MATH 239: Introduction to Combinatorics

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Notes written from Bertrand Guenin's lectures.

1 Introduction, Permutations, and Combinations

1.1 Course Structure

The grading scheme is 50% final, 30% midterm, 10% quizzes, and 10% assignments. There are ten quizzes and ten assignments. The quizzes will not be announced in advance, and each quiz consists of a single clicker question. Assignments are typically due on Friday mornings at 10 am. in the dropboxes outside of MC 4066. The midterm exam is scheduled for March 7, 2013, from 4:30 pm - 6:20 pm. There is no textbook for the course, but there are course notes available at Media.doc in MC, and they're *highly* recommended.

MATH 239 is split into two parts: counting (weeks 1-5) and graph theory (weeks 6-12).

See the course syllabus for more information – it's available on Waterloo LEARN.

1.2 Sample Counting Problems

Problem 1 *How many ways are there to cut a string of length 5 into parts of sizes 1 and 2?*

Here are a few example cuts:

1	2	3	4	5	5 cuts: size 1, size 1, size 1, size 1, size 1.
1	1	2	2	3	3 cuts: size 2, size 2, size 1.
1	2	2	3	3	3 cuts: size 1, size 2, size 2.

This is a finite problem. You could count all of the possibilities manually in this case. However, this problem could be made more complicated to a point where manually counting all possibilities would become quite cumbersome, as is the case in the next problem.

Problem 2 (Cuts) *How many ways are there to cut a string of size 372,694 into parts of sizes 3, 17, 24, and 96?*

Definition 1 *A positive integer n has a* ***composition*** $(m_1, m_2, \ldots, m_k)$*, where* $m_1, \ldots, m_k$ *are positive integers and where* $n = m_1 + m_2 + \ldots + m_k$*.* $m_1, \ldots, m_k$ *are the* ***parts*** *of the composition.*

Problem 1 could be rephrased as looking for the number of compositions of 5 where all parts are 1 or 2.

Problem 3 *How many compositions of n exist such that all parts are odd?*

Problem 4 (Binary Strings) *Let* $S = a_1, a_2, \ldots, a_n$ *where* $a_i \in \{0, 1\}$*. How many strings S exist?*

For each a_i, there is the choice between 0 or 1, and that choice is independent for each character of the string (for each a_i). So, there are 2^n binary strings of length n.

Problem 5 *How many binary strings of size n exist that do not include the substring* 1100*?*

For example: $1010\underline{1100}101 \notin S$.

Problem 6 *How many binary strings of size n exist such that there is no odd-length sequences of zeroes?*

For example: $100100\underline{100}11 \notin S$.

Problem 7 (Recurrences) *How many times does a recursive function get called for a particular input n?*

1.3 Sample Graph Theory Problems

Problem 8 *For any arbitrary map of regions, color the regions such that no two touching boundaries do not have the same color, with the least number of colors possible.*

The **four-color theorem** (proven later in the course) states that you can always do this with four colors. It's also always possible to color these regions with five colors. It's *sometimes* possible to color the regions with three or fewer colors, depending on the layout of the regions and their boundaries.

1.4 Permutations and Combinations

1.4.1 Set Notation

The usual set and sequence notation is used in this course. $(1, 2, 3)$ is a sequence (where order matters), and $\{1, 2, 3\}$ is a set (where order does not matter).

We will also be using one piece of notation you may not be familiar with: $[n] := \{1, 2, \ldots, n\}$.

1.4.2 Permutations

Definition 2 *A **permutation** of $[n]$ is a rearrangement of the elements of $[n]$. The number of permutations of a set of n objects is $n \times (n-1) \times \ldots \times 1 = n!$.*

For example: the number of permutations of 6 objects is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ permutations.

Why is this the case? Simple: there are n choices for the first position, $(n-1)$ choices for the second position, $(n-2)$ choices for the third position, and so on, until there's 1 choice for the nth position.

Definition 3 *A k-**subset** is a subset of size k.*

Problem 9 *How many k-subsets of $[n]$ exist?*

Let's consider a more specific case: how many 4-subsets of 6 are there? $\frac{6 \times 5 \times 4 \times 3}{4!}$.

For simplicity's sake, we will introduce notation for this, which we will refer to as a **combination**, denoted as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ where $n, k \in \mathbb{Z} \geq 0$.

Proposition 1 *There are $\binom{n}{k}$ k-subsets of $[n]$.*