Measure Theory

Kaizhao Liu

September 4, 2025

Contents

Chapter 1

Real Analysis

1.1

1.2 Differentiation and Integration

For conceptual simplicity, we study $\mathbb{R}$ instead of $\mathbb{R}^n$ in this section. Let us first recall what we learned in elementary calculus.

Theorem 1.2.1. *Let f be a continuous function on $[a,b]$, and F be the function defined, for all x in $[a,b]$, by*

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Then F is uniformly continuous on $[a,b]$ and differentiable on (a,b), and

$$F'(x) = f(x).$$

Theorem 1.2.2 (Newton-Leibniz)**.** *Let f be a Riemann integrable function on $[a,b]$, and F a continuous function on $[a,b]$ which is an antiderivative of f in (a,b):*

$$F'(x) = f(x).$$

Then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Definition 1.2.3 (Total Variation)**.** The total variation of a function f defined on $[a,b]$ is the quantity

$$V_a^b(f) = \sup_{P\in\mathcal{P}} \sum_{i=0}^{n_P-1} |f(x_{i+1}) - f(x_i)|,$$

where

$$\mathcal{P} = \{P = \{x_0, \cdots, x_{n_P}\} | P \text{ is a partition of } [a,b] \text{ satisfying } x_i \leq x_{i+1} \text{ for } 0 \leq i \leq n_P - 1\}.$$

If $V_a^b(f) < +\infty$, then f is said to be of bounded variation on $[a,b]$.

Example 1.2.4. The continuous function

$$f(x) = \begin{cases} 0, & x = 0 \\ x\sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

is not of bounded variation on $[0, 2/\pi]$. Just consider the partition

$$P = \{0, \frac{2}{2n-1}, \frac{2}{2n-3}, \cdots, \frac{2}{3}, 1\}.$$

Theorem 1.2.5 (Jordan Decomposition Theorem)**.**

$$f(x) = g(x) - h(x)$$

where $g(x)$ and $h(x)$

Bounded variable functions says that it has a length

Bounded variable functions are a.e. differentiable But it does not satisfies the desirable property: Length is equal to the integral of derivative, which is also known as the **fundamental theorem of calculus**. It turns out that absolute continuous functions is exactly the set of functions that satisfies this property.

Example 1.2.6.

Definition 1.2.7 (Absolute Continuity)**.** A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if for every positive number ϵ, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) satisfies

$$\sum_{k=1}^{N}(y_k - x_k) < \delta$$

then

$$\sum_{k=1}^{N}|f(y_k) - f(x_k)| < \epsilon.$$

Absolutely continuous functions are of bounded variation.

Proposition 1.2.8. *If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a, b]$.*

Theorem 1.2.9. *If f is absolute continuous function on $[a, b]$, then*

$$f(x) - f(a) = \int_a^x f'(t)\mathrm{d}t, \quad x \in [a, b].$$

It is beneficial to compare it with the "classical" version (Theorem 1.2.2).

Chapter 2

Abstract Measure Theory

2.1

In this section, we are working on a set E.

Definition 2.1.1 (σ-algebra)**.**

Remark 2.1.2. Greek letters σ and δ are often used when countable unions and countable intersections are involved. For example, topologists call F_σ every countable union of closed sets in a topological space (F standing possibly for the French word fermé, closed) and G_δ every countable intersection of open sets (G standing for the German word Gebiet, domain, connected open set). The letters σ and δ are often given as Greek abbreviations of German words: σ as S in Summe for sum (in the sense of sum of sets, that is, union) and δ as D in Durchschnitt for intersection, both countable. Thus, in the context of measure theory, the letter σ refers to the stability of a collection of subsets by countable union.

Definition 2.1.3 (generator of σ-algebra)**.**

$$\sigma(\mathcal{A}) = \{A \subset E : A \in \mathcal{E} \quad \forall \mathcal{E} \supset\}$$

Example 2.1.4. Borel σ-algebra

Definition 2.1.5 (π-system)**.** $\mathcal{A}$ is a collection of subsets of E. Then $\mathcal{A}$ is called a π-system if

1. $\emptyset \in \mathcal{A}$;
2. $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Definition 2.1.6 (d-system)**.** $\mathcal{A}$ is a collection of subsets

1. $E \in \mathcal{A}$;
2. $A, B \in \mathcal{A}$, $A \subset B$, then $B \setminus A \in \mathcal{A}$;
3. If $A_n \subset \mathcal{A}$ such that $A_n \subset A_{n+1}$, then $\cup_n A_n \in \mathcal{A}$.

Remark 2.1.7. d-system is also referred as λ-system.

Proposition 2.1.8. *$\mathcal{A}$ is a σ-algebra if and only if it is a π-system and a d-system.*

Proof. A σ-algebra is a π system because
Conversely, □

Lemma 2.1.9 (Dynkin's π-system lemma)**.** *Let $\mathcal{A}$ be a π-system. Then any d-system that contains $\mathcal{A}$ also contains $\sigma(\mathcal{A})$.*

This lemma can reduce the problem of studying a σ-algebra to the study of a π-system.

Proof. Let $\mathcal{D}$ be the intersection of all d-system containing $\mathcal{A}$. We now prove that $\mathcal{D}$ is a σ-algebra. As $\mathcal{D}$ is already a d-system, by Proposition 2.1.8, we only need to prove that $\mathcal{D}$ is a π-system.

(i) First we show the following property for $\mathcal{D}$: If $B \in \mathcal{D}$ and $A \in \mathcal{A}$, then $B \cap A \subset \mathcal{D}$. Let

$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{A}\}.$$

To show that $\mathcal{D}' = \mathcal{D}$, we only need to check that $\mathcal{D}'$ is a d-system containing $\mathcal{A}$. As $\mathcal{A}$ is a π-system, $D' \supset \mathcal{A}$. Now, we can check

- $E \in \mathcal{D}'$; this is because $\mathcal{A} \subset \mathcal{D}$.
- $B_1, B_2 \in \mathcal{D}'$, $B_1 \subset B_2$, then $B_2 \setminus B_1 \in \mathcal{D}'$; this is because for any $A \in \mathcal{A}$, $(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus (B_1 \cap A) \in \mathcal{D}$.
- If $B_n \subset \mathcal{D}'$ such that $B_n \subset B_{n+1}$, then $\cup_n B_n \in \mathcal{D}'$; this is because for any $A \in \mathcal{A}$, $(\cup_n B_n) \cap A = \cup_n (B_n \cap A) \in \mathcal{D}$.

(ii) If $A, B \in \mathcal{D}$, then $B \cap A \in \mathcal{D}$. Let

$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{D}\}.$$

One can see that $\mathcal{D}''$ is a d-system by noting that the argument holds for any $\mathcal{A} \subset \mathcal{D}$. The fact that $\mathcal{D}''$ contains $\mathcal{A}$ follows from property (i). Therefore, $\mathcal{D}'' = \mathcal{D}$ and $\mathcal{D}$ is a π-system. $\square$

Definition 2.1.10 (set function)**.** $\mathcal{A}$ be a collection of subsets of E with $\emptyset \in \mathcal{A}$. A set function $\mu : \mathcal{A} \to [0, \infty]$ is a function such that $\mu(\emptyset) = 0$.

Definition 2.1.11 (increasing set function)**.** $A \subset B$, we have $\mu(A) \leq \mu(B)$.