Lecture Notes in Multivariate Data Analysis

Kaizhao Liu

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1 The Multivariate Normal Distributions

1.1 Asymptotic Distributions of Sample Means and Covariance Matrices

1.2 The Noncentral χ^2 and F Distributions

Definition 1.1 (generalized hypergeometric functions)**.** The generalized hypergeometric function is

$$ {}_p\mathrm{F}_q(a_1,\cdots a_p;b_1,\cdots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!} $$

where $(a)_k=a(a+1)\cdots(a+k-1)$

For our purpose we will make use of the results in the following two lemmas. The first gives a special integral for ${}_0F_1$.

Lemma 1.1.

$$ \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}\int_0^{\pi}e^{z\cos\theta}\sin^{n-2}\theta\mathrm{d}\theta={}_0\mathrm{F}_1(\frac{n}{2};\frac{z^2}{4}) $$

The second lemma shows that ${}_{p+1}\mathrm{F}_q$ is essentially a Laplace transform of ${}_p\mathrm{F}_q$.

Lemma 1.2.

$$ \int_0^{\infty}e^{-zt}t^{a-1}{}_p\mathrm{F}_q(a_1,\cdots,a_p;b_1,\cdots,b_p;kt)\mathrm{d}t=\Gamma(a)z^{-a}{}_{p+1}\mathrm{F}_q(a_1,\cdots,a_p,a;b_1,\cdots,b_q;kz^{-1}) $$

for $p<q,\Re(a)>0,\Re(z)>0$ or $p=q,\Re(a)>0,\Re(z)>\Re(k)$.

Theorem 1.1. *If X is $N_n(\mu,I_n)$ then the random variable $Z=X^TX$ has the density function*

$$ e^{-\frac{\delta}{2}}{}_0\mathrm{F}_1(\frac{n}{2};\frac{\delta z}{4})\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}e^{-\frac{z}{2}}z^{\frac{n}{2}-1}\quad(z>0) $$

where $\delta=\mu^T\mu$. Z is said to have the noncentral χ^2 distribution with n degrees of freedom and noncentrality parameter δ, to be written as $\chi^2_n(\delta)$.

Corollary 1.1.1. *If Z is $\chi_n^2(\delta)$ then its density function can be expressed as*

$$\sum_{k=0}^{\infty} P(K=k) g_{n+2k}(z) \quad (z>0)$$

where K is a Poisson random variable with mean $\frac{\delta}{2}$, and

$$g_r(z) = \frac{1}{2^{\frac{r}{2}}\Gamma(\frac{r}{2})} e^{-\frac{z}{2}} z^{\frac{r}{2}-1}$$

the density function of the χ_r^2 distribution.

Theorem 1.2. *If Z is $\chi_n^2(\delta)$ then its characteristic function is*

$$\varphi_z(t) = (1-2it)^{\frac{n}{2}} e^{\frac{it\delta}{1-2it}}$$

Corollary 1.2.1. $E(Z) = n+\delta$ *and* $Var(Z) = 2n+4\delta$

Corollary 1.2.2. *If Z_1 is $\chi_{n_1}^2(\delta_1)$, Z_2 is $\chi_{n_2}^2(\delta_2)$, and Z_1 and Z_2 are independent, then Z_1+Z_2 is $\chi_{n_1+n_2}^2(\delta_1+\delta_2)$.*

We now turn to the noncentral F distribution. Recall that the usual central F distribution is obtained by taking the ratio of two independent χ^2 variables divided by their degrees of freedom. The noncentral F distribution is obtained by allowing the numerator variable to be noncentral χ^2.

Theorem 1.3. *If Z_1 is $\chi_{n_1}^2(\delta)$,Z_2 is $\chi_{n_2}^2$, and Z_1 and Z_2 are independent, then*

$$F = \frac{Z_1/n_1}{Z_2/n_2}$$

has the density function

$$e^{-\frac{\delta}{2}} {}_1\mathrm{F}_1\left(\frac{n_1+n_2}{2};\frac{n_1}{2};\frac{\frac{n_1}{2n_2}\delta z}{1+\frac{n_1}{n_2}z}\right) \times \frac{\Gamma(\frac{n_1+n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} \frac{z^{\frac{n_1}{2}-1}(\frac{n_1}{n_2})^{\frac{n_1}{2}}}{(1+\frac{n_1}{n_2}z)^{\frac{n_1+n_2}{2}}} \quad z>0$$

F is said to have the noncentral F distribution with n_1 and n_2 degrees of freedom and noncentrality parameter δ, to be written as $F_{n_1,n_2}(\delta)$.

Corollary 1.3.1. $E(F) = \frac{n_2(n_1+\delta)}{n_1(n_2-2)}(n_2>2)$ *and* $Var(F) = 2(\frac{n_2}{n_1})^2 \frac{(n_1+\delta)^2+(n_1+2\delta)(n_2-2)}{(n_2-2)^2(n_2-4)}(n_2>4)$

1.3 Quadratic Forms

Theorem 1.4. *If X is $N_m(\mu,\Sigma)$, where Σ is nonsingular, then*
(i) $(X-\mu)^T\Sigma^{-1}(X-\mu)$ is χ_m^2
(ii) $X^T\Sigma^{-1}X$ is $\chi_m^2(\delta)$ where $\delta = \mu^T\Sigma^{-1}\mu$

Theorem 1.5. *If X is $N_m(\mu, I_m)$ and B is an $m\times m$ symmetric matrix, then X^TBX has an noncentral χ^2 distribution if and only if B is idempotent.*

1.4 Spherical and Elliptical Distributions

Definition 1.2. A $m\times 1$ random vector X is said to have a spherical distribution if X and OX have the same distribution for all $O \in O(n)$.

Theorem 1.6. *Let X be $E_m(\mu, V)$, where V is diagonal. If $X_1,\cdots,X_m$ are all independent then X is normal.*

Proof. WLOG assume $\mu=0$. Then the characteristic function of X has the form $\phi(t) = \psi(t^TVt) = \psi(\sum_{i=1}^m t_i^2 v_{ii})$. Since $X_1,\cdots,X_m$ are independent we have This equation is known as Hamel's euqation and its only continuous solution is $\psi(z) = e^{kz}$ for some constant k. Hence the characteristic function of X has the form $\phi(t) = e^{kt^TVt}$, and because it is a characteristic function, we must have $k\leq 0$ which implies that X has a normal distribution. □

2 Multivariate Integration

2.1 Exterior Products

For any matrix X, $\mathrm{d}(X)$ denotes the matrix of differentials $(\mathrm{d}x_{ij})$.

For an arbitary $n \times m$ matrix X, the symbol $(\mathrm{d}X)$ will denote the exterior product of the mn distinct elements of $\mathrm{d}X$:

$$(\mathrm{d}X) := \bigwedge_{j=1}^{m} \bigwedge_{i=1}^{n} \mathrm{d}x_{ij}$$

For a symmetric $m \times m$ matrix X, the symbol $(\mathrm{d}X)$ will denote the exterior product of the $\frac{m(m+1)}{2}$ distinct elements of $\mathrm{d}X$:

$$(\mathrm{d}X) := \bigwedge_{1 \le i \le j=1 \le m} \mathrm{d}x_{ij}$$

2.2 The Multivariate Gamma Function

2.3 Miscellaneous Jacobians

2.4 Invariant Measures