Measure Theory

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Chapter 1

Real Analysis

1.1

1.2 Differentiation and Integration

For conceptual simplicity, we study $\mathbb{R}$ instead of $\mathbb{R}^n$ in this section. Let us first recall what we learned in elementary calculus.

Theorem 1.2.1. *Let f be a continuous function on $[a, b]$, and F be the function defined, for all x in $[a, b]$, by*

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Then F is uniformly continuous on $[a, b]$ and differentiable on (a, b), and

$$F'(x) = f(x).$$

Theorem 1.2.2 (Newton-Leibniz)**.** *Let f be a Riemann integrable function on $[a, b]$, and F a continuous function on $[a, b]$ which is an antiderivative of f in (a, b):*

$$F'(x) = f(x).$$

Then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Definition 1.2.3 (Total Variation)**.** The total variation of a function f defined on $[a, b]$ is the quantity

$$V_a^b(f) = \sup_{P\in\mathcal{P}} \sum_{i=0}^{n_P-1} |f(x_{i+1}) - f(x_i)|,$$

where

$$\mathcal{P} = \{P = \{x_0, \cdots, x_{n_P}\}|P \text{ is a partition of } [a, b] \text{ satisfying } x_i \leq x_{i+1} \text{ for } 0 \leq i \leq n_P - 1\}.$$

If $V_a^b(f) < +\infty$, then f is said to be of bounded variation on $[a, b]$.

Example 1.2.4. The continuous function

$$f(x) = \begin{cases} 0, & x = 0 \\ x\sin(\frac{1}{x}), & x \neq 0 \end{cases}$$

is not of bounded variation on $[0, 2/\pi]$. Just consider the partition

$$P = \{0, \frac{2}{2n-1}, \frac{2}{2n-3}, \cdots, \frac{2}{3}, 1\}.$$

Theorem 1.2.5 (Jordan Decomposition Theorem)**.**

$$f(x) = g(x) - h(x)$$

where $g(x)$ and $h(x)$

Bounded variable functions says that it has a length

Bounded variable functions are a.e. differentiable But it does not satisfies the desirable property: Length is equal to the integral of derivative, which is also known as the **fundamental theorem of calculus**. It turns out that absolute continuous functions is exactly the set of functions that satisfies this property.

Example 1.2.6.

Definition 1.2.7 (Absolute Continuity)**.** A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if for every positive number ϵ, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) satisfies

$$\sum_{k=1}^{N}(y_k - x_k) < \delta$$

then

$$\sum_{k=1}^{N}|f(y_k) - f(x_k)| < \epsilon.$$

Absolutely continuous functions are of bounded variation.

Proposition 1.2.8. *If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a, b]$.*

Theorem 1.2.9. *If f is absolute continuous function on $[a, b]$, then*

$$f(x) - f(a) = \int_a^x f'(t)\mathrm{d}t, \quad x \in [a, b].$$

It is beneficial to compare it with the "classical" version (Theorem 1.2.2).