Geometry

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An affine connection $\nabla_{\cdot}(\cdot)$ is a map from $\Gamma(M) \times \Gamma(M)$ to $\Gamma(M)$

Definition 2.2.1. 1. $C(M)$-linear in the lower slot

2. $\mathbb{R}$-linear in the upper slot

3. $C(M)$-Leibniz rule in the upper slot

2.2.1 Levi-Civita Connection

Let ∇ be an affine connection, then by the basis theorem there are locally defined functions Γ_{ij}^k such that

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k. \tag{2.1}$$

A connection is symmetric if
metric compaibility

Theorem 2.2.2. *Given a metric g, there is a unique connection ∇ that is symmetric and metric compatible. This connection is called the* ***Levi-Civita connection****, and is given by*

2.2.2 Tensor Leibniz Rule

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$$

Proposition 2.2.3.

$$F^{i_1 \dots i_k}_{j_1 \dots j_l; m} = E_m(F^{i_1 \dots i_k}_{j_1 \dots j_l}) + \sum_{s=1}^{k} \Gamma^{i_s}_{mp} F^{i_1 \dots p \dots i_k}_{j_1 \dots j_l} - \sum_{s=1}^{l} \Gamma^{p}_{m j_s} F^{i_1 \dots i_k}_{j_1 \dots p \dots j_l}$$

higher order covariant derivative can be computed by iterating

2.2.3 Along a Curve

2.3 Curvature

The covariant derivative of a (r, s) tensor can be thought of as an $(r, s+1)$ tensor in a natural way. For example, if

Repeating this, we get a $(1, 2)$ tensor $\nabla\nabla V$, which will abbreviate as $\nabla^2 V$.

Using the Leibniz rule, we see that

$$\nabla_X(\nabla_Y V) = \nabla^2_{X,Y} V + \nabla_{\nabla_X Y} V$$

The tensor is not necessarily symmetric in the two lower slots. In fact, the curvature comes in

$$
\begin{aligned}
\nabla^2_{X,Y} V - \nabla^2_{Y,X} V &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{\nabla_X Y - \nabla_Y X} V \\
&= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{[X,Y]} V \\
&= R(Y, X)V
\end{aligned}
$$

This is known as the **Ricci identity**.

Definition 2.3.1 (Riemann curvature)**.**

2.3.1 Symmetries