Lecture Notes in Functional Analysis

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Contents

0.1 Prologue

In analysis, examples and counterexamples are important.

Chapter 1

Basic Topology

To faciliate the study of convergence on an aribitary space

We use metric space as example, for the sake of functional analysis.

1.1 Basic Topology

1.1.1 Definitions of Topology

neighborhood, open set, closed set,

Neighborhood

Let X be a (possibly empty) set. Let $\mathcal{N}$ be a function assigning to each $x \in X$ a non-empty collection $\mathcal{N}(x)$ of subsets of X. The elements of $\mathcal{N}(x)$ will be called neighbourhoods of x.

(i) If $N \in \mathcal{N}(x)$, then $x \in N$.

(ii) If $N \subset M$ and $N \in \mathcal{N}(x)$, then $M \in \mathcal{N}(x)$.

(iii) If $N_1, N_2 \in \mathcal{N}(x)$, then $N_1 \cap N_2 \in \mathcal{N}(x)$.

(iv) If $N \in \mathcal{N}(x)$, then there exists $M \in \mathcal{N}(x)$ such that $M \subset N$, and for any $y \in M$ we have $N \in \mathcal{N}(y)$.

Interior

Given a neighborhood structure, we can define the interior

$$\operatorname{Int}(A) := \{x \in A | A \in \mathcal{N}(x)\}$$

(i) $\operatorname{Int}(A) \subset A$.

(ii) $\operatorname{Int}(A) \cap \operatorname{Int}(B) = \operatorname{Int}(A \cap B)$.

(iii) $\operatorname{Int}(\operatorname{Int}(A)) = \operatorname{Int}(A)$.

(iv) $\operatorname{Int}(X) = X$.

Conversely, given an interior structure, we can define a neighborhood structure by

$$\mathcal{N}(x) = \{A \subset X | x \in \operatorname{Int} A\}.$$

Open Sets

Given a neighborhood structure, we can define open sets.

Definition 1.1.1 (Open)**.** A set U is **open** if for any $x \in U$, $U \in \mathcal{N}(x)$.

By the equivalence of neighborhood and interior structure,

Proposition 1.1.2. *A set U is open if and only if* $\operatorname{Int}(U) = U$.

Denote

$$\mathcal{T} = \{U \subset X | U \text{ is open}\}.$$

(i) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.

(ii) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.

(iii) If $\{U_\lambda, \lambda \in \Lambda\} \subset \mathcal{T}$, then $\cup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$.

Given a topological structure, we cana define a neighborhood

Definition 1.1.3. Let $(X, \mathcal{T})$ be a topological space, $x \in X$, and $N \subset X$. If there exists $U \in \mathcal{T}$ such that $x \in U \subset N$, then N is a neighborhood of x.

The neighborhood structure can be defined by

$$\mathcal{N}(x) = \{N \subset X | \exists U \in \mathcal{T} \text{ such that } x \in U \subset N\}.$$

Closed Sets

Definition 1.1.4 (Closed)**.** Let F be a set in a topological space $(X, \mathcal{T})$. It is **closed** if F^c is open.

1.1.2 Comparing Different Topologies

Definition 1.1.5. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. We say $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$.

1.1.3 Convergence and Continuity

TODO: add examples

The notion of convergence can be defined when given a topology only.

Definition 1.1.6 (Convergence)**.** Let x_n be a sequence in a topological space X. If there exists $x_0 \in X$ such that for any neighborhood A of x_0 there exists k such that $x_n \in A$ when $n > k$, then the sequence x_n **converges** to x_0, denoted by $x_n \to x_0$.

Definition 1.1.7 (Continuity and Sequential Continuity)**.** Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$

1. continuity

2. sequential continuity

Theorem 1.1.8. *continuity implies sequential continuity*

Definition 1.1.9 (Open and Closed Mappings)**.** Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$

(i) If the image $f(U)$ of any open set U is open in Y, then f is called an **open mapping**.

(ii) If the image $f(F)$ of any closed set F is closed in Y, then f is called an **closed mapping**.

1.1.4 Topological Basis

$$\mathcal{T}_\mathcal{B} = \{U \subset X | \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$$

for which $\mathcal{B}$, $\mathcal{T}_\mathcal{B}$ defined as above is a topology.

$$\forall x \in X, \exists B \in \mathcal{B} \text{ such that } x \in B.$$

$$\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B \in \mathcal{B} \text{ such that } x \in B \subset B_1 \cap B_2$$

Theorem 1.1.10.

$$\mathcal{T}_\mathcal{B} = \{\cup_{B \in \mathcal{B}'} B | \mathcal{B}' \subset \mathcal{B}\}.$$

1.1.5 Constructing New

Definition 1.1.11 (Subspace Topology)**.** Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. The set

$$\mathcal{T}_Y := \{U \cap Y | U \in \mathcal{T}\}$$

is a toplogy on Y, called the **subspace** topology.

Definition 1.1.12 (Product Topology)**.** Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. The set

$$\mathcal{T}_{X\times Y} := \{W \subset X \times Y | \forall (x, y) \in W, \exists U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \text{ such that } (x, y) \in U \times V \subset W\}$$

is a topology on $X \times Y$, called the **product toplogy**.

Definition 1.1.13. Let $(Y_\alpha, \mathcal{T}_\alpha)$ be a family of, let

$$\mathcal{F} = f_\alpha : X \to (Y_\alpha, \mathcal{T}_\alpha)$$

be a family of mappings. The the weakest topology on X such that any f_α is continuous

Example 1.1.14 (Weak and Weak* Topology)**.** Let X be a topological vector space and X^* be its dual,

$$X^* = \{f : X \to \mathbb{K} | f \text{ linear and continuous}\}.$$

(i) weak topology: X^*-induced topology

(ii) weak* topology: $\{\mathrm{ev}_x | x \in X\}$-induced topology

1.1.6 Example: Topologies on Mappings

For any set X and topological space Y, we can consider

$$\mathcal{M}(X, Y) := \{f : X \to Y\}.$$

We would like to study the topologies on $\mathcal{M}(X, Y)$ that is associated with the mapping structure. As $\mathcal{M}(X, Y) = Y^X$, by the product stucture we can define two topologies on $\mathcal{M}(X, Y)$:

(i) product topology, in this case also the **pointwise convergence topology** $\mathcal{T}_{p.c.}$;

(ii) box topology, which is not as useful when studying continuous maps.

Now suppose Y is a metric space, then the metric d_Y on Y induce a metric on $\mathcal{M}(X, Y)$:

$$d_u(f, g) := \sup_{x\in X} \frac{d_Y(f(x), g(x))}{1 + d_Y(f(x), g(x))}.$$

We can see that f_n converges to f in this metric if and only if f_n converges uniformly to f. Therefore,

Definition 1.1.15 (Uniform Convergence Topology)**.** d_u is called

Consider the space of continous mappings

$$\mathcal{C}(X, Y) := \{f \in \mathcal{M}(X, Y) | f \text{ continuous}\}.$$

In general, $\mathcal{C}$ is not closed in $(\mathcal{M}(X, Y), \mathcal{T}_{p.c.})$.

Example 1.1.16.

But we have

Proposition 1.1.17. *$\mathcal{C}$ is closed in $(\mathcal{M}(X, Y), d_u)$.*

On $\mathcal{C}(X, Y)$, all of the above topologies are not satisfying:

Example 1.1.18. Let $X = Y = \mathbb{R}$, then

1. Consider $\mathcal{T}_{p.c.}$: the sequence of continuous functions $f_n(x) = e^{-nx^2}$ converges to $f_0(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$, which is not continuous. This shows that $\mathcal{T}_{p.c.}$ is too weak to ensure the continuity of the limit.

2. Consider $\mathcal{T}_{u.c.}$ or $\mathcal{T}_{box}$: the sequence of continuous functions $f_n(x) = x^2/n$ does not converge, where it converges to a continuous function $f_0(x) = 0$ under $\mathcal{T}_{p.c.}$. This shows that these topologies are too strong for sequences to converge.

What is a good topology on $\mathcal{C}(X, Y)$? We will answer this question in Section 1.2.3.
compact convergence $\mathcal{T}_{c.c.}$
when Y is not a metric space compact open

1.1.7 Sets in Topological Space

Definition 1.1.19 (Limit and Sequential Limit)**.** Let X be a topological space, $A \subset X$, and $x \in X$.

(i) If for any neighborhood U of x,
$$U \cap (A \setminus \{x\}) \neq \emptyset,$$
then x is a **limit point (accumulation point)** of A. The set of all limit points of A is called the **derived set** of A, denoted by A'.

(ii) If there exists a sequence $a_n \in A$ such that $a_n \to x$, then x is a **sequential limit point** of A.

A closed set contains all its sequential limit points.

Proposition 1.1.20. *Let $A \subset X$ be closed. If $x_n \in A$ and $x_n \to x \in X$, then $x \in A$.*

Proof. We prove by contradiction. Suppose $x \in A^c$, as A^c is open, we can find an open neighborhood U of x_0 such that $U \subset F^c$. By the definition of convergence, there exists N such that for any $n > N$, $x_n \in U$. This is contradictory to $x_n \in F$. □

But the converse is not true in general.

Example 1.1.21. Consider the space $X = \mathcal{M}([0,1], \mathbb{R})$ with the pointwise convergence topology. [**kz:** to do 1.119]

A closed set also contains all its limit point. More precisely, we have:

Proposition 1.1.22. *$A \subset X$ is closed if and only if $A' \subset A$.*

Proof. If A is closed, for any $x \in A^c$ there exists an open neighborhood U of x such that $U \subset A^c$. This means that $U \cap A = \emptyset$, so $x \notin A'$ and $A' \subset A$.

If $A' \cap A$, for any $x \in A^c$ there exists a neighborhood U of x such that $U \cap (A \setminus \{x\}) = \emptyset$. So $U \subset A^c$ and A is closed. □

closure $\overline{A} := A \cup A'$

Theorem 1.1.23. *For any $A \subset X$, $A \cup A'$ is closed.*

Corollary 1.1.24.
$$A \cup A' = \bigcup_{A \subset F \text{ closed}} F.$$

The following proposition provides a way to verify if a point is in the closure of a set.

Proposition 1.1.25. *$x \in \overline{A} \iff$ for any open set U contains x, $U \cap A \neq \emptyset$.*

Proof. □

We can use the closure to characterize continuous mappings.

Proposition 1.1.26.

interior is the dual of closure

Definition 1.1.27 (Dense and Nowhere Dense)**.** Let X be a topological space and A be a subset.

(i) If $\overline{A} = X$, then A is called **dense**.

(ii) If $\mathring{\overline{A}} = \emptyset$, then A is called **meager(nowhere dense)**.

(iii) If A is a countable union of meager sets, then A is called a **Baire first category** set.

Definition 1.1.28.

1.2 Compactness

Definition 1.2.1 (Compactness and Sequential Compactness)**.** Let $(X, \mathcal{T})$ be a topological space.

(i) If every open covering of X has a finite sub-covering, then X is **compact**.

(ii) If every sequence in X has a convergent subsequence, then X is **sequentially compact**.

Leveraging the duality of open and closed set, we can immediately
What about subsets? The canonical definition is the following

Remark 1.2.2. Let $A \subset X$ be a subset. If $(A, \mathcal{T}_{subspace})$ is compact, then we say A is compact in X.

We can see that the above definition reduce to the common

Proposition 1.2.3. *Let $A \subset X$ be a subset. A is compact in X if and only if*

For sequential compactness, the definition is straightforward.
Compactness is preserved under continuous mappings.

Theorem 1.2.4. *Let $f : X \to Y$ be a continuous map.*

(i) If $A \subset X$ is compact, then $f(A) \subset Y$ is compact.

(ii) If $A \subset X$ is sequentially compact, then $f(A) \subset Y$ is sequentially compact.

Corollary 1.2.5. *quotient space*

Compactness is inherited for closed subsets.

Theorem 1.2.6. *Let $A \subset X$ be closed.*

(i) If X is compact, then A is compact.

(ii) If X is sequentially compact, then A is sequentially compact.

1.2.1 Product of Compact Spaces: Tychonoff Theorem

Theorem 1.2.7 (Tychonoff)**.** *If X_α is compact for all α, then $(\prod_\alpha X_\alpha, \mathcal{T}_{product})$ is also compact.*

1.2.2 Example: Metric Spaces

Let (X, d) be a metric space. Compared with a general topological space, a metric space has the following nice properties:

(i) **first countable**, because. Thus,

- $F \subset X$ is closed if and only if it contains all sequential limit points.
- For any topological space Y, $f : X \to Y$ is continuous if and only if f is sequentially continuous.

(ii) **Hausdorff**. Thus,

- Compact sets are closed.
- A convergent sequence has a unique limit.

Definition 1.2.8 (Total Boundedness)**.** Let (X, d) be a metric space. If for any $\epsilon > 0$, there exists a finite number of ϵ balls that cover X.

Definition 1.2.9 (ϵ net)**.**

Total boundedness is equivalent to finite ϵ net

Proposition 1.2.10. *A metric space X is totally bounded if and only if for any $\epsilon > 0$, there is a finite ϵ-net in X.*

A topological view of completeness.

Definition 1.2.11 (Absolutely Closed)**.**

Proposition 1.2.12. *A metric space is absolutely closed if and only if it is complete.*

Proof. Suppose (X, d) is complete, and (X, d) can be ? into (Y, d_Y). Then X is closed in (Y, d_Y) because it contains all □

Theorem 1.2.13. *In a metric space (X, d), TFAE:*

(i) A is compact.

(ii) A is sequentially compact.

*(iii) A is **totally** bounded and **absolutely** closed.*

Remark 1.2.14. This theorem is a natural generalization of the different characterizations of compactness on $\mathbb{R}$, which we have learned in mathematical analysis.

1.2.3 Example: Topologies on Mappings

This subsection continues the discussion of Section 1.1.6. We can try to find a topology on $\mathcal{M}(X, Y)$ that describes "uniform convergence on every compact set". We can find it by comparing with the construction of $\mathcal{T}_{p.c.}$ and $\mathcal{T}_{u.c.}$:

- The topological basis of $\mathcal{T}_{u.c.}$ are the balls

$$B(f; X; \epsilon) = \{g \in \mathcal{M}(X, Y) | \sup_{x \in X} d(f(x), g(x)) < \epsilon\}.$$

- The topological basis of $\mathcal{T}_{p.c.}$, i.e. "uniform convergence on every finite set", are the balls

$$B(f; x_1, \ldots, x_m; \epsilon) = \{g \in \mathcal{M}(X, Y) | \sup_{1 \leq i \leq m} d(f(x_i), g(x_i)) < \epsilon\}.$$

Thus for a compact set $K \subset X$, we define

$$B(f; K; \epsilon) = \{g \in \mathcal{M}(X, Y) | \sup_{x \in K} d(f(x), g(x)) < \epsilon\}.$$

Lemma 1.2.15 (Compact Convergence Topology)**.** *Let X be a topological space and Y be a metric space. Then*

$$\mathcal{B}_{c.c.} = \{B(f; K; \epsilon) | f \in \mathcal{M}(X, Y), K \subset X \text{ is compact}, \epsilon > 0\}$$

*is a topological basis of $\mathcal{M}(X, Y)$. The topology $\mathcal{T}_{c.c.}$, called the **compact convergence topology**, it generates satisfies:*

$$f_n \text{ converges to } f \text{ on every compact subset} \Longleftrightarrow f_n \text{ converges to } f \text{ in } \mathcal{T}_{c.c.}.$$

Back to our original problem. Let $f_n \in \mathcal{C}(X, Y)$ and $f_n \to f_0$ under $\mathcal{T}_{c.c.}$, then f_0 is continuous on every compact subset. But is f_0 continuous?

Example 1.2.16.

To remedy, we introduce the following concept

Definition 1.2.17 (Locally Compact Space)**.** If every point in X has a compact neighborhood, then X is called a **locally compact space**.

Proposition 1.2.18. *If X is a locally compact space, then $\mathcal{C}(X, Y)$ is closed in $(\mathcal{M}(X, Y), \mathcal{T}_{c.c.})$.*

In most applications, locally compact spaces are also Hausdorff. A locally compact Hausdorff space is often shorthanded by a **LCH** space.

Proposition 1.2.19. *Let X be a LCH space, K be a compact set, $U \supset K$ be an open set. Then there exists an open set V such that $\overline{V}$ is compact and*

$$K \subset V \subset \overline{V} \subset U.$$

Although locally compactness , it is not the weakest condition that we can come up with.

Definition 1.2.20 (Compactly Generated Space)**.**

When Y is a general topological space (not a metric space any more), we cannot define compact convergence toplogy on $\mathcal{M}(X, Y)$. However, using standard topological arguments, we can define

Definition 1.2.21 (Compact-Open Topology)**.**

Now we begin to study the compactness of

Definition 1.2.22 (Relatively Compact (Precompact))**.** $A \subset X$. If $\bar{A}$ is compact, we say that A is **precompact(relatively compact)**.

Definition 1.2.23. Let $\mathcal{F} \subset \mathcal{C}(X, Y)$ be a family of continuous maps. For $a \in X$, let $\mathcal{F}_a := \{f(a) | f \in \mathcal{F}\}$.

(i) $\mathcal{F}$ is **pointwise bounded** if for any $a \in X$ $\mathcal{F}_a$ is bounded in Y.

(ii) $\mathcal{F}$ is **pointwise precompact** if for any $a \in X$ $\mathcal{F}_a$ is precompact in Y.

Theorem 1.2.24 (Arzela-Ascoli)**.** *Let X be a compact space, (Y, d) be a metric space, and $\mathcal{F} \subset \mathcal{C}(X, Y)$ be equipped with $\mathcal{T}_{c.c.}$.*

(i) If $\mathcal{F}$ is equicontinuous and pointwise precompact, then $\mathcal{F}$ is precompact in $(\mathcal{C}(X, Y), \mathcal{T}_{c.c.})$.

(ii) If in addition X is a LCH, then the converse is also true.

Proof. □

Remark 1.2.25. Be careful that the conclusion stated compactness only, instead of sequential compactness. Thus in general we can not obtain the existence of a convergent subsequence.

However, when X *is compact*, then $\mathcal{T}_{c.c.} = \mathcal{T}_{u.c.}$. As $\mathcal{T}_{u.c.}$ is induced by a metric, compactness is equivalent to sequential compactness in this case.
When X is only a LCH, by the diagonal argument

Definition 1.2.26 (σ-Compact Space)**.** If a topological space X can be written as a countable union of compact subsets, then X is called **σ-compact**.

1.2.4 Application: Stone-Weierstrass Theorem

1.3 Countability Axioms

Definition 1.3.1. Let $(X, \mathcal{T})$ be a topological space

1. If each point of X has a countable neighbourhood basis, then X is called a **first-countable** space.

2. If X has a countable topological basis, then X is called a **second-countable** space.

Remark 1.3.2. second countability $\Longrightarrow$ first countability

benefit of first cou

Definition 1.3.3 (Separability)**.** Let $(X, \mathcal{T})$ be a topological space. If X contains a countable dense subset, then X is called a **separable** space.

Theorem 1.3.4 (Second Countability $\Longrightarrow$ Separability)**.**

Proof. Let $\{U_n\}$ be the countable topological basis. For each n we can choose $x_n \in U_n$. Let $A = \{x_n\}$. A is countable. For any $x \in X$ and an open neighborhood U of x, there exists n such that $x \in U_n \subset U$. Thus $U \cap A \neq \emptyset$ and $\overline{A} = X$ by Proposition 1.1.25. □

Proposition 1.3.5. *A metric space (X, d) is second countable if and only if it is separable.*

1.4 Separation Axioms

TODO:This section is too abstract please give some examples

Definition 1.4.1. Suppose $(X, \mathcal{T})$ is a topological space.

1. If for any $x \neq y$, there exists open sets U, V such that
$$x \in U \setminus V \text{ and } y \in V \setminus U,$$
then X is called a **Frechet(T1)** space.
2. If for any $x \neq y$, there exists open sets U, V such that
$$x \in U,\ y \in V, \text{ and } U \cap V = \emptyset,$$
then X is called a **Hausdorff(T2)** space.
3. If for any closed set A and $x \notin A$, there exists open sets U, V such that
$$A \subset U,\ x \in V, \text{ and } U \cap V = \emptyset,$$
then X is called a **regular(T3)** space.
4. If for any closed sets $A \cap B = \emptyset$, there exists open sets U, V such that
$$A \subset U,\ B \subset V, \text{ and } U \cap V = \emptyset,$$
then X is called a **normal(T4)** space.

T1+T4 -¿ T3, T1+T3 -¿T2

1.4.1 Urysohn Metrization

1.5 Connectedness

Definition 1.5.1. Let $(X, \mathcal{T})$ be a topological space. If there exists nonempty subsets $A, B \subset X$ such that
$$X = A \cup B \text{ and } A \cap \overline{B} = \overline{A} \cap B = \emptyset,$$
then X is *not* connected.

Proposition 1.5.2. *TFAE:*

1. *X is not connected.*
2. *There exists disjoint open sets $A, B \subset X$ such that $X = A \cup B$.*
3. *There exists disjoint closed sets $A, B \subset X$ such that $X = A \cup B$.*
4. *There exists $A \neq \emptyset, A \neq X$ such that A is both open and closed in X.*
5. *There exists a continuous surjection $f : X \to \{0, 1\}$.*

Chapter 2

Banach Spaces I

We can regard linear functional analysis as an extension of linear algebra to infinite-dimensional cases, with $\mathbb{F} = \mathbb{R}$, or $\mathbb{F} = \mathbb{C}$. In classical linear algebra we utilize the powerful concept of dimension to derive the structure of linear spaces and linear transformations. Here to cope with infinity, we must resort to techniques from mathematical analysis.

2.1 Basic Definitions and Examples

Definition 2.1.1 (Seminorm)**.** If V is a vector space over $\mathbb{F}$, a seminorm is a function $p : V \to [0, \infty)$ with:
(i) $p(x + y) \le p(x) + p(y)$, $\forall x, y \in V$
(ii) $p(\alpha x) = |\alpha|\, p(x)$, $\forall \alpha \in \mathbb{F}$, $\forall x \in V$.

Note that (ii) implies $p(0) = 0$.

Definition 2.1.2 (Norm)**.** A norm is a seminorm p s.t. $x = 0$ if $p(x) = 0$. Usually a norm is denoted by $\|\cdot\|$. A normed space is a vector space endowed with with a norm. A Banach space is a normed space that is complete w.r.t. the metric defined by the norm.

Lemma 2.1.3 (Continuity)**.** *If V is a normed space, then:*
(i) $V \times V \to V$, $(x, y) \mapsto x + y$ is continuous.
(ii) $\mathbb{F} \times V \to V$, $(\alpha, x) \mapsto \alpha x$ is continuous.

Proof. By the definiton of continuity and the triangle inequality of norm. □

Lemma 2.1.4. *If p and q are seminorms on V, TFAE:*
(i) $p \le q$
(ii) $p < 1$ whenever $q < 1$
(iii) $p \le 1$ whenever $q \le 1$
(iv) $p \le 1$ whenever $q < 1$.

Proof. Only need to show (iv) implies (i).
Suppose (iv). Fix any x, and set $\alpha = q(x)$. Then for any $\epsilon > 0$, $q(\frac{1}{\alpha+\epsilon}x) < 1$. So $p(\frac{1}{\alpha+\epsilon}x) \le 1$ and $p(x) \le q(x) + \epsilon$. The arbitariness of ϵ implies $p \le q$. □

Definition 2.1.5 (Equivalent Norms)**.** If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on V, they are said to be equivalent norms if they define the same topology on V.

Lemma 2.1.6.

2.2 Linear Operators on Normed Spaces

$B(V, W)$ = all continuous linear transformations from V to W.

Lemma 2.2.1. *If V and W are normed spaces and $T : V \to W$ is a linear transformation, TFAE:*
(i) $T \in B(V, W)$.
(ii) T is continuous at 0.
(iii) T is continuous at some point.
(iv) $\exists c > 0$ s.t. $\forall x \in V$ $\|Tx\| \le c\|x\|$.

2.3 Finite Dimensional Normed Spaces

Theorem 2.3.1. *If V is a finite dimensional vector space, then any two norms on V are equivalent.*

Proof. Let $\{e_1, \cdots, e_d\}$ be a Hamel basis for V. For $x = \sum_{j=1}^{d} x_j e_j$, define $\|x\|_\infty := \max\{|x_j| : 1 \leq j \leq d\}$. Then $\|x\|_\infty$ is a norm. Let $\|\cdot\|$ be any norm on V.If $x = \sum_{j=1}^{d} x_j e_j$, then $\|x\| \leq C\|x\|_\infty$, where $C = \sum_{j=1}^{d} \|e_j\|$. To show the other inequality, we need a technique from analysis (or topology).
??? $\square$

Corollary 2.3.2. *A finite dimensional linear manifold M in a normed space V is closed.*

Proof. Choose a Hamel basis for M and define a norm $\|\cdot\|_\infty$ as above. Then M is complete w.r.t $\|\cdot\|_\infty$, thus complete w.r.t the original norm by the above theorem. Hence it is closed. $\square$

Corollary 2.3.3. *A linear transformation T from a finite dimensional normed space V to any normed space W is continuous.*

Proof. Since all norms are equivalent on V, we may choose a Hamel basis and define $\|\cdot\|_\infty$ as above. Thus for $x = \sum_{j=1}^{d} x_j e_j$, $\|Tx\| = \|\sum_j x_j Te_j\| \leq \sum_j |\xi_j| \|Te_j\| \leq C\|x\|_\infty$, where $C = \sum_j \|Te_j\|$. Hence T is bounded and continuous. $\square$

2.4 Quotients and Products of Normed Spaces

Let V be a normed space, let M be a linear manifold in V, and let $Q : V \to V/M$ be the natural map $Qx = x + M$. Our goal is to make V/M into a normed space, so define

$$\|x + M\| = \inf\{\|x + y\| : y \in M\} = \operatorname{dist}(x, M).$$

This defines a seminorm on V/M, but if M is not closed, it can not define a norm.

Theorem 2.4.1. *If $M \leq V$ and $\|\cdot\|$ is defined above, then it is a norm on V/M. Also:*
(i)

2.5 The Hahn-Banach Theorem

We first state the Hahn-Banach Theorem for real spaces and then extend it to complex spaces. Thereafter, we discuss several corollary of it. We postpone the proof to the end of this section.

Definition 2.5.1 (Sublinear Functional)**.** If V is a vector space over $\mathbb{R}$, a seminorm is a function $q : V \to \mathbb{R}$ with:
(i) $q(x + y) \leq q(x) + q(y)$, $\forall x, y \in V$
(ii) $q(\alpha x) = \alpha q(x)$, $\forall \alpha \mathcal{E} 0$, $\forall x \in V$.

Theorem 2.5.2 (Hahn-Banach)**.** *Let V be a vector space over $\mathbb{R}$ and let q be a sublinear functional on V. If M is a linear manifold in V and $f : M \to \mathbb{R}$ is a linear functional s.t. $\forall x \in M$, $f(x) \leq q(x)$, then there is a linear functional $F : V \to \mathbb{R}$ s.t. $F|M = f$ and $\forall x \in V$, $F(x) \leq q(x)$.*

Note that the essence of the theorem is not the extension exists but that an extension can be found that remains dominated by q. Just to find an extension, we can simply take a Hamel basis.

Lemma 2.5.3 (Complexification)**.** *Let V be a linear space over $\mathbb{C}$.*
(i) If $f : V \to \mathbb{R}$ is an $\mathbb{R}$-linear functional, then $\tilde{f}(x) = f(x) - if(ix)$ is a $\mathbb{C}$-linear functional and $f = \Re\tilde{f}$.
(ii) If $g : V \to \mathbb{C}$ is $\mathbb{C}$-linear, $f = \Re g$, and $\tilde{f}$ is defined as in (i), then $\tilde{f} = g$.
(iii) If p is a seminorm on V, and f and $\tilde{f}$ are as in (i), then $\forall x$ $|f(x)| \leq p(x)$ if and only if $\forall x$ $|\tilde{f}(x)| \leq p(x)$.
(iv) If V is furthermore a normed space, and f and $\tilde{f}$ are as in (i), then $\|f\| = \|\tilde{f}\|$.

Theorem 2.5.4 (Seperation of Convex Sets)**.**

2.6 The Opening Mapping Theorem

Theorem 2.6.1 (Open Mapping Theorem)**.** *Let X be a Banach space and Y be a normed linear space. If $T : X \to Y$ is a bounded linear operator and* $\operatorname{im} T$ *is a Baire second category set in Y, then T is a surjective open mapping.*

Corollary 2.6.2. *If X and Y are Banach spaces and $T : X \to Y$ is a bounded linear surjection, then T is an open mapping.*

Theorem 2.6.3 (Inverse Mapping Theorem)**.** *If X is a Banach space, Y is a normed linear space, $T : X \to Y$ is a bounded linear operator that is bijective, and* $\operatorname{im} T$ *is a Baire second category set in Y then Y is in fact a Banach space, and T^{-1} is bounded linear operator defined everywhere on Y.*

Proof. By Theorem 2.6.1, T is surjective, and hence T^{-1} is defined everywhere on Y. Further, T is an oepn mapping, so T^{-1} is bounded. Therefore Y is also a Banach space. □

Definition 2.6.4 (isomorphism between Banach spaces)**.** If V and W are Banach spaces, an isomorphism of V and W is a linear bijection that is a homeomorphism. Say V and W are isomorphic if there is an isomorphism of V onto W.

Remark 2.6.5. The use of the word 'isomorphism' is counter to the spirit of category theory, but it is traditional in Banach space theory. The inverse mapping theorem just says that a continuous bijection is an isomorphism.

Theorem 2.6.6 (Closed Graph Theorem)**.** *If V and W are Banach spaces and $T : V \to W$ is a linear transformation s.t. the graph of T,*

$$G = \{x \oplus Tx \in V \oplus_1 W : x \in V\}$$

is closed, then T is continuous.

Proof. Since $V \oplus_1 W$ is a Banach space and G is closed, G is a Banach space. Define $P_1 : G \to V$ by $P_1(x \oplus Tx) = x$ and define $P_2 : G \to W$ by $P_2(x \oplus Tx) = Tx$. P_1 and P_2 are bounded. Moreover, P_1 is bijective. By the inverse mapping theorem, P_1^{-1} is continuous. Thus $T = P_2 \circ P_1^{-1}$ is continuous. □

2.7 The Principle of Uniform Boundness

The principle of uniformly boundedness asserts that every pointwise bounded family of bounded linear operators on a Banach space is uniformly bounded.

Let $\{T_\lambda : \lambda \in \Lambda\} \subset B(X, Y)$ be a family of bounded linear operators from X to Y. It is **pointwise bounded** on $E \subset X$ if $\forall x \in E$, $\sup_{\lambda\in\Lambda} \|T_\lambda x\| < \infty$. It is **uniformly bounded** if $\sup_{\lambda\in\Lambda} \|T_\lambda\| < \infty$.

Theorem 2.7.1 (Principle of Uniform Boundedness)**.** *Let X, Y be normed linear spaces, $\{T_\lambda : \lambda \in \Lambda\} \subset B(X, Y)$ be a family of bounded linear operators. If $\{T_\lambda : \lambda \in \Lambda\}$ is pointwise bounded on a Baire second category set, then it is uniformly bouneded.*

Corollary 2.7.2. *Let X be a Banach space and Y a normed space, $\{T_\lambda : \lambda \in \Lambda\} \subset B(X, Y)$ be a family of bounded linear operators. If $\{T_\lambda : \lambda \in \Lambda\}$ is pointwise bounded, then it is uniformly bouneded.*

The proof of Theorem 2.7.1 relies on the following lemmas.

Lemma 2.7.3 (Osgood)**.** *Let X be a normed linear space, $\{f_\lambda : \lambda \in \Lambda\}$ be a family of lower semicontinuous functions on X. If $\{f_\lambda : \lambda \in \Lambda\}$ is pointwise bounded on a Baire second category set $E \subset X$, then there exists $M > 0$ and a nonempty open set U such that*

$$f_\lambda(x) \le M, \quad \forall \lambda \in \Lambda, \quad x \in U.$$

Proof. Let

$$E_{\lambda,n} = \{x \in X : f_\lambda(x) \le n\}, \quad E_n = \bigcap_{\lambda\in\Lambda} E_{\lambda n}.$$

By lower semicontinuity, each $E_{\lambda,n}$ is closed and hence E_n is closed. As $E \subset \bigcup_{n=1}^{\infty} E_n$ and E is a Baire second category set, $\bigcup_{n=1}^{\infty} E_n$ is also a Baire second category set. Therefore there exists n_0 such that $U = \overset{\circ}{E}_{n_0} \neq \emptyset$. So,

$$f_\lambda(x) \leq n_0, \quad \forall \lambda \in \Lambda, \quad x \in U.$$

□

Lemma 2.7.4. *Let X, Y be normed linear spaces, $\{T_\lambda : \lambda \in \Lambda\} \subset B(X,Y)$ be a family of bounded linear operators, and U be a nonempty open set. If there exists $a > 0$ such that $\sup_{\lambda\in\Lambda} \|T_\lambda x\| \leq a$, then there exists $M > 0$ such that $\sup_{\lambda\in\Lambda} \|T_\lambda\| < M$.*

Proof. We can assume $0 \in U$, without any loss of generality. Now [**kz:** notation for a ball] □

Proof of Theorem 2.7.1. Notice that $f_\lambda = \|T_\lambda x\|$ is a continous function of x. Combining the above lemmas yield the desired result. □

Corollary 2.7.5 (Banach-Steinhaus)**.**

2.8 weak and weak*

weaker topologies, easier to converge

Definition 2.8.1 (weakly bounded)**.** $E \subset X$ is a weakly bounded set if for any $f \in X^*$, there exists $M_f > 0$ such that

$$|f(x)| \leq M_f, \quad \forall x \in E.$$

Theorem 2.8.2. *weak bounded if and only if bounded (in norm)*

Theorem 2.8.3. *$x_n \xrightarrow{w} x$ if and only if*

(i) $\{\|x_n\|\}$ is bounded;

(ii) $\exists G \subset X^*$, $\operatorname{span} G$ is dense in X^* and $\forall f \in G$,

$$f(x_n) \to f(x).$$

weakly sequential compact

Theorem 2.8.4. *Every convex subset E of a Banach space X that is closed (in the strong topology) is closed in the weak topology.*

2.9 The Adjoint of a Linear Operator

2.10 The Banach-Stone Theorem

2.11 Compact Operators

Definition 2.11.1. If V and W are Banach spaces and $T : V \to W$ is a linear transformation, then T is compact if ??? is compact in W.

Chapter 3

Hilbert Spaces

3.1 Elementary Properties and Examples

3.2 The Riesz Representation Theorem

3.3 Orthogonality

3.4 Isomorphisms

3.5 Direct Sum

3.6 Operators

3.7 The Adjoint of an Operator

3.8 Projections and Idempotents

3.9 Compact Operators

3.10 Discussion

This chapter follows Conway [2019]

Chapter 4

Sobolev Spaces

We establish a proper setting in which to apply ideas of functional analysis to glean information concerning PDE

4.1 Hölder Spaces

Assume $U \subset \mathbb{R}^n$ is open and $0 < \gamma \leq 1$.

Definition 4.1.1 (Hölder continuous)**.** A function is said to be Hölder continuous with exponent $\gamma \in (0, 1]$ if $\forall x, y \in U$

$$|u(x) - u(y)| \leq C|x - y|^\gamma \tag{4.1}$$

for some constant C.

Definition 4.1.2. (i) If $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\bar{U})} := \sup_{x \in U} |u(x)|$$

(ii) The γth-Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{x,y \in U, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}$$

and the γth-Hölder norm is

$$\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$$

Definition 4.1.3. The Hölder space

$$C^{k,\gamma}(\bar{U})$$

consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [u]_{C^{0,\gamma}(\bar{U})}$$

is finite.

Theorem 4.1.4 (Hölder spaces as function spaces)**.** *$C^{k,\gamma}(\bar{U})$ is a Banach space.*

Proof. The construction of $\| \cdot \|_{C^{k,\gamma}(\bar{U})}$ ensures that it is a norm. In addition, each Cauchy sequence converges. □

4.2 Sobolev Spaces

The Hölder spaces are unfortunately not often suitable settings for elementary PDE theory, as we usually cannot make good enough analytic estimates to demonstrate that the solutions we construct actually belong to such spaces. What are needed rather are some kind of spaces containing less smooth functions.

4.2.1 Weak Derivatives

We start off by substantially weakening the notion of partial derivatives.

Notation 4.2.1. Let $C_c^\infty(U)$ denote the space of infinitely differentiable function $\varphi : U \to \mathbb{R}$, with compact support in U. We will sometimes call a function belonging to $C_c^\infty(U)$ a test function.

Definition 4.2.2. Suppose

Lemma 4.2.3 (uniqueness of weak derivatives)**.** *A weak αth-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Example 4.2.4. Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x & x \in (0,1] \\ 1 & x \in [1,2) \end{cases}$$

has weak derivative

$$v(x) = \begin{cases} x & x \in (0,1] \\ 1 & x \in [1,2) \end{cases}.$$

But

$$u(x) = \begin{cases} x & x \in (0,1] \\ 2 & x \in (1,2) \end{cases}$$

does not have a weak derivative.

4.2.2 Definition of Sobolev Spaces

Fix $1 \le p \le \infty$ and let k be a nonnegative integer.

Definition 4.2.5. The Sobolev space

$$W^{k,p}(U)$$

consists of all locally summable function $u : U \to \mathbb{R}$ s.t. for each multiindex α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Notation 4.2.6. If $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U)$$

H is used since $H^k(U)$ is a Hilbert space.

Definition 4.2.7. If $u \in W^{k,p}(U)$, we define the norm to be

$$\begin{cases} (\sum_{|\alpha|\le k} \int_U |D^\alpha u|^p \mathrm{d}x)^{\frac{1}{p}} & p \in [1,\infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_U |D^\alpha u| & p = \infty \end{cases} \tag{4.2}$$

Notation 4.2.8.

Notation 4.2.9. We denote by

$$W_0^{k,p}(U)$$

the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

We interpret $W_0^{k,p}(U)$ as comprising those functions $u \in W^{k,p}(U)$ s.t.

$$D^\alpha = 0 \text{ on } \partial U \quad \forall |\alpha| \le k - 1.$$

Example 4.2.10. If $n = 1$ and U is an open interval in $\mathbb{R}^1$

4.2.3 Properties

Theorem 4.2.11 (weak derivatives)**.** *Assume* $u, v \in W^{k,p}(U)$, $|\alpha| \leq k$. *Then*
(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$, *and* $D^\alpha(D^\beta u) = D^{\alpha+\beta} u \ \forall |\alpha| + |\beta| \leq k$.
(ii) For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ *and* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \leq k$.
(iii) If V *is an open subset of* U, *then* $u \in W^{k,p}(V)$.
(iv) If $\zeta \in C_c^\infty(U)$, *then* $\zeta u \in W^{k,p}(U)$ *and*

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} C_\alpha^\beta D^\beta \zeta D^{\alpha-\beta} u$$

Theorem 4.2.12 (Sobolev space is Banach)**.** *For each* $k \in \mathbb{Z}_{\mathcal{E}1}$ *and* $1 \leq p \leq \infty$, *the Sobolev space* $W^{k,p}(U)$ *is a Banach space.*

Proof. Note that the completeness is encoded in the definition. If $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence, then for $|\alpha| \leq k$, $\{D^\alpha u_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(U)$ and $L^p(U)$ is complete. □

4.3 Approximation

We need to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions. The method of mollifiers provides such a tool.

Notation 4.3.1. If $U \subset \mathbb{R}^n$ is open and $\epsilon > 0$, we write

$$U_\epsilon := \{x \in U | \mathrm{dist}(x, \partial U) > \epsilon\}$$

Definition 4.3.2. Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x|\mathcal{E}1 \end{cases}$$

C is selected s.t. $\int \eta = 1$. We call η the standard mollifier.
For each $\epsilon > 0$, set

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$$

Definition 4.3.3. If $U \to \mathbb{R}$ is locally integrable, define its mollification

$$f^\epsilon = \eta_\epsilon * f \quad \mathrm{in} U_\epsilon$$

Theorem 4.3.4 (properties of mollifiers)**.** *(i)* $f^\epsilon \in C^\infty(U_\epsilon)$
(ii) $f^\epsilon \to f$ *a.e. as* $\epsilon \to 0$
(iii) If $f \in C(U)$, *then* $f^\epsilon \to f$ *uniformly on compact subsets of* U.
(iv) If $1 \leq p < \infty$ *and* $f \in L^p_{loc}(U)$, *then* $f^\epsilon \to f$ *in* $L^p_{loc}(U)$.

4.4 Extensions

Our goal is to extend functions in the Sobolev space $W^{1,p}(U)$ to become functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$. We must invent a way to extend u which preserves the weak derivatives accross ∂U.

Theorem 4.4.1 (extension theorem)**.** *Assume* U *is bounded and* ∂U *is* C^1. *Select a bounded open set* V *s.t.* $U \subset\subset V$. *Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

s.t. for each $u \in W^{1,p}(U)$*:*
(i) $Eu = u$ *a.e. in* U
(ii) Eu *has support within* V
(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$, *where the constant* C *depends on* p, U, V.

4.5 Traces

Theorem 4.5.1 (trace theorem)**.** *Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \to L^p(\partial U)$$

s.t.
(i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$
(ii) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$

Definition 4.5.2. We will call Tu the trace of u on ∂U.

4.6 Compactness

4.7 Sobolev Inequalities

Our goal in this section is to discover embeddings of various Sobolev spaces into others. The crucical analytic tools here will be so-called Sobolev-type Inequalities, which we prove below for smooth functions. These will then establish the estimates for arbitary functions in the various relevant Sobolev spaces, since smooth functions are dense.

We will consider first only the Sobolev space $W^{1,p}(U)$ and ask the following

4.7.1 $1 \leq p < n$

Let us first ask whether we can establish an estimate of the form

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

for certain constants $C > 0$, $1 \leq p^* < \infty$ and all functions $u \in C_c^\infty(\mathbb{R}^n)$. We should expect that differentiation of a function increases the strength of its local singularities and improves its decay at infinity. Thus, if $Du \in L^p(\mathbb{R}^n)$, it is reasonable to expect that $f \in L^{p^*}(\mathbb{R}^n)$ for some $p^* > p$. By a simple dimensional analysis, we can obtain the only possible p^*:

Definition 4.7.1. If $1 \leq p < n$, the Sobolev conjugate of p is

$$p^* = \frac{np}{n-p}$$

Theorem 4.7.2 (Gagliardo-Nirenberg-Sobolev inequality)**.**

Theorem 4.7.3 (estimates for $W^{1,p}$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}$$

the constant C depending only on p, n, and U.

The Gagliardo-Nirenberg-Sobolev inequality implies the embedding of $W^{1,p}(U)$ into $L^{p^*}(U)$ for $1 \leq p < n$. We now demonstrate that $W^{1,p}(U)$ is in fact **compactly** embedded in $L^q(U)$ for $1 \leq q < p^*$. This compactness will be fundamental for our applications of functional analysis to PDE.

Definition 4.7.4. Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y, written

$$X \subset\subset Y$$

provided
(i) $\|u\|_Y \leq C\|u\|_X$ $(u \in X)$ for some constant C and
(ii) each bounded sequence in X is precompact in Y.

Theorem 4.7.5 (Rellich-Kondrachov Compactness Theorem)**.**

4.7.2 $n < p \leq \infty$

In the previous section, we saw that if the weak derivative of a function that vanishes at infinity belongs to $L^p(\mathbb{R}^n)$ with $p < n$, then the function has improved integrability properties and belongs to $L^{p^*}(\mathbb{R}^n)$. In this regime the Sobolev conjugate of p is negative. Therefore we should expect Even though the function is weakly differentiable, it need not be continuous. In this section, we show that if the derivative belongs to $L^p(\mathbb{R}^n)$ with $p > n$ then the function (or a pointwise a.e. equivalent version of it) is continuous, and in fact Hölder continuous.

Theorem 4.7.6 (Morrey's inequality)**.**

4.7.3 $p = n$

Owing to the Gagliardo-Nirenberg-Sobolev inequality, and the fact that $p^* \to \infty$ as $p \to n$, we might expect $u \in L^\infty(U)$ provided $u \in W^{1,n}(U)$. This is however false if $n > 1$.

Example 4.7.7. If $U = B^0(0,1)$, the function $u = \log\log(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(U)$ but not to $L^\infty(U)$.

4.7.4 General Sobolev inequalities

Now we deal with $W^{k,p}(U)$ with general k.

4.8 Poincare's inequalities

Notation 4.8.1. $(u)_U =$ is the average of u over U.

4.9 Fourier Transform

We employ the Fourier transform to give an alternate characterization of the spaces $H^k(\mathbb{R}^n)$. For this section all functions are complex-valued.

Theorem 4.9.1 (characterization of H^k via Fourier transform)**.** *Let k be a nonnegative integer.* *(i) A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n)$$

(ii) In addition, there exists a positive constant C s.t.

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}$$

4.10 The Space H^{-1}

It it important to have an explicit characterization of the dual space of H_0^1

4.11 Discussion

Evans [2010]

Bibliography

John B Conway. *A Course in Functional Analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer Nature, Netherlands, 2nd ed. 1990. corr. 4th printing 1994. edition, 2019. ISBN 1475743831. 17

Lawrence C. Evans. *Partial differential equations*. Graduate studies in mathematics, v. 19. American Mathematical Society, Providence, R.I, 2nd ed. edition, 2010. ISBN 9780821849743. 23