L3: Curse of Dimensionality: Convex Hulls

Given a set of points $X \subset \mathbb{R}^d$, the *convex hull* $CH(X)$ is

- the smallest convex set which contains X
- if you put nails in a board, and surround with rubber band – it is inside the rubber band
- line segment between any pair of points is in $CH(X)$. Recursively, line segments between points on these line segments in $CH(X)$. In d dimensions, recurse up to d times…
- point $p \in CH(X)$ if there exists $\alpha \in [0,1]^d$ so $\alpha_j > 0$ and $\sum_j \alpha_j = 1$; then $p = \sum_j \alpha_j x_j$ (where $X = \{x_j\}_j$)
- point $p \notin CH(X)$ if and only if there exists a hyperplane H which can separate p from all points in X.

- the intersection of all balls (and hence all halfspaces) which contain X.

The convex hull $CH(X)$ is the most spatially compact way to represent X
… and intimately tied to *linear classification*.

Convex Hull for $d = 2, 3$

For $d = 2$, store sequence of edges connecting pairs of vertices as one "walks" around boundary.
Takes $O(n \log n)$ time, $O(n)$ space.

For $d = 3$, it is more complicated. Need to store edges, and also *faces*:

- 2-dimensional objects with edges on boundary.

- in general position, these are filled triangles.

How hard, big is it to construct?

- Storing all points, edges, faces is $O(n)$ space.
  Planar graph (wrapped on sphere)
- Can build in $O(n \log n)$ time

Convex Hull for High Dimensions

How do we store/represet $CH(X)$ in high-dimensions?

- just points?
- just faces?
- all j-dimensional (for $j \in [0, 1, \ldots, d-1]$) facets?

Moment curve: $x(t) = (t^1, t^2, \ldots, t^d) \in \mathbb{R}^d$ for $t \in \mathbb{R}$.
For n (lets say $n > 2d$) points X on moment curve, they form a *cyclic polytope*, for which all subsets of size $j+1$ for $j < \lfloor d/2 \rfloor$ define a j-dimensional face on the convex hull.
For $j = \lfloor d/2 \rfloor - 1$, there are $\binom{n}{j+1} = \Theta(n^{\lfloor d/2 \rfloor})$ such j-dimensional facets.
Can be shown the number of j-dimensional facets for $j > \lfloor d/2 \rfloor$ is also $\Theta(n^{\lfloor d/2 \rfloor})$.
Including $\Theta(n^{\lfloor d/2 \rfloor})$ of the faces (the $(d-1)$-dimensional facets).
Upper Bound Theorem: This is the most possible.

Approximate Convex Hulls

Can we break exponential dependence on d is we approximate?

Most common definition: *ε-kernel coreset.*
A subset $S \subset X$ approximates all directions within $(1 + \varepsilon)$.

Unit direction $u \in \mathbb{R}^d$ such that $\|u\| = 1$. or say $u \in \mathbb{S}^{d-1}$
Width in direction u is $wid_u(X) = \max_{x \in X} \langle x, u \rangle - \min_{x \in X} \langle x, u \rangle$.

Goal for S is for **all** directions u that

$$wid_u(X) - wid_u(S) \leq \varepsilon \cdot wid_u(X)$$

Hardest case is ball B.
In $d = 2$ it requires $\Omega(1/\sqrt{\varepsilon})$ points
$\varepsilon = 0.01 = 1/100$ (so 1% error) needs only about 10 points.

In high dimensions it requires $\Omega(1/\varepsilon^{\lfloor d/2 \rfloor})$ points.
Exist algorithm to compute ε-kernel of size $O(1/\varepsilon^{\lfloor d/2 \rfloor})$ points.

$(1 + \varepsilon)$-width approximation by faces also requires $\Omega(1/\varepsilon^{\lfloor d/2 \rfloor})$ faces.

John Ellipsoid Approximation

An *ellipsoid* is a ball in $\mathbb{R}^d$ after any affine transformation.
Let $A \in \mathbb{R}^{d \times d}$ full-rank matrix.
An affine transformation of $x \in \mathbb{R}^d$ by A is simply $x' = Ax + t$ for $t \in \mathbb{R}^d$.
Thus an ellipse $E_{A,t}$ is the set of points $x' \in \mathbb{R}^d$ such that $\{x' = Ax + t \mid x \in B\}$ where B is a unit ball.

An ellipsoid is not any "round" shape.
It is controlled by an orthonormal basis $U = [u_1, u_2, \ldots, u_d]$ where each $\|u_j\| = 1$ and each pair $u_j, u_{j'}$ are orthogonal so $\langle u_j, u_{j'} \rangle = 0$.
Then each j is associated with a scaling λ_j.
(indeed, these are the eigenvectors u_j and values λ_j of A)
For an ellipsoid E with those basis and scaling, and center t, then a point $x \in \mathbb{R}^d$ is in E if

$$\sum_{j=1}^{d} \langle u_j, x - t \rangle^2 / \lambda_j^2 \leq 1.$$

How well can we approximate a convex set K (e.g., $K = CH(X)$) with an ellipsoid?

Lowner-John's Ellipsoid theorem says for any convex set $K \subset \mathbb{R}^d$ there exists an ellipsoid E so

$$(1/d)E \subset K \subset E$$

This E is the minimum volume ellipsoid which circumscribes K. This is the best possible dilation ratio.

Making $CH(X)$ Fat

Compute Lowner-John Ellipsoid $E_{A,t}$ for $CH(X)$.
Invert data X by A^{-1}; that is

$$X' = \{x' = A^{-1}x \mid x \in X\}$$

Now X' is *d-fat*, this means that

$$\frac{\max_{u \in \mathbb{S}^{d-1}} wid_u(X')}{\min_{u \in \mathbb{S}^{d-1}} wid_u(X')} \leq d$$