1)(2x_0^3 + 2x_1^3 - 5x_3^3) dx_1 \wedge dx_2 \wedge dx_3 \\
\omega_2 = {} & -2x_3(2x_0^4 + 3x_0^3 x_1 - 3x_0 x_1^3 - 2x_1^4) dx_0 \wedge dx_1 \wedge dx_2 \\
& - 8x_2(2x_0^4 + 3x_0^3 x_1 - 3x_0 x_1^3 - 2x_1^4) dx_0 \wedge dx_1 \wedge dx_3 \\
& + x_1(3x_0 + 2x_1)(2x_0^3 + 2x_1^3 - 5x_2^3) dx_0 \wedge dx_2 \wedge dx_3 \\
& + x_0(2x_0 + 3x_1)(2x_0^3 + 2x_1^3 - 5x_2^3) dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}$$

then $\omega_1, \omega_2 \in \ker(P_5\Omega^3 \xrightarrow{df\wedge} P_9\Omega^4)/\operatorname{im}(P_1\Omega^2 \xrightarrow{df\wedge} P_5\Omega^3)$ and

$$\{x_0^2\omega_1, x_0 x_2 \omega_1, x_0 x_3 \omega_1, x_1^2 \omega_1, x_1 x_2 \omega_1, x_1 x_3 \omega_1, x_2^2 \omega_1, x_3^2 \omega_1, x_0^2 \omega_2, x_1^2 \omega_2\}$$

is a basis for $H^3(K_f^\bullet)_7$. However this basis leaves something to be desired since the de Rham differential of each element is not always a combination of our 44 monomials. For example

$$d(x_1^2\omega_1) = (-10x_0^4 x_1^2 - 10x_0^3 x_1^3 - 10x_0 x_1^5 + 25x_0 x_1^2 x_3^3 - 10x_1^6 + 25x_1^3 x_3^3) dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$$

and the monomial $x_0 x_1^5$ is not on our list of 44 monomials. This is not a problem because we can write $x_0 x_1^5$ as a combination of the 44 monomials plus $df\wedge$ some 3-form and then use this to calculate a basis for the cohomology group on the E_2 page of corresponding degree, $E_2^{4,6}$. Or we can add an element in the image of Koszul to $x_1^2\omega_1$ and see if d of the resulting 3-form has only the monomials on our list of 44, basis for $H_4(K_f^\bullet)_6$. It turns out that such a basis exists $\{b_i\}_{i=1}^{10}$ where

$$
\begin{aligned}
b_1 &= \frac{1}{5}(x_0^2 + x_1^2)\omega_1 - \frac{1}{3}x_3 df \wedge (x_0^2 dx_0 \wedge dx_2 + x_1^2 dx_1 \wedge dx_2)\\
b_2 &= -\frac{1}{5}x_1^2\omega_1 + \frac{1}{3}x_3 df \wedge (x_0^2 dx_0 \wedge dx_2 + x_1^2 dx_1 \wedge dx_2)\\
b_3 &= \frac{4}{5}x_2(x_0 - x_1)\omega_1 + df \wedge \left(x_0x_2(3x_0 + 4x_1)dx_0 \wedge dx_1 - \frac{2}{3}x_3^3 dx_0 \wedge dx_2 \right)\\
b_4 &= \frac{1}{20}x_0x_3\omega_1 + df \wedge \left(\frac{3}{4}x_0^2x_3 + \frac{1}{2}x_0x_1x_3 \right) dx_0 \wedge dx_1\\
b_5 &= \frac{2}{5}x_1x_2\omega_1 - df \wedge \left(x_0x_1x_2 dx_0 \wedge dx_1 - \frac{1}{3}x_3^3 dx_0 \wedge dx_2 + \frac{1}{3}x_1x_2x_3 dx_1 \wedge dx_2 \right)\\
b_6 &= \left(\frac{1}{2}x_1x_3 + \frac{12}{5}x_2^2 \right)\omega_1 + df \wedge \left(\left(5x_0x_1x_3 + \frac{3}{2}x_0x_2^2 + \frac{15}{2}x_1^2x_3 - \frac{3}{2}x_1x_2^2 \right) dx_0 \wedge dx_1 \right.\\
&\quad \left. - 5x_2^2x_3 dx_0 \wedge dx_2 + \left(\frac{5}{3}x_1x_3^2 - 5x_2^2x_3 \right) dx_1 \wedge dx_2 \right)\\
b_7 &= \left(\frac{2}{5}x_0x_2 - \frac{2}{5}x_1x_2 - \frac{1}{20}x_3^2 \right)\omega_1 + df \wedge \left(\left(2x_0x_2(x_0 + x_1) + \frac{1}{8}x_3^2(x_0 - x_1) \right) dx_0 \wedge dx_1 \right.\\
&\quad \left. - \frac{1}{3}x_3^3 dx_0 \wedge dx_2 - \frac{1}{9}x_3^3 dx_1 \wedge dx_2 \right)\\
b_8 &= -\frac{1}{10}x_1x_3\omega_1 - df \wedge \left(x_1x_3\left(x_0 + \frac{3}{2}x_1 \right) dx_0 \wedge dx_1 - x_2^2x_3 dx_0 \wedge dx_2 + \frac{1}{3}x_1x_3^2 dx_1 \wedge dx_2 \right)\\
b_9 &= \frac{1}{25}x_0^2(\omega_1 - \omega_2)\\
b_{10} &= \frac{1}{25}x_1^2(\omega_2 - \omega_1).
\end{aligned}
$$

Let $\omega = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$, then below are the de Rham differentials of each b_i

$$
\begin{aligned}
d(b_1) &= (2x_0^6 - 5x_0^4x_1^2 - 5x_0^3x_3^3 + 5x_0^2x_1^4 - 5x_0^2x_1x_3^3 + 5x_0x_1^2x_3^3 - 2x_1^6 + 5x_1^3x_3^3)\omega\\
d(b_2) &= (2x_0^5x_1 + 5x_0^4x_1^2 + 2x_0^3x_1^3 - 3x_0^2x_1^4 - 5x_0x_1^2x_3^3 + 2x_1^6 - 5x_1^3x_3^3)\omega\\
d(b_3) &= (2x_0^5x_2 - 12x_0^3x_1x_3^2 - 6x_0^2x_1^3x_2 - 18x_0^2x_1^2x_3^2 + 3x_0^2x_2^4 + 4x_0x_1x_2^4 + 8x_1^5x_2)\omega\\
d(b_4) &= (2x_0^5x_3 + 2x_0^4x_1x_3 - 3x_0^2x_2^3x_3 - 2x_0x_1x_2^3x_3)\omega\\
d(b_5) &= (2x_0^4x_1x_2 + 6x_0^3x_1x_3^2 + 3x_0^2x_1^3x_2 + 9x_0^2x_1^2x_3^2 - 2x_0^2x_1x_2^2x_3 - x_0x_1x_2^4 - 4x_1^5x_2)\omega\\
d(b_6) &= (18x_0^4x_2^2 - 30x_0^2x_1^3x_3 + 20x_0^2x_1x_2x_3^2 - 30x_0^2x_2^3x_3 - 20x_0x_1x_2^3x_3 + 3x_0x_2^5 + 20x_1^5x_3 - 18x_1^4x_2^2 - 3x_1x_2^5)\omega\\
d(b_7) &= (2x_0^4x_3^2 - 4x_0^3x_1x_3^2 - 4x_0^2x_1^3x_2 - 6x_0^2x_1^2x_3^2 + 2x_0^2x_2^4 + 2x_0x_1x_2^4 - x_0x_2^3x_3^2 + 4x_1^5x_2 - 2x_1^4x_3^2 + x_1x_2^3x_3^2)\omega\\
d(b_8) &= (6x_0^3x_1x_2^2 + 6x_0^2x_1^3x_3 + 9x_0^2x_1^2x_2^2 - 4x_0^2x_1x_2x_3^2 + 4x_0x_1x_2^3x_3 - 4x_1^5x_3)\omega\\
d(b_9) &= (x_0^3x_2^3 - x_0^3x_3^3 + x_0^2x_1x_2^3 - x_0^2x_1x_3^3)\omega\\
d(b_{10}) &= (x_0x_1^2x_2^3 - x_0x_1^2x_3^3 + x_1^3x_2^3 - x_1^3x_3^3)\omega.
\end{aligned}
$$

As mentioned above the advantage of this basis $\{b_1, b_2, \ldots, b_{10}\}$ is that the de Rham differential of each of these elements is a linear combination of the 44 monomials for our basis of $H^4(K_f^\bullet)_6$. This in turn makes the matrix representation of $d : H^3(K_f^\bullet)_7 \to H^4(K_f^\bullet)_6$ easy to construct since we only have to read off the coefficients of each $d(b_i)$ which are also all integers.

$$
\left[\begin{array}{cccccccccccccccccccccccccccccccccccc}
2 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 & -6 & 0 & 0 & -18 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 3 & 0 & 0 & 9 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -30 & 0 & 0 & 0 & 0 & 20 & 0 & 0 & -30 & 0 & 0 & 0 & -20 & 0 & 3 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & -4 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 9 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.
$$

$$
\left.\begin{array}{ccccccccccc}
\cdots & -2 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
 & 2 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\
\cdots & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 20 & -18 & 0 & 0 & 0 & -3 & 0 & 0 \\
\cdots & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0
\end{array}\right]
$$

The 10×44 matrix above, call it M, is the representation of the de Rham differential $d : H^3(K_f^\bullet)_7 \to H^4(K_f^\bullet)_6$. It has rank 10 so the space $E_2^{4,6}$ has dimension $44 - 10 = 34$. One way to find these 34 basis elements is to augment M with rows of the identity matrix of size 44, I_{44}. Starting with the first row of I_{44}, $e_1 = (1\ \ 0\ \ 0 \ldots 0)$, we have the 11×44 matrix

$$\begin{bmatrix} M \\ e_1 \end{bmatrix}$$

which has rank 11 and therefore we use the first element in the basis $H^4(K_f^\bullet)_6$. Next we try the second row of I_{44}, $e_2 = (0\ \ 1\ \ 0 \ldots 0)$. The matrix

$$\begin{bmatrix} M \\ e_1 \\ e_2 \end{bmatrix}$$

has rank 12 so our basis for $E_2^{4,6}$ will contain the elements

$$\{x_0^6, x_0^5 x_1\} dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3.$$

Continuing in this fashion we find that if we add rows 1 through 21, 23, 24, 26 through 29, 32 through 35, 38, 39, and 44 of I_{44} to M, then the resulting 44×44 matrix will have rank 44. Therefore if we omit entries 22, 25, 30, 31, 36, 37, 40, 41, 42, and 43 of our list of 44 monomials (for $H^4(K_f^\bullet)_6$) then we will have a basis for $E_2^{4,6}$.

x_0^6	$x_0^5x_1$	$x_0^5x_2$	$x_0^5x_3$	$x_0^4x_1^2$	$x_0^4x_1x_2$	$x_0^4x_1x_3$	$x_0^4x_2^2$
$x_0^4x_3^2$	$x_0^3x_1^3$	$x_0^3x_1^2x_2$	$x_0^3x_1^2x_3$	$x_0^3x_1x_2^2$	$x_0^3x_1x_3^2$	$x_0^3x_2^3$	$x_0^3x_3^3$
$x_0^2x_1^4$	$x_0^2x_1^3x_2$	$x_0^2x_1^3x_3$	$x_0^2x_1^2x_2^2$	$x_0^2x_1^2x_3^2$	$x_0^2x_1x_2^2x_3$	$x_0^2x_1x_2x_3^2$	$x_0^2x_2^4$
$x_0^2x_2^3x_3$	$x_0x_1^2x_2^3$	$x_0x_1^2x_3^3$	$x_0x_1x_2^2x_3^2$	$x_0x_2^5$	$x_0x_2^3x_3^2$	x_1^6	$x_1^4x_2^2$
$x_1^4x_3^2$	x_2^6						