Worksheet 2

Introduction

The aim of the following sheet is to use DERIVE to enhance your skills in manipulating and solving equations of the form $f(x) = 0$. This will build on what you did last week, so you may have to refer to that sheet if you have forgotten how to do something.

We consider three problems, the first involving the solution of a quadratic equation, the second the solution of an equation involving algebraic and trigonometric functions, and the third involving an integral.

Problem 1

An exhibition space is to take up half the area of the hall and is to be placed in a reserved strip of width x metres around the edge.

If the length of the hall is 12 metres (i.e. $b = 12$ in the diagram) and the width is 5 metres (i.e. $a = 5$ in the diagram) show that

$$4x^2 - 34x + 30 = 0.$$

(Hints at the end of the work sheet, but try it yourself first.)

- Factorize the quadratic to find the values of x.

  (Try this using DERIVE using the **Simplify**, **Factorize** facility.)

- Write down the one feasible solution. (Warning — problems often can give rise to extra unphysical solutions, usually because you missed out some obvious constraint such as insisting x lies between 0 and half the minimum of a and b)

A more general approach is to use the parameters a and b, rather than the numerical values given above, and instead of creating a display area equal to half the total area we introduce a third parameter α such that the display area $= \alpha \times$(total area).

- Complete the following: $\ldots . \leq \alpha \leq \ldots .$; i.e., write down the maximum feasible range for α.
- Show that for the more general problem x satisfies the following quadratic:

$$4x^2 - 2(a + b)x + \alpha ab = 0. \tag{1}$$

To simplify the analysis of the problem the problem is made non-dimensional by introducing two new symbols λ and y, where we set $b = \lambda a$ and $x = ya$. (Here λ is the aspect ratio of the hall, and hence dimensionless, while y is dimensionless as it is the ratio of two lengths.)

- Use DERIVE to substitute $b = \lambda a$ and $x = ya$ into equation (1) and simplify as follows:

  First author expression (1). To enter greek letters (in this case alpha) use the buttons at the bottom left.

  Use "**Simplify** → **Variable Substitution ...**" and for x substitute ya. Repeat this proceedure to substitute λa for b.

  Then use **Simplify**, **Factor**, **Factor**.

  Write down the simplified equation cancelling out any common factor.

....

- Show that the above problem has a real solution provided that $\alpha \leq \frac{(1+\lambda)^2}{4\lambda}$. (See Hints if needed)

Since $0 \leq \alpha \leq 1$ we now investigate whether or not it is possible for the aspect ratio λ to be such that there is no real value of x for the problem. This can only happen if $\frac{(1+\lambda)^2}{4\lambda}$ is less than 1 for some value of λ. (Recall that λ is positive.)

• Plot $\frac{(1+\lambda)^2}{4\lambda}$ for $\lambda > 0$ and use DERIVE to obtain its minimum value. This final step establishes that the problem will always have a solution.

Problem 2

In the diagram set $\theta = \angle AOB$. Set the radius of the circle $OA = a$. The problem is to find the value of θ such that the area of the shaded portion $= \alpha\times$(area of the circle).

- Show that $\theta - \sin\theta = 2\alpha\pi$. (See Hints)
- Graphically find the value of θ that satisfies this equation for $\alpha = \frac{1}{2}$ as follows. Use DERIVE to Plot $\theta - \sin\theta$ and π. Hence read off the point of intersection of these two graphs and give the solution correct to 1 decimal place.

  $$\theta = \ldots\ldots\ldots\ldots$$

  Is this the value that you expected? What value of θ would you expect if $\alpha = 1$? ......

An alternative way of visualizing this problem is to plot $\frac{\theta - \sin\theta}{2\pi}$ using DERIVE. The range of the vertical axis should be set from -0.5 to 1.5 since α is in the range 0 to 1 and the horizontal axis from -1 to 10 since θ is in the range 0 to 2π. Working with these ranges will give a better graph.

- Use the graph to read off the values of θ, correct to 1 decimal place, for the following values of α.

α	0.125	0.25	0.5	0.75
θ				

We can now solve this problem for any value of α in the range 0 to 1 using the **Solve** and **Approximate** facility of DERIVE. For example, to solve the above problem with $\alpha = \frac{1}{2}$:

- Author the expression $\theta - \sin\theta - \pi$.
- Select **Solve** $\rightarrow$ **Numerically**.
- Set the lower bound to 0 and the upper to 10.
- **Simplify**.
- Repeat this for the other values of α used above to complete the following table, with values now correct to 4 decimal places.

α	0.125	0.25	0.5	0.75
θ				

Problem 3

A slight variation on problem 2 considerably increases the mathematical complexity. If we consider an ellipse being cut by a chord at right angles to its major axis then finding the position of the chord so that a certain proportion of the ellipse is cut off requires the use of calculus.

The equation of the ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is our job to find the value of c such that the shaded area $= \alpha\times$(total area). The shaded area is given by the formula

$$\text{area} = 2b\int_c^a \sqrt{1-\frac{x^2}{a^2}}\,dx.$$

Initially set $a = 1$, $b = \frac{1}{2}$ and $\alpha = \frac{1}{4}$. Evaluate this integral with DERIVE as follows:

- Author expression sqrt$(1 - x^2)$ or $\surd(1 - x^2)$ using the $\surd$ from the bottom right buttons.
- Select **Calculus**, **Integrate**.
- Select **Definite Integral**. It will select x as the variable.
- Enter **Lower Limit** as c.
- Enter **Upper Limit** as 1.
- **OK**.
- **Simplify** $\rightarrow$ **Basic**.

You should now have an expression for the shaded area in terms of c. Put $c = -1$ (using DERIVE if you wish) into this expression to show that the area of the ellipse is $\frac{\pi}{2}$. (In general the area of the ellipse is πab.)

We now use this expression for the shaded area to find the value of c such that the shaded area is one quarter of the total area.

- Author the expression # line number of expression for shaded area $-\frac{\pi}{8}$.

  (This enters the expression to be solved for c.)
- Proceed as in problem 2 to solve this equation numerically using **Solve**, **Numerically**.

  $$c = \ldots\ldots\ldots\ldots$$
- One final check is to rerun the problem with $\alpha = \frac{1}{2}$ and see if this gives $c = 0$.

Hints

- The length of the hall is $b = 12$, the length of the rectangle in the middle (the bit of the hall that isn't exhibition space) is .... The width of the area in the midddle is .... The area of the hall is $ab = 5 \times 12 = 60$, the are of the bit in the middle is .... The area of the exhibition space = area of hall − area of bit in middle = ...
- Remember that a quadratic equation $ax^2 + bx + c = 0$ has roots

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

  and for the roots to be real $b^2 - 4ac \geq 0$. Here your quadratic is in y not x, and the a and b are not those in the problem.
- What is the area of the segment of the circle AOB? It will be proportional to θ, giving area 0 when $\theta = 0$ and area πa^2 when $\theta = 2\pi$.

  What is the area of the triangle AOB? The area will be half the length of the line AB multiplied by the distance from O to this line. The distance from O to this line is $a\cos(\theta/2)$ — can you see why? Similarly the half the length of AB is $a \ldots\ldots (\theta/2)$ so the area of the triangle is .... You can simplify this using the standard result $\sin 2A = 2\sin A\cos A$ (or try using DERIVE?). You should now be able to write down the area of the shaded region as $\frac{a^2\theta - a^2\sin\theta}{2}$, and so derive the result given.