3

Mathematical Expectations

'Belief is no substitute for arithmetic'

3.1 INTRODUCTION

Random variable characterises a random phenomenon by listing the range and the corresponding probability distribution (that is, pmf in the discrete case or pdf in the continuous case). For example we can find a pdf for the water consumption of a city for any given day. Here, we are interested in finding a most probable value for this random event (consumption of water); that is on "expected" consumption of water. As we can find this in another example say if p is the price of a certain commodity and S_i its total sales, then we may be interested in finding the expected receipts ($R = PS$) for the commodity. This section deals with expectation $E(X)$ expectation of a random variable X or more generally expectation $E[u(x)]$ of a random variable (or more variables, as in the preceeding example $E(R) = E((PS))$.

Definition

If X is a discrete random variable and $f(x)$ is the value of its pmf at x, then the expected value of X is

$$E(X) = \sum_{x} x \cdot f(x)$$

provided the sum in RHS exists. Otherwise, the mathematical expectation is undefined.

Correspondingly, if X is a continuous random variable and $f(x)$ is the value of its pdf at x, the expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

provided the integral in RHS exists. Otherwise, the mathematical expectation is undefined.

Infact, the concept of a mathematical expectation defines a value, the random variable is expected to assume an "average" value, when the random experiment is repeated under the given conditions. A mathematical expectations arose in connection with the games of chance. Consider the following situation: A man draws 3 balls from an urn containing 5 white and 7 black balls.

He gets Rs. 10 for each white ball and Rs. 5 for each black ball. If we wish to find the player's "expected" amount he wins then we must know the various ways his sampling could be from the urn.

In particular if X is the number of white balls in his selection of 3 balls, then the range of X is $\{0, 1, 2, 3\}$. Equivalently if Y is the amount he gets then Y assumes the values 15 (3 black balls), 20, 25 and 30 respectively. Hence $p(X = 3) = p(Y = 15) = \frac{7}{44}\left(\frac{7C_3}{12C_3}\right.$ and the probability distribution of Y is

$Y = y$	15	20	25	30
$p(Y = y)$	$\frac{7}{44}$	$\frac{21}{44}$	$\frac{7}{22}$	$\frac{1}{22}$

$\therefore$ on the average he would win $(15)\left(\frac{7}{44}\right) + (20)\left(\frac{21}{44}\right) + (25)\left(\frac{7}{22}\right) + (30)\left(\frac{1}{22}\right) =$ Rs. 21.25 his idea is formulated in the preceeding definition.

■ Example 3.1

If X is the number of points rolled with a balanced die, find the expected value of X.

Let us find the pmf of X. Since each possible outcome $\{1, 2, 3, 4, 5, 6\}$ has the probability $\frac{1}{6}$, we have

$X = x$	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned} \therefore \quad E(X) = \Sigma x\ f(x) &= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) 4 + \left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) \\ &= \frac{7}{2} \end{aligned}$$

■

■ Example 3.2

Find the expected value of the random variable X whose pdf is given by

$$f(x) = \begin{cases} \frac{1}{8}(x+1) & 2 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Since X is a continuous random variable,

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \cdot f(x)dx \\
&= \int_{2}^{4} x \cdot \frac{1}{8}(x+1)dx = \frac{1}{8}\int_{2}^{4} \left(x^2+x\right)dx \\
&= \frac{1}{8}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)_2^4 = \frac{1}{8}\left[\left(\frac{4^3}{3}+\frac{4^2}{2}\right)-\left(\frac{2^3}{3}+\frac{2^2}{2}\right)\right] \\
&= \frac{1}{8}\left[\frac{64}{3}+\frac{16}{2}-\frac{8}{3}-\frac{4}{2}\right] \\
&= \frac{1}{8}\left[\frac{56+18}{3}\right] = \frac{74}{24}
\end{aligned}$$

From the Examples 3.1 and 3.2 we can observe that the definition of expectation for a continuous random variable is an analegous form of the discrete case where summation is replaced by integration. ■

3.2 EXPECTATION OF FUNCTIONS OF A RANDOM VARIABLE

Let X be a random variable with probability distribution $f(x)$. There are many problems in which we are interested not only in the expected value of a random variable X but also in the expected values of random variables related to X That is we might be interested in the random variable Y. Whose values are related to the value of X by means of a relation (function) $Y = g(x)$. Let us define the expectation of $g(X)$ as follows.

$$E[g(X)] = \begin{cases} \sum_{x} g(x)f(x) & X \text{ is discrete RV} \\ \int_{-\infty}^{\infty} g(x)f(x)dx & X \text{ is continuous RV} \end{cases}$$

Some authors prove the above statement and hence adopt it in the numerical problems we assume the above statement as a formula to find $E[g(X)]$ and apply it in the subsequent discussions, and in numerical examples.

■ Example 3.3

If X is the number, obtained when a balanced die is rolled, find the expected value of $Y = g(X) = X^2 + 2X + 1$.

Since X is a discrete random variable with $\boldsymbol{A} = \{1, 2, 3, 4, 5, 6\}$ and each of these possible outcome has the probability $\frac{1}{6}$, we have,

$X = x$	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned}
E[g(X)] &= \sum_x g(x) f(x) \quad \text{(According to the definition)}\\
&= \sum_x \left(X^2 + 2X + 1\right) f(x)\\
&= \left(1^2 + 2(1) + 1\right) f(1) + \left(2^2 + 2(2) + 1\right) f(2) + \left(3^2 + 2(3) + 1\right) f(3) +\\
&\quad \left(4^2 + 2(4) + 1\right) f(4) + \left(5^2 + 2(5) + 1\right) f(5) + \left(6^2 + 2(6) + 1\right) + f(6)\\
&= \frac{139}{6}
\end{aligned}$$

■

■ Example 3.4

If X has the pmf

$$f(x) = \begin{cases} \frac{1}{6} & 0 < x < 6 \\ 0 & \text{elsewhere} \end{cases}$$

find the expected value of $Y = g(X) = X^3 + 2X$.

Since X is a continuous random variable

$$\begin{aligned}
E[g(X)] &= \int_{-\infty}^{\infty} g(x) f(x)\, dx = \int_0^6 \left(X^3 + 2X\right)\frac{1}{6}\, dx\\
&= \frac{1}{6}\left(\frac{x^4}{4} + \frac{2x^2}{2}\right)_0^6 = 60
\end{aligned}$$

In the above two examples we denote $Y = g(X)$ to define a function of the random variable X. Now we extend the concept of a mathematical expectation to the cases involving more than one random variable say X and Y. Therefore in our subsequent problems we may not include Y in defining a function of the random variable. Further, we establish some of the properties of expectation of one (or more) random variables so that calculations in Examples 3.3 and 3.4 can be simplified. Our next section explains this ideas. We start with a definition.

■

Definition

If X and Y are random variables and $f(x, y)$ is the joint probability distribution, then the expected value of $g(x, y)$, an arbitrary function of X and Y is

$$E[g(x, y)] = \begin{cases} \sum_x \sum_y g(x, y) f(x, y) & X \text{ and } Y \text{are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx dy & X \text{ and } Y \text{are continuous} \end{cases}$$

■ Example 3.5

Let X and Y each take on either value 1 or –1 with their joint pmf $\frac{1}{4}$. Find $E(X+Y)$.

The joint pmf of X and Y is as shown in the table.

$X \diagdown Y$	-1	1
-1	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

$$\begin{aligned} E(X+Y) &= \sum_x \sum_y (X+Y)\, f(x,y) \\ &= (-1)(-1)f(-1,-1) + (-1)(1)f(-1,1) + 1(-1) \\ &\quad f(1,-1) + (1)(1)f(1,1) \\ &= \frac{1}{4}(1-1-1+1) = 0. \end{aligned}$$

■

■ Example 3.6

If the joint pdf of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{y} & 0 < x < y,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find the expected value of $\frac{X}{Y}$.

$$\begin{aligned} E(X/Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{X}{Y}\right) f(x,y)\, dydx \\ &= \int_{y=0}^{1}\int_{x=0}^{y} \frac{x}{y}\cdot\frac{1}{y}\, dxdy \\ &= \int_{y=0}^{1} \left(\frac{x^2}{2}\right)_0^y \cdot \frac{1}{y^2}\, dy = \frac{1}{2}\int_0^1 \left(y^2 - 0\right)\frac{1}{y^2}\, dy = \frac{1}{2} \end{aligned}$$

■

Now let us prove certain useful properties of expectation of a random variable X which find applications in subsequent work. We will revisit Examples 3.3, 3.4 and 3.5 to show how this properties are useful in determining the expectations of functions of a random variable from known or easily computed expectations. In the derivation of the properties we will prove for the continuous case and for the discrete case is quite analgous to continuous case. That is for the discrete case, one may have to replace integration by summation in the proof of continuous case.

Properties

Property 3.1: If a is a constant then $E(aX) = aE(X)$. In particular $E(a) = a$.

***Proof*:**

$$E(aX) = \int_{-\infty}^{\infty} aX\, f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x\, f(x)\,dx = aE(X)$$

Also,
$$E(a) = \int_{-\infty}^{\infty} a\, f(x)\,dx = a\int_{-\infty}^{\infty} f(x)dx = a(1) \quad (\because \quad f(x) \text{ is the pdf of } X)$$

$$E(a) = a$$

Property 3.2: If a and b are constants, then $E(aX+b) = aE(X)+b$.

***Proof*:**

$$E(aX+b) = \int_{-\infty}^{\infty} (aX+b)\, f(x)\,dx$$

$$= \int_{-\infty}^{\infty} [ax\, f(x)\,dx + bf(x)]\,dx$$

$$= a\int_{-\infty}^{\infty} xf(x)\;dx + b\int_{-\infty}^{\infty} f(x)\;dx$$

$$= aE(X)+b.$$

Property 3.3: If $C_1, C_2, \cdots, C_n$ are constants and $g_1(x), g_2(x), \cdots, g_n(x)$ are any functions of X, then

$$E\left[\sum_{i=1}^{n} C_i g_i(X)\right] = \sum_{i=1}^{n} C_i E\left[g_i(X)\right]$$

LHS:
$$E\left[\sum_{i=1}^{n} C_i g_i(X)\right] = E\left[C_1g_1(X)+C_2g_2(X)+\cdots+C_ng_n(X)\right]$$

$$= \int_{-\infty}^{\infty} [C_1g_1(X)+C_2g_2(X)+\cdots+C_ng_n(X)]\, f(x)dx$$

$$= C_1\int_{-\infty}^{\infty} g_1(X)f(x)dx + C_2\int_{-\infty}^{\infty} g_2(X)f(x)dx + \cdots + C_n\int_{-\infty}^{\infty} g_n(X)f(x)dx$$

$$= \sum_{i=1}^{n} C_i E\left[g_i(x)\right]$$

Property 3.4: If X and Y are random variables then $E(X+Y) = E(X) + E(Y)$.

***Proof*:** Let $f(x, y)$ be the joint pdf of X and Y and $f_X(x)$ and $f_Y(y)$ be their respective marginal pdfs. Then by definition,

$$\begin{aligned}
E(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f(x, y)\, dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\ f(x, y) dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\ f(x, y)\, dxdy \\
&= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f(x, y) dy\right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f(x, y) dx\right] dy \\
&= \int_{-\infty}^{\infty} x\ f_X(x)\, dx + \int_{-\infty}^{\infty} y\ f_Y(y)\, dy \\
&= E(X) + E(Y).
\end{aligned}$$

<u>Note:</u> Property 4.4 can be extended to n variables as

$$E\left(X_1 + X_2 + \cdots + X_n\right) = E\left(X_1\right) + E\left(X_2\right) + \cdots + E\left(X_n\right)$$

Property 3.5: If X and Y are independent random variables, then $E(XY) = E(X) \cdot E(Y)$.

***Proof*:** Let X and Y be two random variables with joint pdf $f(x, y)$ and the respective marginal pdfs be $f_X(x)$ and $f_Y(y)$.

$$\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} XY\ f(x, y)\, dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f_X(x) \cdot f_Y(y)\, dxdy \\
&= \int_{-\infty}^{\infty} x f_X(d)\, dx \cdot \int_{-\infty}^{\infty} y\ f_Y(y)\, dy \\
&= E(X) \cdot E(Y)
\end{aligned}$$

<u>Note:</u> Property 3.5 can be extended to n independent variables as

$$E\left(X_1, X_2, \cdots, X_n\right) = E\left(X_1\right) \cdot E\left(X_2\right) \cdots E\left(X_n\right)$$

Property 3.6: Telescope formula.

For a random variable X taking its values in natural number system that is $\{1, 2, 3 \cdots\}$ then

$$E(X) = \sum_{n=1}^{\infty} P(X \geq n)$$

***Proof*:** By definition, since X is a discrete random variable,

$$\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x f(x) \\
&= 1f(1) + 2f(2) + 3f(3) + \cdots \\
&= f(1) + f(2) + f(3) + f(4) \\
&\qquad + f(2) + f(3) + f(4) \\
&\qquad\qquad + f(3) + f(4) \\
&\qquad\qquad\qquad + f(4) + \cdots \quad \text{adding row wise,} \\
&= p(X \geq 1) + p(X \geq 2) + p(X \geq 3) + p(X \geq 4) + \cdots \\
&= \sum_{n=1}^{\infty} p(X \geq n).
\end{aligned}$$

Remark

Refer Example 3.36

We continue the list of properties after introducing a notion called moments which we shall do later in this Chapter. As we discussed earlier we solve Examples 3.3, 3.4, 3.5 using these properties.

Example 3.3 to 3.5 revisit

■ Example 3.3

$$\begin{aligned}
E[g(X)] &= E\left(X^2 + 2X + 1\right) \\
&= E\left(X^2\right) + 2E(X) + E(1) \quad \text{(Property 3.3)} \qquad (1) \\
E(X) &= \sum_x x f(x) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) \\
&= \left(\frac{1}{6}\right)(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2} \\
E\left(X^2\right) &= \sum x^2 f(x) = \frac{91}{6} \\
\therefore \quad (1) \quad \Rightarrow \quad E[g(X)] &= \frac{91}{6} + 2\left(\frac{7}{2}\right) + 1 \quad \text{[Since } E(1) = 1 \text{ by Property 3.1]} \\
&= \frac{139}{6}
\end{aligned}$$

■

■ Example 3.4

$$\begin{aligned}
g(X) &= X^3 + 2X \\
E\left(X^3\right) &= \int_0^6 X^3\left(\frac{1}{6}\right) dx = \frac{1}{6}\left(\frac{X^4}{4}\right)_0^6 = 54 \\
E(X) &= \int_0^6 x\left(\frac{1}{6}\right) dx = \frac{1}{6}\left(\frac{x_2}{2}\right)_0^6 = 3 \\
\therefore \quad E[g(X)] &= E\left(X^3 + 2X\right) \\
&= E\left(X^3\right) + 2E(X) = 54 + 2(3) = 60.
\end{aligned}$$

■

■ Example 3.5

Marginal pdfs of X and Y.

$X = x$	-1	1
$f_X(x)$	$\frac{1}{2}$	$\frac{1}{2}$

$Y = y$	-1	1
$f_Y(y)$	$\frac{1}{2}$	$\frac{1}{2}$

$$\begin{aligned}
g(X) &= X + Y \\
E(X+Y) &= E(X) + E(Y) \\
&= \Sigma x\, f_X(x) + \Sigma y\, f_Y(y) \\
&= (-1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) = 0.
\end{aligned}$$

These properties are useful in simplifying the calculation effort one has to do while calculating expectations. It should be noted that all such calculations (Refer Example 3.6) need not to have the application of one or more these properties.

The following is an extension of property 3.3 that is we generalise it to two variables X and Y. If $C_1, C_2, \cdots, C_n$ are real constants and $g_1(X, Y); g_2(X, Y) \cdots g_n(X, Y)$ are any functions in X and Y then,

$$E\left[\sum_{i=1}^{n} C_i g_i(X, Y)\right] = \sum_{i=1}^{n} C_i E[g_i(X, Y)]$$

Let us complete this section with some simple examples. ■

■ Example 3.7

Given a random variable X:

$X = x$	-3	-2	-1	0	1	2	3
$f(x)$	0.1	0.15	0.20	0.05	0.25	0.20	0.05

Find (a) $E(X)$; (b) $E(3X \pm 2)$; (c) $E(X^2)$ and (d) $E(2X+1)^2$.

(a)
$$\begin{aligned} E(X) &= \Sigma x f(X) \\ &= (-3)(0.1) + (-2)(0.15) + (-1)(0.2) + 0(0.05) \\ &\quad +(1)(0.25) + (2)(0.20) + 3(0.05) \\ &= -0.3 - 0.3 - 0.2 + 0 + 0.25 + 0.4 + 0.15 = 0. \end{aligned}$$

(b)
$$\begin{aligned} E(3X \pm 2) &= 3E(X) \pm E(2) \\ &= 3(0) \pm 2 = 2. \quad \text{(using Properties)} \end{aligned}$$

(c)
$$\begin{aligned} E(X^2) &= \Sigma x^2 f(X) \\ &= (-3)^2(0.1) + (-2)^2(0.15) + (-1)^2(0.2) + 0^2(0.05) \\ &\quad +(1)^2(0.25) + (2)^2(0.2) + 3^2(0.05) \\ &= 0.9 + 0.6 + 0.2 + 0 + 0.25 + 0.8 + 0.45 = 3.2. \end{aligned}$$

(d)
$$\begin{aligned} E(2X+1)^2 &= E(4X^2 + 1 + 4X) \\ &= 4E(X^2) + E(1) + 4E(X) \\ &= 4(3.2) + 1 + 4(0) = 13.8. \end{aligned}$$

■

■ Example 3.8

If the probability density of X is given by

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find $E(2X^2 - 1)$

$$\begin{aligned} E(2X^2 - 1) &= 2E(X^2) - E(1) \\ &= 2E(X^2) - 1 \qquad (1) \\ E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 2x\,dx \\ &= 2\left(\frac{x^4}{4}\right)_0^1 = \frac{1}{2} \\ \therefore\ (1) \Rightarrow\ E(2X^2 - 1) &= 2\left(\frac{1}{2}\right) - 1 = 0. \end{aligned}$$

■

■ Example 3.9

Let X and Y have a bivariate distribution given by $f(x, y) = \frac{x+3y}{24}$ where $(x, y) = (1, 1)$ $(1, 2)$, $(2, 1)$, $(2, 2)$ and 0 elsewhere find $E(XY)$.

Let us write the joint pdf as follows:

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy\, f(x, y) \\ &= (1)(1)\left(\frac{1}{6}\right) + (1)(2)\left(\frac{7}{24}\right) + (2)(1)\left(\frac{5}{24}\right) \\ &\quad +(2)(2)\left(\frac{8}{24}\right) = \frac{5}{2} \end{aligned}$$

$X \diagdown Y$	1	2
1	$\frac{1}{6}$	$\frac{7}{24}$
2	$\frac{5}{24}$	$\frac{8}{24}$

In Example 3.9 it is impossible to apply Property 3.5, since X and Y are not independent (check this). ■

■ Example 3.10

The joint pdf of a random variable (X, Y) is given by

$$f(x, y) = \begin{cases} \frac{x^3y^3}{16} & 0 \le x \le 2; 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E(X + Y)$ and $E(X - Y)$.

By property we have $E(X + Y) = E(X) + E(Y)$ and $E(X - Y) = E(X) - E(Y)$ and hence the problem is to calculate $E(X)$ and $E(Y)$.

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \quad \text{and} \quad E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$\text{where} \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$\begin{aligned} f_X(x) &= \int_0^2 \frac{x^3y^3}{16}\, dy = \frac{x^3}{16}\left(\frac{y^4}{4}\right)_0^2 \\ &= \begin{cases} \frac{x^3}{4} & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases} \\ f_Y(y) &= \begin{cases} \frac{y^3}{4} & 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases} \quad \text{(By symmetry)} \end{aligned}$$

$$\therefore \quad E(X) = \int_0^2 x\left(\frac{x^3}{4}\right) dx = \frac{1}{4}\left(\frac{x^5}{5}\right)_0^2 = \frac{8}{5}$$

$$\text{and} \quad E(Y) = \frac{8}{5} \quad \text{(By symmetry)}$$

$$\therefore \quad E(X + Y) = \frac{8}{5} + \frac{8}{5} = \frac{16}{5} \quad \text{and} \quad E(X - Y) = 0.$$

■

3.3 MOMENTS

3.3.1 Univariate Distributions

We define moments of a random variable in two ways of which the first definition is important in statistics because they serve to describe the shape of the distribution of a random variable.

Definition 1: The r^{th} moment above the mean (μ) of a random variable X, denoted by μ_r or m_r is

$$\mu_r = \begin{cases} \sum\limits_{x} (x-\mu)^r\ f(x) & \text{if } X, \text{ is discrete rv} \\ \int\limits_{-\infty}^{\infty} (x-\mu)^r\ f(x)\,dx & \text{if } X \text{ is continuous rv} \end{cases}$$

for $r = 0, 1, 2, 3, \cdots$ that is μ_r is defined as the expected value of $(X-\mu)^r$ or

$$\mu_r = E[(X-\mu)^r]$$

Definition 2: The r^{th} moment about the origin of a random variable X, denoted by μ'_r or m'_r is the expected value of X^r that is $\mu'_r = E(X^r)$.

$$\mu'_r = \begin{cases} \sum\limits_{x} x^r\ f(x) & X \text{ is discrete rv} \\ \int\limits_{-\infty}^{\infty} x^r f(x)\,dx & X \text{ is continuous rv} \end{cases}$$

for $r = 0, 1, 2, 3 \cdots$.

We can observe from the definitions that

$$\mu_0 = E[(X-\mu)^0] = E(1) = 1 \quad \text{and} \quad \mu'_0 = E[X^0] = E(1) = 1$$

for any random variable X.

Also $\mu'_1 = E(X)$ which is the expected value of the random variable X which we define as the mean of the distribution of X or simply mean of X and it is denoted as μ or μ_X. Hence,

$$\begin{aligned} \mu_1 &= E(X-\mu) \\ &= E(X) - E(-\mu) \\ &= \mu - \mu \qquad (\because \ \mu \text{ is a constant}) \\ \mu_1 &= 0 \end{aligned}$$

We shall derive an expression which calculates μ_r in terms of μ'_r and define some special moments μ_2, μ_3 and μ_4 which are of special importance in statistics.

Relationship between μ_r and μ'_r

Let us derive from $r = 2$ since we had already discussed for $r = 1$.

$$\begin{aligned} \text{Let} \qquad r &= 2 \\ \therefore \qquad \mu_2 &= E\left[(x-\mu)^2\right] \\ &= E\left[X^2 + \mu^2 - 2\mu X\right] \end{aligned}$$

$$
\begin{aligned}
&= E\left(X^2\right)+E\left(\mu^2\right)-2\mu E(X)\\
&= E\left(X^2\right)+\mu^2-2\mu\cdot\mu\\
&= E[X^2]-\mu^2\\
&= \mu_2'-\left(\mu_1'\right)^2 \quad \text{or} \quad E(X^2)-E(X)^2\\
\therefore\quad \mu_2 &= E\left(X^2\right)-E\left(X^2\right) \quad \text{or} \quad \text{equivalently}\\
\mu_2 &= \mu_2'-\left(\mu_1'\right)^2=\mu_2'-\mu^2
\end{aligned}
$$

Let $r = 3$

$$
\begin{aligned}
\mu_3 &= E\left[(X-\mu)^3\right]\\
&= E\left[X^2-3X^2\mu+3X\mu^2-\mu^3\right]\\
&= E\left(X^3\right)-3\mu E\left(X^2\right)+3\mu^2E(X)-\mu^3\\
&= E\left(X^3\right)-3\mu E\left(X^2\right)+3\mu^2\mu-\mu^3\\
\mu_3 &= E\left(X^3\right)-3E(X)E\left(X^2\right)+2E(X)^3 \quad \text{or equivalently}\\
\mu_3 &= \mu_3'-3\mu_2\mu_1'+2\left(\mu_2'\right)=\mu_3'-3\mu_2\mu+2\mu^3
\end{aligned}
$$

Similarly, we can derive an expression for μ_4 and in general μ_r using μ_r'. (Refer Excercise 27).

The second moment μ_2 is called the variance of X and it is denoted by σ^2, Var(X) or simply $V(X)$. Also $+\sqrt{V(X)}$, positive square root of the variance, is called the standard deviation (σ) of X that is $\sigma = +\sqrt{V(X)}$. This measure is indicating the dispersion of the distribution of the random variable X.

■ Example 3.11

Find the mean and variance of following random variables.

(a) $f(x) = \begin{cases} \frac{x+2}{14} & x = 0, 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

(a) $\boldsymbol{A} = \{0, 1, 2, 3\}$, hence X is a discrete random variable. So we write its pmf as

$X = x$	0	1	2	3
$f(x)$	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{5}{14}$

$$
\begin{aligned}
\mu_1' &= \mu=\Sigma x\, f(x)=0\cdot\frac{1}{7}+1\cdot\frac{3}{14}+2\cdot\frac{2}{7}+3\cdot\frac{15}{14}=\frac{13}{7}\\
\mu_2' &= E(X^2)=\Sigma x^2\, f(x)=0\cdot\frac{1}{7}+1\cdot\frac{3}{14}+4\cdot\frac{2}{7}+9\cdot\frac{15}{14}=\frac{32}{7}\\
\therefore\quad V(x)=\mu_2'-\mu^2 &= \frac{32}{7}-\left(\frac{13}{7}\right)^2=\frac{55}{49}
\end{aligned}
$$

(b) Here $\boldsymbol{A} = \{x/0 < x < 1\}$ hence X is a continuous random variable.

$$\mu_1' = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x \cdot 3x^2 dx = 3\left(\frac{x^4}{4}\right)_0^1 = \frac{3}{4}$$

$$\mu_2' = E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^1 x^2 3x^2 dx = 3\left(\frac{x^5}{5}\right)_0^1 = \frac{3}{5}$$

$$\therefore \quad V(X) = \mu_2' - \mu^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$$

■

■ Example 3.12

Find the four moments for the following distributions.

$$\text{(a)} \quad f(x) = \begin{cases} Kx & x = 1, 2, 3, 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{(b)} \quad f(x) = \begin{cases} K & 1 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) In this case $\boldsymbol{A} = \{1, 2, 3, 4\}$ and X is a discrete RV.

$$\text{Hence} \quad \sum_x f(x) = 1 \quad \Rightarrow \quad K(1) + K(2) + K(3) + K(4) = 1$$

$$\Rightarrow \quad 10K = 1; \quad K = \frac{1}{10}$$

Hence pmf of X is

$X = x$	1	2	3	4
$f(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

$$\mu = \mu_1' = E(X) = \Sigma x f(x) = 1\left(\frac{1}{10}\right) + 2\left(\frac{2}{10}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{4}{10}\right) = 3$$

$$\mu_2' = E\left(X^2\right) = \Sigma x^2 f(x) = 1^2\left(\frac{1}{10}\right) + 2^2\left(\frac{2}{10}\right) + 3^2\left(\frac{3}{10}\right) + 4^2\left(\frac{4}{10}\right) = 10$$

$$\mu_3' = E\left(X^3\right) = \Sigma x^3 f(x) = 1^3\left(\frac{1}{10}\right) + 2^3\left(\frac{2}{10}\right) + 3^3\left(\frac{3}{10}\right) + 4^3\left(\frac{4}{10}\right) = 35.4$$

$$\mu_4' = E\left(X^4\right) = \Sigma x^4 f(x) = 1^4\left(\frac{1}{10}\right) + 2^4\left(\frac{2}{10}\right) + 3^4\left(\frac{3}{10}\right) + 4^4\left(\frac{4}{10}\right) = 130$$

$$\mu_1 = 0 \quad \text{(Always)} \quad \mu_2 = \mu_2' - \mu^2 = 10 - 3^2 = 1$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3 = 35.4 - 3(10)(3) + 2(3)^3 = -0.6$$

$$\mu_4 = \mu_4' - 4\mu_3' X\mu + 6\mu_2'\mu^2 - 3\mu^4 = 130 - 4(35.4)(3) + 6(10)(3)^2 - 3(3)^4 = 2.2$$

(b) In this case $\boldsymbol{A} = \{x/1 \leq x \leq 4\}$, X is a continuous random variable

Hence $$\int_{-\infty}^{\infty} f(x)dx = 1 \quad \Rightarrow \quad \int_{1}^{4} K dx = 1 \quad \Rightarrow \quad K = \frac{1}{3}$$

$$\therefore \qquad f(x) = \begin{cases} \frac{1}{3} & 1 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu = \mu_1' = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{1}^{4} x\left(\frac{1}{3}\right)\, dx = \frac{1}{3}\left(\frac{x^2}{2}\right)_1^4 = \frac{5}{2}$$

$$\mu_2' = E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 f(x)\, d(x) = \int_{1}^{4} x^2\left(\frac{1}{3}\right)\, dx = 7$$

$$\mu_3' = E\left(X^3\right) = \int_{-\infty}^{\infty} x^3 f(x)\, dx = \int_{1}^{4} x^3\left(\frac{1}{3}\right)\, dx = \frac{85}{4}$$

$$\mu_4' = E\left(X^4\right) = \int_{-\infty}^{\infty} x^4 f(x)\, dx = \int_{1}^{4} x^4\left(\frac{1}{3}\right)\, dx = \frac{341}{5}$$

$$\mu_1 = 0 \quad \text{(Always)}$$

$$\mu_2 = \mu_2' - \mu^2 = 7 - \left(\frac{5}{2}\right)^2 = \frac{3}{4}$$

$$\mu^3 = \mu_3' - 3\mu_2'\mu + 2\mu^3 = \frac{85}{4} - 3(7)\left(\frac{5}{2}\right) + 2\left(\frac{5}{2}\right)^3 = 0$$

$$\begin{aligned} \mu_4 &= \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4 \\ &= \frac{341}{5} - 4\left(\frac{85}{4}\right)\left(\frac{5}{2}\right) + 6(7)\left(\frac{5}{2}\right)^2 - 3\left(\frac{5}{2}\right)^4 = \frac{81}{80} \end{aligned}$$

Property 3.7: $V(C) = 0$ (C is any constant)

$$\begin{aligned} \text{Now} \quad V(C) &= E\left(C^2\right) - (E(C))^2 \\ &= C^2 - C^2 = 0. \end{aligned}$$

Property 3.8:

$$\begin{aligned} V(aX+b) &= a^2 V(X) \\ V(aX+b) &= E\left((aX+b)^2\right) - (E(aX+b))^2 \\ &= E\left(a^2X^2 + b^2 + 2abX\right) - (aE(X) + E(b))^2 \\ &= \left[a^2E\left(X^2\right) + E\left(b^2\right) + 2abE(X)\right] - \left[a^2E(X)^2 + b^2 + 2abE(X)\right] \\ &= a^2\left[E\left(X^2\right) - (E(X))^2\right] = a^2V(X) \end{aligned}$$

Now we consider two more quantities which are defined as measure of skewness and measure of Kurtosis. Skewness is the degree of asymmetry of a distribution. Kurtosis is the degree of peakedness of a distribution. A measure of skewness is defined as $\gamma_1 = \mu_3^2/\mu_2^2$ and a measure of Kurtosis is defined as $\beta_2 = \mu_4/\mu_2^2$. γ_1 could be negative, zero, and positive for a distribution. In that case the distribution is said to be skewed to the left, not skewed (symmetric) and skewed to the right, respectively. A distribution with a high peak ($\beta_2 > 3$) is called *leptokurtic*, a flat-topped curve ($\beta_2 < 3$) is called *platykurtic* and neither of these ($\beta_2 = 3$) is called the *normal curve* or *mesokurtic*.

■ Example 3.12 (Continued)

Let us calculate the measures of skewness and kurtosis.

(a)
$$\begin{aligned} \gamma_1 &= \frac{\mu_3}{\mu_2^3} = \frac{(-0.6)^2}{1} = (-0.6)^2 = 0.36 \\ \beta_2 &= \frac{\mu_4}{\mu_2^2} = \frac{2.2}{1^2} = 2.2 \end{aligned}$$

So the distribution skewed to left and platykurtic (since $\mu_3 < 0$, the sign of γ_1 is < 0).

(b)
$$\begin{aligned} \gamma_1 &= \frac{\mu_3^2}{\mu_2^3} = 0 \\ \beta_2 &= \frac{\mu_4}{\mu_2^2} = \frac{81/80}{9/16} = 1.8 \end{aligned}$$

So the distribution is symmetric (about mean) but platykurtic. ■

■ Example 3.13

Find the measures of skewness and kurtosis for the distribution $f(x) = \frac{3(1-x^2)}{4} |x| < 1$.

$$
\begin{aligned}
\boldsymbol{A} &= \{x/|x| < 1\} = (-1, 1) \\
\mu &= \mu_1' = \int_{-\infty}^{\infty} x f(x) dx = \frac{3}{4}\int_{-1}^{1} x\left(1 - x^2\right) dx = 0 \quad (\because \text{ the integrand is odd}) \\
\mu_2' &= \int_{-\infty}^{\infty} x^2 f(x) dx = \frac{3}{4}\int_{-1}^{1} x^2\left(1 - x^2\right) dx \quad (\because \text{ the integrand is even}) \\
&= \frac{3}{2}\int_{0}^{1}\left(x^2 - x^4\right) dx = \frac{1}{5} \\
\mu_3' &= \int_{-\infty}^{\infty} x^3 f(x) dx = \frac{3}{4}\int_{-1}^{1} x^3\left(1 - x^2\right) dx = 0 \\
\mu_4' &= \int_{-\infty}^{\infty} x^4 f(x) dx = \frac{3}{4}\int_{-1}^{1} x^4\left(1 - x^2\right) dx \\
&= \frac{3}{2}\int_{0}^{1}\left(x^4 - x^6\right) dx = \frac{3}{35} \\
\mu_2 &= \mu_2' - \mu^2 = \frac{1}{5} \\
\mu_3 &= \mu_3' - 3\mu_2'\mu + 2\mu^3 = 0 \\
\mu_4 &= \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 2\mu^4 = \frac{3}{35} \\
\therefore \quad \text{Skewness } \gamma_1 &= \frac{\mu_3^2}{\mu_2^3} = 0 \\
\text{Kurtosis } \beta_2 &= \frac{\mu_4}{\mu_2^2} = \frac{3/35}{1/25} = 2.14.
\end{aligned}
$$

■

3.3.2 Product Moments – Moments of Bivariate Distribution

Let us extend the definition of moments for the two random variable case. Let us define the product moments of two random variables.

Definition 1: The r^{th} and s^{th} product moment about the means of the random variables X and Y denoted by μ_{rs} is the expected value of $(X - \mu_X)^r (Y - \mu_Y)^s$; that is

$$\mu_{rs} = E\left[(X - \mu_X)^r (Y - \mu_Y)^s\right]$$

$$= \begin{cases} \sum\limits_{x}\sum\limits_{y} (X-\mu_X)^r (Y-\mu_Y)^s f(x,y) & \text{when } X \text{ and } Y \text{ are discrete rvs} \\ \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} (x-\mu_X)^r (y-\mu_Y)^s f(x,y)\, dxdy & \text{when } X \text{ and } Y \text{ are continuous rvs} \end{cases}$$

for $r = 0, 1, 2 \cdots n; s = 0, 1, 2 \cdots n$.

Definition 2: The r^{th} and s^{th} product moment about the origin of the random variables X and Y, denoted by μ'_{rs} is the expected value of $X^r Y^s$; that is

$$\begin{aligned} \mu'_{r,s} &= E\left[X^r Y^s\right] \\ &= \begin{cases} \sum\limits_{x}\sum\limits_{y} x^r y^s f(x,y) & X \text{ and } Y \text{ are discrete rvs} \\ \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} x^r y^s f(x,y) = dxdy & X \text{ and } Y \text{ are continuous} \end{cases} \end{aligned}$$

for $r = 0, 1, 2, \cdots$ and $s = 0, 1, 2, \cdots$

In both cases, if $r = s = 0$ then

$$\begin{aligned} \mu'_{0,0} &= E\left(X^0 Y^0\right) = 1 \quad \text{and} \\ \mu_{0,0} &= E\left[(X-\mu_X)^0 (Y-\mu_Y)^0\right] = 1 \end{aligned}$$

Further $\mu_{1,1}$ is of special importance in statistics because it indicates the relationship between the values of the two random variables X and Y. We define this as covariance of X and Y and it is denoted by σ_{XY}, Cov(X, Y) or $C(X, Y)$ that is Cov$(X, Y) = \mu_{11} = E\left[(X-\mu_X)(Y-\mu_Y)\right]$. Let us write μ_X as $E(X)$ and μ_Y as $E(Y)$ respective means of X and Y, so that Cov$(X, Y) = E\left[(X-E(X))(Y-E(Y))\right]$. Let us now prove one more property which may be useful in the computation of Cov(X, Y).

Property 3.9: Cov$(X, Y) = E(XY) - E(X) \cdot E(Y)$

By the definition,

$$\begin{aligned} \text{Cov}(X, Y) &= E\left[(X-E(X))(Y-E(Y))\right] \\ &= E\left[XY - XE(X) - YE(X) + E(X)E(Y)\right] \\ &= E(XY) - E(XE(Y)) - E(YE(X)) + E(E(X)E(Y)) \\ &= E(XY) - E(X) \cdot E(Y) - E(Y) \cdot E(X) + E(X)E(Y) \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

Observe that in the above derivation we use Property 3.1 and extension of Property 3.4.

We can deduce an interesting aspect of relationship between two variables combining Property 3.5 and Property 3.9. That is if X and Y are independent then, $E(XY) = E(X) \cdot E(Y)$ so that

$$E(XY) - E(X) \cdot E(Y) = 0 \quad \text{or} \quad \text{Cov}(X, Y) = 0$$

$\boxed{\text{Independence of two random variables implies zero covariance.}}$

But a caution! zero covariance does not necessarily imply independence. Let us find this with the help of the following example.

■ Example 3.14

Let the joint pdf of a random variable be

$X \diagdown Y$	-1	0	1
0	0	$\frac{1}{6}$	$\frac{1}{12}$
1	$\frac{1}{4}$	0	$\frac{1}{2}$

Find $\text{Cov}(X, Y)$.
The marginal pdfs of X and Y are

$X = x$	0	1
$f_X(x)$	$\frac{1}{4}$	$\frac{3}{4}$

$Y = y$	-1	0	1
$f_Y(y)$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{7}{12}$

$$
\begin{aligned}
E(XY) &= \Sigma\Sigma xyf(x, y) = (0)(-1)(0) + (0)(0)\frac{1}{6} + (0)(1)\frac{1}{12} + \\
&\qquad (1)(-1)\frac{1}{4} + (1)(0)(0) + (1)(1)\frac{1}{2} = \frac{1}{4} \\
E(X) &= \sum_x xf_X(x) = \frac{3}{4}; \quad E(Y) = \Sigma y f_Y(y) = \frac{1}{3} \\
\therefore \quad \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) = \frac{1}{4} - \left(\frac{3}{4}\right)\left(\frac{1}{3}\right) = 0
\end{aligned}
$$

But X and Y are not independent since $f(0, -1) = 0$ where as $f_X(0) = \frac{1}{4}$ and $f_Y(-1) = \frac{1}{4}$ hence $f(0, -1) \neq f_X(0) \cdot f_Y(-1)$.

However if $\text{Cov}(X, Y) = 0$ it leads to a special cases of distributions that is uncorrelated variables. Before go further, let us define the correlation coefficient as

$$\rho_{XY} = \frac{\text{Cov}\,(X, Y)}{\sigma_X \cdot \sigma_Y}$$

where σ_X and σ_Y are the standard deviation of X and Y respectively.

Hence, now we can say that if the variables are correlated in some way, the covariance will be non-zero. Infact, if $\text{Cov}(X, Y) > 0$, then Y tends to increase (decrease) as X increases (decreases). If $\text{Cov}(X, Y) < 0$, then Y tends to increase (decreases) as X decreases (increases). Let us find these measures in the following cases, and observe a fact that $|\rho_{XY}| \leq 1$ (we prove this later). Also the sign of ρ_{XY} is that of $\text{Cov}(X, Y)$ under the assumption that σ_X, are σ_Y are positive. ■

■ Example 1.15

If X and Y have the joint pmf

$$f(x, y) = \frac{1}{4} \text{ for } (x, y) = (-1, -1), (-1, 10), (-1, 0); (0, -1); (0, 0)$$

Find $\text{Cov}(X, Y)$.

Here $\quad \boldsymbol{A} = \{(x, y)/(x, y) = (-1, -1), (-1, 0), (0, -1), (0, 0)\}$

The joint pdf of X and Y is

$X \diagdown Y$	-1	0
-1	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{4}$	$\frac{1}{4}$

Marginal pdf of X:

$X = x$	-1	0
$f_X(x)$	$\frac{1}{2}$	$\frac{1}{2}$

Marginal pdf of Y:

$Y = y$	-1	0
$f_Y(y)$	$\frac{1}{2}$	$\frac{1}{2}$

It is obvious that $f(x, y) = f_X(x) \cdot f_Y(y)$ for all pairs (x, y) in $\boldsymbol{A}$.
That is X and Y are independent.

$$\therefore \quad \text{Cov}(x, y) = 0$$

■

■ Example 3.16

Find the covariance between X and Y if their joint pmf is

$$f(x, y) = \begin{cases} C\left(x^2 + y^2\right) & x = -1, 0, 1, 3, \quad y = -1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \text{Here} \quad \boldsymbol{A} = \{&(x, y)/(x, y) = (-1, -1), (-1, 2), (-1, 3), (0, -1), (0, 12) \\ &(0, 3), (1, -1), (1, 2), (1, 3), (3, -1), (3, 2), (3, 3)\} \end{aligned}$$

Therefore, to find C, we use

$$\begin{aligned} \sum_x \sum_y C f(x, y) &= 1 \text{ for } (x, y) \epsilon \boldsymbol{A} \\ C\,[2 + 5 + 10 + 1 + 4 + 9 + 2 + 5 + 10 + 10 + 13 + 18] &= 1 \\ C &= \frac{1}{89} \end{aligned}$$

Therefore, the joint pdf of X and Y and hence their marginal pdfs can be written as,

$X \diagdown Y$	-1	2	3
-1	$\frac{2}{89}$	$\frac{5}{89}$	$\frac{10}{89}$
0	$\frac{1}{89}$	$\frac{4}{89}$	$\frac{9}{89}$
1	$\frac{2}{89}$	$\frac{5}{89}$	$\frac{10}{89}$
3	$\frac{10}{89}$	$\frac{13}{89}$	$\frac{18}{89}$

$X = x$	-1	0	1	3
$f_X(x)$	$\frac{17}{89}$	$\frac{14}{89}$	$\frac{17}{89}$	$\frac{41}{89}$

$Y = y$	-1	2	3
$f_Y(y)$	$\frac{15}{89}$	$\frac{27}{89}$	$\frac{47}{89}$

$$\begin{aligned}
E(XY) &= \Sigma\Sigma xyf(x,y) \\
&= (-1)(-1)\frac{2}{89} + (-1)(2)\frac{5}{89} + (-1)(3)\left(\frac{10}{89}\right) + (1)(-1)\frac{2}{89} + (1)(2) \\
&\quad \left(\frac{5}{89}\right) + (1)(3)\left(\frac{10}{89}\right) + (3)(-1)\left(\frac{10}{89}\right) + (3)(2)\left(\frac{13}{89}\right) + (3)(3)\left(\frac{18}{89}\right) \\
&= \frac{210}{89} \\
E(X) = \Sigma x f_X(x) &= (-1)\frac{17}{89} + (1)\frac{17}{89} + (3)\frac{41}{89} = \frac{123}{89} \\
E(Y) &= \Sigma y f_Y(y) = (-1)\frac{15}{89} + (2)\frac{27}{89} + (3)\frac{47}{89} = \frac{180}{89} \\
\therefore \quad \text{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\
&= \frac{210}{89} - \left(\frac{123}{89}\right)\left(\frac{180}{89}\right) = -0.436
\end{aligned}$$

■

■ Example 3.17

Find the correlation coefficient between X and Y if their joint pmf is

$$f(x,y) = \begin{cases} \frac{x+y}{4} & (x,y) = (0,0), (0,1)(1,0)(1,1) \\ 0 & \text{elsewhere} \end{cases}$$

The joint pmf of X and Y and their marginal pdfs are

XY	0	1
0	0	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$

$X = x$	0	1
$f_X(x)$	$\frac{1}{4}$	$\frac{3}{4}$

$Y = y$	0	1
$f_Y(y)$	$\frac{1}{4}$	$\frac{3}{4}$

$$\begin{aligned}
E(XY) &= \Sigma\Sigma xy f(x, y) = \frac{1}{2} \\
E(X) &= \frac{3}{4}; \quad E(Y) = \frac{3}{4} \\
\text{Cov}(X, Y) &= E(XY) - E(X)E(Y) = \frac{1}{2} - \left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = -\frac{1}{16} \\
E\left(X^2\right) &= \Sigma x^2 f_X(x) = \frac{3}{4} \\
E\left(Y^2\right) &= \Sigma y^2 f_Y(y) = \frac{3}{4}
\end{aligned}$$

$$\therefore \quad V(X) = E\left(X^2\right) - E(X)^2 = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{16}$$

$$\text{and} \quad V(Y) = E\left(Y^2\right) - E(Y)^2 = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{16}$$

$$\therefore \quad \rho_{XY} = \frac{\text{Cov}(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{-1/16}{\sqrt{3/16}\sqrt{3/16}} = \frac{-1}{3}$$

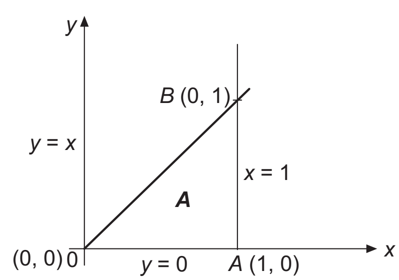
■

■ Example 3.18

If the joint pdf of (X, Y) is $f(x, y) = 24y(1 - x)$ space $0 \le y \le x \le 1$ find ρ_{XY}.

In this case $\qquad \boldsymbol{A} = \{(x, y)/0 \le y \le x \le 1\}$

First let us find the marginal pdfs.



$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x, y)\, dy \\
&= \int_{y=0}^{x} 24y(1 - x)\, dy \\
&= 24\left(\frac{y^2}{2}\right)_0^1 (1 - x) \\
&= \begin{cases} 12x^2(1 - x) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases} \\
f_Y(y) &= \int_{-\infty}^{\infty} f(x, y) dx = \int_{y}^{1} 24y(1 - x)\, dx
\end{aligned}$$

$$
\begin{aligned}
&= 24y\left(x - \frac{x^2}{2}\right)_y^1 = 24y\left[\left(1 - \frac{1}{2}\right) - \left(y - \frac{y^2}{2}\right)\right] \\
&= 24y\left(\frac{y^2}{2} - y + \frac{1}{2}\right) = \frac{24y\,(1-y)^2}{2} \\
&= \begin{cases} 12y(1-y)^2 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases} \\
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f(x,y)\,dx\,dy = \int_0^1\int_y^1 xy\;24y(1-x)\,dx\,dy \\
&= 24\int_0^1\int_y^1 xy^2(1-x)\,dx\,dy \\
&= 24\int_0^1 y^2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_y^1 dx \\
&= 24\int_0^1 y^2\left(\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right) dx \\
E(XY) &= \frac{4}{15} \\
E(X) &= \int_{-\infty}^{\infty} x\; f_X(x)\,dx = \int_0^1 x\;12x^2(1-x)\,dx \\
&= 12\int_0^1 \left(x^3 - x^4\right)\,dx = \frac{3}{5} \\
E(Y) &= \int_{-\infty}^{\infty} y\; f_Y(y)\,dy = \int_0^1 y \cdot 12y\,(1-y)^2\;dy \\
&= 12\int_0^1 y^2\,(1-y)^2\;dy \\
&= 12\int_0^1 y^{3-1}\,(1-y)^{3-1}\,dy \\
&= 12\,\beta(3,3) = \frac{2}{5} \\
\operatorname{Cov}(X,Y) &= E(XY) - E(X)E(Y)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4}{15} - \left(\frac{3}{5}\right)\left(\frac{2}{5}\right) = \frac{2}{75} \\
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = 12\int_0^1 x^4(1-x)dx \\
&= 12\int_0^1 x^{5-1}(1-x)^{2-1}dx \\
&= 12\,\beta(5,2) = \frac{4}{5} \\
E\left(Y^2\right) &= \int_{-\infty}^{\infty} y^2 f_Y(y)dy = 12\int_0^1 y^3(1-y)^2dy \\
&= 12\,\beta(4,3) = \frac{6}{5} \\
V(X) &= E\left(X^2\right) - E(X)^2 \\
&= \frac{4}{5} - \left(\frac{3}{5}\right)^2 = \frac{11}{25} \\
V(Y) &= E\left(Y^2\right) - E(Y)^2 \\
&= \frac{6}{5} - \left(\frac{2}{5}\right)^2 = \frac{26}{25} \\
\rho_{XY} &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)}\cdot\sqrt{V(Y)}} \\
&= \frac{2/75}{\sqrt{11/25}\sqrt{26/25}} \\
&= 0.018.
\end{aligned}
$$

■

3.3.3 Moments of Linear Combination of Random Variables

The previous two sections in moments introduce the important constants of a random variable namely mean, variance and covariance between two random variables and their statistical significance. In this section, we shall derive expressions for the mean and the variance of a linear combination of n random variables and the covariance of two linear combination of n random variables. Indeed Property 3.4 is a special case of a Property 3.10, which we shall derive in this section.

There are two things before we start to derive the properties. Even though the following properties could be stated and proved for 'n' random variables, we prove only for $n = 2$ variables, since our treatment is restricted with 2 variables. Secondly, we can continue the properties list of of expectations which we started in our earlier section itself.

Property 3.10: If X_1 and X_2 are two random variables and a_1 and a_2 are constants, then $E\,(a_1X_1 + a_2X_2) = a_1E\,(X_1) + a_2E\,(X_2)$.

Let us prove for continuous random variable (proof is similar to discrete case).

Let $f\,(X_1, X_2)$ be the joint pdf of X_1 and X_2 and $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ be their respective marginal pdfs.

Now

$$\begin{aligned}
E\,(a_1X_1 + a_2X_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (a_1X_1 + a_2X_2) f\,(x_1, x_2)\,dX_1dX_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_1X_1 f\,(x_1, x_2)\,dx_1dx_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_2X_2 f\,(x_1x_2)\,dx_1dx_2 \\
&= a_1 \int_{-\infty}^{\infty} x_1 \left[\int_{-\infty}^{\infty} f\,(x_1x_2)\,dx_2\right] dx_1 + a_2 \int_{-\infty}^{\infty} x_2 \left[\int_{-\infty}^{\infty} f\,(x_1x_2)\,dx_1\right] dx_2 \\
&= a_1 \int_{-\infty}^{\infty} x_1 f_{X_1}\,(x_1)\,dx_1 + a_2 \int_{-\infty}^{\infty} x_2 f_{X_2}\,(x_2)\,dx_2 \\
&= a_1E\,(X_1) + a_2E\,(X_2)
\end{aligned}$$

This property can be generalised to n random variables as

$$E\left[\sum_{i=1}^{n} a_iX_i\right] = \sum_{i=1}^{n} a_iE\,(X_i)$$

Also taking $a_1 = a_2 = 1$, this reduces to Property 3.4.

Property 3.11: Let X_1, X_2 be random variables and a_1, a_2 be constants. Then,

$$\begin{aligned}
V\,(a_1X_1 + a_2X_2) &= a_1^2V\,(X_1) + a_2^2V\,(X_2) + 2ab\,\mathrm{Cov}\,(X_1, X_2) \\
\text{Let } Y &= a_1X_1 + a_2X_2 \\
V\,(Y) &= E\left(Y^2\right) - E\,(Y)^2 \\
&= E\left[(a_1X_1 + a_2X_2)^2\right] - \left[E\,(a_1X_1 + a_2X_2)\right]^2 \\
&= E\left[a_1^2X_1^2 + a_2^2X_2^2 + 2a_1a_2X_1X_2\right] - \left[a_1E\,(X_1) + a_2E\,(X_2)\right]^2 \\
&= \left[a_1^2E\left(X_1^2\right) + a_2^2E\left(X_2^2\right) + 2a_1a_2E\,(X_1X_2)\right] \\
&\quad - \left[a_1^2E\,(X_1)^2 + a_2^2E\,(X_2)^2 + 2a_1a_2E\,(X_1)\,E\,(X_2)\right]
\end{aligned}$$

$$\begin{aligned}
&= a_1^2\left[E\left(X_1^2\right) - E\left(X_1\right)^2\right] + a_2^2\left[E\left(X_2^2\right) - E\left(X_2\right)^2\right] \\
&\quad +2a_1a_2\left[E\left(X_1X_2\right) - E\left(X_1\right)E\left(X_2\right)\right] \\
&= a_1^2V\left(X_1\right) + a_2^2V\left(X_2\right) + 2ab\,\text{Cov}\left(X_1, X_2\right)
\end{aligned}$$

Corollary: If the random variables X_1, X_2 are independent then Property 3.11 becomes

$$V\left(a_1X_1 + a_2X_2\right) = a_1^2V\left(X_1\right) + a_2^2V\left(X_2\right)$$

Since X_1, $X2$ are independent.
$\text{Cov}(X_1X_2) = 0$ and hence Property 3.11 becomes,

$$V\left(a_1X_1 + a_2X_2\right) = a_1^2V\left(X_1\right) + a_2^2V\left(X_2\right)$$

In particular if we take $a_2 = 0$, then the property is exactly what we derived in Property 3.8. The generalisation of Property 3.11 is as follows. If $X_1, X_2 \cdots X_n$ are random variables, $a_1, a_2 \cdots a_n$ are constants then

$$V\left(\sum_{i=1}^{n} a_iX_i\right) = a_i^2V\left(X_i\right) + 2\sum_{1\le i<}\sum_{j\le n} a_ia_j \quad \text{Cov}\left(X_i, X_j\right)$$

Particularly, if $X_1X_2 \cdots X_n$ are independent then

$$V\left(\sum_{i=1}^{n} a_iX_i\right) = \sum_{i=1} a_i^2V\left(X_i\right)$$

Property 3.12: If X_1, X_2 are random variables a_1, a_2, b_1, b_2 are constants then

$\text{Cov}\left(a_1X_1 + a_2X_2, b_1X_1 + b_2X_2\right)$

$$\begin{aligned}
&= a_1b_1V\left(X_1\right) + a_2b_2V\left(X_2\right) + \left(a_1b_2 + a_2b_1\right)\text{Cov}\left(X_1, X_2\right) \\
\text{Let } u &= a_1X_1 + a_2X_2 \quad \text{and} \quad V = b_1X_1 + b_2X_2 \\
\therefore\ \text{Cov}(u, v) &= E[uv] - E(u)E(v) \qquad (1) \\
E[uv] &= E\left[\left(a_1X_1 + a_2X_2\right)\left(b_1X_1 + b_2X_2\right)\right] \\
&= E\left[a_1b_1X_1^2 + a_1b_2X_1X_2 + a_2b_1X_2X_1 + a_2b_2X_2^2\right] \\
&= a_1b_1E\left[X_1^2\right] + \left(a_1b_2 + a_2b_1\right)E\left(X_1X_2\right) + a_2b_2E\left(X_2^2\right) \qquad (2) \\
E[u]\,E[v] &= E\left[\left(a_1X_1 + a_2X_2\right)\right]\cdot E\left[\left(b_1X_1 + b_2X_2\right)\right] \\
&= \left(a_1E\left(X_1\right) + a_2E\left(X_2\right)\right)\left(b_1E\left(X_1\right) + b_2E\left(X_2\right)\right) \\
&= a_1b_1E\left(X_1\right)^2 + a_1b_2E\left(X_1\right)E\left(X_2\right) + a_2b_1E\left(X_2\right)E\left(X_1\right) \\
&\quad +a_2b_2\left(E\left(X_2\right)\right)^2 \\
&= a_1b_1E\left(X_1^2\right) + \left(a_1b_2 + a_2b_1\right)E\left(X_1\right)E\left(X_2\right) + a_2b_2E\left(X_2\right)^2 \qquad (3)
\end{aligned}$$

Now if we substitute (2) and (3) in (1) and collect the coefficients respectively in each term of (2) and (3) we get

$$\begin{aligned}\text{Cov}(u, v) &= a_1 b_1 \left[E\left(X_1^2\right) - E\left(X_1\right)^2\right] + (a_1 a_2 + a_2 b_1) \\ &\quad \left[E\left(X_1 X_2\right) - E\left(X_1\right) E\left(X_2\right)\right] + a_2 b_2 \left[E\left(X_2^2\right) - E\left(X_2\right)^2\right] \\ &= a_1 b_1 V\left(X_1\right) + (a_1 b_2 + a_2 b_1)\,\text{Cov}\left(X_1, X_2\right) + a_2 b_2 V\left(X_2\right)\end{aligned}$$

Corollary: If the random variables X_1, X_2 are independent, then

$$\text{Cov}\left(a_1 X_1 + a_2 X_2, b_1 X_1 + b_2 X_2\right) = a_1 b_1 V\left(X_1\right) + a_2 b_2 V\left(X_2\right)$$

Since Cov $(X_1 + X_2) = 0$, proof is immediate from Property 3.12.

We can generalise Property 3.12 and its corollary for n random variables as follows. If $X_1, X_2 \cdots, X_n$ are random variables; $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots b_n$ are constants, we define

$$\begin{aligned}u &= \sum_{i=1}^{n} a_i X_i, \quad v = \sum_{i=1}^{n} b_i X_i \quad \text{then} \\ \text{Cov}(u, v) &= \sum_{i=1}^{n} a_i b_i V(X_i) + \sum_{1 \le i < j \le n} \left(a_i b_j + a_j b_i\right)\text{Cov}\left(X_i, X_j\right)\end{aligned}$$

In particular, if $X_1, X_2, \cdots, X_n$ are independent random variables, then

$$\text{Cov}(u, v) = \sum_{i=1}^{n} a_i b_i v(X_i)$$

Eventhough our discussion is restricted to two variables, in the subsequent examples we may try to apply the of these results generalised versions also so that we can extend these ideas more naturally.

■ Example 3.19

If the independent random variables X and Y have the variances 64 and 25 respectively, find ρ_{uv} where $u = X + Y$ and $v = X - Y$.

$$\begin{aligned}\rho_{uv} &= \frac{\text{Cov}(u, v)}{\sigma_u \sigma_v} \qquad (1) \\ \text{Cov}(u, v) &= \text{Cov}(X + Y, X - Y) \\ &= (1)(1)V(X) + (1)(-1)V(Y) \quad \text{(using Property 3.12)} \\ &= 64 - 25 = 39 \\ \text{Var}(u) &= V(X + Y) \\ &= (1)^2 V(X) + (1)^2 V(Y) \quad \text{(using Property 3.11)} \\ &= 64 + 25 = 89 \\ \text{Var}(v) &= V(X - Y) \\ &= (1)^2 V(X) + (-1)^2 V(Y) = 89 \\ \therefore \text{ (1) becomes, } \rho_{uv} &= \frac{39}{\sqrt{89}\sqrt{89}} = 0.44. \end{aligned}$$

■

■ Example 3.20

If $Y = \frac{X-\mu_X}{\sigma_X}$ where μ_X and σ_X are the mean and standard deviation of X, find μ_Y and σ_Y.

Using the appropriate properties,

$$\begin{aligned}
\mu_Y &= E(Y) = E\left[\frac{X-\mu_X}{\sigma_X}\right] \\
&= \frac{1}{\sigma_X^2} E\left[X-\mu_X\right] \\
&= \frac{1}{\sigma_X^2}\left(E(X)-\mu_X\right) \\
&= \frac{1}{\sigma_X^2}\left(\mu_X-\mu_X\right) = 0 \\
V(Y) &= V\left(\frac{X-\mu_X}{\sigma_X}\right) = \frac{1}{\sigma_X^2} V(X) \\
&= \frac{1}{\sigma_X^2}\sigma_X^2 = 1.
\end{aligned}$$

■

■ Example 3.21

If X and Y are independent RVs with means 4 and 5 and variance 1 and 2 respectively, find the mean and variance of $Z = 3X - 2Y$.

Using appropriate properties,

$$\begin{aligned}
\mu_Z &= E(Z) = E\left(3X-2Y\right) = 3E(X) - 2E(Y) \\
&= 3(4) - 2(5) \\
&= 12 - 10 = 2 \\
V(Z) &= V(3X-2Y) = 3^2 V(X) + (-2)^2 V(Y) \\
&= 9(1) + 4(2) = 17.
\end{aligned}$$

■

■ Example 3.22

If X is a random variable with $E(X) = 20$ and $V(X) = 36$. Find the positive values of a and b such that $Y = aX - b$ has mean 0 and variance 1.

$$\begin{aligned}
E(Y) &= aE\left(X\right) - b \\
\Rightarrow \quad 0 &= 20a - b \qquad (1) \\
V(Y) &= a^2 V(X) \\
1 &= a^2 36 \\
\therefore \quad a &= \frac{1}{\sqrt{36}} = \frac{1}{6} \\
\text{equation (1)} \Rightarrow \quad b &= 20a = \frac{10}{3}
\end{aligned}$$

■

■ Example 3.23

Let X_1, X_2 and X_3 be three random variables with $V(X_1) = 5$, $V(X_2) = 4$, $V(X_3) = 7$ $\text{Cov}(X_1, X_2) = 3$; $\text{Cov}(X_1, X_3) = -2$ and X_2 and X_3 are independent. Find the covariance of $Y_y = X_1 - 2X_2 + 3X_3$ and $Z = -2X_1 + 3X_2 + 4X_3$.

(Since we have 3 variables, we apply the extension of Property 3.12)

$$\begin{aligned}
\text{Cov}\,(Y, Z) &= \text{Cov}\,(X_1 - 2X_2 + 3X_3, -2X_1 + 3X_2 + 4X_3) \\
&= [(1)(-2)V(X_1) + (-2)(3)V(X_2) + (3)(4)V(X_3)] \\
&\quad + [(1)(3) + (-2)(-2)]\,\text{Cov}(X_1, X_2) + [(1)(4) + (3)(-2)] \\
&\quad \text{Cov}(X_1, X_3) + [(-2)(4) + (3)(3)]\,\text{Cov}(X_2, X_3) \\
&= (-2)5 + (-6)(4) + 12(7) + (7)(3) + (-2)(-2) + (1)(0)
\end{aligned}$$

■

Since X_2 and X_3 are independent $\text{Cov}(X_2, X_3) = 0$.

3.4 CONDITIONAL EXPECTATIONS

In the previous chapter, we discussed the methods of obtaining conditional probabilities and conditional *pdf* of discrete type or continuous type of random variables. In this section, we define conditional expectations of random variables in terms of their conditional distributions.

Definition 1: Let X, Y be any two random variables. $f(x/y)$ is the value of the conditional *pdf* of X given $Y = y$ at x. Let $u(X)$ be a function of X. Then, the conditional expectation of $u(X)$ given $Y = y$ denoted by $E[u(X)/y]$ is defined as

$$E[u(X)/Y = y] = \begin{cases} \sum\limits_{x} u(X) f(x/y) & \text{for discrete case} \\ \int\limits_{-\infty}^{\infty} u(X) f(x/y) dx & \text{for continuous case} \end{cases}$$

Such conditional expectation of $\phi(Y)$ (some function of Y) given $X = x$ can be defined in a similar way.

Two special cases of $u(X)$ (similarly $\phi(Y)$) are of more important that is $u(X) = X$ and $u(X) = X^2$.

Definition 2: The conditional mean of the random variable X given $Y = y$ is defined as $E[X/Y = y]$ and

$$E[X/Y = y] = \begin{cases} \sum\limits_{x} x f(x/y) & \text{for discrete case} \\ \int\limits_{-\infty}^{\infty} x f(x/y) dx & \text{for continuous case} \end{cases}$$

Definition 3: The conditional variance of X given $Y = y$ is defined as

$$V(X/Y = y) = E[(X - E(X/Y = y))^2/Y = y]$$

Remember that we use the expression $V(X) = E(X^2) - E(X)^2$ to compute variance of a random variable X. In a similar way, the conditional variance can be calculated using an expression

$$E\left[(X - E(X/Y = y))^2/Y = y\right] = E[X^2/Y = y] - (E[X/Y = y])^2$$

where $E[X^2/Y = y]$ is obtained from definition (1) with $u(X) = X^2$. All these measures can be defined for conditional expectations of Y given $X = x$. One should not find it difficult to write the appropriate expressions. Following examples illustrate the above ideas.

■ Example 3.24

Let X and Y have the joint pdf $f(x, y) = x + y, 0 < x < 1, 0 < y < 1$; zero elsewhere. Find the conditional mean and variance of Y given $X = x$.

Let us first find the conditional *pdf* of Y given X.

$$\begin{aligned}
f(y/x) &= \frac{f(x,y)}{f_X(x)} \\
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \int_0^1 (x+y)\,dy = \left(xy + \frac{y^2}{2}\right)_0^1 = \begin{cases} x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \\
\therefore \quad f(y/x) &= \begin{cases} \frac{x+y}{x+\frac{1}{2}} & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases} \\
E[Y/X = x] &= \int_{-\infty}^{\infty} y\, f(y/x)\,dy = \int_0^1 y\frac{x+y}{x+\frac{1}{2}}\,dy = \frac{1}{x+\frac{1}{2}}\left(x\frac{y^2}{2} + \frac{y^3}{3}\right)_0^1 \\
&= \frac{2}{2x+1}\left(\frac{x}{2} + \frac{1}{3}\right) = \frac{3x+2}{6x+3} \quad 0 < x < 1 \\
E[Y^2/X = x] &= \int_{-\infty}^{\infty} y^2\, f(y/x)\,dy = \int_0^1 y^2\frac{x+y}{x+\frac{1}{2}}\,dy \\
&= \frac{1}{x+\frac{1}{2}}\left(x\frac{y^3}{3} + \frac{y^4}{4}\right)_0^1 = \frac{4x+3}{6(2x+1)} \\
\therefore \quad V[Y/X = x] &= E[Y^2/X = x] - E[Y/X = x]^2 \\
&= \frac{4x+3}{6(2x+1)} - \frac{(3x+2)^2}{9(2x+1)^2} \\
&= \frac{6x^2+6x+1}{18(2x+1)^2} \quad 0 < x < 1
\end{aligned}$$

■

■ Example 3.25

Let X and Y have the joint *pdf* as follows $p(1,1)=\frac{1}{9},\ p(2,1)=\frac{1}{3},\ p(3,1)=\frac{1}{9};\ p(1,2)=\frac{1}{9}, p(2,2)=0, p(3,2)=\frac{1}{18};\ p(1,3)=0,\ p(2,3)=\frac{1}{6},\ p(3,3)=\frac{1}{9}$. Find $E[X/Y=i]$ $i=1,2,3$.

Let us write the *joint pdf* X and Y and their *marginal pdf* in our usual table form.

$X \diagup Y$	1	2	3
1	$\frac{1}{9}$	$\frac{1}{9}$	0
2	$\frac{1}{3}$	0	$\frac{1}{6}$
3	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{9}$

$X=x$	1	2	3
$f_X(x)$	$\frac{2}{9}$	$\frac{1}{2}$	$\frac{5}{18}$

$Y=y$	1	2	3
$f_Y(y)$	$\frac{5}{9}$	$\frac{1}{6}$	$\frac{5}{18}$

$$f(X/Y=i)=\frac{f(x,i)}{f_Y(i)}\ i=1,2,3$$

When $i=1$

$X=x$	1	2	3
$f(x/y=1)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

When $i=2$

$X=x$	1	2	3
$f(x/y=2)$	$\frac{2}{3}$	0	$\frac{1}{3}$

When $i=3$

$X=x$	1	2	3
$f(x/y=1)$	0	$\frac{3}{5}$	$\frac{2}{5}$

$$
\begin{aligned}
E[X/Y=i] &= \sum_x x f(x/y=i) \quad \text{for } i=1,2,3 \\
\therefore \qquad E[X/Y=1] &= \sum_x x f(x/y=1) = 1\left(\frac{1}{5}\right)+2\left(\frac{3}{5}\right)+3\left(\frac{1}{5}\right)=2 \\
E[X/Y=2] &= \sum_x x f(x/y=2) \\
&= 1\left(\frac{2}{3}\right)+2(0)+3\left(\frac{1}{3}\right)=\frac{5}{3} \\
E[X/Y=3] &= \sum_x x f(x/y=3) \\
&= 1(0)+2\left(\frac{3}{5}\right)+3\left(\frac{2}{5}\right)=\frac{12}{5}
\end{aligned}
$$

■

More on Conditional Mean

Let $f(x, y)$ denote the joint pdf of two random variables X and Y and if $f_X(x)$ is the marginal pdf of X then we have the conditional pdf of Y given $X = x$ is

$$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)}$$

and the conditional mean of Y given $X = x$ is given by

$$E[Y/X = x] = \int_{-\infty}^{\infty} y\ f_{Y/X}(y/x)\,dy \quad \text{(or)} \quad \sum_y y\ f_{Y/X}(y/x)$$

accordingly (X, Y) is continuous or discrete. This conditional mean of Y, given $X = x$ is a function of x alone. Similarly, the conditional mean X, given $Y = y$ is a function of y alone.

The conditional mean $E(Y/X = x)$ for a continuous distribution is called the regression function of Y on X and the graph of this function of x is known as the regression curve of Y on X or sometimes it is called as the regression curve for the mean of Y. Similarly, we define $E(X/Y = y)$ as the regression curve of X on Y or regression curve for the mean of X.

If this curve is linear (straight line), it is called the line of regression between the variables. Otherwise regression is said to be curvilinear.

Further, the regression lines are obtained by the following two equations.

(a) Regression line of Y on X is

$$Y = E(Y/X) = \bar{Y} + r\frac{\sigma_Y}{\sigma_X}(X - \bar{X}) \quad \text{and}$$

(b) Regression line of X on Y is

$$X = E(X/Y) = \bar{X} + r\frac{\sigma_X}{\sigma_Y}(Y - \bar{Y}) \quad \text{where}$$

$$\bar{X} = E(X);\ \bar{Y} = E(Y) \quad \text{and} \quad r = \rho_{XY}$$

■ Example 3.25 (Continued)

Obtain the regression curves of X on Y and Y on X.

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x, y)\, dx \\ &= \int_0^1 (x + y)\, dx \\ &= \begin{cases} y + \frac{1}{2} & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

$\therefore$ The conditional pdf of X on Y is

$$f_{X/Y}(x/y) = \frac{f(x, y)}{f_Y(y)} = \begin{cases} \frac{x+y}{y+\frac{1}{2}} & 0 < x < 1; 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ The conditional mean of X on Y is

$$\begin{aligned} E[X/Y = y] &= \int_{-\infty}^{\infty} x\ f_{X/Y}(x/y)\, dx \\ &= \int_0^1 x\frac{x + y}{y + \frac{1}{2}}\, dx \\ &= \frac{1}{y + \frac{1}{2}}\left(\frac{x^3}{3} + y\frac{x^2}{2}\right)_0^1 \\ &= \frac{3y + 2}{6y + 3} \qquad 0 < y < 1 \end{aligned}$$

[Observe that $E(X/Y = y)$ is a function of y and $E(Y/X = x)$ is a function of x].

Hence, the regression curve of X on Y is

$$x = \frac{3Y+2}{6Y+3} \qquad 0 < y < 1$$

and the regression curve of Y on X is

$$Y = \frac{3x+2}{6x+3} \qquad 0 < x < 1$$

$$\begin{aligned} \text{Also,} \qquad E(X) &= \int_0^1 x\left(x+\frac{1}{2}\right)dx = \frac{7}{12} \\ E(X^2) &= \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \frac{5}{12} \\ V(X) &= \frac{11}{144} \end{aligned}$$

Similarly, $E(Y) = \frac{7}{12}$ and $V(Y) = \frac{11}{144}$ (By Symmetry)

$$\begin{aligned} \text{Also,} \qquad E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)\,dxdy \\ &= \int_0^1\int_0^1 xy(x+y)\,dxdy = \frac{1}{3} \\ \therefore \qquad \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ &= \frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = -\frac{1}{144} \\ \therefore \qquad \rho &= \frac{\text{cov}(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = -\frac{1}{11} \end{aligned}$$

$\therefore$ Regression lines of (i) X on Y is

$$\begin{aligned} X &= \frac{7}{12} + \left(\frac{-1}{11}\right)\frac{\sqrt{\frac{11}{144}}}{\sqrt{\frac{11}{144}}}\left(Y - \frac{7}{12}\right) \\ &= \frac{7}{12} - \frac{1}{11}\left(Y - \frac{7}{12}\right) \\ X &= -\frac{1}{11}Y + \frac{7}{11} \end{aligned}$$

and (ii) Y on X is $Y = -\frac{1}{11}X + \frac{7}{11}$

Remark

$b_{XY} = r\frac{\sigma_X}{\sigma_Y}$ and $b_{YX} = r\frac{\sigma_Y}{\sigma_X}$ are called the regression coefficients of X on Y and Y on X respectively.

Properties of regression coefficients

1. b_{XY} and b_{YX} have always same sign.
2. Correlation coefficient is the geometric mean of regression coefficients that is $\rho_{XY} = \pm\sqrt{b_{XY} \cdot b_{YX}}$ where the sign of ρ is the same as that of the regression coefficients.
3. Both coefficients cannot exceed one simultaneously.
4. Regression coefficients are independent of the change of origin but not of scale.

Remark

The two regression lines are not reversible (Refer Example (3.22) (continued)). Also, the mean $\left(\bar{X}, \bar{Y}\right)$ can be obtained as the point of intersection of the two regression lines.

If the conditional mean of Y given $X = x$ is linear function of x say $a + bx$, we say that the conditional mean of Y is linear in x. In such cases, we write the conditional mean as

$$E[Y/X = x] \quad = \quad E(Y) + \frac{\text{Cov}(X, Y)}{V(X)}(X - E(X))$$

Similarly, the conditional mean of X given $Y = y$ is linear in y then we write

$$E[X/Y = y] = E(X) + \frac{\text{Cov}(X, Y)}{V(Y)}(Y - E(Y))$$

Since, the above statements follow directly from the definition of conditional mean, we omit the proof here. Following example illustrates this relationship.

Consider the pdf

$$f(x, y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

One can easily calculate the following: mpdf of X and Y are

$$f_X(x) \quad = \quad \begin{cases} e^{-x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) \quad = \quad \begin{cases} ye^{-Y} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

so that the cpdfs are

$$f_{X/Y}(x/y) = \begin{cases} \frac{1}{y} & 0 < x < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

and
$$f_{Y/X}(y/x) = \begin{cases} e^{x-y} & 0 < x < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Subsequently, we have

$$E[X/Y = y] = \frac{y}{2}\ y > 0 \quad \text{and} \quad E[Y/X = x] = x + 1\ x > 0$$

where conditional mean of X given Y is linear in Y and conditional mean of Y given X is linear x.

Now, we shall find the regression lines with the values $E(X) = 1;\ E(X^2) = 2;\ V(X) = 1$ and $E(Y) = 2;\ E(Y^2) = 6$ and $V(Y) = 2$. Also $\text{Cov}(X, Y) = 1$ and $\rho = \frac{1}{\sqrt{2}}$.

$\therefore$ Regression lines are

$$\begin{aligned} X &= \bar{X} + \rho\frac{\sigma_X}{\sigma_Y}(Y - \bar{Y}) \\ &= 1 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(Y - 2) \\ X &= \frac{Y}{2} \end{aligned}$$

and
$$\begin{aligned} Y &= \bar{Y} + \rho\frac{\sigma_Y}{\sigma_X}(X - \bar{X}) \\ &= 2 + \frac{1}{\sqrt{2}}\sqrt{2}(X - 1) \\ Y &= X + 1 \end{aligned}$$

Thus, both methods yield the same equations.

Remark

In the above example both conditional means are linear. But, it does not however in general true. Check this with Exercise (18)(c).

3.5 MOMENT GENERATING FUNCTION

In this section we define another special mathematical expectation, called the *moment-generating function* (MGF) of a random variable X. It is defined as $E\left(e^{tx}\right)$ and is denoted by $M_x(t)$. That is

$$\begin{aligned} M_X(t) &= E\left(e^{tx}\right) \\ &= \begin{cases} \sum_x e^{tx} f(x) & X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx & X \text{ is continuous} \end{cases} \end{aligned}$$

for $-h < t < h$; $(h > 0)$ and the summation or integral should exist. We will see later that not every distribution has a MGF. But if it exists, it is unique and completely determines the distribution of the random variable. Let us consider two examples to calculate this special expectation.

■ Example 3.26

Let the *pdf* of a random variable be

$$\begin{aligned}
f(x) &= \begin{cases} \frac{x}{15} & x = 1, 2, 3, 4, 5 \\ 0 & \text{elsewhere} \end{cases} \text{, find the MGF of } X. \\
, M_X(t) &= E\left(e^{tx}\right) = \sum_x e^{tx} f(x) \\
&= e^t f(1) + e^{2t} f(2) + e^{3t} f(3) + e^{4t} f(4) + e^{5t} f(5) \\
&= \frac{1}{15}\left[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t}\right]
\end{aligned}$$

■

■ Example 3.27

If the *pdf* of a random variable is

$$\begin{aligned}
f(x) &= \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \\
M_X(t) &= E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\
&= \int_0^1 e^{tx} 3x^2\, dx \\
&= 3\left[x^2 \frac{e^{tx}}{t} - 2x\left(\frac{e^{tx}}{t^2}\right) + 2\left(\frac{e^{tx}}{t^3}\right)\right]_0^1 \\
&= 3\left[\frac{e^t}{t} - \frac{2e^t}{t^2} + \frac{2e^t}{t^3} - \frac{1}{t^3}\right]
\end{aligned}$$

■

Now let us prove two results from the definition of MGF and realize why we refer to this function as a *moment generating function*.

Now
$$\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \frac{(tx)^4}{4!} + \cdots\right] \\
&= E(1) + E\left(\frac{tx}{1!}\right) + E\left(\frac{(tx)^2}{2!}\right) + E\left(\frac{(tx)^3}{3!}\right) + \cdots \\
&= 1 + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \frac{t^4}{4!}E(X^4) + \cdots \qquad (1)
\end{aligned}$$

Equation (1) helps us to calculate r^{th} moment about zero (μ_r') of the random variable X. In fact there are two ways to derive the moments from MGF of X.

Result 1: $E(X^r) =$ coefficient of $\frac{t^r}{r!}$ in $M_X(t)$

Proof is straight forward from equation (1).

Result 2:
$$E(X^r) = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}$$

Proof is by induction. Differentiate equation (1) with respect to t,

$$\frac{d}{dt} M_X(t) = E(X) + \frac{2t}{2!} E(X^2) + \frac{3t^2}{3!} E(X^3) + \cdots \qquad (2)$$

Put $t = 0$,

$$\begin{aligned} \left[\frac{d}{dt} M_X(t)\right]_{t=0} &= E(X) + 0 + 0 + \cdots \\ \therefore \qquad E(X) &= \left[\frac{d}{dt} M_X(t)\right]_{t=0} \end{aligned}$$

Differentiate equation (2) with respect to t,

$$\frac{d^2}{dt^2} M_X(t) = E(X^2) + \left(\frac{2.3}{3!}\right) t \ E(X^3) + \left(\frac{4.3}{4!}\right) t^2 \ E(X^4) + \cdots$$

put $t = 0$,

$$\begin{aligned} \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} &= E(X^2) + 0 + 0 + \cdots \\ \therefore \qquad E(X^2) &= \left[\frac{d}{dt^2} (M_X(t))\right]_{t=0} \end{aligned}$$

Similarly extending this idea we get,

$$E(X^r) = \left[\frac{d^r}{dt^r} (M_X(t))\right]_{t=0}$$

This method of obtaining moments or special cases namely mean and variance illustrated in the following set of examples.

■ Example 3.28

Find the MGF of X and hence its mean and variance if the pdf is

$$f(x) = \begin{cases} 2\left(\frac{1}{3}\right)^x & x = 1, 2, 3, \cdots \\ 0 & \text{elsewhere} \end{cases}$$

$$
\begin{aligned}
M_X(t) &= E\left(e^{tx}\right) = \sum_x e^{tx} f(x) \\
&= 2\sum_{x=1}^{\infty} e^{tx}\left(\frac{1}{3}\right)^x \\
&= 2\sum_{x=1}^{\infty}\left(\frac{1}{3}e^t\right)^x \\
&= 2\left[\frac{1}{3}e^t + \left(\frac{1}{3}e^t\right)^2 + \left(\frac{1}{3}e^t\right)^3 + \cdots\right] \\
&= 2\left(\frac{2}{3}\right)e^t\left[1 + \left(\frac{1}{3}\right)e^t + \left(\frac{1}{3}e^t\right)^2 + \cdots\right] \\
M_X(t) &= \left(\frac{2}{3}\right)\frac{e^t}{1 - 1/3e^t} = \frac{2e^t}{3 - e^t}
\end{aligned}
$$

Differentiate $M_X(t)$ with respect to t,

$$
\begin{aligned}
M'_X(t) &= 2\left[\frac{(3 - e^t)e^t - e^t(-e^t)}{(3 - e^t)^2}\right] \\
&= 2\left[\frac{3e^t}{(3 - e^t)^2}\right] \\
M'_X(t) &= 6\left[\frac{e^t}{(3 - e^t)^2}\right] \qquad (1)
\end{aligned}
$$

Put $t = 0$ in equation (1), we have

$$
\begin{aligned}
\left[M'_X(t)\right]_{t=0} &= E(X) \\
&= 6\left[\frac{1}{(3 - 1)^2}\right] = \frac{3}{2}
\end{aligned}
$$

Differentiate equation (1) with respect to t,

$$
M''_X(t) = 6\left[\frac{(3 - e^t)^2 e^t - 2(3 - e^t)(-e^t)e^t}{(3 - e^t)^4}\right]
$$

Putting $t = 0$, we have

$$
\begin{aligned}
\left[M''_X(t)\right]_{t=0} &= E(X^2) = 3 \\
\therefore \quad V(X) &= E(X^2) - E(X)^2 \\
&= 3 - \frac{9}{4} = \frac{3}{4}
\end{aligned}
$$

■

■ Example 3.29

If the pdf of a random variable X is

$$f(x) = \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find its MGF and hence its mean and variance.

$$\begin{aligned}
M_X(t) &= E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}\, f(x)\, dx \\
&= \int_{-1}^{1} \frac{1}{2}\left(e^{tx}\right)\, dx = \left(\frac{e^{tx}}{2t}\right)_{-1}^{1} \\
M_X(t) &= \frac{e^{t} - e^{-t}}{2t} \\
&= \frac{1}{2t}\left[\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) - \left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\right] \\
&= \frac{1}{2t}\left[\frac{2t}{1!} + \frac{2t^3}{3!} + \frac{2t^5}{5!} + \cdots\right] \\
&= 1 + \frac{1}{3!}t^2 + \frac{1}{5!}t^4 + \cdots \\
E(X) &= \text{coefficient of } \left(\frac{t}{1!}\right) \text{ in } M_X(t) = 0 \\
E(X^2) &= \text{coefficient of } \left(\frac{t^2}{2!}\right) \text{ in } M_X(t) = \frac{1}{3} \\
\therefore\quad \text{Mean} &= 0 \text{ and variance } = \frac{1}{3}
\end{aligned}$$

In this example differentiating $M_X(t)$ with respect to t and assigning $t = 0$ will not bring a finite value, so it is necessary to use the expansion principle to obtain mean and variance of X. ■

Properties of MGF

1. If c is a constant, $M_{cX}(t) = M_X(ct)$

   Proof: By definition,

$$\begin{aligned}
M_{cX}(t) &= E\left[e^{(cX)t}\right] \\
&= E\left[e^{X(ct)}\right] \\
&= M_X(ct)
\end{aligned}$$

2. Let X_1 and X_2 be two independent random variables. Then $M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$.
Proof: By definition,

$$\begin{aligned} M_{X_1+X_2}(t) &= E\left[e^{(X_1+X_2)t}\right] \\ &= E\left[e^{X_1t+X_2t}\right] \\ &= E\left[e^{X_1t} \cdot e^{X_2t}\right] \\ &= E\left[e^{X_1t}\right] \cdot E\left[e^{X_2(t)}\right] \\ &= M_{X_1}(t) \cdot M_{X_2}(t) \end{aligned}$$

Hence the proof.

This property states that the sum of two independent random variables is equal to the product of their respective moment generating functions. Also this idea can be generalised to n independent random variables

$$M_{X_1+\cdots+X_n}^{(t)} = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$

3. If $M_X(t)$ is the MGF of a random variable X then $M_U(t) = e^{-at/h} M_X\left(\frac{t}{h}\right)$ where $U = \frac{X-a}{h}$, a and h are constants.

$$\begin{aligned} M_U(t) &= E\left[e^{tU}\right] = E\left[e^{t\left(\frac{X-a}{h}\right)}\right] \\ &= E\left[e^{\frac{tX}{h}-\frac{ta}{h}}\right] = E\left[e^{\frac{tX}{h}} \cdot e^{\frac{-at}{h}}\right] \\ &= e^{-\frac{at}{h}} E\left[e^{\frac{tX}{h}}\right] = e^{-\frac{at}{h}} e\left[e^{\frac{t}{h}X}\right] \\ &= e^{-\frac{at}{h}} M_X\left(\frac{t}{h}\right) \end{aligned}$$

This property establishes the effect of change of origin and scale on MGF.

4. Let $K_X(t) = \log_e M_X(t)$, provided the right-hand side can be expanded as a convergent series in power of t. The coefficient of $\left(\frac{t^r}{r!}\right)$ in this series is called the r^{th} cumulant and it is denoted by K_r. This cumulants are helpful in finding central moments, infact without finding non-central moments.

$$\begin{aligned} K_X(t) &= \log\left[M_X(t)\right] \\ &= \log\left[1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots\right] \\ &= \left[\left(\frac{t}{1!}\right)E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots\right] - \frac{1}{2}\left[\frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \cdots\right]^2 + \\ &\quad \frac{1}{3}\left[\frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^3) + \frac{t^3}{3!}E(X^3) + \cdots\right]^3 - \cdots \\ &= \frac{t}{1!}E(X) + \frac{t^2}{2!}\left[E(X^2) - E(X)^2\right] + \frac{t^3}{3!}\left[E(X^3) - 3E(X)E(X^2) + \cdots + 2E(X^3)\right] \end{aligned}$$

That is

$$K_1 t + K_2\left(\frac{t^2}{2!}\right) + K_3\left(\frac{t^3}{3!}\right) + \cdots = \left(\frac{t}{1!}\right)E(X) + \left(\frac{t^2}{2!}\right)(\mu_2) + \left(\frac{t^3}{3!}\right)(\mu_3) + \cdots$$

Comparing the coefficients of $\frac{t^r}{r!}$, we have

$$K_1 = E(X) = \text{Mean}; \quad K_2 = V(X); \quad K_3 = \mu_3; \quad k_4 = \mu_4 - 3k_2^2 \cdots$$

Remark

If we differentiate $K_X(t)$ with respect to $t,' r'$times and then putting $t = 0$, we get the cumulant of r^{th} order.

That is

$$K_r = \left[\frac{d^r}{dt^r} K_X(t)\right]_{t=0}$$

Since if $r = 1$,

$$\begin{aligned}
K_1 &= \left[\frac{d}{dt} K_X(t)\right]_{t=0} \\
&= \left[\frac{d}{dt} \log M_X(t)\right]_{t=0} \\
&= \left[\frac{1}{M_X(t)} M'_X(t)\right]_{t=0} = \frac{\mu'_1}{1} \qquad \text{Since } M_X(0) = E(e^0) = 1 \\
\therefore \qquad K_1 &= \mu'_1 = E(X), \text{Mean of} X
\end{aligned}$$

Similarly if $r = 2$,

$$\begin{aligned}
K_2 &= \left[\frac{d^2}{dt^2} K_X(t)\right]_{t=0} \\
&= \left[\frac{M_X(t)M''_X(t) - \left(M'_X(t)\right)^2}{[M_X(t)]^2}\right]_{t=0} \\
&= \frac{\mu_2 - \left(\mu'_1\right)^2}{1} \\
&= V(X)
\end{aligned}$$

This idea can be extended to a general $'r'$. That is

$$K_r = \left[\frac{d^r}{dt^r} K_X(t)\right]_{t=0}$$

■ Example 3.30

Find the r^{th} cumulant of the random variable X if its pdf is

$$
\begin{aligned}
f(x) &= \begin{cases} Ce^{-Cx} & C > 0 \text{ and } x > 0 \\ 0 & \text{elsewhere} \end{cases} \\
M_X(t) &= E(e^{tx}) \\
&= \int_0^\infty Ce^{-ex} e^{tx}\, dx \\
&= C\int_0^\infty e^{-(C-t)x} dx \\
&= C\left[\frac{e^{-(c-t)x}}{-(c-t)}\right]_0^\infty \\
&= \frac{C}{C-t} \\
K_x(t) &= \log M_X(t) \\
&= \log\left(\frac{C}{C-t}\right) = \log C - \log(C-t) \\
K_x'(t) &= \frac{1}{C-t}; \quad K_x''(t) = \frac{1}{(C-t)^2}; \\
K_X'''(t) &= \frac{2}{(C-t)^3}, \quad K_X^{(iv)}(t) = \frac{3!}{(C-t)^4} \quad \text{(and so on)} \\
K^{(r)}(t) &= \frac{(r-1)!}{(C-t)^r} \\
\text{Now } K^{\text{th}} \text{ cumulant } K_r &= \left[\frac{d^r}{dt^r} k_X(t)\right]_{t=0} \\
&= \frac{(r-1)!}{C^r}
\end{aligned}
$$

■

There are so many distributions that do not have MGF. One such distribution is a random variable X with pdf

$$f(x) = \begin{cases} \frac{1}{x^2} & 1 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

For details, refer the Appendix.

Instead, we would define a function called the characteristic function of the distribution. That is, $\phi_X(t) = E(e^{itx})$ where i is the imaginary unit or $i^2 = -1$ and t is an arbitrary real number. Infact, every distribution has a unique characteristic function, and to each characteristic function there corresponds a unique distribution. Now we state some properties of $\phi_X(t)$ without proof.

(a) $\phi_{CX}(t) = \phi_X(Ct)$ C is a constant.

(b) If X_1 and X_2 are independent random variables then $\phi_{X_1+X_2}(t) = \phi_{X_1}(t).\phi_{X_2}(t)$

(c) $E(X^r) = (-i)^r \left[\frac{d^r}{dt^r}\phi_X(t)\right]_{t=0}$

■ Example 3.31

Find the characteristic function of the random variable X whose pdf is

$$\begin{aligned}
f(x) &= \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{elsewhere} \end{cases} \\
\phi_X(t) &= E(e^{itx}) \\
&= \int_{-\infty}^{\infty} e^{itx} f(x) dx \\
&= \int_{0}^{\infty} e^{itx} 2e^{-2x} dx \\
&= 2\int_{0}^{\infty} e^{-x(2-it)} dt \\
&= 2\left[\frac{e^{-x(2-it)}}{-(2-it)}\right]_0^{\infty} = \frac{2}{2-it}
\end{aligned}$$

Differentiate with respect to t,

$$\begin{aligned}
\phi'_X(t) &= \frac{2i}{(2-it)^2} \text{ and} \\
\phi''_X(t) &= \frac{4i^2}{(2-it)^3} \\
E(X) &= (-i)\left[\frac{2i}{(2-it)^2}\right]_{t=0} = \frac{1}{2} \\
E(X^2) &= (-i)^2\left[\frac{4i^2}{(2-it)^3}\right]_{t=0} \\
&= \frac{4}{8} = \frac{1}{2} \\
\text{Hence,} \quad V(X) &= E(X^2) - E(X)^2 \\
\Rightarrow \quad V(X) &= \frac{1}{4}
\end{aligned}$$

Let us now extend the idea of MGF in a bivariate case. Let $f(x, y)$ be the joint pdf of X and Y. Then the moment generating function of the joint-distribution of X and Y is defined as $M_{XY}(t_1, t_2) = E\left[e^{t_1X+t_2Y}\right]$ for $-h_1 < t_1 < h_2$, $-h_2 < t_2 < h_2$ where h_1 and h_2 are positive. We state following properties without proof.

(a) $M_{XY}(t_1, 0) = M_X(t_1)$; $M_{XY}(0, t_2) = M_Y(t_2)$

(b) $\left[\dfrac{\partial^{K+M}}{\partial t_1^K \partial t_2^M} M_{XY}(t_1, t_2)\right]_{t_1=0=t_2} = E\left[X^K y^M\right]$

(c) If X and Y are independent random variables, then $M_{XY}(t_1, t_2) = M_X(t_1, 0)\, M_Y(0, t_2)$
Hence from Property 3.2, we have

$$\begin{aligned}
\mu_1 &= E(X) = \left[\frac{\partial}{\partial t_1} M_{XY}(t_1 t_2)\right]_{(0,0)} \\
\mu_2 &= E(Y) = \left[\frac{\partial}{\partial t_2} M_{XY}(t_1 t_2)\right]_{(0,0)} \\
\sigma_1^2 &= \left[\frac{\partial^2 M_{XY}}{\partial t_1^2}\right]_{(0,0)} - \mu_1^2; \quad \sigma_2^2 = \left[\frac{\partial^2}{\partial t_2^2} M_{XY}\right]_{(0,0)} - \mu_2^2 \\
\text{and} \quad \operatorname{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= \left[\frac{\partial^2}{\partial t_1 \partial t_2}(M_{XY})\right]_{(0,0)} - \mu_1 \mu_2.
\end{aligned}$$

■

■ Example 3.32

Find the MGF of the following bivariate distribution and hence ρ_{XY}.

$X \backslash Y$	0	1
0	0.1	0.3
1	0.2	0.4

$$\begin{aligned}
M_{XY}(t_1, t_2) &= E\left(e^{t_1 X + t_2 Y}\right) = \sum_x \sum_y e^{(t_1 x + t_2 y)} f(x, y) \\
&= (0.1) + (0.3)e^{t_2} + (0.2)e^{t_1} + (0.4)e^{t_1+t_2} \\
M_X(t_1) &= M_X(t_1, 0) = 0.4 + (0.6)e^{t_1} \\
M_X(t_2) &= M_X(0, t_2) = 0.3 + (0.7)e^{t_2} \\
\mu_1 &= E(X) = \left[\frac{\partial}{\partial t_1} M_X(t_1)\right]_{t_1=0} = 0.6 \\
\mu_2 &= E(Y) = \left[\frac{\partial}{\partial t_2} M_X(t_2)\right]_{t_2=0} = 0.7 \\
\operatorname{Cov}(X, Y) &= \left[\frac{\partial^2}{\partial t_1 \partial t_2} M_X(t_1 t_2)\right]_{t_1=0, t_2=0} - \mu_1 \mu_2 = 0.4 - (0.6)(0.7) = -0.02 \\
\sigma_1^2 &= \frac{\partial^2}{\partial t_1^2} M_{XY}(0, 0) - \mu_1^2 = 0.6 - (0.6)^2 = 0.24 \\
\sigma_2^2 &= \frac{\partial^2}{\partial t_2^2} M_{XY}(0, 0) - \mu_2^2 = 0.7 - (0.7)^2 = 0.21 \\
\therefore \quad \rho_{XY} &= \frac{\operatorname{Cov}(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{-0.02}{\sqrt{0.24}\sqrt{0.21}} \\
\rho_{XY} &= -0.0891.
\end{aligned}$$

■

Let us conclude this section with one more expectation of a function of random variable, X. That is, if A is any arbitrary value then let us find the expectation of $(X - A)^r$ where r is any positive integer. Indeed, A could be a number in the middle of the distribution though the idea is true in general. Let us define the following

$$E\left[(X-A)^r\right] = \begin{cases} \sum\limits_x (x-A)^r f(x) & X \text{isdiscrete} \\ \int\limits_{-\infty}^{\infty} (x-A)^r f(x) & X \text{iscontinous} \end{cases}$$

By a suitable choice of A (as zero or mean (μ) of X) the above definition reduces to the non-central and central moments respectively. Also, from these expectation values one can evaluate the moments using the binomial expansion of positive index (r). Consider the following example.

■ Example 3.33

The first four moments about $x = 4$ are 1, 4, 10, and 45 respectively. Show that the mean is 5, variance is 3. Also show that the distribution is symmetric and platykurtic.

Given that the first four moment about $X = 4$ are 1, 4, 10, and 45.

$$\begin{array}{ll} E[X-4] = 1; & E\left[(X-4)^2\right] = 4 \\ E\left[(X-4)^3\right] = 10; & E\left[(X-4)^4\right] = 45 \end{array}$$

Using the properties of expectations, we have

$$\begin{aligned} E\,[X-4] &= 1 \\ E\,[X]-4 &= 1 \\ \therefore \quad E(X) &= 5 \qquad (1) \\ E\left[(X-4)^2\right] &= 4 \\ E\left[X^2-8X+16\right] &= 4 \\ E\left[X^2\right]-8E(X)+16 &= 4 \\ E\left(X^2\right)-40+16 &= 4 \quad [\text{use}(1)] \\ \therefore \quad E\left(X^2\right) &= 28 \qquad (2) \\ E\left[(X-4)^3\right] &= 10 \\ E\left[X^3-12X^2+48X-64\right] &= 10 \\ E\left(X^3\right)-12E\left(X^2\right)+48E(X)-64 &= 10 \end{aligned}$$

Using (1) and (2), we get

$$E\left(X^3\right) = 170 \tag{3}$$

$$E\left[(X-4)^4\right] = 45$$

$$E\left(X^4\right) - 16E\left(X^3\right) + 96E\left(X^2\right) - 256E(X) + 256 = 45$$

Using (1), (2) and (3), we get

$$E\left(X^4\right) = 1101 \tag{4}$$

$$\begin{aligned}
\text{Now, mean is } \mu &= E(X) \\
&= 5 \quad \text{[Refer (1)]} \\
\text{Variance is } \mu_2 &= E\left(X^2\right) - E(X)^2 \\
&= 3 \quad \text{[Refer (1) and (2)]} \\
\mu_3 &= E\left(X^3\right) - 3E\left(X^2\right) E(X) + 2E(X)^3 \\
&= 0 \quad \text{[Refer(1), (2)and(3)]} \\
\mu_4 &= E\left(X^4\right) - 4E\left(X^3\right) E\,(X) + 6E\left(X^2\right) E(X)^2 - 3E(X)^4 \\
&= 26 \quad \text{[Refer(1), (2), (3)and(4)]}
\end{aligned}$$

Hence skewness = 0 and kurtosis = 2.9 so that the distribution is symmetric and platykurtic. ■

<u>*Note:*</u> In the above example we obtain moments about mean the moments about origin However, a direct manipulation of moments about mean [using (1)] is possible. Recall that $E(X-\mu) = 0$.

$$\left[E(X-4)^2\right] = 4$$

$$E\left[(X-5+1)^2\right] = 4 \quad [\mu = 5,\ \text{refer (1)}]$$

$$E\left[(X-5)^2 + 1 + 2(X-5)\right] = 4$$

$$\therefore \quad E\left[(X-5)^2\right] + 1 + 2E(X-5) = 4$$

$$\therefore \quad E\left[(X-5)^2\right] = 3 \quad \text{or} \quad \sigma^2 = 3$$

Similarly the other moments (μ_3 and μ_4) can be calculated directly using the expansions of $(a+b)^3$ and $(a+b)^4$.

We present here a chart which summarizes the various aspects of a random variable (both univariable and bivariate) and their moments.

ADDITIONAL EXAMPLES

■ Example 3.34

An integer is chosen between 1 and 100. Find the expected value.

Let X be a random variable that represents any integer between 1 and 100.

$$\therefore \quad \boldsymbol{A} = \{x \epsilon z / 1 \le x \le 100\} = \{1, 2, 3, \cdots, 100\}$$

so that X is a discrete RV. Since, the choice is random all these numbers are equally likely. Hence, the pdf of X is

$$f(x) = \begin{cases} \frac{1}{100} & 1 \le x \le 100 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad E(X) = \sum_x x f(x)$$

$$= \sum_{x=1}^{100} x \cdot \frac{1}{100} = \frac{1}{100} \sum_{x=1}^{100} x = \frac{1}{100}[1 + 2 + 3 + \cdots + 100]$$

$$= \frac{1}{100} \left(\frac{100 \times 101}{2} \right) = 50.5. \qquad \blacksquare$$

■ Example 3.35

Find the mean and variance of the random variable X if its pdf is

$$f(x) = \begin{cases} \frac{1}{3}(1 + 4x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$\boldsymbol{A} = \{x / 0 < x < 1\} = (0, 1)$, X is a continuous RV.

$$\therefore \quad \mu = E(X) = \int_{-\infty}^{\infty} x\ f(x)\ dx = \int_0^1 x \frac{1}{3}(1 + 4x)\ dx$$

$$= \frac{1}{3} \left(\frac{x^2}{2} + \frac{4x^3}{3} \right)_0^1 = \frac{11}{18}$$

$$\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\ dx = \int_0^1 x^2 \frac{1}{3}(1 + 4x)\ dx$$

$$= \frac{1}{3} \left(\frac{x^3}{3} + \frac{4x^4}{4} \right)_0^1 = \frac{4}{9}$$

$$\therefore \quad V(X) = \mu_2' - \mu^2 = \frac{4}{9} - \left(\frac{11}{18} \right)^2 = \frac{23}{324} \qquad \blacksquare$$

■ Example 3.36

If the pdf of a random variable X is

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find the mean and variance of X.

$$\begin{aligned}
\boldsymbol{A} &= \{x/0 < x < 2\} = (0, 2) \\
\mu = E(X) &= \int_{-\infty}^{\infty} x\ f(x)\ dx = \int_0^1 x(x)\ dx + \int_1^2 x(2-x)\ dx = 1 \\
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\ dx = \int_0^1 x^2(x)\ dx + \int_1^2 x^2(2-x)\ dx = \frac{7}{6} \\
\therefore \qquad V(X) &= \mu_2' - \mu_2 = \frac{7}{6} - 1^2 = \frac{1}{6}
\end{aligned}$$

■

■ Example 3.37

Find the mean and variance of RV X whose pmf is

$$f(x) = \begin{cases} \frac{x+2}{25} & x = 1, 2, 3, 4, 5 \\ 0 & \text{elsewhere} \end{cases}$$

$\boldsymbol{A} = \{1, 2, 3, 4, 5\}$, X is a discrete RV.

$X = x$	1	2	3	4	5
$f(x)$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{5}{25}$	$\frac{6}{25}$	$\frac{7}{25}$

$$\begin{aligned}
\mu = E(X) &= \sum_{x \epsilon \boldsymbol{A}} x\ f(x) = 1\left(\frac{3}{25}\right) + 2\left(\frac{4}{25}\right) + 3\left(\frac{5}{25}\right) + 4\left(\frac{6}{25}\right) + 5\left(\frac{7}{25}\right) \\
&= \frac{17}{5} \\
\mu_2' = E(X^2) &= \Sigma x^2 f(x) = 1^2\left(\frac{3}{25}\right) + 2^2\left(\frac{4}{25}\right) + 3^2\left(\frac{5}{25}\right) + 4^2\left(\frac{6}{25}\right) + 5^2\left(\frac{7}{25}\right) \\
&= \frac{67}{5} \\
V(X) = \mu_2' - \mu^2 &= \left(\frac{67}{5}\right) - \left(\frac{17}{5}\right)^2 = \frac{46}{25}
\end{aligned}$$

■

■ Example 3.38

A coin is tossed until a tail appears. What is the expectation and variance of the number of tosses?

Let X be the number of tosses. Hence, its probability distribution is

$X=x$	1	2	3	4	5	$\cdots$
$f(x)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$	$\cdots$

$$
\begin{aligned}
\mu = E(X) &= \sum_{x=1}^{\infty} x\, f(x) \\
&= 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \cdots \\
&= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots\right] \\
&= \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2} = \frac{1}{2}(4) = 2 \\
\mu_2' = E(X^2) &= \sum_{x=1}^{\infty} x^2 f(x) \\
&= 1^2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{2}\right)^2 + 3^2\left(\frac{1}{2}\right)^3 + 4^2\left(\frac{1}{2}\right)^4 + \cdots \\
&= \frac{1}{2}\left[1^2 + 2^2\left(\frac{1}{2}\right) + 3^2\left(\frac{1}{2}\right)^2 + 4^2\left(\frac{1}{2}\right)^3 + \cdots\right] \\
&= \frac{1}{2}\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)^{-3} = 6 \\
\therefore \qquad V(X) = \mu_2' - \mu^2 &= 6 - 4 = 2.
\end{aligned}
$$

Note: Compare this method with exercise 11(c). ■

■ Example 3.39

Let X be a non-negative continuous random variable with pdf $f(x)$ and distribution function $F(x)$. Show that

$$
\begin{aligned}
E(X) &= \int_0^{\infty} [1 - F(x)]\, dx \\
E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx
\end{aligned}
$$

Since X is non-negative variable,

$$\begin{aligned}
E(X) &= \int_0^\infty x f(x)\, dx = \int_0^\infty x\, dF(x) \\
&= [xF(x)]_0^\infty - \int_0^\infty F(x)\, dx \\
&= \int_0^\infty dx - \int_0^\infty F(x)\, dx \qquad (\because F(\infty) = 1 \text{ and } F(0) = 0) \\
&= \int_0^\infty [1 - F(x)]\, dx \qquad \text{(or)} \qquad \int_0^\infty p(X > x)\, dx
\end{aligned}$$

Remark

This form of $E(X)$ in the above example is analogue of Property 3.6.

■ Example 3.40

Find the expected value of $Y = (X - 1)^2$ if the pdf of X is

$$\text{(a)} \quad f(x) = \begin{cases} Kx^2 & x = 0, \pm 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{(b)} \quad f(x) = \begin{cases} Kx^2 & -1 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) $\boldsymbol{A} = \{-1, 0, 1\}$ and X is a discrete RV.

$$\therefore \quad \Sigma f(x) = 1 \quad \Rightarrow \quad \sum_{x=-1,0,1} Kx^2 = 1 \quad \Rightarrow \quad K = \frac{1}{2}$$

$\therefore$ pdf of X is

$X = x$	-1	0	1
$f(x)$	$\frac{1}{2}$	0	$\frac{1}{2}$

$$
\begin{aligned}
\mu &= E(X) = \Sigma x \ f(x) = 0 \\
\mu_2' &= E(X^2) = \Sigma x^2 \ f(x) = 1 \\
\therefore \quad V(X) &= \mu_2' - \mu^2 = 1 \\
\text{Now} \quad Y = (X-1)^2 &= X^2 - 2X + 1 \\
\therefore \quad E(Y) &= E(X^2 - 2X + 1) = E(X^2) - 2E(X) + 1 \qquad (1) \\
\therefore \quad E(Y) &= 1 - 2(0) + 1 = 2.
\end{aligned}
$$

(b) $\boldsymbol{A} = \{x / -1 \leq x \leq 1\}$ so that X is a continuous Rv

$$
\begin{aligned}
\therefore \quad \int_{-\infty}^{\infty} f(x)d(x) &= 1 \quad \Rightarrow \quad \int_{-1}^{1} Kx^2 dx = 1 \quad \Rightarrow \quad K = \frac{3}{2} \\
\therefore \quad f(x) &= \begin{cases} \left(\frac{3}{2}\right) x^2 & -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases} \qquad (\because \text{ Integrand is odd}) \\
\mu = E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{1} x\left(\frac{3}{2}\right) x^2 dx = 0 \\
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-1}^{1} x^2 \left(\frac{3}{2}\right) x^2 dx = 0 \\
&= \left(\frac{3}{2}\right)\left(\frac{2}{5}\right) = \frac{3}{5} \\
\therefore \quad \text{equation (1) implies} \quad E(Y) &= \frac{3}{5} - 2(0) + 1 = \frac{8}{5}
\end{aligned}
$$

■

■ Example 3.41

If X and Y have a joint pdf

$$
f(x, y) = \begin{cases} C|x+y| & x = -2, -1, 0, 1, 2 \\ & y = -2, -1, 0, 1, 2 \\ 0 & \text{elsewhere} \end{cases}
$$

Find (a) $E(XY)$, (b) $E(X+Y)$

$$
\boldsymbol{A} = \{(x, y)/x = -2, -1, 0, 1, 2; \quad y = -2, -1, 0, 1, 2\}
$$

Hence (X, Y) is a discrete RV.

$$
\begin{aligned}
\therefore \quad \sum_x \sum_y f(x, y) &= 1 \\
\Rightarrow \quad \sum_x \sum_y C|x+y| &= 1
\end{aligned}
$$

$$\Rightarrow \qquad C\sum_x\sum_y |x+y| = 1$$

$$\Rightarrow \qquad C = \frac{1}{40}$$

$\therefore$ Joint pdf is

$X \diagdown Y$	-2	-1	0	1	2
-2	$\frac{1}{10}$	$\frac{3}{40}$	$\frac{1}{20}$	$\frac{1}{40}$	0
-1	$\frac{3}{40}$	$\frac{1}{20}$	$\frac{1}{40}$	0	$\frac{1}{40}$
0	$\frac{1}{20}$	$\frac{1}{40}$	0	$\frac{1}{40}$	$\frac{1}{20}$
1	$\frac{1}{40}$	0	$\frac{1}{40}$	$\frac{1}{20}$	$\frac{3}{40}$
2	0	$\frac{1}{40}$	$\frac{1}{20}$	$\frac{3}{40}$	$\frac{1}{10}$

Now (a)

$$\begin{aligned} E(XY) &= \sum_x\sum_y xyf(x,y) \\ &= (-2)(-2)\frac{1}{10} + (-2)(-1)\frac{3}{40} + 0 + (-2)(1)\frac{1}{40} + 0 + \\ &\quad (-1)(-2)\frac{3}{40} + (-1)(-1)\frac{1}{20} + 0 + 0 + (-1)(2)\frac{1}{40} + \\ &\quad 0 + (1)(-2)\frac{1}{40} + 0 + 0 + (1)(1)\frac{1}{20} + (1)(2)\frac{3}{40} + \\ &\quad 0 + (2)(-1)\frac{1}{40} + 0 + (2)(1)\frac{3}{40} + (2)(2)\frac{1}{10} = \frac{13}{10} \end{aligned}$$

(b)

$$\begin{aligned} E(X+Y) &= \sum_x\sum_y (X+Y)f(x,y) \\ &= (-4)\frac{1}{10} + (-3)\frac{3}{40} + (-2)\frac{1}{20} + (-1)\frac{1}{40} + 0 + \\ &\quad (-3)\frac{3}{40} + (-2)\frac{1}{20} + (-1)\frac{1}{40} + 0 + (1)\frac{1}{40} + \\ &\quad (-2)\frac{1}{20} + (-1)\frac{1}{40} + 0 + (1)\frac{1}{40} + (2)\frac{1}{20} + \\ &\quad (-1)\frac{1}{40} + 0 + (1)\frac{1}{40} + (2)\frac{1}{20} + (3)\frac{3}{40} + \\ &\quad 0 + (1)\frac{1}{40} + (2)\frac{1}{20} + (3)\frac{3}{40} + (4)\frac{1}{10} = 0. \end{aligned}$$

■

■ Example 3.42

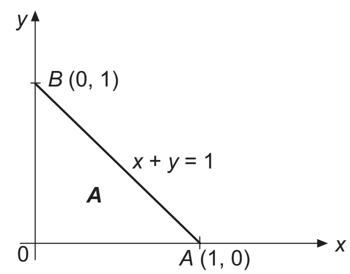
Repeat Example 3.41 for the distribution.

$$f(x, y) = \begin{cases} Kxy & x \geq 0, y \geq 0; x + y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\boldsymbol{A} = \{(x, y)/x \geq 0, y \geq 0; \quad x + y \leq 1\}$$

To find K:

$$\iint\limits_{\boldsymbol{A}} f(x, y)\, dxdy = 1$$



$$\Rightarrow \quad \int_{y=0}^{1} \int_{x=0}^{1-y} Kxy\, dxdy = 1$$

$$\Rightarrow \quad K \int_{y=0}^{1} \left(\frac{x^2}{2}\right)_0^{1-y} y\, dy = 1$$

$$\Rightarrow \quad K\frac{1}{2} \int_{y=0}^{1} (1-y)^2 y\, dy = 1$$

$$\Rightarrow \quad \frac{K}{2} \int_{y=0}^{1} y^{2-1}(1-y)^{3-1} dy = 1$$

$$\Rightarrow \frac{K}{2}[\beta(2, 3)] = 1 \quad \Rightarrow \quad K = 2\left[\frac{1}{\beta(2, 3)}\right] = 24$$

$\therefore$ The joint pdf is

$$f(x, y) = \begin{cases} 24xy & x > 0, y \geq 0; x + y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) $$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y)\, dxdy$$

$$= \iint\limits_{\boldsymbol{A}} xy(24xy)\, dxdy = 24 \int_{y=0}^{1} \int_{x=0}^{1-y} x^2y^2\, dxdy$$

$$= 8 \int_{y=0}^{1} y^2(1-y)^3 dy = 8\beta(3, 4) = 8\left(\frac{1}{120}\right) = \frac{1}{15}$$

(b)
$$
\begin{aligned}
E(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X+Y)f(x,y)\,dxdy \\
&= \iint_{A}(x+y)24xy\,dxdy \\
&= 24\left[\int_0^1\int_0^{1-y} x^2y\,dxdy + \int_0^1\int_0^{1-y} xy^2dxdy\right] \\
&= 24\left[\frac{1}{3}\int_0^1 (1-y)^3y\,dy + \frac{1}{2}\int_0^1 (1-y)^2y^2dy\right] \\
&= 8\beta(2,4) + 12\beta(3,3) = \frac{4}{5}
\end{aligned}
$$

Remark

In the Examples 3.41 and 3.42 we can compute $E(X+Y)$, using the property $E(X+Y) = E(X) + E(Y)$. But this requires the mpdf of X and Y, which we avoided in our method. But in Part (a) of both examples, we cannot use the property of expectation. (Why?)

■ Example 3.43

Two independent random variables X and Y have their pdfs

$$
f_X(x) = \begin{cases} 4ax & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 4ay & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}
$$

show that $U = X + Y$ and $V = X - Y$ are uncorrelated.

Since X and Y are inddependent, $\text{Cov}(X, Y) = 0$.

Hence, by Property 3.12 and letting $a_1 = 1, b_1 = 1, a_2 = 1$ and $b_2 = -1$ we have,

$$
\text{Cov}(U, V) = V(X) - V(Y) \tag{1}
$$

Let us find a and b, since $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, we have $\int_0^1 4ax\,dx = 1 \Rightarrow 4a\left(\frac{x^2}{2}\right)_0^1 \Rightarrow a = \frac{1}{2}$ and similarly $b = \frac{1}{2}$. Hence $f_X(x) = 2x,\ 0 \le x \le 1$ zero elsewhere and $f_Y(y) = 2y,\ 0 \le y \le 1$ zero elsewhere. Hence

$$
E(X) = \int_0^1 x(2x)dx = \frac{2}{3} = E(Y) \quad \text{(Why)}
$$

$$E(X^2) = \int_0^1 x^2(2x)\,dx = \frac{1}{2} = E(y^2)$$

$$\therefore \quad V(X) = E(X)^2 - E(X)^2 = \frac{1}{18} = V(Y)$$

$$\therefore \quad V(X) - V(Y) = 0$$

equation (1) $\Rightarrow$ U and V are uncorrelated. ■

■ Example 3.44

If the joint pdf of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 < x < 1;\, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the variance of $Z = 3X + 4Y - 5$.

Using Property 3.11, $$V(Z) = 3^2 V(X) + 4^2 V(Y) + 2(3)(4)\,\mathrm{Cov}(X,Y)$$
$$= 9V(X) + 16V(Y) + 24\,\mathrm{Cov}(X,Y) \qquad (1)$$

First let us find Mpdf of X and Mpdf of Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \frac{1}{3}(x+y)\,dy$$

$$f_X(x) = \begin{cases} \frac{2}{3}(x+1) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{1}{3}(x+y)\,dx$$

$$f_Y(y) = \begin{cases} \frac{1}{6}(1+2y) & 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x\ f_X(x)\,dx = \frac{2}{3}\int_0^1 x(x+1)\,dx = \frac{5}{9}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2\frac{2}{3}(x+1)\,dx = \frac{7}{12}$$

$$V(X) = E(X^2) - E(X)^2 = \left(\frac{7}{12}\right) - \left(\frac{5}{9}\right)^2 = \frac{89}{324}$$

$$E(Y) = \int_{-\infty}^{\infty} Y f_y \, dy = \frac{1}{6}\int_0^2 y(1+2y)\, dy = \frac{11}{9}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \frac{1}{6}\int y^2(1+2y)\, dy = \frac{16}{9}$$

$$V(Y) = E(Y^2) - E(Y)^2 = \left(\frac{16}{9}\right) - \left(\frac{11}{9}\right)^2 = \frac{23}{81}$$

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f(x,y)\, dxdy \\ &= \frac{1}{3}\int_0^2\int_0^1 xy\ (x+y)\, dxdy = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{Cov}(X,Y) &= E(XY) - E(X)\,E(Y) \\ &= \left(\frac{2}{3}\right) - \left(\frac{5}{9}\right)\left(\frac{11}{9}\right) = -\frac{1}{81} \end{aligned}$$

equation (1) $\Rightarrow$

$$\begin{aligned} V(Z) &= 9\left(\frac{89}{324}\right) + 16\left(\frac{23}{81}\right) + 24\left(\frac{-1}{81}\right) \\ &= 6.72. \end{aligned}$$

■

■ Example 3.45

Find the MGF of a R.V X if its pdf is $f(x) = \frac{1}{2a}\, e^{-\frac{|x-a|}{a}}$; $a > 0 - \infty < x < \infty$ and hence its mean and variance.

$$\begin{aligned} M_X(t) &= E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}\ f(x)\, dx \\ &= \frac{1}{2a}\int_{-\infty}^{\infty} e^{tx}\ e^{-\frac{|x-a|}{a}}\, dx \\ &= \frac{1}{2a}\left[\int_{-\infty}^{a} e^{tx}\ e^{-\frac{1}{a}[-(x-a)]}dx + \int_a^{\infty} e^{tx}\ e^{-\frac{1}{a}(x-a)}\, dx\right] \\ &= \frac{1}{2a}\left[\int_{-\infty}^{a} e^{\frac{x-a}{a}+tx}dx + \int_a^{\infty} e^{-\left(\frac{x-a}{a}-tx\right)}dx\right] \\ &= \frac{1}{2a}\left[e^{-1}\int_{-\infty}^{a} e^{x\left(t+\frac{1}{a}\right)}dx + e\int_a^{\infty} e^{-x\left(\frac{1}{a}-t\right)}dx\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2a}\left[e^{-1}\left(\frac{e^{x\left(\frac{1}{a}+t\right)}}{\frac{1}{a}+t}\right)_{-\infty}^{a} + e\left(-\frac{e^{-x\left(\frac{1}{a}-t\right)}}{\frac{1}{a}-t}\right)_{a}^{\infty}\right] \\
&= \frac{1}{2a}\left[e^{-1}a\,\frac{e^{a\left(\frac{1}{a}+t\right)}}{1+at} + e\,a\frac{e^{-a\left(\frac{1}{a}-t\right)}}{1-at}\right] \\
&= \frac{1}{2a}e^{at}\left[\frac{a}{1+at}+\frac{a}{1-at}\right] = \frac{e^{at}}{2a}a\left[\frac{1-at+1+at}{1-a^2t^2}\right] \\
M_X(t) &= \frac{e^{at}}{2}\,\frac{2}{1-a^2t^2} = \frac{e^{at}}{1-a^2t^2}
\end{aligned}
$$

To find mean and variance of X from MGF, we use cumulates of X.

$$
\begin{aligned}
\text{i.e.,} \qquad K_X(t) &= \log_e M_X(t) = \log_e\left(\frac{e^{at}}{1-a^2t^2}\right) \\
&= \log\, e^{at} - \log(1-a^2t^2) \\
&= at - \log(1-a^2t^2)
\end{aligned}
$$

Differentiating twice with respect to 't' we have

$$
\begin{aligned}
K_X'(t) &= a + \frac{2a^2t}{1-a^2t^2} \\
K_X''(t) &= 2a^2\frac{1+a^2t^2}{(1-a^2t^2)^2} \\
\text{Hence,} \qquad \text{mean} &= \mu = \left[K_X'(t)\right]_{t=0} = a \\
\text{Variance} &= \left[K_X''(t)\right]_{t=0} = 2a^2
\end{aligned}
$$

■

■ Example 3.46

Find the MGF of X and hence its mean and variance if its pdf of X is

$$
f(x) = \begin{cases} \left(\frac{2}{3}\right)x^3e^{-2x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}
$$

$$
\begin{aligned}
M_X(t) &= \int_0^\infty \left(\frac{2}{3}\right)x^3\, e^{-2x}\, e^{tx}\, dx \\
&= \frac{2}{3}\int_0^\infty x^3\, e^{-x(2-t)}\, dx \qquad \text{put} \quad (2-t)x = z\; dx = \frac{dz}{2-t} \\
&= \frac{2}{3}\frac{1}{(2-t)^4}\int_0^\infty z^3\, e^{-z}\, dz \qquad t < 2 \\
&= \frac{2}{3(2-t)^4}\,\Gamma 4 = \frac{16}{(2-t)^4} \quad \text{(using Gamma Integral)}
\end{aligned}
$$

Differentiating $M_X(t)$ with respect to $'t'$ twice, we get

$$M'_X(t) = \frac{64}{(2-t)^5} \quad \text{and} \quad M''_X(t) = \frac{320}{(2-t)^6}$$

$$\begin{aligned} \therefore \qquad \text{Mean} = \mu &= \left[M'_X(t)\right]_{t=0} = 2 \\ \mu'_2 &= \left[M''_X(t)\right]_{t=0} = 5 \\ \therefore \qquad V(X) &= 1 \end{aligned}$$

■

■ Example 3.47

Find μ'_r of the distribution that has the MGF $M_X(t) = (1-t)^{-3}\ t < 1$.

We can differentiate 'r' times with respect to t and get $M_X(t)$ (putting $t = 0$). But we do the following alternative method in this Example.

Let us try to expand $M_X(t) = (1-t)^{-3}$, we have

$$\begin{aligned}
(1-t)^{-1} &= \sum_{n=0}^{\infty} t^n \quad \text{Differentiating this series, we get} \\
(1-t)^{-2} &= \sum_{n=1}^{\infty} n\ t^{n-1} \\
(1-t)^{-3} &= \sum_{n=2}^{\infty} \frac{n(n-1)}{2}\ t^{n-2} \\
\therefore \qquad M_X(t) &= \sum_{n=2}^{\infty} \frac{n(n-1)}{2}\ t^{n-2} \\
&= 1 + 3t + 6t^2 + \cdots + \frac{(r+2)(r+1)}{2}\left(t^r\right) + \cdots \\
&= 1 + \frac{3t}{1!} + \frac{12t^2}{2!} + \cdots + \frac{r!(r+2)(r+1)}{2}\left(\frac{t^r}{r!}\right) + \cdots \\
&= 1 + \frac{3t}{1!} + \frac{12t^2}{2!} + \cdots + \frac{(r+2)!}{2}\left(\frac{t^r}{r!}\right) + \cdots \\
\therefore \qquad \mu'_r &= \text{coefficient of } \left(\frac{t^r}{r!}\right) \text{ in } M_X(t) \\
&= \frac{(r+2)!}{2}
\end{aligned}$$

■

■ Example 3.48

Find the joint MGF of (X, Y) and hence determine $E(XY)$; $E(X)$; $E(Y)$, and Cov (X, Y). If the joint pdf is

$$f(x, y) = \begin{cases} 6e^{-(2x+3y)} & x > 0,\ y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

We can easily prove that X and Y are independent random variables, since $f(x, y) = 6e^{-(2x+3y)}$ $= 2e^{-2x} \cdot 3e^{-3y}$

$$\begin{aligned}
M_{XY}(t_1 t_2) &= E\left(e^{t_1 x + t_2 y}\right) = E(e^{t_1 x}) \cdot E(e^{t_2 y}) \\
&= M_X(t_1) \cdot M_Y(t_2) \\
M_X(t) &= \int_0^\infty 2e^{-2x}\, e^{t_1 x}\, dx \\
&= 2\int_0^\infty e^{-x(2-t_1)}\, dx = \frac{2}{(2-t_1)}
\end{aligned}$$

Similarly,
$$M_Y(t_2) = \frac{3}{(3-t_2)}$$

$$\begin{aligned}
\therefore \quad M_{XY}(t_1\, t_2) &= \frac{6}{(2-t_1)(3-t_2)} \\
\frac{\partial}{\partial t_1} M_{XY}(t_1 t_2) &= \frac{6}{(3-t_2)} \frac{1}{(2-t_1)^2} \\
\therefore \quad E(X) &= \left[\frac{\partial}{\partial t_1} M(t_1 t_2)\right]_{(0,0)} = \frac{6}{12} = \frac{1}{2} \\
\frac{\partial}{\partial t_2} M_{XY}(t_1 t_2) &= \frac{6}{2-t_1} \cdot \frac{1}{(3-t_2)^2}
\end{aligned}$$

At (0,0), we have

$$\begin{aligned}
E(Y) &= \frac{6}{18} = \frac{1}{3} \\
E(XY) &= \left[\frac{\partial^2}{\partial t_1 \partial t_2} M_{XY}(t_1 t_2)\right]_{(0,0)} \\
&= \left[\frac{\partial}{\partial t_1} \frac{6}{(2-t_1)(3-t_2)^2}\right]_{(0,0)} \\
&= \left[\frac{6}{(2-t_1)^2(3-t_2)^2}\right]_{(0,0)} \\
&= \frac{1}{6}
\end{aligned}$$

Hence,
$$\text{Cov}(X, Y) = \frac{1}{6} - \frac{1}{3} \cdot \frac{1}{2} = 0.$$

Remark

In the above example, X and Y are independent random variable and hence $\text{Cov}(X, Y) = 0$. But the feature of $M_{XY}(t_1 t_2)$ is brought out by this example. However, we indirectly use the fact of independence in this example.

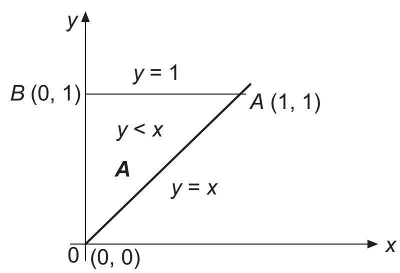
■ Example 3.49

Let
$$f(x, y) = \begin{cases} 2 & 0 < x < y,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

be the joint pdf of X and Y. Find the conditional means and conditional variances.

$$\boldsymbol{A} = \{(x, y)/0 < x < y < 1\}$$

Mpdf:



$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{x}^{1} 2\, dy \\ &= \begin{cases} 2(1 - x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \\ f_Y(y) &= \int_{-\infty}^{\infty} f(x, y)\, dx = \int_{0}^{Y} 2\, dx = 2(x)_0^Y \\ f_Y(y) &= \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

Cpdf:

$$\begin{aligned} f_{X/Y}(x/y) &= \frac{f(x, y)}{f_Y(y)} \\ &= \begin{cases} \frac{2}{2Y} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases} \\ f_{X/Y}(x/y) &= \begin{cases} \frac{1}{y} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases} \\ f_{Y/X}(y/x) &= \frac{f(x, y)}{f_X(x)} \\ &= \begin{cases} \frac{2}{2(1-x)} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases} \\ &= \begin{cases} \frac{1}{1-x} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

Conditional means

$$\begin{aligned}
E[X/Y=y] &= \int_{-\infty}^{\infty} x\ f_{X/Y}(x/y)\,dx \\
&= \int_{0}^{y} x\left(\frac{1}{y}\right)dx = \frac{1}{y}\left(\frac{x^2}{2}\right)_0^y \\
&= \frac{y}{2},\ 0<y<1 \\
E[Y/X=x] &= \int_{-\infty}^{\infty} y\cdot f_{Y/X}(y/x)\,dy = \int_{x}^{1} y\left(\frac{1}{1-x}\right)dy \\
&= \left(\frac{1}{1-x}\right)\left(\frac{y^2}{2}\right)_x^1 = \frac{1+x}{2}\ 0<x<1.
\end{aligned}$$

Conditional variances

$$\begin{aligned}
E[X^2/Y=y] &= \int_{-\infty}^{\infty} x^2 f_{X/Y}(x/y)\,dx = \int_{0}^{y} x^2\left(\frac{1}{y}\right)dx \\
&= \left(\frac{1}{y}\right)\left(\frac{y^3}{3}\right) = \frac{y^2}{3},\ 0<y<1 \\
E[Y^2/X=x] &= \int_{-\infty}^{\infty} y^2 f_{Y/X}(y/x)\,dy = \int_{x}^{1} y^2\left(\frac{1}{1-x}\right)dy \\
&= \left(\frac{1}{1-x}\right)\frac{(1-x^3)}{3} = \frac{1+x^2+x}{3}
\end{aligned}$$

Now

$$\begin{aligned}
V[X/Y=y] &= E[X^2/Y=y]-E[X/Y=y]^2 \\
&= \frac{y^2}{3}-\frac{y^2}{4} = \frac{y^2}{12},\ 0<y<1 \\
V[Y/X=x] &= E[Y^2/X=x]-E[Y/X=x]^2 \\
&= \frac{1+x^2+x}{3}-\left(\frac{1+x}{2}\right)^2 = \frac{(1-x)^2}{12},\ 0<x<1.
\end{aligned}$$

■

■ Example 3.50

Repeat Example 3.49 for a bivariate distribution whose joint pmf is

$$f(x,y) = \begin{cases} \frac{x+2y}{18} & x=1,2;\ y=1,2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\boldsymbol{A} = \{(x,y)/\ x=1,2;\ y=1,2\ \}$$

Jpdf is

$X \diagdown Y$	1	2
1	$\frac{1}{6}$	$\frac{5}{18}$
2	$\frac{2}{9}$	$\frac{1}{3}$

Mpdfs are

$X = x$	1	2
$f_X(x)$	$\frac{4}{9}$	$\frac{5}{9}$

$Y = y$	1	2
$f_Y(y)$	$\frac{7}{18}$	$\frac{11}{18}$

Cpdfs:

$$f_{X/Y}(x/y) = \frac{f(x, y)}{f_Y(y)}$$

Let $Y = 1$

$X = x$	1	2
$f_{X/Y}(x/Y = 1)$	$\frac{3}{7}$	$\frac{4}{7}$

Let $Y = 2$

$X = x$	1	2
$f_{X/Y}(x/Y = 2)$	$\frac{5}{11}$	$\frac{6}{11}$

$$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)}$$

Let $X = 1$

$Y = y$	1	2
$f_{Y/X}(y/X = 1)$	$\frac{3}{8}$	$\frac{5}{8}$

Let $X = 2$

$Y = y$	1	2
$f_{Y/X}(y/X = 2)$	$\frac{2}{5}$	$\frac{3}{5}$

Conditional means

$$E(X/Y = y) = \sum_x x\ f_{X/Y}(x/y)$$

If $y = 1$, $$E[X/Y = 1] = 1\left(\frac{3}{7}\right) + 2\left(\frac{4}{7}\right) = \frac{11}{7}$$

If $Y = 2$, $$E[X/Y = 2] = 1\left(\frac{5}{11}\right) + 2\left(\frac{6}{11}\right) = \frac{17}{11}$$

$$E[Y/X = x] = \sum_y y\ f_{Y/X}(y/x)$$

$$\text{If } X = 1, \qquad E[Y/X = 1] = 1\left(\frac{3}{8}\right) + 2\left(\frac{5}{8}\right) = \frac{13}{8}$$

$$\text{If } X = 2, \qquad E[Y/X = 2] = 1\left(\frac{2}{5}\right) + 2\left(\frac{3}{5}\right) = \frac{8}{5}$$

Conditional variances

$$\begin{aligned}
& E[X^2/Y = y] &=& \Sigma x^2 f_{X/Y}(x/y) \\
\text{If } Y = 1, \quad & E[X^2/Y = 1] &=& 1^2\left(\frac{3}{7}\right) + 2^2\left(\frac{4}{7}\right) = \frac{19}{7} \\
\text{If } Y = 2, \quad & E[X^2/Y = 2] &=& 1^2\left(\frac{5}{11}\right) + 2^2\left(\frac{6}{11}\right) = \frac{29}{11} \\
& V[X/Y = y] &=& E[X^2/Y = y] - E[X/Y = y]^2 \\
\text{If } Y = 1, \quad & V(X/Y = 1) &=& \frac{19}{7} - \left(\frac{11}{7}\right)^2 = \frac{12}{49} \\
\text{If } y = 2, \quad & V(X/Y = 2) &=& \frac{29}{11} - \left(\frac{17}{11}\right)^2 = \frac{30}{121} \\
& E[Y^2/X = x] &=& \Sigma y^2 f_{Y/X}(y/x) \\
\text{If } X = 1, \quad & E[Y^2/X = 1] &=& 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{5}{8}\right) = \frac{23}{8} \\
\text{If } X = 2, \quad & E[Y^2/X = 2] &=& 1^2\left(\frac{2}{5}\right) + 2^2\left(\frac{3}{5}\right) = \frac{14}{5} \\
& V[Y/X = x] &=& E[Y^2/X = x] - E[Y/X = x]^2 \\
\text{If } X = 1, \quad & V(Y/X = 1) &=& \frac{23}{8} - \left(\frac{13}{8}\right)^2 = \frac{15}{64} \\
\text{If } X = 2, \quad & V(Y/X = 2) &=& \frac{14}{5} - \left(\frac{8}{5}\right)^2 = \frac{6}{25}
\end{aligned}$$

■

■ Example 3.51

Let the joint density of X and Y be

$$f(x, y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the regression curves of (a) Y on X, (b) X on Y.

$$f_X(x) = \int_{-\infty}^{\infty} 8xy\,dy = \int_{x}^{1} 8xy\,dy$$

$$= \begin{cases} 4x(1 - x^2) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} 8xy\, dx \quad = \quad \int_{0}^{y} 8xy\, dx$$

$$= \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

cpdf:

$$f_{Y/X}(y/x) \quad = \quad \frac{f(x, y)}{f_X(x)}$$

$$= \begin{cases} \frac{8xy}{4x(1-x^2)} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{2y}{1-x^2} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_{X/Y}(x/y) \quad = \quad \frac{f(x, y)}{f_Y(y)}$$

$$= \begin{cases} \frac{2x}{y^2} & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Regression curves:
(a) X on Y

$$E(X/Y = y) \quad = \quad \int_{-\infty}^{\infty} x f_{X/Y}(x/y) dx$$

$$= \int_{0}^{y} x \left(\frac{2x}{y^2} \right) dx$$

$$= \left(\frac{2}{y^2} \right) \left(\frac{x^3}{3} \right)_0^y$$

$$= \left(\frac{2y}{3} \right) \quad 0 < y < 1.$$

(b) Y on X

$$E(Y/X = x) \quad = \quad \int_{-\infty}^{\infty} y f_{Y/X}(y/x)\, dy$$

$$= \int_{x}^{1} y \left(\frac{2y}{1 - x^2} \right) dy$$

$$
\begin{aligned}
&= \left(\frac{2}{1-x^2}\right)\left(\frac{y^3}{3}\right)_x^1 \\
&= \frac{2}{3(1-x^2)}(1-x^3) = \frac{2(1-x)(1+x^2+x)}{3(1-x)(1+x)} \\
&= \frac{2}{3}\left(\frac{1+x+x^2}{1+x}\right), \quad 0 < x < 1.
\end{aligned}
$$
■

■ Example 3.52

Let the joint pdf of (X, Y) be $f(x, y) = 3x^2 - 8xy + 6y^2;\ 0 \le (x, y) \le 1$. Find the regression lines and regression curves for the means.

$$\boldsymbol{A} = \left\{(x, y)\big/0 \le x \le 1,\ 0 \le y \le 1\right\}$$

Mpdfs:

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^1 \left(3x^2 - 8xy + 6y^2\right)\, dy \\
\Rightarrow \quad f_X(x) &= \begin{cases} 3x^2 - 4x + 2 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases} \\
f_Y(y) &= \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^1 \left(3x^2 - 8xy + 6y^2\right)\, dx \\
f_Y(y) &= \begin{cases} 6y^2 - 4y + 1 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}
\end{aligned}
$$

cpdfs:

$$
\begin{aligned}
f_{X/Y}(x/y) &= \frac{f(x, y)}{f_Y(y)} = \begin{cases} \frac{3x^2-8xy+6y^2}{6y^2-4y+1} & 0 \le x \le 1;\ 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases} \\
f_{Y/x}(y/x) &= \frac{f(x, y)}{f_X(x)} = \begin{cases} \frac{3x^2-8xy+6y^2}{3x^2-4x+2} & 0 \le x \le 1; \quad 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}
\end{aligned}
$$

Conditional means: (Regression curves for means)

$$
\begin{aligned}
E(X/Y = y) &= \int_{-\infty}^{\infty} (x) f_{X/Y}(X/Y = y)\, dx \\
&= \int_0^1 (x) \frac{3x^2 - 8xy + 6y^2}{6y^2 - 4y + 2}\, dx
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{6y^2-44+1}\int_0^1 \left(3x^3-8x^2y+6xy^2\right)\,dx\\
&= \frac{1}{6y^2-44+1}\left[\left(\frac{3}{4}\right)-\left(\frac{8}{3}\right)y+3y^2\right]\\
&= \frac{36y^2-32y-9}{12(6y^2-4y+1)},\quad 0\le y\le 1\\
E[Y/X=x] &= \int_{-\infty}^{\infty}(y)\ f_{Y/X}(y/x)\,dy\\
&= \int_0^1 (y)\ \frac{3x^2-8xy+6y^2}{3x^2-4x+2}\,dy\\
&= \frac{1}{3x^2-4x+2}\int_0^1\left(3x^2y-8xy^2+6y^3\right)\,dy\\
&= \frac{1}{3x^2-4x+2}\left[\left(\frac{3}{2}\right)x^2-\left(\frac{8x}{3}\right)+\left(\frac{3}{2}\right)\right]\\
&= \frac{9x^2-16x+9}{6(3x^2-4x+2)}0\le x\le 1
\end{aligned}$$

Now to find the regression lines,

$$\begin{aligned}
E(X)=\int_{-\infty}^{\infty} x\ f_x(x)\,dx &= \int_0^1 x\left(3x^2-4x+2\right)\,dx\\
&= \left(\frac{3x^4}{4}-\frac{4x^3}{3}+\frac{2x^2}{2}\right)_0^1=\frac{5}{12}\\
E(Y)=\int_{-\infty}^{\infty} y\ f_Y(y)\,dy &= \int_0^1 y(6y^2-4y+1)\,dy\\
&= \left(\frac{6y^4}{4}-\frac{4y^3}{3}+\frac{y^2}{2}\right)_0^1=\frac{2}{3}\\
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f(x,y)\,dxdy\\
&= \int_0^1\int_0^1 xy\ \left(3x^2-8xy+6y^2\right)\,dxdy
\end{aligned}$$

$$= \int_0^1 \left[\left(\frac{3x^4}{4}\right)y - \left(\frac{8x^3}{3}\right)y^2 + \left(\frac{6x^2}{2}\right)y^3\right]_0^1 dy$$

$$= \int_0^1 \left[\left(\frac{3}{4}\right)y - \left(\frac{8}{3}\right)y^2 + (3)y^3\right] dy$$

$$= \left[\left(\frac{3}{4}\right)\frac{y^2}{2} - \left(\frac{8}{3}\right)\frac{y^3}{3} + \frac{3y^4}{4}\right]_0^1 = \frac{17}{72}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{17}{72} - \left(\frac{5}{12}\right)\left(\frac{2}{3}\right) = -\frac{1}{24}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2\left(3x^2 - 4x + 2\right)dx$$

$$= \left(\frac{3x^5}{5} - \frac{4x^4}{4} + \frac{2x^3}{3}\right)_0^1 = \frac{4}{15}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2\, f_Y(y)\,dy = \int_0^1 y^2\left(6y^2 - 4y + 1\right)dy$$

$$= \left(\frac{6y^5}{5} - \frac{4y^4}{4} + \frac{y^3}{3}\right)_0^1 = \frac{8}{15}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{4}{15} - \left(\frac{5}{12}\right)^2 = \frac{67}{720}$$

$$V(Y) = E(Y^2) - E(Y)^2 = \frac{8}{15} - \left(\frac{2}{3}\right)^2 = \frac{4}{45}$$

Regression lines:

(a) Regression line of Y on X is

$$Y - \bar{Y} = \frac{\text{Cov}(X, Y)}{V(X)}(X - \bar{X})$$

$$Y - \left(\frac{2}{3}\right) = \frac{-1/24}{67/720}\left(X - \frac{5}{12}\right)$$

$$Y - \left(\frac{2}{3}\right) = -\left(\frac{30}{67}\right)\left(X - \frac{5}{12}\right) \quad \text{or} \quad y = \frac{-30}{67}X + \frac{343}{402}$$

(b) Regression line of X on Y is

$$X - \bar{X} = \frac{\text{Cov}(X, Y)}{V(Y)}(Y - \bar{Y})$$

$$X - \left(\frac{5}{12}\right) = -\left(\frac{15}{32}\right)\left(Y - \frac{2}{3}\right) \quad \text{(or)} \quad X = -\frac{15}{32}y + \frac{35}{48}$$

■

■ Example 3.53

Prove that $|\rho_{XY}| \leq 1$.

The variance of any quantity is always non-negative. So,

$$V\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) \geq 0$$

$$\Rightarrow \quad \left(\frac{1}{\sigma_X^2}\right) V(X) + \left(\frac{1}{\sigma_Y^2}\right) V(Y) + \left(\frac{2}{\sigma_X\sigma_Y}\right) \text{Cov}(X, Y) \geq 0 \qquad \text{(by Property 3.11)}$$

$$\Rightarrow \quad 1 + 1 + 2\left(\frac{\text{Cov}(X, Y)}{\sigma_X\, \sigma_Y}\right) \geq 0$$

$$\Rightarrow \quad 2 + 2\ (\rho_{XY}) \geq 0$$

$$\therefore \quad \rho_{XY} \geq -1 \qquad (1)$$

Similarly,

$$V\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) \geq 0$$

$$\Rightarrow \quad \left(\frac{1}{\sigma_X^2}\right) V(X) + \left(\frac{1}{\sigma_Y^2}\right) V(Y) - \left(\frac{2}{\sigma_X\sigma_Y}\right) \text{Cov}(X, Y) \geq 0$$

$$\Rightarrow \quad 2 - 2\ (\rho_{XY}) \geq 0$$

$$\Rightarrow \quad \rho_{XY} \leq 1. \qquad (2)$$

Combining equations (1) and (2), we have

$$-1 \leq \rho_{XY} \leq 1$$

$$\Rightarrow \quad |\rho_{XY}| \leq 1$$

■

Remark

If $\rho_{XY} = 0$ (variables are uncorrelated), the two lines of regression are perpendicular. If $r = \pm 1, \theta = 0$ i.e, the two lines coincide, so that the variables have higher degree of correlation. Hence, if the angle between the lines is smaller then correlation is high. [Refer Exercise (26)]

Exercises 3

1. Find the mean and variance, if they exist, in each of the following distributions.

(a) $f(x) = \begin{cases} K(1-x) & |x| \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} 4x^3 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^3}{36} & x = 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$

(d) $f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & \text{elsewhere} \end{cases}$

(e) $f(x) = \begin{cases} \frac{x}{15} & x = 1, 2, 3, 4, 5 \\ 0 & \text{elsewhere} \end{cases}$

2. Let X have the pdf $f(x) = \begin{cases} K(x+2) & -2 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

   Find $E\left[(X-2)^3\right]$ and $E\left[6X + 2(X-3)^2\right]$.

3. If $f(x) = \begin{cases} 3(1-x)^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ is the pdf of a random variable X.

   Find $E\left(6X + 3X^2\right) E(2X \pm 3)$; $V(2X \pm 3)$.

4. If $f(x) = \begin{cases} K(4-x) & |x| < 2 \\ 0 & \text{elsewhere} \end{cases}$ is the pdf of X.

   Find $E\left(X^3 + 2X^2 - 3X + 1\right)$ and $E\left((2X+1)^2\right)$.

5. If $f(x, y) = \begin{cases} 2 & x > 0, y > 0; \quad x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$

   find $E(XY)E(2X + 3Y)$.

6. If $f(x, y) = \begin{cases} K(x+2y) & 0 < x < 1; \quad 1 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$

   is the joint pdf of (X, Y) then find $E\left(\frac{X}{Y}\right)$.

7. The joint pdf of (X, Y) is $f(x, y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$

   find $E(X)$, $E(Y)$ and $E(XY)$ and hence show that $E(XY) \neq E(X) \cdot E(Y)$ in general.

8. The joint pdf of X and Y is $f(x, y) = K(x + y)$ where X and Y assume all integers between 0 and 4. Find $\mathrm{Cov}(X, Y)$.

9. Find the correlation coefficient of X and Y in each case.

   (a) $f(x, y) = \begin{cases} K(x+2y) & (x, y) = (1, 1), (2, 2), (1, 2), (2, 1) \\ 0 & \text{elsewhere} \end{cases}$.

(b) $f(x, y) = \begin{cases} 2(x+y) & 0 \le x < y \le 1 \\ 0 & \text{elsewhere} \end{cases}$

(c) $f(x, y) = \begin{cases} Kxy & 0 \le x \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$

10. Find the moments upto order 4 and hence find the mean, variance, measures of skewness and kurtosis in each of the following.

(a) $f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} K\left(1 - x^2\right) & |x| < 1 \\ 0 & \text{elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} \frac{3x+1}{8} & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

11. Find MGF and hence find an expression for μ_r'.

(a) $f(x) = \begin{cases} Kx^n & 0 < x < 1 \quad n > 0, \text{ an integer} \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(x) = \begin{cases} Kx^n e^{-ax} & x > 0;\ a > 0\ n \text{ is a positive integer} \\ 0 & \text{elsewhere} \end{cases}$

(c) $f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & x = 1, 2, 3, \cdots \\ 0 & \text{elsewhere} \end{cases}$

(d) $f(x) = Ke^{-|x|} \quad -\infty < x < \infty$

12. Suppose that a class contains 15 boys and 10 girls, and if 7 students are to be selected at random from the class. Let X denote the number of boys and Y denote the number of girls that are selected. Find (a) $E(X+Y)$; (b) $E(X-Y)$.

13. If X and Y are independent random variables for which $V(X) = 2$ and $V(Y) = 3$. Find (a) $V(X-Y)$, (b) $V(3X - 2Y + 5)$ and (c) $\text{Cov}(X - Y, X + Y)$.

14. If the joint pdf of X and Y is given by

$$f(x) = \begin{cases} K(x+y) & 1 < x < 2;\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $V(4X + 3Y - 2)$.

15. Obtain the variance of $2X + 3Y + 4Z$ if X, Y and Z are three random variables with means 3, 4, 5 respectively, variances 10, 20, 30 respectively and $\text{Cov}(X, Y) = 0 = \text{Cov}(Y, Z)$ and $\text{Cov}(X, Z) = 5$.

16. Let X and Y have the joint pdf described as follows. Find their joint MGF and hence calculate their means, variances and ρ_{XY}.

(a) $f(x, y) = \begin{cases} \frac{2x+y}{27} & x = 0, 1, 2;\ \ y = 0, 1, 2 \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(x, y) = \begin{cases} 4e^{-2y} & 0 < x < y < \infty \\ 0 & \text{elsewhere} \end{cases}$

17. Compute the conditional means and variances in each of the following cases.

(a) $f(x, y) = \begin{cases} 28x^3y^2 & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(x, y) = \begin{cases} 2 - x - y & 0 \le x \le 1;\ \ 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$

(c)

$Y \diagdown X$	1	3	5
-1	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$
0	$\frac{1}{4}$	$\frac{1}{4}$	0
1	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$

(d) $f(x, y) = \begin{cases} \frac{x+y}{21} & x = 1, 2, 3;\ \ y = 1, 2 \\ 0 & \text{elsewhere} \end{cases}$

(e) $f(x, y) = \begin{cases} 6(1 - x - y) & x > 0, y > 0;\ \ x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$

18. Obtain the regression curves (a) of Y on X and (b) of X on Y. Are the regression linear?

(a) $f(x, y) = \begin{cases} \frac{1}{2} & 0 < x < y < 4 \\ 0 & \text{elsewhere} \end{cases}$

(b) $f(x, y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$

(c) $f(x, y) = \begin{cases} 1 & -x < y < x;\ 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

(d) $f(x, y) = \begin{cases} xe^{-x(y+1)} & x > 0;\ y > 0 \\ 0 & \text{elsewhere} \end{cases}$

19. Let the joint density function of X and Y be given by

$$f(x, y) = \begin{cases} \frac{x+y}{3} & 0 < x < 1;\ 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the regression lines of Y on X and X on Y and hence ρ_{XY}.

20. If X and Y are uncorrelated random variables and if $U = X + Y$ are and $V = X - Y$, find ρ_{UV}. Show that U and V are uncorrelated if $V(X) = V(Y)$.

21. If the independent random variables X on Y and Z have the means 4, 9, and 3 and variances 3, 7, 5 respectively, obtain ρ_{UV} if $U = 2X - 3Y + 4Z$ and $V = X + 2Y - Z$.

22. The joint pdf of (X, Y) is

$$f(x, y) = \begin{cases} \left(\frac{4}{3}\right) xy & 0 < x < 1; 1 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find ρ_{UV} where $U = X + Y$ and $V = X - Y$. (Hint: Check whether X and Y are independent).

23. If X and Y are uncorrelated random variables such that $E(X) = E(Y) = 0$ and equal variances. Let $Z = \frac{X+Y}{2}$, find $\text{Cov}(X, Z)$ and $\text{Cov}(Y, Z)$.

24. Two random variables X and Y which are uncorrelated having zero means and equal variance σ^2. If $U = X\cos\alpha + Y\sin\alpha$ and $V = X\sin\alpha - Y\cos\alpha$, then show that $\text{Var}(U) = \text{Var}(V) = \sigma^2$ and U and V are uncorrelated.

25. If $X_i (i = 1, 2, 3)$ are three uncorrelated random variables having standard deviations σ_1, σ_2 and σ_3 respectively, obtain ρ_{UV}, ρ_{UW} and ρ_{VW} where

$$U = X_1 + X_2 \quad V = X_2 + X_3 \quad \text{and} \quad W = X_3 + X_1$$

26. Find the angle θ, between the two lines of regression and interpret the cases when $\rho = 0$ and $\rho = \pm 1$.

27. Find an expression for μ_r (r^{th} moment about mean) in terms of moments about origin M_r'.