Geometry

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Chapter 1

Smooth Manifold

Lie derivative

1.1 Tensor Algebra

Let $\Gamma^{r,s}(M)$ denote the space of (r, s) tensors on M.

Given a coordinate $A \in \Gamma^{r,s}$ can be written as

$$A = A^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}, \tag{1.1}$$

tangent vectors $\Gamma(M) = \Gamma^{1,0}(M)$
one-forms $\Gamma^*(M) = \Gamma^{0,1}(M)$

Chapter 2

Riemannian Manifold

2.1 Riemannian Metrics

raising and lowering indices

Example 2.1.1 (Gradient)**.**

$$\nabla f = \sum_{i,j} g^{ij} \partial_j f$$

$$g(\nabla f, V) = V(f)$$

2.2 Affine Connections

An affine connection $\nabla_{\cdot}(\cdot)$ is a map from $\Gamma(M) \times \Gamma(M)$ to $\Gamma(M)$

Definition 2.2.1.

1. $C(M)$-linear in the lower slot
2. $\mathbb{R}$-linear in the upper slot
3. $C(M)$-Leibniz rule in the upper slot

2.2.1 Levi-Civita Connection

Let ∇ be an affine connection, then by the basis theorem there are locally defined functions Γ_{ij}^k such that

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k. \tag{2.1}$$

A connection is symmetric if
metric compaibility

Theorem 2.2.2. *Given a metric g, there is a unique connection ∇ that is symmetric and metric compatible. This connection is called the* ***Levi-Civita connection****, and is given by*

2.2.2 Tensor Leibniz Rule

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$$

Proposition 2.2.3.

$$F^{i_1 \ldots i_k}_{j_1 \ldots j_l ; m} = E_m(F^{i_1 \ldots i_k}_{j_1 \ldots j_l}) + \sum_{s=1}^{k} \Gamma^{i_s}_{mp} F^{i_1 \ldots p \ldots i_k}_{j_1 \ldots j_l} - \sum_{s=1}^{l} \Gamma^{p}_{m j_s} F^{i_1 \ldots i_k}_{j_1 \ldots p \ldots j_l}$$

higher order covariant derivative can be computed by iterating $\nabla : \Gamma^{r,s} \to \Gamma^{r,s+1}$

2.2.3 Along a Curve

2.3 Curvature

Definition 2.3.1. The Riemann curvature is a (1,3) tensor given by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z. \tag{2.2}$$

The covariant derivative of a (r,s) tensor can be thought of as an $(r,s+1)$ tensor in a natural way. For example, if

Repeating this, we get a $(1,2)$ tensor $\nabla\nabla V$, which will abbreviate as $\nabla^2 V$.

Using the Leibniz rule, we see that

$$\nabla_X(\nabla_Y V) = \nabla^2_{X,Y} V + \nabla_{\nabla_X Y} V \tag{2.3}$$

The tensor is not necessarily symmetric in the two lower slots. In fact, the curvature comes in

$$\begin{aligned}
\nabla^2_{X,Y} V - \nabla^2_{Y,X} V &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{\nabla_X Y - \nabla_Y X} V \\
&= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{[X,Y]} V \\
&= R(Y,X)V
\end{aligned}$$

This is known as the **Ricci identity**.

Definition 2.3.2 (Riemann curvature)**.**

2.3.1 Symmetries

2.3.2

The *Ricci curvature* is a $(0,2)$ tensor given by taking the trace of R.

Definition 2.3.3 (Ricci Curvature)**.**

A manifold is said to be *Einstein* with Einstein constant $\lambda \in \mathbb{R}$ if

The trace of the Ricci tensor is the *scalar curvature*:

Definition 2.3.4 (Scalar Curvature)**.**

Definition 2.3.5 (Sectional curvature)**.**

A Riemannian manifold has *constant sectional curvature* κ if

Lemma 2.3.6 (Schur Lemma)**.**

$$\mathrm{d}S = 2 \operatorname{div} \operatorname{Ric} \tag{2.4}$$

2.4 Submanifolds

If (N,g) is a Riemannian manifold and

$$\phi : M \to N$$

is an immersion, then $\phi^* g$ gives a metric on M.

$$T_{\phi(p)} N = d\phi(T_p M) \oplus [d\phi(T_p M)]^\perp$$

induced connection

Definition 2.4.1 (Second Fundamental Form)**.**

$$A(V,W) = (\nabla_V W)^\perp \tag{2.5}$$

M is said to be totally geodesic if $A \equiv 0$. totally geodesic

Definition 2.4.2 (Mean Curvature Vector)**.** The trace of the second fundamental form is called the mean curvature vector $\mathbf{H}$.

M is said to be *minimal* if $\mathbf{H} = 0$.

2.4.1 Gauss and Codazzi Equations

The Gauss equation relates the curvature of the submanifold to the curvature via the second fundamental form

Theorem 2.4.3 (Gauss Equation)**.**

Theorem 2.4.4 (Codazzi Equation)**.**

2.4.2 Hypersurfaces

An immersed submanifold $M^m \subset N^n$ is a *hypersurface* when $m = n - 1$. *unit normal* **n**

Using **n**, the second fundamental form

Chapter 3

Geodesics and Minimal Submanifolds

3.1 Variational Theory of Geodesics

Jacobi equation

Definition 3.1.1 (Conjugate Points)**.** Suppose γ is a geodesic. We say that $\gamma(t_2)$ is conjugate to $\gamma(t_1)$ along γ if there is a non-zero Jacobi field J along γ so that $J(t_1) = J(t_2) = 0$

The energy of a piece-wise smooth curve $\gamma : [0, a] \to M$

$$\mathbf{E}(\gamma) = \int_0^a |\gamma'|^2 \mathrm{d}t. \tag{3.1}$$

3.2 Variational Theory of Minimal Submanifolds

Given an isometrically embedded submanifold $\Sigma \subset M$, a variation is a map

$$F : \Sigma \times (-\epsilon, \epsilon) \to M$$

so that $F(x, 0) = x$. The variation vector field $F_s = \mathrm{d}F(\partial_s)$ describes the motion of points in Σ under the variation.

3.2.1 First Variation

Proposition 3.2.1.

$$\partial_s \overline{dv} = (\mathrm{div}(F_s^T) - \langle F_s^\perp, \mathbf{H} \rangle)\overline{dv} \tag{3.2}$$

3.2.2 Monotonicity

3.2.3 Second Variation

We now investigate the second derivative of volume for a hypersurface

$$\Sigma^n \subset M^{n+1},$$

whose calculations are substantially simpler than a general submanifold.

Lemma 3.2.2. *Given a normal variation $F_s = u\mathbf{n}$, the derivative of $\mathbf{H}$ at 0 is*

$$\mathbf{H}' = (\Delta u + |A|^2 u + \mathrm{Ric}(\mathbf{n}, \mathbf{n}))\mathbf{n} - \langle \mathbf{H}, \mathbf{n} \rangle \nabla u. \tag{3.3}$$

Chapter 4

The Laplacian

4.1 Divergence and Laplacian

Let M^n be a manifold with a metric g and associated Levi-Civita connection ∇.
derivative of $u \in C(M)$:

- gradient $\nabla u \in \Gamma^{1,0}(M)$
- differential $du \in \Gamma^{0,1}(M)$.

they are dual via the metric

$$\langle \nabla u, V \rangle = du(V) = V(u).$$

Definition 4.1.1 (Hessian)**.** The Hessian is a (0,2) tensor defined by ∇du, which satisfies

$$\mathrm{Hess}_u(V, W) = \langle \nabla_V \nabla u, W \rangle . \tag{4.1}$$

Proposition 4.1.2. Hess_u *is symmetric.*

We will give two proofs.

Proof 1. First,

$$\mathrm{Hess}_u(\partial_i, \partial_j) = u_{ij} - \sum_k \Gamma_{ij}^k u_k$$

□

Proof 2.

$$\begin{aligned} \mathrm{Hess}_u(V, W) - \mathrm{Hess}_u(W, V) &= \langle \nabla_V \nabla u, W \rangle - \langle \nabla_W \nabla u, V \rangle \\ &= V \langle \nabla u, W \rangle - \langle \nabla u, \nabla_V W \rangle - W \langle \nabla u, V \rangle + \langle \nabla u, \nabla_W V \rangle \\ &= V(W(u)) - W(V(u)) - \langle \nabla u, [V, W] \rangle = 0. \end{aligned}$$

□

Definition 4.1.3 (Divergence)**.** The divergence $\operatorname{div} V$ of a vector field V is the trace of the $(1,1)$ tensor ∇V.

In an orthonormal frame

$$\operatorname{div} V = \sum_i \langle \nabla_{e_i} V, e_i \rangle$$

In a local coordinate,

$$\operatorname{div} V = \partial_i V^i + \Gamma_{ij}^j V^i$$

But we also have

Proposition 4.1.4.

$$\operatorname{div} V = \frac{1}{\sqrt{\det g}} \sum_i \partial_i (\sqrt{\det g} V^i). \tag{4.2}$$

Proof. This is shown by

$$\frac{\partial_i \sqrt{\det g}}{\sqrt{\det g}} = \sum_j \Gamma_{ij}^j$$

□

Proposition 4.1.4 makes it possible to extend the Euclidean divergence theorem to manifolds. Recall the volume element $dv_g = \sqrt{\det g} \mathrm{d}x$, we have

$$\mathrm{div}(V) dv_g = \mathrm{div}_{\mathbb{R}^n}(V\sqrt{\det g})\mathrm{d}x. \tag{4.3}$$

From Leibniz rule

$$\mathrm{div}(uV) = u\,\mathrm{div}(V) + \langle \nabla u, V\rangle \tag{4.4}$$

Definition 4.1.5 (Laplacian)**.** The Laplacian is the divergence of ∇u.

Therefore, we have in an orthonormal frame

$$\begin{aligned}
\Delta u &= \sum_i \langle \nabla_{e_i} \nabla u, e_i\rangle \\
&= \sum_i \nabla_{e_i} \langle \nabla u, e_i\rangle - \langle \nabla u, \nabla_{e_i} e_i\rangle \\
&= \sum_i \nabla_{e_i} \nabla_{e_i} u - \nabla_{\nabla_{e_i} e_i} u
\end{aligned}$$

which is in accordance with Equation (2.3)

4.1.1 Bochner Formula

Proposition 4.1.6. *If $u \in C(M)$, then*

$$\Delta \nabla u = \nabla(\Delta u) + \mathrm{Ric}(\nabla u, \cdot) \tag{4.5}$$

Proof 1. □

geodesic normal coordinates for $p \in M$ fixed

- $g_{ij}(p) = \delta_{ij}$, meaning that ∂_i are orthonormal at p
- $\Gamma_{ij}^k = 0$, meaning that $\nabla \partial_i(p) = 0$

Proof 2. □

Corollary 4.1.7.

$$\Delta |\nabla u|^2 = |\operatorname{Hess}_u|^2 + \langle \nabla(\Delta u), \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u) \tag{4.6}$$

4.1.2 Lichnerowicz Theorem

4.2 Submanifold Divergence and Laplacian

In this section, $\Sigma \subset M$ is a submanifold with the induced connection $\bar{\nabla}$.

Proposition 4.2.1. *The submanifold Hessian is given by*

$$\overline{\mathrm{Hess}}_u(V, W) = \mathrm{Hess}_u(V, W) + \left\langle A(V, W), \nabla^\perp u\right\rangle \tag{4.7}$$

Proof.

$$\begin{aligned}
\overline{\mathrm{Hess}}_u(V, W) &= \left\langle \nabla_V \nabla^T u, W\right\rangle \\
&= \langle \nabla_V \nabla u, W\rangle - \left\langle \nabla_V \nabla^\perp u, W\right\rangle \\
&= \mathrm{Hess}_u(V, W) + \left\langle \nabla^\perp u, \nabla_V W\right\rangle \\
&= \mathrm{Hess}_u(V, W) + \left\langle A(V, W), \nabla^\perp u\right\rangle
\end{aligned}$$

□

4.3 Laplacian Comparison

4.3.1 Distance Function

Definition 4.3.1 (Cut Point)**.**

Theorem 4.3.2. *If $q \in \mathrm{Cut}(p)$, then either*

1. *q is the first conjugate to p along a minimizing geodesic, or*
2. *q is the first point along a minimizing geodesic where there is a second, different, minimizing geodesic from p.*

Now we show that d is smooth away from $\mathrm{Cut}(p)$.

Proposition 4.3.3.

$\triangle|x|^2$ on $\mathbb{R}^n$
$\nabla|x|^2 = 2|x|\nabla|x|$
$\triangle|x| = \frac{n-1}{|x|}$ on $\mathbb{R}^n \setminus \{0\}$
Another way $\mathrm{Hess}_{|x|}$
Suppose $\mathrm{Ric} \geq 0$

4.3.2 Laplacian Comparison

We first state the Laplacian comparison for smooth points

Theorem 4.3.4. *Suppose that $\mathrm{Ric} \geq 0$ and $r(x) = d(p, x)$ for p fixed. Away from $\mathrm{Cut}(p) \cup \{p\}$, we have that*

$$\Delta r \leq \frac{n-1}{r}.$$

Proof. First, Hess_r has rank at most $n-1$, so

$$|\operatorname{Hess}_r|^2 \geq \frac{(\Delta r)^2}{n-1}.$$

By Bochner formula (Corollary 4.1.7), along γ,

$$\begin{aligned} 0 &= \frac{1}{2}\Delta|\nabla r|^2 \\ &= |\operatorname{Hess}_r|^2 + \langle \nabla(\Delta r), \nabla r\rangle + \mathrm{Ric}(\nabla r, \nabla r) \\ &\geq \frac{(\Delta r)^2}{n-1} + (\Delta r)' \end{aligned}$$

∎

Lemma 4.3.5. *If $f(t)$ is a function on $(0, T]$ with $(n-1)f' \leq -f^2$, then*

$$f(T) \leq \frac{n-1}{T}.$$

Proof. Notice that

$$\left(\frac{1}{f(t)}\right)' = -\frac{f'(t)}{f(t)^2}$$

∎

Definition 4.3.6 (barrier)**.** We say that $\Delta f \geq g$ at $p \in \Omega$ in the *barrier sense* if for every $\epsilon > 0$ there exists a C^2 function h_ϵ and an open set U_ϵ containing p so that

1. $f(p) = h_\epsilon(p)$ and $h_\epsilon \leq f$ in U_ϵ.
2. $\Delta h_\epsilon(p) \geq g(p) - \epsilon$.

Definition 4.3.7 (viscosity)**.** We say that $\Delta f \geq g$ at $p \in \Omega$ in the *viscosity sense* if for every open set U containing p and C^2 function φ on U with $f(p) = \varphi(p)$ and $\varphi \geq f$ in U, we have that $\Delta\varphi(p) \geq g(p)$.

Proposition 4.3.8. *If $\Delta f \geq g$ at p in the barrier sense, then it also holds in the viscosity sense. If $\Delta f \geq g$ at p in the viscosity sense, then it also holds in the distributional sense.*

Theorem 4.3.9. *If $\mathrm{Ric} \geq 0$ and $d(x) := d(p, x)$ for some fixed p, then $\Delta d \leq \frac{n-1}{d}$ in the barrier sense on $M \setminus \{p\}$.*

4.3.3 Bishop-Gromov Volume Comparison

Theorem 4.3.10. *If* $\mathrm{Ric} \geq 0$, $p \in M^n$, *and* $0 < r_1 < r_2$, *then*

$$\frac{\mathrm{Vol}(B_{r_2}(p))}{r_2^n} \leq \frac{\mathrm{Vol}(B_{r_1}(p))}{r_1^n}.$$

4.3.4 Dirichlet Poincare Inequality

4.4 Gradient Estimates

Theorem 4.4.1. *If* $B_R(p) \subset M^n$ *has* $\mathrm{Ric} \geq 0$, $\Delta u = 0$, *and* $u > 0$, *then*

$$\sup_{B_{R/2}(p)} |\nabla \log u| \leq \frac{C}{R},$$

where C *depends just on* n.

differential Harnack Inequality

Corollary 4.4.2.

Bernstein technique

Lemma 4.4.3. $w = \log u$ *satisfies*

- $\Delta w = -|\nabla w|^2$
- $\frac{1}{2}\Delta|\nabla w|^2 \geq \frac{1}{n}|\nabla w|^4 - \left\langle \nabla w, \nabla |\nabla w|^2 \right\rangle$

But $|\nabla w\|^2$ may not have an interior max. Use cut-off

Proof. □

Meanvalue Inequality
for harmonic is evident due to Harnack, extend it to sub-harmonic

Proof. ϕ cutoff 1 on $B_{2R}(p)$, reverse Poincare

$$\int_{B_{2R}(p)} |\nabla v|^2 \leq 4 \int_{B_{4R}(p)} v^2 |\nabla \phi|^2.$$

Choose ϕ such that $|\nabla \phi| \leq \frac{1}{2R}$,

$$\int_{B_{2R}(p)} |\nabla v|^2 \leq \frac{1}{R^2} \int_{B_{4R}(p)} v^2.$$

$$\mathrm{div}(\phi^2 v \nabla v) = \phi^2 |\nabla v|^2 + \phi^2 v \Delta v +$$

solve for u on $B_{2R}(p)$ such that $\Delta u = 0$ in $B_{2R}(p)$ and $u = v$ on $\partial B_{2R}(p)$.

$$v \leq u.$$

use Harnack inequality on u □

4.5

Theorem 4.5.1 (Colding-Minicozzi)**.** $\mathrm{Ric} \geq 0$, $\dim \mathcal{H}^d(M^n) \leq C d^{n-1}$

Theorem 4.5.2. $\exists C = C(n)$ *such that if* $u_1, \ldots, u_N$ *are* $L^2(B_{2r})$ *orthonormal and* $\Delta u_i = 0$, *and* $\int_{B_r} u_i^2 \geq \alpha > 0$, *then* $N \leq \frac{C}{\alpha}$.

Lemma 4.5.3. *Given* $x \in B_{2r}$, $\exists y \in S^{N-1}$ *such that* $w = \sum_{i=1}^N y_i u_i$ *has* $\sum u_i^2(x) = w^2(x)$

Proof. Define $f : S^{n-1} \to \mathbb{R}$, $f(y) = \sum_{i=1}^N y_i u_i(x)$, max achieved when. □

Proof. given $x \in B_r$, $\sum_{i=1}^N u_i^2(x) = w^2(x)$ where w is from the lemma.

$$w^2(x) \le \frac{C}{\mathrm{Vol}(B_r(x))} \int_{B_r(x)} w^2 \le \frac{C}{\mathrm{Vol}(B_r(x))}$$

Bishop-Gromov: □

Theorem 4.5.4. *If $v_1, \dots, v_{2N} \in \mathcal{H}^d(M^n)$ and are linearly independent. Then $\exists R > 0$ and $u_1, \dots, u_N$ in the span of the v_i's such that $\int_{B_{2R}} u_i u_j = \delta_{ij}$ and $\int_{B_R} u_i^2 > 2^{-4(d+n)}$*

Lemma 4.5.5. *$0 < F \le Cr^d$ on $[1, \infty)$, then $\exists \infty$ many $k \in \mathbb{N}$ such that*

$$\frac{F(2^{k+1})}{F(2^k)} \le 2^{d+\epsilon}.$$

Proof. by contradiction □

Proof. Define

$$\Lambda_j = \{v_1, \dots, v_{j-1} \subset \mathcal{H}^d\}.$$

Given r, define $w_{j,r}$ to be the $L^2(B_r)$ projection of v_j onto Λ_j. Define $f_j(r) = \int_{B_r} (v_j - w_{j,r})^2 \le \int_{B_r} (v_j - w)^2$ □