One Equation To Derive Them All

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In Calculus 2 they teach you about power series and in particular the power series for e^x which is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1}$$

So the number $e \approx 2.718281828459045$ is exactly equal to

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots .$$

If we replace x with ix in the power series for e^x (1) then we get

$$\begin{aligned} e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} &= \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(i^2)^n x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \cos(x) + i\sin(x). \end{aligned} \tag{2}$$

Note that the two sums on the right hand of (2) run through the even and odd positive integers respectively. More importantly, in the chain of equalities above we can equate the first and last expressions to arrive at Euler's Formula.

$$\boxed{e^{ix} = \cos(x) + i\sin(x)} \tag{3}$$

With this one formula we can derive all of the classical trig identities. That's where the title of these notes comes from. For starters let's consider e^{-ix} which is also equal to $e^{i(-x)}$. Just to be clear, the parenthesis are not needed, but are added for emphasis because by Euler's Formula (3) we have that e to the i times any real number is equal to cosine of that real number plus i times sine of that real number. Therefore

$$e^{-ix} = e^{i(-x)} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x) \tag{4}$$

In the last equality of (4) we have used the fact that cosine and sine are even and odd functions respectively. With equations (3) and (4) we can derive the following identity

$$\begin{aligned} 1 = e^{ix}e^{-ix} = (\cos(x) + i\sin(x))(\cos(x) - i\sin(x)) = \cos^2(x) + \sin^2(x) \\ \implies \boxed{\cos^2(x) + \sin^2(x) = 1} \end{aligned}$$

Next let us derive the sum angle formulas for cosine and sine. Consider $e^{i(x+y)}$. By definition this is equal to

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y).$$

We can also split this product up and use Euler's formula to yield

$$\begin{aligned} e^{i(x+y)} = e^{ix}e^{iy} &= (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) \\ &= \cos(x)\cos(y) + i\cos(x)\sin(y) + i\sin(x)\cos(y) - \sin(x)\sin(y) \\ &= \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)). \end{aligned}$$

Recall that two complex numbers are equal if and only if their real and imaginary parts are equal we have that

$$\boxed{\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)}$$

$$\boxed{\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)}$$

We can also derive other identities by writing $\cos(x)$ and $\sin(x)$ in terms of complex exponentials. Consider

$$\begin{gathered} e^{ix} + e^{-ix} = \cos(x) + i\sin(x) + \cos(x) - i\sin(x) = 2\cos(x) \\ \cos(x) = \frac{e^{ix} + e^{-ix}}{2}. \end{gathered}$$

Similarly

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

Now let's use the above to calculate $\cos^2(x)$.

$$\begin{gathered} \cos^2(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{1}{4}(e^{2ix} + 2 + e^{-2ix}) = \frac{1}{2} + \frac{e^{2ix} + e^{-2ix}}{4} = \frac{1}{2} + \frac{1}{2}\cos(2x) \\ \implies \boxed{\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)} \end{gathered}$$

Can you derive similar identities for $\sin^2(x)$ and $\cos^3(x)$?