01_01_Phugoid_Theory_sage

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```
%auto
typeset_mode(true)
```

1 01_01_Phugoid_Theory as SageMathCloud worksheet

Based on `numerical-mooc/lessons/01_phugoid/01_01_Phugoid_Theory.ipynb` from `MAE6286`.

Numbers in parentheses are equation numbers in the IPython notebook.


```
# (1) equation of lift

# L      lift force
# S      surface area
# rho    density of air
# v      forward velocity
# C_L    coefficient of lift

%var L,S,rho,v, C_L
eq_L = L == C_L * S * (1/2) * rho * v^2
eq_L
```

$$L = \frac{1}{2} C_L S \rho v^2$$

```
# (2) equation of drag


# D      drag force
# C_D    coefficient of drag

%var D,C_D
eq_D = D == C_D * S * (1/2) * rho * v^2
eq_D
```

$$D = \frac{1}{2} C_D S \rho v^2$$

```
# (3) equation of force perpendicular to the trajectory

# W     weight of airplane
# theta glide angle

%var W,theta
eq_fprp = L == W * cos(theta)
eq_fprp
```

$$L = W \cos(\theta)$$

```
# (3) equation of force parallel to the trajectory

eq_fpar = D == W * sin(theta)
eq_fpar
```

$$D = W \sin(\theta)$$

```
# (4) at trim velocity, lift matches weight

# v_t   trim velocity (when L==W)

%var v_t
eq_L2 = eq_L.subs(v = v_t,L = W)
eq_L2
```

$$W = \frac{1}{2} C_L S \rho v_t^2$$

```
# (5) lift ratio as function of flight velocity

eq_lr = eq_L / eq_L2
eq_lr
```

$$\frac{L}{W} = \frac{v^2}{v_t^2}$$

```
# (6) balance centripetal force from curve of plane's path and gravity

# g     acceleration of Earth's gravity
# R     radius of curvature of trajectory

%var g,R

eq_gbal = (L - W * cos(theta) == (W / g) * (v^2 / R)).add_to_both_sides(W\
    * cos(theta))
eq_gbal
```

$$L = W \cos(\theta) + \frac{W v^2}{R g}$$

```
# (7) phugoid equation in terms of velocity

eq_phv = (eq_gbal / W).subs_expr(eq_lr).factor().expand().\
   add_to_both_sides(- cos(theta))
```

```
eq_phv
```

$$\frac{v^2}{v_t^2} - \cos(\theta) = \frac{v^2}{Rg}$$

```
# (8) simplify - no friction, lift does no work, total energy is constant
# also set zero energy point at reference horizontal (z == 0)

# z     depth of plane below reference horizontal

%var z,z_t

eq_ze = (1/2) * v^2 - g * z == 0
eq_ze
eq_zt = eq_ze.subs(v = v_t, z = z_t)
eq_zt
```

$$\frac{1}{2}\,v^2 - gz = 0$$

$$\frac{1}{2}\,v_t^2 - gz_t = 0$$

```
# rearrange z equation

eq_ze2 = eq_ze.solve(v^2)[0]
eq_ze2
```

$$v^2 = 2\,gz$$

```
# rearrange z_t equation

eq_zt2 = eq_zt.solve(v_t^2)[0]
eq_zt2
```

$$v_t^2 = 2\,gz_t$$

```
# rewrite phugoid equation in terms of z, step 1

eq_p2 = eq_phv.subs_expr(eq_ze2)
eq_p2
```

$$\frac{2\,gz}{v_t^2} - \cos(\theta) = \frac{2\,z}{R}$$

```
# (9) rewrite phugoid equation in terms of z, step 2

eq_phz = (eq_p2 * eq_zt2).expand().subs_expr(eq_zt2).multiply_both_sides\
   (1/(2*g*z_t)).expand()
eq_phz
```

$$\frac{z}{z_t} - \cos(\theta) = \frac{2\,z}{R}$$

```
# derivatives - sage notation is preceded by LaTeX version

# (10a) derivative of theta with respect to s, with chain rule

%var s,R
z = function('z')
theta = function('theta')

show('1/R=\\frac{d{\\theta}}{ds}=\\frac{d{\\theta}}{dz}\\frac{dz}{ds}')
eq10a = 1/R == (theta(z(s))).derivative(s)
eq10a
```

$$1/R = \frac{d\theta}{ds} = \frac{d\theta}{dz}\frac{dz}{ds}$$

$$\frac{1}{R} = D[0]\left(\theta\right)\left(z\left(s\right)\right)D[0]\left(z\right)\left(s\right)$$

```
# (10b) derivative of z with respect to s

show('\\frac{dz}{ds}=-sin{\\theta(s)}')
eq10b = z(s).derivative(s) == -sin(theta(s))
eq10b
```

$$\frac{dz}{ds} = -sin\theta(s)$$

$$D[0]\left(z\right)\left(s\right) = -\sin\left(\theta\left(s\right)\right)$$

```
# (11) combine previous two steps for derivative of theta with respect to\
    z

show('1/R=-sin{\\theta}\\frac{d{\\theta}}{dz}')
eq11 = eq10a.subs(eq10b)
eq11
```

$$1/R = -sin\theta\frac{d\theta}{dz}$$

$$\frac{1}{R} = -\sin\left(\theta\left(s\right)\right)D[0]\left(\theta\right)\left(z\left(s\right)\right)$$

```
# (12) multiply phugoid equation (9) by 1/(2*sqrt(z))

# z and theta are now functions

eq_phz
eq_phzf = eq_phz.subs(z=z(s),theta=theta(z(s)))
eq_phzf
eq_phzf2 = eq_phzf.multiply_both_sides(1/(2*sqrt(z(s)))).simplify_radical\
   ().simplify()
eq_phzf2
```

$$\frac{z}{z_t} - \cos\left(\theta\right) = \frac{2\,z}{R}$$

$$\frac{z\left(s\right)}{z_t} - \cos\left(\theta\left(z\left(s\right)\right)\right) = \frac{2\,z\left(s\right)}{R}$$

$$-\frac{z_t \cos\left(\theta\left(z\left(s\right)\right)\right) - z\left(s\right)}{2\, z_t \sqrt{z\left(s\right)}} = \frac{\sqrt{z\left(s\right)}}{R}$$

```
# (13) substitute for 1/R in (12)

eq_phzf3 = eq_phzf2.expand().add_to_both_sides(cos(theta(z(s)))/(2*sqrt(z\
    (s))))
eq_phzf3
eq_phzf4 = eq_phzf3.subs(eq11)
eq_phzf4
```

$$\frac{\sqrt{z\left(s\right)}}{2\, z_t} = \frac{\cos\left(\theta\left(z\left(s\right)\right)\right)}{2\, \sqrt{z\left(s\right)}} + \frac{\sqrt{z\left(s\right)}}{R}$$

$$\frac{\sqrt{z\left(s\right)}}{2\, z_t} = -\sin\left(\theta\left(s\right)\right) \sqrt{z\left(s\right)} D[0]\left(\theta\right)\left(z\left(s\right)\right) + \frac{\cos\left(\theta\left(z\left(s\right)\right)\right)}{2\, \sqrt{z\left(s\right)}}$$

```
# text uses "z" for both a variable and a function of s
# we use two symbols for those in Sage
# "zeta" for the variable and "z" for the function of s


%var zeta

eq_phzf5 = eq_phzf4.substitute(z(s)== zeta)
eq_phzf5
```

$$\frac{\sqrt{\zeta}}{2\, z_t} = -\sqrt{\zeta} \sin\left(\theta\left(s\right)\right) D[0]\left(\theta\right)\left(\zeta\right) + \frac{\cos\left(\theta\left(\zeta\right)\right)}{2\, \sqrt{\zeta}}$$

```
# (14) rhs of previous equation (eq_phzf5) is an exact derivative!

h(zeta) = sqrt(zeta) * cos(theta(zeta))
h(zeta)
h.derivative(zeta)
```

$$\sqrt{\zeta} \cos\left(\theta\left(\zeta\right)\right)$$

$$\zeta \mapsto -\sqrt{\zeta} \sin\left(\theta\left(\zeta\right)\right) D[0]\left(\theta\right)\left(\zeta\right) + \frac{\cos\left(\theta\left(\zeta\right)\right)}{2\, \sqrt{\zeta}}$$

```
# to be continued
```