Geometry

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Chapter 1

Riemannian Geometry

1.1 Riemannian Metrics

1.2 Tensor Algebra

1.3 Connection

1.3.1 Levi-Civita Connection

A connection is symmetric if

metric compaibility

Theorem 1.3.1. *Given a metric g, there is a unique connection ∇ that is symmetric and metric compatible. This connection is called the* ***Levi-Civita connection****, and is given by*

1.3.2 Tensor Leibniz

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k$$

Proposition 1.3.2.

$$F^{i_1 \dots i_k}_{j_1 \dots j_l;m} = E_m(F^{i_1 \dots i_k}_{j_1 \dots j_l}) + \sum_{s=1}^{k} \Gamma^{i_s}_{mp} F^{i_1 \dots p \dots i_k}_{j_1 \dots j_l} - \sum_{s=1}^{l} \Gamma^{p}_{mj_s} F^{i_1 \dots i_k}_{j_1 \dots p \dots j_l}$$

higher order covariant derivative can be computed by iterating

1.3.3 Along a Curve

1.4 Curvature

The covariant derivative of a (r, s) tensor can be thought of as an $(r, s+1)$ tensor in a natural way. For example, if

Repeating this, we get a $(1, 2)$ tensor $\nabla\nabla V$, which will abbreviate as $\nabla^2 V$.

Using the Leibniz rule, we see that

$$\nabla_X(\nabla_Y V) = \nabla^2_{X,Y} V + \nabla_{\nabla_X Y} V$$

The tensor is not necessarily symmetric in the two lower slots. In fact, the curvature comes in

$$\begin{aligned} \nabla^2_{X,Y} V - \nabla^2_{Y,X} V &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{\nabla_X Y - \nabla_Y X} V \\ &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{[X,Y]} V \\ &= R(Y, X) V \end{aligned}$$

This is known as the **Ricci identity**.

Definition 1.4.1 (Riemann curvature)**.**

1.4.1 Symmetries