Worksheet 9

Introduction

This sheet deals with the problem of finding the maximum and minimum values of a given function, either over some finite range of values or over all real values. We will begin with a review of the basic theory that we will need.

Background

There are two distinct concepts of max and min:

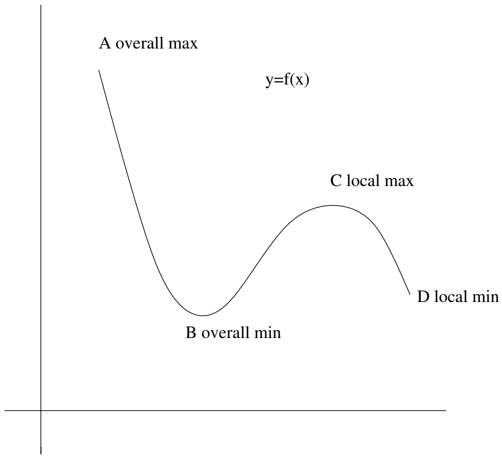
a) First the ideas of an *overall max* and *overall min*. These correspond to the usual meanings of max or min and are used to refer to the largest or smallest value of the function on the given interval, or over all values if no interval is specified.

b) Second the ideas of *local max* and *local min*. A local max is a point on the curve where all neighbouring points on the curve are lower. Similarly a local min is a point on the curve where all neighbouring points on the curve are higher.

It follows immediately that an overall max (min) is also a local max (min)

Consider the example shown in the diagram:

- A is the overall max value of the function on the given interval and hence is also a local max since there are no values to its left and it is bigger than the values to its right.
- B is the overall min value of the function on the given interval and is also a local min, as all values in the neighbourhood of B are higher than B.
- C is a local max since all points in the neighbourhood of C are lower than C
- D is a local min since there are no points to the right of D and all points to the left are higher than D.



For obvious reasons the points B and C are called *turning points*.

To investigate the max and min property of a given function we need to:

- Locate the turning points.
- Classify the turning points as local max and min.
- Evaluate the function at these points.
- If the function is defined on an interval, evaluate the function at the end points of the interval.
- If the function is defined for all real numbers, consider $f(x)$ as $x \to \pm\infty$

Locating turning points

From the diagram it is obvious that at a turning point:

- The gradient of the tangent is zero (tangent horizontal), i.e. $f'(x) = 0$. Such a point is referred to as a *stationary point*.
- If the turning point is a max then the gradient changes from positive, $f'(x) > 0$, to negative, $f'(x) < 0$, as we pass from left to right through the point.
- If the turning point is a min then the gradient changes from negative, $f'(x) < 0$, to positive, $f'(x) > 0$, as we pass from left to right through the point.

Problem 35

Given $f(x) = x^2 + 4x + 2$ we can locate the possible turning points by solving $f'(x) = 0$, i.e. $2x + 4 = 0$. This has the single solution $x = -2$.
To the left of $x = -2$, say at $x = -3$, we have $f'(-3) = 2(-3) + 4 = -2$. Thus the gradient is negative, and $f(x)$ is decreasing.
To the right of $x = -2$, say at $x = -1$, we have $f'(-1) = 2(-1) + 4 = 2$. Thus the gradient is positive, and $f(x)$ is increasing.
Thus the point $x = -2$ is a turning point which is a local minimum. Since $f(x)$ is defined over all the real numbers we can see that as $x \to \pm\infty$ $f(x)$ tends to $+\infty$. Thus the turning point is an overall minimum.
Note: In this example $f(x)$ has no maximum value.

- Sketch the graph of $f(x)$ using DERIVE to verify that the above calculations are correct.

Problem 36

For each of the functions given below:

- Use DERIVE to differentiate $f(x)$, and solve $f'(x) = 0$.
- Evaluate $f(x)$ at the turning points, and $f'(x)$ either side of the turning points.
- Hence classify the turning points and find the maximum and minimum values of the functions on the given intervals, or for all real values of x for those examples where no interval is given. Say if the function does not have a maximum or minimum value.
- Plot the functions:

  (a) $f(x) = x^3 + 2x^2 - 7x + 1 \quad -3 \leq x \leq 3$ (b) $f(x) = \frac{\sin x}{x} \quad 0 < x \leq 3\pi$.

  (c) $f(x) = 15x^4 + 2x^3 - 51x^2 + 36x + 5 \quad -2 \leq x \leq 2$

  (d) $f(x) = 1 - e^{-(x-1)^2}$. (e) $f(x) = x \sin x \quad -\pi \leq x \leq \pi$

  (f) $f(x) = 8x^5 - 33x^4 + 44x^3 - 14x^2 - 12x + 2 \quad -1 \leq x \leq 2$.

  In (f) we observe that at $x = 1$ the function is stationary, i.e., $f'(1) = 0$ but the point is not a turning point. This stationary point is an example of a *point of inflection*.

Problem 37

Locate the stationary points of the following functions and classify each such point as a local max, local min, or point of inflection:

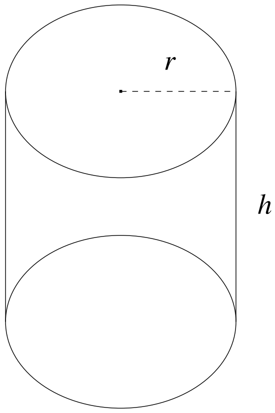
(a) $f(x) = x^3$ (b) $f(x) = 2(x-1)\sin x - (x^2 - 2x - 1)\cos x \quad -3\pi \le x \le 3\pi$.
(c) $f(x) = 4x^5 - 5x^4 - 5x^3 + 10x^2 - 5x$ (d) $f(x) = 12x^5 - 45x^4 + 20x^3 + 240x$.

Using your results try to get DERIVE to plot (d) in such a way that it shows the point of inflection. Also plot (a), (b), and (c).

In the final pair of problems, we will illustrate how stationary points can be applied to various real-life calculations.

Problem 38

A soft drinks manufacturer wishes to produce a cylindrical can for its product. If the can is to have a volume of 300ml, find the dimensions of the can that will require the least amount of material.

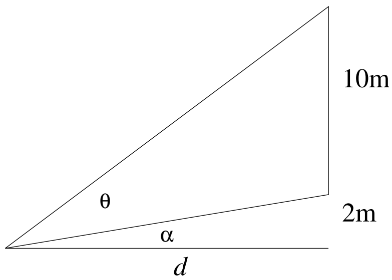


This problem appears to have two variables, the radius r and the height h. The volume V of the can is given by $V = \pi r^2 h$ and the surface area S by $S = 2\pi r^2 + 2\pi r h$.

- Show that $S = 2\pi r^2 + \frac{600}{r}$.
- Find the value of r such that S is stationary.
- Show that the value of r obtained above makes S an overall minimum.
- Find the corresponding value of h.

Problem 39

The lower edge of a 10m high movie theatre screen is 2m above the viewer's eye. Determine how far from the screen should the observer sit to obtain the most favourable view. (In other words, what value of d maximises θ?)



Hint:
From the diagram we see that $\tan\alpha = \frac{2}{d}$ and $\tan(\theta+\alpha) = \frac{12}{d}$. Now use the angle sum rule for $\tan(\theta+\alpha)$ to show that

$$\tan\theta = \frac{10d}{(24 + d^2)}$$

.
Hence obtain $\dfrac{d\theta}{dd}$.