Measure Theory

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Chapter 1

Real Analysis

1.1

Continuity of Translation

Theorem 1.1.1. *Suppose $f \in L^1(\mathbb{R}^n)$. Then*

$$\|f_h - f\|_{L^1} \to 0 \quad h \to 0.$$

Proof. The result is clearly true The proof is a simple consequence of the approximation of integrable functions by continuous functions of compact support. □

1.2 Differentiation and Integration

For conceptual simplicity, we study $\mathbb{R}$ instead of $\mathbb{R}^n$ in this section. Let us first recall what we learned in elementary calculus.

Theorem 1.2.1. *Let f be a continuous function on $[a, b]$, and F be the function defined, for all x in $[a, b]$, by*

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Then F is uniformly continuous on $[a, b]$ and differentiable on (a, b), and

$$F'(x) = f(x).$$

Theorem 1.2.2 (Newton-Leibniz)**.** *Let f be a Riemann integrable function on $[a, b]$, and F a continuous function on $[a, b]$ which is an antiderivative of f in (a, b):*

$$F'(x) = f(x).$$

Then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Definition 1.2.3 (Total Variation)**.** The total variation of a function f defined on $[a, b]$ is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} |f(x_{i+1}) - f(x_i)|,$$

where

$$\mathcal{P} = \{P = \{x_0, \cdots, x_{n_P}\} | P \text{ is a partition of } [a, b] \text{ satisfying } x_i \leq x_{i+1} \text{ for } 0 \leq i \leq n_P - 1\}.$$

If $V_a^b(f) < +\infty$, then f is said to be of bounded variation on $[a, b]$.

Example 1.2.4. The continuous function

$$f(x) = \begin{cases} 0, & x = 0 \\ x\sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

is not of bounded variation on $[0, 2/\pi]$. Just consider the partition

$$P = \{0, \frac{2}{2n-1}, \frac{2}{2n-3}, \cdots, \frac{2}{3}, 1\}.$$

Theorem 1.2.5 (Jordan Decomposition Theorem)**.**

$$f(x) = g(x) - h(x)$$

where $g(x)$ *and* $h(x)$

Bounded variable functions says that it has a length

Bounded variable functions are a.e. differentiable But it does not satisfies the desirable property: Length is equal to the integral of derivative, which is also known as the **fundamental theorem of calculus**. It turns out that absolute continuous functions is exactly the set of functions that satisfies this property.

Example 1.2.6.

Definition 1.2.7 (Absolute Continuity)**.** A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if for every positive number ϵ, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) satisfies

$$\sum_{k=1}^{N} (y_k - x_k) < \delta$$

then

$$\sum_{k=1}^{N} |f(y_k) - f(x_k)| < \epsilon.$$

Absolutely continuous functions are of bounded variation.

Proposition 1.2.8. *If* $f : [a, b] \to \mathbb{R}$ *is absolutely continuous, then it is of bounded variation on* $[a, b]$*.*

Theorem 1.2.9. *If* f *is absolute continuous function on* $[a, b]$*, then*

$$f(x) - f(a) = \int_a^x f'(t)\mathrm{d}t, \quad x \in [a, b].$$

It is beneficial to compare it with the “classical” version (Theorem 1.2.2).

Chapter 2

Abstract Measure Theory

2.1

In this section, we are working on a set E.

Definition 2.1.1 (σ-algebra)**.**

Remark 2.1.2. Greek letters σ and δ are often used when countable unions and countable intersections are involved. For example, topologists call F_σ every countable union of closed sets in a topological space (F standing possibly for the French word fermé, closed) and G_δ every countable intersection of open sets (G standing for the German word Gebiet, domain, connected open set). The letters σ and δ are often given as Greek abbreviations of German words: σ as S in Summe for sum (in the sense of sum of sets, that is, union) and δ as D in Durchschnitt for intersection, both countable. Thus, in the context of measure theory, the letter σ refers to the stability of a collection of subsets by countable union.

Definition 2.1.3 (generator of σ-algebra)**.**

$$\sigma(\mathcal{A}) = \{A \subset E : A \in \mathcal{E} \quad \forall \mathcal{E} \supset\}$$

Example 2.1.4. Borel σ-algebra

Definition 2.1.5 (π-system)**.** $\mathcal{A}$ is a collection of subsets of E. Then $\mathcal{A}$ is called a π-system if

1. $\emptyset \in \mathcal{A}$;
2. $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Definition 2.1.6 (d-system)**.** $\mathcal{A}$ is a collection of subsets

1. $E \in \mathcal{A}$;
2. $A, B \in \mathcal{A}$, $A \subset B$, then $B \setminus A \in \mathcal{A}$;
3. If $A_n \subset \mathcal{A}$ such that $A_n \subset A_{n+1}$, then $\cup_n A_n \in \mathcal{A}$.

Remark 2.1.7. d-system is also referred as λ-system.

Proposition 2.1.8. *$\mathcal{A}$ is a σ-algebra if and only if it is a π-system and a d-system.*

Proof. A σ-algebra is a π system because

Conversely, □

Lemma 2.1.9 (Dynkin's π-system lemma)**.** *Let $\mathcal{A}$ be a π-system. Then any d-system that contains $\mathcal{A}$ also contains $\sigma(\mathcal{A})$.*

This lemma can reduce the problem of studying a σ-algebra to the study of a π-system.

Proof. Let $\mathcal{D}$ be the intersection of all d-system containing $\mathcal{A}$. We now prove that $\mathcal{D}$ is a σ-algebra. As $\mathcal{D}$ is already a d-system, by Proposition 2.1.8, we only need to prove that $\mathcal{D}$ is a π-system.

(i) First we show the following property for $\mathcal{D}$: If $B \in \mathcal{D}$ and $A \in \mathcal{A}$, then $B \cap A \subset \mathcal{D}$. Let

$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{A}\}.$$

To show that $\mathcal{D}' = \mathcal{D}$, we only need to check that $\mathcal{D}'$ is a d-system containing $\mathcal{A}$. As $\mathcal{A}$ is a π-system, $D' \supset \mathcal{A}$. Now, we can check

- $E \in \mathcal{D}'$; this is because $\mathcal{A} \subset \mathcal{D}$.
- $B_1, B_2 \in \mathcal{D}'$, $B_1 \subset B_2$, then $B_2 \setminus B_1 \in \mathcal{D}'$; this is because for any $A \in \mathcal{A}$, $(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus (B_1 \cap A) \in \mathcal{D}$.
- If $B_n \subset \mathcal{D}'$ such that $B_n \subset B_{n+1}$, then $\cup_n B_n \in \mathcal{D}'$; this is because for any $A \in \mathcal{A}$, $(\cup_n B_n) \cap A = \cup_n (B_n \cap A) \in \mathcal{D}$.

(ii) If $A, B \in \mathcal{D}$, then $B \cap A \in \mathcal{D}$. Let

$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{D}\}.$$

One can see that $\mathcal{D}''$ is a d-system by noting that the argument holds for any $\mathcal{A} \subset \mathcal{D}$. The fact that $\mathcal{D}''$ contains $\mathcal{A}$ follows from property (i). Therefore, $\mathcal{D}'' = \mathcal{D}$ and $\mathcal{D}$ is a π-system. □

2.2 Caratheodory Extension Theorem

Definition 2.2.1 (set function)**.** $\mathcal{A}$ be a collection of subsets of E with $\emptyset \in \mathcal{A}$. A set function $\mu : \mathcal{A} \to [0, \infty]$ is a function such that $\mu(\emptyset) = 0$.

Definition 2.2.2 (increasing set function)**.** $A \subset B$, we have $\mu(A) \leq \mu(B)$.

Definition 2.2.3 (Additive Set Function)**.** A set function is additve if whenever $A, B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}, A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Definition 2.2.4 (Countably Additive Set Function)**.** A set function is countably additive if whenever $(A_n) \subset \mathcal{A}$ such that $A_n \cap A_m = \emptyset$ for all $n \neq m$ and $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$, then $\mu(\cup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$.

Definition 2.2.5 (Countably subadditive Set Function)**.** A set function is countably subadditive if whenever $(A_n) \subset \mathcal{A}$ and $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$, then $\mu(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$.

Definition 2.2.6 (ring)**.** A collection of subsets $\mathcal{A}$ of a subset E is a ring if

1. $\emptyset \in \mathcal{A}$
2. $\forall A, B \in \mathcal{A}$, we have $B \setminus A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

Definition 2.2.7 (algebra)**.** A collection of subsets $\mathcal{A}$ of a subset E is an algebra if

1. $\emptyset \in \mathcal{A}$
2. $\forall A, B \in \mathcal{A}$, we have $A^c \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

Theorem 2.2.8 (Caratheodory Extension Theorem)**.** *Let $\mathcal{A}$ be a ring on E, and μ be a countably additive set function on $\mathcal{A}$. Then μ extends to a measure on $\sigma(\mathcal{A})$.*

Theorem 2.2.9. *Suppose that μ_1, μ_2 are measures on $(E, \mathcal{E})$ with $\mu_1(E) = \mu_2(E) < \infty$. If $\mathcal{A}$ is a π-system with $\sigma(\mathcal{A}) = \mathcal{E}$ and $\mu_1|_{\mathcal{A}} = \mu_2|_{\mathcal{A}}$, then $\mu_1 = \mu_2$.*

Proof. Let $\mathcal{D} = \{A \in \mathcal{E} : \mu_1(A) = \mu_2(A)\}$, then $\mathcal{D} \supset \mathcal{A}$. By Dynkin's lemma, it suffices to show that $\mathcal{D}$ is a d-system. We can check

- $E \in \mathcal{D}$.
- Let $A, B \in \mathcal{D}$ with $A \subset B$. (Note that finiteness is important here.)
-

□

Example 2.2.10. Let $E = \mathbb{Z}$, $\mathcal{E} =$. let $\mathcal{A} = \{\{x, x+2, x+2, \dots\} : x \in E\} \cup \{\emptyset\}$. $\mathcal{A}$ is a π-system such that $\mathcal{E} = \sigma(\mathcal{A})$. Let $\mu_1(A)$ be the number of elements of A and $\mu_2 = 2\mu_1$.

Definition 2.2.11 (Borel σ-algebra)**.** Let E be a topological space. $\mathcal{B}(E) = \sigma(\{U \subset E : U \text{ open}\})$.

Definition 2.2.12 (borel and Radon Measure)**.** A measure μ on $(E, \mathcal{B}(E))$ is called a Borel measure. If $\mu(K) < \infty$ for all $K \subset E$ compact, then μ is called a Radon measure.

2.3 Lebesgue Measure

Theorem 2.3.1. *There exists a unique Borel measure μ on $\mathcal{B}(\mathbb{R})$ with $\mu([a,b]) = b - a$.*

Remark 2.3.2. This measure is called the Lebesgue measure.

Proof. **Uniqueness.** Suppose $\tilde{\mu}$ is another measure. For any $n \in \mathbb{Z}$, we set $\mu_n(A) = \mu(A \cap (n+1, n])$. The previous theorem imples that $\mu_n = \tilde{\mu}_n$ for all $n \in \mathbb{Z}$. Firx $A \in \mathcal{B}(\mathbb{R})$, $A = \cup_{n\in\mathbb{Z}}(A \cap (n, n+1])$, $\mu(A) = \sum_{n\in\mathbb{Z}} \mu_n(A) = \sum_{n\in\mathbb{Z}} \tilde{\mu}_n(A) = \tilde{\mu}(A)$.

Existence. $\mathcal{A}$ be the collection of finite, disjoint unions of the following form $A = \cup_{n\in\mathbb{Z}}(a_i, b_i]$. $\mathcal{A}$ is a ring on $\mathbb{R}$ and $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. Define $\mu(A) = \sum_{i=1}^n (b_i - a_i)$. It is well defined and additive. Suppose $A = \cup_{n=1}^\infty A_n \in \mathcal{A}$, $\mu(A) = \sum_{i=1}^n \mu(A_i) + \mu(A \setminus \sum_{j=1}^n A_j)$. Let $B_n = A \setminus \sum_{j=1}^n A_j$.

Argue by contradiction,$\exists \epsilon > 0, n_0 \in \mathbb{N}$ such that $\mu(B_n) \geq 2\epsilon \; \forall n \geq n_0$. $\forall n \in \mathbb{N}$, $\exists C_n \in \mathcal{A}$ such that $\overline{C_n} \subset B_n$ and $\mu(B_n \setminus C_n) \leq \frac{\epsilon}{2^n}$. Then we have $\mu(B_n) - \mu(\cap_{m=1}^n \overline{C_m}) = \mu(B_n \setminus \cap_{m=1}^n C_m) \leq \mu(\cup_{m=1}^n (B_m \setminus C_m)) \leq \sum_{m=1}^n \mu(B_m \setminus C_m) \leq \epsilon$. The finite intersection property □

Lemma 2.3.3 (Finite intersection property of compact sets of $\mathbb{R}$)**.** *If*

Definition 2.3.4 (σ-finite measure)**.** Let $(E, \mathcal{E})$ be a measurable space, and μ a measure on $\mathcal{E}$. We say that μ is σ-finite if $E = \cup_{n\in\mathbb{N}} E_n, E_n \in \mathcal{E}, \mu(E_n) < \infty \forall n$.

It is easy to see that Lebesgue measure is translation invariant using the above theorem. Just take

Theorem 2.3.5. *Let $\tilde{\mu}$ be a Borel measure on $\mathbb{R}$ that is translation invariant and $\tilde{\mu}([0,1]) = 1$. Then $\tilde{\mu}$ is the Lebesgue measure.*

Proof. First show no singleton. $\tilde{\mu}(\{a\}) \leq \tilde{\mu}([a, a + \frac{1}{n})) = \tilde{\mu}([0, \frac{1}{n})) \leq \frac{1}{n}$.

Can find $p_n, q_n \in \mathbb{Q}$ such that $p_n \downarrow a$, $q_n \uparrow b$ as $n \to \infty$. □

2.4 Probability Measure

Definition 2.4.1. Let $(E, \mathcal{E})$ be a measure space with the property $\mu(E) = 1$. Then μ is called a probability measure and $(E, \mathcal{E}, \mu)$ a probability space.

2.5 Radon Measure

Every Borel measure on $\mathbb{R}^n$ is regular and locally finite, which is Radon.

There are basically two approach es to mesure theory, one via simpler sets like ring, the other via outer measure and define measurable sets.