

Geometry

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Chapter 1

Smooth Manifold

Lie derivative

1.1 Tensor Algebra

Chapter 2

Riemannian Manifold

2.1 Riemannian Metrics

2.2 Affine Connections

An affine connection $\nabla(\cdot)$ is a map from $\Gamma(M) \times \Gamma(M)$ to $\Gamma(M)$

Definition 2.2.1. 1. $C(M)$ -linear in the lower slot

2. \mathbb{R} -linear in the upper slot

3. $C(M)$ -Leibniz rule in the upper slot

2.2.1 Levi-Civita Connection

Let ∇ be an affine connection, then by the basis theorem there are locally defined functions Γ_{ij}^k such that

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k. \quad (2.1)$$

A connection is symmetric if
metric compaibility

Theorem 2.2.2. *Given a metric g , there is a unique connection ∇ that is symmetric and metric compatible. This connection is called the **Levi-Civita connection**, and is given by*

2.2.2 Tensor Leibniz Rule

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$$

Proposition 2.2.3.

$$F_{j_1 \dots j_l; m}^{i_1 \dots i_k} = E_m(F_{j_1 \dots j_l}^{i_1 \dots i_k}) + \sum_{s=1}^k \Gamma_{mp}^{i_s} F_{j_1 \dots j_l}^{i_1 \dots p \dots i_k} - \sum_{s=1}^l \Gamma_{mj_s}^p F_{j_1 \dots p \dots j_l}^{i_1 \dots i_k}$$

higher order covariant derivative can be computed by iterating

2.2.3 Along a Curve

2.3 Curvature

The covariant derivative of a (r, s) tensor can be thought of as an $(r, s+1)$ tensor in a natural way. For example, if

Repeating this, we get a $(1, 2)$ tensor $\nabla \nabla V$, which will abbreviate as $\nabla^2 V$.

Using the Leibniz rule, we see that

$$\nabla_X(\nabla_Y V) = \nabla_{X,Y}^2 V + \nabla_{\nabla_X Y} V$$

The tensor is not necessarily symmetric in the two lower slots. In fact, the curvature comes in

$$\begin{aligned}
\nabla_{X,Y}^2 V - \nabla_{Y,X}^2 V &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{\nabla_X Y - \nabla_Y X} V \\
&= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{[X,Y]} V \\
&= R(Y, X)V
\end{aligned}$$

This is known as the **Ricci identity**.

Definition 2.3.1 (Riemann curvature).

2.3.1 Symmetries

2.3.2

The *Ricci curvature* is a $(0, 2)$ tensor given by taking the trace of R .

Definition 2.3.2 (Ricci Curvature).

A manifold is said to be *Einstein* with Einstein constant $\lambda \in \mathbb{R}$ if
The trace of the Ricci tensor is the *scalar curvature*:

Definition 2.3.3 (Scalar Curvature).

Definition 2.3.4 (Sectional curvature).

A Riemannian manifold has *constant sectional curvature* κ if

Lemma 2.3.5 (Schur Lemma).

$$dS = 2 \operatorname{div} \operatorname{Ric} \tag{2.2}$$

2.4 Submanifolds

If (N, g) is a Riemannian manifold and

$$\phi : M \rightarrow N$$

is an immersion, then ϕ^*g gives a metric on M .

$$T_{\phi(p)}N = d\phi(T_p M) \otimes [d\phi(T_p M)]^\perp$$

induced connection

Definition 2.4.1 (Second Fundamental Form).

M is said to be totally geodesic if $A \equiv 0$. totally geodesic

Definition 2.4.2 (Mean Curvature Vector). The trace of the second fundamental form is called the mean curvature vector \mathbf{H} .

M is said to be *minimal* if $\mathbf{H} = 0$.