

# Geometry

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# Chapter 1

## Smooth Manifold

Lie derivative

### 1.1 Tensor Algebra

Let  $\Gamma^{r,s}(M)$  denote the space of  $(r,s)$  tensors on  $M$ .

Given a coordinate  $A \in \Gamma^{r,s}$  can be written as

$$A = A_{j_1, \dots, j_s}^{i_1, \dots, i_r} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}, \quad (1.1)$$

tangent vectors  $\Gamma(M) = \Gamma^{1,0}(M)$

one-forms  $\Gamma^*(M) = \Gamma^{0,1}(M)$



# Chapter 2

## Riemannian Manifold

### 2.1 Riemannian Metrics

raising and lowering indices

**Example 2.1.1** (Gradient).

$$\nabla f = \sum_{i,j} g^{ij} \partial_j f$$

$$g(\nabla f, V) = V(f)$$

### 2.2 Affine Connections

An affine connection  $\nabla(\cdot)$  is a map from  $\Gamma(M) \times \Gamma(M)$  to  $\Gamma(M)$

**Definition 2.2.1.** 1.  $C(M)$ -linear in the lower slot

2.  $\mathbb{R}$ -linear in the upper slot

3.  $C(M)$ -Leibniz rule in the upper slot

#### 2.2.1 Levi-Civita Connection

Let  $\nabla$  be an affine connection, then by the basis theorem there are locally defined functions  $\Gamma_{ij}^k$  such that

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k. \quad (2.1)$$

A connection is symmetric if  
metric compatibility

**Theorem 2.2.2.** *Given a metric  $g$ , there is a unique connection  $\nabla$  that is symmetric and metric compatible. This connection is called the **Levi-Civita connection**, and is given by*

#### 2.2.2 Tensor Leibniz Rule

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$$

**Proposition 2.2.3.**

$$F_{j_1 \dots j_l; m}^{i_1 \dots i_k} = E_m(F_{j_1 \dots j_l}^{i_1 \dots i_k}) + \sum_{s=1}^k \Gamma_{mp}^s F_{j_1 \dots j_l}^{i_1 \dots p \dots i_k} - \sum_{s=1}^l \Gamma_{mj_s}^p F_{j_1 \dots p \dots j_l}^{i_1 \dots i_k}$$

higher order covariant derivative can be computed by iterating  $\nabla : \Gamma^{r,s} \rightarrow \Gamma^{r,s+1}$

### 2.2.3 Along a Curve

## 2.3 Curvature

**Definition 2.3.1.** The Riemann curvature is a (1,3) tensor given by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z. \quad (2.2)$$

The covariant derivative of a  $(r, s)$  tensor can be thought of as an  $(r, s+1)$  tensor in a natural way. For example, if

Repeating this, we get a (1, 2) tensor  $\nabla \nabla V$ , which will abbreviate as  $\nabla^2 V$ .

Using the Leibniz rule, we see that

$$\nabla_X(\nabla_Y V) = \nabla_{X,Y}^2 V + \nabla_{\nabla_X Y} V \quad (2.3)$$

The tensor is not necessarily symmetric in the two lower slots. In fact, the curvature comes in

$$\begin{aligned} \nabla_{X,Y}^2 V - \nabla_{Y,X}^2 V &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{\nabla_X Y - \nabla_Y X} V \\ &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{[X, Y]} V \\ &= R(Y, X)V \end{aligned}$$

This is known as the **Ricci identity**.

**Definition 2.3.2** (Riemann curvature).

### 2.3.1 Symmetries

### 2.3.2

The *Ricci curvature* is a (0, 2) tensor given by taking the trace of  $R$ .

**Definition 2.3.3** (Ricci Curvature).

A manifold is said to be *Einstein* with Einstein constant  $\lambda \in \mathbb{R}$  if

The trace of the Ricci tensor is the *scalar curvature*:

**Definition 2.3.4** (Scalar Curvature).

**Definition 2.3.5** (Sectional curvature).

A Riemannian manifold has *constant sectional curvature*  $\kappa$  if

**Lemma 2.3.6** (Schur Lemma).

$$dS = 2 \operatorname{div} \operatorname{Ric} \quad (2.4)$$

## 2.4 Submanifolds

If  $(N, g)$  is a Riemannian manifold and

$$\phi : M \rightarrow N$$

is an immersion, then  $\phi^* g$  gives a metric on  $M$ .

$$T_{\phi(p)} N = d\phi(T_p M) \oplus [d\phi(T_p M)]^\perp$$

induced connection

**Definition 2.4.1** (Second Fundamental Form).

$$A(V, W) = (\nabla_V W)^\perp \quad (2.5)$$

$M$  is said to be totally geodesic if  $A \equiv 0$ . totally geodesic

**Definition 2.4.2** (Mean Curvature Vector). The trace of the second fundamental form is called the mean curvature vector  $\mathbf{H}$ .

$M$  is said to be *minimal* if  $\mathbf{H} = 0$ .

### 2.4.1 Gauss and Codazzi Equations

The Gauss equation relates the curvature of the submanifold to the curvature via the second fundamental form

**Theorem 2.4.3** (Gauss Equation).

**Theorem 2.4.4** (Codazzi Equation).

### 2.4.2 Hypersurfaces

An immersed submanifold  $M^m \subset N^n$  is a *hypersurface* when  $m = n - 1$ . *unit normal*  $\mathbf{n}$

Using  $\mathbf{n}$ , the second fundamental form



# Chapter 3

## Geodesics and Minimal Submanifolds

### 3.1 Variational Theory of Geodesics

Jacobi equation

**Definition 3.1.1** (Conjugate Points). Suppose  $\gamma$  is a geodesic. We say that  $\gamma(t_2)$  is conjugate to  $\gamma(t_1)$  along  $\gamma$  if there is a non-zero Jacobi field  $J$  along  $\gamma$  so that  $J(t_1) = J(t_2) = 0$

The energy of a piece-wise smooth curve  $\gamma : [0, a] \rightarrow M$

$$\mathbf{E}(\gamma) = \int_0^a |\gamma'|^2 dt. \quad (3.1)$$

### 3.2 Variational Theory of Minimal Submanifolds

Given an isometrically embedded submanifold  $\Sigma \subset M$ , a variation is a map

$$F : \Sigma \times (-\epsilon, \epsilon) \rightarrow M$$

so that  $F(x, 0) = x$ . The variation vector field  $F_s = dF(\partial_s)$  describes the motion of points in  $\Sigma$  under the variation.

#### 3.2.1 First Variation

**Proposition 3.2.1.**

$$\partial_s \overline{dv} = (\operatorname{div}(F_s^T) - \langle F_s^\perp, \mathbf{H} \rangle) \overline{dv} \quad (3.2)$$

#### 3.2.2 Monotonicity

#### 3.2.3 Second Variation

We now investigate the second derivative of volume for a hypersurface

$$\Sigma^n \subset M^{n+1},$$

whose calculations are substantially simpler than a general submanifold.

**Lemma 3.2.2.** *Given a normal variation  $F_s = u\mathbf{n}$ , the derivative of  $\mathbf{H}$  at 0 is*

$$\mathbf{H}' = (\Delta u + |A|^2 u + \operatorname{Ric}(\mathbf{n}, \mathbf{n}))\mathbf{n} - \langle \mathbf{H}, \mathbf{n} \rangle \nabla u. \quad (3.3)$$



# Chapter 4

## The Laplacian

### 4.1 Divergence and Laplacian

Let  $M^n$  be a manifold with a metric  $g$  and associated Levi-Civita connection  $\nabla$ .  
derivative of  $u \in C(M)$ :

- gradient  $\nabla u \in \Gamma^{1,0}(M)$
- differential  $du \in \Gamma^{0,1}(M)$ .

they are dual via the metric

$$\langle \nabla u, V \rangle = du(V) = V(u).$$

**Definition 4.1.1** (Hessian). The Hessian is a  $(0,2)$  tensor defined by  $\nabla du$ , which satisfies

$$\text{Hess}_u(V, W) = \langle \nabla_V \nabla u, W \rangle. \quad (4.1)$$

**Proposition 4.1.2.**  $\text{Hess}_u$  is symmetric.

We will give two proofs.

*Proof 1.* First,

$$\text{Hess}_u(\partial_i, \partial_j) = u_{ij} - \sum_k \Gamma_{ij}^k u_k$$

□

*Proof 2.*

$$\begin{aligned} \text{Hess}_u(V, W) - \text{Hess}_u(W, V) &= \langle \nabla_V \nabla u, W \rangle - \langle \nabla_W \nabla u, V \rangle \\ &= V \langle \nabla u, W \rangle - \langle \nabla u, \nabla_V W \rangle - W \langle \nabla u, V \rangle + \langle \nabla u, \nabla_W V \rangle \\ &= V(W(u)) - W(V(u)) - \langle \nabla u, [V, W] \rangle = 0. \end{aligned}$$

□

**Definition 4.1.3** (Divergence). The divergence  $\text{div } V$  of a vector field  $V$  is the trace of the  $(1,1)$  tensor  $\nabla V$ .

In an orthonormal frame

$$\text{div } V = \sum_i \langle \nabla_{e_i} V, e_i \rangle$$

In a local coordinate,

$$\text{div } V = \partial_i V^i + \Gamma_{ij}^j V^i$$

But we also have

**Proposition 4.1.4.**

$$\text{div } V = \frac{1}{\sqrt{\det g}} \sum_i \partial_i (\sqrt{\det g} V^i). \quad (4.2)$$

*Proof.* This is shown by

$$\frac{\partial_i \sqrt{\det g}}{\sqrt{\det g}} = \sum_j \Gamma_{ij}^j$$

□

Proposition 4.1.4 makes it possible to extend the Euclidean divergence theorem to manifolds. Recall the volume element  $dv_g = \sqrt{\det g} dx$ , we have

$$\operatorname{div}(V)dv_g = \operatorname{div}_{\mathbb{R}^n}(V\sqrt{\det g})dx. \quad (4.3)$$

From Leibniz rule

$$\operatorname{div}(uV) = u \operatorname{div}(V) + \langle \nabla u, V \rangle \quad (4.4)$$

**Definition 4.1.5** (Laplacian). The Laplacian is the divergence of  $\nabla u$ .

Therefore, we have in an orthonormal frame

$$\begin{aligned} \Delta u &= \sum_i \langle \nabla_{e_i} \nabla u, e_i \rangle \\ &= \sum_i \nabla_{e_i} \langle \nabla u, e_i \rangle - \langle \nabla u, \nabla_{e_i} e_i \rangle \\ &= \sum_i \nabla_{e_i} \nabla_{e_i} u - \nabla_{\nabla_{e_i} e_i} u \end{aligned}$$

which is in accordance with Equation (2.3)

#### 4.1.1 Bochner Formula

**Proposition 4.1.6.** If  $u \in C(M)$ , then

$$\Delta \nabla u = \nabla(\Delta u) + \operatorname{Ric}(\nabla u, \cdot) \quad (4.5)$$

*Proof 1.*

□

geodesic normal coordinates for  $p \in M$  fixed

- $g_{ij}(p) = \delta_{ij}$ , meaning that  $\partial_i$  are orthonormal at  $p$
- $\Gamma_{ij}^k = 0$ , meaning that  $\nabla \partial_i(p) = 0$

*Proof 2.*

□

**Corollary 4.1.7.**

$$\Delta |\nabla u|^2 = |\operatorname{Hess}_u|_{}^2 + \langle \nabla(\Delta u), \nabla u \rangle + \operatorname{Ric}(\nabla u, \nabla u) \quad (4.6)$$

#### 4.1.2 Lichnerowicz Theorem

### 4.2 Submanifold Divergence and Laplacian

In this section,  $\Sigma \subset M$  is a submanifold with the induced connection  $\bar{\nabla}$ .

**Proposition 4.2.1.** The submanifold Hessian is given by

$$\overline{\operatorname{Hess}}_u(V, W) = \operatorname{Hess}_u(V, W) + \langle A(V, W), \nabla^\perp u \rangle \quad (4.7)$$

*Proof.*

$$\begin{aligned} \overline{\operatorname{Hess}}_u(V, W) &= \langle \nabla_V \nabla^T u, W \rangle \\ &= \langle \nabla_V \nabla u, W \rangle - \langle \nabla_V \nabla^\perp u, W \rangle \\ &= \operatorname{Hess}_u(V, W) + \langle \nabla^\perp u, \nabla_V W \rangle \\ &= \operatorname{Hess}_u(V, W) + \langle A(V, W), \nabla^\perp u \rangle \end{aligned}$$

□

## 4.3 Laplacian Comparison

### 4.3.1 Distance Function

**Definition 4.3.1** (Cut Point).

**Theorem 4.3.2.** If  $q \in \text{Cut}(p)$ , then either

1.  $q$  is the first conjugate to  $p$  along a minimizing geodesic, or
2.  $q$  is the first point along a minimizing geodesic where there is a second, different, minimizing geodesic from  $p$ .

Now we show that  $d$  is smooth away from  $\text{Cut}(p)$ .

**Proposition 4.3.3.**

$$\Delta|x|^2 \text{ on } \mathbb{R}^n$$

$$\nabla|x|^2 = 2|x|\nabla|x|$$

$$\Delta|x| = \frac{n-1}{|x|} \text{ on } \mathbb{R}^n \setminus \{0\}$$

Another way  $\text{Hess}_{|x|}$

Suppose  $\text{Ric} \geq 0$

### 4.3.2 Laplacian Comparison

We first state the Laplacian comparison for smooth points

**Theorem 4.3.4.** Suppose that  $\text{Ric} \geq 0$  and  $r(x) = d(p, x)$  for  $p$  fixed. Away from  $\text{Cut}(p) \cup \{p\}$ , we have that

$$\Delta r \leq \frac{n-1}{r}.$$

*Proof.* First,  $\text{Hess}_r$  has rank at most  $n-1$ , so

$$|\text{Hess}_r|^2 \geq \frac{(\Delta r)^2}{n-1}.$$

By Bochner formula (Corollary 4.1.7), along  $\gamma$ ,

$$\begin{aligned} 0 &= \frac{1}{2} \Delta |\nabla r|^2 \\ &= |\text{Hess}_r|^2 + \langle \nabla(\Delta r), \nabla r \rangle + \text{Ric}(\nabla r, \nabla r) \\ &\geq \frac{(\Delta r)^2}{n-1} + (\Delta r)' \end{aligned}$$

□

**Lemma 4.3.5.** If  $f(t)$  is a function on  $(0, T]$  with  $(n-1)f' \leq -f^2$ , then

$$f(T) \leq \frac{n-1}{T}.$$

*Proof.* Notice that

$$\left( \frac{1}{f(t)} \right)' = -\frac{f'(t)}{f(t)^2}$$

□

**Definition 4.3.6** (barrier). We say that  $\Delta f \geq g$  at  $p \in \Omega$  in the *barrier sense* if for every  $\epsilon > 0$  there exists a  $C^2$  function  $h_\epsilon$  and an open set  $U_\epsilon$  containing  $p$  so that

1.  $f(p) = h_\epsilon(p)$  and  $h_\epsilon \leq f$  in  $U_\epsilon$ .

2.  $\Delta h_\epsilon(p) \geq g(p) - \epsilon$ .

**Definition 4.3.7** (viscosity). We say that  $\Delta f \geq g$  at  $p \in \Omega$  in the *viscosity sense* if for every open set  $U$  containing  $p$  and  $C^2$  function  $\varphi$  on  $U$  with  $f(p) = \varphi(p)$  and  $\varphi \geq f$  in  $U$ , we have that  $\Delta\varphi(p) \geq g(p)$ .

**Proposition 4.3.8.** If  $\Delta f \geq g$  at  $p$  in the barrier sense, then it also holds in the viscosity sense. If  $\Delta f \geq g$  at  $p$  in the viscosity sense, then it also holds in the distributional sense.

**Theorem 4.3.9.** If  $\text{Ric} \geq 0$  and  $d(x) := d(p, x)$  for some fixed  $p$ , then  $\Delta d \leq \frac{n-1}{d}$  in the barrier sense on  $M \setminus \{p\}$ .

### 4.3.3 Bishop-Gromov Volume Comparison

**Theorem 4.3.10.** *If  $\text{Ric} \geq 0$ ,  $p \in M^n$ , and  $0 < r_1 < r_2$ , then*

$$\frac{\text{Vol}(B_{r_2}(p))}{r_2^n} \leq \frac{\text{Vol}(B_{r_1}(p))}{r_1^n}.$$

### 4.3.4 Dirichlet Poincare Inequality

## 4.4 Gradient Estimates

**Theorem 4.4.1.** *If  $B_R(p) \subset M^n$  has  $\text{Ric} \geq 0$ ,  $\Delta u = 0$ , and  $u > 0$ , then*

$$\sup_{B_{R/2}(p)} |\nabla \log u| \leq \frac{C}{R},$$

where  $C$  depends just on  $n$ .

Bernstein

**Lemma 4.4.2.**  $w = \log u$  satisfies

- $\Delta w = -|\nabla w|^2$
- $\frac{1}{2}\Delta|\nabla w|^2 \geq \frac{1}{n}|\nabla w|^4 - \langle \nabla w, \nabla|\nabla w|^2 \rangle$

But  $|\nabla w|^2$  may not have an interior max. Use cut-off

*Proof.*

□