Worksheet 5

Introduction

This sheet looks further at the solution of simultaneous equations, both geometrically and algebraically, considering the conditions for the existence and uniqueness of solutions. We introduce methods for manipulating matrices. Initially we consider the equations of two straight lines of the form:

$$ax + by = c \qquad \text{and} \qquad dx + ey = f.$$

These are two simultaneous equations in the unknowns x and y.

Problem 13

- Plot the following three pairs of equations using DERIVE:
  (a) $2x - 3y = 1$ and $5x - 4y = 6$.
  (b) $3x - 4y = 1$ and $8y - 6x = 2$.
  (c) $6x - 4y = 2$ and $6y - 9x = -3$.
- In case (a) write down from your diagram the point of intersection of the two lines. That is to say, use the diagram to write down the solution to the two equations. Check that this result is correct using the method of elimination (which you should have seen in lectures).
- Explain the result in (b). What does this mean in terms of the existence and uniqueness of the solution of the two equations? Try and solve these equations by elimination — what happens?
- Explain the result in (c). What does this mean in terms of the existence and uniqueness of the solution of the two equations? Can you write down more than one solution to the equations? If we introduce a parameter t into the problem and set $y = t$ then write down the value of x in terms of t. This gives a parametric solution to the problem.

The results of Problem 13 show us that there may be a unique solution to the problem, there may be no solution to the problem, or there may be infinitely many solutions to the problem. In the case of an infinite number of solutions a parametric form is always possible. In the next Problem we will consider the general case.

Problem 14

In this Problem we will consider the two equations

$$ax + by = c \qquad \text{and} \qquad dx + ey = f.$$

- Show by eliminating x that
$$(ae - bd)y = af - dc.$$
- Hence deduce that
$$y = \frac{(af - dc)}{(ae - bd)}.$$
- Similarly deduce that
$$x = \frac{(ce - bf)}{(ae - bd)}.$$
- Write down a condition involving a, b, d, and e such that the equations have a unique solution.
- Using the above, write down the solution to each of the following pairs of equations:
  (a) $2x + 3y = 2$ and $4x - 3y = 5$.
  (b) $x - y = 2$ and $2x - 3y = 6$.
  (c) $3x - 4y = 1$ and $-x + 3y = 1$.

Matrices

A *matrix* is a rectangular array of elements, usually numbers. If the matrix has m rows and n columns then it is said to be an m by n (or $m \times n$) matrix.

We now introduce matrices to write down and solve sets of simultaneous equations. In the problem

$$\begin{aligned} 2x + 3y &= 2 \\ 4x - 3y &= 5 \end{aligned} \tag{1}$$

It is only the coefficients of x and y and the right-hand side of each equation that are important, not the actual variables x and y. This idea is strengthened by the introduction of matrix notation where we separate out the three different types of quantities, namely the coefficients of x and y, x and y themselves, and the two values on the right-hand side of the equations. We write (1) as:

$$\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}. \tag{2}$$

This uses the 2×2 matrix $\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix}$ and the 2×1 matrices $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ to describe the equations. A further simplification in notation is now made by setting

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

With this notation we now write (2) as

$$A\mathbf{x} = \mathbf{b}.$$

Problem 15

Write down the equations in Problem 14 in matrix form as in (2), giving the matrices A and $\mathbf{b}$ in each case.

Solving equations by 'division'

Single Equation

To solve an equation of the form

$$ax = b$$

we divide by a to give

$$x = \frac{b}{a}$$

or alternatively we multiply by a^{-1} to give

$$x = a^{-1}b.$$

Matrix Equation

To solve an equation of the form

$$A\mathbf{x} = \mathbf{b}$$

we cannot divide by a matrix as this does not make sense, but we can use a special matrix denoted A^{-1} which gives the solution as

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

We call A^{-1} the *inverse* of A.

Problem 16

DERIVE allows us to compute A^{-1} and hence solve sets of equations. It will do this for more than just two equations in two unknowns. For the problem

$$\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

we enter the matrices $\begin{pmatrix} 2 & 3 \\ 4 & -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ into DERIVE as follows:

- **Author** → **Matrix** Change to 2 rows and 2 columns **OK**.
- Enter the matrix values **OK**.

This should display the matrix on line number 1 denoted #1 (if it is on another line then, obviously, enter the number of that line instead of 1). We now enter the right-hand side of the equation, namely the matrix $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

- **Author** → **Matrix** Change to 2 rows and 1 column **OK**.
- Enter the matrix values **OK**.

This should display the matrix on line number 2, denoted #2. Next we calculate $A^{-1}\mathbf{b}$:

- **Author** → **Expression** Then enter #1^(-1).#2 (Note: you *must* include the dot!) **OK**.
- **Simplify** → **Basic** **OK**.

This should display the two solutions in matrix form: i.e., you should have $\begin{pmatrix} \frac{7}{6} \\ -\frac{1}{9} \end{pmatrix}$.

Problem 17

Use the above method to solve the following pairs of equations:

(a) $2x - 3y = 1$ and $x - 4y = 7$.

(b) $-x + 3y = 2$ and $7x - 4y = -1$.

(c) $5x + 2y = 2$ and $4x - 2y = 8$.

(d) $4x + 2y = 2$ and $2x + y = 2$.

Explain why DERIVE failed in (d). Write down the solution to (d), if possible.

If you have some time left over you can investigate other methods for solving systems of equations in DERIVE. Try **solve** → **System** and enter 2 as the number of equations. Then enter one of pair of equations from above in the next boxes. The solution variables should be x and y. Then click on **Solve**. This may be quicker, but doesn't introduce matrices!