Lecture Notes in Stochastic Process

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Contents

1 Discrete Space Discrete Time Markov Chain

1.1 Basic Theory

We begin by summarizing the concepts mathematicans are most interested in, and introducing their notations.

Definition 1.1 (initial distribution)**.**

Definition 1.2 (transition matrix)**.**

Remark. *Transition matrix provides a way to calculate the probability that after n steps the Markov chain is in a given state.*

Next we explain the markov property.

Definition 1.3 (markov property)**.**

Definition 1.4 (strong markov property)**.**

Next we explain the class structure of markov chain.

Definition 1.5 (lead to)**.** We say i leads to j and write $i \to j$ if $P_i(X_n = j \text{ for some } n \geqslant 0) > 0$.

Definition 1.6 (communicate with)**.** We say i communicate with j and write $i \leftrightarrow j$ if both $i \to j$ and $j \to i$.

Theorem 1.1 (communicating classes)**.** *Communication is an equivalence relation on I, thus partition I into communicating classes.*

Definition 1.7 (closed class)**.** We say a class C is closed if

$$i \in C, i \to j \Longrightarrow j \in C$$

A state i is called absorbing if $\{i\}$ is a closed class.

Definition 1.8 (irreducibility)**.** A markov chain with only one class is called irreducible.

1.2 Recurrence and Transience

A markov chain starting from a state can visit other state. We are interested in the timing of a visit.

Definition 1.9 (hitting time)**.** Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix P. The hitting time of a subset A of I is the random variable

$$H^A(\omega) = \inf\{n \geqslant 0 : X_n(\omega) \in A\}$$

.

Definition 1.10 (first passage time)**.** The first passage time to state i is the random variable T_i defined by

$$T_i(\omega) = \inf\{n \geqslant 1 : X_n(\omega) = i\}$$

Definition 1.11 (rth passage time)**.** We define rth passage time inductively by $T_i^{(0)}(\omega) = 0, T_i^{(1)}(\omega) = T_i(\omega)$ and for $r = 0, 1, 2, \cdots$,

$$T_i^{(r+1)}(\omega) = \inf\left\{n \geqslant T_i^{(r)}(\omega) + 1 : X_n(\omega) = i\right\}$$

A state can never be visited as well.

Definition 1.12 (absorption probability)**.** The probability starting from i that $(X_n)_{n\geqslant 0}$ ever hits A is then

$$h_i^A = P_i(H^A < \infty)$$

When A is a closed class, h_i^A is called the absorption probability.

Remark. *A less formal notation is* $h_i^A = \mathbb{P}(hit\ A)$*.*

Absorption probability can be calculated by first-step analysis.

Definition 1.13. The mean time taken for $(X_n)_{n\geqslant 0}$ to reach A from i is given by

$$k_i^A = E_i(H^A) = \sum_{n<\infty} nP(H^A = n) + \infty\mathbb{P}(H^A = \infty)$$

Remark. *A less formal notation is* $k_i^A = E_i(time\ to\ hit\ A)$*.*

We can classify the states into three categories according to whether a state is visited i.o. by a markov chain.

Definition 1.14 (recurrent)**.** We say a state i is recurrent if $P_i(X_n = i \text{ i.o. }) = 1$.

Definition 1.15 (transient)**.** We say a state i is transient if $P_i(X_n = i \text{ i.o. }) = 0$.

Definition 1.16 (positive recurrent)**.** A state i is postive recurrent if the expected return time $m_i = E_i(T_i)$ is finite. A recurrent state which fails to have this stronger property is called null recurrent.

The next result is immediate once we combine the strong markov property and the property of the geometric distribution.

Theorem 1.2. *y is recurrent if and only if $\mathbb{E}_y N(y) = \infty$.*

This is also called Green's function.
The next fact help us identify recurrent state when the state space is finite.

Theorem 1.3. *Let C be a finite closed set. Then C contains a recurrent state. If C is irreducible then all states in C are recurrent.*

Theorem 1.4 (Decomposition theorem)**.**

1.3 Invariant Measures

Definition 1.17 (invariant measure)**.** A measure is any row vector $(\lambda_i : i \in I)$ with non-negative entries. We say λ is invariant if

$$\lambda P = \lambda$$

Remark. *Invariant measure is also called stationary measure.*

Definition 1.18 (detailed balance)**.**

Definition 1.19 (reversible measure)**.** A measure that satisfies the detailed balance condition is said to be a reversible measure.

These quantities can be computed by definition.

Example 1.1 (Chip-Firing/Sand-Pile)**.** *Every time, pick a random τ such that $a_\tau \geqslant 2$ and do the update*

$$\begin{cases} a_\tau^{t+1} = a_\tau^t - 2 \\ a_{\tau+1}^{t+1} = a_{\tau+1}^t + 1 \\ a_{\tau-1}^{t+1} = a_{\tau-1}^t + 1 \end{cases}$$

The next theorem explain why reversible measure is reversible.

Theorem 1.5 (dual transition probability)**.**

A necessary and sufficient condition for a chain to have a reversible measure is given below. This condition can be checked if the transition probability is given.

Theorem 1.6 (Kolmogorov's cycle condition)**.**

Only special chains have reversible measures, but the next result shows that many markov chains have stationary measures by relating it to the expected number of visits during an excursion.

Theorem 1.7. *Let x be an recurrent state and let $T = \inf\{n \geqslant 1 : X_n = x\}$. Then*

$$\mu_x(y) = \mathbb{E}_x(\sum_{n=0}^{T-1} 1_{X_n=y})$$

defines a stationary measure.

Theorem 1.8. *If p is irreducible and recurrent, then the stationary measure is unique up to constant multiples.*

Having examine the existence and uniqueness of stationary measures, we turn our attention to stationary distributions. Stationary measures may exist for transient chains, e.g. random walks in $d \geqslant 3$, but stationary distribution only exist for recurrent chains, as the following theorem shows:

Theorem 1.9. *If there is a stationary distribution, then all states y with $\pi(y) > 0$ are recurrent.*

We can relate stationary distributions to the expected time of an excursion

Theorem 1.10. *If p is irreducible and has stationary distribution π, then*

$$\pi(x) = \frac{1}{\mathbb{E}_x T_x}$$

Recall the concept of positive recurrent and null recurrent. The next result

Theorem 1.11. *If p is irreducible, then TFAE:*

- *Some x is positive recurrent.*
- *There is a stationary distribution.*
- *All states are positive recurrent.*

1.4 Ergodic Theorems

The first topic in this section is to investigate

Let

$$N_n(y) = \sum_{m=1}^{n} 1_{X_m=y}$$

be the number of visits to y by time n.

Theorem 1.12. *Suppose y is recurrent. For any $x \in S$, as $n \to \infty$*

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y} 1_{T_y<\infty}$$

Proof. a

Step 1 Suppose first we start at $x = y$. Let $R(k) = \min\{n \geqslant 1 : N_n(y) = k\}$ and $t_k = R(k) - R(k-1)$, where $R(0) = 0$. Then t_i are i.i.d. and SLLN implies

$$\frac{R(k)}{k} \to \mathbb{E}_y T_y$$

Step 2 Then we generalize to $x \neq y$. The result is obviously true if $T_y = \infty$. $\square$

Example 1.2.

Definition 1.20 (period)**.** let d_x by the greatest common divisor of I_x. d_x is called the period of x.

The next lemma says that period is a class property.

Lemma 1.1. *If $\rho_{xy} > 0$, then $d_y = d_x$.*

Theorem 1.13. *Suppose p is irreducible, aperiodic, and has stationary distribution π. Then as $n \to \infty$, $p^n(x, y) \to \pi(y)$.*

Proof. The proof technique is called **coupling**. Let $S^2 = S \times S$. Define a transition probability $\bar{p}$ on $S \times S$ by

$$\bar{p} = p((x_1, y_1), (x_2, y_2)) = p(x_1, x_2)p(y_1, y_n)$$

i.e. each coordinate moves independently.

$\bar{p}$ is irreducible. This is the only step that requires aperiodicity.

$\bar{p}$ is recurrent. This is becasue $\bar{\pi}(a, b) = \pi(a)\pi(b)$ defines a stationary distribution for $\bar{p}$.

Now let (X_n, Y_n) denote the cahin on $S \times S$, and let T be the first time that this chain hits the diagonal $\{(y, y) : y \in S\}$. Let $T_{(x,x)}$ be the hitting time of (x, x). Since $\bar{p}$ is irreducible and recurrent, $T_{(x,x)} < \infty$ a.s. and hence $T < \infty$ a.s..

Now we want to show the following key identity:

$$\sum_y |\mathbb{P}(X_n = y) - \mathbb{P}(Y_n = y)| \leqslant 2\mathbb{P}(T > n).$$

This is because on $\{T \leqslant n\}$, the two coordinates X_n and Y_n have the same distribution.

Finally we let $X_0 = x$ and $Y_0 \sim \pi$, then Y_n has distribution π, and it follows that

$$\sum_y |p^n(x, y) - \pi(y)| \leqslant 2\mathbb{P}(T > n) \to 0.$$

$\square$

Remark. *Some shortcuts exist for helping to determine when a Markov chain is ergodic, i.e. irreducible, aperiodic, and positive recurrent.*

1. *A Markov chain with a finite number of states has only transient and positive recurrent states. Only a Markov chain with an infinite number of states can be null recurrent.*
2. *A sufficient test for a state to be aperiodic is that it has a self-loop.*
3. *In an irreducible, finite state Markov chain, the presence of one aperiodic state guarantees ergodicity.*

1.5 Introduction to Markov Chain Mixing

Contraction

Canonical Path

Cheeger's Constant

2 Introduction to Markov Random Fields

2.1 Basics

2.2 Coloring

Glauber Dynamics Every time pick a random vertex and assign a random color which is not used by any neighbors.

2.3 1D Ising Model without Magnetic Field

The Ising model is a theoretical model in statistical physics that was originally developed to describe ferromagnetism, a property of certain materials such as iron. A system of magnetic particles can be modeled as a linear chain in one dimension or a lattice in two dimension, with one molecule or atom at each lattice site i. To each molecule or atom a magnetic moment is assigned that is represented in the model by a discrete variable σ_i. Each 'spin' can only have a value of $\sigma_i = \pm 1$. The two possible values indicate whether two spins σ_i and σ_j are align and thus parallel ($\sigma_i \cdot \sigma_j = +1$) or anti-parallel ($\sigma_i \cdot \sigma_j = +1$)

A system of two spins is considered to be in a lower energetic state if the two magnetic moments are aligned. If the magnetic moments points in opposite directions they are consider to be in a higher energetic state. Due to this interaction the system tends to align all magnetic moments in one direction in order to minimise energy. If nearly all magnetic moments point in the same direction the arrangement of molecules behaves like a macroscopic magnet.

A phase transition in the context of the Ising model is a transition from an ordered state to a disordered state. A ferromagnet above the critical temperature T_C is in a disordered state. In the Ising model this corresponds to a random distribution of the spin values. Below the critical temperature T_C (nearly) all spins are aligned, even in the absence of an external applied magnetic field H. If we heat up a cooled ferromagnet, the magnetization vanishes at T_C and the ferromagnet switches from an ordered to a disordered state. This is a phase transition of second order.

2.4 1D Ising Model with Magnetic Field

2.5 2D Ising Model: Peirls Proof

The Hamiltonion of the system is

The idea of Peirls proof is very similar to the idea of reflection principle, which map

3 Electrical Networks

3.1 The Correspondence

Assume a unit voltage is charged on a and b is grounded, i.e. $\varphi(a) = 1$ and $\varphi(b) = 0$.

The Correspondence between Reversible Markov Chains and Electrical Networks Given an electrical network, we can define a corresponding reversible Markov chain as follows. Let the **transition probability** be

$$p(x, y) = \frac{C_{xy}}{C_x},$$

where

$$C_x = \sum_y C_{xy}.$$

This chain is indeed reversible because we can define a measure as

$$\pi(x) = C_x,$$

then the detailed balance condition is satisdied as

$$\pi(x)p(x, y) = C_{xy} = C_{yx} = \pi(y)p(y, x).$$

Conversely, given a reversible Markov chain with reversible measure $\pi(x)$, i.e. $\pi(x)p(x, y) = \pi(y)p(y, x)$, we can define a corresponding electrical network as follows. Let

$$C_{xy} = \pi(x)p(x, y),$$

then it is indeed a well-defined electrical network as

$$C_{xy} = \pi(x)p(x, y) = \pi(y)p(y, x) = C_{yx}.$$

Voltage

Theorem 3.1 (Voltage as). $\varphi(x) = \mathbb{P}_x(\tau_a < \tau_b)$

Proof. $\varphi(x)$ satisfies the same harmonic equation as $\mathbb{P}_x(\tau_a < \tau_b)$. Moreover, the boundary condition is also the same. □

Theorem 3.2 (Electrical Current as Edge Crossings). $I(x,y) = ?$

Proof.

$$I(x,y) = C_{xy}(\varphi(x) - \varphi(y))$$

□

Effective Resistance

Definition 3.1 (effective resistance). $R_{\text{eff}} = \frac{1}{I(a^+)}$

Theorem 3.3. $\mathbb{P}_b(\tau_a < \sigma_b) = \frac{1}{C_b R_{eff}}$

Proof.

$$\begin{aligned}
\mathbb{P}_b(\tau_a < \sigma_b) &= \sum_x p(b,x)\varphi(x) \\
&= \sum_x \frac{C_{bx}\varphi(x)}{C_b} \\
&= \sum_x \frac{I(b,x)}{C_b} \\
&= \frac{1}{C_b R_{\text{eff}}}
\end{aligned}$$

□

Thomson Principle Nash-Williams Criterion

Dirichlet Principle

3.2 Random Walks

4 Introduction to Random Walks

4.1 1D Simple Random Walk

Let $S_n = \sum_{i=1}^n \xi_i$ where ξ_i's are i.i.d. Rademacher random varaibles.

Reflection Principle

Lemma 4.1. $\mathbb{P}_0(\tau_i < n, S_n = i + j) = \mathbb{P}_0(\tau_i < n, S_n = i - j)$

Proof. Note that when $\tau_i < n$ and $S_n = i + j$, the trajectory must cross the line i. Reflect the trajectory with respect to i yields a trajectory which satisfies $\tau_i < n$ and $S_n = i - j$, and vice versa. Therefore a bijection between these two events is constructed. Noting that the weights of all these trajectories induced by the probability distribution are equal. □

Remark. *Note that* $\mathbb{P}_0(\tau_i < n, S_n = i + j) = \mathbb{P}_0(S_n = i + j)$.

Theorem 4.1 (Ballot Theorem). $\mathbb{P}_0(\tau_i = n | S_n = i) = \frac{i}{n}$

Proof.

$$
\begin{aligned}
\mathbb{P}_0(\tau_i = n) &= \mathbb{P}_0(S_n = i) - \mathbb{P}_0(\tau_i < n, S_n = i) \\
&= \mathbb{P}_0(S_n = i) - \frac{1}{2}\mathbb{P}_0(\tau_i < n, S_{n-1} = i-1) - \frac{1}{2}\mathbb{P}_0(\tau_i < n, S_{n-1} = i+1) \\
&= \mathbb{P}_0(S_n = i) - \mathbb{P}_0(\tau_i < n, S_{n-1} = i+1) \\
&= \mathbb{P}_0(S_n = i) - \mathbb{P}_0(S_{n-1} = i+1) \\
&= C_n^{\frac{n+i}{2}} \frac{1}{2^n} - C_{n-1}^{\frac{n+i}{2}} \frac{1}{2^{n-1}} \\
&= \frac{i}{n} C_n^{\frac{n+i}{2}} \frac{1}{2^n} \\
&= \frac{i}{n} \mathbb{P}_0(S_n = i)
\end{aligned}
$$

□

Remark. *Another interpretation of this result is that consider the trajectory of. The first time that it reaches There is exactly i such time. Therefore,*

Theorem 4.2. $\mathbb{P}_0(\tau_1 > 2n-1) = \mathbb{P}_0(S_{2n} = 0)$

Proof.

$$
\begin{aligned}
\mathbb{P}_0(\tau_1 \leqslant 2n) &= \sum_i \mathbb{P}_0(\tau_1 \leqslant 2n, S_{2n} = i) \\
&= \sum_{i \geqslant 1} \mathbb{P}_0(\tau_1 \leqslant 2n, S_{2n} = i) + \sum_{i \leqslant 0} \mathbb{P}_0(\tau_1 \leqslant 2n, S_{2n} = i) \\
&= 2 \sum_{i \geqslant 1} \mathbb{P}_0(S_{2n} = i) \\
&= 1 - \mathbb{P}_0(S_{2n} = 0)
\end{aligned}
$$

□

Remark. $\mathbb{P}_0(S_1 \neq 0, \cdots, S_{2n} \neq 0) = \mathbb{P}_0(S_{2n} = 0)$

Theorem 4.3.

Corollary 4.3.1. $\mathbb{P}_0(\tau_1 < \infty) = 1, \mathbb{E}_0\, \tau_1 = \infty$

Arcsin Laws Let $u_{2n} = \mathbb{P}_0(S_{2n} = 0)$. We first describe the arcsin law for the last visit to 0.

Lemma 4.2. *Let* $\sigma_{2n} = \sup\{m \leqslant 2n : S_m = 0\}$. *Then*

$$\mathbb{P}(\sigma_{2n} = 2k) = u_{2k} u_{2n-2k}$$

Proof. $\mathbb{P}(\sigma_{2n} = 2k) = \mathbb{P}(S_{2k} = 0, S_{2k+1} \neq 0, \cdots, S_{2n} \neq 0)$. □

Theorem 4.4. *For* $0 < a < b < 1$,

$$\mathbb{P}_0(a \leqslant \frac{\sigma_{2n}}{2n} \leqslant b) \to \int_a^b \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}$$

Corollary 4.4.1. $\lim_{n\to\infty} \mathbb{P}_0(S_r \neq 0 \quad \forall \delta n < r \leqslant n) = \frac{2}{\pi} \arcsin \sqrt{\delta}$

Remark. *Anti-concentration Ineqaulity*

Next, we prove the arcsin law for the time above 0.

Lemma 4.3. *Let* π_{2n} *be the number of segments* $(k-1, S_{k-1}) \to (k, S_k)$ *that lie above the axis, i.e. in* $\{(x, y) : y \geqslant 0\}$. *Then*

$$\mathbb{P}_0(\pi_{2n} = 2k) = u_{2k} u_{2n-2k}.$$

Corollary 4.4.2. *For* $0 < a < b < 1$,

$$\mathbb{P}_0(a \leqslant \frac{\pi_{2n}}{2n} \leqslant b) \to \int_a^b \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}$$

Maringale Now we want to calculate the moments of τ. Our method involves differentiating the exponential martingales. For a simple random walk,

$$Y_n(t) := \frac{e^{tS_n}}{(\cosh t)^n}$$

is a martingale. Thus its derivatives are also martingales.

$$\frac{\mathrm{d}Y_n(t)}{\mathrm{d}t} = \frac{e^{tS_n}}{(\cosh t)^{n+1}}(S_n \cosh t - n \sinh t)$$

so $\frac{\mathrm{d}Y_n(t)}{\mathrm{d}t}|_{t=0} = S_n$ is a martingale, as expected.

$$\frac{\mathrm{d}^2Y_n(t)}{\mathrm{d}t^2} = \frac{e^{tS_n}}{(\cosh t)^{n+1}}((S_n^2 - n)\cosh t - 2nS_n \sinh t) + n(n+1)\frac{e^{tS_n}\sinh^2 t}{(\cosh t)^{n+2}}$$

so $\frac{\mathrm{d}^2Y_n(t)}{\mathrm{d}t^2}|_{t=0} = S_n^2 - n$ is a martingale, as expected.

4.2 Lamplighter

Example 4.1. *Consider*

reversibility

transience

recurrence

5 Introduction to Branching Process

5.1 Galton-Watson Process

Let $\{\xi_{ni} : n \geqslant 0, i \geqslant 0\}$ be a set of independent and identically-distributed natural number-valued random variables. A Galton-Watson process is a stochastic process $\{X_n\}$ which evolves according to the recurrence formula $X_0 = 1$ and

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_{ni}.$$

Our goal is to analysis the properties of this process.

Mean and Variance As the Galton-Watson process is tree-like, it possesses many recurrence structure. The first one we would utilize is the martingale property. We have

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n \mathbb{E}\,\xi,$$

so that

$$\frac{X_n}{(\mathbb{E}\,\xi)^n}$$

is a martingale. By the property of martingales, we have

$$\mathbb{E}\,X_n = (\mathbb{E}\,\xi)^n.$$

If we further assume that $\mathrm{Var}\xi < \infty$, we can calculate the variance of X_n similarly. We have

$$\begin{aligned}\mathrm{Var}(X_{n+1}) &= \mathrm{Var}(\mathbb{E}(X_{n+1}|\mathcal{F}_n)) + \mathbb{E}\,\mathrm{Var}(X_{n+1}|\mathcal{F}_n)\\ &= (\mathbb{E}\,\xi)^2\mathrm{Var}(X_n) + \mathbb{E}\,X_n\mathrm{Var}\xi\\ &= (\mathbb{E}\,\xi)^2\mathrm{Var}(X_n) + (\mathbb{E}\,\xi)^n\mathrm{Var}\xi\end{aligned}$$

and we can solve this update formula by noting that $\mathrm{Var}X_1 = \mathrm{Var}\xi$, which yields

$$\mathrm{Var}X_n = \mathrm{Var}\xi \sum_{i=n}^{2n-1} (\mathbb{E}\,\xi)^n.$$

The Extinction Probability The key quantity we are interested in is the extinction probability (i.e. the probability of final extinction), which is given by

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0).$$

Note that once $X_n = 0$, then $X_{n+k} = 0$ for all $k \geqslant 1$, so 0 is an absorbing state.

Theorem 5.1 (subcritical)**.** *If $\mu < 1$, then X_n extincts.*

Proof. $\mathbb{P}(X_n > 0) = \mathbb{P}(X_n \geqslant 1) \leqslant \mathbb{E}\, X_n = \mu^n \to 0$. □

Remark. *The extinction rate when $\mu < 1$ is exponentially fast.*

Theorem 5.2 (critical)**.** *If $\mu = 1$ and $\mathbb{P}(\xi = 1) < 1$, then X_n extincts.*

Proof. When $\mu = 1$, X_n itself is a nonnegative martingale. So X_n converges to a finite limit X_∞ a.s.. As X_n is integer valued, we must have $X_n = X_\infty$ for large n. However, for any $k > 0$, $\mathbb{P}(X_n = k \quad \forall n \geqslant N) = 0$ for any N because $\mathbb{P}(\xi = 1) < 1$. So we must have $X_\infty = 0$. □

In this case, the extinction rate is at most $\frac{1}{n}$ by an easy second moment estimate.

Theorem 5.3. *If $\mu = 1$, then $\mathbb{P}(X_n \geqslant 1) \geqslant \frac{1}{1+n\,Var(\xi)}$.*

Proof. $\mathbb{P}(X_n \geqslant 1) \geqslant \frac{(\mathbb{E}\, X_n)^2}{\mathbb{E}\, X_n^2} = \frac{1}{1+n\mathrm{Var}(\xi)}$. □

This rate cannot be improved.

Example 5.1. *Let $\xi \sim Geo(\frac{1}{2})$. Then $\mathbb{P}(X_n \geqslant 1) = \frac{1}{n+1}$.*

Proof. The distribution of X_n can be described by its generating function $f_{X_n}(s) = f^{(n)}(s)$ where $f(s) = \frac{1}{2-s}$. So $f(0) = \frac{1}{2}$, $f^{(2)}(0) = \frac{2}{3}$, and $f^{(n)} = \frac{n}{n+1}$. Now $\mathbb{P}(X_n = 0) = f_{X_n}(0) = f^{(n)}(0)$, so $\mathbb{P}(X_n \geqslant 1) = 1 - \mathbb{P}(X_n = 0) = \frac{1}{n+1}$. □

Remark. *Another view of this result. This is the probability that a simple random walk*

For $s \in [0, 1]$, let $f(s) = \sum_{k\geqslant 0} p_k s^k$ be the generating function of ξ. Then $f(1) = 1$ and $f'(s) \geqslant 0$.

Lemma 5.1. *The generating function for X_n*

Theorem 5.4 (supercritical)**.** *If $\mu > 1$, then $\lim_{n\to\infty} \mathbb{P}(X_n = 0) = \rho$, where ρ is the only solution of $f(\rho) = \rho$ in $[0, 1)$.*

Proof. This is basically because a tree extinct if and only if all children trees extinct. □

5.2 Biased Random Walk on a Galton-Watson Tree

Let $\mathbb{Q}(\cdot) = \mathbb{P}(\cdot||\mathbb{T}| = \infty)$ be the measure conditioned on all Galton-Watson trees that survive.

Lemma 5.2. *On the event of nonextinction, $X_n \to \infty$ almost surely, provided $p_1 \neq 1$.*

Proof. If 0 is the only nontransient state state of X_n, then the result follows. Thus we want to show that any state other than 0 is transient.

Now for any $k > 0$, eventually returning to k requires not immediately being extinct, which has probability $\leqslant 1 - p_0^k$. If $p_0 > 0$, then k is transient. If $p_0 = 0$, as $p_1 \neq 1$, k is also transient. □

Definition 5.1 (inherited property)**.** Call a property of trees inherited if
(i) every finite tree has this property and
(ii) if whenever a tree has this property, so do all the descendant trees of the children of the root.

Theorem 5.5 (0-1 law for inherited properties)**.** *Every inherited property has conditional probability either 0 or 1 given nonextinction.*

Proof. Let A be the set of trees possessing a given inherited property. For a tree T with k children of the root, let $T(1), \ldots, T(k)$ be the descendant trees of these children. Then

$$\mathbb{P}(A) = \mathbb{E}[\mathbb{P}[T \in A \,|\, X_1]] \leqslant \mathbb{E}[\mathbb{P}[T(1) \in A, \ldots, T(X_1) \in A \,|\, Z_1]]$$

by definition of inherited. Since $T(1), \ldots, T(Z_1)$ are i.i.d. given Z_1, the last quantity above is equal to $\mathbb{E}[\mathbb{P}(A)^{Z_1}] = f(\mathbb{P}(A))$. Thus, $\mathbb{P}(A) \leqslant f(\mathbb{P}(A))$. On the other hand, $\mathbb{P}(A) \geqslant q$ since every finite tree is in A. It follows upon inspection of a graph of f that $\mathbb{P}(A) \in [q, 1]$, from which the desired conclusion follows. □

Corollary 5.5.1. *Either $W = 0$ a.s. or $W > 0$ a.s. on nonextinction*

Theorem 5.6 (The Seneta-Heyde Theorem)**.** *If $1 < \mu < 1$, then there exist constants c_n such that*

(i) $\lim \frac{Z_n}{c_n}$ *exists almost surely in $[0, \infty)$;*

(ii) $\mathbb{P}[\lim \frac{Z_n}{c_n} = 0] = \rho$;

(iii) $\frac{c_{n+1}}{c_n} \to m$.

Proof. □

Let η be the (corresponds to ξ)

$$q_k = \sum_{l=0}^{\infty} p_{k+l} C_{k+l}^{k} \rho^{k-1} (1-\rho)^l \quad k \geqslant 1$$

We will often want to consider random trees produced by a Galton-Watson branching process. For a precise formulation of tree-valued random variables, one is referred to.

$$\frac{1}{R} = \sum_{i=1}^{\eta} \frac{1}{\lambda + \lambda R^{(i)}}$$

therefore $R = \infty$ if and only if $R^{(i)} = 0\ \forall i$.

Lemma 5.3 (0-1 law)**.** $\mathbb{Q}(R = \infty) = 0$ *or* 1

Theorem 5.7. *When $\lambda \geqslant m$, recurrent*

Theorem 5.8. *When $\lambda < m$, transient*

Example 5.2 (3-1 tree)**.**

Definition 5.2 (Branching Number)**.** The branching number of a tree T is the supremum of those λ that admit a positive total amount of water to flow through T ; denote this critical value of λ by br$T$

Precolation

Theorem 5.9. *Let T be the family tree of a Galton-Watson process with mean $\mu > 1$. Then $p_c(T) = \frac{1}{\mu}$ almost surely, given nonextinction.*

The basic intuition goes like this. If T is an n-ary tree, then the cluster of the root under percolation is a Galton-Watson tree with progeny distribution Binomial(n, p). Thus, this cluster is infinite with positive probability if and only if $np > 1$, whence $p_c(T) = \frac{1}{n}$.

Proof. Let T be a given tree, and write K for the cluster of the root of T after percolation on T with the survival parameter p. When T has the law of a Galton-Watson tree with mean μ, we claim that K has the law of another Galton-Watson tree having mean μp: if Y_i represent i.i.d. Bin$(1, p)$ random variables that are also independent of L, then

$$\mathbb{E}\left[\sum_{i=1}^{L} Y_i\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{L} Y_i \,|\, L\right]\right] = \mathbb{E}\left[\sum_{i=1}^{L} \mathbb{E}[Y_i]\right] = \mathbb{E}\left[\sum_{i=1}^{L} p\right] = \mu p.$$

Hence, K is finite almost surely if and only if $mp \leqslant 1$. Since

$$\mathbb{E}[\mathbb{P}(|K| < \infty \,|\, T)] = \mathbb{P}(|K| < \infty), \tag{1}$$

this means that for almost every Galton-Watson tree T, the cluster of its root is finite almost surely if $p \leqslant \frac{1}{\mu}$. On the other hand, for fixed p, the property $\{T; P_p(|K| < \infty) = 1\}$ is inherited, so has probability ρ or 1. If it has probability 1, then Equation (1) shows that $mp \leqslant 1$. That is, if $mp > 1$, this property has probability ρ, so that the cluster of the root of T will be infinite with positive probability almost surely on the event of nonextinction. Considering a sequence $p_n \to \frac{1}{\mu}$, we see that this holds almost surely on the event of nonextinction for all $p > \frac{1}{\mu}$ at once, not just for a fixed p. We conclude that $p_c(T) = \frac{1}{\mu}$ almost surely on nonextinction.

□

Corollary 5.9.1. *If T is a Galton-Watson tree with mean $\mu > 1$, then $br(T) = \mu$ almost surely, given nonextinction.*

Theorem 5.10. *For an independent percolation and adapted conductances on the same tree, we have*

$$\frac{C(o \leftrightarrow 1)}{1 + C(o \leftrightarrow 1)} \leqslant \mathbb{P}[o \leftrightarrow 1].$$

Corollary 5.10.1. *For any locally finite infinite tree T,*

$$p_c(T) = \frac{1}{brT}$$

6 Measurable Space Discrete Time Markov Chain

Now we develop a more formal theory for discrete time Markov Chains by means of measure theory. Let $(S, \mathcal{S}) \to \mathbb{R}$ be a measurable space. This is the state space for our Markov chain.

7 Discrete Space Continuous Time Markov Chain

Definition 7.1 (continuous-time random process)**.** Let I be a countable set. A continuous-time random process

$$(X_t)_{t \geqslant 0} = (X_t : 0 \leqslant t \leqslant \infty)$$

with values in I is a family of random variables $X_t : \Omega \to I$.

We are going to consider ways in which we might specify the probabilistic behavior of $(X_t)_{t\geqslant 0}$. To avoid uncountable union, we shall restrict our attention to processes $(X_t)_{t\geqslant 0}$ which are right-continuous.

Definition 7.2 (right continuous)**.** In the context of discrete space continuous time, a right-continuous process means $\forall \omega \in \Omega$ and $t \geqslant 0$, $\exists \epsilon > 0$ s.t.

$$X_s(\omega) = X_t(\omega) \quad t \leqslant s \leqslant t + \epsilon$$

Definition 7.3 (increment)**.** If $(X_t)_{t\geqslant 0}$ is a real-valued process, we can consider its increment $X_t - X_s$ over any interval $(s, t]$.

Definition 7.4 (stationary)**.** We say that $(X_t)_{t\geqslant 0}$ has stationary increments if the distribution of $X_{s+t} - X_s$ depends only on $t \geqslant 0$.

Definition 7.5 (independent)**.** We say that $(X_t)_{t\geqslant 0}$ has independent increments if its increments over amy finite collection of disjoint intervals are independent.

Definition 7.6 (Q-matrix)**.** A Q-matrix on I is a matrix $Q = (q_{ij} : i, j \in I)$ satisfying the following conditions:
(i) $\forall i \quad 0 \leqslant -q_{ii} < \infty$
(ii) $\forall i \neq j \quad q_{ij} \geqslant 0$
(iii) $\forall i \quad \sum_{j \in I} q_{ij} = 0$

7.1 Review: Properties of Exponential Distribution

Definition 7.7. A random variable $T : \Omega \to [0, \infty]$ has an exponential distribution of parameter λ $(0 \leqslant \lambda < \infty)$ if

$$\mathbb{P}(T > t) = e^{-\lambda t} \quad \text{for all } t \geqslant 0.$$

We write $T \sim \mathrm{E}(\lambda)$ for short. If $\lambda > 0$, then T has a density function

$$f_T(t) = \lambda e^{-\lambda t} 1_{\{t \geqslant 0\}}.$$

Remark. *The mean of T is given by*

$$\mathbb{E}(T) = \int_0^\infty P(T > t)\, dt = \lambda^{-1}.$$

Theorem 7.1 (memoryless property)**.** *A random variable $T : \Omega \to (0,\infty]$ has an exponential distribution if and only if it has the following memoryless property:*

$$\mathbb{P}(T > s+t \,|\, T > s) = \mathbb{P}(T > t) \quad \text{for all } s,t \geqslant 0.$$

Proof. Suppose $T \sim \mathrm{E}(\lambda)$, then

$$\mathbb{P}(T > s+t \,|\, T > s) = \frac{\mathbb{P}(T > s+t)}{\mathbb{P}(T > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(T > t).$$

On the other hand, suppose T has the memoryless property whenever $P(T > s) > 0$. Then $g(t) = P(T > t)$ satisfies

$$g(s+t) = g(s)g(t) \quad \text{for all } s,t \geqslant 0.$$

We assumed $T > 0$ so that $g\left(\frac{1}{n}\right) > 0$ for some n. Then, by induction,

$$g(1) = g(\frac{1}{n} + \cdots + \frac{1}{n}) = g\left(\frac{1}{n}\right)^n > 0,$$

so $g(1) = e^{-\lambda}$ for some $0 \leqslant \lambda < \infty$. By the same argument, for integers $p,q \geqslant 1$,

$$g\left(\frac{p}{q}\right) = g\left(\frac{1}{q}\right)^p = g(1)^{p/q},$$

so $g(r) = e^{-\lambda r}$ for all rationals $r > 0$. For real $t > 0$, choose rationals $r,s > 0$ with $r \leqslant t \leqslant s$. Since g is decreasing,

$$e^{-\lambda r} = g(r) \geqslant g(t) \geqslant g(s) = e^{-\lambda s}$$

and, since we can choose r and s arbitrarily close to t, this forces $g(t) = e^{-\lambda t}$, so $T \sim \mathrm{E}(\lambda)$. □

Theorem 7.2 (infimum)**.** *Let I be a countable set and let T_k $k \in I$ be independent random variables with $T_k \sim E(q_k)$ and $0 < q := \sum_{k\in I} q_k < \infty$. Set $T = \inf_k T_k$. Then this infimum is attained at a unique random value K of k a.s.. Moreover, T and K are independent, with $T \sim E(q)$ and $\mathbb{P}(K = k) = \frac{q_k}{q}$.*

Proof. Set $K = k$ if $T_k < T_j$ for all $j \neq k$, otherwise, let K be undefined. Then

$$\begin{aligned}
\mathbb{P}(K = k \text{ and } T \geqslant t) &= \mathbb{P}(T_k \geqslant t \text{ and } T_j > T_k \text{ for all } j \neq k) \\
&= \int_t^\infty q_k e^{-q_k s} \mathbb{P}(T_j > s \text{ for all } j \neq k)\, ds \\
&= \int_t^\infty q_k e^{-q_k s} \prod_{j\neq k} e^{-q_j s}\, ds \\
&= \int_t^\infty q_k e^{-qs}\, ds = \frac{q_k}{q} e^{-qt}.
\end{aligned}$$

Hence, $\mathbb{P}(K = k \text{ for some } k) = 1$, and T and K have the claimed joint distribution. □

Theorem 7.3. *Let $S_1, S_2, \dots$ be a sequence of independent random variables with $S_n \sim E(\lambda_n)$ and $0 < \lambda_n < \infty$ for all n.*
(i) If $\sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty$, then $\mathbb{P}\left(\sum_{n=1}^\infty S_n < \infty\right) = 1$.
(ii) If $\sum_{n=1}^\infty \frac{1}{\lambda_n} = \infty$, then $\mathbb{P}\left(\sum_{n=1}^\infty S_n = \infty\right) = 1$.

Proof. (i) Suppose $\sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty$. Then, by monotone convergence

$$\mathbb{E}\left(\sum_{n=1}^\infty S_n\right) = \sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty$$

so

$$\mathbb{P}\left(\sum_{n=1}^\infty S_n < \infty\right) = 1.$$

(ii) Suppose instead that $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$. Then $\prod_{n=1}^{\infty}(1 + \frac{1}{\lambda_n}) = \infty$. By monotone convergence and independence

$$\mathbb{E}\left[\exp\left\{-\sum_{n=1}^{\infty} S_n\right\}\right] = \prod_{n=1}^{\infty} \mathbb{E}\left[\exp\{-S_n\}\right] = \prod_{n=1}^{\infty}(1+\lambda_1 n)^{-1} = 0$$

so

$$\mathbb{P}\left(\sum_{n=1}^{\infty} S_n = \infty\right) = 1.$$

□

Theorem 7.4. *For independent random variables $S \sim E(\lambda)$ and $R \sim E(\mu)$ and for $t \geqslant 0$, we have*

$$\mu\mathbb{P}(S \leqslant t < S + R) = \lambda\mathbb{P}(R \leqslant t < R + S).$$

Proof. We have

$$\mu\mathbb{P}(S \leqslant t < S + R) = \int_0^t \int_{t-s}^{\infty} \lambda\mu e^{-\lambda s} e^{-\mu r}\, dr\, ds = \lambda\mu \int_0^t e^{-\lambda s} e^{-\mu(t-s)}\, ds$$

from which the identity follows by symmetry.

□

7.2 Poisson Process

We begin with a definition of Poisson process in terms of jump chain and holding times, and then relate it to the infinitesimal definition and transition probability definition.

Definition 7.8. A right-continuous process $(X_t)_{t\leqslant 0}$ with values in $\mathbb{N}_{\geqslant 0}$ is a Poisson process of rate $\lambda \in (0, \infty)$ if its holding times $S_1, S_2, \cdots$ are i.i.d. exponential random variables of mean λ and its jump chain is given by $Y_n = n$.

Theorem 7.5. *Let $(X_t)_{t\geqslant 0}$ be an increasing, right-continuous integer-valued process starting from 0. Let $\lambda \in (0, \infty)$. TFAE:*
(i) (jump chain holding time definition) the holding times $S_1, S_2, \cdots$ of $(X_t)_{t\geqslant 0}$ are i.i.d. exponential random variables of mean λ and the jump chain is given by $Y_n = n$.
(ii) (infinitesimal definition) $(X_t)_{t\geqslant 0}$ has independent increments and as $h \downarrow 0$, uniformly in t,

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h), \quad \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$$

(iii) (incremental definition) $(X_t)_{t\geqslant 0}$ has stationary independent increments and for each t, X_t has Poisson distribution of parameter λt.

Theorem 7.6. *Let $(X_t)_{t\geqslant 0}$ be a Poisson process. Then, conditional on $(X_t)_{t\geqslant 0}$ having exactly one jump in the interval $[s, s+t]$, the time at which that jump occurs is uniformly distributed on $[s, s+t]$.*

Theorem 7.7. *Let $(X_t)_{t\geqslant 0}$ be a Poisson process. Then, conditional on the event $\{X_t = n\}$, the jump times $J_1, \cdots, J_n$ have joint density function*

$$f(t_1, \cdots, t_n) = n! 1_{0\leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant t}$$

Remark. *Thus, conditional on $\{X_t = n\}$, the jump times $J_1, \cdots, J_n$ have the same distribution as an ordered sample of size n from the uniform distribution on $[0, t]$.*

An Approximation Scheme for Poisson Process In the same spirit as Donsker's invariance principle,

8 Continuous Time Martingale

8.1 Stopping Times

Definition 8.1 (stopping time)**.** Let τ be a random time. If $\{\tau \leqslant t\} \in \mathcal{F}_t$ for every $t \geqslant 0$, then τ is called a stopping time.

Definition 8.2 (optional time)**.** Let T be a random time. If $\{T < t\} \in \mathcal{F}_t$ for every $t \geqslant 0$, then T is called a stopping time.

Lemma 8.1. *T is an optional time of the filtration $\{\mathcal{F}_t\}$ if and only if it is a stopping time of the right-continuous filtration $\{\mathcal{F}_{t+}\}$.*

Corollary 8.0.1. *Every stopping time is optional, and the two concepts coincide if the filtration is right-continuous.*

Lemma 8.2. *If T is optional and θ is a positive constant, then $T + \theta$ is a stopping time.*

Lemma 8.3. *If τ, σ are stopping times, then so are $\tau \wedge \sigma$, $\tau \vee \sigma$, $\tau + \sigma$.*

Proof. The first two assertions are trivial.
For the third, start with the decomposition ☐

Lemma 8.4. *Let T, S be optional times; then $T + S$ is optional.*
Moreover, it is a stopping time if

Lemma 8.5. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of optional times; then the random times*

$$\sup_{n\geqslant 1} T_n \quad \inf_{n\geqslant 1} T_n \quad \limsup_{n\to\infty} T_n \quad \liminf_{n\to\infty} T_n$$

are all optional.
Moreover, if the T_n's are stopping times, then so is $\sup_{n\geqslant 1} T_n$.

Definition 8.3 (σ-field of events determined prior to a stopping time)**.** Let τ be a stopping time of the filtration $\{\mathcal{F}_t\}$. The σ-field of events determined prior to the stopping time T consists of those events $A \in \mathcal{F}$ for which $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$ for every $t \geqslant 0$.

Lemma 8.6. *τ is $\mathcal{F}_\tau$-measurable.*

Proof. $\{\tau \leqslant t\} \cap \{\tau \leqslant t\} = \{\tau \leqslant t\} \in \mathcal{F}_t$, so $\{\tau \leqslant t\} \in \mathcal{F}_\tau$. ☐

Theorem 8.1. *For any two stopping time and τ, σ a random time s.t. $\sigma \leqslant \tau$ on Ω, we have $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.*

Proof. For every stopping time τ and positive constant t, $\tau \wedge t$ is an $\mathcal{F}_t$-measurable random variable because $\mathcal{F}_{\tau\wedge t} \subset \mathcal{F}_t$. Therefore, $\{\sigma \wedge t \leqslant \tau \wedge t\} \in \mathcal{F}_t$. Then for any $A \in \mathcal{F}_\sigma$ we have $A \cap \{\sigma \leqslant \tau\} \in \mathcal{F}_\tau$, because

$$A \cap \{\sigma \leqslant \tau\} \cap \{\tau \leqslant t\} = (A \cap \{\sigma \leqslant t\}) \cap \{\tau \leqslant t\} \cap \{\sigma \wedge t \leqslant \tau \wedge t\}$$

Finally notice that $\{\sigma \leqslant \tau\} = \Omega$. ☐

Remark. *We have proved a stronger result, namely for any $A \in \mathcal{F}_\sigma$ we have $A \cap \{\sigma \leqslant \tau\} \in \mathcal{F}_\tau$.*

Theorem 8.2. *Let σ and τ be stopping times. Then $\mathcal{F}_{\tau\wedge\sigma} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.*
Moreover, $\{\tau < \sigma\}$, $\{\tau > \sigma\}$, $\{\tau \leqslant \sigma\}$, $\{\tau \geqslant \sigma\}$, $\{\tau = \sigma\}$ belongs to $\mathcal{F}_\tau \cap \mathcal{F}_\sigma$.

Proof. From the above theorem, $\mathcal{F}_{\tau\wedge\sigma} \subset \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.
For $A \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$, $A \cap \{\tau \wedge \sigma \leqslant t\} = A \cap (\{\tau \leqslant t\} \cup \{\sigma \leqslant t\}) \in \mathcal{F}_t$. ☐

Theorem 8.3. *Let τ, σ be stopping times and X an integrable random variable. We have*
(i) $\mathbb{E}(X|\mathcal{F}_\tau) = \mathbb{E}(X|\mathcal{F}_{\sigma\wedge\tau})$ a.s. on $\{\tau \leqslant \sigma\}$.
(ii) $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_\tau)|\mathcal{F}_\sigma) = \mathbb{E}(X|\mathcal{F}_{\sigma\wedge\tau})$ a.s.. (i) $\mathbb{E}(X|\mathcal{F}_\tau) = \mathbb{E}(X|\mathcal{F}_{\sigma\wedge\tau})$ a.s. on $\{\tau \leqslant \sigma\}$.
(ii) $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_\tau)|\mathcal{F}_\sigma) = \mathbb{E}(X|\mathcal{F}_{\sigma\wedge\tau})$ a.s..

Proof. (i) Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau \leqslant \sigma\}$ belongs to both $\mathcal{F}_\tau$ and $\mathcal{F}_\sigma$, and therefore to $\mathcal{F}_\tau \cap \mathcal{F}_\sigma$. So

$$\int_A 1_{\tau\leqslant\sigma}\, \mathbb{E}(X|\mathcal{F}_{\tau\wedge\sigma})\mathrm{d}\mathbb{P} = \int \mathbb{E}(1_A 1_{\tau\leqslant\sigma} X|\mathcal{F}_{\tau\wedge\sigma})\mathrm{d}\mathbb{P} = \int_A 1_{\tau\leqslant\sigma} X\mathrm{d}\mathbb{P}$$

(ii) On $\{\tau \leqslant \sigma\}$ we have $\mathbb{E}(X|\mathcal{F}_\tau) = \mathbb{E}(X|\mathcal{F}_{\sigma\wedge\tau})$ a.s. by (i), so $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_\tau)|\mathcal{F}_\sigma) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_{\sigma\wedge\tau})|\mathcal{F}_\sigma) = \mathbb{E}(X|\mathcal{F}_{\sigma\wedge\tau})$. Similarly on $\{\sigma \leqslant \tau\}$ we have $\mathbb{E}(\mathbb{E}(X|\mathcal{F}_\tau)|\mathcal{F}_\sigma) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_\tau)|\mathcal{F}_{\sigma\wedge\tau}) = \mathbb{E}(X|\mathcal{F}_{\sigma\wedge\tau})$. ☐

Theorem 8.4. *Let $X = \{X_t, \mathcal{F}_t\}$ be a progressively measurable process, and let τ be a stopping time of the filtration $\mathcal{F}_t$. Then the random variable X_τ defined on $\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable, and the stopped process $\{X_{\tau\wedge t}, \mathcal{F}_t\}$ is progressively measurable.*

8.2 From Discrete to Continuous

In this subsection, we generalize inequalities and convergence results for discrete time martingales to continuous time martingales.
Let X_t be a submartingale adapted to $\{\mathcal{F}_t\}$ whose paths are right-continuous. Let $[\sigma, \tau]$ be a subinterval of $[0, +\infty)$, and let $a < b$, $\lambda > 0$ be real numbers.

Theorem 8.5 (Doob's inequality)**.** *Let $A = \left\{\sup_{\sigma\leqslant t\leqslant\tau} X_t^+ \geqslant \lambda\right\}$, then*

$$\lambda\mathbb{P}(A) \leqslant EX_\tau 1_A \leqslant EX_\tau^+$$

Proof. Let the finite set $\mathcal{S}$ consist of σ, τ and a finite subset of $[\sigma, \tau] \cap \mathbb{Q}$.

By considering an increasing sequence $\{\mathcal{S}_n\}_{n=1}^\infty$ of finite sets whose union is the whole of $([\sigma, \tau] \cap \mathbb{Q}) \cup \{\sigma, \tau\}$, we may replace S by this union in the preceding discrete version of the inequality. □

Theorem 8.6 (upcrossing inequality)**.**

$$(b-a)EU_{[\sigma,\tau]} \leqslant \mathbb{E}(X_\tau - a)^+ - \mathbb{E}(X_\sigma - a)^+$$

Theorem 8.7 (L^p maximum inequality)**.** $\bar{X}_\tau = \sup_{\sigma\leqslant t\leqslant\tau} X_t^+$*, then for $1 < p < \infty$,*

$$\mathbb{E}(\bar{X}_\tau^p) \leqslant (\frac{p}{p-1})^p E(X_\tau^+)^p$$

For the remainder of this subsection, we deal only with right-continuous processes, usually imposing no condition on the filtration $\mathcal{F}_t$.

Theorem 8.8 (submartingale convergence)**.** *Assume* $\sup_{t\geqslant 0} \mathbb{E}(X_t^+) < \infty$*. Then $X_\infty = \lim_{t\to\infty} X_t$ exists a.s., and $\mathbb{E}\,|X_\infty| < \infty$. Assume* $\sup_{t\geqslant 0} \mathbb{E}(X_t^+) < \infty$*. Then $X_\infty = \lim_{t\to\infty} X_t$ exists a.s., and $\mathbb{E}\,|X_\infty| < \infty$.*

Proof. □

Theorem 8.9 (optional sampling)**.** *Assume the submartingale has a last element X_∞, and let $S \leqslant T$ be two optional times of the filtration. We have*

$$\mathbb{E}(X_T|\mathcal{F}_{S^+}) \geqslant X_S \quad a.s.$$

$$\mathbb{E}(X_T|\mathcal{F}_{S^+}) \geqslant X_S \quad a.s.$$

If S is a stopping time, then $\mathcal{F}_S$ can replace $\mathcal{F}_{S^+}$ above.

Proof. Consider the sequence of random times

$$S_n(\omega) = \begin{cases} +\infty & S(\omega) = +\infty \\ \frac{k}{2^n} & \frac{k-1}{2^n} \leqslant S(\omega) < \frac{k}{2^n} \end{cases}$$

and similarly defined sequences $\{T_n\}$. These are stopping times. For every fixed integer $n \geqslant 1$, both S_n and T_n take on a countable number of values and we also have $S_n \leqslant T_n$. □

8.3 Doob-Meyer Decomposition

Definition 8.4 (increasing process)**.** An adapted process A is called increasing if for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have
(i) $A_0(\omega) = 0$
(ii) $t \mapsto A_t(\omega)$ is a nondecreasing, right-continuous function, and $EA_t < \infty$ holds for every $t \in [0, \infty)$.
An increasing process is called integrable if $EA_\infty < \infty$.

Definition 8.5. An increasing process A is called natural if for every bounded, right-continuous martingale $\{M_t, \mathcal{F}_t; 0 \leqslant t < \infty\}$ we have

$$\mathbb{E}\int_{(0,t]} M_s \mathrm{d}A_s = \mathbb{E}\int_{(0,t]} M_{s^-} \mathrm{d}A_s \quad \forall 0 < t < \infty$$

$$\mathbb{E}\int_{(0,t]} M_s \mathrm{d}A_s = \mathbb{E}\int_{(0,t]} M_{s^-} \mathrm{d}A_s \quad \forall 0 < t < \infty$$

Lemma 8.7. *If A is an increasing process and $\{M_t, \mathcal{F}_t; 0 \leqslant t < \infty\}$ is a bounded right-continuous martingale, then*

$$\mathbb{E}(M_t A_t) = \mathbb{E} \int_{(0,t]} M_s \mathrm{d} A_s$$

$$\mathbb{E}(M_t A_t) = \mathbb{E} \int_{(0,t]} M_s \mathrm{d} A_s$$

The following concept is a strengthening of the notion of uniform integrablity for submartingales.

Definition 8.6 (class DL)**.**

Theorem 8.10. *Let $\{\mathcal{F}_t\}$ satisfies the usual conditions. If the right-continuous submartingale $X =$ is of class DL, then it admits the decomposition as the summation if a right-continuous martingale*

8.4 Square Integrable Martingales

9 BM

9.1 Construction

Definition 9.1 (d-dimensional Brownian motion)**.** A d-dimensional Brownian motion $B = (B_t)_{t \geqslant 0}$ is a stochastic process indexed by $[0, \infty)$ taking values in $\mathbb{R}^d$ s.t.
(i) $B_0(\omega) = 0$

9.2 Sample Path Properties

Theorem 9.1. *Almost surely, for all $0 < a < b < \infty$, BM is not monotone on the interval $[a, b]$.*

Lemma 9.1. *Almost surely,*

$$\limsup_{t \to \infty} \frac{B(t)}{\sqrt{t}} = +\infty \quad \liminf_{t \to \infty} \frac{B(t)}{\sqrt{t}} = -\infty$$

Lemma 9.2. *Fix $t \geqslant 0$. Then almost surely, BM is not differentiable at t. Moreover, $D^* B(t) = +\infty$ and $D_* B(t) = -\infty$.*

These two lemmas can be strengthened to the law of iterated logarithm and nowhere differentiability.

Theorem 9.2. *Almost surely,*

$$\limsup_{t \to \infty} \frac{B(t)}{\sqrt{2t \log \log t}} = 1$$

Proof. □

Theorem 9.3. *Almost surely, BM is nowhere differentiable. Furthermore, almost surely, for all t, either $D^* B(t) = +\infty$ or $D_* B(t) = -\infty$ or both.*

Next we consider modulus of continuity.

Lemma 9.3. *There exists a constant $C > 0$ such that, almost surely, for every sufficiently small $h > 0$ and all $0 < t \leqslant 1 - h$,*

$$|B(t+h) - B(t)| \leqslant C \sqrt{h \log \frac{1}{h}}$$

Lemma 9.4. *For every constant $c < \sqrt{2}$, almost surely, for every $\epsilon > 0$ there exist $0 < h < \epsilon$ and $t \in [0, 1-h]$ with*

$$|B(t+h) - B(t)| \geqslant c \sqrt{h \log \frac{1}{h}}$$

Theorem 9.4. *Almost surely,*

$$\limsup_{h \downarrow 0} \sup_{0 \leqslant t \leqslant 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log \frac{1}{h}}} = 1$$

9.3 Brownian Local Time

We denote by $D(a,b,t)$ the number of downcrossings of the interval $[a,b]$ before time t. Note that $D(a,b,t)$ is almost surely finite by the uniform continuity of Brownian motion on the compact interval $[0,t]$.

Theorem 9.5. *There exists a stochastic process $\{L(t) : t \geqslant 0\}$ called the local time at zero such that for all sequence $a_n \uparrow 0$ and $b_n \downarrow 0$ with $a_n < b_n$, a.s.,*

$$\lim_{n\to\infty} 2(b_n - a_n)D(a_n, b_n, t) = L(t) \quad \forall t > 0.$$

Moreover, this process is almost surely locally γ-Holder continuous for any $\gamma < \frac{1}{2}$.

Theorem 9.6. $\{L(t) : t \geqslant 0\} \overset{d}{=} \{M(t) : t \geqslant 0\}$

Theorem 9.7 (Occupation time representation of the local time at zero)**.** *For all sequence $a_n \uparrow 0$ and $b_n \downarrow 0$ with $a_n < b_n$, almost surely,*

$$\lim_{n\to\infty} \frac{1}{b_n - a_n} \int_0^t 1_{a_n \leqslant B(s) \leqslant b_n} \mathrm{d}s = L(t) \quad \forall t > 0$$

Theorem 9.8 (Ray-Knight theorem)**.** *Suppose $a > 0$ and $\{B(t) : 0 \leqslant t \leqslant T\}$ is a liinear BM started at a and stopped at time $T = \int\{t \geqslant 0 : B(t) = 0\}$. Then*

$$\{L^x(T) : 0 \leqslant x \leqslant a\} \overset{d}{=} \{|W(x)|^2 : 0 \leqslant x \leqslant a\},$$

where $\{W(x) : x \geqslant 0\}$ is a standard planar BM.

10 Stochastic Integration

10.1

10.2 Martingale Characterization of BM

Theorem 10.1 (Levy)**.**

10.3 Representations of Martingales by BM

Theorem 10.2 (time-change for martingales)**.**

Theorem 10.3 (representation of square-integrable martingales by BM via Ito's integral)**.**

10.4 The Girsanov Theorem

11 The PDE Connection

12 Stochastic Differential Equations

13 Diffusions

13.1 Kolmogorov's Theory

13.2 Ito's theory