

Notes in Game Theory

Kaizhao Liu

December 2022

Contents

1	Fundamental Concepts	1
1.1	Basic Assumptions	1
1.2	Utility	1
1.3	Descriptions of Games	2
2	Equilibria	3
2.1	Nash Equilibria	3
2.1.1	games in strategic form	3
2.1.2	games in extensive form	3
2.1.3	Bayesian games	4
2.2	Refinements of Nash Equilibrium	4
2.2.1	Perfect Equilibrium	4
2.2.2	Proper Equilibrium	4
2.2.3	Correlated Equilibrium	4
2.3	Subgame-Perfect Equilibrium	4
2.4	Perfect Bayesian Equilibria	5
3	Applications	5
3.1	Bargaining	5
3.2	Auctions	5
3.3	Mechanism Design	6
4	Miscellanea	6
4.1	Repeated Games	6
4.2	Signaling Games	6
4.3	Building a Reputation	6
4.4	Information Transmission and Cheap Talk	6
4.5	Two-Person Zero-Sum Game	6
4.6	Two-Person Bargaining Problems and the Nash Bargaining Solution	7
4.7	Universal Belief Space	7

1 Fundamental Concepts

1.1 Basic Assumptions

Definition 1.1 (common knowledge). Let A be a knowledge. If all players know A , and know that their opponents know A , and know their opponents know that they know, so on ad infinitum. Then A is called common knowledge.

Definition 1.2 (sequential rationality).

Definition 1.3 (perfect recall). No players ever forgets any information he once knew, and all players know the actions they have chosen previously.

1.2 Utility

Utility is a quantification of preference.

Theorem 1.1 (Von Neumann-Morgenstern utility theorem).

1.3 Descriptions of Games

For static games, we model them with strategic form. For dynamic games, we model them with extensive form. If the game, dynamic or not, has incomplete information, we model them with Bayesian

Definition 1.4 (game in strategic form). A game in strategic form consist of the following information:

(the set of players) $i \in I$, where I is taken to be the finite set $\{1, 2, \dots, I\}$

(the pure-strategy space) S_i for each $i \in I$

(payoff functions) $u_i = u_i(s)$, where $s = (s_1, \dots, s_I)$ is the profile of strategies.

Remark. *The finiteness can be removed by utilizing measure theory.*

Definition 1.5 (mixed strategy). A mixed strategy σ_i is a probability distribution over pure strategies. The space of player i 's mixed strategies is denoted by Σ_i , where $\sigma_i(s_i)$ is the probability that σ_i assigns to s_i .

Games in strategic form do not make explicit time. To model games with dynamic structures, for example multistage games, we propose the extensive form, which is a generalization of decision trees. Games in extensive form make explicit the order in which players move, and what each player knows when making each of his decisions.

Definition 1.6 (game in extensive form). A game in strategic form consist of the following information:

(the set of players) (the order of moves)

(payoff functions)

Next we relate strategies of extensive-form games to strategies of strategic-form games. Maps from extensive-form games to strategies of strategic-form games can be defined. Normal representation by von Neumann and multiagent representation by Selten are two.

Definition 1.7 (normal representation).

Definition 1.8 (multiagent representation).

Now let us examine mixed-strategies for extensive-form games closely. The above two different give rise to two different ways to define randomized strategy profiles for any given extensive form game.

Definition 1.9 (behavior strategy). Let $\Delta(A(h_i))$ be the probability distributions on $A(h_i)$. A behavior strategy for player i , denoted by b_i , is an element of $\prod_{h_i \in H_i} \Delta(A(h_i))$. That is, a behavior strategy specifies a probability distribution over actions at each h_i , and the probability distributions at different information sets are independent.

Theorem 1.2 (Kuhn's theorem). *If Γ^e is a game with perfect recall. then any two mixed strategies that are behaviorally equivalent are also payoff equivalent.*

Example 1.1.

To model games with incomplete information, Harsanyi introduced a common prior, i.e. belief, and type of players to simplify the analysis. We call these kind of model Bayesian games. Note that there are also other models of games with incomplete information, for example the Aumann model and the belief space model.

Definition 1.10 (belief).

Definition 1.11 (consistent).

Theorem 1.3 (Aumann). *Agreeing to disagree.*

Definition 1.12 (Bayesian game).

On the one hand, a Bayesian game can be described as an extensive-form game.

Theorem 1.4. *Every Bayesian game can be described as an extensive-form game (with moves of chance and information sets).*

This extensive-form game can be further represented by a strategic-form game (normal or multiagent). On the other hand, a Bayesian game can be described as another kind of strategic-form game.

Definition 1.13 (type-agent representation).

2 Equilibria

In game theory, we focus on the concept of equilibrium. When rational people play a game, the final result must be a Nash equilibrium. The theory of equilibrium can be regarded as a ultimate theory for super-human, who are really rational. The process of how real-human in our current myopia update their strategies, is somewhat explained by evolutionary game theory and the theory of learning in games.

2.1 Nash Equilibria

2.1.1 games in strategic form

Definition 2.1 (Nash Equilibrium of an strategic form game). A mixed-strategy profile σ^* is a Nash equilibrium if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \quad \forall s_i \in S_i \quad \forall i \in I$$

Theorem 2.1. *Every finite strategic-form game has a mixed-strategy equilibrium.*

Lemma 2.1 (Kakutani's fixed point theorem). *Let S be a non-empty, compact, convex subset of some finite-dimensional Euclidean space,*

*Proof.*¹ □

*Proof.*² The idea of the proof is to apply Kakutanils fixed point theorem to the players' reaction correspondences.

Player i 's reaction correspondence r_i maps each strategy profile σ to the set of mixed strategies that maximize player i 's payoff when his opponents play σ_{-i} . Note that although r_i only depends on σ_{-i} , we write it as a function of the strategies of all players, because later we will look for a fixed point in the space Σ of strategy profiles. Define the correspondence $r : \Sigma \rightarrow \Sigma$ be the Cartesian product of the r_i 's. Then a fixed point of r is a Nash equilibrium. □

Now let us examine how to find a Nash equilibrium.

2.1.2 games in extensive form

If we define an equilibrium of an extensive form game to be an equilibrium of its normal representation in the strategic form, then some games would have an enormous multiplicity of equilibria that are all behaviorally equivalent.

Example 2.1.

To avoid this problem, we may consider defining an equilibrium of an extensive-form game to be an equilibrium of its multiagent representation, but such a definition could create a more serious problem of nonsensical equilibria in which a player fails to coordinate his moves across different information states.

Example 2.2.

To avoid both problems, we have the following definition.

Definition 2.2 (Nash Equilibrium of an extensive form game). A Nash equilibrium of an extensive form game, or an equilibrium in behavioral strategies, is defined to be any Nash equilibrium σ of the multiagent representation such that the the mixed representation of σ is also an equilibrium of the normal representation.

Now let us examine how to find an equilibrium for a game in extensive form.

Theorem 2.2. *If Γ^e is an extensive-form game with perfect recall and τ is an equilibrium of the normal representation of Γ^e , then any behavioral representation of τ is an equilibrium of the multiagent representation of Γ^e .*

Corollary 2.2.1. *For any extensive-form game with perfect recall, a Nash equilibrium in behavioral strategies exists.*

So to find an equilibrium of Γ^e , it suffices to find an equilibrium of the normal representation of Γ^e , because any equilibrium of the normal representation corresponds to an equilibrium of the multiagent representation.

¹Shizuo Kakutani. 1941. A generalization of Brouwer's fixed point theorem. Duke Math. J. 8(3): 457-459.

²John F. Nash, Jr. 1950. Equilibrium points in n-person games. Proc. Natl. Acad. Sci. U.S.A. 36 (1): 48-49.

2.1.3 Bayesian games

As we pointed out, Bayesian games may be analyzed at two separate points in time: at the ex ante stage, before the players know their types, and at the interim stage, after they have learned what types they are. Accordingly, two different types of equilibria can be defined. The first equilibrium concept, which is Nash equilibrium in Bayesian games, poses the requirement that no player can profit by a unilateral deviation before knowing his type. The second equilibrium concept, called Bayesian equilibrium, poses the requirement that no player i can profit by deviating at the interim stage, after learning his type t_i .

Definition 2.3 (Bayesian equilibrium). A Bayesian equilibrium in a game of incomplete information is defined to be any Nash equilibrium of the type-agent representation in strategic form.

To state it formally,

Definition 2.4.

One can verify that these two formulations of Bayesian equilibrium are equivalent.

Theorem 2.3. Every Bayesian game in which the set of types is finite and the set of actions of each type is finite has a Nash equilibrium in behavior strategies.

Proof. As every game with incomplete information can be described as an extensive-form game, every finite extensive-form game has a Nash equilibrium in mixed strategies, and every mixed strategy is equivalent to a behavior strategy, the result is obvious. \square

Theorem 2.4. Every Bayesian game in which the set of types is finite and the set of actions of each type is finite has a Bayesian equilibrium.

Proof. The type-agent representation of the game has a Nash equilibrium. \square

As already noted, the two definitions of equilibrium presented in this section (Nash equilibrium and Bayesian equilibrium) express two different perspectives on the game. The next theorem shows that these two definitions are in fact equivalent.

Theorem 2.5 (Harsanyi). In a game with incomplete information in which the number of types of each player is finite, every Bayesian equilibrium is also a Nash equilibrium, and conversely every Nash equilibrium is also a Bayesian equilibrium.

Proof. Because the expected payoff of player i in a game with incomplete information is the expectation of the conditional expected payoff of all of his types t_i , and because the probability of each type is positive, it follows that every deviation that increases the expected payoff of any single type of player i also increases the overall payoff for player i in the game. In the other direction, if there is a deviation that increases the total expected payoff of player i in the game, it must necessarily increase the conditional expected payoff of at least one type t_i . \square

2.2 Refinements of Nash Equilibrium

2.2.1 Perfect Equilibrium

2.2.2 Proper Equilibrium

2.2.3 Correlated Equilibrium

2.3 Subgame-Perfect Equilibrium

Nash equilibrium itself can not explain everything. In a game, there may exist too many Nash equilibria, and many of them look ridiculous. This subsection concern some refinements of Nash equilibria. Let us first look at an example.

Example 2.3 (5 Pirates and 100 Gold Coins). There are 5 pirates, they must decide how to distribute 100 gold coins among them. The pirates have seniority levels, the senior-most is A, then B, then C, then D, and finally the junior-most is E.

Rules of distribution are:

1. The most senior pirate proposes a distribution of coins.
2. All pirates vote on whether to accept the distribution.
3. The distribution is approved if at least half of the pirates agree (including the proposer)

4. If the distribution is accepted, the coins are disbursed and the game ends. If not, the proposer is thrown and dies, and the next most senior pirate makes a new proposal to begin the system again.
5. In case of a tie vote, the proposer can have the casting vote.

Rules every pirate follows:

1. Every pirate wants to survive
2. Given survival, each pirate wants to maximize the number of gold coins he receives.

Question: What is the maximum number of coins that pirate A might get?

Answer: 98

1. Consider the situation when A, B, and C die, only D and E are left. E knows that he will not get anything (D is senior and will make a distribution of (100, 0)). So E would be fine with anything greater than 0.
2. Consider the situation when A and B die, C, D, and E are left. D knows that he will not get anything (C will make a distribution of (99, 0, 1) and E will vote in favor of C).
3. Consider the situation when A dies. B, C, D, and E are left. To survive, B only needs to give 1 coin to D. So distribution is (99, 0, 1, 0).
4. Similarly, A knows about point 3, so he just needs to give 1 coin to C and 1 coin to E to get them in favor. So distribution is (98, 0, 1, 0, 1).

The idea we use to analysis this illustrative example is exactly backward induction. The equilibrium concept behind backward induction is just subgame-perfect equilibrium.

Definition 2.5 (subgame-perfect equilibrium). A strategy profile s of a multi-stage game with observed actions is a subgame-perfect equilibrium if

$$s|h^k$$

Theorem 2.6 (one-stage-deviation principle for finite-horizon games).

Definition 2.6. A game is continuous at infinity if for each player i the utility function u_i satisfies

$$\sup_{h, \tilde{h} \text{ s.t. } h^t = \tilde{h}^t} |u_i(h) - u_i(\tilde{h})| \rightarrow 0 \text{ as } t \rightarrow \infty$$

Example 2.4. If we introduce a discount factor $\delta \in (0, 1)$ to discount future payoffs in a game, then this game must be continuous at infinity.

Theorem 2.7 (one-stage-deviation principle for infinite-horizon games).

2.4 Perfect Bayesian Equilibria

Now let us examine how to find a perfect Bayesian equilibrium.

One way is to first find all the profiles of strategies in the Bayesian game that are Bayesian Nash equilibria; then we can systematically check for each Bayesian Nash equilibrium to see whether we can find a system of beliefs so that together they constitute a perfect Bayesian equilibrium.

Now let us consider refinements of perfect Bayesian equilibrium.

Definition 2.7 (consistent).

Definition 2.8 (sequential equilibrium).

3 Applications

Game theory roots in reality and applies to reality. Here we discuss some simplified models. However, when analysing real life problems, note that what we are analysing is not an isolated game, but is a subgame of an enormous game.

3.1 Bargaining

3.2 Auctions

The concept of Bayesian equilibrium is important in the analysis of auctions.

We consider goods with independent private values first.

Theorem 3.1 (revenue equivalence theorem).

Theorem 3.2. *In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation.*

Next, let us consider goods with public

3.3 Mechanism Design

Theorem 3.3 (revelation principle).

4 Miscellanea

4.1 Repeated Games

First we consider

Theorem 4.1. *A strategy vector at which in each stage the players play a one-stage equilibrium is an equilibrium of the T -stage game.*

Theorem 4.2 (Folk theorem).

Proof. □

Next let us consider infinitely repeated games.

Theorem 4.3 (Folk theorem for the infinitely repeated game). *The set of equilibrium payoffs of Γ_∞ is the set $F \cap V$.*

Remark. *Note that the Folk Theorem for Γ_∞ and the Folk theorem for Γ_T differ in two aspects.*

Now let us examine infinitely repeated games with discount factor δ .

4.2 Signaling Games

4.3 Building a Reputation

4.4 Information Transmission and Cheap Talk

4.5 Two-Person Zero-Sum Game

This is actually the earliest work in game theory, which was first considered by von Neumann.

Theorem 4.4. *(σ_1, σ_2) is an equilibrium of a finite two-person zero-sum game if and only if*

$$\sigma_1 \in \arg \max_{\tau_1 \in \Sigma(S_1)} \min_{\tau_2 \in \Sigma(S_2)} u_1(\tau_1, \tau_2)$$

and

$$\sigma_2 \in \arg \min_{\tau_2 \in \Sigma(S_2)} \max_{\tau_1 \in \Sigma(S_1)} u_1(\tau_1, \tau_2)$$

Furthermore, if (σ_1, σ_2) is an equilibrium of this game, then

$$u_1(\sigma_1, \sigma_2) = \max_{\tau_1 \in \Sigma(S_1)} \min_{\tau_2 \in \Sigma(S_2)} u_1(\tau_1, \tau_2) = \min_{\tau_2 \in \Sigma(S_2)} \max_{\tau_1 \in \Sigma(S_1)} u_1(\tau_1, \tau_2)$$

Remark. *Note that we have always assumed that the players in a strategic form game choose their strategies simultaneously and independently, so a change of plans by one player cannot affect the strategy chosen by the other player. Suppose for a moment, however, that player 1 is afraid that, whatever randomized strategy he might choose, 2 would be able to discover that he had chosen this and would respond in the way that is worst for 1. The theorem asserts that player 1's equilibrium strategy in a two-people zero-sum game also maximizes 1's expected payoff under this pessimistic assumption.*

Proof. ³ □

³von Neumann, 1928. Zur theorie der gesellschaftsspiele. Math. Ann. 100, 295-320.

4.6 Two-Person Bargaining Problems and the Nash Bargaining Solution

Definition 4.1. A two-person bargaining problem to consist of a pair (F, v) .
(feasible set) F is a closed convex subset of \mathbb{R}^2
(disagreement point) $v = (v_1, v_2) \in \mathbb{R}^2$ is a vector
(technical) the set $F \cap \{(x_1, x_2) | x_1 \geq v_1, x_2 \geq v_2\}$ is nonempty and bounded.

The axioms for Nash's bargaining solution can be written as follows:

1. (strong efficiency) $\phi(F, v)$ is an allocation on F , and for any $x \in F$ with $x \geq \phi(F, v)$, $x = \phi(F, v)$.
2. (individual rationality) $\phi(F, v) \geq v$.
3. (scale covariance)
4. (independence of irrelevant alternatives)
5. (symmetry) If $v_1 = v_2$ and $\{(x_2, x_1) | (x_1, x_2) \in F\} = F$, then $\phi_1(F, v) = \phi_2(F, v)$.

Theorem 4.5. *There is a unique solution function $\phi(\cdot, \cdot)$ that satisfies the above axioms. This solution function satisfies*

$$\phi(F, v) \in \arg \max_{x \in F, x \geq v} (x_1 - v_1)(x_2 - v_2)$$

Proof. ⁴

□

4.7 Universal Belief Space

In this section, we prove that under some technical assumptions, there exist type sets that are large enough to include all possible states of all players. This construction is purely algebraic, but it reveals how expressive is the model of Bayesian game.

⁴John F. Nash, Jr. 1950. The bargaining problem. *Econometrica*. 18 (2): 155-162.