2

Random Variables

"No Human Investigation can be called Real Science if it cannot be demonstrated Mathematically"

In the previous chapter we have discussed the idea of sample space which characterizes completely a random experiment. However, sample space may be tedious to describe if it is non-numeric. Also, in most applied problems involving probabilities one is interested only in a particular aspect (or more) of the outcomes of random experiment. We shall now discuss how we may formulate a rule or a set of rules by which the elements of a sample space $\boldsymbol{S}$ may be represented by numbers (Section 2.1) or ordered pairs of numbers (Section 2.2). We start this discussion with a definition of a function over the points of a sample space which is so called *random variable*.

2.1 UNIVARIATE RANDOM VARIABLES

2.1.1 Definition

Give a random experiment with sample space $\boldsymbol{S}$. A function X, which assigns to each element s in $\boldsymbol{S}$ one and only one real number $X(s) = x$, is called a random variable. The range or space of X is the set of real numbers $\boldsymbol{A}$ such that $\boldsymbol{A} = \{x \epsilon R / X(x) = s, s \epsilon \boldsymbol{S}\}$

If the set $\boldsymbol{S}$ has elements which are themselves real numbers, then we could write $X(c) = c$, so that $\boldsymbol{A} = \boldsymbol{S}$. On the other hand, if $\boldsymbol{S}$ has non-numerical elements then a random variable assigns a numerical value (of course, unique) to each of the elements of $\boldsymbol{S}$.

■ Example 2.1

Let a coin be tossed thrice

$$\therefore \quad \boldsymbol{S} = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}$$

For simplicity we assume that $\boldsymbol{S}_i (i = 1 \text{ to } 8)$ denotes the i^{th} outcomes i.e., i^{th} elements in $\boldsymbol{S}$. If our interest is in the number of heads in the i^{th} outcome (i = 1 to 8).

$$\begin{aligned} \text{i.e.,} \qquad X(s_i) &= \text{Number of heads in the } i^{\text{th}} \text{ outcome} \\ \therefore \qquad X(s_1) &= 3 \quad X(s_2) = X(s_3) = X(s_4) = 2 \\ X(s_5) &= X(s_6) = X(s_7) = 1 \quad X(s_8) = 0 \end{aligned}$$

Hence $\qquad A = \{0, 1, 2, 3\}.$ ■

■ Example 2.2

Let us choose a real number randomly in an interval say (0, 1). In this case S itself has real numbers as its elements.

$$\therefore \quad \text{we define} \quad X : S \to R \quad \text{s.t}$$
$$X(s) = s \quad \text{so that} \quad \boldsymbol{A} = (0, 1) \text{ or } S.$$ ■

■ Example 2.3

If a person randomly answers a 'True or False' question set then the sample space $\boldsymbol{S} = \{$'True', 'False'$\}$. In this case, define $X : S \to R$ as X(True) = 1 and X(False) = 0 so that $\boldsymbol{A} = \{0, 1\}$. ■

■ Example 2.4

In Example 2.3, if we are interested to count the number of times the person answered 'True' thus $\boldsymbol{S} = \{0, 1, 2, 3 \cdots\}$ (assuming that number of Questions is unknown). So that $\boldsymbol{S}$ has real numbers.

$$\therefore \quad \boldsymbol{A} = \boldsymbol{S}$$ ■

Discussion

1. Examples 2.1 and 2.3 show how non-numerical elements are assigned with real numbers and Examples 2.2 and 2.4 are for numerical elements in sample space.
2. Examples 2.1 and 2.3 have finite space $\boldsymbol{A}$ and Example 2.4 has a countably infinite space where as Example 2.2 has uncountably infinite space $\boldsymbol{A}$. This idea formulates the two different types of random variables, as

1. Discrete random variables

Let X be a random variable with one-dimensional space $\boldsymbol{A}$. If $\boldsymbol{A}$ has only a finite number of different values $x_1, x_2 \cdots x_k$ or at most countably infinite sequence of different values $x_1, x_2 \cdots$ then X is said to be discrete random variables.

2. Continuous random variables

Let X be a random variable with one-dimensional space $\boldsymbol{A}$. If $\boldsymbol{A}$ is a uncountably infinite set (as every value in an interval) then X is said to be continuous random variable.

In general, countable quantities can be regarded as discrete random variables whereas measurable quantities (such as height, weight, voltage) can be considered as continuous random variables, of course, the variable in the problem possesses random nature.

Also, one can classify the nature of a random variable based only on the range $\boldsymbol{A}$ of X.

2.1.2 Probability Functions

For each random variable X we have a set of real numbers $\boldsymbol{A}$. X assumes its value in $\boldsymbol{A}$ we could calculate the probability that X takes its value in $\boldsymbol{A}$. The collection of these probabilities is the distribution of X. Also, these distributions can be described by what we will call as probability Density Function (pdf).

For any random variable X and its space $\boldsymbol{A}$ we call a new function $f(x)$ for all $x \epsilon \boldsymbol{A}$ which has to satisfy the two conditions viz. (a) non-negativity, (b) Totality of $f(x) = 1$. Now, let use define precisely for the two types of random variables.

1. Discrete random variables

Let X be a discrete random variable with space $\boldsymbol{A}$. Then $f(x)$ is defined on $\boldsymbol{A}$ such that

(a) $f(x) \geq 0$,

(b) $\sum\limits_{x \epsilon A} f(x) = 1$.

2. Continuous random variables

Let X be a continuous random variable with space $\boldsymbol{A}$. Then $f(x)$ is defined on $\boldsymbol{A}$ such that

(a) $f(x) \geq 0$,

(b) $\int\limits_{A} f(x)dx = 1$.

***Note*:** We can simplify the way in which we denote the pdf and range of a random variable (continuous or discrete).

i.e., if X is a random variable with pdf $f(x)$ and space $\boldsymbol{A}$, then we may treat the entire real axis $(-\infty, \infty)$ as its space.

$$\therefore \quad \text{we replace } \int\limits_{A} f(x)dx \text{ by } \int\limits_{-\infty}^{\infty} f(x)dx \text{ and } \sum\limits_{A} f(x) \text{ by } \sum\limits_{x} f(x)$$

In practice, the probability function $f(x)$ of a discrete random variable is called as probability mass function (pmf) and that of a continuous random variable we call it as a probability density function (pdf).

It is seen that whether the random variable X is of the discrete type or of the continuous type, the probability $p(X \epsilon A)$ where $A \subseteq \boldsymbol{A}$ is completely determined by a function $f(x)$. On either case, we may work exclusively with the probability function $f(x)$.

Calculating normalizing constants

The first calculation based on pdfs is finding an unknown constant which may exist in a random variable. Such constants can be calculated easily by using (b) at in the definition of a random variable.

■ Example 2.5

If the pmf or pdf of a random variable X is as follows:

(a) $$f(x) = \begin{cases} Cx^2 & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$f(x) = \begin{cases} Ce^{-2x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(c) $$f(x) = \begin{cases} Cx & x = 1, 2, 3, 4, 5 \\ 0 & \text{elsewhere} \end{cases}$$

(d) $$f(x) = \begin{cases} C & x = 1, 2, 3, 10 \\ 0 & \text{elsewhere} \end{cases}$$

► Solution

(a) X is a continuous random variable since $\boldsymbol{A} = \{1, 2\}$.

$$\therefore \quad \int_{\boldsymbol{A}} f(x)dx = 1 \quad \Rightarrow \quad \int_1^2 Cx^2 dx = 1$$

$$\Rightarrow \quad C\left(\frac{x^3}{3}\right)_1^2 = 1$$

$$\Rightarrow \quad C\left(\frac{8-1}{3}\right) = 1$$

$$\Rightarrow \quad C = \frac{3}{7}$$

(b) In this case $\boldsymbol{A} = (0, \infty)$, so that X is a continuous random variable, so that

$$\int_{\boldsymbol{A}} f(x)dx = 1 \quad \Rightarrow \quad \int_0^\infty Ce^{-2x} dx = 1$$

$$\Rightarrow \quad C\left(\frac{e^{-2x}}{-2}\right)_0^\infty = 1$$

$$\Rightarrow \quad C\left(\frac{1-0}{2}\right) = 1$$

$$\Rightarrow \quad C = 2$$

(c) Here $\boldsymbol{A} = \{1, 2, 3, 4, 5\}$, hence X is a discrete random variable, so that

$$\sum_{x \epsilon A} f(x) = 1 \quad \Rightarrow \quad C(1) + C(2) + C(3) + C(4) + C(5) = 1$$

$$\Rightarrow \quad C[1 + 2 + 3 + 4 + 5] = 1$$

$$\Rightarrow \quad C = \frac{1}{15}$$

(d) Here $\boldsymbol{A} = \{1, 2, 3, \cdots, 10\} X$ is a discrete random variable so that $\sum_{x \epsilon A} f(x) = 1$

$$\Rightarrow \sum_{x=1}^{10} C = 1 \quad \Rightarrow \quad C = \frac{1}{10}$$

In our subsequent discussions, finding this unknown normalizing constant may not be a part of a problem. In spite of this, we need to calculate such constants. Also, it is customary to denote the probability distribution of a discrete random variable in a tabulated arrangements as follows.

Consider Examples 2.5 (c) and (d)

(c)

$X = x$	1	2	3	4	5
$f(x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(d)

$X = x$	1	2	3	4	5	6	7	8	9	10
$f(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

■

Finding probabilities

Let X be a random variable with space $\boldsymbol{A}$. Let A be a subset of A. We define $p(X \epsilon A)$ as the probability that X takes only on the values of A.

i.e., with A a subset of $\boldsymbol{A}$, let S be that subset of the sample space $\boldsymbol{S}$ such that

$$\boldsymbol{S} = \{x \epsilon S / X(x) \epsilon A\}$$

Thus s has as its elements all outcomes in S for which the random variable X has a value that is in A. Thus $p(X \epsilon A)$ is an assignment of probability to a set A, which is a subset of the space $\boldsymbol{A}$ associated with the random variable X. This assignment is determined by the pdf of X. This idea can be illustrated now.

Consider Example 2.5 (a) where $\boldsymbol{A} = [1, 2]$. Let $A = (1, 1.5)$. Then $P(X \epsilon A)$ is the probability that X takes values between 1 and 1.5. Also, if we consider Example 2.5 (d) where $\boldsymbol{A} = \{1, 2, 3, \cdots, 10\}$. Let $A = \{2, 4, 6, 8, 10\}$ a subset of $\boldsymbol{A}$. Then $p(X \epsilon A)$ is the probability that X takes values in A i.e., even numbers from 1 to 10.

We may summarize the various ways through which probabilities of an event can be calculated.

If X is a continuous random variable, and a and b any two real constants, then

$$\begin{aligned} (1) \quad pr(a < X < b) &= p(a \leq X < b) = p(a < X \leq b) \\ &= p(a \leq X \leq b) = \int_a^b f(x) dx \end{aligned}$$

$$(2) \quad p(X < a) = p(X \le a) = \int_{-\infty}^{a} f(x)dx$$

$$(3) \quad p(X > a) = p(X \ge a) = \int_{a}^{\infty} f(x)dx$$

For a discrete random variable the integration in the above formulas may be replaced with summation over the appropriate range. But it is important to note that strict inequalities and other inequalities play different roles in a discrete random variable.

If $A = \{x; x = a\}$ and X is a discrete random variable then $p(A) = p(X = a)$ may be defined. But if X is a continuous random variable then $p(A) = \int_a^a f(x)dx = 0$. So that all the results in (1), (2), (3) are valid only for continuous random variable.

■ Example 2.6

Let
$$f(x) = \begin{cases} Kx & x = 1, 2, 3, 4 \\ 0 & \text{elsewhere} \end{cases}$$

be the pmf of X. Find $P(X = 1 \text{ or } 2)$, $P\left(\frac{1}{2} < X < \frac{5}{2}\right)$, $P(1 \le X \le 2)$

First let us find 'K'. Here $\boldsymbol{A} = \{1, 2, 3, 4\}$ so that X is discrete.

$$\therefore \quad \sum_{x \epsilon A} f(x) = 1 \Rightarrow \sum_{x=1}^{4} Kx = 1$$
$$\Rightarrow K(1) + K(2) + K(3) + K(4) = 1$$
$$\Rightarrow K = \frac{1}{10}$$

$\therefore$ Probability distribution of X is

$X = x$	1	2	3	4
$f(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Now
$$P(X = 1 \text{ or } 2) = P(X = 1) + P(X = 2)$$
$$= \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$$
$$P\left(\frac{1}{2} < X < \frac{5}{2}\right) = P(X = 1, 2) = \frac{3}{10}$$
$$P(1 \le X \le 2) = P(X = 1, 2) = \frac{3}{10}$$

■

■ Example 2.7

Let X be a random variable with space $\boldsymbol{A} = \{1, 2, 3, \cdots\}$ and let

$$f(x) = \begin{cases} K^x & x \epsilon \boldsymbol{A} \\ 0 & \text{elsewhere} \end{cases}$$

Then find (a) $p(X$ is even), (b) $p(X$ is a multiple of 3), (c) $p(X \epsilon A)$, $\boldsymbol{A} = \{1, 3, 5, 7 \cdots\}$.

To find K: X is discrete random variable

$$\begin{aligned} \therefore \quad \sum_{X \epsilon \boldsymbol{A}} f(x) = 1 \quad &\Rightarrow \sum_{x=1,2,\cdots}^{\infty} K^x = 1 \\ &\Rightarrow K + K^2 + K^3 + \cdots = 1 \\ &\Rightarrow K\left[1 + K + K^2 + \cdots\right] = 1 \\ &\Rightarrow K\frac{1}{1-K} = 1 \\ &\Rightarrow K = 1 - K \\ &\Rightarrow K = \frac{1}{2} \end{aligned}$$

$\therefore$ The pdf of X is

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & x = 1, 2, 3, \cdots \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \text{(a)} \quad p(X \text{ is even}) &= p(X = 2, 4, 6, \cdots) \\ &= p(X = 2) + p(X = 4) + p(X = 6) + \cdots \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots \\ &= \left(\frac{1}{2}\right)^2 \left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots\right] \\ &= \frac{1}{4}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right] \\ &= \frac{1}{4}\left[\frac{1}{1 - 1/4}\right] = \frac{1}{3} \end{aligned}$$

(b) $p(X$ is a multiple of $3) = p(X = 3, 6, 9, 12, \cdots)$

$$\begin{aligned} &= p(X = 3) + p(X = 6) + p(X = 9) + \cdots \\ &= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \cdots \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{2}\right)^3\left[1+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^6+\cdots\right]\\
&= \frac{1}{8}\left[1+\frac{1}{8}+\left(\frac{1}{8}\right)^2+\cdots\right]\\
&= \frac{1}{8}\left[\frac{1}{1-1/8}\right]=\frac{1}{7}
\end{aligned}$$

$$\begin{aligned}
\text{(c)}\quad p(X\epsilon A) &= p(X=1,3,5,7,\cdots)\\
&= p(X=1)+p(X=3)+p(X=5)+\cdots\\
&= \frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^7+\cdots\\
&= \frac{1}{2}\left[1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^6+\cdots\right]=\frac{2}{3}
\end{aligned}$$

■

■ Example 2.8

A random variable has the following probability distribution.

$X=x$	-2	-1	0	1	2	3
$f(x)$	a	a	$2a$	$2a$	$3a$	a

Find the constant a and $p(X>0)$ $p(X$ is atleast 0) $p(X$ is atmost 2).
Here $\boldsymbol{A}=\{-2,-1,0,1,2,3\}$ so that X is a discrete random variable.

$$\begin{aligned}
\therefore\quad \sum_{x\epsilon \boldsymbol{A}} f(x) &= 1\\
\therefore\quad a+a+2a+2a+3a+a &= 1\\
a &= \frac{1}{10}
\end{aligned}$$

$\therefore$ Probability distribution of X is

$X=x$	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{10}$

$$\begin{aligned}
p(X>0) &= p(X=1,2,3)=p(X=1)+p(X=2)+p(X=3)\\
&= \frac{2}{10}+\frac{3}{10}+\frac{1}{10}=\frac{6}{10}
\end{aligned}$$

$$\begin{aligned} p(X \text{is atleast } 0) &= p(X \geq 0) = \frac{8}{10} \\ p(X \text{is atmost } 2) &= p(X \leq 2) = p(X = -2, -1, 0, 1, 2) \\ &= 1 - p(X > 2) = 1 - p(X = 3) \\ &= 1 - \frac{1}{10} = \frac{9}{10} \end{aligned}$$

■

■ Example 2.9

Let the probability distribution of a random variable X be as follows.

$\boldsymbol{A} = \{0, 1, 2\}$ and $f(0) = 3C^3$. $f(1) = 4C - 10C^2$ $f(2) = 5C - 1$ for some $C > 0$ find $p(X < 2) p(1 < X \leq 2)$.

Given $\boldsymbol{A} = \{0, 1, 2\}$ so that X is a discrete random variable.

$$\begin{aligned} \therefore \quad \sum_{x \epsilon \boldsymbol{A}} f(x) = 1 \quad &\Rightarrow \quad 3C^3 + 4C - 10C^2 + 5C - 1 = 1 \\ &\Rightarrow \quad 3C^3 + 9C - 10C^2 - 2 = 0 \end{aligned}$$

Solving this cubic equation, we get

$$C = 1, 2, \frac{1}{3}$$

$C = 1$ or 2 results with impossible probability distribution (since if $C = 1$, $f(0) = 3C^3 = 3$ which is impossible).

$\therefore \quad C = \frac{1}{3}$, hence the probability distribution at X is

$X = x$	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$

Now
$$\begin{aligned} p(X < 2) &= \frac{3}{9} \quad \text{and} \\ p(1 < X \leq 2) &= p(X = 2) = \frac{2}{3} \end{aligned}$$

■

■ Example 2.10

Consider the probability distribution of a random variable X $f(x) = \frac{K}{2^x}$ $x = 0, 1, 2, 3, 4$. Find $p(X \geq 2/X < 4)$; $p(X \leq 3/X > 1)$.

Here $\boldsymbol{A} = \{0, 1, 2, 3, 4\}$ so that X is a discrete random variable, so that

$$\sum_{x \epsilon \boldsymbol{A}} f(x) = 1 \quad \Rightarrow \quad \sum_{x=0}^{4} \frac{K}{2^x} = 1$$

$$\Rightarrow \quad K + \frac{K}{2} + \frac{K}{4} + \frac{K}{8} + \frac{K}{16} = 1$$

$$\Rightarrow \quad K = \frac{16}{31}$$

$\therefore$ The probability distribution of X is

$X = x$	0	1	2	3	4
$f(x)$	$\frac{16}{31}$	$\frac{8}{31}$	$\frac{4}{31}$	$\frac{2}{31}$	$\frac{1}{31}$

$$\begin{aligned} p(X \geq 2/X < 4) &= \frac{p(X \geq 2 \cap X < 4)}{p(X < 4)} = \frac{p(2 \leq X < 4)}{P(X < 4)} \\ &= \frac{p(X = 2, 3)}{1 - p(X = 4)} = \frac{6}{30} \\ p(X \leq 3/X > 1) &= \frac{p(X \leq 3 \cap X > 1)}{p(X > 1)} = \frac{p(1 < X \leq 3)}{p(X > 1)} \\ &= \frac{p(X = 2, 3)}{1 - p(X \leq 1)} = \frac{6}{7} \end{aligned}$$

■

■ Example 2.11

If the pdf of a random variable is

$$f(x) = \begin{cases} Ax^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability that (a) X lies between 0.2 and 0.5, (b) X is less than 0.3 and (c) X is greater than $\frac{3}{4}$ given that X is greater than $\frac{1}{2}$.

Here $\boldsymbol{A} = (0, 1)$ so that X is a continuous random variable.

$$\therefore \quad \int_{x \in \boldsymbol{A}} f(x) = 1 \quad \Rightarrow \quad \int_0^1 Ax^2 dx = 1$$

$$\Rightarrow \quad A\left(\frac{x^3}{3}\right)_0^1 = 1$$

$$\Rightarrow \quad A\left(\frac{1}{3}\right) = 1 \quad \Rightarrow \quad A = 3$$

$$\therefore \quad f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) $$p(0.2 < X < 0.5) = \int_{0.2}^{0.5} 3x^2 dx = 3\left(\frac{x^3}{3}\right)_{0.2}^{0.5}$$
$$= (0.5)^3 - (0.2)^3 = 0.117$$

(b) $$p(X < 0.3) = \int_{0}^{0.3} 3x^2 dx = 3\left(\frac{x^3}{3}\right)_{0}^{0.3} = 0.027$$

(c) $$p\left(X > \frac{3}{4} \diagup X > \frac{1}{2}\right) = \frac{p\,(X > 3/4 \cap X > 1/2)}{p\,(X > 1/2)} = \frac{p\,(X > 3/4)}{p\,(X > 1/2)}$$
$$= \frac{\int_{3/4}^{1} 3x^2 dx}{\int_{1/2}^{1} 3x^2 dx} = \frac{37/64}{7/8} = \frac{37}{56}$$

■

■ Example 2.12

Consider the pdf

$$f(x) = \begin{cases} Kx & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find (a) $p\left(X < \frac{1}{2}\right)$, (b) $p\left(\frac{1}{4} < X < \frac{1}{2}\right)$, (c) $p\left(X > \frac{3}{4} \diagup X > \frac{1}{2}\right)$ and (d) $p\left(X < \frac{3}{4} \diagup X > \frac{1}{2}\right)$.

Here $\boldsymbol{A}(0, 1)$ so that X is a continuous random variable.

$$\therefore \quad \int_{x \epsilon \boldsymbol{A}} f(x) dx = 1 \quad \Rightarrow \quad \int_{0}^{1} Kx dx = 1$$
$$\Rightarrow \quad K\left(\frac{x^2}{2}\right)_0^1 = 0$$
$$\Rightarrow \quad K = 2$$
$$\therefore \quad f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) $$p\left(X < \frac{1}{2}\right) = \int_{0}^{1/2} 2x dx = 2\left(\frac{x^2}{2}\right)^{1/2} = \frac{1}{4}$$

(b) $$p\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{1/4}^{1/2} 2x dx = 2\left(\frac{x^2}{2}\right)_{1/4}^{1/2}$$
$$= \left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{3}{16}$$

(c) $$p\left(X > \frac{3}{4} \diagup X > \frac{1}{2}\right) = \frac{p\,(X > 2/4 \cap X > 1/2)}{p(X > 1/2)} = \frac{p\,(X > 3/4)}{p\,(X > 1/2)} = \frac{7/16}{3/4} = \frac{7}{12}$$

(d) $$p\left(X < \frac{3}{4} \diagup X > \frac{1}{2}\right) = \frac{p\,(X < 3/4 \cap X > 1/2)}{p\,(X > 1/2)} = \frac{p\left(\frac{1}{2} < X < 3/4\right)}{p\,(X > 1/2)} = \frac{5/16}{3/4} = \frac{5}{12}$$

■

■ Example 2.13

For each of the following, find the constant K so that $f(x)$ satisfies the conditions of being a pdf of a random variable X. Also, for each of the functions compute $p\,(|X| < 1)$.

(a) $$f(x) = \begin{cases} Kx^2e^{-2x} & 0 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$f(x) = \begin{cases} Kx^2 & -3 > x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(c) $$f(x) = \begin{cases} K(x+2) & -2 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Here $\boldsymbol{A} = (0, \infty)$ so that X is a continuous random variable

$$\therefore \int\limits_{x \epsilon \mathcal{A}} f(x)dx = 1 \Rightarrow \int\limits_0^\infty Kx^2e^{-2x}dx = 1$$

$$2x = Z, \quad x = \frac{Z}{2}, \quad dx = \frac{dZ}{2}$$

$$\Rightarrow K\int\limits_0^\infty \left(\frac{Z}{2}\right)^2 e^{-z}\frac{dZ}{2} = 1$$

$$\Rightarrow K\frac{1}{8}\int\limits_0^\infty Z^2e^{-z}dZ = 1$$

$$\Rightarrow \frac{K}{8}\sqrt{3} = 1$$

$$\Rightarrow K = \frac{8}{2}$$

$$\Rightarrow K = A$$

Now

$$
\begin{aligned}
p(|X| < 1) &= p(-1 < X < 1) \\
&= \int_0^1 4x^2 e^{-2x}\, dx \quad (\text{since } X > 0) \\
&= 4\left[x^2\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right)\right]_0^1 \\
&= 4\left[\left(\frac{e^{-2}}{-2} - \frac{2e^{-2}}{4} - \frac{2}{8}e^{-2}\right) - \left(0 - 0 - \frac{2}{8}\right)\right] \\
&= 4[0.081] = 0.324
\end{aligned}
$$

(b) Here $\mathcal{A} = (-3, 3)$ so that X is a continuous random variable.

$$
\begin{aligned}
\therefore \quad \int_{x\epsilon\mathcal{A}} f(x)dx = 1 \quad &\Rightarrow \int_{-3}^{3} Kx^2 dx = 1 \\
&\Rightarrow K\left(\frac{x^3}{3}\right)_{-3}^{3} = 1 \\
&\Rightarrow K\left(\frac{3^3 - (-3)^3}{3}\right) = 1 \quad K = \frac{1}{18}
\end{aligned}
$$

$$
\therefore \quad f(x) = \begin{cases} \frac{1}{18}x^2 & -3 < x < 3 \\ 0 & \text{elsewhere} \end{cases}
$$

$$
\begin{aligned}
\therefore \quad p(|X| < 1) &= p(-1 < x < 1) \\
&= \int_{-1}^{1} \frac{1}{18}x^2 dx = \frac{1}{18}\left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{27}
\end{aligned}
$$

(c) Here $\mathcal{A} = (-2, 4)$ so that X is a continuous random variable.

$$
\begin{aligned}
\therefore \quad \int_{x\epsilon\mathcal{A}} f(x)dx = 1 \quad &\Rightarrow \int_{-2}^{4} K(x+2)dx = 1 \quad K\left[\frac{x^2}{2} + 2x\right]_{-2}^{4} = 1 \\
&\Rightarrow K\left[\left(\frac{4^2}{2} + 8\right) - \left(\frac{(-2)^2}{2} + 2(-2)\right)\right] = 1 \quad K = \frac{1}{18}
\end{aligned}
$$

$$
\therefore \quad f(x) = \begin{cases} \frac{x+2}{18} & -2 < x < 4 \\ 0 & \text{elsewhere} \end{cases}
$$

$$
\begin{aligned}
\therefore \quad p(|X| < 1) &= p(-1 < X < 1) \\
&= \int_{-1}^{1} \frac{x+2}{18} dx = \frac{1}{18}\left(\frac{x^2}{2} + 2x\right)_{-1}^{1} = \frac{2}{9}
\end{aligned}
$$

■

■ Example 2.14

A random variable which can assume values between $x = 3$ and $x = 6$ has a density function given by $f(x) = K(1+x)$. Find $p(4 < X < 5)$.

Here $\mathcal{A} = [3, 6]$ so that X is a continuous random variable.

$$\therefore \quad \int_{x \epsilon \mathcal{A}} f(x)dx = 1 \quad \Rightarrow \quad \int_3^6 K(1+x)dx = 1$$

$$\Rightarrow \quad K\left(x + \frac{x^2}{2}\right)_3^6 = 1 \quad K = \frac{2}{33}$$

$$\therefore \quad f(x) = \begin{cases} \frac{2}{33}(1+x) & 3 \le x \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

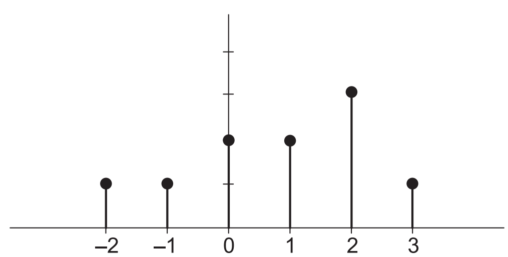
$$p(4 < X < 5) = \int_4^5 \frac{2}{33}(1+x)dx = \frac{2}{33}\left(x + \frac{x^2}{2}\right)_4^5$$

$$= \frac{2}{33}\left(\frac{11}{2}\right) = \frac{1}{3}$$

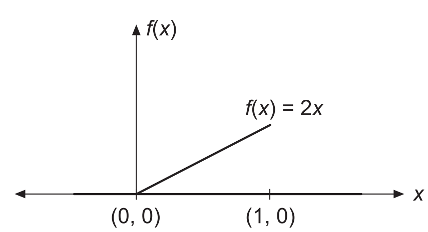
■

It is often helpful to visualize probability distributions by means of graphs such as a histogram or a bar chart. A typical pmf is sketched (Refer Example 2.8) in which each vertical segment represents the value of $f(x)$ corresponding to a possible value x. The sum of the heights of the vertical segments must be 1.

Graph of the probability distribution of a discrete random variable.



Graph of pdf of a continuous R.V (Refer Example 2.12)



There are many problems in which we are interested not only in $p(X = x)$ but also in the probability that the value of a R.V is less than or equal to x. In our next discussion we formally define this as distribution function.

2.1.3 The Distribution Function

Let X be a random variable with space set $\boldsymbol{A}$. Let A be such that $A = [-\infty, x]$ where x is any real number i.e., A is an unbounded set from $-\infty$ to x, including the point x itself. For all such sets $p(A) = p(X \epsilon A) = p(X \leq x)$. This probability depends on x. This is denoted as $F(x)$ and called as *distribution function* (or, cummulative distribution function, cdf) of the random variable X.

$$\begin{aligned} \therefore \quad F(x) &= p(X \leq x) \\ &= \begin{cases} \sum\limits_{t \leq x} f(t) & \text{if } X \text{ is Discrete random variable} \\ \int\limits_{-\infty}^{x} f(t)dt & \text{if } X \text{ is continuous random variable} \end{cases} \end{aligned}$$

■ Example 2.15

Find cdf of X if its pmf is

$$f(x) = \begin{cases} \frac{x}{10} & x = 1, 2, 3, 4 \\ 0 & \text{elsewhere} \end{cases}$$

X is a discrete random variable with space $\mathcal{A} = \{1, 2, 3, 4\}$. i.e., $f(x) > 0$ for all $x \epsilon \mathcal{A}$ and $f(x) = 0 \;\; x \notin \mathcal{A}$.

$$\therefore \quad f(x) = 0 \quad \text{if } x \neq 1, 2, 3, 4$$

Case (i): Let $x < 1$

$$\therefore \quad F(x) = \sum_{t<1} f(t) = 0$$

Case (ii): Let $1 \leq x < 2$

$$\begin{aligned} \therefore \quad F(x) &= \sum_{t \leq x} f(t) \\ &= f(1) \qquad (\because \quad f(x) = 0, \text{ at all points in this interval}) \\ &= \frac{1}{10} \end{aligned}$$

Case (iii): Let $2 \leq x < 3$

$$\begin{aligned} \therefore \quad F(x) &= \sum_{t \leq x} f(t) = f(1) + f(2) \\ &= \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \end{aligned}$$

Case (iv): Let $3 \leq x < 4$

$$F(x) = \sum_{t \leq x} f(t) \quad = \quad f(1) + f(2) + f(3) = \frac{6}{10}$$

Case (v): Let $4 \leq x < 5$

$$F(x) \quad = \quad 1$$

Case (vi): Let $x \geq 5$

$$\begin{aligned} F(x) &= f(1) + f(2) + f(3) + f(4) = 1 \\ \therefore \qquad F(x) &= \begin{cases} 0 & x < 1 \\ \frac{1}{10} & 1 \leq x < 1 \\ \frac{3}{10} & 2 \leq x < 3 \\ \frac{6}{10} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases} \end{aligned}$$

■

■ Example 2.16

Find cdf of a random variable X whose pdf is

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Here $\mathcal{A} = (0, 1)$ and X is a continuous RV.

$$\therefore \qquad f(x) > 0 \quad \forall x \epsilon \mathcal{A} \quad \text{and} \quad f(x) = 0 \quad \forall x \notin \mathcal{A}$$

$$\therefore \qquad f(x) = 0 \quad \text{if} \quad x \leq 0 \quad \text{or} \quad x \geq 1$$

Case (i): Let $x \leq 0$

$$F(x) = \int_{-\infty}^{x} f(t)dt = 0$$

Case (ii): Let $0 < x < 1$

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt \\ &= 0 + \int_{0}^{x} 3t^2 dt = (t^3)_0^x = x^3 \end{aligned}$$

Case (iii): Let $x \geq 1$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt$$

$$= 0 + 1 + 0 = 1$$

$$\therefore \quad F(x) = \begin{cases} 0 & x \leq 1 \\ x^3 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

■

Remark

- ◇ In Example 2.18, Case (iii) we take $\int_{0}^{1} f(t)dt = 1$. Since $f(x)$ is a pdf of x in its range (0, 1) and the total probability is 1.
- ◇ In both Examples 2.17 and 2.18 we could observed that if $x \notin \boldsymbol{A}$ then $F(x) = 0$ or 1. The reason is: if $x \notin \boldsymbol{A}$ but $x < a$ (or even $x \leq a$) $\forall a \epsilon \boldsymbol{A}$ then $f(x) = 0$ so that $\sum_{t \leq x} f(t) = 0$ or $\int_{t \leq x} f(t)dt = 0$ for all such x. Similarly if $x \notin \boldsymbol{A}$ but $x > a$ (or $x \geq a$) $\forall a \epsilon A$ then $F(x) = p(x \leq x) = p(x \epsilon \boldsymbol{A})$ where $\boldsymbol{A}$ contains all points of $\boldsymbol{A}$ also. This includes the total probability of x over its space $\boldsymbol{A}$ which is 1 always.

$$\therefore \quad F(x) = 1 \quad \text{for all such } x.$$

For this reason in our subsequent problems we may not calculate $F(x)$ for $x \notin A$ in detailed manner. Instead, we can consider its value as 0 or 1 accordingly.

We now state some properties of cdf without proof. Let X be discrete or continuous RV.

1. $F(\infty) = \lim_{x \to \infty} F(x) = 1.$
2. $F(-\infty) = \lim_{x \to \infty} F(-x) = 0.$
3. $0 \leq F(x) \leq 1 \ \forall x \epsilon R.$
4. $F(x)$ is a non-decreasing function.

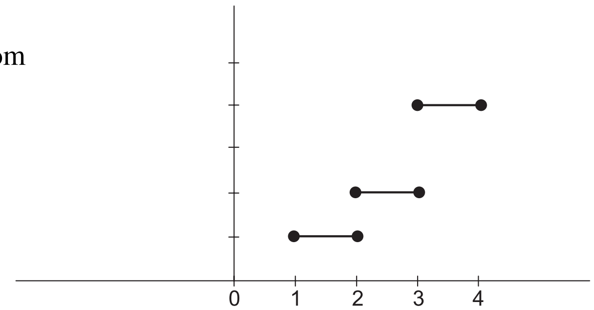
   i.e., if $x_1 < x_2$ then $F(x_1) \leq F(x_2)$.
5. If $a < b$ then $p(a < X \leq b) = F(b) - F(a)$.
6. $F(x)$ is continuous to the right at every point $x = a$.

   i.e., $F(a) = F(a + 1)$ where $F(a+)$ is the right hand limit of $F(x)$ at $x = a$.
7. If X is a continuous RV then its pdf can be obtained from cdf as

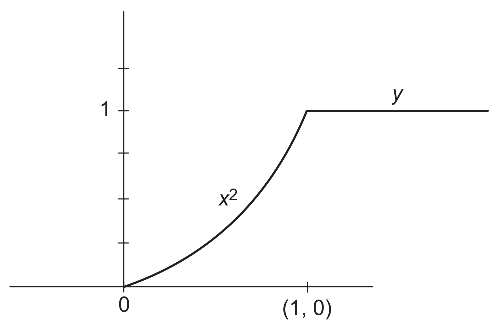
$$f(x) = F'(x) = \frac{d}{dx} F(x)$$

8. If $F(x)$ is continuous at $x = a$ then $p(x = a) = 0$ and if it is not continuous then $p(X = a) = F(a) - F(a-)$ i.e., for every value of $X = xp(X = x) = F(x) - F(x-)$.
9. If X is a discrete random variable then the cdf $F(x)$ will have a jump at each value x_i of X and it is constant between every pair of successive jump.

Graph of cdf of a discrete random variable (Refer Example 2.17)



Graph of the cdf of a continuous random variable (Refer Example 2.20)



☐ Result

If the space of a random variable X consists of the values $x_1 < x_2 < \cdots < x_n$ then
$f(x_1) = F(x_1) - F(x_{i-1})$ for $i = 2, 3, \cdots n$

Since $\quad F(x_i) = \sum_{t \le x_i} f(t)$ and $F(x_{i-1}) = \sum_{t \le x_{i-1}} f(t)$

We have $F(x_i) - F(x_i - 1)$

$$\begin{aligned} &= [f(x_1) + f(x_2) + \cdots + f(x_i)] - \left[f(x_1) + f(x_2) + \cdots + f(x_{i-1})\right] \\ &= f(x_i) \end{aligned}$$

Hence proved.
Using the above result, we can prove the following two results.

(i)
$$\begin{aligned} p(X > x_i) &= 1 - F(x_i) \quad i = 1, 2, 3, \cdots, n \\ \text{Since} \quad p(X > x_i) &= 1 - p(X \le x_i) \\ &= 1 - F(x_i) \quad i = 1, 2, 3, \cdots, n \end{aligned}$$

$$\begin{aligned}
\text{(ii)} \qquad p(X \geq x_i) &= 1 - F(x_{i-1}) \quad i = 1, 2, 3, \cdots, n \\
\text{Since} \quad p(X \geq x_i) &= 1 - p(x < x_i) \\
&= 1 - p(X \leq x_{i-1}) \quad \left[\because \quad f(x) = 0 \; x_{i-1} < x < x\right] \\
&= 1 - F(x_{i-1})
\end{aligned}$$

In particular $p(x \geq x_i) = 1$.
Refer Examples 2.25 and 2.26 later in this Chapter.

■ Example 2.17

Give the c.d.f of a R.V X

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+2}{4} & -1 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

Find (a) $p\left(-\frac{1}{2} < X \leq \frac{1}{2}\right)$, (b) $p(X = 0)$, (c) $p(X = 1)$ and (d) $p(2 < X \leq 3)$.

$$\begin{aligned}
\text{(a)} \quad p\left(-\frac{1}{2} < x \leq \frac{1}{2}\right) &= F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) \\
&= \frac{1/2 + 2}{4} - \frac{-1/2 + 2}{4} = \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \qquad p(X = 0) &= 0 \quad (\text{since } F(x) \text{ is continuous at } x = 0) \\
\text{(c)} \qquad p(X = 1) &= F(1) - F(1-) \\
&= 1 - \frac{1+2}{4} \quad (\text{since } F(x) \text{ is not continuous at } x = 1) \\
&= \frac{1}{4} \\
\text{(d)} \quad p(2 < X \leq 3) &= F(3) - F(2) = 0.
\end{aligned}$$

■

■ Example 2.18

Find the cdf of a R.V X whose pdf is

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$F(x) = p(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} 2(t)dt = x^2$$

$$\therefore \quad F(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

If $F(x)$ is the cdf of a continuous random variable and is strictly increasing on some interval and that $F(x) = 0$ to the left of I and $F(x) = 1$ to the right of I then its inverse function $F^{-1}(x)$ is defined. This will help us in finding p^{th} *quantile of the distribution*.

That is the p^{th} quantile or (100p)th percentile is defined as a value x_p such that $F(x_p) = p$ or $p(X \leq x_P) = p$. Under the assumption stated above x_p is uniquely determined. Special cases of quantile are $p = \frac{1}{2}$ and $\frac{1}{4}$ and $\frac{3}{4}$. Indeed, if $p = \frac{1}{2}$ it is said to be median and if $p = \frac{1}{4}$ and $p = \frac{3}{4}$ then it is said to be *lower* and *upper quatiles* of X. This calculation of quantiles can be either using integration or using the graph of $F(x)$.

Let us formally introduce two terms here. A median of a distribution of one random variable X is a value x such that $p(X < x) \leq \frac{1}{2}$ and $p(x \leq x) \geq \frac{1}{2}$. If there is only one such x, it is called the *median of the distribution*.

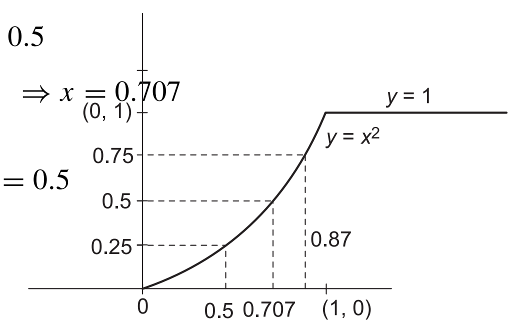
With a continuous random variable X, $p(X < x_B) = p(X \leq x)$ and hence common value must equal to $\frac{1}{2}$ that is $p(X \leq x) = \frac{1}{2}$ or, $F(x) = \frac{1}{2}$ which coinlides with the above notion. In fact, the p^{th} percentile p of a continuous random variable is such that $p(X < x_p) \geq p$ and $p(X \leq x_p) \leq p$ and hence $p(X \leq x_p) = p$ that is $F(x_p) = p$. In this case of a discrete random variable we find p^{th} percentile as a value x_p such that $p(X < x_p) \leq \frac{1}{2}$ and $p(X \leq x) \geq \frac{1}{2}$.

We define a mode of a distribution of one random variable X is a value x that maximizes the pdf $f(x)$. If there is only one such x, it is called the *mode of the distribution*. ■

■ Example 2.19

Find the median, lower and upper quartiles of the random variable defined in Example 2.20.

$$\begin{aligned}
\text{Median} = x_{1/2} \quad &\Rightarrow \quad F\left(x_{\frac{1}{2}}\right) = 0.5 \\
&\Rightarrow \quad x^2 = 0.5; \quad \Rightarrow x = 0.707 \\
\text{Similarly } F\left(x_{1/4}\right) \quad &= \quad 0.25 \\
x^2 \quad &= \quad 0.25 \Rightarrow x = 0.5 \\
\text{and upper quartile} \quad &= \quad 0.866.
\end{aligned}$$



Now, let us consider its graph and identify the quantiles. ■

■ Example 2.20

Let the pdf of a random variable the

$$f(x) = \begin{cases} 4x^3 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the 20^{th} percentile of the distribution.

20^{th} percentile means $p = 0.2$ (loop $= 20 \Rightarrow p = 0.2$).

Let us find the distribution function of X

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} 4t^3 dt = x^4$$

$$\therefore \quad F(x) = \begin{cases} 0 & x \leq 0 \\ x^4 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

To find 20$^{\text{th}}$ percentile:

$$F(x_{0.2}) = 0.2 \quad \Rightarrow \quad x^4 = 0.2$$
$$\therefore \quad x = 0.67$$

■

2.1.4 Additional Examples

■ Example 2.21

Let X denote the random variable that is defined as the sum of the numbers in the two fair dice. Find the space and pdf of X.

X can take on any integral value between 2 and 12, its space is $\boldsymbol{A} = \{2, 3, 4, \cdots 12\}$, since, the sum of the first pair is 2 [(1,1)] and so on.

$$\therefore \quad p(X = 2) = p[(1, 1)] = \frac{1}{36}$$
$$p(X = 3) = p[(1, 2); (2, 1)] = \frac{2}{36}$$
$$p(X = 4) = p[(1, 3); (2, 2); (3, 1)] = \frac{3}{36}$$
$$p(X = 12) = p[(6, 6)] = \frac{1}{36}$$

Hence the pdf of X is

$X = x$	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

■

■ Example 2.22

A bag contains 5 red 3 white and 2 blue chips. Two chips are randomly selected. Let X represent the number of white chips selected. What is the space of X and the probabilities associated with the values of X?

The total number of chips in the bag is 10.

$\therefore$ Number of ways selecting 2 chips from the bag is $10C_2 = 45$.

The selected two chips are non-white or one of them is white or both of them are white.
$\therefore$ X assumes the values 0, 1, 2. That is the space of X is $\boldsymbol{A} = \{0, 1, 2\}$.

$$\begin{aligned}
p(X=0) &= p(\text{No white}) \\
&= \frac{5C_2}{10C_2} + \frac{5C_1 \times 2C_1}{10C_2} + \frac{2C_2}{10C_2} = \frac{21}{45} \\
p(X=1) &= \frac{3C_1 \times 5C_1}{10C_2} + \frac{3C_1 \times 2C_1}{10C_2} = \frac{21}{45} \\
p(X=2) &= \frac{3C_2}{10C_2} = \frac{3}{45}
\end{aligned}$$

Hence the pdf of X

$X = x$	0	1	2
$f(x)$	$\frac{21}{45}$	$\frac{21}{45}$	$\frac{3}{45}$

■

■ Example 2.23

For each of the following find K so that the function can serve as the probability distribution of a random variable.

(a) $f(x) = k\left(\frac{1}{3}\right)^x \quad x = 1, 2, 3, \cdots$

(b) $f(x) = k\ x^2 \quad x = 1, 2, 3, \cdots, 10$

(c) $f(x) = k^x(1-k) \quad x = 0, 1, 2, \cdots$

(a) Since, the space of X is $\boldsymbol{A} = \{1, 2, 3, \cdots, \} X$ is discrete

$$\therefore \sum_x f(x) = 1$$

$$\begin{aligned}
\therefore \quad \sum_{x=1}^{\infty} k\left(\frac{1}{3}\right)^x = 1 \quad &\Rightarrow \quad K\left[\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \cdots\right] = 1 \\
&\Rightarrow \quad K\left(\frac{1}{3}\right)\left[\frac{1}{1-(1/3)}\right] = 1 \\
&\Rightarrow \quad K\left(\frac{1}{3}\right) \times \left(\frac{3}{2}\right) = 1 \quad \Rightarrow \quad K = 2
\end{aligned}$$

(b) Space of X is $\boldsymbol{A} = \{1, 2, 3, \cdots, 10\}$. X is discrete.

$$\begin{aligned}
\therefore \quad \sum_x f(x) = 1 \quad &\Rightarrow \quad \sum_{x=1}^{10} K\ x^2 = 1 \\
&\Rightarrow \quad K\left(1^2 + 2^2 + 3^2 + \cdots + 10^2\right) = 1 \\
&\Rightarrow \quad K\frac{10(11)(21)}{6} = 1; \quad K = \frac{1}{385}
\end{aligned}$$

(c) In this case also X is discrete

$$\therefore \quad \sum_x f(x) = 1 \quad \Rightarrow \quad \sum_{x=0}^{\infty} K^x (1-K) = 1$$
$$\Rightarrow \quad (1-K)\left[1 + K + K^2 + \cdots\right] = 1$$
$$\Rightarrow \quad (1-K)\frac{1}{1-K} = 1$$

The above identity exists only for $0 < K < 1$.
$\therefore$ For any K s.t $0 < K < 1$ $f(x)$ is an admissable pdf of a discrete random variable X. ■

■ Example 2.24

(a) $f(x) = \begin{cases} \dfrac{K}{\sqrt{x}} & 1 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} Kx(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} Kx\left(e^{-x^2}\right) & x > 0 \\ 0 & \text{elsewhere} \end{cases}$

(d) $f(x) = \begin{cases} Kx\left(e^{-x}\right) & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$

In (a) to (d), the space of X is respectively $\{x/1 < x < 4\}$; $\{x/0 < x < 1\}$; $\{x/x > 0\}$ and $\{x/x \geq 0\}$ which are uncountably infinite. So, X is a continuous random variable in each case.

(a) $$\int_{-\infty}^{\infty} f(x)dx = 1 \quad \Rightarrow \quad \int_1^4 \frac{k}{\sqrt{x}}dx = 1 \Rightarrow k\left(2\sqrt{x}\right)_1^4 = 1$$
$$\Rightarrow \quad 2k\left(\sqrt{4} - \sqrt{1}\right) = 1 \Rightarrow k = \frac{1}{2}$$

(b) $$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \quad \int_0^1 kx(1-x)dx = 1 \quad \Rightarrow \quad k\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = 1$$
$$\Rightarrow \quad k\left(\frac{1}{2} - \frac{1}{3}\right) = 1; \quad k = 6$$

(c) $$\int_{-\infty}^{\infty} f(x)dx = 1 \quad \Rightarrow \quad \int_{0}^{\infty} kx\, e^{-x^2} dx = 1 \qquad \begin{matrix} x^2 = 1 \\ 2x dx = dt \end{matrix}$$

$$\Rightarrow \quad \frac{k}{2}\int_{0}^{\infty} e^{-t} dt = 1$$

$$\Rightarrow \quad \frac{k}{2}\left(-e^{-t}\right)_{0}^{\infty} = 1; \quad k = 2$$

(d) $$\int_{-\infty}^{\infty} f(x)dx = 1 \quad \Rightarrow \quad \int_{0}^{\infty} k\, x\, e^{-x} dx = 1$$

$$\Rightarrow \quad k\int_{0}^{\infty} x^{2-1} e^{-x} dx = k\sqrt{2} = 1$$

$$\Rightarrow \quad k = 1$$

■

■ Example 2.25

The probability distribution of X, the weekly power shutdown in a certain city is given by $f(0) = 0.45$, $f(1) = 0.25$, $f(2) = 0.25$ $f(3) = 0.05$. Find the distribution function of X. Find the probability that there will be atleast two shutdown in any one week, using (a) the original probabilities and (b) the distribution function.

The pdf of X is

$X = x$	0	1	2	3
$f(x)$	0.45	0.25	0.25	0.05

X is a discrete random variable with space $\boldsymbol{A} = \{0, 1, 2, 3\}$

Case (i): Let $x < 0$ $\therefore$ $F(x) = \sum_{t<x} f(t) = 0$

Case (ii): Let $0 \le x < 1$ $F(x) = \sum_{t \le x} f(t)$
$= f(0) = 0.45$

Case (iii): Let $1 \le x < 2$ $\therefore$ $F(x) = f(0) + f(1)$
$= 0.45 + 0.25 = 0.70$

Case (iv): Let $2 \leq x < 3$ $F(x) = f(0) + f(1) + f(2)$

$$= 0.45 + 0.25 + 0.25 = 0.95$$

Case (v): Let $3 \leq x$ $F(x) = f(0) + f(1) + f(2) + f(3) = 1$

$$\therefore \quad F(x) = \begin{cases} 0 & x < 0 \\ 0.45 & 0 \leq x < 1 \\ 0.70 & 1 \leq x < 2 \\ 0.95 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

(a) $$p(X \geq 2) = p(X = 2) + p(X = 3) = f(2) + f(3) = 0.25 + 0.05 = 0.30$$

(b) $$p(X \geq 2) = 1 - f(1) = 1 - 0.70 = 0.30.$$

■

■ Example 2.26

If the pdf of a random variable X is $f(1) = \frac{1}{3}$ $f(4) = \frac{1}{6}$ $f(6) = \frac{1}{3}$ $f(10) = \frac{1}{6}$; find the distribution function of X and hence find (a) $p(2 < X \leq 6)$, (b) $p(X = 4)$, (c) $p(x \geq 1)$, (d) $p(x \geq 4)$, (e) $p\left(\frac{1}{4} < X < 4\right)$ and (f) $p(x < 3)$.

Let $F(x)$ be the distribution of X. The pdf of X is

$X = x$	1	4	6	10
$f(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{3} & 1 \leq x < 4 \\ \frac{1}{2} & 4 \leq x < 6 \\ \frac{5}{6} & 6 \leq x < 10 \\ 1 & x \geq 10 \end{cases}$$

(a) $$p(2 < x \leq 6) = F(6) - F(2) = \frac{5}{6} - \frac{1}{3} = \frac{1}{2}$$

(b) $$p(X = 4) = F(4) - F(4-) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(c) $p(X \geq 1) = 1$ since $\boldsymbol{A} = \{1, 4, 6, 10\}$

(d) $$\begin{aligned} p(X \geq 4) &= 1 - F(1) \\ &= 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

(e) $$p\left(\frac{1}{4} < X < 4\right) = F(1) = \frac{1}{3}$$

(f) $$\begin{aligned} p\,(X < 3) &= F(3) - p(X = 3) \\ &= \frac{1}{3} - 0 = \frac{1}{3} \end{aligned}$$

■

■ Example 2.27

Two ideal dice are thrown. Let X be the maximum of the scores in the dice. Find the probability distribution of X. Also find (a) $p(X \text{ is even})$, (b) $p\left\{\frac{3}{2} < X < \frac{7}{2}/X > 2\right\}$ and (c) the distribution function of X.

The space of X is $\boldsymbol{A} = \{1, 2, 3, 4, 5, 6\}$

Its pdf is

$$\begin{aligned} p(X = 1) &= p\,\{(1, 1)\} = \frac{1}{36} \\ p(X = 2) &= p\,\{(1, 2), (2, 1), (2, 2)\} = \frac{3}{36} \\ p(X = 3) &= p\,\{(1, 3), (2, 3), (3, 1), (3, 2), (3, 3)\} = \frac{5}{36} \quad \text{and so on} \end{aligned}$$

$X = x$	1	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

(a) $$\begin{aligned} p(X \text{ is even}) &= p(X = 2 \text{ or } 4 \text{ or } 6) \\ &= \frac{3}{36} + \frac{7}{36} + \frac{11}{36} = \frac{21}{36} = \frac{7}{12} \end{aligned}$$

(b) $$\begin{aligned} p\left[\frac{3}{2} < X < \frac{7}{2}x > 2\right] &= \frac{p\left[\left(\frac{3}{2} < X < \frac{7}{2}\right) \cap (X > 2)\right]}{p\,(X > 2)} \\ &= p\frac{[(X = 2, 3) \cap (X = 3, 4, 5, 6)]}{1 - p(X \leq 2)} \\ &= \frac{p\,[X = 3]}{1 - [p(X = 1) + p(X = 2)]} \\ &= \frac{5/36}{1 - [4/36]} = \frac{5}{32} \end{aligned}$$

(c) Let $F(x)$ be the distribution of X.

$$F(x) = \begin{cases} 0 & X < 1 \\ \frac{1}{36} & 1 \le X < 2 \\ \frac{1}{9} & 2 \le X < 2 \\ \frac{1}{4} & 3 \le X < 4 \\ \frac{4}{9} & 4 \le X < 5 \\ \frac{25}{36} & 5 \le X < 6 \\ 1 & 6 \le X \end{cases}$$

■

■ Example 2.28

If a random variable X takes the values 1, 2, 3 and 4 such that $3p(X = 1) = 2p(X = 2) = 5p(X = 3) = p(X = 4)$. Find the distribution function of X.

Since, the space of X is $\boldsymbol{A} = \{1, 2, 3, 4\}$ X is a discrete random variable.

Let $f(x)$ be pdf of X. Given $3f(1) = 2f(2) = 5f(3) = f(4)$ and we know that $\sum_{i=1}^{4} f(i) = 1$

$$\Rightarrow \quad f(1) + f(2) + f(3) + f(4) = 1$$

$$\Rightarrow \quad \frac{1}{3}f(4) + \frac{1}{2}f(4) + \frac{1}{5}f(4) + f(4) = 1$$

$$\Rightarrow \quad f(4)\left[\frac{1}{3} + \frac{1}{2} + \frac{1}{5} + 1\right] = 1$$

$$\Rightarrow \quad f(4) = \frac{30}{61}$$

$$\therefore \quad f(1) = \frac{10}{61}; \quad f(2) = \frac{15}{61}; \quad f(3) = \frac{6}{61}$$

$X = x$	1	2	3	4
$f(x)$	$\frac{10}{61}$	$\frac{15}{61}$	$\frac{6}{61}$	$\frac{30}{61}$

Let $F(x)$ be the distribution function of the random variable X

$$\therefore \quad F(x) = \begin{cases} 0 & x < 1 \\ \frac{10}{61} & 1 \le X < 2 \\ \frac{25}{61} & 2 \le X < 3 \\ \frac{31}{61} & 3 \le X < 4 \\ 1 & 4 \le X \end{cases}$$

■

■ Example 2.29

Let X be a random variable with pdf

$$f(x) = \begin{cases} k(1 - x^2) & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) What is the value of K? (b) Find the distribution function of X.
Since, $\boldsymbol{A} = \{x / -1 < x < 1\}$, X is a continuous random variable.

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = 1 \quad \Rightarrow \quad k\int_{-1}^{1}(1-x^2)dx = 1$$

$$\Rightarrow \quad k\left(x - \frac{x^3}{3}\right)_{-1}^{1} = 1$$

$$\Rightarrow \quad k\left[\left(1-\frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right)\right] = 1$$

$$\Rightarrow \quad k\left[\frac{2}{3} + \frac{2}{3}\right] = 1; \quad k = \frac{3}{4}$$

$$\therefore \quad f(x) = \begin{cases} \frac{3}{4}\left(1-x^2\right) & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Let $F(x)$ be the distribution function of X.

$$\therefore \quad F(x) = \int_{-\infty}^{x} f(t)dt$$

Since $f(x) = 0$, when $x \notin \boldsymbol{A}$
$F(x) = 0$ if $x \leq -1$ and $F(x) = 1$ if $x \geq 1$. If $-1 < x < 1$ that is when $x \epsilon \boldsymbol{A}$

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt = \int_{-1}^{x} \frac{3}{4}\left(1-t^2\right)dt \\ &= \frac{3}{4}\left(t - \frac{t^3}{3}\right)_{-1}^{x} \\ &= \frac{3}{4}\left[\left(x - \frac{x^3}{3}\right) - \left(-1 - \frac{-1}{3}\right)\right] \\ &= \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right) \end{aligned}$$

$$\therefore \quad F(x) = \begin{cases} 0 & x \leq -1 \\ \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right) & -1 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

■

■ Example 2.30

If the density funciton of X is

$$f(x) = \begin{cases} Ce^{-2x} & 0 < x < \infty \\ 0 & x \leq 0 \end{cases}$$

Find (a) $p(X > 2)$, (b) $p\left(2 < X < 3 \diagup X > \frac{5}{2}\right)$ and (c) $p(X > 2 \diagup X \leq 5)$.
Space of X is $\boldsymbol{A} = \{x/0 < x < \infty$ so that X is a continuous random variable.
To find constant C:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx = 1 &\Rightarrow \int_0^{\infty} Ce^{-2x}dx = 1 \\
&\Rightarrow C\left(\frac{e^{-2x}}{-2}\right)_0^{\infty} = 1 \\
&\Rightarrow \frac{C}{-2}(0-1) = 1 \\
&\Rightarrow C = 2 \\
\therefore \quad f(x) &= \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}
\end{aligned}$$

(a) $$p(X > 2) = \int_2^{\infty} f(x)dx = \int_2^{\infty} 2e^{-2x}dx = 2\left(\frac{e^{-2x}}{-2}\right)_2^{\infty} = e^{-4} = 0.0183$$

(b) $$\begin{aligned}
p\left(2 < X < 3 \diagup X > \frac{5}{2}\right) &= \frac{p\left[(2 < X < 3) \cap (X > 5/2)\right]}{p\,(X > 5/2)} \\
&= \frac{p\,(5/2 < X < 3)}{p\left(X > \frac{5}{2}\right)} = \frac{\int_{5/2}^{3} 2e^{-2x}dx}{\int_{5/2}^{\infty} e^{-2x}dx} \\
&= \frac{2\left(\frac{e^{-2x}}{-2}\right)_{5/2}^{3}}{2\left(\frac{e^{-2x}}{-2}\right)_{5/2}^{\infty}} = \frac{e^{-5} - e^{-6}}{e^{-5}} = \frac{0.0043}{0.0067} = 0.64.
\end{aligned}$$

■

■ Example 2.31

The pdf of the random variable X is given by

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the distribution function and hence evaluate $p(0.8 < X < 1.2)$; $p(X > 1)$.

Since X is a continuous random variable, its distribution function $F(x) = \int_{-\infty}^{x} f(t)dt$.

Here $\boldsymbol{A} = \{x/0 < x < 2\}$.

$\therefore$ If $x \notin \boldsymbol{A}$ $f(x) = 0$ hence if $x \leq 0$

$$F(x) = 0 \quad \text{and if} \quad x \geq 2 \; F(x) = 1$$

Now, let $x \epsilon \boldsymbol{A}$. Also, $\boldsymbol{A}$ is dovided into two sets

$$\boldsymbol{A}_1 = \{x/0 < x < 1\} \quad \boldsymbol{A}_2 = \{x/1 \leq x < 2\}$$

Case (i): Let $0 < x < 1$

$$\begin{aligned} F(x) = \int_{-\infty}^{x} f(t)dt &= \int_{0}^{x} f(t)dt \\ &= \int_{0}^{x} (t)dt = \left(\frac{t^2}{2}\right)_0^x = \frac{x^2}{2} \end{aligned}$$

Case (ii): Let $1 \leq x < 2$

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} f(t)dt \\ &= \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt = \int_{0}^{1} (t)dt + \int_{1}^{x} (2-t)dt \\ &= \left(\frac{t^2}{2}\right)_0^1 + \left(2t - \frac{t^2}{2}\right)_1^x \\ &= \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right) \\ &= 2x - \frac{x^2}{2} - 1 \end{aligned}$$

$$\therefore \quad F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

To evaluate the probabilities using $F(x)$.

$$\begin{aligned} p(0.8 < X < 1.2) &= F(1.2) - F(0.8) \\ &= \left[2(1.2) - \frac{(1.2)^2}{2} - 1\right] - \left[\frac{(0.8)^2}{2}\right] \end{aligned}$$

$$
\begin{aligned}
&= 2.4 - 0.32 = 0.36 \\
p(X > 1) &= 1 - p(X \le 1) = 1 - F(1) \\
&= 1 - \left[2(1) - \frac{1^2}{2} - 1\right] = \frac{1}{2}
\end{aligned}
$$

■

■ Example 2.32

The distribution function of the random variable X is given by

$$
F(x) = \begin{cases} 1 - (1+x)e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}
$$

(a) Find $p(X \le 2)$; $p(1 < X < 3)$; $p(X > 4)$

(b) Find the pdf of X and hence do (a).

Since X is a continuous random variable, its pdf $f(x)$ can be calculated from $F(x)$ using the relation $f(x) = \frac{d}{dx}F(x)$.

$$
\begin{aligned}
\therefore \quad f(x) &= \begin{cases} \frac{d}{dx}\left[1 - (1+x)e^{-x}\right] & x > 0 \\ 0 & x \le 0 \end{cases} \\
&= \begin{cases} (x)e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}
\end{aligned}
$$

(a) Computing propabilities using distribution function $F(x)$.

$$
\begin{aligned}
p(X \le 2) &= F(2) = 1 - (1+2)e^{-2} \\
&= 1 - 3e^{-2} = 0.594 \\
p(1 < X < 3) &= F(3) - F(1) \\
&= \left[1 - (1+3)e^{-3}\right] - \left[1 - (1+1)e^{-1}\right] \\
&= (1-4)e^{-3} - (1+2)e^{-1} \\
&= 2e^{-1} - 4e^{-3} = 0.537 \\
p(X > 4) &= 1 - p(X \le 4) \\
&= 1 - F(4) \\
&= 1 - [1 - (1+4)e^{-4}] \\
&= 5e^{-4} = 0.0916
\end{aligned}
$$

(b) Computing above probabilities using the pdf of X

$$
\begin{aligned}
p(X \le 2) &= \int_{-\infty}^{2} f(x)dx = \int_{0}^{2} xe^{-x}dx \\
&= \left[x(-e^{-x}) - e^{-x}\right]_0^2
\end{aligned}
$$

$$
\begin{aligned}
&= 1-(2e^{-2}+e^{-2}) \\
&= 1-0.406 = 0.594 \\
p(1 < X < 3) &= \int_1^3 f(x)dx = \int_1^3 (x)e^{-x}dx \\
&= \left[(x) - e^{-x} - (1)e^{-x}\right]_1^3 \\
&= (e^{-1}+e^{-1}) - (3e^{-3}+e^{-3}) \\
&= 2e^{-1} - 4e^{-3} = 0.537 \\
p(X > 4) &= \int_4^\infty f(x)dx = \int_4^\infty xe^{-x}dx \\
&= \left[(x)\diagup - e^{-x} - (1)e^{-x}\right]_4^\infty \\
&= (4e^{-4}+e^{-4}) - (0) \\
&= 5e^{-4} = 0.0916.
\end{aligned}
$$

■

Remark

Examples 2.31 and 2.32 illustrate the easier way of computing probabilites using distribution function.

■ Example 2.33

The distribution function of a random variable X is given by

$$
F(x) = \begin{cases} 0 & x < -2 \\ \frac{x+4}{8} & -2 \le x < 2 \\ 1 & x \ge 2 \end{cases}
$$

Find $p(X = -2)$; $p(X = 2)$; $p(-2 < X < 1)$; $p(0 \le X \le 2)$.
Since X is not continuous at $X = -2$.

$$
\begin{aligned}
p(X = -2) &= F(-2) - F\left[(-2) - 1\right] \\
&= \frac{-2+4}{8} - 0 = \frac{1}{4} \\
p(X = 2) &= F(2) - F(2-) \\
&= 1 - \frac{2+4}{8} = \frac{1}{4}
\end{aligned}
$$

$$\begin{aligned} p(-2 < z < 1) &= F(1) - F(-2) \\ &= \frac{1+4}{8} - \frac{-2+4}{8} = \frac{3}{8} \\ p(0 \le z \le 2) &= F(2) - F(0) \\ &= 1 - \frac{0+4}{8} = \frac{1}{2} \end{aligned}$$

■

■ Example 2.34

Let a distribution function of a random variable X is

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x+1}{2} & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

Find $p\left(-3 < X \le \frac{1}{2}\right)$; $p(X = 0)$

$$\begin{aligned} p\left(-3 < X \le \frac{1}{2}\right) &= F\left(\frac{1}{2}\right) - F(-3) \\ &= \frac{1/2+1}{2} - 0 = \frac{3}{4} \\ p(X = 0) &= F(0) - F(0-) \quad \text{(since } F(x) \text{ is not continuous at } x = 0) \\ &= \frac{1}{2} - 0 = \frac{1}{2} \end{aligned}$$

■

■ Example 2.35

In each of the following case find the median of the distribution.

$$\text{(a)} \quad f(x) = \begin{cases} \frac{1}{3} & x = 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{(b)} \quad f(x) = \begin{cases} \frac{1}{x^2} & x > 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Since $\boldsymbol{A} = \{1, 2, 3\}$, X is a discrete random variable. Let $F(x)$ be the distribution funciton of X

$$\therefore \quad F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{3} & 1 \le x < 2 \\ \frac{2}{3} & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

Median is a value x such that $p(X < x) \le \frac{1}{2}$ and $p(X \le x) \ge \frac{1}{2}$.
From $F(x)$ we find that $x = 2$.
$\therefore$ Median of the distribution is 2.

(b) Since $\boldsymbol{A} = \{x/x > 1\}$, X is a continuous random variable. Let $F(x)$ be the distribution function of X.

$$\begin{aligned}
\therefore \quad F(x) &= 0 \quad x \le 1 \\
\text{If} \quad x > 1, \quad F(x) &= \int_{-\infty}^{x} f(t)dt \\
&= \int_{1}^{x} \frac{1}{t^2}dt = 1 - \frac{1}{x} \\
\therefore \quad F(x) &= \begin{cases} 0 & x \le 1 \\ 1 - \frac{1}{x} & x > 1 \end{cases} \\
p(X \le x) &= \frac{1}{2} \\
F(x) &= \frac{1}{2} \\
1 - \frac{1}{x} &= \frac{1}{2}; \quad x = 2
\end{aligned}$$

Compare this with Examples 2.19 and 2.20. ■

■ Example 2.36

Find the mode of each of the following distributions:

$$\text{(a)} \quad f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & x = 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{(b)} \quad f(x) = \begin{cases} 12x^2(1-x) & 0 < x1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Since $f(x)$ is maximum at $x = 1$ mode of X is $x = 1$.

(b) Since X is continuous we find maximum of $f(x)$ as follows,

$$f(x) = 12x^2(1-x) = 12(x^2 - x^3)$$

$$\begin{aligned}
f'(x) = 0 \quad &\Rightarrow \quad 12(2x - 3x^2) = 0 \\
&\Rightarrow \quad x = 0 \quad \text{or} \quad x = \frac{2}{3}
\end{aligned}$$

Also, $f''(x) = 12(2 - 6x), \quad \Rightarrow \quad f''\left(\frac{2}{3}\right) = -24 < 0 f(x)$ attains maximum at $x = \frac{2}{3}$

$$\therefore \quad \text{mode of } X = \frac{2}{3}$$ ■

Exercise 2.1

In the following problems readers are advised to classify the random variables and also to compute any unknown constants.

1. Let $f(x) = Kx$; $x = 1, 2, 3, 4, 5, 6$; zero elsewhere be the pdf of X. (i) $p(X = 1 or 2)$; (ii) $p(1/2 < X < 5/2)$ and (iii) $p(1X \leq 3)$.
2. Find the distribution function of X in Exercise 1 and hence compute the probabilities.
3. If $f(x) = K\left(4x - 2x^2\right)$; $0 < x < 2$; zero elsewhere is the pdf of a random variable X find (i) $p(1/2 < X < 3/2)$; (ii) $p(1/2 < X < 3/2 \diagup X > 1)$; (iii) $p(X < 3/2 \diagup X > 1)$.
4. Find the distribution function of X in Exercise 3.
5. If a coin is biased so that it has a probability 0.7 of coming up heads. It is tossed 3 times. Let X be the number of heads that appear in the three tosses. Find the probability distribution and distribution function of X. Also find what is the smallest value of x for which $p(X \leq x) > 0.5$.
6. If X has the distribution function

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4} & 1 \leq x < 3 \\ \frac{1}{2} & 3 \leq x < 5 \\ \frac{3}{4} & 5 \leq x < 7 \\ 1 & x \geq 7 \end{cases}$$

   Find (a) $p(X \leq 5)$; (b) $p(X = 5)$; (c) $p(X < 5)$; (d) $p(X \geq 3)$; (e) $p(1.4 < X < 6)$ and (f) $p(X = 7)$.
7. Find the pdf of the random variable X in Exercise 6 and hence calculate the probabilities.
8. The pdf of a random variable X is given by $f(x) = K$; $-2 < x < 2$; zero elsewhere. Find $p(|x| < 1)$. Also, find its distribution function and reevaluate the probability.
9. If $f(x) = Kx^2(1 - x)$, $0 < x < 1$; zero elsewhere is the pdf of X find $p(X < 1/4)$ and $p(X > 1/2)$. Find the distribution and re-evaluate the probability.
10. Find the distribution function of the random variable X whose pdf is given by

$$f(x) = \begin{cases} kx & 0 < x \leq 1 \\ k & 1 < x \leq 2 \\ k(3 - x) & 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

   Find the distribution function of X and hence evaluate (a) $p(X = 1)$; (b) $p(1 < X < 3)$; (c) $p(X > 2)$.

11. The distribution function of X is given by

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Find the pdf of X.

12. Find the median and mode of a distribution of a random variable X whose pdf is

(a) $f(x) = \dfrac{1}{\pi\left(1 + x^2\right)}; \quad -\infty < x < \infty$

(b) $f(x) = \dfrac{x}{15}; \qquad x = 1, 2, 3, 4, 5;$

zero elsewhere.

13. The daily consumption of milk (in millions of liters) is a random variable whose pdf is given by

$$f(x) = \begin{cases} \frac{1}{4}(x)e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

What are the probabilities that on a given day (a) the consumption in this city is not more than 5 million liters; (b) milk supply is inadequate if the daily capacity of this city is 9 million liters ?

14. The lifetime of a certain brand of tyre (in thousands of kilometers) is a random variable whose pdf is proportional to $e^{-x/20}$ for $x > 0$. Find the probabilities that one of these tyres will last (a) atmost 1500 kilometers; (b) anywhere from 20000 to 35000 kilometers and (c) atleast 40000 kilometers.

15. If the lifetime of an electronic component is a random variable whose distribution function is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/10} & x \geq 0 \end{cases}$$

Find the probability that the lifetime is (a) less than 10; (b) between 5 and 15 and (c) atleast 8.

2.2 BIVARIATE DISTRIBUTIONS

2.2.1 Probability Functions

We generalise the concept of probability distribution of a random variable to the joint distribution of two random variables. The need of such generalization is often experiments are conducted where two random variables are observed simultaneously in order to determine not only their individual behaviour but also the degree of relationship between them.

Let X and Y be two random variables whose space be $\boldsymbol{A}_x$ and $\boldsymbol{A}_y$ respectively. Let $\boldsymbol{A} = \{(x, y)/x\epsilon \boldsymbol{A}_x$ and $y\epsilon \boldsymbol{A}_y\}$. Then $\boldsymbol{A}$ is the space of the random variable (x, y) in R^2 $\boldsymbol{A}$ is finite or infinite accordingly as $\boldsymbol{A}_x$ and $\boldsymbol{A}_y$ are finite or infinite.

If $f(x, y)$ is a function defined on $\boldsymbol{A}$ is defined as the pdf of the random variable (X, Y) if it satisfies the two conditions namely non-negativity and the total probability is one we define it more formally as follows.

Let X and Y be two discrete random variables and let $\boldsymbol{A}_x = \{x_1, x_2, \cdots, x_m, \cdots\}$ and $\boldsymbol{A}_y\{y_1, y_2, \cdots, y_n, \cdots\}$. So that $\boldsymbol{A} = \boldsymbol{A}_x \times \boldsymbol{A}_y = \{(x_i, y_j)/x_i\epsilon A_x; y_j\epsilon A_y\}$. Then a function $f(x, y)$ on $\boldsymbol{A}$ is defined as a pdf of the R.V (x, y) if

(a) $f(x_i, y_j) \geq 0 \qquad \forall(x_i, y_j)$

(b) $\displaystyle\sum_i \sum_j f(x_i, y_j) = 1 \quad (x_i, y_j) \in \boldsymbol{A}$

On the other hand, if x and y are continuous random variables with spaces $\boldsymbol{A}_x$ and $\boldsymbol{A}_y$ respectively, then $\boldsymbol{A} = \boldsymbol{A}_x \times \boldsymbol{A}_y = \{(x, y)/x\epsilon \boldsymbol{A}_x, y\epsilon \boldsymbol{A}_y\}$ is the space of the R.V (x, y). A function $f(x, y)$ defined on $\boldsymbol{A}$ is defined as a pdf of (x, y) if

(a) $f(x, y) \geq 0 \qquad \forall(x, y)$

(b) $\displaystyle\int\int_{(x,y)\,\in A} f(x, y) = 1$

We may extend the definition of a pdf of $f(x, y)$ over the entire xy plane so that the references to the space $\boldsymbol{A}$ can be avoided. Once this is done we replace the (ii) condition of the above two definitions as

$$\sum_{x_i} \sum_{y_j} f(x_i, y_j) \quad \text{and} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy$$

respectively for the discrete and continuous random variables. At this point we can compare the definition of pdf of a 1-dimensional or 2-dimensional R.V (discrete or continuous), essentially it satisfies the conditions of being a pdf (a) f is defined and is not negative for all real values of its arguments, (b) its integral or sum (according to continuous or discrete) over all real values of arguments is 1. This idea will help us to generalise or extend the notion of pdf of a random variable to even more than 2 variables. Let us consider following examples relating to the preceeding points.

■ Example 2.37

Consider a familiar experiment of tossing 3 coins simultaneously. Let X denote the number of heads occuring and let Y denote the longest string of heads occuring. Let us find the joint pdf of X and Y.

Clearly the space of X is $\boldsymbol{A}_x = \{0, 1, 2, 3\}$ and that of Y is $\boldsymbol{A}_y = \{0, 1, 2, 3\}$. (Here string of heads means consecutive occurance of heads). Define a pdf $f(x, y)$ on # its space, $\boldsymbol{A} = \{(x, y)/x = 0, 1, 2, 3;\ y = 0, 1, 2, 3\}$.

Now let us consider two particular cases:

(a) $$\begin{aligned} f(2, 2) &= p(x = 2 \text{ and } y = 2) \\ &= p(\text{Number of heads} = 2 \text{ and longest string of heads} = 2) \\ &= \frac{2}{8} \text{ (corresponding outcomes are HHT, THH)} \end{aligned}$$

(b) $$\begin{aligned} f(2, 1) &= p(x = 2) \text{ and } y = 1 \\ &= \frac{1}{8} \text{ (HTH)} \end{aligned}$$

Let us represent all such calculations in a more convenient tabular form.

$x \diagdown y$	0	1	2	3
0	$\frac{1}{8}$	0	0	0
1	0	$\frac{3}{8}$	0	0
2	0	$\frac{1}{8}$	$\frac{2}{8}$	0
3	0	0	0	$\frac{1}{8}$

One can easily check that two conditions of a pmf are satisfied. ■

■ Example 2.38

Three coins are tossed. Let X be the number of heads on the first two coins. Y be the number of tails on the last two. Let us form the joint pdf of X and Y.

Spaces of X and Y are $\boldsymbol{A}_x = \boldsymbol{A}_y = \{0, 1, 2\}$ (Remember the sample space if the experiment continuous 8 points).

$\therefore$ The space of $\boldsymbol{A} = \boldsymbol{A}_x \times \boldsymbol{A}_y = \{(x, y)/x = 0, 1, 2;\ y = 0, 1, 2\}$

Forinstance, $$\begin{aligned} f(1, 2) &= p(x = 1, y = 2) \\ &= P(\text{Number of heads on the first two coin} = 1 \\ &\quad \text{and number of tails in the last two coins} = 2) \\ &= \frac{1}{8} \text{ (HTT)} \end{aligned}$$

Similarly, $$f(1, 1) = \frac{2}{8} \text{ (HTH, THT)}$$

$\therefore$ Joint pdf is summarized as,

$Y \diagdown X$	0	1	2
0	0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	$\frac{1}{8}$	$\frac{1}{8}$	0

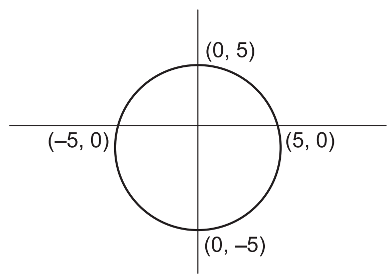
Note that the table can be constructed in any manner an as in previous example. The choice of the values of X and Y should be made correctly to understand the joint probabilities of X and Y. ■

■ Example 2.39

In this example let us consider a bivariate pdf in continuous case. If we wish to select a point randomly from inside a circle $x^2 + y^2 = 25$, let us determine the joint pdf of X and Y.

Here $\boldsymbol{A}_x = [-5, 5]$ and $\boldsymbol{A}_y = [-5, 5]$

$$\begin{aligned} \therefore \quad \boldsymbol{A} &= \boldsymbol{A}_x \times \boldsymbol{A}_y \text{ and interior of the circle } x^2 + y^2 = 25 \\ &= \left\{ (x, y) \Big/ \begin{array}{c} -5 \le x \le 5, -5 \le y \le 5 \\ \text{and } x^2 + y^2 < 25 \end{array} \right\} \end{aligned}$$



is the space of (X, Y) which represents a random point inside the circle $X^2 + Y^2 = 25$.
i.e., the joint pdf of X and Y is constant over $\boldsymbol{A}$ and 0 outside.

$$\text{i.e.,} \qquad f(x, y) = \begin{cases} C & (x, y) \in \boldsymbol{A} \\ 0 & \text{elsewhere} \end{cases}$$

But we must have

$$\begin{aligned} \int\limits_{(x,y)} \int\limits_{\in \boldsymbol{A}} f(x, y)dxdy = 1 \quad &\Rightarrow \quad C \int\limits_{(x,y)} \int\limits_{\in \boldsymbol{A}} f(x, y)dxdy = 1 \\ &\Rightarrow \quad C(\text{Area of the circle}) = 1 \\ &\Rightarrow \quad C \cdot 25\pi = 1 \\ &\Rightarrow \quad C = \frac{1}{25\pi} \end{aligned}$$

$$\therefore \quad f(x, y) = \begin{cases} \frac{1}{25\pi} & |x| \le 5, |y| \le 5 \\ & \text{and } x^2 + y^2 \le 25 \\ 0 & \text{elsewhere} \end{cases}$$

This is a continuous random variable one can easily check the conditions of a pdf. The representation we used to denote the joint pdf of a discrete R.V (X, Y) can be generalised as:

$X \diagdown Y$	y_1	y_2	$\cdots$	y_n	$\cdots$
x_1	p_{11}	p_{12}	$\cdots$	p_{1n}	$\cdots$
x_2	p_{21}	p_{22}	$\cdots$	p_{2n}	$\cdots$
$\vdots$					
x_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}	$\cdots$
$\vdots$					

Here we denote $p_{ij} = p\left(X = x_i, Y = y_j\right) = f\left(x_i, y_j\right)$ as the probability that X assumes 'x_i' and y assumes y_j. ■

Calculating normalizing constant

■ Example 2.40

The joint pdf of X and Y is

$$f(x, y) = \begin{cases} K(x + y) & 0 < x < 2 \quad \text{and} \quad 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find 'K' since (X, Y) is continuous R.V.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1 \quad \Rightarrow \quad \int_{y=0}^{2}\int_{x=0}^{2} K(x + y)dxdy = 1$$

$$\Rightarrow \quad K\int_{y=0}^{2}\left(\frac{x^2}{2} + xy\right)_{x=0}^{2} dy = 1$$

$$\Rightarrow \quad K\int_{y=0}^{2}(2 + 2y)dy = 1$$

$$\Rightarrow \quad K\left(2y + y^2\right)_0^2 = 1$$

$$\Rightarrow \quad 8K = 1; \quad \therefore \quad K = \frac{1}{8}$$

$\therefore$ The pdf is

$$f(x) = \begin{cases} \frac{1}{8}(x+y) & 0 < x, y < 2 \\ 0 & \text{elsewhere} \end{cases}$$ ■

■ Example 2.41

The joint pdf of (X, y) is

$$f(x, y) = \begin{cases} K(x+2y) & x = 1, 2, 3; \quad y = 0, 1, 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the constant 'K'.

Since X and Y are discrete the R.V (X, Y) is also a discrete random variable.

$$\therefore \quad \sum_{x=1}^{3}\sum_{y=0}^{2} K(x+2y) = 1 \quad \Rightarrow \quad K\left[\sum_{x=1}^{3} x + \sum_{x=1}^{3}(x+2) + \sum_{x=1}^{3}(x+4)\right] = 1$$

$$\Rightarrow \quad K\,[(1+2+3) + (3+4+5) + (5+6+7)] = 1$$

$$K(36) = 1$$

$$\therefore \quad K = \frac{1}{36}$$

$\therefore$ Joint pdf can be summarized as:

$X \diagdown Y$	0	1	2
1	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$
2	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{6}{36}$
3	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$

■

2.2.2 Marginal Probability Functions

Let $f(x, y)$ be the pdf of two R.Vs X and Y. By assuming X and Y as R.Vs in one variable they have pdfs on their own satisfying the two conditions. This pdfs can be found from the joint pdf of (X, Y). Such pdfs are said to be marginal pdf of X and that of Y. We define them as follows.

M.p.d.f of X:

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f(x, y)\,dy & \text{for the continuous case} \\ \sum_{y} f(x, y) & \text{for the discrete case} \end{cases}$$

Similarly, the mpdf of Y is

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f(x, y)dx & \text{for continuous case} \\ \sum_x f(x, y) & \text{for discrete case} \end{cases}$$

■ Example 2.42

Find the marginal pdf of X and Y in each of the following cases.

Let the joint pdf of X and Y be

$$\text{(a)} \quad f(x, y) = \begin{cases} K(x_0 + y) & x = 1, 2, 3; \quad y = 1, 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{(b)} \quad f(x, y) = \begin{cases} y^2 & 0 \leq x \leq 2, 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) (X, Y) is a discrete R.V

$X \diagdown Y$	1	2
1	$2K$	$3K$
2	$3K$	$4K$
3	$4K$	$5K$

$X \diagdown Y$	1	2
1	$\frac{2}{21}$	$\frac{3}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$
3	$\frac{4}{21}$	$\frac{5}{21}$

$$\therefore \quad \Sigma\Sigma f(x, y) = 1 \quad \Rightarrow \quad 2K + 3K + 3K + 4K + 4K + 5K = 1$$

$$K = \frac{1}{21}$$

To find mpdfs:

(1)

$X = x$	1	2	3
$f_X(x)$	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$

(2)

$Y = y$	1	2
$f_Y(y)$	$\frac{9}{21}$	$\frac{12}{21}$

Note that pdf of X of Y $f_X(x)$ and $f_Y(y)$ satisfy the required conditions as a pdf of one variable.

(b) Let us first find the value of 'C'

$\because$ (X, Y) is continuous.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1 \quad \Rightarrow \quad \int_{y=0}^{1}\int_{x=0}^{2} Cy^2dxdy = 1$$

$$\Rightarrow \quad C\int_{y=0}^{1} y^2(x)_0^2\, dy = 1$$

$$\Rightarrow \quad 2C\left(\frac{Y^3}{3}\right)_0^1 = 1$$

$$\Rightarrow \quad C = \frac{3}{2}$$

$\therefore$ Joint pdf is

$$f(x, y) = \begin{cases} \frac{3}{2}y^2 & 0 \le x \le 2; \quad 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

To find mpdfs:

$$\text{(i)} \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 \left(\frac{3}{2}\right) y^2dy$$

$$= \frac{3}{2}\left(\frac{y^3}{3}\right)_0^1 = \frac{1}{2}$$

$$\therefore \quad f_X(x) = \begin{cases} \frac{1}{2} & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{(ii)} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^2 \left(\frac{3}{2}\right) y^2dx$$

$$= \left(\frac{3}{2}\right) y^2(x)_0^2$$

$$= \begin{cases} (3)y^2 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

One can easily find that $f_X(x)$ and $f_Y(y)$ are satisfying the conditions of a pdf in one variable case. ■

2.2.3 Stochastic Independence

Let the random variables X and Y have the joint pdf $f(x, y)$ and the marginal probability density functions $f_X(x)$ and $f_Y(y)$ respectively. The random variables X and Y are said to be stochastically independent if and only if $f(x, y) = f(x) \cdot f(y)$.

Remark

- ◇ Random variables that are not independent are said to be stochastically dependent.
- ◇ If the spaces of X and Y are $\boldsymbol{A}_x$ and $\boldsymbol{A}_y$ then $f_X(x)$ is positive on, and only on $\boldsymbol{A}_x$ and $f_Y(y)$ is on $\boldsymbol{A}_y$. Hence, their product $f_X(x) \cdot f_Y(y)$ is positive on and only on $\boldsymbol{A} = \boldsymbol{A}_x \times \boldsymbol{A}_y = \{(x, y)/x\epsilon\boldsymbol{A}_x;\ y\epsilon\boldsymbol{A}_y\}$. i.e., we define such spaces $\boldsymbol{A}$ as the product space of $\boldsymbol{A}_x$ and $\boldsymbol{A}_y$.
- ◇ If there exists a pair $\left(x_i, y_j\right)$ in $\boldsymbol{A}$ such that $f\left(x_i, y_j\right) \neq f_X\left(x_i\right) \cdot f_Y\left(y_j\right)$ then X and Y are not independent variables, where (X, Y) is a discrete R.V.

■ **Example 2.43**

Let the joint pdf of (X, Y) is

$$f(x, y) = \begin{cases} \frac{1}{4} & (x, y) : (1, 1), (1, 0), (0, 1), (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

We can easily construct the tabular form as

$Y \backslash X$	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

and $\quad \boldsymbol{A} = \{(x, y)/x = 0, 1;\ y = 0, 1\}$

Then the mpdf of X and Y are:

$X{:}x$	0	1
$f_X(x)$	$\frac{1}{2}$	$\frac{1}{2}$

$Y{:}y$	0	1
$f_Y(y)$	$\frac{1}{2}$	$\frac{1}{2}$

Now $\quad f_X(0) \cdot f_Y(0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = f(0, 0)$

$$\begin{aligned} f_X(0)\cdot f_Y(1) &= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = f(0,1) \\ f_X(1)\cdot f_Y(0) &= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = f(1,0) \\ f_X(1)\cdot f_Y(1) &= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = f(1,1) \\ \therefore \qquad f(x,y) &= f_X(x)\cdot f_Y(y) \quad \forall (x,y)\epsilon \boldsymbol{A} \end{aligned}$$

$\therefore$ X and Y are independent. ■

■ Example 2.44

Let the joint pdf of X and Y be:

$X \backslash Y$	0	1
0	K	$2K$
1	$4K$	$2K$
2	K	0

Check whether X and Y are independent. First let us find the value of 'K'

$$\Sigma\Sigma f(x,y) = 1 \quad \Rightarrow \quad K + 2K + 4K + 2K + K + 0 = 1$$

$$K = \frac{1}{10}$$

$\therefore$ Joint pdf is:

$X \backslash Y$	0	1
0	$\frac{1}{10}$	$\frac{2}{10}$
1	$\frac{4}{10}$	$\frac{2}{10}$
2	$\frac{1}{10}$	0

Marginal pdf of x and y

$X = x$	0	1	2
$f_X(x)$	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{1}{10}$

$Y = y$	0	1
$f_Y(y)$	$\frac{6}{10}$	$\frac{4}{10}$

$$\begin{aligned} f_X(0) \cdot f_Y(0) = \frac{3}{10} \cdot \frac{6}{10} = \frac{18}{100} &= 0.18 \\ &\neq f(0,0) \end{aligned}$$

$\therefore$ X, Y are not independent. ■

■ Example 2.45

If the joint probability density of two random variables is

$$f(x, y) = \begin{cases} (6)e^{-2x-3y} & x, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Check whether X and Y are independent.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y)dy \\ &= \int_0^{\infty} (6)e^{-2x-3y}dy = (6)e^{-2x}\left(\frac{e^{-3y}}{-3}\right)_0^{\infty} \\ &\qquad = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x, y)dx \\ &= \int_0^{\infty} (6)e^{-2x-3y}dx = (6)e^{-3y}\left(\frac{e^{-2x}}{-2}\right)_0^{\infty} \\ &\qquad = \begin{cases} 3e^{-3y} & y > 0 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

Now
$$\begin{aligned} f_X(x) \cdot f_Y(y) &= (2)e^{-2x} \cdot (3)e^{-3y} \\ &= (6)e^{-2x-3y} = f(x, y) \end{aligned}$$

$\therefore$ X and Y are independent. ■

■ Example 2.46

Let the joint pdf of (X, Y) is

$$f(x, y) = \begin{cases} \frac{1}{4}(1 + xy) & |x| < 1; |y| < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find whether X and Y are not independent.

Marginal pdfs of X and Y:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y)dy = \int_{-1}^{1} \frac{1}{4}(1 + xy)dy \\ &= \frac{1}{4}\left(y + \frac{xy^2}{2}\right)_{-1}^{1} \\ f_X(x) &= \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \\ f_Y(y) &= \int_{-\infty}^{\infty} f(x, y)dx = \int_{-1}^{1} \frac{1}{4}(1 + xy)dx \\ &= \frac{1}{4}\left(x + \frac{x^2 y}{2}\right)_{-1}^{1} \\ f_Y(y) &= \begin{cases} \frac{1}{2} & -1 < y < 1 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

Now $f_X(x) \cdot f_Y(y) \neq f(x, y)$.

Hence X and Y are not independent.

Next let us consider a result (without proof) to check whether the random variables X and Y are independent without computing the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

■

□ Result

Let X and Y have the joint pdf $f(x, y)$. Then X and Y are stochastically independent if and only if $f(x, y)$ can be written as a product of non-negative function of x alone and a non-negative function of y alone.

i.e., $f(x, y) = g(x)h(y)$ where $g(x) > 0 \; \forall x \epsilon \boldsymbol{A}_x$ and $h(y) > 0 \; y \epsilon \boldsymbol{A}_y$ and 0 elsewhere. However, this result is applicable if the space $\boldsymbol{A}$ of (X, Y) is the product space of $\boldsymbol{A}_x$ and $\boldsymbol{A}_y$.

For example, if we consider the joint pdf of (X, Y) is

$$f(x, y) = \begin{cases} e^{-(x+y)} & x, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Here $\boldsymbol{A}_x = (0, \infty)$ and $\boldsymbol{A}_y = (0, \infty)$ so that $\boldsymbol{A} = \{(x, y)/x\epsilon(0, \infty);\ y\epsilon(0, \infty)\}$.

$$\text{Now} \qquad \begin{aligned} f(x, y) &= e^{-x} \cdot e^{-y} \\ &= h(x) \cdot g(y) \end{aligned}$$

$\therefore$ X and Y are independent.

Consider another case, the pdf of (X, Y) is $f(x, y) = 8xy,\ 0 < x < y < 1$ (Marginal pdfs we shall calculate later in this chapter).

But $8xy$ might suggest to some that X and Y are independent [$f(x, y)$ can be expressed as a product of two functions $g(x)$ and $h(y)$]. However X and Y are not independent since $\boldsymbol{A} = \{(x, y)/0 < x < y < 1\}$ is not a product space.

We may discuss some other results to check whether two random variables X and Y are independent (without computing marginal pdfs) in this chapter.

2.2.4 Conditional Distributions

We shall now define the notion of a conditional pdf. Let X and Y denote random variables which have the joint pdf $f(x, y)$ with space $\boldsymbol{A}$. Let $f_X(x)$, $f_Y(y)$ be their marginal probability density function respectively. Then the conditional probability that $Y = y$ given that $X = x$ is defined as

$$f(x/y) = \frac{f(x, y)}{f_X(x)}$$

where $f_X(x) > 0$.

[Remember the definition of conditional probability $p(A/B) = \frac{p(A \cap B)}{p(B)}$]

In a similar way we can define the conditional pdf of $X = x$ given that $Y = y$ as

$$f_{Y/X}(x/y) = \frac{f(x, y)}{f_Y(y)}$$

provided $f_Y(y) > 0$.

Depending upon the nature of the random variable (X, Y) as discrete or continuous random variable, the computation of $f_X(x)$ and $f_Y(y)$ will be summation or integration (we discussed in our earlier section).

■ Example 2.47

Let X and Y have the joint pdf

$$f(x, y) = \begin{cases} x + y & 0 < x < 1; \\ & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional pdf of X given Y and Y given X.

First let us find the mpdfs of X and Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 (x+y)dy$$
$$= \left(xy + \frac{y^2}{2}\right)_0^1$$
$$= \begin{cases} x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 (x+y)dx$$
$$= \left(\frac{x^2}{2} + yx\right)_0^1$$
$$= \begin{cases} y + \frac{1}{2} & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Now the conditional pdfs can be computed using respective formulas.

$$f_{X/Y}(x/y) = \frac{f(x, y)}{f_Y(y)}$$
$$= \begin{cases} \frac{x+y}{y+1/2} & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$
$$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} \frac{x+y}{x+1/2} & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

■

■ Example 2.48

Let X and Y have the joint pdf of

$$f(x, y) = \frac{x + 2y}{18} \quad (x, y) = (1, 1), (1, 2), (2, 1), (2, 2);\ \text{zero elsewhere.}$$

Find the conditional pdfs.

(Remark: Since (X, y) is discrete we need to compute the conditional pdfs for each value of X and that of Y.)

Let the joint pdf be expressed as:

$X \diagdown Y$	1	2
1	$\frac{3}{18}$	$\frac{5}{18}$
2	$\frac{4}{18}$	$\frac{6}{18}$

$\therefore$ The marginal pdf of X and Y are:

$X = x$	1	2
$f_X(x)$	$\frac{8}{18}$	$\frac{10}{18}$

$Y = y$	1	2
$f_Y(y)$	$\frac{7}{18}$	$\frac{11}{18}$

The conditional pdf of X given Y.

$$f_{X/Y}(X = 1/Y = 1) = \frac{f(1,1)}{f_Y(1)} = \frac{3/18}{7/18} = \frac{3}{7}$$
$$f_{X/Y}(X = 2/Y = 1) = \frac{f(2,1)}{f_Y(1)} = \frac{4/18}{7/18} = \frac{4}{7}$$

$f_{X/Y}(X/Y = 1)$:

$X = x$	1	2
$f_{X/Y}(X/1)$	$\frac{3}{7}$	$\frac{4}{7}$

$$f_{X/y}(X = 1/Y = 2) = \frac{f(1,2)}{f_Y(2)} = \frac{5/18}{11/18} = \frac{5}{11}$$
$$f_{X/Y}(X = 2/Y = 2) = \frac{f(2,2)}{f_Y(2)} = \frac{6/18}{11/18} = \frac{6}{11}$$

$\therefore$ $f_{X/Y}(X/Y = 2)$:

$X = x$	1	2
$f_{X/Y}(X/2)$	$\frac{5}{11}$	$\frac{6}{11}$

The conditional pdf of Y given X:

$$f_{Y/X}(Y = 1/X = 1) = \frac{f(1,1)}{f_X(1)} = \frac{3/18}{8/18} = \frac{3}{8}$$
$$f_{Y/X}(Y = 2/X = 1) = \frac{f(1,2)}{f_X(1)} = \frac{5/18}{8/18} = \frac{5}{8}$$

$X = x$	1	2
$f_{Y/X}(y/1)$	$\frac{3}{8}$	$\frac{5}{8}$

The conditional pdf of Y given X:

$$f_{Y/X}(y = 1/X = 2) = \frac{f(2,1)}{f_X(2)} = \frac{4/18}{10/18} = \frac{4}{10}$$
$$f_{Y/X}(y = 2/X = 2) = \frac{f(2,2)}{f_X(2)} = \frac{6/18}{10/18} = \frac{6}{10}$$

$Y = y$	1	2
$f_{Y/X}(y/2)$	$\frac{4}{10}$	$\frac{6}{10}$

■

Remark

In the above two examples one can observe that $f_{X/Y}(x/y)$ is a function of x and it satisfies the conditions of a pdf.

$$\left[f_{X/Y}(x/y) = \frac{x+y}{y+1/2} \quad \text{so that} \quad \int_{-\infty}^{\infty} f(x/y)dx = \int_0^1 \left(\frac{x+y}{y+1/2}\right) dx = 1 \right]$$

Also, $f_{Y/X}(y/x)$ is a function of y and it satisfies the conditions of a pdf.

$$\left[f_{Y/X}(y/x) = \frac{x+y}{x+1/2} \quad \text{so that} \quad \int_{-\infty}^{\infty} f_{Y/x}(y/x)dy = \int_0^1 \frac{x+y}{x+1/2} dy = 1 \right]$$

In a similar way is the proceeding discrete case we can easily find that $f_{X/Y}(x/y)$ and $f_{Y/X}(Y/x)$ (for each choice of X and Y) are functions of x and y respectively and they satisfy the conditions of a pdf. This idea will be applied in our later chapters.
A word about marginal pdfs: We observe the importance of marginal pdfs in a Bivariate probability distributions. If the joint pdf is symmetry function in X and Y that is $f(X, Y) = f(Y, X)$ then the marginal pdfs will also be symmetric function in X and Y (of course, this idea will help in computations in the subsequent chapters).
For symmetric pdfs refer Examples 2.43 (Discrete case) and Example 2.46 (Continuous case).

2.2.5 Joint Distributions Function

We shall extend the concept of *commulative distribution function* (cdf) to the 2-variable case. We write $F(x, y)$ as the probability that X takes on a value less than or equal to x (i.e., $X \leq x$) and Y

takes on a value less than or equal to y (i.e., $Y \le y$) and we refer to the corresponding function F as the joint commulative function of the 2 random variables.

$$\therefore \quad F(x, y) = p(X \le x, Y \le y)$$

In case of continuous case, $F(x, y)$ can be formulated as,

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v)\, du dv$$

(we use u and v as dummy variables for integration purpose).

■ Example 2.49

Find the joint distribution function of the two random variables x, y whose joint pdf is

$$f(x, y) = \begin{cases} (6)e^{-2x-3y} & x, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Applying the definition of joint cdf

$$\begin{aligned} F(x, y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} (u, v) du dv \\ &= \int_{0}^{y} \int_{0}^{x} (6)e^{-24-3v} du dv \\ &= 6 \int_{0}^{y} \left(\frac{e^{-24}}{-2} \right)_{0}^{x} e^{-3v} dv \\ &= 3 \int_{0}^{y} \left(1 - e^{-2x}\right) e^{-3v} dv \\ &= 2\left(1 - e^{-2x}\right) \left(\frac{e^{-3v}}{-3} \right)_{0}^{y} \\ F(x, y) &= \begin{cases} (1 - e^{-2x})(1 - e^{-3y}) & x > 0;\, y > 0 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

■

Marginal distribution functions

From the knowledge of joint distribution function $F(x, y)$ it is possible to obtain the distribution function of X and Y which are said to be cdf of X and Y respectively.

$$\text{i.e.,} \qquad \begin{aligned} F_X(x) = p(X \le x) &= p(X \le x, Y < \infty) \\ &= F(x, \infty) = \lim_{y \to \infty} F(x, y) \end{aligned}$$

Similarly, $$\begin{aligned} F_Y(y) = p(Y \le y) &= p(X < \infty,\ Y \le y) = F(\infty, y) \\ &= \lim_{x\to\infty} F(x, y) \end{aligned}$$

Let us use this definition to the proceeding example.

$$\begin{aligned} F_X(x) &= \lim_{y\to\infty} F(x, y) \\ &= \lim_{y\to\infty} \left(1 - e^{-2x}\right)\left(1 - e^{-3y}\right) \\ &= 1 - e^{-2x} \end{aligned}$$

and $$\begin{aligned} F_Y(y) &= \lim_{x\to\infty} F(x, y) \\ &= \lim_{x\to\infty} \left(1 - e^{-2x}\right)\left(1 - e^{-3y}\right) \\ &= 1 - e^{-3y} \end{aligned}$$

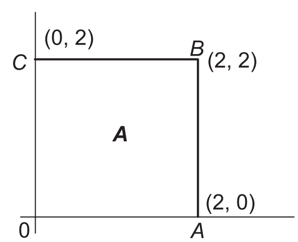
■ Example 2.50

Let the joint pdf of (X, Y) is

$$f(x, y) = \begin{cases} \frac{x^3 y^3}{16} & 0 \le x \le 2; \quad 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint distribution 8 X.

$$\boldsymbol{A} = \{(x, y)/0 \le x \le 2; 0 \le y \le 2\}$$



The joint distribution function is,

$$F(x, y) = p(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u, v)\,dv\,du$$

If $(x, y) \epsilon \boldsymbol{A}$ then

$$\begin{aligned} F(x, y) &= \int_0^x\int_0^y \frac{u^3 v^3}{16} dv\,du \\ &= \frac{1}{16}\int_0^x u^3 \left(\frac{v^4}{4}\right)_0^y du = \frac{1}{64} y^4 \int_0^x u^3 du \end{aligned}$$

Indeed $F(x, y)$ has to be calculated for every point in the xy-plane.

Here $$A = \{(x, y)/0 \le x \le 2;\ 0 \le y \le 2\}$$

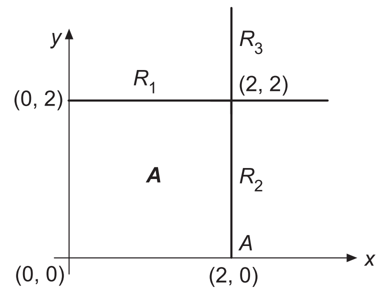
i.e., $$f(x, y) = 0 \quad \forall (x, y) \notin A$$

Let us consider the following 3 cases with $x > 0$, $y > 0$ (for if $x < 0$, $y < 0$, $F(x, y) = 0$).

(i) $0 \le x \le 2$ and $y > 2$

(ii) $x > 2$ and $0 \le y \le 2$

(iii) $x > 2$ and $y > 2$



Case (i): The region of integration is $R_1 \cup A$

$$\begin{aligned}
\therefore \quad F(x, y) &= p(X \le x,\ Y \le y) \\
&= \int_{u=0}^{x}\int_{v=0}^{y} f(u, v)\,dv\,du \quad \text{(as per definition)} \\
&= \int_{u=0}^{x}\int_{v=0}^{2} f(u, v)\,dv\,du + \int_{u=0}^{x}\int_{v=0}^{y} f(u, v)\,dv\,du \\
&= \int_{u=0}^{x}\int_{v=0}^{2} \frac{u^3v^3}{16}\,dv\,du + 0 \quad [\because\ (x, y) \in R_1,\quad f(x, y) = 0] \\
F(x, y) &= \frac{x^4}{16} \quad \text{if} \quad (x, y) \in R_1
\end{aligned}$$

Case (ii): The region of integration is $A \cup R_2$

$$\begin{aligned}
\therefore \quad F(x, y) &= p(X \le x, Y \le y) = \int_{v=0}^{y}\int_{u=0}^{x} f(u, v)\,du\,dv \\
&= \int_{v=0}^{y}\int_{u=0}^{2} f(u, v)\,du\,dv + \int_{v=0}^{y}\int_{u=2}^{x} f(u, v)\,du\,dv \\
&= \int_{v=0}^{y}\int_{u=0}^{2} \frac{1}{16}u^3v^3\,du\,dv + 0 \quad (\because\ \text{second integral is over } R_2) \\
F(x, y) &= \frac{1}{16}y^4 \quad \text{if} \quad (x, y) \in R_2
\end{aligned}$$

Case (iii): The region of integration is $\boldsymbol{A} \cup R_3$

$$
\begin{aligned}
\therefore \qquad F(x, y) &= \int_0^x \int_0^y f(u, v) dv du = \int_0^2 \int_0^2 f(u, v) dv du + \int_2^x \int_2^y f(u, v) dv du \\
&= 1 + 0 = 1 \\
\therefore \qquad F(x, y) &= \begin{cases} 0 & x < 0;\ y < 0 \\ \left(\frac{1}{256}\right) x^4 y^4 \quad [(x, y) \epsilon \boldsymbol{A}] & 0 \le x \le 2;\ 0 \le y \le 2 \\ \left(\frac{1}{16}\right) x^4 & 0 \le x \le 2;\ y > 2 \\ \left(\frac{1}{16}\right) y^4 & x > 2;\ 0 \le y \le 2 \\ 1 & x > 2;\ y > 2 \end{cases} \\
\text{Now} \qquad F_X(x) &= \lim_{y \to \infty} F(x, y) \\
&= \lim_{y \to 2} F(x, y) = \lim_{y \to 2} \frac{x^4 y^4}{256} = \frac{x^4}{16} \\
F_X(x) &= \begin{cases} 0 & x < 1 \\ x^4/16 & 0 \le x \le 2 \\ 1 & x > 2 \end{cases} \\
\text{III}^{ey} \qquad F_Y(y) &= \begin{cases} 0 & y < 1 \\ y^4/16 & 0 \le y \le 2 \\ 1 & y > 2 \end{cases}
\end{aligned}
$$

Above example illustrates than the idea of computing marginal cdf from joint cdf. When we let $x \to \infty$ or $y \to \infty$ to space X and Y as well as the space of (X, Y) is be considered.

PROPERTIES OF JOINT DISTRIBUTIONS

(1) (i) $F(-\infty, y) = 0$, (ii) $F(x, -\infty) = 0$ (iii) $F(-\infty, -\infty)$ (iv) $F(\infty, \infty) = 1$.

(2) If the A.V is continuous then its j.p.d.f $f(X, Y)$ obtained from its joint p.d.f $F(x, y)$

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

(3) For the real numbers a_1 b_1 a_2 and $p\ (a_1 < X \le b_1, a_2 < Y \le b_2) = F\ (a_1 a_2) + F\ (b_1 b_2) - F\ (a_1 b_2) - F\ (b_1 a_2)$.

Summary

The ideals so far we gained expressed as a flow diagram to remember and follow their relationships.

Remark

Following examples will provide a set of illustration for the ideas we completed so far, in 2-D random variable cases. Further, in continuous cases if emphazises the importance of calculus.

2.2.6 Additional Examples

■ Example 2.51

Let the joint pdf of (X, Y) is

$Y \backslash X$	1	2	3
1	K	K	$2K$
2	$2K$	$3K$	K

(a) Find the value of K.
(b) Find the conditional pdf of X given $Y = 2$.
(c) Find $p(X \leq 2, Y = 2)$.
(d) Find $p(X < 3)$; $p(Y = 2)$.

(a) To find K:
Since (X, Y) is discrete, $\sum\limits_x \sum\limits_y f(x, y) = 1$

$$\Rightarrow \quad K + K + 2K + 2K + 3K + K = 1$$

$$\therefore \quad K = \frac{1}{10} = 0.1$$

Joint pdf of X and Y:

$Y \backslash X$	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Marginal pdf of X and Y:

$X = x$	1	2	3
$f_X(x)$	0.3	0.4	0.3

$Y = y$	1	2
$f_Y(y)$	0.4	0.6

(b)
$$f_{X/Y}(X/Y=2) = \frac{f(x,2)}{f_Y(2)}$$
$$f_{X/Y}(X=1/Y=2) = \frac{f(1,2)}{f_Y(2)} = \frac{0.2}{0.6} = \frac{1}{3}$$
$$f_{X/Y}(X=2/Y=2) = \frac{f(2,2)}{f_Y(2)} = \frac{0.3}{0.6} = \frac{1}{2}$$
$$f_{X/Y}(X=3/Y=2) = \frac{f(3,2)}{f_Y(2)} = \frac{0.1}{0.6} = \frac{1}{6}$$

$\therefore \quad f_{X/Y}(x/y)$ is:

$X=x$	1	2	3
$f_{X/Y}(x/2)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

(c)
$$\begin{aligned} p(X \le 2, Y=2) &= p\,[(X=1 \text{ or } 2), Y=2] \\ &= p(X=1, Y=2) + p(X=2, Y=2) \\ &= 0.2 + 0.3 = 0.5 \end{aligned}$$

(d)
$$\begin{aligned} p(X<3) &= p(X=1) + p(X=2) \\ &= 0.7 \quad [\text{from } f_X(x)] \end{aligned}$$

(e) $p(Y=2) = 0.6 \quad [\text{from } f_Y(y)]$ ■

■ Example 2.52

Let
$$f(x,y) = \begin{cases} Kxy & 0<x<1;\ 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$
be the pdf of X and Y. Find $p(0<X<\frac{1}{2}, \frac{1}{4}<Y<1)$; $p(X<Y)$

To find K: Since (X, Y) is continuous

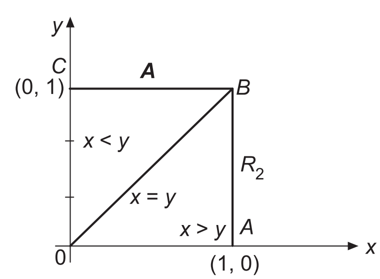
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \quad \Rightarrow \quad \int_0^1\int_0^1 Kxy\,dx\,dy = 1$$
$$\Rightarrow \quad (K)\int_0^1 \left(\frac{x^2}{2}\right)_0^1 y\,dy = 1$$
$$\Rightarrow \quad \frac{K}{2}\left(\frac{y^2}{2}\right)_0^1 = 1$$
$$\frac{K}{2}\left(\frac{1}{2}\right) = 1 \quad \therefore \quad K = 4$$

The joint pdf of

$$f(x, y) = \begin{cases} 4xy & 0 < x < 1; 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a)
$$\begin{aligned} p\left(0 < X < \frac{1}{2}, \frac{1}{4} < Y < 1\right) &= \int_0^{1/2}\int_{1/4}^{1} (4)xydydx \\ &= 4\int_0^{1/2} x\left(\frac{y^2}{2}\right)_{1/4}^{1} dy \\ &= 2\int_0^{1/2} x\left(1 - \frac{1}{16}\right) dy \\ &= \frac{15}{8}\left(\frac{x^2}{2}\right)_0^{1/2} = \frac{15}{64} \end{aligned}$$

(b)
$$\begin{aligned} p(X < Y) &= \int_{\Delta OBC}\int f(x, y)dxdy \\ &= \int_{y=0}^{1}\int_{x=0}^{y} (4)xydxdy = 4\int_0^1 y\left(\frac{x^2}{2}\right)_0^y dy \\ &= 2\int_0^1 y^3 dy = 2\left(\frac{y^4}{4}\right)_0^1 = \frac{1}{2} \end{aligned}$$



■

Remark

In the preceeding example one can easily verify that $p(X \le Y) = p(X < Y) = \frac{1}{2}$.

■ Example 2.53

Let
$$f(x, y) = \begin{cases} K\, x^2y^3 & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

be the joint pdf of (X, Y). Find the conditional pfds of X given Y and Y given X.

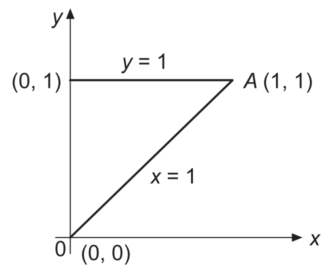
First let us find the value of K.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \quad \Rightarrow \quad \int_{\Delta OAB}\int f(x,y)dxdy = 1$$

$$\Rightarrow \quad \int_{x=0}^{1}\int_{y=x}^{1} K\, x^2y^3dydx = 1$$

$$\Rightarrow \quad K\int_0^1 x^2\left(\frac{y^4}{4}\right)_{y=x}^{1} dx = 1$$

$$\Rightarrow \quad \frac{K}{4}$$



$$\int_0^1 x^2\left(1-x^4\right)dx = 1 \quad \Rightarrow \quad \frac{K}{4}\left(\frac{x^3}{3}-\frac{x^7}{7}\right)_0^1 = 1$$

$$\Rightarrow \quad \frac{K}{4}\left(\frac{4}{21}\right) = 1 \quad \Rightarrow K = 21$$

$$\therefore \quad f(x,y) = \begin{cases} 21x^2y^3 & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

To find marginal pdfs

(a) $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{y=1}^{1}(21)x^2y^3\,dy$$

$$= (21)x^2\left(\frac{y^4}{4}\right)_x^1$$

$$= \begin{cases} \frac{21}{4}x^2(1-x^4) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$f_Y(Y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{x=0}^{y}(21)x^2y^3dx$$

$$= 21y^3\left(\frac{x^3}{3}\right)_0^y$$

$$f_Y(y) = \begin{cases} 7y^6 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Now the conditional pdf of X given $Y = y$.

$$f_{X/Y}(x/y) = \frac{f(x, y)}{f_Y(y)}$$

$$= \begin{cases} \frac{(21)x^2y^3}{(7)y^6} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \left\{ \frac{3x^2}{y^3} \quad 0 < x < y < 1 \right.$$

Then the conditional pdf of Y gives $X = x$ is

$$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} \frac{(21)x^2Y^3}{(\frac{21}{4})x^2(1-x^4)} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{4y^3}{1-x^4} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

■ Example 2.54

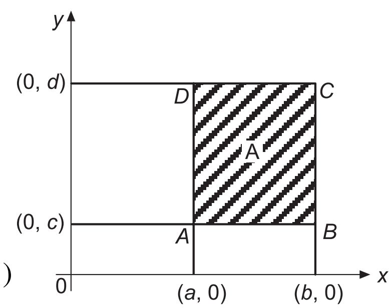
Prove that if $F(x, y)$ is the joint cdf of X and Y then

$p(a \le x \le b, c \le y \le d) = F(a, c) + F(b, d) - F(a, d) - F(b, c)$ where a, b, c, d are real numbers.

Now the probability $p(a \le X \le b, c \le Y \le d)$ is the probability of (X, Y) over the rectangle

$$\left\{ (x, y)/ \begin{array}{c} a \le x \le b; \\ c \le y \le d \end{array} \right\}$$

$$p(a \le X \le b, c \le Y \le d) = \int\limits_{ABCD}\int f(x, y)\, dy\, dx$$



$$\begin{aligned} \text{Now } F(b, d) &= p(-\infty < x \le b,\ -\infty < y \le d) \\ &= p(-\infty < x \le a,\ -\infty < y \le d) \\ &\quad + p(-\infty < x < b, -\infty < y < c) \\ &\quad + p(a \le x \le b,\ c \le y \le d) \\ &\quad - p(-\infty < x \le a,\ -\infty < y \le c) \quad (1) \end{aligned}$$

$$= \int_{-\infty}^{a}\int_{-\infty}^{d} f(x, y)dydx + \int_{-\infty}^{b}\int_{-\infty}^{c} f(x, y)dydx + \int_{a}^{b}\int_{c}^{d} f(x, y)dydx - \int_{-\infty}^{a}\int_{-\infty}^{c} f(x, y)dydx$$

$$= F(a, d) + F(b, c) + \int\limits_{ABCD}\int f(x, y)dydx - F(a, c)$$

$$\therefore \int\limits_{ABCD}\int f(x, y)dydx = F(a, c) + F(b, d) - F(a, d) - F(b, c) \qquad (2)$$

(1) and (2)

$$\Rightarrow \quad p(a \le X \le b,\ c \le Y \le d) \quad = \quad F(a,c) + F(b,d) - F(a,d) - F(b,c)$$

■

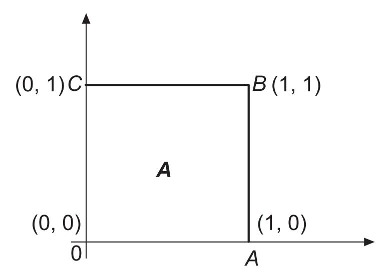
■ Example 2.55

If two random variables have the joint density

$$f(x,y) = \begin{cases} K(x+y^2) & 0 < x < 1; 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find (a) $p(0.2 < X < 0.5, 0.4 < Y < 0.6)$, (b) the joint distribution function and repeat (a) using distribution function.

Since (X, Y) is continuous,



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \quad \Rightarrow \quad K\int_0^1\int_0^1 K(x+y^2)dxdy = 1$$

$$\Rightarrow \quad K\int_0^1 \left(\frac{x^2}{2} + xy^2\right)_0^1 dy = 1$$

$$\Rightarrow \quad K\int_0^1 \left(\frac{1}{2} + y^2\right) dy = 1$$

$$\Rightarrow \quad K\left(\frac{1}{2}Y + \frac{Y^3}{3}\right)_0^1 = 1$$

$$\Rightarrow \quad K\left(\frac{1}{2} + \frac{1}{3}\right) = 1$$

$$\Rightarrow \quad K = \frac{6}{5}$$

$$\therefore \quad f(x,y) \quad = \quad \begin{cases} \frac{6}{5}(x+y^2) & 0 < x < 10 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Now, $$p(0.2 < X < 0.5, 0.4 < Y < 0.6) = \int_{0.2}^{0.5}\int_{0.4}^{0.6} f(x,y)dydx$$

$$= \frac{6}{5}\int_{0.2}^{0.5}\int_{0.4}^{0.6} \frac{6}{5}(x+y^2)dydx$$

$$= \frac{6}{5}\int_{0.2}^{0.5}\left(xy+\frac{y^3}{3}\right)_{0.4}^{0.6} dx$$

$$= \frac{6}{5}\int_{0.2}^{0.5}(0.2x+0.051)\ dx$$

$$= \frac{6}{5}\left(0.2\frac{x^2}{2}+0.051x\right)_{0.2}^{0.5}$$

$$= \frac{6}{5}(0.036) = 0.04$$

(b) To find the distribution function. If $(x, y)\epsilon \boldsymbol{A}$

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)dudv$$

$$= \int_{0}^{y}\int_{0}^{x}\frac{6}{5}(u+v^2)dudv$$

$$= \frac{6}{5}\int_{0}^{y}\left(\frac{u^2}{2}+uv^2\right)_0^x dv$$

$$= \frac{6}{5}\int_{0}^{y}\left(\frac{x^2}{2}+xv^2\right)dv = \frac{6}{5}\left(\frac{x^2}{2}v+x\frac{v^3}{3}\right)_0^y$$

$$= \frac{6}{5}\left(\frac{x^2y}{2}+\frac{xy^3}{3}\right)$$

$$\therefore \quad F(x,y) = \frac{6}{5}\left(\frac{x^2y}{2}+\frac{xy^3}{3}\right) \quad 0 < x < 1, 0 < y < 1$$

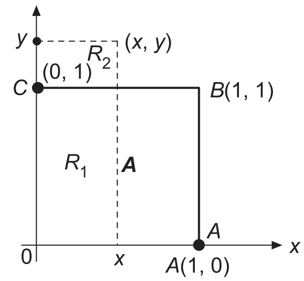
If $(x, y) \notin \boldsymbol{A}$ then the following possibilities may occur.

(i) $0 < x < 1; y \geq 1$

(ii) $x \geq 1; 0 < y < 1$

(iii) $x \geq 1; y \geq 1$

(i) $0 < x < 1;\ y \geq 1$



$$F(x, y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f(x, y)\, dv\, du$$

$$= \int_{u=0}^{x} \int_{v=0}^{1} f(u, v)\, dv\, du + \int_{u=0}^{x} \int_{v=1}^{y} f(u, v)\, dv\, du$$

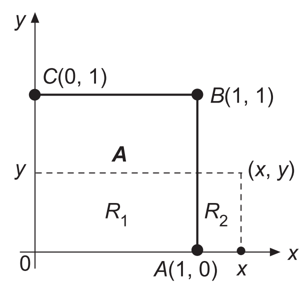
Refer diagram.

$$= \int_{u=0}^{x} \int_{v=0}^{1} \frac{6}{5}(u + v^2)\, dv\, du + 0 \qquad (\because\ f(x, y) = 0,\ (x, y) \notin \boldsymbol{A})$$

$$= \left(\frac{3}{5}\right) x^2 + \left(\frac{2}{5}\right) x \qquad \text{(after integrating)}$$

(ii) $x \geq 1;\ 0 < y < 1$

Similar to case (i)

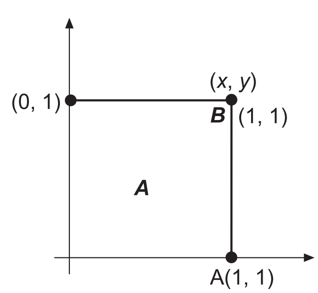


$$F(x, y) = \int_{u=0}^{y} \int_{u=0}^{1} f(u, v)\, du\, dv + \int_{v=0}^{y} \int_{u=1}^{x} f(u, v)\, du\, dv$$

$$= \int_{v=0}^{y} \int_{u=0}^{1} \frac{6}{5}(u + v^2)\, du\, dv + 0$$

$$= \left(\frac{3}{5}\right) y + \left(\frac{2}{5}\right) y^3$$

(iii) $x \geq 1,\ y \geq 1$



$$\therefore \quad F(x, y) = \int_{0}^{x} \int_{0}^{y} f(u, v)\, dv\, du = 1 \quad \text{(Refer diagram)}$$

$$\therefore \quad F(x, y) = \begin{cases} 0 & x \leq 0;\ y \leq 0 \\ \frac{6}{5}\left(\frac{x^2 y}{2} + \frac{x y^3}{3}\right) & 0 < x < 1;\ 0 < y < 1 \\ \left(\frac{3}{5}\right) x^2 + \left(\frac{2}{5}\right) x & 0 < x < 1;\ y \geq 1 \\ \left(\frac{3}{5}\right) y + \left(\frac{2}{5}\right) y^3 & 0 < y < 1;\ x \geq 1 \\ 1 & x \geq 1;\ y \geq 1 \end{cases}$$

$$
\begin{aligned}
p(0.2 < X < 0.5, 0.4 < Y < 0.6) & \\
&= F(0.2,\ 0.4) + F(0.5,\ 0.6) - F(0.2,\ 0.6) - F(0.5,\ 0.4) \\
F(0.2,\ 0.4) &= \frac{6}{5}\left(\frac{(0.2)^2 0.4}{2} + \frac{0.2(0.4)^3}{3}\right) \\
&= \frac{6}{5}(0.008 + 0.004) = 0.014 \\
\text{Similarly,} \quad F(0.5,\ 0.6) &= 0.133 \\
F(0.2,\ 0.6) &= 0.032 \\
F(0.5,\ 0.4) &= 0.073 \\
\therefore \quad (1) \quad &\Rightarrow \ p(0.2 < X < 0.5, 0.4 < Y < 0.6) \\
&= 0.014 + 0.133 - 0.032 - 0.073 = 0.04
\end{aligned}
$$

■

Note: Hence probabilities involving random variables (more than one) can be computed using joint pdf or joint cdf.

We can conclude the following to find the distribution function of (X, Y) if $\boldsymbol{A}$ is a finite product space.

$$\text{i.e.,} \qquad \boldsymbol{A} = \{(x, y)/a \le x \le b,\ c \le y \le d\}$$

Case (i): $(x, y) \notin \boldsymbol{A}$ and $x < a$ and $y < c$

$$F_1(x, y) = 0$$

Case (ii): $(x, y) \epsilon \boldsymbol{A}$ compute $F(x, y)$ using the definition.

Case(iii): $a \le x \le b;\ \ y > d$

$$\text{then} \qquad F_2(x, y) = F(x, d)$$

Similarly that is replace y by d in $F(x, y)$ computed in Case (ii)

Case(iv): $x > b;\ \ c \le y \le d$

$$\text{then} \qquad F_3(x, y) = F(b, y)$$

Case(v): $x > b;\ \ y > d$

$$\text{then} \qquad F_4(x, y) = 1$$

Compare this cases with the computations in Examples 2.50 and 2.55. In our subsequent examples such elaborate integrations may be avoided.

■ Example 2.56

Two discrete random variables X and Y have

$$p(X=1,\ Y=1)=2K \qquad p(X=1,\ Y=2)=K$$
$$p(x=2,\ Y=1)=2K \qquad p(X=2,\ Y=2)=4K$$

Examine whether X and Y are independent

The joint pdf of (X, Y) is

$X \diagdown Y$	1	2
1	$2K$	K
2	$2K$	$4K$

To find K:

$$\begin{aligned} \sum_x \sum_y f(x, y) &= 1 \\ \Rightarrow \quad 2K + K + 2K + 4K &= 1 \\ K &= \frac{1}{9} \end{aligned}$$

$\therefore$ Joint pdf can be written as

$X \diagdown Y$	1	2
1	$\frac{2}{9}$	$\frac{1}{9}$
2	$\frac{2}{9}$	$\frac{4}{9}$

Marginal pdf of X and Y:

$X = x$	1	2
$f_X(x)$	$\frac{3}{9}$	$\frac{6}{9}$

$Y = y$	1	2
$f_Y(y)$	$\frac{4}{9}$	$\frac{5}{9}$

$$\begin{aligned} f_X(1) \cdot f_Y(1) = \frac{3}{9} \cdot \frac{4}{9} &= \frac{4}{27} \\ &\neq f(1, 1) \end{aligned}$$

$\therefore$ X and Y are not independent. ■

■ Example 2.57

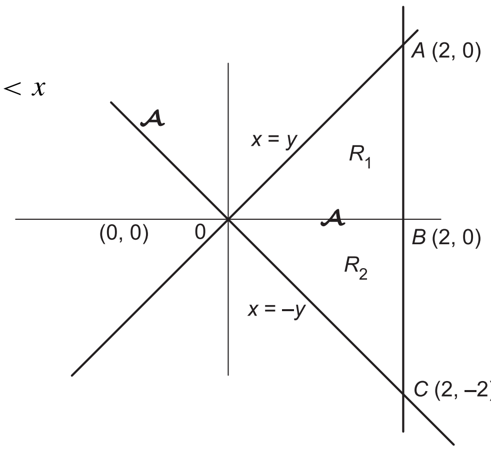
If the random variables X and Y have the joint pdf

$$f(x, y) = \begin{cases} Kx(x-y) & 0 < x < 2; -x < y < x \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional pdf of X given Y and Y given X. Also find whether X and Y are independent.

Given jpdf of (x, y) is

$$f(x, y) = \begin{cases} kx(x-y) & 0 < x < 2, -x < y < x \\ 0 & \text{elsewhere} \end{cases}$$



(Refer diagram for identifying $\boldsymbol{A}$)

$$\begin{aligned} \boldsymbol{A} &= R_1 \cup R_2 \\ &= \Delta OAC \end{aligned}$$

To find K:

$$\iint\limits_{\boldsymbol{A}} f(x, y)\,dy\,dx = 1$$

$$\int_{x=0}^{2}\int_{y=-x}^{x} kx(x-y)\,dy\,dx = 1 \Rightarrow K\int_0^2 \left(x^2y - \frac{xy^2}{2}\right)_{-x}^{x} dx = 1$$

$$\Rightarrow K\int_0^2 2x^3\,dx = 1$$

$$\Rightarrow 2K\left(\frac{x^4}{4}\right)_0^2 = 1; \quad K = \frac{1}{8}$$

$$\therefore \quad f(x, y) = \begin{cases} \frac{1}{8}x(x-y) & 0 < x < 2;\ -x < y < x \\ 0 & \text{elsewhere} \end{cases}$$

To find marginal pdf of X and Y

$$\begin{aligned} f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy &= \int_{-x}^{x} \frac{1}{8}x(x-y)\,dy \\ &= \frac{1}{8}\left(x^2y - \frac{xy^2}{2}\right)_{-x}^{x} \end{aligned}$$

$$f_X(x) = \begin{cases} \frac{x^3}{4} & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

$$= \begin{cases} \int_{-y}^{2} \frac{x}{8}(x - y)\, dx & -2 < y < 0 \text{ (over } R_2) \\ \int_{y}^{2} \frac{x}{8}(x - y)\, dx & 0 \le y < 2 \text{ (over } R_1) \end{cases}$$

$$= \begin{cases} \frac{1}{8}\left(\frac{x^3}{3} - \frac{x^2 y}{2}\right)_{-y}^{2} & -2 < y < 0 \\ \frac{1}{8}\left(\frac{x^3}{3} - \frac{x^2 y}{2}\right)_{y}^{2} & 0 \le y < 2 \end{cases}$$

$$= \begin{cases} \frac{1}{48}(16 - 12y + 5y^3) & -2 < y < 0 \\ \frac{1}{48}(16 - 12y + y^3) & 0 \le y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

To find the conditional pdf.

$$f_{X/Y}(x/y) = \frac{f(x, y)}{f_Y(y)}$$

$$= \begin{cases} \frac{6x(x-y)}{16-12y+5y^3} & -2 < y < 0,\ -y < x < 2 \\ \frac{6x(x-y)}{16-12y+y^3} & 0 \le y < 2,\ y < x < 2 \end{cases}$$

$$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)}$$

$$= \begin{cases} \frac{x-y}{2x^2} & 0 < x < 2,\ -y < x < y \\ 0 & \text{elsewhere} \end{cases}$$

Also, X and Y are not independent (How?) ■

■ Example 2.58

Let X and Y be two independent random variables with the pdf of each is as follows.

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $p(X + Y \le 1)$

(Similar cases arise such as when X and Y are two independent measurements made of the rainfall during a given period of time and the question could be to find the probability that the total rainfall is ≤ 1 unit).

Solution: Since X and Y are having identical distributions, their pdfs can be

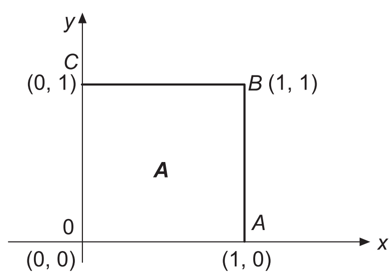
$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

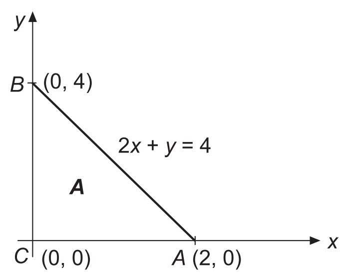
and $$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Given X and Y are independent. Therefore, the joint pdf of (X, Y) is

$$\begin{aligned} f(x, y) &= f_X(x) \cdot f_Y(y) \\ &= 2x \cdot 2y \\ &= \begin{cases} 4xy & 0 \le x \le 1 \quad 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

(observe that $\boldsymbol{A} = \boldsymbol{A}_X \times \boldsymbol{A}_Y$ the product space of $\boldsymbol{A}_x = [0, 1]$ and $\boldsymbol{A}_y = [0, 1]$).





Now to find $p(X + Y < 1)$.

$$\begin{aligned} &= \int_A \int f(x, y)dxdy = \int_{y=0}^{1} \int_{x=0}^{1-y} 4xydxdy \\ &= 4\int_0^1 \left(\frac{x^2}{2}\right)_0^{1-y} ydy \\ &= 2\int_0^1 y(1-y)^2 dy = \frac{1}{6} \end{aligned}$$

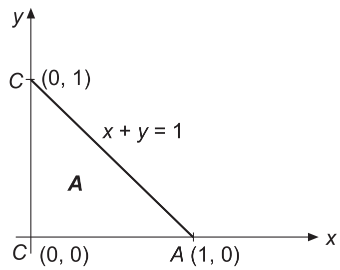
■

■ Example 2.59

If the joint pdf of (X, Y) is as follows:

$$f(x, y) = \begin{cases} K(4 - 2x - y) & x > 0 \quad y > 0 \\ & 2x + y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find (a) $f(Y/X)$ and (b) $p(Y \geq 2/X = 1/2)$



To find K:

$$\iint\limits_{A} f(x, y)dy\,dx = 1 \Rightarrow \int_{x=0}^{2}\int_{y=0}^{4-2x} K(4-2x-y)dydx = 1$$

$$\Rightarrow K\int_{x=0}^{2}\left(4y-2xy-\frac{y^2}{2}\right)_0^{4-2x} dx = 1$$

$$\Rightarrow K\int_{x=0}^{2}\left[4(4-2x)-2x(4-2x)-\frac{(4-2x)^2}{2}\right]dx = 1$$

$$\Rightarrow K\int_{x=0}^{2}(8-8x+2x^2)dx = 1$$

Solving, $\Rightarrow K = \dfrac{3}{16}$

$$\therefore \quad f(x) = \begin{cases} \frac{3}{16}(4-2x-y) & x > 0; \quad y > 0; 2x + y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) To find $f_{Y/X}(y/x)$

Now $\quad f_{Y/X}(y/x) = \dfrac{f(x, y)}{f_X(x)}$

Let us find $\quad f_X(x) = \displaystyle\int_{-\infty}^{\infty} f(x, y)\,dy$

$$= \int_{Y=0}^{4-2x} f(x, y)\,dy$$

$$= \int_{Y=0}^{4-2x} \frac{3}{16}(4-2x-y)dy = \begin{cases} \frac{3}{8}(x-2)^2 & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad f_{Y/X}(y/x) = \begin{cases} \frac{4-2x-y}{2(x-2)^2} & x > 0,\ y > 0;\quad 2x + y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

To find

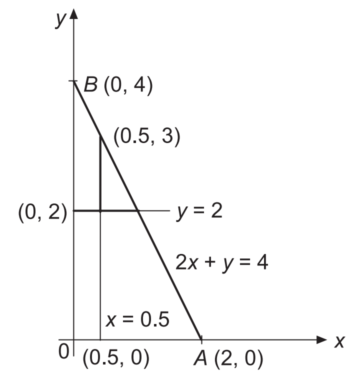
(b) $p(Y \geq 2/X = 0.5)$

$$\text{Now} \quad f_{Y/X}(y/0.5) = \frac{4-2(0.5)-y}{2(0.5-2)^2}$$

$$= \begin{cases} \frac{2}{9}(3-4) & 0 < y < 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{Now} \quad p(Y \geq 2/X = 0.5) = \int_{2}^{3} \frac{2}{9}(3-4)\,dy$$

$$= \frac{2}{9}\left(3y - \frac{y^2}{2}\right)_2^3 = \frac{1}{9}$$

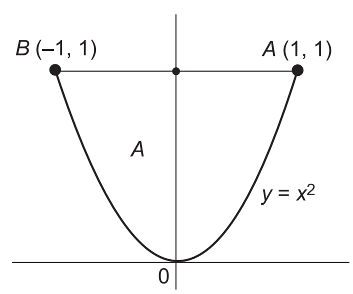


■

■ Example 2.60

If the joint pdf of X and Y is

$$f(x, y) = \begin{cases} kx^2y & x^2 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$



Find $p(X \geq Y)$.

Let us first define $\boldsymbol{A}$

$\because \quad x^2 \geq 0$ (always), we have

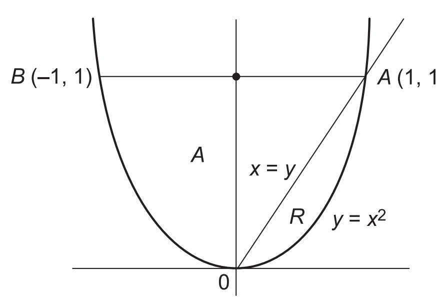
$$0 \leq x^2 \leq y \leq 1$$

i.e., $0 \leq y \leq 1$; $-1 \leq x \leq 1$ and $0 \leq x^2 \leq y \leq 1$.

Refer the diagram for the space A.

To find K

$$\iint_A f(x, y)dydx = 1 \Rightarrow \int_{-1}^{1}\int_{x^2}^{1} kx^2ydydx = 1$$



$$\Rightarrow \quad k\int_{-1}^{1} x^2\left(\frac{y^2}{2}\right)_{x^2}^{1} dx = 1$$

$$\Rightarrow \quad \frac{k}{2}\int_{-1}^{1} x^2\left(1-x^4\right) dx = 1$$

$$\Rightarrow \quad \frac{k}{2}\left(\frac{8}{21}\right) = 1; \quad \Rightarrow k = \frac{21}{4}$$

Now, $$p(X \geq y) = \iint_R f(x,y)dydx = \int_0^1\int_{x^2}^{x} \frac{21}{4}x^2 ydydx$$

$$= \frac{21}{4}\int_0^1 x^2\left(\frac{y^2}{2}\right)_{x^2}^{x} dx$$

$$= \frac{21}{8}\int_0^1 x^2\left(x^2-x^4\right) dx = \frac{3}{20}$$ ■

■ Example 2.61

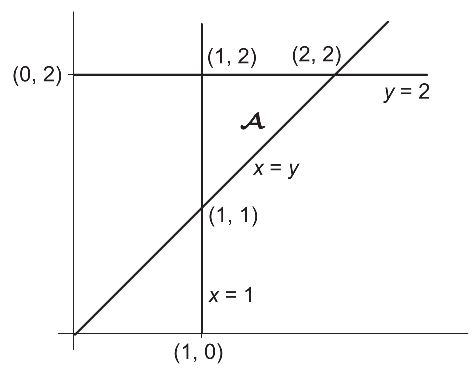
The joint pdf of R.V (X, Y) is

$$f(x,y) = \begin{cases} \left(\frac{8}{9}\right)xy & 1 \leq x \leq y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional density function of Y given $X = x$ and X given $Y = y$.

The space $\boldsymbol{A}$ is indicated in the figure.

To find mpdf:



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$= \int_x^2 \left(\frac{8}{9}\right)xydy = \frac{8}{9}x\left(\frac{y^2}{2}\right)_x^2$$

$$= \begin{cases} \frac{4}{9}x\left(4-x^2\right) & 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_1^y \left(\frac{8}{9}\right)xydx = \frac{8}{9}y\left(\frac{x^2}{2}\right)_1^y$$

$$= \begin{cases} \left(\frac{4}{9}\right)y\left(y^2-1\right) & 1 \leq y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

To find cpdf:

$$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{(8/9)xy}{(4/9)y\left(y^2-1\right)} & (x,y) \in \boldsymbol{A} \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{2x}{y^2-1} & 1 \le x \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{(8/9)xy}{(4/9)x\left(4-x^2\right)} & (x,y) \in \boldsymbol{A} \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{2y}{4-x^2} & 1 \le x \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

■

■ Example 2.62

If the joint cdf of (X, Y) is

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-2x} - e^{-3y} + e^{-(2x+3y)} & x > 0;\ y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find (a) the joint pdf of (X, Y), (b) marginal cdf of X and Y, (c) $p(X \le 1 \cap Y \le 1)$ and (d) $p(1 < X < 3, 1 < Y < 2)$

(a) Let $f(x, y)$ be the joint pdf of (X, Y)

$$\begin{aligned} f(x,y) &= \frac{\partial^2}{\partial x \partial y} F(x,y) \\ &= \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-2x} - e^{-3y} + e^{-(2x+3y)}\right] \\ &= \frac{\partial}{\partial x}\left[3e^{-3y} - 3e^{-(2x+3y)}\right] \\ &= 0 + (6)e^{-(2x+3y)} \\ \therefore\ f(x,y) &= \begin{cases} 6e^{-(2x+3y)} & x > 0;\quad y > 0 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

(b) Let $F_X(x)$ and $F_Y(y)$ be the marginal cdf of X and Y respectively.

$$\begin{aligned} \therefore\ F_X(x) &= \lim_{Y \to \infty} F_{XY}(x,y) \\ &= \lim_{Y \to \infty}\left[1 - e^{-2x} - e^{-3y} + e^{-(2x+3y)}\right] \\ &= 1 - e^{-2x} - 0 + e^{-2x}(0) \\ F_X(x) &= \begin{cases} 1 - e^{-2x} & x > 0 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

Similarly,
$$F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y)$$
$$= \begin{cases} 1-e^{-3y} & y>0 \\ 0 & \text{elsewhere} \end{cases}$$

Hence, if $f_X(x)$ and $f_Y(y)$ are the mpdf of X and Y respectively,

$$f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} 2e^{-2x} & x>0 \\ 0 & \text{elsewhere} \end{cases}$$
$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} 3e^{-3y} & y>0 \\ 0 & \text{elsewhere} \end{cases}$$

(c) $p(X \le 1 \cap Y \le 1) = p(0 < X \le 1 \cap 0 < Y \le 1)$ (refer the space $\boldsymbol{A}$)

$$= F(1,1)$$
$$= 1-e^{-2}-e^{-3}+e^{-5} = (1-e^{-2})(1-e^{-3})$$
$$= 0.822$$

(d) $p(1 < X < 3, 1 < y < 2) = F(1,1) + F(3,2) - F(1,2) - F(3,1)$

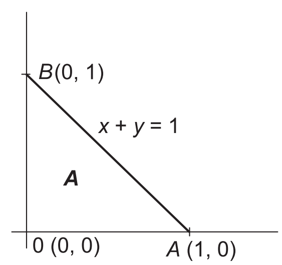
$$= \left(1-e^{-2}-e^{-3}+e^{-5}\right) + \left(1-e^{-6}-e^{-6}+e^{-12}\right) - \left(1-e^{-2}-e^{-6}+e^{-8}\right) - \left(1-e^{-6}-e^{-3}+e^{9}\right)$$
$$= 0.006$$

■

■ Example 2.63

If the joint pdf of (X, Y) is

$$f(x,y) = \begin{cases} k & x \ge 0 \quad y \ge 0; \quad x+y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$



Find the conditional distributions.
To find k:

$$\iint_{\boldsymbol{A}} f(x,y)\,dx\,dy = 1 \quad \Rightarrow \quad \int_0^1 \int_0^{1-y} k\,dx\,dy = 1$$
$$\Rightarrow \quad k\int_0^1 (x)_0^{1-y}\,dy = 1$$
$$\Rightarrow \quad k\int_0^1 (1-y)\,dy = 1$$

$$\Rightarrow \quad k\left(y - \frac{y^2}{2}\right)_0^1 = 1; \quad \Rightarrow k = 2$$

$$\therefore \quad f(x, y) = \begin{cases} 2 & (x, y) \in A \\ 0 & \text{elsewhere} \end{cases}$$

To find mpdf:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{0}^{1-x} 2dy$$

$$= \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_{0}^{1-y} 2dx$$

$$= \begin{cases} 2(1-y) & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

To find cpdf:

$$f_{X/Y}(x/y) = \frac{f(x, y)}{f_Y(y)}$$

$$= \begin{cases} \frac{1}{1-y} & (x, y) \in \boldsymbol{A} \\ 0 & \text{elsewhere} \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} \frac{1}{1-x} & (x, y) \in \boldsymbol{A} \\ 0 & \text{elsewhere} \end{cases}$$

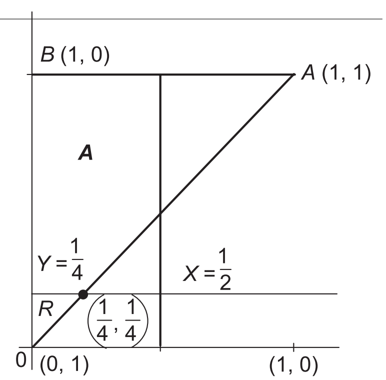
■

■ Example 2.64

Let the joint pdf of X and Y be

$$f(x, y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find (a) $p\left(X < \frac{1}{2} \cap Y < \frac{1}{4}\right)$, (b) $p\left(X \le \frac{1}{4} \big/ \frac{1}{2} < Y \le 1\right)$ and (c) $p(X + Y < 1$.

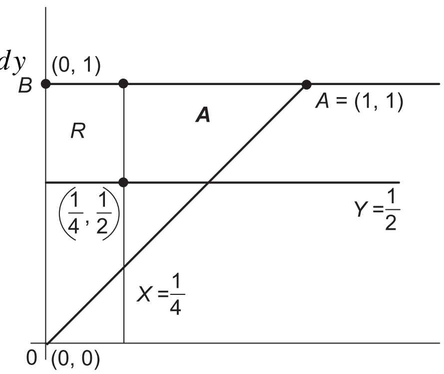


(a)
$$p\left(X < \frac{1}{2} \cap Y < \frac{1}{4}\right) = \iint_R f(x, y)dxdy = \int_{y=0}^{1/4}\int_{x=0}^{y} f(x, y)dxdy$$

$$= \int_{y=0}^{1/4}\int_{0}^{y} 8xydxdy = 8\int_{0}^{1/4}\left(\frac{x^2}{2}\right)_0^y ydy = 4\int_{0}^{1/4} y^3dy = \frac{1}{256}$$

(b) $$p\left(X \le \frac{1}{4} \Big/ \frac{1}{2} < Y \le 1\right) = \frac{p\left(X \le \frac{1}{4} \cap \frac{1}{2} < Y \le 1\right)}{p\left(\frac{1}{2} < Y \le 1\right)} \quad (1)$$

Now $$p\left(X \le \frac{1}{4} \cap \frac{1}{2} < Y \le 1\right) = \iint_R f(x, y)dxdy$$



$$= \int_0^{\frac{1}{4}} \int_{\frac{1}{2}}^{1} 8xydxdy$$

$$= 4 \int_0^{\frac{1}{4}} x \left(y^2\right)_0^{1/4} dx$$

$$= \frac{1}{4}\left(\frac{x^2}{2}\right)_0^{1/4} = \frac{1}{128}$$

To find $$p\left(\frac{1}{2} < Y \le 1\right) = \int_{\frac{1}{2}}^{1} f_Y(y)dy$$

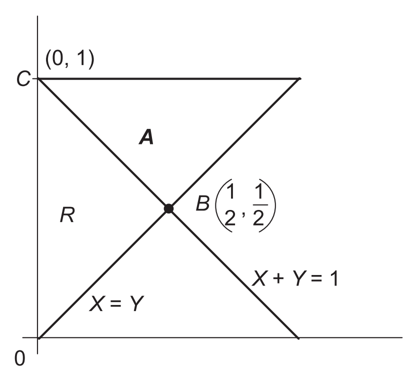
Now $$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\ dx$$

$$= \int_0^{y} 8x, ydx = 8y\left(\frac{x^2}{2}\right)_0^{y}$$

$$= \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore\quad p\left(\frac{1}{2} < Y \le 1\right) = \int_{\frac{1}{2}}^{1} 4y^3 dy = \frac{15}{16}$$

$$\therefore\quad (1) \Rightarrow\quad p\left(X \le \frac{1}{4} \Big/ \frac{1}{2} < Y \le 1\right) = \frac{1/128}{15/16} = \frac{1}{120}$$



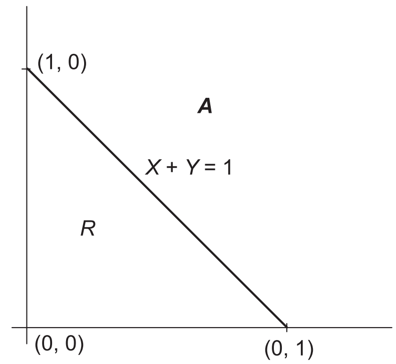
(c) $$p(X + Y < 1) = \iint_R f(x, y)dydx$$

$$= \int_{x=0}^{\frac{1}{2}} \int_{y=x}^{1-x} 8xydydx = \frac{1}{6}$$

■

■ Example 2.65

The joint pdf of the R.V (X, Y) is $f(x, y) = e^{-(x+y)}$ $0 \leq x, y < \infty$. Find (a) $p(X+Y) < 1$ and (b) check whether X and Y are independent.



$$\begin{aligned}
p(X+Y<1) &= \iint_R f(x, y)dxdy \\
&= \int_{y=0}^{1}\int_{x=0}^{1-y} e^{-(x+y)}dxdy \\
&= \int_{y=0}^{1} e^{-y}\left(-e^{-x}\right)_0^{1-y} dy \\
&= \int_{y=0}^{1} e^{-y}\left(1-e^{-(1-y)}\right) dy \\
&= 1-2e^{-1} = 0.26
\end{aligned}$$

To find marginal pdf of X and Y

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x, y)dy = \int_0^{\infty} e^{-(x+y)}dy \\
&= \int_0^{\infty} e^{-x}\left(-e^{-y}\right) \\
&= \begin{cases} e^{-x} & x > 0 \\ 0 & \text{elsewhere} \end{cases} \\
f_Y(y) &= \int_{-\infty}^{\infty} f(x, y)\ dx = \int_0^{\infty} e^{-(x+y)}dx \\
&= \begin{cases} e^{-y} & y > 0 \\ 0 & \text{elsewhere} \end{cases} \\
\therefore\quad f_X(x)\cdot f_Y(y) &= f(x, y)
\end{aligned}$$

$\therefore$ X and Y are independent. ■

■ Example 2.66

Let the joint pdf of X and Y be

$$f(x, y) = \begin{cases} k(1-x-y) & x > 0;\ y > 0;\ x+y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional pdf of X given $Y = y$.

To find k:

$$\iint_A f(x, y)dxdy = 1 \quad \Rightarrow \quad \int_{y=0}^{1}\int_{x=0}^{1-y} k(1-x-y)dxdy = 1$$

$$\Rightarrow \quad k\int_{y=0}^{1}\left(x - \frac{x^2}{2} - xy\right)_0^{1-y} dy = 1$$

$$\Rightarrow \quad k\int_{y=0}^{1}\left[1 - y - \frac{(1-y)^2}{2} - y\,(1-y)\right]dy = 1$$

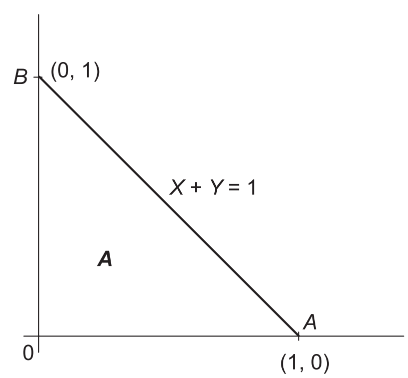
$$\Rightarrow \quad k\int_{y=0}^{1}\frac{(1-y)^2}{2}dy = 1$$

$$\Rightarrow \quad k = 6$$

$$\therefore \quad f(x, y) = \begin{cases} 6(1-x-y) & 0 < x;\ y > 0;\ x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Now, the conditional pdf of X given $Y = y$.

$$f_{X/Y}(x/y) = \frac{f(x, y)}{f_Y(y)}$$



$\therefore$ To find $f_Y(y)$:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\ dx$$

$$= \int_{x=0}^{1-y} 6(1-x-y)\ dx$$

$$= 6\left(x - \frac{x^2}{2} - xy\right)_0^{1-y}$$

$$= \begin{cases} 3(1-y)^2 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad f_{X/Y}(x/y) = \frac{6\,(1-x-y)}{3(1-y)^2}$$

$$= \begin{cases} \frac{2(1-x-y)}{(1-y)^2} & x > 0;\ y > 0;\ x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

■

Exercise 2.2

1. Two discrete random variables X and Y have

$$p\,(X = 1,\ Y = 1) = \frac{1}{4};\quad p\,(X = 1,\ Y = 2) = \frac{1}{3}$$
$$p\,(X = 2,\ Y = 1) = \frac{1}{6};\quad p\,(X = 2,\ Y = 2) = \frac{1}{4}$$

   Examine whether X and Y are independent.

2. A fair coin is tossed three times. Let X denote the number of heads on the first toss and Y the total number of heads. Check whether X and Y are independent. Also find (a) $p(X \leq 1, Y = 0)$; (b) $p(X = 1, Y \leq 1)$.

3. If the joint probability distributions of X and Y is given by $f(x, y) = K\left(x^2 + y^2\right) x = 0, 1, 3;$ $y = -1, 2, 3$, find (a) $p(X \leq 1, Y > 2)$; (b) $p(X = 0, Y \leq 2)$. Also check whether X and Y are independent.

4. If the joint probability distributions of X and Y is given by $f(x, y) = kxy$ for $x = 1, 2, 3;$ $y = 1, 2, 3$ find the joint *distribution function* of X and Y.

5. Two students are selected at random from a class containing four boys, three girls. If X and Y are respectively, the numbers of boys and girls included among the two students selected from the class, find the probabilities associated with all possible pairs of values of X and Y. Also, find the probability that the selection has more girls.

6. Consider two independent random variables X and Y with the domain (1, 1), (1, 2), (2, 1), (2, 2), and the joint density function $f(x, y)$. If $f(1, 1) = \frac{2}{10}$ and $f_X(1) = \frac{4}{10}$ find the table showing the joint pdf of X and Y.

7. If X is the number of tails and Y the number of tails minus the number of heads obtained in three tosses of a balanced coin, construct a table showing the probability distribution of X and Y. Also find (a) $p(0 < X < 3, Y \leq 0)$; (b) $p(X > Y)$.

8. X and Y have a bivariate distribution given by $f(x, y) = k(x + 3y)$ $(x, y) = (1, 1), (1, 2),$ (2,1), (2, 2). Find the conditional distributions of Y given $X = 2$ and X given $Y = 1$.

9. Find the conditional distribution of X and Y and Y given X, if the joint pdf of X and Y is

$$f(x, y) = k(x + y) \quad x = 1, 2, 3 \quad \text{and} \quad y = 1, 2$$

10. Given the joint pdf

$$f(x, y) = \begin{cases} kx(y + x) & 0 < x < 1;\ 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

   of two random variables X and Y, find $p\left(0 < x < \frac{1}{2}, 1 < y < 2\right)$.

11. If the joint pdf of X and Y is

$$f(x, y) = \begin{cases} cy^2 & 0 \leq x \leq 2;\ 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find (a) $p(X + Y > 2)$; (b) $p(Y < 1/2)$; (c) $p\,(X \leq 1)$.

12. If the joint pdf of a random variable (x, y) is constant on the rectangle where $0 \leq x \leq 2$ and $0 \leq y \leq 1$ and if the pdf iz zero outside this rectangle then find $p(X \geq Y)$.

13. Check whether X and Y are independent if their joint pdf is

(a) $$f(x, y) = \begin{cases} k\,x\,e^{-y} & 0 \leq x \leq 1;\ y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$f(x, y) = \begin{cases} k\,x\,y & x \geq 0; y \geq 0;\ x + y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

14. If the joint pdf of X and Y is

$$f(x, y) = \begin{cases} k(x + y) & 0 < x < 2;\ 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint distribution of X and Y and hence the marginal distribution of X and Y.

15. If

$$f(x, y) = \begin{cases} x + y & 0 < x < 1;\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

is the joint pdf of X and Y, find the joint distribution of X and Y, find the joint distribution of X and Y. Hence find (a) $p\left(\frac{1}{2} < X < \frac{3}{4}, \frac{1}{4}, < Y < \frac{1}{2}\right)$ and (b) $p\left(X < \frac{1}{2}, Y \geq 1\right)$.

16. If the joint pdf of X and Y is given by

$$f(x, y) = \begin{cases} k\,x\,y & x > 0; \quad y > 0;\ x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find (a) $p\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right)$, (b) $p(X > 2Y)$ and (c) $p\left(X + Y > \frac{1}{3}\right)$.

17. If the joint pdf of X and Y is given by

$$f(x, y) = \begin{cases} k\,y(1 - x - y) & \text{for} \quad x > 0;\ y > 0;\ x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Check whether X and Y are independent.

18. The joint density of two random variables is given by

$$f(x, y) = \begin{cases} k\left(\frac{20-x}{x}\right) & 10 < x < 20;\ \frac{x}{2} < y < x \\ 0 & \text{elsewhere} \end{cases}$$

find (a) The conditional density of Y given $X = 12$ and (b) $p(X \geq 12, Y \geq 8)$.

19. The two random variables X and Y have the joint pdf

$$f(x, y) = \begin{cases} \frac{k}{x^2 y} & 1 \leq x < \infty;\ \frac{1}{x} < y < x \\ 0 & \text{elsewhere} \end{cases}$$

obtain the conditional distributions of Y given X and X given Y.

20. If the random variables X and Y have the joint pdf

$$f(x, y) = \begin{cases} k & 0 < x < 2;\ -x < y < x \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional distribution of Y given X and X given Y.