Partial Fraction Decomposition TRICKS

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Given two polynomials, say $p(x)$ and $q(x)$ with $\deg(p(x)) < \deg(q(x))$, the Partial Fraction Decomposition of $\frac{p(x)}{q(x)}$ is the operation/process of writing this quotient as the sum of fractions with simpler denominators. For example

$$\frac{x+4}{x^2+3x+2} = \frac{3}{x+1} - \frac{2}{x+2}. \tag{1}$$

There are a couple of ways to find the identity above and the purpose of these notes is to cover a quick way which, in my experience, seems to be unknown to most students. First, let's give some motivation as to why identities like (1) are useful. If you are asked to integrate the function on the left hand side of (1), then partial fraction decomposition is helpful in carrying out this calculation since

$$\int \frac{x+4}{x^2+3x+2}dx = 3\int \frac{dx}{x+1} - 2\int \frac{dx}{x+2} = 3\ln|x+1| - 2\ln|x+2| + C.$$

Another application of partial fraction decomposition is in finding the inverse Laplace Transform of a function, which is useful in Electrical Engineering.

$$\mathscr{L}^{-1}\left\{\frac{s+4}{s^2+3s+2}\right\} = 3\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} = 3e^{-t} - 2e^{-2t}$$

Now let us consider how to find the partial fraction decomposition of the LHS of equation (1). Starting with the denominator we see that it factors as $x^2+3x+2 = (x+1)(x+2)$. Hence we wish to find constants A and B to give us the following decomposition

$$\frac{x+4}{x^2+3x+2} = \frac{x+4}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}. \tag{2}$$

From here most students are only taught to clear the denominator. So in this case we multiply the whole equation by the quantity $(x+1)(x+2)$ which yields

$$x+4 = A(x+2) + B(x+1)$$
$$x+4 = x(A+B) + 2A + B$$

Equating the coefficients of the x terms and the constant terms gives us the equations

$$A + B = 1 \tag{3}$$
$$2A + B = 4 \tag{4}$$

Subtracting $(4) - (3)$ gives $A = 3 \Rightarrow B = -2$ and therefore

$$\frac{x+4}{(x+1)(x+2)} = \frac{3}{x+1} - \frac{2}{x+2}$$

as originally stated.

So this is one method of finding the partial fraction decomposition of a function, but the "trick" method is faster, and here it is. Starting with (2), instead of clearing the denominator, let us just multiply by $x+1$

$$\frac{x+4}{x+2} = A + \frac{B(x+1)}{x+2}. \tag{5}$$

Now the equation above holds for all values except $x=-2$, so let us evaluate it at $x=-1$

$$\left.\frac{x+4}{x+2}\right|_{x=-1} = A + \left.\frac{B(x+1)}{x+2}\right|_{x=-1}.$$

The advantage of doing this is that the term containing B vanishes since it has -1 as a zero

$$\left.\frac{x+4}{x+2}\right|_{x=-1} = A + \left.\cancel{\frac{B(x+1)}{x+2}}\right|_{x=-1}.$$

Therefore

$$A = \left.\frac{x+4}{x+2}\right|_{x=-1} = \frac{-1+4}{-1+2} = 3.$$

Similarly

$$B = \left.\frac{x+4}{x+1}\right|_{x=-2} = \frac{-2+4}{-2+1} = -2$$

which are the same values that we calculated above by using the other method. Notice that in this "trick" method we did not have to solve a system of linear equations, just evaluate an expression. You might say that this trick isn't that much faster, but as the degree of the denominator increases, this trick saves more and more time. As an example consider

$$\frac{2x^3-3x^2+13}{(x+2)(x-1)(x-2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x-3}. \tag{6}$$

Solving for the constants A through D by clearing the denominator is laborious since you have to expand the following

$$A(x-1)(x-2)(x-3) + B(x+2)(x-2)(x-3) + C(x+2)(x-1)(x-3) + D(x+2)(x-1)(x-2)$$

and then group all of the x^3 terms, x^2 terms, and so forth. After gathering like terms you then have to solve the four by four linear system of equations seen below.

$$\begin{aligned} A + B + C + D &= 2 \\ 6A + 3B + 2C + D &= 3 \\ 11A - 4B - 5C - 4D &= 0 \\ -6A + 12B + 6C + 4D &= 13 \end{aligned}$$

Or...you can immediately solve for A by multiplying (6) by $x+2$ and then evaluating the resulting equation at $x=-2$ to get

$$A = \left.\frac{2x^3-3x^2+13}{(x-1)(x-2)(x-3)}\right|_{x=-2} = \frac{-16-12+13}{-3(-4)(-5)} = \frac{15}{3\cdot4\cdot5} = \frac{1}{4}.$$

Continuing in this fashion one finds that

$$\frac{2x^3-3x^2+13}{(x+2)(x-1)(x-2)(x-3)} = \frac{1}{4}\cdot\frac{1}{x+2} + \frac{2}{x-1} - \frac{17}{4}\cdot\frac{1}{x-2} + \frac{4}{x-3}.$$

This trick can also be used with complex numbers. For example in the partial fraction decomposition of

$$\frac{x^3+10x^2+9x+4}{(x^2+1)(x+3)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3} + \frac{D}{x-1}$$

one can solve for A and B by multiplying the above equation by x^2+1 and then evaluating the result at $x=i$ (or $x=-i$ since both are roots of x^2+1).

$$\left.\frac{x^3+10x^2+9x+4}{(x+3)(x-1)}\right|_{x=i} = Ax + B + \left.\cancel{\frac{C(x^2+1)}{x+3}} + \cancel{\frac{D(x^2+1)}{x-1}}\right|_{x=i}$$

$$\begin{aligned}
\Rightarrow Ai+B = \left.\frac{x^3+10x^2+9x+4}{(x+3)(x-1)}\right|_{x=i} &= \frac{-i-10+9i+4}{(i+3)(i-1)} \\
&= \frac{-6+8i}{-1-i+3i-3} \\
&= \frac{-6+8i}{-4+2i} \\
&= \frac{-3+4i}{-2+i}\cdot\frac{-2-i}{-2-i} \\
&= \frac{6+3i-8i+4}{4+1} \\
&= 2-i
\end{aligned}$$

$$\Rightarrow Ai+B = 2-i$$

Recall that two complex numbers are equal if and only if their corresponding real and imaginary parts are equal. This coupled with the fact that A and B are both real gives us the solution $A=-1$ and $B=2$. We omit the calculations for C and D, but it can be verified that

$$\frac{x^3+10x^2+9x+4}{(x^2+1)(x+3)(x-1)} = \frac{-x+2}{x^2+1} - \frac{1}{x+3} + \frac{3}{x-1}.$$

Lastly this trick does have limitations, specifically in the cases where a factor of the denominator is raised to a power greater than one. As our last example consider, the partial fraction decomposition of

$$\frac{x^3+6x^2-90x-41}{(x+1)^2(x-2)(x-5)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x-5}. \tag{7}$$

The term $x+1$ is raised to the second power, so one can multiply (7) by $(x+1)^2$ and find A. But one cannot multiply (7) by $x+1$ and then evaluate the resulting equation at $x=-1$ because this is a pole of the LHS and the fraction containing A. In other words we would be dividing by zero, which is undefined.

$$\frac{x^3+6x^2-90x-41}{(x+1)(x-2)(x-5)} = \frac{A}{x+1} + B + \frac{C(x+1)}{x-2} + \frac{D(x+1)}{x-5}$$

Instead we use the trick method to find that $A=3$, $C=7$, and $D=-2$, so we have

$$\frac{x^3+6x^2-90x-41}{(x+1)^2(x-2)(x-5)} = \frac{3}{(x+1)^2} + \frac{B}{x+1} + \frac{7}{x-2} - \frac{2}{x-5}$$

and now we clear the denominator to solve for B.

$$x^3+6x^2-90x-41 = 3(x-2)(x-5) + B(x+1)(x-2)(x-5) + 7(x+1)^2(x-5) - 2(x+1)^2(x-2)$$

We can solve for B by looking at the x^n terms for $n = 0, 1, 2, 3$. Let's choose the x^3 terms since we can just read those off.

$$\begin{aligned} 1 &= 0 + B + 7 - 2 \\ \Rightarrow B &= -4 \end{aligned}$$

Altogether we have

$$\frac{x^3 + 6x^2 - 90x - 41}{(x+1)^2(x-2)(x-5)} = \frac{3}{(x+1)^2} - \frac{4}{x+1} + \frac{7}{x-2} - \frac{2}{x-5}$$