Lecture Notes in Functional Analysis

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Contents

0.1 Prologue

In analysis, examples and counterexamples are important.

Chapter 1

Basic Topology

To faciliate the study of convergence on an aribitary space

We use metric space as example, for the sake of functional analysis.

1.1 Basic Topology

1.1.1 Definitions of Topology

neighborhood, open set, closed set,

Neighborhood

Let X be a (possibly empty) set. Let $\mathcal{N}$ be a function assigning to each $x \in X$ a non-empty collection $\mathcal{N}(x)$ of subsets of X. The elements of $\mathcal{N}(x)$ will be called neighbourhoods of x.

Interior

Open

Closed

1.1.2 Comparing Different Topologies

Definition 1.1.1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. We say $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$.

1.1.3 Constructing New

Subspace Topology

Product Topology

1.1.4 Topological Basis

1.1.5 Sets in Topological Space

1.2 Compactness

Definition 1.2.1 (Compact)**.** Every open covering has finite sub covering.

Leveraging the duality of open and closed set, we can immediately

Definition 1.2.2 (Sequentially Compact)**.** Every sequence has a convergent subsequence.

What about subsets? The canonical definition is the following

Remark 1.2.3. Let $A \subset X$ be a subset. If $(A, \mathcal{T}_{subspace})$ is compact, then we say A is compact in X.

We can see that the above definition reduce to the common

Proposition 1.2.4. *Let $A \subset X$ be a subset. A is compact in X if and only if*

1.2.1 Constructing New

Subspace of Compact

Product of Compact

Theorem 1.2.5 (Tychonoff). *If X_α is compact for all α, then $(\prod_\alpha X_\alpha, \mathcal{T}_{product})$ is also compact.*

1.2.2 Compactness in Metric Space

Theorem 1.2.6. *In a metric space (X, d), TFAE:*

(i) A is compact

(ii) A is sequentially compact

*(iii) A is **totally** bounded and complete.*

1.3 Separability

Definition 1.3.1 (first countability).

Definition 1.3.2 (second countability).

Chapter 2

Banach Spaces I

We can regard linear functional analysis as an extension of linear algebra to infinite-dimensional cases, with $\mathbb{F} = \mathbb{R}$, or $\mathbb{F} = \mathbb{C}$. In classical linear algebra we utilize the powerful concept of dimension to derive the structure of linear spaces and linear transformations. Here to cope with infinity, we must resort to techniques from mathematical analysis.

2.1 Basic Definitions and Examples

Definition 2.1.1 (Seminorm)**.** If V is a vector space over $\mathbb{F}$, a seminorm is a function $p : V \to [0, \infty)$ with:
(i) $p(x + y) \le p(x) + p(y)$, $\forall x, y \in V$
(ii) $p(\alpha x) = |\alpha|\, p(x)$, $\forall \alpha \in \mathbb{F}$, $\forall x \in V$.

Note that (ii) implies $p(0) = 0$.

Definition 2.1.2 (Norm)**.** A norm is a seminorm p s.t. $x = 0$ if $p(x) = 0$. Usually a norm is denoted by $\|\cdot\|$. A normed space is a vector space endowed with with a norm. A Banach space is a normed space that is complete w.r.t. the metric defined by the norm.

Lemma 2.1.3 (Continuity)**.** *If V is a normed space, then:*
(i) $V \times V \to V$, $(x, y) \mapsto x + y$ is continuous.
(ii) $\mathbb{F} \times V \to V$, $(\alpha, x) \mapsto \alpha x$ is continuous.

Proof. By the definiton of continuity and the triangle inequality of norm. □

Lemma 2.1.4. *If p and q are seminorms on V, TFAE:*
(i) $p \le q$
(ii) $p < 1$ whenever $q < 1$
(iii) $p \le 1$ whenever $q \le 1$
(iv) $p \le 1$ whenever $q < 1$.

Proof. Only need to show (iv) implies (i).
Suppose (iv). Fix any x, and set $\alpha = q(x)$. Then for any $\epsilon > 0$, $q(\frac{1}{\alpha+\epsilon}x) < 1$. So $p(\frac{1}{\alpha+\epsilon}x) \le 1$ and $p(x) \le q(x) + \epsilon$. The arbitariness of ϵ implies $p \le q$. □

Definition 2.1.5 (Equivalent Norms)**.** If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on V, they are said to be equivalent norms if they define the same topology on V.

Lemma 2.1.6.

2.2 Linear Operators on Normed Spaces

$B(V, W)$ = all continuous linear transformations from V to W.

Lemma 2.2.1. *If V and W are normed spaces and $T : V \to W$ is a linear transformation, TFAE:*
(i) $T \in B(V, W)$.
(ii) T is continuous at 0.
(iii) T is continuous at some point.
(iv) $\exists c > 0$ s.t. $\forall x \in V$ $\|Tx\| \le c\|x\|$.

2.3 Finite Dimensional Normed Spaces

Theorem 2.3.1. *If V is a finite dimensional vector space, then any two norms on V are equivalent.*

Proof. Let $\{e_1, \cdots, e_d\}$ be a Hamel basis for V. For $x = \sum_{j=1}^d x_j e_j$, define $\|x\|_\infty := \max\{|x_j| : 1 \leq j \leq d\}$. Then $\|x\|_\infty$ is a norm. Let $\|\cdot\|$ be any norm on V.If $x = \sum_{j=1}^d x_j e_j$, then $\|x\| \leq C\|x\|_\infty$, where $C = \sum_{j=1}^d \|e_j\|$. To show the other inequality, we need a technique from analysis (or topology).
???
□

Corollary 2.3.2. *A finite dimensional linear manifold M in a normed space V is closed.*

Proof. Choose a Hamel basis for M and define a norm $\|\cdot\|_\infty$ as above. Then M is complete w.r.t $\|\cdot\|_\infty$, thus complete w.r.t the original norm by the above theorem. Hence it is closed. □

Corollary 2.3.3. *A linear transformation T from a finite dimensional normed space V to any normed space W is continuous.*

Proof. Since all norms are equivalent on V, we may choose a Hamel basis and define $\|\cdot\|_\infty$ as above. Thus for $x = \sum_{j=1}^d x_j e_j$, $\|Tx\| = \|\sum_j x_j T e_j\| \leq \sum_j |\xi_j| \|Te_j\| \leq C\|x\|_\infty$, where $C = \sum_j \|Te_j\|$. Hence T is bounded and continuous. □

2.4 Quotients and Products of Normed Spaces

Let V be a normed space, let M be a linear manifold in V, and let $Q : V \to V/M$ be the natural map $Qx = x + M$. Our goal is to make V/M into a normed space, so define

$$\|x + M\| = \inf\{\|x + y\| : y \in M\} = \text{dist}(x, M).$$

This defines a seminorm on V/M, but if M is not closed, it can not define a norm.

Theorem 2.4.1. *If $M \leq V$ and $\|\cdot\|$ is defined above, then it is a norm on V/M. Also:*
(i)

2.5 The Hahn-Banach Theorem

We first state the Hahn-Banach Theorem for real spaces and then extend it to complex spaces. Thereafter, we discuss several corollary of it. We postpone the proof to the end of this section.

Definition 2.5.1 (Sublinear Functional)**.** If V is a vector space over $\mathbb{R}$, a seminorm is a function $q : V \to \mathbb{R}$ with:
(i) $q(x + y) \leq q(x) + q(y)$, $\forall x, y \in V$
(ii) $q(\alpha x) = \alpha q(x)$, $\forall \alpha \mathcal{E} 0$, $\forall x \in V$.

Theorem 2.5.2 (Hahn-Banach)**.** *Let V be a vector space over $\mathbb{R}$ and let q be a sublinear functional on V. If M is a linear manifold in V and $f : M \to \mathbb{R}$ is a linear functional s.t. $\forall x \in M$, $f(x) \leq q(x)$, then there is a linear functional $F : V \to \mathbb{R}$ s.t. $F|M = f$ and $\forall x \in V$, $F(x) \leq q(x)$.*

Note that the essence of the theorem is not the extension exists but that an extension can be found that remains dominated by q. Just to find an extension, we can simply take a Hamel basis.

Lemma 2.5.3 (Complexification)**.** *Let V be a linear space over $\mathbb{C}$.*
(i) If $f : V \to \mathbb{R}$ is an $\mathbb{R}$-linear functional, then $\tilde{f}(x) = f(x) - if(ix)$ is a $\mathbb{C}$-linear functional and $f = \Re\tilde{f}$.
(ii) If $g : V \to \mathbb{C}$ is $\mathbb{C}$-linear, $f = \Re g$, and $\tilde{f}$ is defined as in (i), then $\tilde{f} = g$.
(iii) If p is a seminorm on V, and f and $\tilde{f}$ are as in (i), then $\forall x$ $|f(x)| \leq p(x)$ if and only if $\forall x$ $|\tilde{f}(x)| \leq p(x)$.
(iv) If V is furthermore a normed space, and f and $\tilde{f}$ are as in (i), then $\|f\| = \|\tilde{f}\|$.

Theorem 2.5.4 (Seperation of Convex Sets)**.**

2.6 The Opening Mapping Theorem

Theorem 2.6.1 (Open Mapping Theorem)**.** *If V and W are Banach spaces and $T : V \to W$ is a continuous linear surjection, then $T(G)$ is open in W whenever G is open in V.*

Theorem 2.6.2 (Inverse Mapping Theorem)**.** *If V and W are Banach spaces and $T : V \to W$ is a bounded linear transformation that is bijective, then T^{-1} is bounded.*

Proof. As T is continuous and surjective, by the open mapping theorem T is open and hence a homeomorphism. ☐

Definition 2.6.3 (isomorphism between Banach spaces)**.** If V and W are Banach spaces, an isomorphism of V and W is a linear bijection that is a homeomorphism. Say V and W are isomorphic if there is an isomorphism of V onto W.

Remark 2.6.4. The use of the word 'isomorphism' is counter to the spirit of category theory, but it is traditional in Banach space theory. The inverse mapping theorem just says that a continuous bijection is an isomorphism.

Theorem 2.6.5 (Closed Graph Theorem)**.** *If V and W are Banach spaces and $T : V \to W$ is a linear transformation s.t. the graph of T,*

$$G = \{x \oplus Tx \in V \oplus_1 W : x \in V\}$$

is closed, then T is continuous.

Proof. Since $V \oplus_1 W$ is a Banach space and G is closed, G is a Banach space. Define $P_1 : G \to V$ by $P_1(x \oplus Tx) = x$ and define $P_2 : G \to W$ by $P_2(x \oplus Tx) = Tx$. P_1 and P_2 are bounded. Moreover, P_1 is bijective. By the inverse mapping theorem, P_1^{-1} is continuous. Thus $T = P_2 \circ P_1^{-1}$ is continuous. ☐

2.7 The Principle of Uniform Boundness

Theorem 2.7.1. *Let X be a Banach space and Y a normed space. If $\mathcal{T} \subset B(X, Y)$ s.t. $\forall x \in V$, $\sup\{\|Tx\| : T \in \mathcal{T}\} < \infty$, then $\sup\{\|T\| : T \in \mathcal{T}\} < \infty$.*

2.8 The Adjoint of a Linear Operator

2.9 The Banach-Stone Theorem

2.10 Compact Operators

Definition 2.10.1. If V and W are Banach spaces and $T : V \to W$ is a linear transformation, then T is compact if ??? is compact in W.

Chapter 3

Hilbert Spaces

3.1 Elementary Properties and Examples

3.2 The Riesz Representation Theorem

3.3 Orthogonality

3.4 Isomorphisms

3.5 Direct Sum

3.6 Operators

3.7 The Adjoint of an Operator

3.8 Projections and Idempotents

3.9 Compact Operators

3.10 Discussion

This chapter follows Conway [2019]

Chapter 4

Sobolev Spaces

We establish a proper setting in which to apply ideas of functional analysis to glean information concerning PDE

4.1 Hölder Spaces

Assume $U \subset \mathbb{R}^n$ is open and $0 < \gamma \leq 1$.

Definition 4.1.1 (Hölder continuous)**.** A function is said to be Hölder continuous with exponent $\gamma \in (0, 1]$ if $\forall x, y \in U$

$$|u(x) - u(y)| \leq C|x - y|^\gamma \tag{4.1}$$

for some constant C.

Definition 4.1.2. (i) If $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\bar{U})} := \sup_{x \in U} |u(x)|$$

(ii) The γth-Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{x,y \in U, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}$$

and the γth-Hölder norm is

$$\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$$

Definition 4.1.3. The Hölder space

$$C^{k,\gamma}(\bar{U})$$

consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [u]_{C^{0,\gamma}(\bar{U})}$$

is finite.

Theorem 4.1.4 (Hölder spaces as function spaces)**.** *$C^{k,\gamma}(\bar{U})$ is a Banach space.*

Proof. The construction of $\| \cdot \|_{C^{k,\gamma}(\bar{U})}$ ensures that it is a norm. In addition, each Cauchy sequence converges. $\square$

4.2 Sobolev Spaces

The Hölder spaces are unfortunately not often suitable settings for elementary PDE theory, as we usually cannot make good enough analytic estimates to demonstrate that the solutions we construct actually belong to such spaces. What are needed rather are some kind of spaces containing less smooth functions.

4.2.1 Weak Derivatives

We start off by substantially weakening the notion of partial derivatives.

Notation 4.2.1. Let $C_c^\infty(U)$ denote the space of infinitely differentiable function $\varphi : U \to \mathbb{R}$, with compact support in U. We will sometimes call a function belonging to $C_c^\infty(U)$ a test function.

Definition 4.2.2. Suppose

Lemma 4.2.3 (uniqueness of weak derivatives)**.** *A weak αth-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Example 4.2.4. Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x & x \in (0, 1] \\ 1 & x \in [1, 2) \end{cases}$$

has weak derivative

$$v(x) = \begin{cases} x & x \in (0, 1] \\ 1 & x \in [1, 2) \end{cases}.$$

But

$$u(x) = \begin{cases} x & x \in (0, 1] \\ 2 & x \in (1, 2) \end{cases}$$

does not have a weak derivative.

4.2.2 Definition of Sobolev Spaces

Fix $1 \le p \le \infty$ and let k be a nonnegative integer.

Definition 4.2.5. The Sobolev space

$$W^{k,p}(U)$$

consists of all locally summable function $u : U \to \mathbb{R}$ s.t. for each multiindex α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Notation 4.2.6. If $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U)$$

H is used since $H^k(U)$ is a Hilbert space.

Definition 4.2.7. If $u \in W^{k,p}(U)$, we define the norm to be

$$\begin{cases} \left(\sum_{|\alpha|\le k} \int_U |D^\alpha u|^p \mathrm{d}x\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_U |D^\alpha u| & p = \infty \end{cases} \tag{4.2}$$

Notation 4.2.8.

Notation 4.2.9. We denote by

$$W_0^{k,p}(U)$$

the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

We interpret $W_0^{k,p}(U)$ as comprising those functions $u \in W^{k,p}(U)$ s.t.

$$D^\alpha = 0 \text{ on } \partial U \quad \forall |\alpha| \le k - 1.$$

Example 4.2.10. If $n = 1$ and U is an open interval in $\mathbb{R}^1$

4.2.3 Properties

Theorem 4.2.11 (weak derivatives). *Assume $u, v \in W^{k,p}(U)$, $|\alpha| \le k$. Then*
(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$, and $D^\alpha(D^\beta u) = D^{\alpha+\beta} u \;\forall |\alpha| + |\beta| \le k$.
(ii) For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \le k$.
(iii) If V is an open subset of U, then $u \in W^{k,p}(V)$.
(iv) If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and

$$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} C_\alpha^\beta D^\beta \zeta D^{\alpha-\beta} u$$

Theorem 4.2.12 (Sobolev space is Banach). *For each $k \in \mathbb{Z}_{\mathcal{E}1}$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.*

Proof. Note that the completeness is encoded in the definition. If $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence, then for $|\alpha| \le k$, $\{D^\alpha u_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(U)$ and $L^p(U)$ is complete. □

4.3 Approximation

We need to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions. The method of mollifiers provides such a tool.

Notation 4.3.1. If $U \subset \mathbb{R}^n$ is open and $\epsilon > 0$, we write

$$U_\epsilon := \{x \in U | \text{dist}(x, \partial U) > \epsilon\}$$

Definition 4.3.2. Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x|\mathcal{E}1 \end{cases}$$

C is selected s.t. $\int \eta = 1$. We call η the standard mollifier.
For each $\epsilon > 0$, set

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$$

Definition 4.3.3. If $U \to \mathbb{R}$ is locally integrable, define its mollification

$$f^\epsilon = \eta_\epsilon * f \quad \text{in} U_\epsilon$$

Theorem 4.3.4 (properties of mollifiers). *(i) $f^\epsilon \in C^\infty(U_\epsilon)$*
(ii) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$
(iii) If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of U.
(iv) If $1 \le p < \infty$ and $f \in L^p_{loc}(U)$, then $f^\epsilon \to f$ in $L^p_{loc}(U)$.

4.4 Extensions

Our goal is to extend functions in the Sobolev space $W^{1,p}(U)$ to become functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$. We must invent a way to extend u which preserves the weak derivatives accross ∂U.

Theorem 4.4.1 (extension theorem). *Assume U is bounded and ∂U is C^1. Select a bounded open set V s.t. $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

s.t. for each $u \in W^{1,p}(U)$:
(i) $Eu = u$ a.e. in U
(ii) Eu has support within V
(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(U)}$, where the constant C depends on p, U, V.

4.5 Traces

Theorem 4.5.1 (trace theorem). *Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \to L^p(\partial U)$$

s.t.
(i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$
(ii) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$

Definition 4.5.2. We will call Tu the trace of u on ∂U.

4.6 Compactness

4.7 Sobolev Inequalities

Our goal in this section is to discover embeddings of various Sobolev spaces into others. The crucical analytic tools here will be so-called Sobolev-type Inequalities, which we prove below for smooth functions. These will then establish the estimates for arbitary functions in the various relevant Sobolev spaces, since smooth functions are dense.

We will consider first only the Sobolev space $W^{1,p}(U)$ and ask the following

4.7.1 $1 \leq p < n$

Let us first ask whether we can establish an estimate of the form

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

for certain constants $C > 0$, $1 \leq p^* < \infty$ and all functions $u \in C_c^\infty(\mathbb{R}^n)$. We should expect that differentiation of a function increases the strength of its local singularities and improves its decay at infinity. Thus, if $Du \in L^p(\mathbb{R}^n)$, it is reasonable to expect that $f \in L^{p^*}(\mathbb{R}^n)$ for some $p^* > p$. By a simple dimensional analysis, we can obtain the only possible p^*:

Definition 4.7.1. If $1 \leq p < n$, the Sobolev conjugate of p is

$$p^* = \frac{np}{n-p}$$

Theorem 4.7.2 (Gagliardo-Nirenberg-Sobolev inequality).

Theorem 4.7.3 (estimates for $W^{1,p}$). *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}$$

the constant C depending only on p, n, and U.

The Gagliardo-Nirenberg-Sobolev inequality implies the embedding of $W^{1,p}(U)$ into $L^{p^*}(U)$ for $1 \leq p < n$. We now demonstrate that $W^{1,p}(U)$ is in fact **compactly** embedded in $L^q(U)$ for $1 \leq q < p^*$. This compactness will be fundamental for our applications of functional analysis to PDE.

Definition 4.7.4. Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y, written

$$X \subset\subset Y$$

provided
(i) $\|u\|_Y \leq C\|u\|_X$ $(u \in X)$ for some constant C and
(ii) each bounded sequence in X is precompact in Y.

Theorem 4.7.5 (Rellich-Kondrachov Compactness Theorem).

4.7.2 $n < p \leq \infty$

In the previous section, we saw that if the weak derivative of a function that vanishes at infinity belongs to $L^p(\mathbb{R}^n)$ with $p < n$, then the function has improved integrability properties and belongs to $L^{p^*}(\mathbb{R}^n)$. In this regime the Sobolev conjugate of p is negative. Therefore we should expect Even though the function is weakly differentiable, it need not be continuous. In this section, we show that if the derivative belongs to $L^p(\mathbb{R}^n)$ with $p > n$ then the function (or a pointwise a.e. equivalent version of it) is continuous, and in fact Hölder continuous.

Theorem 4.7.6 (Morrey's inequality)**.**

4.7.3 $p = n$

Owing to the Gagliardo-Nirenberg-Sobolev inequality, and the fact that $p^* \to \infty$ as $p \to n$, we might expect $u \in L^\infty(U)$ provided $u \in W^{1,n}(U)$. This is however false if $n > 1$.

Example 4.7.7. If $U = B^0(0,1)$, the function $u = \log\log(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(U)$ but not to $L^\infty(U)$.

4.7.4 General Sobolev inequalities

Now we deal with $W^{k,p}(U)$ with general k.

4.8 Poincare's inequalities

Notation 4.8.1. $(u)_U =$ is the average of u over U.

4.9 Fourier Transform

We employ the Fourier transform to give an alternate characterization of the spaces $H^k(\mathbb{R}^n)$. For this section all functions are complex-valued.

Theorem 4.9.1 (characterization of H^k via Fourier transform)**.** *Let k be a nonnegative integer.*
(i) A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if

$$(1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n)$$

(ii) In addition, there exists a positive constant C s.t.

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}$$

4.10 The Space H^{-1}

It it important to have an explicit characterization of the dual space of H^1_0

4.11 Discussion

Evans [2010]

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