ASSIGNMENT 2: HINTS

Let $A_i = \nabla^2 \ell_i(a,b)$ and let $A = \nabla^2 \ell(a,b) = \sum A_i$.

(1b) Show that A_i has two nonnegative eigenvalues.

(1c) Find a basic solution of the system

$$A_i \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(1d) An element of $N(A_i) \cap N(A_j)$ is a solution of the homogeneous system

$$\begin{bmatrix} A_i \\ A_j \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Use the fact that $x_i \neq x_j$ to deduce that this system has only the trivial solution.

(1e) Let $\boldsymbol{x}$ be such that $\boldsymbol{x}^T A \boldsymbol{x} = 0$. We need to show that $\boldsymbol{x} = 0$. Note that

$$\boldsymbol{x}^T A \boldsymbol{x} = \sum \boldsymbol{x}^T A_i \boldsymbol{x}.$$

The $\boldsymbol{x}^T A_i \boldsymbol{x}$ are nonnegative (why?), so $\boldsymbol{x}^T A \boldsymbol{x} = 0$ if and only if $\boldsymbol{x}^T A_i \boldsymbol{x} = 0$ for all i. By the Lemma below, this holds if and only if $\boldsymbol{x} \in N(A_i)$ for all i...

Lemma: If S is a positive semidefinite, symmetric matrix, then

$$\{\boldsymbol{y} : \boldsymbol{y}^T S \boldsymbol{y} = 0\} = N(S)$$

Proof: Clearly, $N(S) \subseteq \{\boldsymbol{y} : \boldsymbol{y}^T S \boldsymbol{y} = 0\}$. Conversely, suppose $\boldsymbol{y}^T S \boldsymbol{y} = 0$. Since S is symmetric and positive semidefinite, there is a matrix R of real numbers such that $S = R^T R$. (This follows from the the Spectral Theorem.) Then

$$0 = \boldsymbol{y}^T S \boldsymbol{y} = \boldsymbol{y}^T R^T R \boldsymbol{y} = (R\boldsymbol{y})^T (R\boldsymbol{y}) = \|R\boldsymbol{y}\|^2,$$

so $R\boldsymbol{y}$ must be zero. Thus, $\boldsymbol{y} \in N(R) \subseteq N(S)$.

(2a) Use the fact that $0 < \sigma(x) < 1$ for all x.

(2b) Show that

$$\lim_{t \to \infty} \ell_i(tv_1, tv_2) = \infty$$

if and only if

- $(v_1 + v_2 x_1) > 0$, if $y_i = 0$.
- $(v_1 + v_2 x_1) < 0$, if $y_i = 1$.

(2c) I'll accept some nicely drawn pictures illustrating what's going on here in lieu of a formal proof.

(2d) Suppose $y_i = y_k = 0$ and $y_j = 1$. By (b),

$$H_i = H(\boldsymbol{w}_i), \quad H_j = H(-\boldsymbol{w}_j), \quad \text{and} \quad H_k = H(\boldsymbol{w}_k),$$

where $\boldsymbol{w}_i = \begin{bmatrix} 1 & x_i \end{bmatrix}^T$. Show that $\boldsymbol{w}_j = a\boldsymbol{w}_i + b\boldsymbol{w}_k$ with $a, b > 0$. Then invoke (c).

(2e) Use (b).