# Linear Algebra & Convex Optimization Review

CS 534: Machine Learning

Slides adapted from Lee Cooper

# Linear Algebra

# Notation

- Vector: $\mathbf{x} \in \mathbb{R}^n$

Hastie et al. book notation

$$\mathbf{x} = X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Matrix: $\mathbf{A} \in \mathbb{R}^{m \times n}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix}$$

# Special Matrices

- Identity Matrix:

$$\mathbf{I} \in \mathbb{R}^{n \times n}, \text{ where } I_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\mathbf{AI} = \mathbf{A} = \mathbf{IA}$$

- Diagonal Matrix:

$$\mathbf{D} = \text{diag}(d_1, d_2, \cdots, d_n) \text{ with } D_{ij} = \begin{cases} d_i, \ i = j \\ 0, \ i \neq j \end{cases}$$

# Matrix Multiplication

$$\text{If } \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{n \times p},$$

$$\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times p}, \text{ where } C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

- Properties
  - Associative

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

  - Distributive

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

Generally not commutative so AB =/= BA

# Transpose

- "Flip" rows and columns of a matrix

$$(A^\top)_{ij} = A_{ji}$$

- Properties
  - $(\mathbf{A}^\top)^\top = A$
  - $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$
  - $(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$

# Trace

- Sum of the diagonal elements in a square matrix

$$\mathrm{Tr}(\mathbf{A}) = \sum_{i=1}^{n} A_{ii}$$

- Properties
  - $\mathrm{Tr}(\mathbf{A}) = \mathrm{Tr}(\mathbf{A}^\top)$
  - $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n},\ \mathrm{Tr}(\mathbf{A} + \mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B})$
  - $\mathbf{A} \in \mathbb{R}^{n \times n},\ t \in \mathbb{R},\ \mathrm{Tr}(t\mathbf{A}) = t\mathrm{Tr}(\mathbf{A})$

  $\mathbf{AB} \in \mathbb{R}^{n \times n},\ \mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$

# Norms

- Norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 4 properties:
  - Non-negativity

$$\text{For all } \mathbf{x} \in \mathbb{R}^n, f(\mathbf{x}) \geq 0$$

  - Definiteness

$$f(\mathbf{x}) = 0 \text{ if and only if } \mathbf{x} = 0$$

  - Homogeneity

$$\text{For all } \mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}, f(t\mathbf{x}) = |t| f(x)$$

  - Triangle Inequality

$$\text{For all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$$

# Common Vector Norms

- Euclidean ($\ell_2$) norm

$$||\mathbf{x}||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

- $\ell_1$ norm

$$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i|$$

- $\ell_\infty$ norm

$$||\mathbf{x}||_\infty = \max_{x_i} |x_i|$$

- $\ell_p$ norm

$$||\mathbf{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

# Common Matrix Norms

- Frobenius norm

$$||\mathbf{A}||_F = \sqrt{\sum_{ij} |A_{ij}|^2} = \sqrt{\mathrm{Tr}(\mathbf{A}^\top \mathbf{A})}$$

- 1-norm

$$||\mathbf{A}||_1 = \max_j \sum_i |A_{ij}|$$

- 2-norm

$$||\mathbf{A}||_2 = \sqrt{\max \mathrm{eig}(\mathbf{A}^\top \mathbf{A})}$$

- p-norm

$$||\mathbf{A}||_p = (\max_{||\mathbf{x}||_p = 1} ||\mathbf{A}\mathbf{x}||_p)^{1/p}$$

# Linear Independence

- Set of vectors are linearly independent if no vector can be represented as a linear combination of the remaining vectors

- Linearly dependent vector:

$$\mathbf{x}_n = \sum_{i=1}^{n-1} \alpha_i \mathbf{x}_i$$

# Rank

- Column rank: size of largest subset of columns of A such that constitute a linearly dependent set
- Row rank: largest number of rows of A that constitute a linearly independent set
- For any matrix in real space, column rank = row rank

# Rank Properties

$\mathbf{A} \in \mathbb{R}^{m \times n}$

- Rank vs dimension

$$\operatorname{rank}(\mathbf{A}) \leq \min(m, n)$$

- Full rank

$$\operatorname{rank}(\mathbf{A}) = \min(m, n)$$

- Rank of transpose

$$\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^\top)$$

# Rank Properties (2)

- Multiplication of two matrices

$$\mathrm{rank}(\mathbf{AB}) \leq \min(\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{B}))$$

- Addition of two same sized matrices

$$\mathrm{rank}(\mathbf{A} + \mathbf{B}) \leq \mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\mathbf{B})$$

# Matrix Inverse

- Unique matrix such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

- A is invertible and non-singular if inverse exists
- A is singular if not invertible
- A must be full rank to have an inverse

# Matrix Inverse Properties

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$

# Pseudo Inverse (Moore-Penrose)

- Generalization of inverse for non-square but full rank
- Criteria:
  - $\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}$
  - $\mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}$
  - $(\mathbf{A}\mathbf{A}^{\dagger})^{\top} = \mathbf{A}\mathbf{A}^{\dagger}$
  - $(\mathbf{A}^{\dagger}\mathbf{A})^{\top} = \mathbf{A}^{\dagger}\mathbf{A}$

# Orthogonal Matrices

- Orthogonal vectors x, y:

$$\mathbf{x}^\top \mathbf{y} = 0$$

- Normalized vector:

$$||\mathbf{x}||_2 = 1$$

- Orthogonal square matrix if all columns are orthogonal to one another

- Orthonormal square matrix if orthogonal matrix and all columns are normalized

# Orthogonal Properties

- Inverse of orthogonal matrix is its transpose

$$\mathbf{U}^\top \mathbf{U} = \mathbf{I} = \mathbf{U}\mathbf{U}^\top$$

- Vector operation will not change its Euclidean norm

$$||\mathbf{U}\mathbf{x}||_2 = ||\mathbf{x}||_2$$

# Range and Nullspace

- Span of a set of vectors is all the vectors that can expressed as linear combination of these vectors

$$\mathrm{span}(\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}) = \left\{\mathbf{v} : \mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i\right\}$$

- Range (columnspace) is the span of the columns of the matrix

$$\mathcal{R}(\mathbf{A}) = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} = \mathbf{A}\mathbf{x}, \mathbf{x} \in \mathbb{R}^n\}$$

- Nullspace is the set of all vectors that equal 0 when multiple by matrix

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = 0\}$$

# Fundamental Subspaces

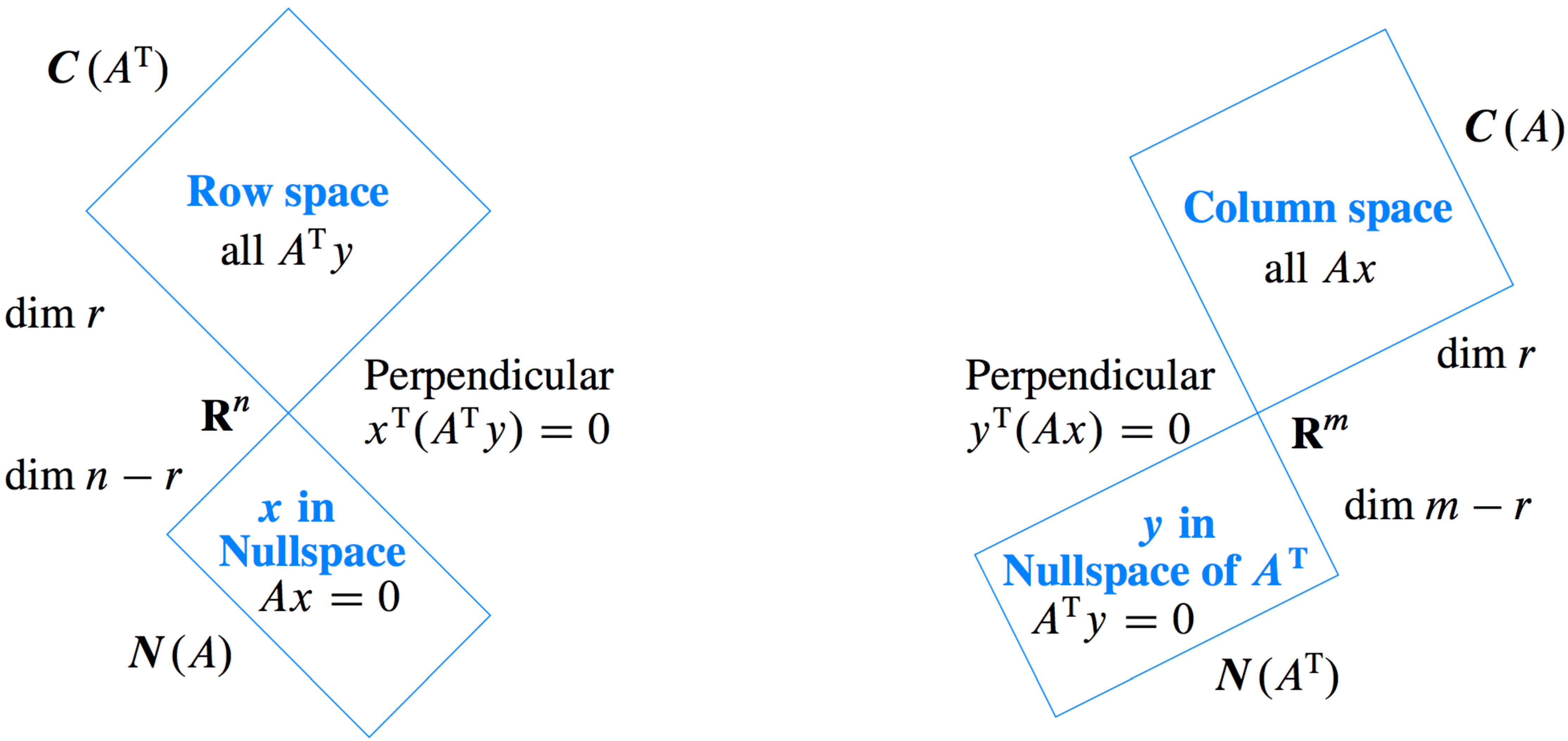



Figure 1: Dimensions and orthogonality for any $m$ by $n$ matrix $A$ of rank $r$.

http://web.mit.edu/18.06/www/Essays/newpaper_ver3.pdf

# Eigenvalues and Eigenvectors

- Instrumental to systems

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

- Analogy: Matrix is a gust of wind (invisible force with visible result)
  - Eigenvector is like a weathervane which tells you the direction the wind is blowing in
  - Eigenvalue is just the scalar coefficient

https://deeplearning4j.org/eigenvector

# Eigenvalue Properties

- Trace of a matrix is sum of its eigenvalues

$$\mathrm{Tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$$

- Determinant of matrix is equal to product of its eigenvalues

$$|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$$

- Rank of matrix is the number of non-zero eigenvalues

- If eigenvectors of matrix are linearly independent, then the matrix is invertible

$$\mathbf{A} = \mathbf{X}\Lambda\mathbf{X}^{-1}$$

# Symmetric Matrix & Eigenvectors

- Two remarkable properties from a symmetric matrix
    - Eigenvalues of the matrix are real
    - Eigenvectors of the matrix are orthonormal

$$\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^\top$$

- Eigenvalues are positive —> positive definite
- Eigenvalues are non-negative —> positive semidefinite

# Convex Optimization Review

# Optimization Problem

- Minimize a function subject to some constraints

$$\begin{aligned}\min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_k(x) \leq 0, k = 1, 2, \cdots, K \\ & h_j(x) = 0, j = 1, 2, \cdots, J\end{aligned}$$

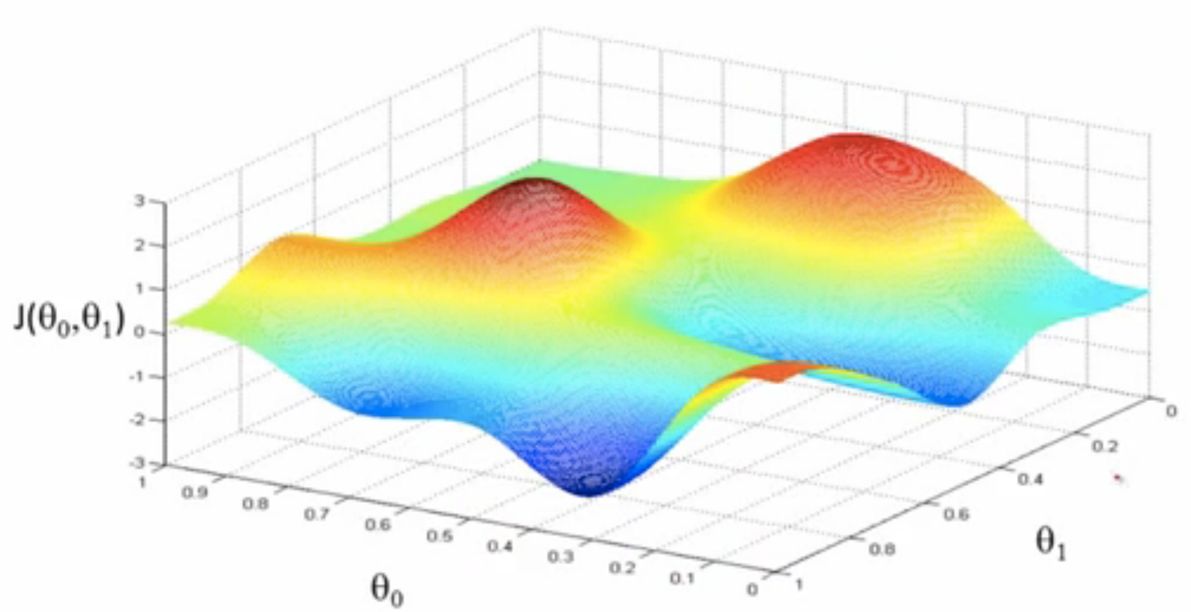
- Example: Minimize the variance of your returns while earning at least $100 in the stock market.

# Machine Learning and Optimization

- Linear regression $$\min_w \; ||Xw - y||^2$$
- Logistic regression $$\min_w \; \sum_i \log(1 + \exp(-y_i x_i^\top w))$$
- SVM $$\min_w \; ||w||^2 + C \sum_i \xi_i$$ $$\text{s.t. } \xi_i \geq 1 - y_i x_i^\top w$$ $$\xi_i \geq 0$$
- And many more ...

# Non-Convex Problems are Everywhere

- Local (non-global) minima
- All kinds of constraints




No easy solution for these problems

Consider convex problems

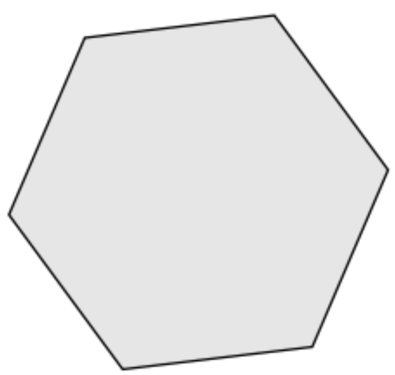
# Why Convex Optimization?

- Achieves global minimum, no local traps
- Highly efficient software available
- Can be solved by polynomial time complexity algorithms
- Dividing line between “easy” and “difficult” problems

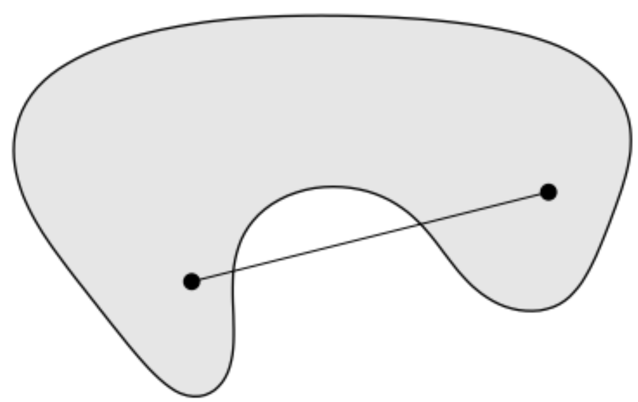
# Convex Sets

Any line segment joining any two elements lies entirely in set

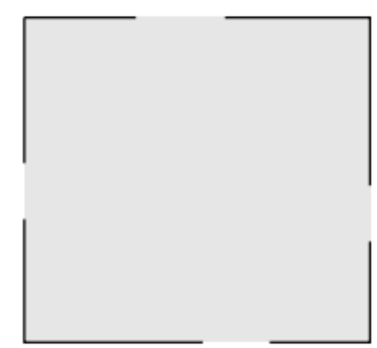
$$x_1, x_2 \in C, 0 \leq \theta \leq 1 \implies \theta x_1 + (1-\theta)x_2 \in C$$



convex



non-convex



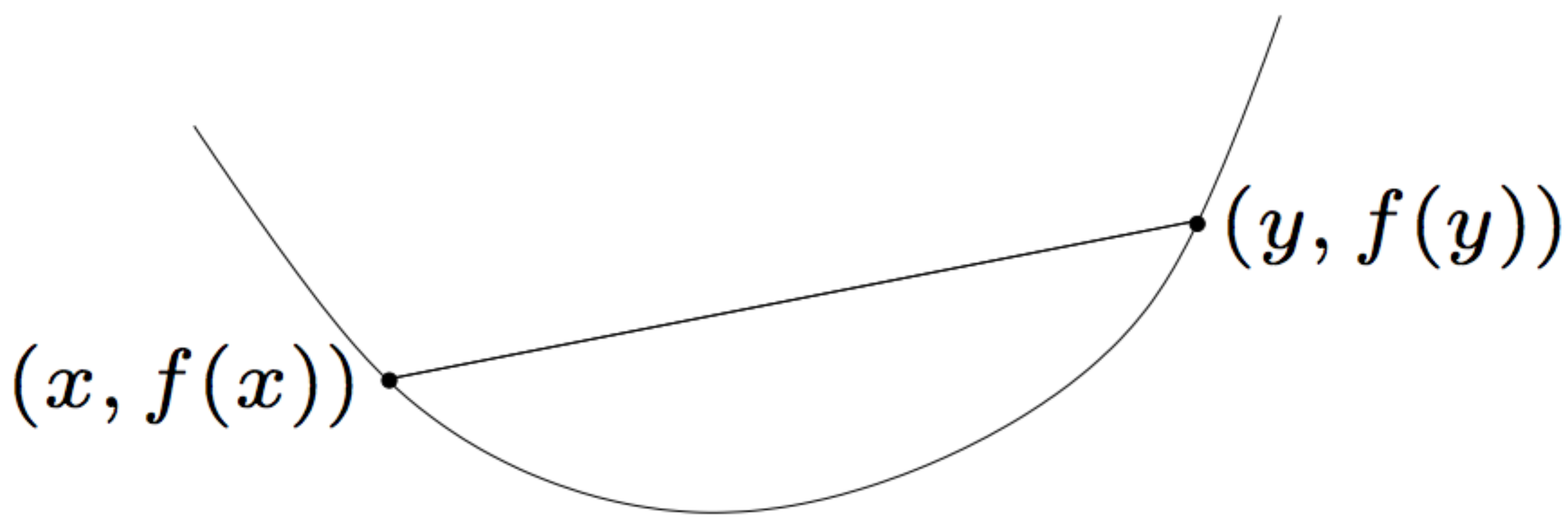
non-convex

# Convex Function

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathbf{dom}\ f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\ f, 0 \leq \theta \leq 1$




f lies below the line segment joining f(x), f(y)

# Properties of Convex Functions

- Convexity over all lines

$$f(x) \text{ is convex } \implies f(x_0 + th) \text{ is convex in } t \text{ for all } x_0, h$$

- Positive multiple

$$f(x) \text{ is convex } \implies \alpha f(x) \text{ is convex for all } \alpha \geq 0$$

- Sum of convex functions

$$f_1(x), f_2(x) \text{ convex } \implies f_1(x) + f_2(x) \text{ is convex}$$

- Pointwise maximum

$$f_1(x), f_2(x) \text{ convex } \implies \max\{f_1(x), f_2(x)\} \text{ is convex}$$

- Affine transformation of domain

$$f(x) \text{ is convex } \implies f(Ax + b) \text{ is convex}$$

# Convex Optimization Problem

Definition:
An optimization problem is **convex** if its objective is a convex function, the inequality constraints are convex, and the equality constraints are affine

| | |
|---|---|
| $\min_{x} \; f_0(x)$ | convex function |
| $\text{s.t.} \; f_k(x) \leq 0, k = 1, 2, \cdots, K$ | convex sets |
| $h_j(x) = 0, j = 1, 2, \cdots, J$ | affine constraints |

# Benefits of Convexity

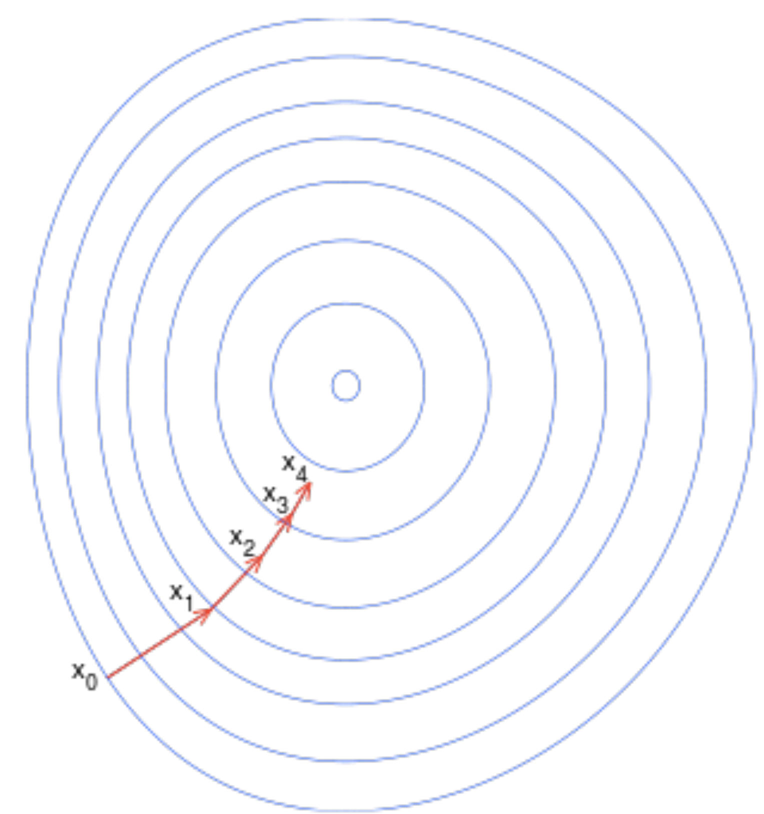
- Theorem: If x is a local minimizer of a convex optimization problem, it is a **global** minimizer

- Theorem: If the gradient at c is zero, then c is the global minimum of f(x)

$$\nabla f(c) = 0 \iff c = x^*$$

# Gradient Descent (Steepest Descent)

- Simplest and extremely popular
- Main Idea: take a step proportional to the negative of the gradient
- Easy to implement
- Each iteration is relatively cheap
- Can be slow to converge



# Gradient Descent Algorithm

**Algorithm 1:** Gradient Descent

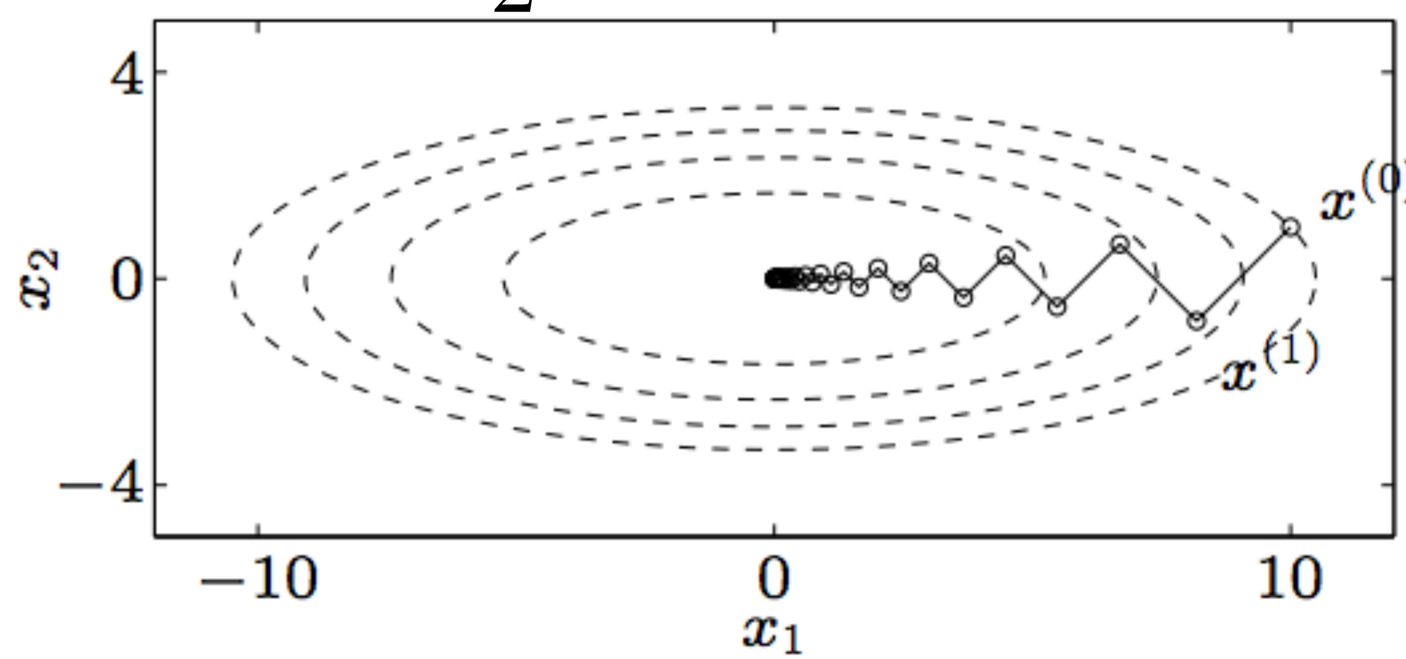
**while** *Not Converged* **do**

$\quad x^{(k+1)} = x^{(k)} - \eta^{(k)} \nabla f(x)$

**end**

return $x^{(k+1)}$

Example: $f(x) = \frac{1}{2}(x_1^2 + \gamma x_2^2),\ \gamma > 0$



# Some Resources for Convex Optimization

- Boyd & Landenberghe's Book on Convex Optimization
  https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf

- Stephen Boyd's Class at Stanford
  http://stanford.edu/class/ee364a/

- Vandenberghe's Class at UCLA
  http://www.seas.ucla.edu/~vandenbe/ee236b/ee236b.html

- Ben-Tai & Nemirovski Lectures on Modern Convex Optimization
  http://epubs.siam.org/doi/book/10.1137/1.9780898718829