Measure Theory

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Chapter 1

Real Analysis

1.1

Continuity of Translation

Theorem 1.1.1. *Suppose $f \in L^1(\mathbb{R}^n)$. Then*

$$\|f_h - f\|_{L^1} \to 0 \quad h \to 0.$$

Proof. The result is clearly true The proof is a simple consequence of the approximation of integrable functions by continuous functions of compact support. □

1.2 Differentiation and Integration

For conceptual simplicity, we study $\mathbb{R}$ instead of $\mathbb{R}^n$ in this section. Let us first recall what we learned in elementary calculus.

Theorem 1.2.1. *Let f be a continuous function on $[a, b]$, and F be the function defined, for all x in $[a, b]$, by*

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Then F is uniformly continuous on $[a, b]$ and differentiable on (a, b), and

$$F'(x) = f(x).$$

Theorem 1.2.2 (Newton-Leibniz)**.** *Let f be a Riemann integrable function on $[a, b]$, and F a continuous function on $[a, b]$ which is an antiderivative of f in (a, b):*

$$F'(x) = f(x).$$

Then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Definition 1.2.3 (Total Variation)**.** The total variation of a function f defined on $[a, b]$ is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} |f(x_{i+1}) - f(x_i)|,$$

where

$$\mathcal{P} = \{P = \{x_0, \cdots, x_{n_P}\} | P \text{ is a partition of } [a, b] \text{ satisfying } x_i \le x_{i+1} \text{ for } 0 \le i \le n_P - 1\}.$$

If $V_a^b(f) < +\infty$, then f is said to be of bounded variation on $[a, b]$.

Example 1.2.4. The continuous function

$$f(x) = \begin{cases} 0, & x = 0 \\ x\sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

is not of bounded variation on $[0, 2/\pi]$. Just consider the partition

$$P = \{0, \frac{2}{2n-1}, \frac{2}{2n-3}, \cdots, \frac{2}{3}, 1\}.$$

Theorem 1.2.5 (Jordan Decomposition Theorem)**.**

$$f(x) = g(x) - h(x)$$

where $g(x)$ and $h(x)$

Bounded variable functions says that it has a length

Bounded variable functions are a.e. differentiable But it does not satisfies the desirable property: Length is equal to the integral of derivative, which is also known as the **fundamental theorem of calculus**. It turns out that absolute continuous functions is exactly the set of functions that satisfies this property.

Example 1.2.6.

Definition 1.2.7 (Absolute Continuity)**.** A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if for every positive number ϵ, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) satisfies

$$\sum_{k=1}^{N} (y_k - x_k) < \delta$$

then

$$\sum_{k=1}^{N} |f(y_k) - f(x_k)| < \epsilon.$$

Absolutely continuous functions are of bounded variation.

Proposition 1.2.8. *If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a, b]$.*

Theorem 1.2.9. *If f is absolute continuous function on $[a, b]$, then*

$$f(x) - f(a) = \int_a^x f'(t)\mathrm{d}t, \quad x \in [a, b].$$

It is beneficial to compare it with the “classical” version (Theorem 1.2.2).

Chapter 2

Abstract Measure Theory

2.1

In this section, we are working on a set E.

Definition 2.1.1 (σ-algebra)**.**

Remark 2.1.2. Greek letters σ and δ are often used when countable unions and countable intersections are involved. For example, topologists call F_σ every countable union of closed sets in a topological space (F standing possibly for the French word fermé, closed) and G_δ every countable intersection of open sets (G standing for the German word Gebiet, domain, connected open set). The letters σ and δ are often given as Greek abbreviations of German words: σ as S in Summe for sum (in the sense of sum of sets, that is, union) and δ as D in Durchschnitt for intersection, both countable. Thus, in the context of measure theory, the letter σ refers to the stability of a collection of subsets by countable union.

Definition 2.1.3 (generator of σ-algebra)**.**

$$\sigma(\mathcal{A}) = \{A \subset E : A \in \mathcal{E} \quad \forall \mathcal{E} \supset\}$$

Example 2.1.4. Borel σ-algebra

Definition 2.1.5 (π-system)**.** $\mathcal{A}$ is a collection of subsets of E. Then $\mathcal{A}$ is called a π-system if

1. $\emptyset \in \mathcal{A}$;
2. $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Definition 2.1.6 (d-system)**.** $\mathcal{A}$ is a collection of subsets

1. $E \in \mathcal{A}$;
2. $A, B \in \mathcal{A}$, $A \subset B$, then $B \setminus A \in \mathcal{A}$;
3. If $A_n \subset \mathcal{A}$ such that $A_n \subset A_{n+1}$, then $\cup_n A_n \in \mathcal{A}$.

Remark 2.1.7. d-system is also referred as λ-system.

Proposition 2.1.8. *$\mathcal{A}$ is a σ-algebra if and only if it is a π-system and a d-system.*

Proof. A σ-algebra is a π system because

Conversely, □

Lemma 2.1.9 (Dynkin's π-system lemma)**.** *Let $\mathcal{A}$ be a π-system. Then any d-system that contains $\mathcal{A}$ also contains $\sigma(\mathcal{A})$.*

This lemma can reduce the problem of studying a σ-algebra to the study of a π-system.

Proof. Let $\mathcal{D}$ be the intersection of all d-system containing $\mathcal{A}$. We now prove that $\mathcal{D}$ is a σ-algebra. As $\mathcal{D}$ is already a d-system, by Proposition 2.1.8, we only need to prove that $\mathcal{D}$ is a π-system.

(i) First we show the following property for $\mathcal{D}$: If $B \in \mathcal{D}$ and $A \in \mathcal{A}$, then $B \cap A \subset \mathcal{D}$. Let

$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{A}\}.$$

To show that $\mathcal{D}' = \mathcal{D}$, we only need to check that $\mathcal{D}'$ is a d-system containing $\mathcal{A}$. As $\mathcal{A}$ is a π-system, $D' \supset \mathcal{A}$. Now, we can check

- $E \in \mathcal{D}'$; this is because $\mathcal{A} \subset \mathcal{D}$.
- $B_1, B_2 \in \mathcal{D}'$, $B_1 \subset B_2$, then $B_2 \setminus B_1 \in \mathcal{D}'$; this is because for any $A \in \mathcal{A}$, $(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus (B_1 \cap A) \in \mathcal{D}$.
- If $B_n \subset \mathcal{D}'$ such that $B_n \subset B_{n+1}$, then $\cup_n B_n \in \mathcal{D}'$; this is because for any $A \in \mathcal{A}$, $(\cup_n B_n) \cap A = \cup_n (B_n \cap A) \in \mathcal{D}$.

(ii) If $A, B \in \mathcal{D}$, then $B \cap A \in \mathcal{D}$. Let

$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \quad \forall A \in \mathcal{D}\}.$$

One can see that $\mathcal{D}''$ is a d-system by noting that the argument holds for any $\mathcal{A} \subset \mathcal{D}$. The fact that $\mathcal{D}''$ contains $\mathcal{A}$ follows from property (i). Therefore, $\mathcal{D}'' = \mathcal{D}$ and $\mathcal{D}$ is a π-system. □

2.2 Caratheodory Extension Theorem

Definition 2.2.1 (set function)**.** $\mathcal{A}$ be a collection of subsets of E with $\emptyset \in \mathcal{A}$. A set function $\mu : \mathcal{A} \to [0, \infty]$ is a function such that $\mu(\emptyset) = 0$.

Definition 2.2.2 (increasing set function)**.** $A \subset B$, we have $\mu(A) \leq \mu(B)$.

Definition 2.2.3 (Additive Set Function)**.** A set function is additve if whenever $A, B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}, A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Definition 2.2.4 (Countably Additive Set Function)**.** A set function is countably additive if whenever $(A_n) \subset \mathcal{A}$ such that $A_n \cap A_m = \emptyset$ for all $n \neq m$ and $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$, then $\mu(\cup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$.

Definition 2.2.5 (Countably subadditive Set Function)**.** A set function is countably subadditive if whenever $(A_n) \subset \mathcal{A}$ and $\cup_{n \in \mathbb{N}} A_n \in \mathcal{A}$, then $\mu(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$.

Definition 2.2.6 (ring)**.** A collection of subsets $\mathcal{A}$ of a subset E is a ring if

1. $\emptyset \in \mathcal{A}$
2. $\forall A, B \in \mathcal{A}$, we have $B \setminus A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

Definition 2.2.7 (algebra)**.** A collection of subsets $\mathcal{A}$ of a subset E is an algebra if

1. $\emptyset \in \mathcal{A}$
2. $\forall A, B \in \mathcal{A}$, we have $A^c \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

Theorem 2.2.8 (Caratheodory Extension Theorem)**.** *Let $\mathcal{A}$ be a ring on E, and μ be a countably additive set function on $\mathcal{A}$. Then μ extends to a measure on $\sigma(\mathcal{A})$.*

Theorem 2.2.9. *Suppose that μ_1, μ_2 are measures on $(E, \mathcal{E})$ with $\mu_1(E) = \mu_2(E) < \infty$. If $\mathcal{A}$ is a π-system with $\sigma(\mathcal{A}) = \mathcal{E}$ and $\mu_1|_{\mathcal{A}} = \mu_2|_{\mathcal{A}}$, then $\mu_1 = \mu_2$.*

Proof. Let $\mathcal{D} = \{A \in \mathcal{E} : \mu_1(A) = \mu_2(A)\}$, then $\mathcal{D} \supset \mathcal{A}$. By Dynkin's lemma, it suffices to show that $\mathcal{D}$ is a d-system. We can check

- $E \in \mathcal{D}$.
- Let $A, B \in \mathcal{D}$ with $A \subset B$. (Note that finiteness is important here.)
-

□

Example 2.2.10. Let $E = \mathbb{Z}$, $\mathcal{E} =$. let $\mathcal{A} = \{\{x, x+2, x+2, \dots\} : x \in E\} \cup \{\emptyset\}$. $\mathcal{A}$ is a π-system such that $\mathcal{E} = \sigma(\mathcal{A})$. Let $\mu_1(A)$ be the number of elements of A and $\mu_2 = 2\mu_1$.

Definition 2.2.11 (Borel σ-algebra)**.** Let E be a topological space. $\mathcal{B}(E) = \sigma(\{U \subset E : U \text{ open}\})$.

Definition 2.2.12 (borel and Radon Measure)**.** A measure μ on $(E, \mathcal{B}(E))$ is called a Borel measure. If $\mu(K) < \infty$ for all $K \subset E$ compact, then μ is called a Radon measure.

2.3 Lebesgue Measure

Theorem 2.3.1. *There exists a unique Borel measure μ on $\mathcal{B}(\mathbb{R})$ with $\mu([a,b]) = b - a$.*

Remark 2.3.2. This measure is called the Lebesgue measure.

Proof. **Uniqueness.** Suppose $\tilde{\mu}$ is another measure. For any $n \in \mathbb{Z}$, we set $\mu_n(A) = \mu(A \cap (n+1, n])$. The previous theorem imples that $\mu_n = \tilde{\mu}_n$ for all $n \in \mathbb{Z}$. Firx $A \in \mathcal{B}(\mathbb{R})$, $A = \cup_{n \in \mathbb{Z}}(A \cap (n, n+1])$, $\mu(A) = \sum_{n \in \mathbb{Z}} \mu_n(A) = \sum_{n \in \mathbb{Z}} \tilde{\mu}_n(A) = \tilde{\mu}(A)$.

Existence. $\mathcal{A}$ be the collection of finite, disjoint unions of the following form $A = \cup_{n \in \mathbb{Z}}(a_i, b_i]$. $\mathcal{A}$ is a ring on $\mathbb{R}$ and $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. Define $\mu(A) = \sum_{i=1}^{n}(b_i - a_i)$. It is well defined and additive. Suppose $A = \cup_{n=1}^{\infty} A_n \in \mathcal{A}$, $\mu(A) = \sum_{i=1}^{n} \mu(A_i) + \mu(A \setminus \sum_{j=1}^{n} A_j)$. Let $B_n = A \setminus \sum_{j=1}^{n} A_j$.

Argue by contradiction,$\exists \epsilon > 0, n_0 \in \mathbb{N}$ such that $\mu(B_n) \geq 2\epsilon \ \forall n \geq n_0$. $\forall n \in \mathbb{N}$, $\exists C_n \in \mathcal{A}$ such that $\overline{C_n} \subset B_n$ and $\mu(B_n \setminus C_n) \leq \frac{\epsilon}{2^n}$. Then we have $\mu(B_n) - \mu(\cap_{m=1}^{n} \overline{C_m}) = \mu(B_n \setminus \cap_{m=1}^{n} C_m) \leq \mu(\cup_{m=1}^{n}(B_m \setminus C_m)) \leq \sum_{m=1}^{n} \mu(B_m \setminus C_m) \leq \epsilon$. The finite intersection property □

Lemma 2.3.3 (Finite intersection property of compact sets of $\mathbb{R}$)**.** *If*

Definition 2.3.4 (σ-finite measure)**.** Let $(E, \mathcal{E})$ be a measurable space, and μ a measure on $\mathcal{E}$. We say that μ is σ-finite if $E = \cup_{n \in \mathbb{N}} E_n, E_n \in \mathcal{E}, \mu(E_n) < \infty \forall n$.

It is easy to see that Lebesgue measure is translation invariant using the above theorem. Just take

Theorem 2.3.5. *Let $\tilde{\mu}$ be a Borel measure on $\mathbb{R}$ that is translation invariant and $\tilde{\mu}([0,1]) = 1$. Then $\tilde{\mu}$ is the Lebesgue measure.*

Proof. First show no singleton. $\tilde{\mu}(\{a\}) \leq \tilde{\mu}([a, a + \frac{1}{n})) = \tilde{\mu}([0, \frac{1}{n})) \leq \frac{1}{n}$.

Can find $p_n, q_n \in \mathbb{Q}$ such that $p_n \downarrow a$, $q_n \uparrow b$ as $n \to \infty$. □

2.4 Probability Measure

Definition 2.4.1. Let $(E, \mathcal{E})$ be a measure space with the property $\mu(E) = 1$. Then μ is called a probability measure and $(E, \mathcal{E}, \mu)$ a probability space.

2.5 Radon Measure

Every Borel measure on $\mathbb{R}^n$ is regular and locally finite, which is Radon.