# Calculating $\pi$ Newton's Way

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In these notes we walk through Newton's Method of calculating $\pi$ as described in Veritasium's video "The Discovery That Transformed Pi" https://www.youtube.com/watch?v=gMlf1ELvRzc. Obviously he did not go too deep into the mathematics, otherwise he would loose his audience, but in these notes we will. Recall that the area of a circle with radius $r$ is $A = \pi r^2$. Therefore the unit circle in Figure 1 has area equal to $\pi$.

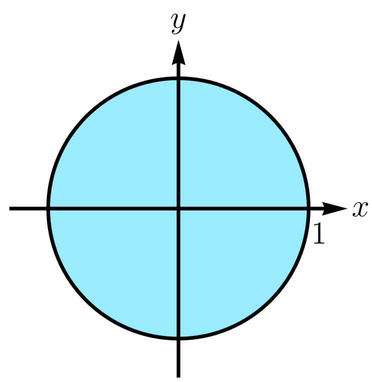


Figure 1: Unit Circle

By symmetry we know that the area of a quarter of the unit circle is equal to $\pi/4$, see Figure 2. In addition we can also express this area as the following integral.

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}dx \tag{1}$$

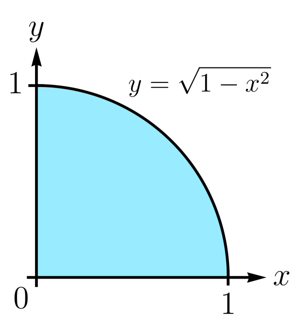


Figure 2: Quarter of the Unit Circle

Using the Taylor Series for $\sqrt{1-x^2}$ together with (1) will give us an infinite series for $\pi$ hence allowing us to calculate the first $n$ digits of $\pi$ by adding more terms of the series. Instead of directly finding the Taylor Series for $\sqrt{1-x^2}$ we first find the Taylor Series for $\sqrt{1+x}$ and then make the substitution $x \mapsto -x^2$. We do this because the $n^{\text{th}}$ derivative of $\sqrt{1+x}$ is much easier to calculate than the $n^{\text{th}}$ derivative of $\sqrt{1-x^2}$. Only the power rule is needed for the former, while the quotient rule is eventually needed for the latter.

Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$, then the first 5 derivatives of $f$ and their evaluation at $x = 0$ are listed below.

$$\begin{aligned}
f'(x) &= \frac{1}{2}(1+x)^{-1/2} & f'(0) &= \frac{1}{2} \\
f''(x) &= -\frac{1}{2^2}(1+x)^{-3/2} & f''(0) &= -\frac{1}{2^2} \\
f'''(x) &= \frac{3}{2^3}(1+x)^{-5/2} & f'''(0) &= \frac{3}{2^3} \\
f^{(4)}(x) &= -\frac{3\cdot 5}{2^4}(1+x)^{-7/2} & f^{(4)}(0) &= -\frac{3\cdot 5}{2^4} \\
f^{(5)}(x) &= \frac{3\cdot 5\cdot 7}{2^5}(1+x)^{-9/2} & f^{(5)}(0) &= \frac{3\cdot 5\cdot 7}{2^5} \\
&\vdots \qquad \vdots & &\vdots \qquad \vdots
\end{aligned}$$

We can see three patterns arising in $f^{(n)}(0)$. First, the sign (eventually) alternates, second, the denominator is $2^n$, and third, the numerator is a product of odd numbers. In order to write the formula for $f^{(n)}(0)$ we must come up with a formula for the product of the first $n$ odd numbers. Let's take a specific example, algebraically manipulate it, and then see if we can generalize this. For instance consider

$$1\cdot 3\cdot 5\cdot 7 = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{2\cdot 4\cdot 6} = \frac{7!}{2^3\cdot 1\cdot 2\cdot 3} = \frac{(2\cdot 3+1)!}{2^3\cdot 3!}. \tag{2}$$

Similarly $1\cdot 3\cdot 5\cdot 7\cdot 9 = \dfrac{(2\cdot 4+1)!}{2^4\cdot 4!}$ and in order to generalize this we first need a remark on notation. Just as a capital sigma, $\Sigma$, is used for the summation of a list of numbers, a capital pi, $\Pi$, denotes the product of a list of numbers, i.e.,

$$\prod_{k=1}^{n} a_k = a_1 a_2 a_3 \ldots a_{n-1} a_n.$$

Generalizing (2) and using this product notation, we get the formula

$$\prod_{k=1}^{n}(2k+1) = \frac{(2n+1)!}{2^n\cdot n!}. \tag{3}$$

Equation (3) can be proved by induction, and this is left as an exercise for the readers who are familiar with proof by induction. Now we can return to writing a formula for $f^{(n)}(0)$. Putting everything we have together gives

$$\left\{\frac{(-1)^{n-1}}{2^n}\cdot\frac{(2n+1)!}{2^n n!}\right\}_{n=0}^{\infty} = \left\{-1, \frac{3}{2}, -\frac{3\cdot 5}{2^2}, \frac{3\cdot 5\cdot 7}{2^3}, -\frac{3\cdot 5\cdot 7\cdot 9}{2^4}, \ldots\right\}. \tag{4}$$

But this does not agree with our calculations from above which show that

$$\{f^{(n)}(0)\}_{n=0}^{\infty} = \left\{1, \frac{1}{2}, -\frac{1}{2^2}, \frac{3}{2^3}, -\frac{3\cdot 5}{2^4}, \ldots\right\}.$$

The trick is to take our sequence in (4) and "shift the numerator" two places to the right. This can be achieved by dividing by the odd numbers $2n-1$ and $2n+1$.

$$f^{(n)}(0) = \frac{(-1)^{n-1}}{2^n} \cdot \frac{(2n+1)!}{2^n n!(2n-1)(2n+1)} = \frac{(-1)^{n-1}(2n+1)(2n)!}{4^n n!(2n-1)(2n+1)} = \frac{(-1)^{n-1}(2n)!}{4^n n!(2n-1)}$$

Recall that the Taylor Series for $f(x)$ centered at $x=0$ is the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $a_n = f^{(n)}(0)/n!$. Hence

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{4^n (n!)^2 (2n-1)} x^n \tag{5}$$

and now we find the values of $x$ for which this series converges. By the Ratio Test we have

$$\begin{aligned}
\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| &= \frac{(2(n+1))!|x|^{n+1}}{4^{n+1}((n+1)!)^2(2(n+1)-1)} \cdot \frac{4^n (n!)^2 (2n-1)}{(2n)!|x|^n} \\
&= \frac{(2n+2)\cancel{(2n+1)}\cancel{(2n)!}}{4 \cdot \cancel{4^n}(n+1)^2\cancel{(n!)^2}\cancel{(2n+1)}} \cdot \frac{\cancel{4^n}\cancel{(n!)^2}(2n-1)}{\cancel{(2n)!}}|x| \\
&= \frac{2\cancel{(n+1)}(2n-1)}{4(n+1)^{\cancel{2}}}|x| \\
&= \frac{2n-1}{2n+2}|x|
\end{aligned}$$

Therefore

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\frac{2n-1}{2n+2}|x| = |x|.$$

Now the Ratio Test tells us that (5) will converge if $\lim_{n\to\infty}|\frac{a_{n+1}x^{n+1}}{a_n x^n}| < 1$. From our work above we know that this limit is equal to $|x|$, so the series in (5) converges for $|x| < 1$. One can also show that this series converges for $x = \pm 1$. Finally we can return to (1), our original integral.

$$\begin{aligned}
\frac{\pi}{4} &= \int_0^1 \sqrt{1-x^2}\,dx \\
&= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{4^n(n!)^2(2n-1)}(-x^2)^n\right) dx \\
&= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{-(2n)!}{4^n(n!)^2(2n-1)}x^{2n}\right) dx \\
&= -\sum_{n=0}^{\infty}\left(\int_0^1 \frac{(2n)!x^{2n}}{4^n(n!)^2(2n-1)}dx\right) \\
&= -\sum_{n=0}^{\infty}\left(\frac{(2n)!}{4^n(n!)^2(2n-1)}\int_0^1 x^{2n}dx\right) \\
&= -\sum_{n=0}^{\infty}\left(\frac{(2n)!}{4^n(n!)^2(2n-1)} \cdot \frac{x^{2n+1}}{2n+1}\Bigg|_0^1\right)
\end{aligned}$$

$$= -\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n-1)(2n+1)}$$

$$= -\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (4n^2-1)}$$

Hence

$$\Rightarrow \boxed{\pi = -4 \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (4n^2-1)}} \tag{6}$$

Newton could have stopped here, but being the genius that he was, he found another infinite series for $\pi$, one that converged much faster. Instead of integrating $\sqrt{1-x^2}$ from 0 to 1 he integrated this function from 0 to 1/2 (Figure 3) and broke up the area as shown in Figure 4.

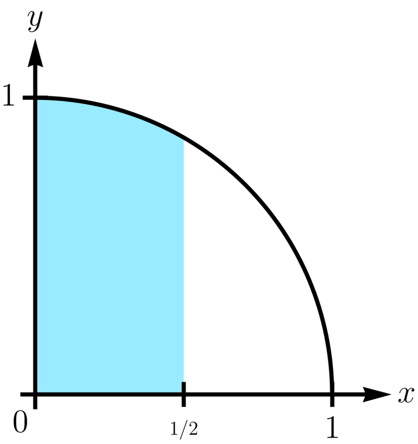


Figure 3: 0 to 1/2

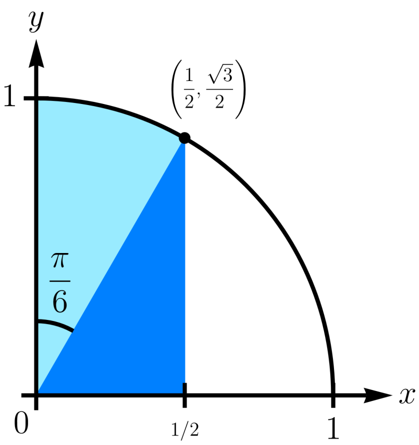


Figure 4: Sector and Triangle

Geometrically we have

$$\begin{aligned}
\int_0^{1/2} \sqrt{1-x^2}dx &= \text{Area of the Sector } + \text{ Area of the Triangle} \\
&= \frac{1}{2}\theta r^2 + \frac{1}{2}bh \\
&= \frac{1}{2} \cdot \frac{\pi}{6} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{8}. \qquad (7)
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\int_0^{1/2} \sqrt{1-x^2}dx &= \int_0^{1/2} \left( \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{4^n(n!)^2(2n-1)}(-x^2)^n \right) dx \\
&= -\sum_{n=0}^{\infty} \left( \frac{(2n)!}{4^n(n!)^2(2n-1)} \cdot \frac{x^{2n+1}}{2n+1}\bigg|_0^{1/2} \right) \\
&= -\sum_{n=0}^{\infty} \left( \frac{(2n)!}{4^n(n!)^2(2n-1)} \cdot \frac{1}{2\cdot 4^n(2n+1)} \right) \\
&= -\frac{1}{2}\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(4n^2-1)}. \qquad (8)
\end{aligned}$$

Equating (7) and (8) yields

$$\begin{aligned}
\frac{\pi}{12} + \frac{\sqrt{3}}{8} &= -\frac{1}{2}\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(4n^2-1)} \\
\pi &= 12\left( -\frac{1}{2}\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(4n^2-1)} - \frac{\sqrt{3}}{8} \right) \\
\pi &= -6\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(4n^2-1)} - \frac{3\sqrt{3}}{2} \qquad (9)
\end{aligned}$$

Equation (9) is the other infinite series for $\pi$ that Newton found and it converges much faster than (6). However, Newton did not have a calculator so we must find an infinite series for $\sqrt{3}$. While it is true that $\sqrt{3} = \sqrt{1+2}$, we cannot plug in $x = 2$ into the Taylor Series for $\sqrt{1+x}$ since this series converges for $|x| \leq 1$. Instead we note that

$$\begin{aligned}
\sqrt{3} = \sqrt{4-1} = \sqrt{4\left(1-\frac{1}{4}\right)} = 2\sqrt{1-\frac{1}{4}} &= 2\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{4^n(n!)^2(2n-1)}\left(-\frac{1}{4}\right)^n \\
&= -2\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(2n-1)}.
\end{aligned}$$

Now we can substitute this series for $\sqrt{3}$ into (9).

$$\begin{aligned}
\pi &= -6\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(4n^2-1)} - \frac{3\sqrt{3}}{2} \\
&= -6\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(4n^2-1)} + 3\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(2n-1)} \\
&= -6\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(4n^2-1)} + 3\sum_{n=0}^{\infty} \frac{(2n)!(2n+1)}{16^n(n!)^2(4n^2-1)} \\
&= 3\sum_{n=0}^{\infty} \frac{(2n)!(2n+1-2)}{16^n(n!)^2(4n^2-1)} \\
&= 3\sum_{n=0}^{\infty} \frac{(2n)!(2n-1)}{16^n(n!)^2(2n-1)(2n+1)} \\
&= 3\sum_{n=0}^{\infty} \frac{(2n)!}{16^n(n!)^2(2n+1)}
\end{aligned}$$

$$\boxed{\pi = 3\sum_{n=0}^{\infty} \frac{(2n)!}{16^n (n!)^2 (2n+1)}}$$

As mentioned earlier, this infinite series for $\pi$ converges much quicker than (6) which can be seen in the table below. Note: $\pi$ to ten places after the decimal is 3.1415926535.

| $k$ | $4\sum_{n=0}^{k} \frac{-(2n)!}{4^n (n!)^2 (4n^2-1)}$ | $3\sum_{n=0}^{k} \frac{(2n)!}{16^n (n!)^2 (2n+1)}$ |
|---|---|---|
| 0 | 4.0000000000 | 3.0000000000 |
| 1 | 3.3333333333 | 3.1250000000 |
| 2 | 3.2333333333 | 3.1390625000 |
| 3 | 3.1976190476 | 3.1411551339 |
| 4 | 3.1802579365 | 3.1415111723 |
| 5 | 3.1703147547 | 3.1415767158 |
| 6 | 3.1640046585 | 3.1415894253 |
| 7 | 3.1597077835 | 3.1415919824 |
| 8 | 3.1566273033 | 3.1415925112 |
| 9 | 3.1543304540 | 3.1415926229 |
| 10 | 3.1525640675 | 3.1415926469 |
| 11 | 3.1511712054 | 3.1415926521 |
| 12 | 3.1500499515 | 3.1415926533 |
| 13 | 3.1491315454 | 3.1415926535 |

Figure 5: Approximations of $\pi$