Lecture Notes in Stochastic Process

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1 Discrete Space Discrete Time Markov Chain

1.1 Theory

We begin by summarizing the concepts mathematicans are most interested in, and introducing their notations.

Definition 1.1 (initial distribution)**.**

Definition 1.2 (transition matrix)**.**

Remark. *Transition matrix provides a way to calculate the probability that after n steps the Markov chain is in a given state.*

Now we focus on the class structure of markov chain.

Definition 1.3 (lead to)**.** We say i leads to j and write $i \to j$ if $P_i(X_n = j \text{ for some } n \geq 0) > 0$.

Definition 1.4 (communicate with)**.** We say i communicate with j and write $i \leftrightarrow j$ if both $i \to j$ and $j \to i$.

Theorem 1.1 (communicating classes)**.** *Communication is an equivalence relation on I, thus partition I into communicating classes.*

Definition 1.5 (closed class)**.** We say a class C is closed if

$$i \in C, i \to j \Longrightarrow j \in C$$

A state i is called absorbing if $\{i\}$ is a closed class.

Definition 1.6 (irreducibility)**.** A markov chain with only one class is called irreducible.

Definition 1.7 (hitting time)**.** Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. The hitting time of a subset A of I is the random variable

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}$$

.

Definition 1.8 (first passage time)**.** The first passage time to state i is the random variable T_i defined by

$$T_i(\omega) = \inf\{n \geq 1 : X_n(\omega) = i\}$$

Definition 1.9 (rth passage time)**.** We define rth passage time inductively by $T_i^{(0)}(\omega) = 0, T_i^{(1)}(\omega) = T_i(\omega)$ and for $r = 0, 1, 2, \cdots$,

$$T_i^{(r+1)}(\omega) = \inf\left\{n \geq T_i^{(r)}(\omega) + 1 : X_n(\omega) = i\right\}$$

Definition 1.10 (absorption probability)**.** The probability starting from i that $(X_n)_{n\geq 0}$ ever hits A is then

$$h_i^A = P_i(H^A < \infty)$$

When A is a closed class, h_i^A is called the absorption probability.

Remark. *A less formal notation is* $h_i^A = P(\textit{hit } A)$.

Definition 1.11. The mean time taken for $(X_n)_{n\geq 0}$ to reach A from i is given by

$$k_i^A = E_i(H^A) = \sum_{n<\infty} nP(H^A = n) + \infty P(H^A = \infty)$$

Remark. *A less formal notation is* $k_i^A = E_i(\textit{time to hit } A)$.

Definition 1.12 (recurrent)**.** We say a state i is recurrent if $P_i(X_n = i \text{ i.o. }) = 1$.

Definition 1.13 (transient)**.** We say a state i is transient if $P_i(X_n = i \text{ i.o. }) = 0$.

Definition 1.14 (positive recurrent)**.** A state i is postive recurrent if the expected return time $m_i = E_i(T_i)$ is finite. A recurrent state which fails to have this stronger property is called null recurrent.

Definition 1.15 (invariant measure)**.** A measure is any row vector $(\lambda_i : i \in I)$ with non-negative entries. We say λ is invariant if

$$\lambda P = \lambda$$

Definition 1.16 (detailed balance)**.**

2 Discrete Space Continuous Time Markov Chain

Definition 2.1 (continuous-time random process)**.** Let I be a countable set. A continuous-time random process

$$(X_t)_{t\geq 0} = (X_t : 0 \leq t \leq \infty)$$

with values in I is a family of random variables $X_t : \Omega \to I$.

We are going to consider ways in which we might specify the probabilistic behavior of $(X_t)_{t\geq 0}$. To avoid uncountable union, we shall restrict our attention to processes $(X_t)_{t\geq 0}$ which are right-continuous.

Definition 2.2 (right continuous)**.** In the context of discrete space continuous time, a right-continuous process means $\forall \omega \in \Omega$ and $t \geq 0$, $\exists \epsilon > 0$ s.t.

$$X_s(\omega) = X_t(\omega) \quad t \leq s \leq t + \epsilon$$

Definition 2.3 (increment)**.** If $(X_t)_{t\geq 0}$ is a real-valued process, we can consider its increment $X_t - X_s$ over any interval $(s, t]$.

Definition 2.4 (stationary)**.** We say that $(X_t)_{t\geq 0}$ has stationary increments if the distribution of $X_{s+t} - X_s$ depends only on $t \geq 0$.

Definition 2.5 (independent)**.** We say that $(X_t)_{t\geq 0}$ has independent increments if its increments over amy finite collection of disjoint intervals are independent.

Definition 2.6 (Q-matrix)**.** A Q-matrix on I is a matrix $Q = (q_{ij} : i, j \in I)$ satisfying the following conditions:
(i) $\forall i \quad 0 \leq -q_{ii} < \infty$
(ii) $\forall i \neq j \quad q_{ij} \geq 0$
(iii) $\forall i \quad \sum_{j\in I} q_{ij} = 0$

2.1 Review: Properties of Exponential Distribution

Definition 2.7.

Theorem 2.1 (memoryless property)**.**

Theorem 2.2 (infimum)**.** *Let I be a countable set and let T_k $k \in I$ be independent random variables with $T_k \sim E(q_k)$ and $0 < q := \sum_{k\in I} q_k < \infty$. Set $T = \inf_k T_k$. Then this infimum is attained at a unique random value K of k a.s.. Moreover, T and K are independent, with $T \sim E(q)$ and $P(K = k) = \frac{q_k}{q}$.*

2.2 Poisson Process

We begin with a definition of Poisson process in terms of jump chain and holding times, and then relate it to the infinitesimal definition and transition probability definition.

Definition 2.8. A right-continuous process $(X_t)_{t\leq 0}$ with values in $\mathbb{N}_{\geq 0}$ is a Poisson process of rate $\lambda \in (0, \infty)$ if its holding times $S_1, S_2, \cdots$ are i.i.d. exponential random variables of mean λ and its jump chain is given by $Y_n = n$.

Theorem 2.3. *Let $(X_t)_{t\geq 0}$ be an increasing, right-continuous integer-valued process starting from 0. Let $\lambda \in (0, \infty)$. TFAE:*
(i) (jump chain holding time definition) the holding times $S_1, S_2, \cdots$ of $(X_t)_{t\geq 0}$ are i.i.d. exponential random variables of mean λ and the jump chain is given by $Y_n = n$.
(ii) (infinitesimal definition) $(X_t)_{t\geq 0}$ has independent increments and as $h \downarrow 0$, uniformly in t,

$$P(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h), \quad P(X_{t+h} - X_t = 1) = \lambda h + o(h)$$

(iii) (incremental definition) $(X_t)_{t\geq 0}$ has stationary independent increments and for each t, X_t has Poisson distribution of parameter λt.

Theorem 2.4. *Let $(X_t)_{t\geq 0}$ be a Poisson process. Then, conditional on $(X_t)_{t\geq 0}$ having exactly one jump in the interval $[s, s+t]$, the time at which that jump occurs is uniformly distributed on $[s, s+t]$.*

Theorem 2.5. *Let $(X_t)_{t\geq 0}$ be a Poisson process. Then, conditional on the event $\{X_t = n\}$, the jump times $J_1, \cdots, J_n$ have joint density function*

$$f(t_1, \cdots, t_n) = n! 1_{0\leq t_1 \leq \cdots \leq t_n \leq t}$$

Remark. *Thus, conditional on $\{X_t = n\}$, the jump times $J_1, \cdots, J_n$ have the same distribution as an ordered sample of size n from the uniform distribution on $[0, t]$.*

3 Continuous Time Martingale

3.1 Stopping Times

Definition 3.1 (stopping time)**.** Let τ be a random time. If $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$, then τ is called a stopping time.

Definition 3.2 (optional time)**.** Let T be a random time. If $\{T < t\} \in \mathcal{F}_t$ for every $t \geq 0$, then T is called a stopping time.

Lemma 3.1. *T is an optional time of the filtration $\{\mathcal{F}_t\}$ if and only if it is a stopping time of the right-continuous filtration $\{\mathcal{F}_{t+}\}$.*

Corollary 3.0.1. *Every stopping time is optional, and the two concepts coincide if the filtration is right-continuous.*

Lemma 3.2. *If T is optional and θ is a positive constant, then $T+\theta$ is a stopping time.*

Lemma 3.3. *If τ, σ are stopping times, then so are $\tau \wedge \sigma$, $\tau \vee \sigma$, $\tau + \sigma$.*

Proof. The first two assertions are trivial.
For the third, start with the decomposition □

Lemma 3.4. *Let T, S be optional times; then $T+S$ is optional.*
Moreover, it is a stopping time if

Lemma 3.5. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of optional times; then the random times*

$$\sup_{n\geq 1} T_n \quad \inf_{n\geq 1} T_n \quad \limsup_{n\to\infty} T_n \quad \liminf_{n\to\infty} T_n$$

are all optional.
Moreover, if the T_n's are stopping times, then so is $\sup_{n\geq 1} T_n$.

Definition 3.3 (σ-field of events determined prior to a stopping time)**.** Let τ be a stopping time of the filtration $\{\mathcal{F}_t\}$. The σ-field of events determined prior to the stopping time T consists of those events $A \in \mathcal{F}$ for which $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Lemma 3.6. *τ is $\mathcal{F}_\tau$-measurable.*

Proof. $\{\tau \leq t\} \cap \{\tau \leq t\} = \{\tau \leq t\} \in \mathcal{F}_t$, so $\{\tau \leq t\} \in \mathcal{F}_\tau$. □

Theorem 3.1. *For any two stopping time and τ, σ a random time s.t. $\sigma \leq \tau$ on Ω, we have $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.*

Proof. For every stopping time τ and positive constant t, $\tau \wedge t$ is an $\mathcal{F}_t$-measurable random variable because $\mathcal{F}_{\tau\wedge t} \subset \mathcal{F}_t$. Therefore, $\{\sigma \wedge t \leq \tau \wedge t\} \in \mathcal{F}_t$. Then for any $A \in \mathcal{F}_\sigma$ we have $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau$, because

$$A \cap \{\sigma \leq \tau\} \cap \{\tau \leq t\} = (A \cap \{\sigma \leq t\}) \cap \{\tau \leq t\} \cap \{\sigma \wedge t \leq \tau \wedge t\}$$

Finally notice that $\{\sigma \leq \tau\} = \Omega$. □

Remark. *We have proved a stronger result, namely for any $A \in \mathcal{F}_\sigma$ we have $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau$.*

Theorem 3.2. *Let σ and τ be stopping times. Then $\mathcal{F}_{\tau\wedge\sigma} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.*
Moreover, $\{\tau < \sigma\}$, $\{\tau > \sigma\}$, $\{\tau \leq \sigma\}$, $\{\tau \geq \sigma\}$, $\{\tau = \sigma\}$ belongs to $\mathcal{F}_\tau \cap \mathcal{F}_\sigma$.

Proof. From the above theorem, $\mathcal{F}_{\tau\wedge\sigma} \subset \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.
For $A \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$, $A \cap \{\tau \wedge \sigma \leq t\} = A \cap (\{\tau \leq t\} \cup \{\sigma \leq t\}) \in \mathcal{F}_t$. □

Theorem 3.3. *Let τ, σ be stopping times and X an integrable random variable. We have*
(i) $E(X|\mathcal{F}_\tau) = E(X|\mathcal{F}_{\sigma\wedge\tau})$ a.s. on $\{\tau \leq \sigma\}$.
(ii) $E(E(X|\mathcal{F}_\tau)|\mathcal{F}_\sigma) = E(X|\mathcal{F}_{\sigma\wedge\tau})$ a.s..

Proof. (i) Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau \leq \sigma\}$ belongs to both $\mathcal{F}_\tau$ and $\mathcal{F}_\sigma$, and therefore to $\mathcal{F}_\tau \cap \mathcal{F}_\sigma$. So

$$\int_A 1_{\tau\leq\sigma} E(X|\mathcal{F}_{\tau\wedge\sigma})\mathrm{d}P = \int E(1_A 1_{\tau\leq\sigma} X|\mathcal{F}_{\tau\wedge\sigma})\mathrm{d}P = \int_A 1_{\tau\leq\sigma} X \mathrm{d}P$$

(ii) On $\{\tau \leq \sigma\}$ we have $E(X|\mathcal{F}_\tau) = E(X|\mathcal{F}_{\sigma\wedge\tau})$ a.s. by (i), so $E(E(X|\mathcal{F}_\tau)|\mathcal{F}_\sigma) = E(E(X|\mathcal{F}_{\sigma\wedge\tau})|\mathcal{F}_\sigma) = E(X|\mathcal{F}_{\sigma\wedge\tau})$. Similarly on $\{\sigma \leq \tau\}$ we have $E(E(X|\mathcal{F}_\tau)|\mathcal{F}_\sigma) = E(E(X|\mathcal{F}_\tau)|\mathcal{F}_{\sigma\wedge\tau}) = E(X|\mathcal{F}_{\sigma\wedge\tau})$. □

Theorem 3.4. *Let $X = \{X_t, \mathcal{F}_t\}$ be a progressively measurable process, and let τ be a stopping time of the filtration $\mathcal{F}_t$. Then the random variable X_τ defined on $\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable, and the stopped process $\{X_{\tau\wedge t}, \mathcal{F}_t\}$ is progressively measurable.*

3.2 From Discrete to Continuous

In this subsection, we generalize inequalities and convergence results for discrete time martingales to continuous time martingales.
Let X_t be a submartingale adapted to $\{\mathcal{F}_t\}$ whose paths are right-continuous. Let $[\sigma, \tau]$ be a subinterval of $[0, +\infty)$, and let $a < b$, $\lambda > 0$ be real numbers.

Theorem 3.5 (Doob's inequality)**.** *Let* $A = \left\{\sup_{\sigma \le t \le \tau} X_t^+ \ge \lambda\right\}$*, then*

$$\lambda P(A) \le EX_\tau 1_A \le EX_\tau^+$$

Proof. Let the finite set $\mathcal{S}$ consist of σ, τ and a finite subset of $[\sigma, \tau] \cap \mathbb{Q}$.

By considering an increasing sequence $\{\mathcal{S}_n\}_{n=1}^\infty$ of finite sets whose union is the whole of $([\sigma, \tau] \cap \mathbb{Q}) \cup \{\sigma, \tau\}$, we may replace S by this union in the preceding discrete version of the inequality. □

Theorem 3.6 (upcrossing inequality)**.**

$$(b-a)EU_{[\sigma,\tau]} \le E(X_\tau - a)^+ - E(X_\sigma - a)^+$$

Theorem 3.7 (L^p maximum inequality)**.** $\bar{X}_\tau = \sup_{\sigma \le t \le \tau} X_t^+$*, then for* $1 < p < \infty$*,*

$$E(\bar{X}_\tau^p) \le (\frac{p}{p-1})^p E(X_\tau^+)^p$$

For the remainder of this subsection, we deal only with right-continuous processes, usually imposing no condition on the filtration $\mathcal{F}_t$.

Theorem 3.8 (submartingale convergence)**.** *Assume* $\sup_{t\ge 0} E(X_t^+) < \infty$*. Then* $X_\infty = \lim_{t\to\infty} X_t$ *exists a.s., and* $E|X_\infty| < \infty$*.*

Proof. □

Theorem 3.9 (optional sampling)**.** *Assume the submartingale has a last element* X_∞*, and let* $S \le T$ *be two optional times of the filtration. We have*

$$E(X_T | \mathcal{F}_{S^+}) \ge X_S \quad a.s.$$

If S *is a stopping time, then* $\mathcal{F}_S$ *can replace* $\mathcal{F}_{S^+}$ *above.*

Proof. Consider the sequence of random times

$$S_n(\omega) = \begin{cases} +\infty & S(\omega) = +\infty \\ \frac{k}{2^n} & \frac{k-1}{2^n} \le S(\omega) < \frac{k}{2^n} \end{cases}$$

and similarly defined sequences $\{T_n\}$. These are stopping times. For every fixed integer $n \ge 1$, both S_n and T_n take on a countable number of values and we also have $S_n \le T_n$. □

3.3 Doob-Meyer Decomposition

Definition 3.4 (increasing process)**.** An adapted process A is called increasing if for P-a.e. $\omega \in \Omega$ we have
(i) $A_0(\omega) = 0$
(ii) $t \mapsto A_t(\omega)$ is a nondecreasing, right-continuous function, and $EA_t < \infty$ holds for every $t \in [0, \infty)$.
An increasing process is called integrable if $EA_\infty < \infty$.

Definition 3.5. An increasing process A is called natural if for every bounded, right-continuous martingale $\{M_t, \mathcal{F}_t; 0 \le t < \infty\}$ we have

$$E\int_{(0,t]} M_s \mathrm{d}A_s = E\int_{(0,t]} M_{s^-} \mathrm{d}A_s \quad \forall 0 < t < \infty$$

Lemma 3.7. *If* A *is an increasing process and* $\{M_t, \mathcal{F}_t; 0 \le t < \infty\}$ *is a bounded right-continuous martingale, then*

$$E(M_t A_t) = E\int_{(0,t]} M_s \mathrm{d}A_s$$

The following concept is a strengthening of the notion of uniform integrablity for submartingales.

Definition 3.6 (class DL)**.**

Theorem 3.10. *Let* $\{\mathcal{F}_t\}$ *satisfies the usual conditions. If the right-continuous submartingale* $X =$ *is of class DL, then it admits the decomposition as the summation if a right-continuous martingale*

3.4 Square Integrable Martingales

4 BM

Definition 4.1 (d-dimensional Brownian motion). A d-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ is a stochastic process indexed by $[0, \infty)$ taking values in $\mathbb{R}^d$ s.t.
(i) $B_0(\omega) = 0$

5 Stochastic Integration

5.1 Martingale Characterization of BM

Theorem 5.1 (Levy).

5.2 Representations of Martingales by BM

Theorem 5.2 (time-change for martingales).

Theorem 5.3 (representation of square-integrable martingales by BM via Ito's integral).

5.3 The Girsanov Theorem

6 The PDE Connection

7 Stochastic Differential Equations