

Geometry

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Chapter 1

Smooth Manifold

Lie derivative

1.1 Tensor Algebra

Let $\Gamma^{r,s}(M)$ denote the space of (r, s) tensors on M .

Given a coordinate $A \in \Gamma^{r,s}$ can be written as

$$A = A_{j_1, \dots, j_s}^{i_1, \dots, i_r} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}, \quad (1.1)$$

tangent vectors $\Gamma(M) = \Gamma^{1,0}(M)$

one-forms $\Gamma^*(M) = \Gamma^{0,1}(M)$

Chapter 2

Riemannian Manifold

2.1 Riemannian Metrics

raising and lowering indices

Example 2.1.1 (Gradient).

$$\nabla f = \sum_{i,j} g^{ij} \partial_j f$$

$$g(\nabla f, V) = V(f)$$

2.2 Affine Connections

An affine connection $\nabla(\cdot)$ is a map from $\Gamma(M) \times \Gamma(M)$ to $\Gamma(M)$

Definition 2.2.1. 1. $C(M)$ -linear in the lower slot

2. \mathbb{R} -linear in the upper slot

3. $C(M)$ -Leibniz rule in the upper slot

2.2.1 Levi-Civita Connection

Let ∇ be an affine connection, then by the basis theorem there are locally defined functions Γ_{ij}^k such that

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k. \quad (2.1)$$

A connection is symmetric if
metric compaibility

Theorem 2.2.2. *Given a metric g , there is a unique connection ∇ that is symmetric and metric compatible. This connection is called the **Levi-Civita connection**, and is given by*

2.2.2 Tensor Leibniz Rule

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$$

Proposition 2.2.3.

$$F_{j_1 \dots j_l; m}^{i_1 \dots i_k} = E_m(F_{j_1 \dots j_l}^{i_1 \dots i_k}) + \sum_{s=1}^k \Gamma_{mp}^{i_s} F_{j_1 \dots j_l}^{i_1 \dots p \dots i_k} - \sum_{s=1}^l \Gamma_{mj_s}^p F_{j_1 \dots p \dots j_l}^{i_1 \dots i_k}$$

higher order covariant derivative can be computed by iterating $\nabla : \Gamma^{r,s} \rightarrow \Gamma^{r,s+1}$

2.2.3 Along a Curve

2.3 Curvature

Definition 2.3.1. The Riemann curvature is a $(1,3)$ tensor given by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z. \quad (2.2)$$

The covariant derivative of a (r, s) tensor can be thought of as an $(r, s + 1)$ tensor in a natural way. For example, if

Repeating this, we get a $(1, 2)$ tensor $\nabla \nabla V$, which will abbreviate as $\nabla^2 V$.

Using the Leibniz rule, we see that

$$\nabla_X (\nabla_Y V) = \nabla_{X, Y}^2 V + \nabla_{\nabla_X Y} V \quad (2.3)$$

The tensor is not necessarily symmetric in the two lower slots. In fact, the curvature comes in

$$\begin{aligned} \nabla_{X, Y}^2 V - \nabla_{Y, X}^2 V &= \nabla_X (\nabla_Y V) - \nabla_Y (\nabla_X V) - \nabla_{\nabla_X Y - \nabla_Y X} V \\ &= \nabla_X (\nabla_Y V) - \nabla_Y (\nabla_X V) - \nabla_{[X, Y]} V \\ &= R(Y, X)V \end{aligned}$$

This is known as the **Ricci identity**.

Definition 2.3.2 (Riemann curvature).

2.3.1 Symmetries

2.3.2

The *Ricci curvature* is a $(0, 2)$ tensor given by taking the trace of R .

Definition 2.3.3 (Ricci Curvature).

A manifold is said to be *Einstein* with Einstein constant $\lambda \in \mathbb{R}$ if

The trace of the Ricci tensor is the *scalar curvature*:

Definition 2.3.4 (Scalar Curvature).

Definition 2.3.5 (Sectional curvature).

A Riemannian manifold has *constant sectional curvature* κ if

Lemma 2.3.6 (Schur Lemma).

$$dS = 2 \operatorname{div} \operatorname{Ric} \quad (2.4)$$

2.4 Submanifolds

If (N, g) is a Riemannian manifold and

$$\phi : M \rightarrow N$$

is an immersion, then ϕ^*g gives a metric on M .

$$T_{\phi(p)}N = d\phi(T_p M) \oplus [d\phi(T_p M)]^\perp$$

induced connection

Definition 2.4.1 (Second Fundamental Form).

$$A(V, W) = (\nabla_V W)^\perp \quad (2.5)$$

M is said to be totally geodesic if $A \equiv 0$. totally geodesic

Definition 2.4.2 (Mean Curvature Vector). The trace of the second fundamental form is called the mean curvature vector \mathbf{H} .

M is said to be *minimal* if $\mathbf{H} = 0$.

2.4.1 Gauss and Codazzi Equations

The Gauss equation relates the curvature of the submanifold to the curvature via the second fundamental form

Theorem 2.4.3 (Gauss Equation).

Theorem 2.4.4 (Codazzi Equation).

2.4.2 Hypersurfaces

An immersed submanifold $M^m \subset N^n$ is a *hypersurface* when $m = n - 1$. *unit normal* \mathbf{n}

Using \mathbf{n} , the second fundamental form

Chapter 3

Geodesics and Minimal Submanifolds

3.1 Variational Theory of Geodesics

Jacobi equation

Definition 3.1.1 (Conjugate Points). Suppose γ is a geodesic. We say that $\gamma(t_2)$ is conjugate to $\gamma(t_1)$ along γ if there is a non-zero Jacobi field J along γ so that $J(t_1) = J(t_2) = 0$

The energy of a piece-wise smooth curve $\gamma : [0, a] \rightarrow M$

$$\mathbf{E}(\gamma) = \int_0^a |\gamma'|^2 dt. \quad (3.1)$$

3.2 Variational Theory of Minimal Submanifolds

Given an isometrically embedded submanifold $\Sigma \subset M$, a variation is a map

$$F : \Sigma \times (-\epsilon, \epsilon) \rightarrow M$$

so that $F(x, 0) = x$. The variation vector field $F_s = dF(\partial_s)$ describes the motion of points in Σ under the variation.

3.2.1 First Variation

Proposition 3.2.1.

$$\partial_s \overline{dv} = (\operatorname{div}(F_s^T) - \langle F_s^\perp, \mathbf{H} \rangle) \overline{dv} \quad (3.2)$$

3.2.2 Monotonicity

3.2.3 Second Variation

We now investigate the second derivative of volume for a hypersurface

$$\Sigma^n \subset M^{n+1},$$

whose calculations are substantially simpler than a general submanifold.

Lemma 3.2.2. *Given a normal variation $F_s = u\mathbf{n}$, the derivative of \mathbf{H} at 0 is*

$$\mathbf{H}' = (\Delta u + |A|^2 u + \operatorname{Ric}(\mathbf{n}, \mathbf{n}))\mathbf{n} - \langle \mathbf{H}, \mathbf{n} \rangle \nabla u. \quad (3.3)$$

Chapter 4

The Laplacian

4.1 Divergence and Laplacian

Let M^n be a manifold with a metric g and associated Levi-Civita connection ∇ .
derivative of $u \in C(M)$:

- gradient $\nabla u \in \Gamma^{1,0}(M)$
- differential $du \in \Gamma^{0,1}(M)$.

they are dual via the metric

$$\langle \nabla u, V \rangle = du(V) = V(u).$$

Definition 4.1.1 (Hessian). The Hessian is a $(0,2)$ tensor defined by ∇du , which satisfies

$$\text{Hess}_u(V, W) = \langle \nabla_V \nabla u, W \rangle. \quad (4.1)$$

Proposition 4.1.2. Hess_u is symmetric.

We will give two proofs.

Proof 1. First,

$$\text{Hess}_u(\partial_i, \partial_j) = u_{ij} - \sum_k \Gamma_{ij}^k u_k$$

□

Proof 2.

$$\begin{aligned} \text{Hess}_u(V, W) - \text{Hess}_u(W, V) &= \langle \nabla_V \nabla u, W \rangle - \langle \nabla_W \nabla u, V \rangle \\ &= V \langle \nabla u, W \rangle - \langle \nabla u, \nabla_V W \rangle - W \langle \nabla u, V \rangle + \langle \nabla u, \nabla_W V \rangle \\ &= V(W(u)) - W(V(u)) - \langle \nabla u, [V, W] \rangle = 0. \end{aligned}$$

□

Definition 4.1.3 (Divergence). The divergence $\text{div } V$ of a vector field V is the trace of the $(1,1)$ tensor ∇V .

In an orthonormal frame

$$\text{div } V = \sum_i \langle \nabla_{e_i} V, e_i \rangle$$

In a local coordinate,

$$\text{div } V = \partial_i V^i + \Gamma_{ij}^j V^i$$

But we also have

Proposition 4.1.4.

$$\text{div } V = \frac{1}{\sqrt{\det g}} \sum_i \partial_i (\sqrt{\det g} V^i). \quad (4.2)$$

Proof. This is shown by

$$\frac{\partial_i \sqrt{\det g}}{\sqrt{\det g}} = \sum_j \Gamma_{ij}^j$$

□

Proposition 4.1.4 makes it possible to extend the Euclidean divergence theorem to manifolds. Recall the volume element $dv_g = \sqrt{\det g} dx$, we have

$$\operatorname{div}(V) dv_g = \operatorname{div}_{\mathbb{R}^n}(V \sqrt{\det g}) dx. \quad (4.3)$$

From Leibniz rule

$$\operatorname{div}(uV) = u \operatorname{div}(V) + \langle \nabla u, V \rangle \quad (4.4)$$

Definition 4.1.5 (Laplacian). The Laplacian is the divergence of ∇u .

Therefore, we have in an orthonormal frame

$$\begin{aligned} \Delta u &= \sum_i \langle \nabla_{e_i} \nabla u, e_i \rangle \\ &= \sum_i \nabla_{e_i} \langle \nabla u, e_i \rangle - \langle \nabla u, \nabla_{e_i} e_i \rangle \\ &= \sum_i \nabla_{e_i} \nabla_{e_i} u - \nabla_{\nabla_{e_i} e_i} u \end{aligned}$$

which is in accordance with Equation (2.3)

4.1.1 Bochner Formula

Proposition 4.1.6. If $u \in C(M)$, then

$$\Delta \nabla u = \nabla(\Delta u) + \operatorname{Ric}(\nabla u, \cdot) \quad (4.5)$$

Proof 1.

□

geodesic normal coordinates for $p \in M$ fixed

- $g_{ij}(p) = \delta_{ij}$, meaning that ∂_i are orthonormal at p
- $\Gamma_{ij}^k = 0$, meaning that $\nabla \partial_i(p) = 0$

Proof 2.

□

Corollary 4.1.7.

$$\Delta |\nabla u|^2 = |\operatorname{Hess}_u|^2 + \langle \nabla(\Delta u), \nabla u \rangle + \operatorname{Ric}(\nabla u, \nabla u) \quad (4.6)$$

4.1.2 Lichnerowicz Theorem

4.2 Submanifold Divergence and Laplacian

In this section, $\Sigma \subset M$ is a submanifold with the induced connection $\bar{\nabla}$.

Proposition 4.2.1. The submanifold Hessian is given by

$$\overline{\operatorname{Hess}}_u(V, W) = \operatorname{Hess}_u(V, W) + \langle A(V, W), \nabla^\perp u \rangle \quad (4.7)$$

Proof.

$$\begin{aligned} \overline{\operatorname{Hess}}_u(V, W) &= \langle \nabla_V \nabla^T u, W \rangle \\ &= \langle \nabla_V \nabla u, W \rangle - \langle \nabla_V \nabla^\perp u, W \rangle \\ &= \operatorname{Hess}_u(V, W) + \langle \nabla^\perp u, \nabla_V W \rangle \\ &= \operatorname{Hess}_u(V, W) + \langle A(V, W), \nabla^\perp u \rangle \end{aligned}$$

□

4.3 Laplacian Comparison

4.3.1 Distance Function

Definition 4.3.1 (Cut Point).

Theorem 4.3.2. *If $q \in \text{Cut}(p)$, then either*

1. *q is the first conjugate to p along a minimizing geodesic, or*
2. *q is the first point along a minimizing geodesic where there is a second, different, minimizing geodesic from p .*

Now we show that d is smooth away from $\text{Cut}(p)$.

Proposition 4.3.3.

$$\begin{aligned} \Delta|x|^2 & \text{ on } \mathbb{R}^n \\ \nabla|x|^2 &= 2|x|\nabla|x| \\ \Delta|x| &= \frac{n-1}{|x|} \text{ on } \mathbb{R}^n \setminus \{0\} \end{aligned}$$

Another way $\text{Hess}_{|x|}$
Suppose $\text{Ric} \geq 0$

4.3.2 Laplacian Comparison

We first state the Laplacian comparison for smooth points

Theorem 4.3.4. *Suppose that $\text{Ric} \geq 0$ and $r(x) = d(p, x)$ for p fixed. Away from $\text{Cut}(p) \cup \{p\}$, we have that*

$$\Delta r \leq \frac{n-1}{r}.$$

Proof. First, Hess_r has rank at most $n-1$, so

$$|\text{Hess}_r|^2 \geq \frac{(\Delta r)^2}{n-1}.$$

By Bochner formula (Corollary 4.1.7), along γ ,

$$\begin{aligned} 0 &= \frac{1}{2} \Delta |\nabla r|^2 \\ &= |\text{Hess}_r|^2 + \langle \nabla(\Delta r), \nabla r \rangle + \text{Ric}(\nabla r, \nabla r) \\ &\geq \frac{(\Delta r)^2}{n-1} + (\Delta r)' \end{aligned}$$

□

Lemma 4.3.5. *If $f(t)$ is a function on $(0, T]$ with $(n-1)f' \leq -f^2$, then*

$$f(T) \leq \frac{n-1}{T}.$$

Proof. Notice that

$$\left(\frac{1}{f(t)} \right)' = -\frac{f'(t)}{f(t)^2}$$

□

Definition 4.3.6 (barrier). We say that $\Delta f \geq g$ at $p \in \Omega$ in the *barrier sense* if for every $\epsilon > 0$ there exists a C^2 function h_ϵ and an open set U_ϵ containing p so that

1. $f(p) = h_\epsilon(p)$ and $h_\epsilon \leq f$ in U_ϵ .
2. $\Delta h_\epsilon(p) \geq g(p) - \epsilon$.

Definition 4.3.7 (viscosity). We say that $\Delta f \geq g$ at $p \in \Omega$ in the *viscosity sense* if for every open set U containing p and C^2 function φ on U with $f(p) = \varphi(p)$ and $\varphi \geq f$ in U , we have that $\Delta \varphi(p) \geq g(p)$.

Proposition 4.3.8. *If $\Delta f \geq g$ at p in the barrier sense, then it also holds in the viscosity sense. If $\Delta f \geq g$ at p in the viscosity sense, then it also holds in the distributional sense.*

Theorem 4.3.9. *If $\text{Ric} \geq 0$ and $d(x) := d(p, x)$ for some fixed p , then $\Delta d \leq \frac{n-1}{d}$ in the barrier sense on $M \setminus \{p\}$.*

4.3.3 Bishop-Gromov Volume Comparison

Theorem 4.3.10. *If $\text{Ric} \geq 0$, $p \in M^n$, and $0 < r_1 < r_2$, then*

$$\frac{\text{Vol}(B_{r_2}(p))}{r_2^n} \leq \frac{\text{Vol}(B_{r_1}(p))}{r_1^n}.$$

4.3.4 Dirichlet Poincare Inequality

4.4 Gradient Estimates

Theorem 4.4.1. *If $B_R(p) \subset M^n$ has $\text{Ric} \geq 0$, $\Delta u = 0$, and $u > 0$, then*

$$\sup_{B_{R/2}(p)} |\nabla \log u| \leq \frac{C}{R},$$

where C depends just on n .

Bernstein

Lemma 4.4.2. *$w = \log u$ satisfies*

- $\Delta w = -|\nabla w|^2$
- $\frac{1}{2} \Delta |\nabla w|^2 \geq \frac{1}{n} |\nabla w|^4 - \langle \nabla w, \nabla |\nabla w|^2 \rangle$

But $|\nabla w|^2$ may not have an interior max. Use cut-off

Proof.

□