Root Systems

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1 Root

Definition 1.1 (Root System)**.** A root system is a finite-dimensional real vector space V with an inner product $\langle \cdot, \cdot \rangle$, together with a finite collection R of nonzero vectors in V satisfying the following properties:
(i) The vectors in R span V.
(ii) $\alpha \in R \Longrightarrow -\alpha \in R$.
(iii) If $\alpha \in R$, then the only multiples of α in R are α and $-\alpha$.
(iv) $\alpha, \beta \in R \Longrightarrow w_\alpha \cdot \beta \in R$, where

$$w_\alpha \cdot \beta = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

(v) $\forall \alpha, \beta \in R$, $2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ is an integer.
$\dim V$ is called the rank of the root system and the elements of R is called roots.

There are some redundancy in this definition since $w_\alpha(\alpha) = -\alpha$, but our definition is more intuitive.

Theorem 1.1 (Weyl Group)**.** *Let (V, R) be a root system, then the Weyl group W of R is the subgroup of the orthogonal group of V, generated by w_α where $\alpha \in R$.*

Theorem 1.2 (Direct Sum)**.** *Suppose (V, R) and (W, S) are root systems. Consider the vector space $V \oplus W$, with the natural inner product. Then $R \cup S$ is a root system in $V \oplus W$, called the direct sum of V and W. Here we assume the natural identification.*

Definition 1.2 (Irreduciblility)**.** A root system (V, R) is called reducible if there exists an orthogonal decomposition $V = V_1 \oplus V_2$ with $\dim V_1 > 0$ and $\dim V_2 > 0$ s.t. every element of R is either in V_1 or in V_2. If no such decomposition exists, (V, R) is called irreducible.

Definition 1.3 (Equivalence)**.** Two root systems (V_1, R_1) and (V_2, R_2) are said to be equivalent if $\exists T : V_1 \to V_2$ which is an invertible linear transformation s.t. T maps R_1 onto R_2 and s.t. $\forall \alpha \in R_1, \beta \in V_1$, we have

$$T(w_\alpha \cdot \beta) = w_{T\alpha} \cdot T\beta.$$

A map T with this property is called an equivalence.

What are the root systems?

Theorem 1.3. *Suppose α, β are noncollinear roots and $\langle\alpha\rangle \geq \langle\beta\rangle$. Then one of the following holds:*
(i) $\langle\alpha, \beta\rangle = 0$
(ii) $\langle\alpha, \alpha\rangle = \langle\beta, \beta\rangle$ *and the angle between α and β is $\frac{\pi}{3}$ or $\frac{2\pi}{3}$*
(iii) $\langle\alpha, \alpha\rangle = 2\langle\beta, \beta\rangle$ *and the angle between α and β is $\frac{\pi}{4}$ or $\frac{3\pi}{4}$*
(iv) $\langle\alpha, \alpha\rangle = 3\langle\beta, \beta\rangle$ *and the angle between α and β is $\frac{\pi}{6}$ or $\frac{5\pi}{6}$*

Proof. Let θ be the angle between α and β.

$$4\frac{\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}\frac{\langle\beta, \alpha\rangle}{\langle\beta, \beta\rangle} = 4\cos^2\theta \in \mathbb{Z}$$

□

Corollary 1.3.1. *Suppose α, β are roots.*
(i) If the angle between α and β is strictly obtuse, then $\alpha + \beta$ is a root.
(ii) If the angle between α and β is strictly acute, then $\alpha - \beta$ is a root.

Proof. Discuss case by case. □

Now we discuss the dual of a root system.

Definition 1.4 (Co-Root)**.** If (V, R) is a root system, then for each root $\alpha \in R$, define the co-root H_α by

$$H_\alpha = 2\frac{\alpha}{\langle\alpha, \alpha\rangle}$$

The set of all co-roots is denoted $R^\vee$ and is called the dual root system to R.

This is actually an inversion with respect to the ball centered at the origin with radius $\sqrt{2}$.

Theorem 1.4 (Duality)**.** $(R^\vee)^\vee = R$

Proof. The dual system can be regarded as an inversion with respect to the ball centered at the origin with radius $\sqrt{2}$, and inversion is a kind of involution. □

Theorem 1.5 (Dual Root System)**.** *$R^\vee$ is a root system and its Weyl group is the same as the that of R.*

Next we construct the base if a root system.

Definition 1.5 (Base)**.** A subset Δ of R is called a base for R if the following conditions hold:
(i) Δ is a basis for V as a vector space.
(ii) Each root can be expressed as a linear combination of elements of Δ with integer coefficients and in such a way that the coefficients are either all non-negative or all nonpositive.

The roots for which the coefficients are non-negative are called positive roots and the others are called negative roots. The set of positive roots relative to a fixed base Δ is denoted R^+. The elements of Δ are called simple positive roots.

Lemma 1.1. *If α, β are distinct elements of a base, then $\langle\alpha, \beta\rangle \leq 0$.*

Proof. Use Corollary 1.3.1. □

Lemma 1.2. *If V is a finite-dimensional real vector space and R is a finite subset of V not containing 0, then there exists a hyperplane M that does not contain any element of R.*

Proof. The union of finite collection of hyperplanes can not be all of V. □

Definition 1.6 (Indecomposability)**.**

Theorem 1.6. *Suppose (R, V) is a root system, W is a hyperplane not containing any element of R, and R^+ is the set of roots lying on the fixed side of V. Then the set of indecomposable elements of R^+ is a base for R.*

Theorem 1.7. *Given any base Δ for R, there exists a hyperplane V s.t. Δ arises as in Theorem 1.6.*

Theorem 1.8 (Base for Dual Root System). *If Δ is a base for R, then the set of all co-roots H_α, $\alpha \in \Delta$, is a base for the dual root system $R^\vee$.*

Finally we explore the integral elements.

Definition 1.7 (Integral Element). An element $v \in V$ is called an integral element if for all $\alpha \in R$, the quantity

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is an integer.

Definition 1.8. If Δ is a base for R, then an integral element μ is called dominant integral if

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \geq 0, \quad \forall \alpha \in \Delta$$

It is called strictly dominant if the inequality is strict.

Remark. *An integral element is dominant if and only if it is contained in closed fundamental Weyl chamber, and strictly dominant if and only if contained in the open fundamental Weyl chamber.*

Lemma 1.3. *If $v \in V$ has the property that*

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is an integer for all $\alpha \in \Delta$, then v us an integral element.

Proof. Use Theorem 1.8. □

Definition 1.9 (Fundamental Weights).

Definition 1.10 (Higher and Lower).

2 Example: Rank 2

3 Example: Rank 3

4 Additional Properties

5 Application: Classical Lie Algebras

6 Dynkin Diagrams

The classification of root systems is given in terms of an object called the Dynkin diagram.

Suppose $\Delta = \{\alpha_1, \cdots, \alpha_r\}$ is a base for a root system R. Then the Dynkin diagram for R is a graph having vertices $v_1, \cdots, v_r$. Between any two vertices, we place either no edge, one edge, two edge, or three edge corresponding to the four cases in Theorem 1.3.

Theorem 6.1. *Every irreducible root system is isomorphic to precisely one root system from the following list:*
(i) A_n, $n \geq 1$
(ii) B_n, $n \geq 2$
(iii) C_n, $n \geq 3$
(iv) D_n, $n \geq 4$
(v) G_2, F_4, E_6, E_7, E_8

7 The Root Lattice