Geometry

Kaizhao Liu

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An affine connection $\nabla_{\cdot}(\cdot)$ is a map from $\Gamma(M) \times \Gamma(M)$ to $\Gamma(M)$

Definition 2.2.1.
1. $C(M)$-linear in the lower slot
2. $\mathbb{R}$-linear in the upper slot
3. $C(M)$-Leibniz rule in the upper slot

2.2.1 Levi-Civita Connection

Let ∇ be an affine connection, then by the basis theorem there are locally defined functions Γ_{ij}^k such that

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k. \tag{2.1}$$

A connection is symmetric if
metric compaibility

Theorem 2.2.2. *Given a metric g, there is a unique connection ∇ that is symmetric and metric compatible. This connection is called the **Levi-Civita connection**, and is given by*

2.2.2 Tensor Leibniz Rule

$$\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$$

Proposition 2.2.3.

$$F_{j_1 \dots j_l; m}^{i_1 \dots i_k} = E_m(F_{j_1 \dots j_l}^{i_1 \dots i_k}) + \sum_{s=1}^{k} \Gamma_{mp}^{i_s} F_{j_1 \dots j_l}^{i_1 \dots p \dots i_k} - \sum_{s=1}^{l} \Gamma_{m j_s}^{p} F_{j_1 \dots p \dots j_l}^{i_1 \dots i_k}$$

higher order covariant derivative can be computed by iterating

2.2.3 Along a Curve

2.3 Curvature

The covariant derivative of a (r, s) tensor can be thought of as an $(r, s+1)$ tensor in a natural way. For example, if

Repeating this, we get a $(1, 2)$ tensor $\nabla\nabla V$, which will abbreviate as $\nabla^2 V$.

Using the Leibniz rule, we see that

$$\nabla_X (\nabla_Y V) = \nabla^2_{X,Y} V + \nabla_{\nabla_X Y} V$$

The tensor is not necessarily symmetric in the two lower slots. In fact, the curvature comes in

$$\begin{aligned}
\nabla^2_{X,Y}V - \nabla^2_{Y,X}V &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{\nabla_X Y - \nabla_Y X}V \\
&= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{[X,Y]}V \\
&= R(Y,X)V
\end{aligned}$$

This is known as the **Ricci identity**.

Definition 2.3.1 (Riemann curvature)**.**

2.3.1 Symmetries

2.3.2

The *Ricci curvature* is a $(0,2)$ tensor given by taking the trace of R.

Definition 2.3.2 (Ricci Curvature)**.**

A manifold is said to be *Einstein* with Einstein constant $\lambda \in \mathbb{R}$ if
The trace of the Ricci tensor is the *scalar curvature*:

Definition 2.3.3 (Scalar Curvature)**.**

Definition 2.3.4 (Sectional curvature)**.**

A Riemannian manifold has *constant sectional curvature* κ if

Lemma 2.3.5 (Schur Lemma)**.**

$$\mathrm{d}\,S = 2\,\mathrm{div}\,\mathrm{Ric} \tag{2.2}$$

2.4 Submanifolds

If (N, g) is a Riemannian manifold and

$$\phi : M \to N$$

is an immersion, then $\phi^* g$ gives a metric on M.

$$T_{\phi(p)}N = d\phi(T_p M) \otimes [d\phi(T_p M)]^\perp$$

induced connection

Definition 2.4.1 (Second Fundamental Form)**.**

M is said to be totally geodesic if $A \equiv 0$. totally geodesic

Definition 2.4.2 (Mean Curvature Vector)**.** The trace of the second fundamental form is called the mean curvature vector $\mathbf{H}$.

M is said to be *minimal* if $\mathbf{H} = 0$.

2.4.1 Gauss and Codazzi Equations

The Gauss equation relates the curvature of the submanifold to the curvature via the second fundamental form

Theorem 2.4.3 (Gauss Equation)**.**

Theorem 2.4.4 (Codazzi Equation)**.**

Chapter 3

$\triangle|x|^2$ on $\mathbb{R}^n$

$\nabla|x|^2 = 2|x|\nabla|x|$

$\triangle|x| = \frac{n-1}{|x|}$ on $\mathbb{R}^n \setminus \{0\}$

Another way $\mathrm{Hess}_{|x|}$

Suppose $\mathrm{Ric} \geq 0$