

Measure Theory

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Chapter 1

Real Analysis

1.1

Continuity of Translation

Theorem 1.1.1. Suppose $f \in L^1(\mathbb{R}^n)$. Then

$$\|f_h - f\|_{L^1} \rightarrow 0 \quad h \rightarrow 0.$$

Proof. The result is clearly true. The proof is a simple consequence of the approximation of integrable functions by continuous functions of compact support. \square

1.2 Differentiation and Integration

For conceptual simplicity, we study \mathbb{R} instead of \mathbb{R}^n in this section. Let us first recall what we learned in elementary calculus.

Theorem 1.2.1. Let f be a *continuous* function on $[a, b]$, and F be the function defined, for all x in $[a, b]$, by

$$F(x) = \int_a^x f(t) dt.$$

Then F is uniformly continuous on $[a, b]$ and differentiable on (a, b) , and

$$F'(x) = f(x).$$

Theorem 1.2.2 (Newton-Leibniz). Let f be a *Riemann integrable* function on $[a, b]$, and F a *continuous* function on $[a, b]$ which is an antiderivative of f in (a, b) :

$$F'(x) = f(x).$$

Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Definition 1.2.3 (Total Variation). The total variation of a function f defined on $[a, b]$ is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P-1} |f(x_{i+1}) - f(x_i)|,$$

where

$$\mathcal{P} = \{P = \{x_0, \dots, x_{n_P}\} | P \text{ is a partition of } [a, b] \text{ satisfying } x_i \leq x_{i+1} \text{ for } 0 \leq i \leq n_P - 1\}.$$

If $V_a^b(f) < +\infty$, then f is said to be of bounded variation on $[a, b]$.

Example 1.2.4. The continuous function

$$f(x) = \begin{cases} 0, & x = 0 \\ x \sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

is not of bounded variation on $[0, 2/\pi]$. Just consider the partition

$$P = \{0, \frac{2}{2n-1}, \frac{2}{2n-3}, \dots, \frac{2}{3}, 1\}.$$

Theorem 1.2.5 (Jordan Decomposition Theorem).

$$f(x) = g(x) - h(x)$$

where $g(x)$ and $h(x)$

Bounded variable functions says that it has a length

Bounded variable functions are a.e. differentiable But it does not satisfies the desirable property: Length is equal to the integral of derivative, which is also known as the **fundamental theorem of calculus**. It turns out that absolute continuous functions is exactly the set of functions that satisfies this property.

Example 1.2.6.

Definition 1.2.7 (Absolute Continuity). A function $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous if for every positive number ϵ , there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) satisfies

$$\sum_{k=1}^N (y_k - x_k) < \delta$$

then

$$\sum_{k=1}^N |f(y_k) - f(x_k)| < \epsilon.$$

Absolutely continuous functions are of bounded variation.

Proposition 1.2.8. *If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a, b]$.*

Theorem 1.2.9. *If f is absolute continuous function on $[a, b]$, then*

$$f(x) - f(a) = \int_a^x f'(t)dt, \quad x \in [a, b].$$

It is beneficial to compare it with the “classical” version (Theorem 1.2.2).

Chapter 2

Abstract Measure Theory

2.1

In this section, we are working on a set E .

Definition 2.1.1 (σ -algebra).

Remark 2.1.2. Greek letters σ and δ are often used when countable unions and countable intersections are involved. For example, topologists call F_σ every countable union of closed sets in a topological space (F standing possibly for the French word fermé, closed) and G_δ every countable intersection of open sets (G standing for the German word Gebiet, domain, connected open set). The letters σ and δ are often given as Greek abbreviations of German words: σ as S in Summe for sum (in the sense of sum of sets, that is, union) and δ as D in Durchschnitt for intersection, both countable. Thus, in the context of measure theory, the letter σ refers to the stability of a collection of subsets by countable union.

Definition 2.1.3 (generator of σ -algebra).

$$\sigma(\mathcal{A}) = \{A \subset E : A \in \mathcal{E} \quad \forall \mathcal{E} \supset \mathcal{A}\}$$

Example 2.1.4. Borel σ -algebra

Definition 2.1.5 (π -system). \mathcal{A} is a collection of subsets of E . Then \mathcal{A} is called a π -system if

1. $\emptyset \in \mathcal{A}$;
2. $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Definition 2.1.6 (d -system). \mathcal{A} is a collection of subsets

1. $E \in \mathcal{A}$;
2. $A, B \in \mathcal{A}$, $A \subset B$, then $B \setminus A \in \mathcal{A}$;
3. If $A_n \subset \mathcal{A}$ such that $A_n \subset A_{n+1}$, then $\cup_n A_n \in \mathcal{A}$.

Remark 2.1.7. d -system is also referred as λ -system.

Proposition 2.1.8. \mathcal{A} is a σ -algebra if and only if it is a π -system and a d -system.

Proof. A σ -algebra is a π system because

Conversely,

□

Lemma 2.1.9 (Dynkin's π -system lemma). Let \mathcal{A} be a π -system. Then any d -system that contains \mathcal{A} also contains $\sigma(\mathcal{A})$.

This lemma can reduce the problem of studying a σ -algebra to the study of a π -system.

Proof. Let \mathcal{D} be the intersection of all d -system containing \mathcal{A} . We now prove that \mathcal{D} is a σ -algebra. As \mathcal{D} is already a d -system, by Proposition 2.1.8, we only need to prove that \mathcal{D} is a π -system.

(i) First we show the following property for \mathcal{D} : If $B \in \mathcal{D}$ and $A \in \mathcal{A}$, then $B \cap A \in \mathcal{D}$. Let

$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{A}\}.$$

To show that $\mathcal{D}' = \mathcal{D}$, we only need to check that \mathcal{D}' is a d -system containing \mathcal{A} . As \mathcal{A} is a π -system, $\mathcal{D}' \supset \mathcal{A}$. Now, we can check

- $E \in \mathcal{D}'$; this is because $\mathcal{A} \subset \mathcal{D}$.
 - $B_1, B_2 \in \mathcal{D}'$, $B_1 \subset B_2$, then $B_2 \setminus B_1 \in \mathcal{D}'$; this is because for any $A \in \mathcal{A}$, $(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus (B_1 \cap A) \in \mathcal{D}$.
 - If $B_n \in \mathcal{D}'$ such that $B_n \subset B_{n+1}$, then $\cup_n B_n \in \mathcal{D}'$; this is because for any $A \in \mathcal{A}$, $(\cup_n B_n) \cap A = \cup_n (B_n \cap A) \in \mathcal{D}$.
- (ii) If $A, B \in \mathcal{D}$, then $B \cap A \in \mathcal{D}$. Let

$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{D}\}.$$

One can see that \mathcal{D}'' is a d -system by noting that the argument holds for any $\mathcal{A} \subset \mathcal{D}$. The fact that \mathcal{D}'' contains \mathcal{A} follows from property (i). Therefore, $\mathcal{D}'' = \mathcal{D}$ and \mathcal{D} is a π -system. \square

2.2 Caratheodory Extension Theorem

Definition 2.2.1 (set function). \mathcal{A} be a collection of subsets of E with $\emptyset \in \mathcal{A}$. A set function $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a function such that $\mu(\emptyset) = 0$.

Definition 2.2.2 (increasing set function). $A \subset B$, we have $\mu(A) \leq \mu(B)$.

Definition 2.2.3 (Additive Set Function). A set function is additive if whenever $A, B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$, $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Definition 2.2.4 (Countably Additive Set Function). A set function is countably additive if whenever $(A_n) \subset \mathcal{A}$ such that $A_n \cap A_m = \emptyset$ for all $n \neq m$ and $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$, then $\mu(\cup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$.

Definition 2.2.5 (Countably subadditive Set Function). A set function is countably subadditive if whenever $(A_n) \subset \mathcal{A}$ such that $A_n \cap A_m = \emptyset$ for all n, m and $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$, then $\mu(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$.

Definition 2.2.6 (ring). A collection of subsets \mathcal{A} of a subset E is a ring if

1. $\emptyset \in \mathcal{A}$
2. $\forall A, B \in \mathcal{A}$, we have $B \setminus A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

Definition 2.2.7 (algebra). A collection of subsets \mathcal{A} of a subset E is an algebra if

1. $\emptyset \in \mathcal{A}$
2. $\forall A, B \in \mathcal{A}$, we have $A^c \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

Theorem 2.2.8 (Caratheodory Extension Theorem). Let \mathcal{A} be a ring on E , and μ be a countably additive set function on \mathcal{A} . Then μ extends to a measure on $\sigma(\mathcal{A})$.

Theorem 2.2.9. Suppose that μ_1, μ_2 are measures on (E, \mathcal{E}) with $\mu_1(E) = \mu_2(E) < \infty$. If \mathcal{A} is a π -system with $\sigma(\mathcal{A}) = \mathcal{E}$ and $\mu_1|_{\mathcal{A}} = \mu_2|_{\mathcal{A}}$, then $\mu_1 = \mu_2$.

Proof. Let $\mathcal{D} = \{A \in \mathcal{E} : \mu_1(A) = \mu_2(A)\}$, then $\mathcal{D} \supset \mathcal{A}$. By Dynkin's lemma, it suffices to show that \mathcal{D} is a d -system. We can check

- $E \in \mathcal{D}$.
- Let $A, B \in \mathcal{D}$ with $A \subset B$. (Note that finiteness is important here.)
-

\square

Example 2.2.10. Let $E = \mathbb{Z}$, $\mathcal{E} =$. let $\mathcal{A} = \{\{x, x+2, x+2, \dots\} : x \in E\} \cup \{\emptyset\}$. \mathcal{A} is a π -system such that $\mathcal{E} = \sigma(\mathcal{A})$. Let $\mu_1(A)$ be the number of elements of A and $\mu_2 = 2\mu_1$.

Definition 2.2.11 (Borel σ -algebra). Let E be a topological space. $\mathcal{B}(E) = \sigma(\{U \subset E : U \text{ open}\})$.

Definition 2.2.12 (borel and Radon Measure). A measure μ on $(E, \mathcal{B}(E))$ is called a Borel measure. If $\mu(K) < \infty$ for all $K \subset E$ compact, then μ is called a Radon measure.

2.3 Lebesgue Measure

Theorem 2.3.1. *There exists a unique Borel measure μ on $\mathcal{B}(\mathbb{R})$ with $\mu([a, b]) = b - a$.*

Remark 2.3.2. This measure is called the Lebesgue measure.

Proof. Uniqueness. Suppose $\tilde{\mu}$ is another measure. For any $n \in \mathbb{Z}$, we set $\mu_n(A) = \mu(A \cap (n+1, n])$. The previous theorem implies that $\mu_n = \tilde{\mu}_n$ for all $n \in \mathbb{Z}$. Fix $A \in \mathcal{B}(\mathbb{R})$, $A = \cup_{n \in \mathbb{Z}} (A \cap (n, n+1])$, $\mu(A) = \sum_{n \in \mathbb{Z}} \mu_n(A) = \sum_{n \in \mathbb{Z}} \tilde{\mu}_n(A) = \tilde{\mu}(A)$.

Existence. \mathcal{A} be the collection of finite, disjoint unions of the following form $A = \cup_{i \in \mathbb{Z}} (a_i, b_i]$. \mathcal{A} is a ring on \mathbb{R} and $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. Define $\mu(A) = \sum_{i=1}^n (b_i - a_i)$. It is well defined and additive. Suppose $A = \cup_{n=1}^{\infty} A_n \in \mathcal{A}$, $\mu(A) = \sum_{i=1}^n \mu(A_i) + \mu(A \setminus \sum_{j=1}^n A_j)$. Let $B_n = A \setminus \sum_{j=1}^n A_j$.

Argue by contradiction, $\exists \epsilon > 0, n_0 \in \mathbb{N}$ such that $\mu(B_n) \geq 2\epsilon \forall n \geq n_0$. $\forall n \in \mathbb{N}$, $\exists C_n \in \mathcal{A}$ such that $\overline{C_n} \subset B_n$ and $\mu(B_n \setminus C_n) \leq \frac{\epsilon}{2^n}$. Then we have $\mu(B_n) - \mu(\cap_{m=1}^n \overline{C_m}) = \mu(B_n \setminus \cap_{m=1}^n C_m) \leq \mu(\cup_{m=1}^n (B_m \setminus C_m)) \leq \sum_{m=1}^n \mu(B_m \setminus C_m) \leq \epsilon$. The finite intersection property \square

Lemma 2.3.3 (Finite intersection property of compact sets of \mathbb{R}). *If*

Definition 2.3.4 (σ -finite measure). Let (E, \mathcal{E}) be a measurable space, and μ a measure on \mathcal{E} . We say that μ is σ -finite if $E = \cup_{n \in \mathbb{N}} E_n, E_n \in \mathcal{E}, \mu(E_n) < \infty \forall n$.

It is easy to see that Lebesgue measure is translation invariant using the above theorem. Just take

Theorem 2.3.5. *Let $\tilde{\mu}$ be a Borel measure on \mathbb{R} that is translation invariant and $\tilde{\mu}([0, 1]) = 1$. Then $\tilde{\mu}$ is the Lebesgue measure.*

Proof. First show no singleton. $\tilde{\mu}(\{a\}) \leq \tilde{\mu}([a, a + \frac{1}{n}]) = \tilde{\mu}([0, \frac{1}{n}]) \leq \frac{1}{n}$.

Can find $p_n, q_n \in \mathbb{Q}$ such that $p_n \downarrow a, q_n \uparrow b$ as $n \rightarrow \infty$. \square

2.4 Probability Measure

Definition 2.4.1. Let (E, \mathcal{E}) be a measure space with the property $\mu(E) = 1$. Then μ is called a probability measure and (E, \mathcal{E}, μ) a probability space.

2.5 Radon Measure

Every Borel measure on \mathbb{R}^n is regular and locally finite, which is Radon.