Worksheet 8

Introduction

This sheet continues the discussion of the problems involved in solving numerically the general equation $f(x) = 0$, using an iterative scheme of the form $x_{n+1} = g(x_n)$. This scheme was used with the trial starting value x_1 and iteration continued until we found a value α such that $\alpha = g(\alpha)$ to a given number of decimal places. We saw in worksheet 7 that the type of rearrangement and the choice of initial value were critical in finding a particular solution. We now introduce Newton's method, a more reliable way of obtaining a suitable $g(x)$ to use in the iteration. Since Newton's method uses calculus and the equation of a tangent to a given curve we begin by reviewing some basic results from calculus.

Background

The diagram shows the curve $y = x^3$ and a tangent drawn to the curve at the point $(2, 8)$.

The gradient of the tangent at a general point x is given by $3x^2$. Thus at $x = 2$ the gradient of the tangent is 12 and the equation of the tangent is given by

$$y - 8 = 12(x - 2)$$

which simplifies to $y = 12x - 16$.

A brief review of calculus results

The gradient of the tangent to the curve $y = f(x)$ is called the *derivative* of $f(x)$ and is denoted by y' or $\frac{dy}{dx}$. We say that y has been *differentiated* to give y'.

We give a list of standard derivatives:

1. If $y = x^n$ then $y' = nx^{n-1}$. (In our example $n = 3$.)
2. If $y = \sin(\alpha x)$ then $y' = \alpha\cos(\alpha x)$.
3. If $y = \cos(\alpha x)$ then $y' = -\alpha\sin(\alpha x)$.
4. If $y = \ln(x)$ then $y' = \frac{1}{x}$, provided that $x > 0$.
5. If $y = e^{\alpha x}$ then $y' = \alpha e^{\alpha x}$.

There are many other results and rules for differentiating more complicated functions. However for the moment we will allow DERIVE to do the work.

Problem 31

Use DERIVE to differentiate the equation $y = x^2 \sin x$ as follows:

- Author the expression $x^2\sin(x)$
- **Calculus** $\rightarrow$ **Differentiate** *(check variable is x and order is 1)* $\rightarrow$ **Simplify**.

Use this method to differentiate the following equations:

(a) $x^3 + 3x^2 + x + 1$, (b) $x^4 + x^2 + 1$, (c) $\dfrac{x+1}{x-1}$, (d) $\sin(x^2)$,
(e) $x^2\cos(x)$. (f) $x\ln(x^2 + 1)$. (g) $\tan(x)$.

Don't forget to check that the expression you author is indeed the correct expression!

Problem 32

For each of the following functions

- Plot the function.
- Evaluate the function at $x = 1$ using
  Simplify $\rightarrow$ **Variable Substitution** *(substitute value)* $\rightarrow$ **Simplify**
- Calculate the derivative of the function.
- Evaluate the derivative at $x = 1$ using DERIVE.
- Use the above information to write down the equation of the tangent to the curve in the form

$$y = mx + c.$$

- Author the equation of the tangent into DERIVE and plot.

(a) $x^5 + x^4 + x^3 + x^2 + x + 1$. (b) $x^4 + 3x^2 - 1$. (c) $\dfrac{x+1}{x^2+1}$. (d) $x\sin(\frac{\pi}{2}x)$.

(e) $x\ln(\sin(\frac{\pi}{2}x) + 1)$. (f) $\frac{e^{x^2}-1}{x}$.

In each case the line you plot should clearly be a tangent to the curve at $x = 1$.

Newton's method

Let $x = a$ be the solution of the equation $f(x) = 0$.
Let x_1 be a first approximation to the solution of $f(x) = 0$.
Draw the tangent to $y = f(x)$ at x_1. This tangent cuts the x-axis at $x = x_2$.
Newton's method assumes that x_2 is closer to the solution at $x = a$ than x_1.

By using the fact that the slope of the tangent is given by $f'(x)$ we can find the coordinates of x_2 in terms of x_1 and the function.
From the diagram the slope of the tangent is equal to $\dfrac{f(x_1)}{x_1 - x_2} = f'(x_1)$.
Rearranging this equation gives *Newton's Iterative Formula*

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Thus we have an iterative scheme of the form $x_2 = g(x_1)$ where now the function $g(x)$ is obtained in the above manner rather than by a simple rearrangement. The advantage of this method is that it gives a sequence that is much more likely to converge, and the rate of convergence is usually rapid compared with that arising from the rearrangement method.

General Newton's method

To solve $f(x) = 0$ given a first approximation x_1. Set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Problem 33

Solve $f(x) = x \sin x - 1 = 0$ starting with $x_1 = 1$. We use DERIVE to carry out all of the calculations including setting up the formula and using iterates to carry out the iteration as follows:

- Author $x \sin x - 1$ line number $\#1$.
- With $\#1$ highlighted: **Calculus** $\rightarrow$ **Differentiate** $\rightarrow$ **Simplify** line number $\#3$.
- Author $x - \#1/\#3$ (this is $g(x)$) line number $\#4$.
- Author `iterates`($\#4$, x, 1, 10)
- **Simplify** $\rightarrow$ **Approximate** $\rightarrow$ **Approximate**.

Obviously, you may have to use different line numbers. You should see rapid convergence in this case.

Problem 34

For the following functions use the above method to set up Newton's iterative scheme to solve $f(x) = 0$. Hence calculate *all* the roots of $f(x) = 0$. You will need to plot the graphs of $f(x)$ in order to find suitable starting values to reach all the roots.

(a) $f(x) = x^3 + 3x^2 - 5x + 2$. (b) $f(x) = x - \sin 2x$.

(c) $f(x) = x^2 + x \sin x - 1$. (d) $f(x) = x\ln\, x - \sin x$.

Note: Inappropriate starting values can still lead to problems. One example would be starting (a) at $x = 1$, although it still eventually converges. In (d) you have the problem when trying to calculate the root at $x = 0$ that if you use a small positive value for x_1 then x_2 will be negative. You are told that $\ln x$ is undefined for $x \le 0$, however DERIVE tries to use the natural extension of logarithms to complex numbers which allows you to find these with $\ln(x) = \ln|x| + i\pi$ for $x < 0$.

End of term test

The end of term test will take place on Wednesday 30$^{\text{th}}$ November in the class. The test from 3 years ago is on the next page. Please note

1. The test is **Open Book**, so you can bring along notes and past example sheets. Make sure you bring along a complete set as spare copies will not be provided on the day.
2. **Calculators are not allowed** – you are meant to be learning to use DERIVE alongside pen and paper.
3. Talking or communicating with each other is not allowed. Anyone doing so will be asked to leave the test.
4. People who arrive late will not be given extra time.