

# Measure Theory

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# Chapter 1

## Real Analysis

### 1.1

Continuity of Translation

**Theorem 1.1.1.** Suppose  $f \in L^1(\mathbb{R}^n)$ . Then

$$\|f_h - f\|_{L^1} \rightarrow 0 \quad h \rightarrow 0.$$

*Proof.* The result is clearly true. The proof is a simple consequence of the approximation of integrable functions by continuous functions of compact support.  $\square$

### 1.2 Differentiation and Integration

For conceptual simplicity, we study  $\mathbb{R}$  instead of  $\mathbb{R}^n$  in this section. Let us first recall what we learned in elementary calculus.

**Theorem 1.2.1.** Let  $f$  be a *continuous* function on  $[a, b]$ , and  $F$  be the function defined, for all  $x$  in  $[a, b]$ , by

$$F(x) = \int_a^x f(t) dt.$$

Then  $F$  is uniformly continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and

$$F'(x) = f(x).$$

**Theorem 1.2.2** (Newton-Leibniz). Let  $f$  be a *Riemann integrable* function on  $[a, b]$ , and  $F$  a *continuous* function on  $[a, b]$  which is an antiderivative of  $f$  in  $(a, b)$ :

$$F'(x) = f(x).$$

Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

**Definition 1.2.3** (Total Variation). The total variation of a function  $f$  defined on  $[a, b]$  is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P-1} |f(x_{i+1}) - f(x_i)|,$$

where

$$\mathcal{P} = \{P = \{x_0, \dots, x_{n_P}\} | P \text{ is a partition of } [a, b] \text{ satisfying } x_i \leq x_{i+1} \text{ for } 0 \leq i \leq n_P - 1\}.$$

If  $V_a^b(f) < +\infty$ , then  $f$  is said to be of bounded variation on  $[a, b]$ .

**Example 1.2.4.** The continuous function

$$f(x) = \begin{cases} 0, & x = 0 \\ x \sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

is not of bounded variation on  $[0, 2/\pi]$ . Just consider the partition

$$P = \{0, \frac{2}{2n-1}, \frac{2}{2n-3}, \dots, \frac{2}{3}, 1\}.$$

**Theorem 1.2.5** (Jordan Decomposition Theorem).

$$f(x) = g(x) - h(x)$$

where  $g(x)$  and  $h(x)$

Bounded variable functions says that it has a length

Bounded variable functions are a.e. differentiable But it does not satisfies the desirable property: Length is equal to the integral of derivative, which is also known as the **fundamental theorem of calculus**. It turns out that absolute continuous functions is exactly the set of functions that satisfies this property.

**Example 1.2.6.**

**Definition 1.2.7** (Absolute Continuity). A function  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous if for every positive number  $\epsilon$ , there is a positive number  $\delta$  such that whenever a finite sequence of pairwise disjoint sub-intervals  $(x_k, y_k)$  satisfies

$$\sum_{k=1}^N (y_k - x_k) < \delta$$

then

$$\sum_{k=1}^N |f(y_k) - f(x_k)| < \epsilon.$$

Absolutely continuous functions are of bounded variation.

**Proposition 1.2.8.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous, then it is of bounded variation on  $[a, b]$ .*

**Theorem 1.2.9.** *If  $f$  is absolute continuous function on  $[a, b]$ , then*

$$f(x) - f(a) = \int_a^x f'(t) dt, \quad x \in [a, b].$$

It is beneficial to compare it with the “classical” version (Theorem 1.2.2).

## Chapter 2

# Abstract Measure Theory

### 2.1

In this section, we are working on a set  $E$ .

**Definition 2.1.1** ( $\sigma$ -algebra).

*Remark 2.1.2.* Greek letters  $\sigma$  and  $\delta$  are often used when countable unions and countable intersections are involved. For example, topologists call  $F_\sigma$  every countable union of closed sets in a topological space ( $F$  standing possibly for the French word fermé, closed) and  $G_\delta$  every countable intersection of open sets ( $G$  standing for the German word Gebiet, domain, connected open set). The letters  $\sigma$  and  $\delta$  are often given as Greek abbreviations of German words:  $\sigma$  as S in Summe for sum (in the sense of sum of sets, that is, union) and  $\delta$  as D in Durchschnitt for intersection, both countable. Thus, in the context of measure theory, the letter  $\sigma$  refers to the stability of a collection of subsets by countable union.

**Definition 2.1.3** (generator of  $\sigma$ -algebra).

$$\sigma(\mathcal{A}) = \{A \subset E : A \in \mathcal{E} \quad \forall \mathcal{E} \supset \mathcal{A}\}$$

**Example 2.1.4.** Borel  $\sigma$ -algebra

**Definition 2.1.5** ( $\pi$ -system).  $\mathcal{A}$  is a collection of subsets of  $E$ . Then  $\mathcal{A}$  is called a  $\pi$ -system if

1.  $\emptyset \in \mathcal{A}$ ;
2.  $A, B \in \mathcal{A}$ , then  $A \cap B \in \mathcal{A}$ .

**Definition 2.1.6** ( $d$ -system).  $\mathcal{A}$  is a collection of subsets

1.  $E \in \mathcal{A}$ ;
2.  $A, B \in \mathcal{A}$ ,  $A \subset B$ , then  $B \setminus A \in \mathcal{A}$ ;
3. If  $A_n \subset \mathcal{A}$  such that  $A_n \subset A_{n+1}$ , then  $\cup_n A_n \in \mathcal{A}$ .

*Remark 2.1.7.*  $d$ -system is also referred as  $\lambda$ -system.

**Proposition 2.1.8.**  $\mathcal{A}$  is a  $\sigma$ -algebra if and only if it is a  $\pi$ -system and a  $d$ -system.

*Proof.* A  $\sigma$ -algebra is a  $\pi$  system because

Conversely,

□

**Lemma 2.1.9** (Dynkin's  $\pi$ -system lemma). Let  $\mathcal{A}$  be a  $\pi$ -system. Then any  $d$ -system that contains  $\mathcal{A}$  also contains  $\sigma(\mathcal{A})$ .

This lemma can reduce the problem of studying a  $\sigma$ -algebra to the study of a  $\pi$ -system.

*Proof.* Let  $\mathcal{D}$  be the intersection of all  $d$ -system containing  $\mathcal{A}$ . We now prove that  $\mathcal{D}$  is a  $\sigma$ -algebra. As  $\mathcal{D}$  is already a  $d$ -system, by Proposition 2.1.8, we only need to prove that  $\mathcal{D}$  is a  $\pi$ -system.

(i) First we show the following property for  $\mathcal{D}$ : If  $B \in \mathcal{D}$  and  $A \in \mathcal{A}$ , then  $B \cap A \in \mathcal{D}$ . Let

$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{A}\}.$$

To show that  $\mathcal{D}' = \mathcal{D}$ , we only need to check that  $\mathcal{D}'$  is a  $d$ -system containing  $\mathcal{A}$ . As  $\mathcal{A}$  is a  $\pi$ -system,  $\mathcal{D}' \supset \mathcal{A}$ . Now, we can check

- $E \in \mathcal{D}'$ ; this is because  $\mathcal{A} \subset \mathcal{D}$ .
- $B_1, B_2 \in \mathcal{D}'$ ,  $B_1 \subset B_2$ , then  $B_2 \setminus B_1 \in \mathcal{D}'$ ; this is because for any  $A \in \mathcal{A}$ ,  $(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus (B_1 \cap A) \in \mathcal{D}$ .
- If  $B_n \in \mathcal{D}'$  such that  $B_n \subset B_{n+1}$ , then  $\cup_n B_n \in \mathcal{D}'$ ; this is because for any  $A \in \mathcal{A}$ ,  $(\cup_n B_n) \cap A = \cup_n (B_n \cap A) \in \mathcal{D}$ .

(ii) If  $A, B \in \mathcal{D}$ , then  $B \cap A \in \mathcal{D}$ . Let

$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{D}\}.$$

One can see that  $\mathcal{D}''$  is a  $d$ -system by noting that the argument holds for any  $\mathcal{A} \subset \mathcal{D}$ . The fact that  $\mathcal{D}''$  contains  $\mathcal{A}$  follows from property (i). Therefore,  $\mathcal{D}'' = \mathcal{D}$  and  $\mathcal{D}$  is a  $\pi$ -system.  $\square$

**Definition 2.1.10** (set function).  $\mathcal{A}$  be a collection of subsets of  $E$  with  $\emptyset \in \mathcal{A}$ . A set function  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is a function such that  $\mu(\emptyset) = 0$ .

**Definition 2.1.11** (increasing set function).  $A \subset B$ , we have  $\mu(A) \leq \mu(B)$ .