

Calculating π Newton's Way

Scott Stetson

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In these notes we walk through Newton's Method of calculating π as described in Veritasium's video "The Discovery That Transformed Pi" <https://www.youtube.com/watch?v=gMlf1ELvRzc>. Obviously he did not go too deep into the mathematics, otherwise he would lose his audience, but in these notes we will. Recall that the area of a circle with radius r is $A = \pi r^2$. Therefore the unit circle in Figure 1 has area equal to π .

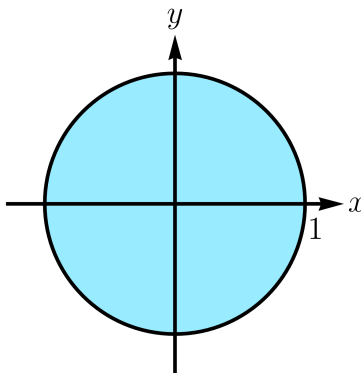


Figure 1: Unit Circle

By symmetry we know that the area of a quarter of the unit circle is equal to $\pi/4$, see Figure 2. In addition we can also express this area as the following integral.

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2} dx \quad (1)$$

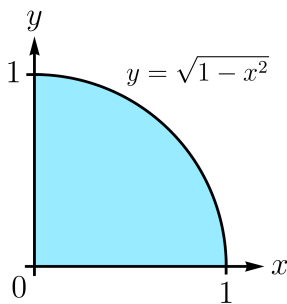


Figure 2: Quarter of the Unit Circle

Using the Taylor Series for $\sqrt{1-x^2}$ together with (1) will give us an infinite series for π hence allowing us to calculate the first n digits of π by adding more terms of the series. Instead of directly finding the Taylor Series for $\sqrt{1-x^2}$ we first find the Taylor Series for $\sqrt{1+x}$ and then make the substitution $x \mapsto -x^2$. We do this because the n^{th} derivative of $\sqrt{1+x}$ is much easier to calculate than the n^{th} derivative of $\sqrt{1-x^2}$. Only the power rule is needed for the former, while the quotient rule is eventually needed for the latter.

Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$, then the first 5 derivatives of f and their evaluation at $x=0$ are listed below.

$$\begin{array}{ll} f'(x) = \frac{1}{2}(1+x)^{-1/2} & f'(0) = \frac{1}{2} \\ f''(x) = -\frac{1}{2^2}(1+x)^{-3/2} & f''(0) = -\frac{1}{2^2} \\ f'''(x) = \frac{3}{2^3}(1+x)^{-5/2} & f'''(0) = \frac{3}{2^3} \\ f^{(4)}(x) = -\frac{3 \cdot 5}{2^4}(1+x)^{-7/2} & f^{(4)}(0) = -\frac{3 \cdot 5}{2^4} \\ f^{(5)}(x) = \frac{3 \cdot 5 \cdot 7}{2^5}(1+x)^{-9/2} & f^{(5)}(0) = \frac{3 \cdot 5 \cdot 7}{2^5} \\ \vdots & \vdots \end{array}$$

We can see three patterns arising in $f^{(n)}(0)$. First, the sign (eventually) alternates, second, the denominator is 2^n , and third, the numerator is a product of odd numbers. In order to write the formula for $f^{(n)}(0)$ we must come up with a formula for the product of the first n odd numbers. Let's take a specific example, algebraically manipulate it, and then see if we can generalize this. For instance consider

$$1 \cdot 3 \cdot 5 \cdot 7 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 4 \cdot 6} = \frac{7!}{2^3 \cdot 1 \cdot 2 \cdot 3} = \frac{(2 \cdot 3 + 1)!}{2^3 \cdot 3!}. \quad (2)$$

Similarly $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = \frac{(2 \cdot 4 + 1)!}{2^4 \cdot 4!}$ and in order to generalize this we first need a remark on notation. Just as a capital sigma, Σ , is used for the summation of a list of numbers, a capital pi, Π , denotes the product of a list of numbers, i.e.,

$$\prod_{k=1}^n a_k = a_1 a_2 a_3 \dots a_{n-1} a_n.$$

Generalizing (2) and using this product notation, we get the formula

$$\prod_{k=1}^n (2k+1) = \frac{(2n+1)!}{2^n \cdot n!}. \quad (3)$$

Equation (3) can be proved by induction, and this is left as an exercise for the readers who are familiar with proof by induction. Now we can return to writing a formula for $f^{(n)}(0)$. Putting everything we have together gives

$$\left\{ \frac{(-1)^{n-1}}{2^n} \cdot \frac{(2n+1)!}{2^n n!} \right\}_{n=0}^{\infty} = \left\{ -1, \frac{3}{2}, -\frac{3 \cdot 5}{2^2}, \frac{3 \cdot 5 \cdot 7}{2^3}, -\frac{3 \cdot 5 \cdot 7 \cdot 9}{2^4}, \dots \right\}. \quad (4)$$

But this does not agree with our calculations from above which show that

$$\{f^{(n)}(0)\}_{n=0}^{\infty} = \left\{ 1, \frac{1}{2}, -\frac{1}{2^2}, \frac{3}{2^3}, -\frac{3 \cdot 5}{2^4}, \dots \right\}.$$

The trick is to take our sequence in (4) and “shift the numerator” two places to the right. This can be achieved by dividing by the odd numbers $2n - 1$ and $2n + 1$.

$$f^{(n)}(0) = \frac{(-1)^{n-1}}{2^n} \cdot \frac{(2n+1)!}{2^n n! (2n-1)(2n+1)} = \frac{(-1)^{n-1} (2n+1)(2n)!}{4^n n! (2n-1)(2n+1)} = \frac{(-1)^{n-1} (2n)!}{4^n n! (2n-1)}$$

Recall that the Taylor Series for $f(x)$ centered at $x = 0$ is the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $a_n = f^{(n)}(0)/n!$. Hence

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} (2n)!}{4^n (n!)^2 (2n-1)} x^n \quad (5)$$

and now we find the values of x for which this series converges. By the Ratio Test we have

$$\begin{aligned} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| &= \frac{(2(n+1))! |x|^{n+1}}{4^{n+1} ((n+1)!)^2 (2(n+1)-1)} \cdot \frac{4^n (n!)^2 (2n-1)}{(2n)! |x|^n} \\ &= \frac{(2n+2)(2n+1)(2n)!}{4 \cdot \cancel{4^n} (n+1)^2 \cancel{(n!)^2} \cancel{(2n+1)}} \cdot \frac{\cancel{4^n} (n!)^2 (2n-1)}{(2n)!} |x| \\ &= \frac{2(n+1)(2n-1)}{4(n+1)^2} |x| \\ &= \frac{2n-1}{2n+2} |x| \end{aligned}$$

Therefore

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \rightarrow \infty} \frac{2n-1}{2n+2} |x| = |x|.$$

Now the Ratio Test tells us that (5) will converge if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| < 1$. From our work above we know that this limit is equal to $|x|$, so (5) converges for $|x| < 1$. It turns out that (5) also converges for $x = \pm 1$, but we omit these calculations because we are not interested in those values of x . Finally we can return to (1), our original integral.

$$\begin{aligned} \frac{\pi}{4} &= \int_0^1 \sqrt{1-x^2} dx \\ &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^{n-1} (2n)!}{4^n (n!)^2 (2n-1)} (-x^2)^n \right) dx \\ &= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{-(2n)!}{4^n (n!)^2 (2n-1)} x^{2n} \right) dx \\ &= - \sum_{n=0}^{\infty} \left(\int_0^1 \frac{(2n)! x^{2n}}{4^n (n!)^2 (2n-1)} dx \right) \\ &= - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{4^n (n!)^2 (2n-1)} \int_0^1 x^{2n} dx \right) \\ &= - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{4^n (n!)^2 (2n-1)} \cdot \frac{x^{2n+1}}{2n+1} \Big|_0^1 \right) \end{aligned}$$

$$\begin{aligned}
&= - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n-1)(2n+1)} \\
&= - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (4n^2 - 1)}
\end{aligned}$$

Hence

$$\Rightarrow \pi = -4 \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (4n^2 - 1)} \quad (6)$$

Newton could have stopped here, but being the genius that he was, he found another infinite series for π , one that converged much faster. Instead of integrating $\sqrt{1-x^2}$ from 0 to 1 he integrated this function from 0 to $1/2$ (Figure 3) and broke up the area as shown in Figure 4.

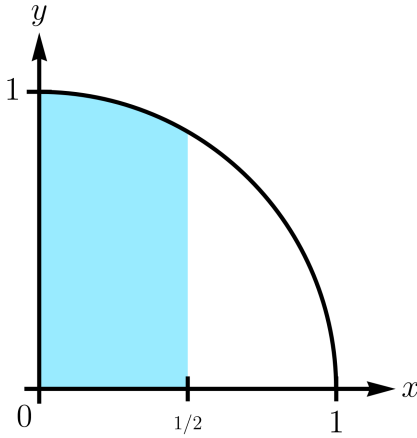


Figure 3: 0 to $1/2$

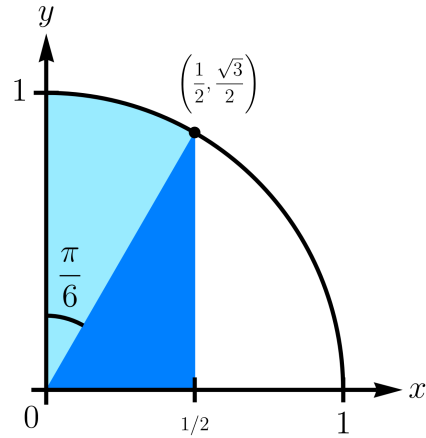


Figure 4: Sector and Triangle

Geometrically we have

$$\begin{aligned}
\int_0^{1/2} \sqrt{1-x^2} dx &= \text{Area of the Sector} + \text{Area of the Triangle} \\
&= \frac{1}{2} \theta r^2 + \frac{1}{2} b h \\
&= \frac{1}{2} \cdot \frac{\pi}{6} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{8}.
\end{aligned} \quad (7)$$

On the other hand we have

$$\begin{aligned}
\int_0^{1/2} \sqrt{1-x^2} dx &= \int_0^{1/2} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n-1} (2n)!}{4^n (n!)^2 (2n-1)} (-x^2)^n \right) dx \\
&= - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{4^n (n!)^2 (2n-1)} \cdot \frac{x^{2n+1}}{2n+1} \Big|_0^{1/2} \right) \\
&= - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{4^n (n!)^2 (2n-1)} \cdot \frac{1}{2 \cdot 4^n (2n+1)} \right) \\
&= - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (4n^2-1)}. \tag{8}
\end{aligned}$$

Equating (7) and (8) yields

$$\begin{aligned}
\frac{\pi}{12} + \frac{\sqrt{3}}{8} &= - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (4n^2-1)} \\
\pi &= 12 \left(- \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (4n^2-1)} - \frac{\sqrt{3}}{8} \right) \\
\pi &= -6 \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (4n^2-1)} - \frac{3\sqrt{3}}{2} \tag{9}
\end{aligned}$$

Equation (9) is the other infinite series for π that Newton found and it converges much faster than (6). However, Newton did not have a calculator so we must find an infinite series for $\sqrt{3}$. While it is true that $\sqrt{3} = \sqrt{1+2}$, we cannot plug in $x = 2$ into the Taylor Series for $\sqrt{1+x}$ since this series converges for $|x| \leq 1$. Instead we note that

$$\begin{aligned}
\sqrt{3} &= \sqrt{4-1} = \sqrt{4 \left(1 - \frac{1}{4} \right)} = 2 \sqrt{1 - \frac{1}{4}} = 2 \sum_{n=0}^{\infty} \frac{(-1)^{n-1} (2n)!}{4^n (n!)^2 (2n-1)} \left(-\frac{1}{4} \right)^n \\
&= -2 \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (2n-1)}.
\end{aligned}$$

Now we can substitute this series for $\sqrt{3}$ into (9).

$$\begin{aligned}
\pi &= -6 \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (4n^2-1)} - \frac{3\sqrt{3}}{2} \\
&= -6 \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (4n^2-1)} + 3 \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (2n-1)} \\
&= -6 \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (4n^2-1)} + 3 \sum_{n=0}^{\infty} \frac{(2n)!(2n+1)}{16^n (n!)^2 (4n^2-1)} \\
&= 3 \sum_{n=0}^{\infty} \frac{(2n)!(2n+1-2)}{16^n (n!)^2 (4n^2-1)} \\
&= 3 \sum_{n=0}^{\infty} \frac{(2n)!(2n-1)}{16^n (n!)^2 (2n-1)(2n+1)} \\
&= 3 \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (2n+1)}
\end{aligned}$$

$$\pi = 3 \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (2n+1)}$$

As mentioned earlier, this infinite series for π converges much quicker than (6) which can be seen in the table below. Note: π to ten places after the decimal is 3.1415926535.

k	$4 \sum_{n=0}^k \frac{-(2n)!}{4^n (n!)^2 (4n^2 - 1)}$	$3 \sum_{n=0}^k \frac{(2n)!}{16^n (n!)^2 (2n+1)}$
0	4.0000000000	3.0000000000
1	3.3333333333	3.1250000000
2	3.2333333333	3.1390625000
3	3.1976190476	3.1411551339
4	3.1802579365	3.1415111723
5	3.1703147547	3.1415767158
6	3.1640046585	3.1415894253
7	3.1597077835	3.1415919824
8	3.1566273033	3.1415925112
9	3.1543304540	3.1415926229
10	3.1525640675	3.1415926469
11	3.1511712054	3.1415926521
12	3.1500499515	3.1415926533
13	3.1491315454	3.1415926535

Figure 5: Approximations of π