# Capacitors and Inductors

**EEE103 ELECTRICAL CIRCUITS (Part 2)**
**Week 6**
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# Content

- Voltage-current relationship of an ideal capacitor
- Current-voltage relationship of an ideal inductor
- Calculating energy stored in capacitors and inductors
- Series and parallel combinations
- Op amp circuits with capacitors

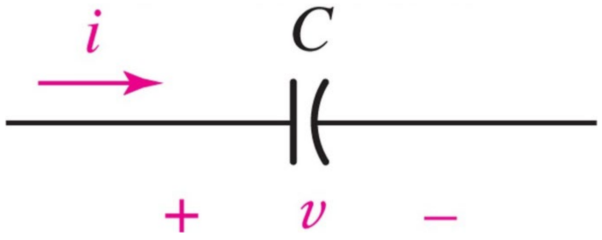
# Active and passive elements

- An active element is an element that is capable of supplying an average power greater than zero to some external device over an infinite time interval (e.g. Ideal sources, the operational amplifier, the transistor)
- A passive element is defined as an element that cannot supply an average power greater than zero over infinite time interval. (e.g. the resistor, more? )

Capacitor and Inductors

# The Capacitor

- The ideal capacitor is a passive element with circuit symbol




- The current-voltage relation is

$$i = C\frac{dv}{dt}$$

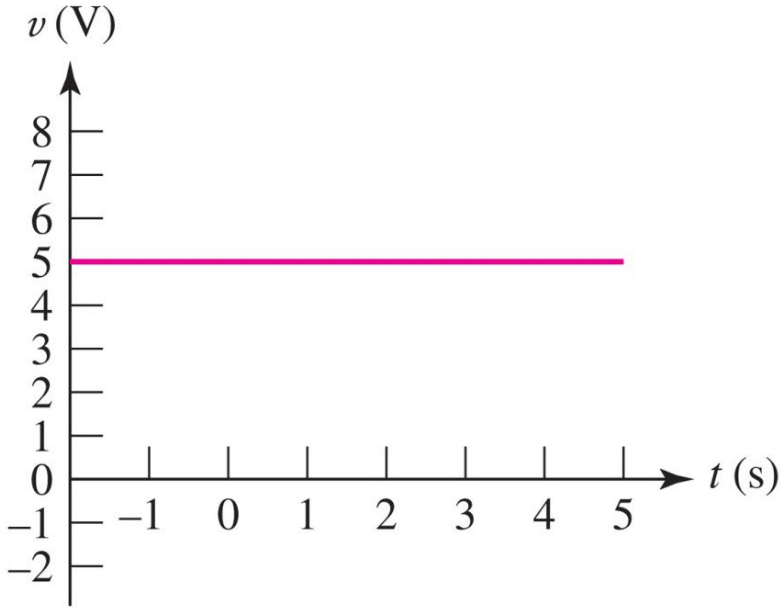
- Capacitance *C* is measured in ampere-second per volt or coulomb per volt or farad (F).

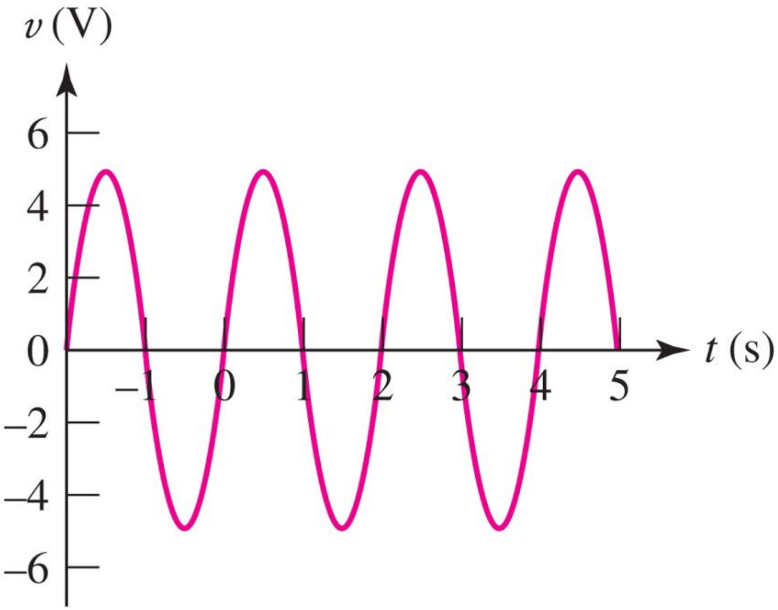
# Example capacitors

- Capacitors vary in size depending on capacitance and voltage tolerance
- Typical values range from pF to µF


# Example 1

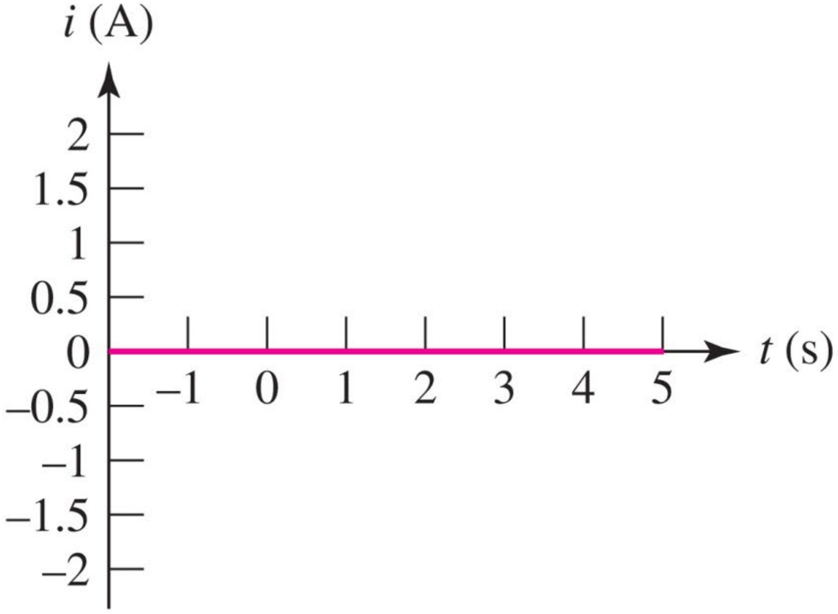
Find $i(t)$ for the voltages shown, if $C = 2F$ .

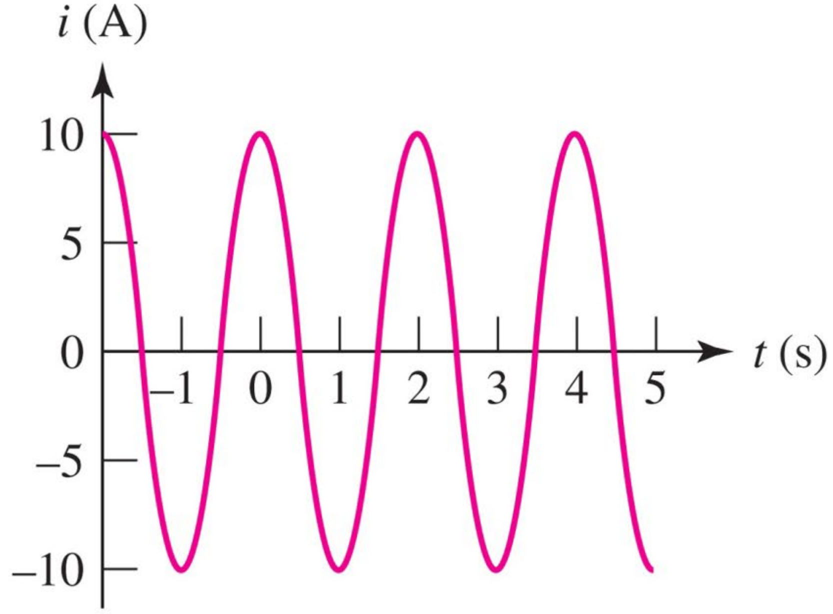


# Solution to Example 1

Apply $i(t) = 2\frac{dv}{dt}$ and graph:



# Key capacitor behaviors

- A constant voltage across a capacitor results in zero current passing it; thus capacitors are "open circuits" to constant dc voltages.

- A sudden jump in the voltage requires an in finite current, which is physically impossible. Therefore the voltage on a capacitor cannot jump.

$$i = C\frac{dv}{dt}$$

# Integral Voltage-Current relationships

- The capacitor voltage-current relationship

$$i = C\frac{dv}{dt} \qquad [1]$$

- Integrating Eq.[1], we first obtain

$$dv = \frac{1}{C} i(t)dt \qquad [2]$$

- Then integrate between times $t_0$ and $t$ and between corresponding voltages $v(t_0)$ and $v(t)$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')dt' + v(t_0) \qquad [3]$$

- Eq.[3] may also be written as an indefinite integral plus a constant of integration:

$$v(t) = \frac{1}{C}\int i dt + k \qquad [4]$$

# Integral Voltage-Current relationships

- Finally, in many situations the voltage initially across the capacitor cannot be discerned. Thus set $t_0 = -\infty$ and $v(-\infty)$ =0， so that

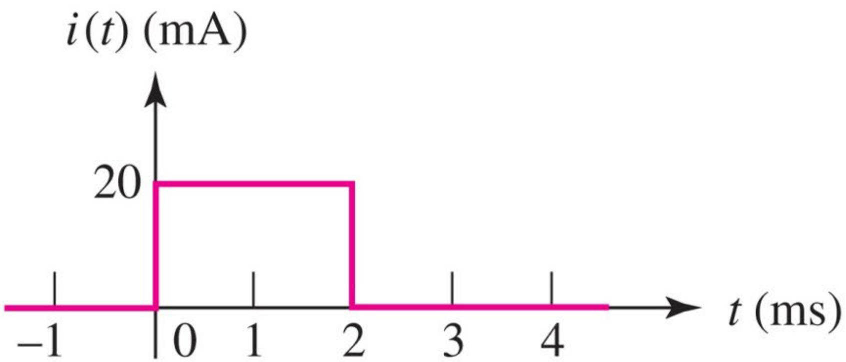
$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i d\, t' \qquad [5]$$

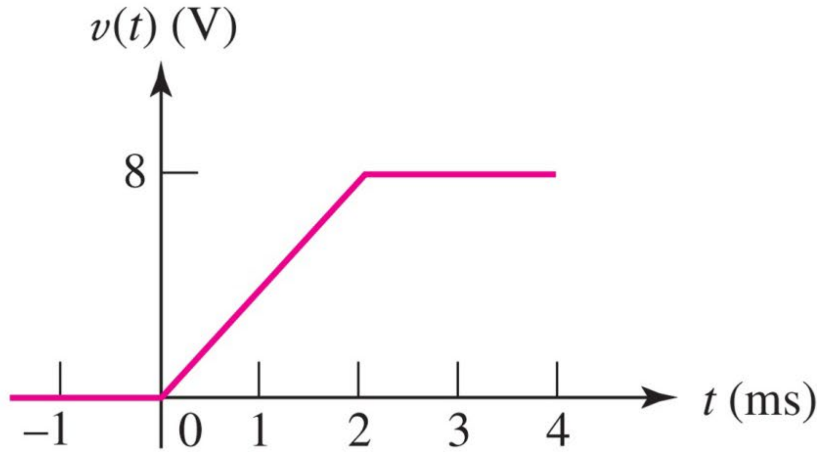
- According to Eq.[5], the charge accumulated on the capacitor plate can be represented as:

$$q(t) = Cv(t) \qquad [6]$$

# Example 2

Show that the following graphs are matching voltage and current graphs for a capacitor of $C = 5\mu F$.



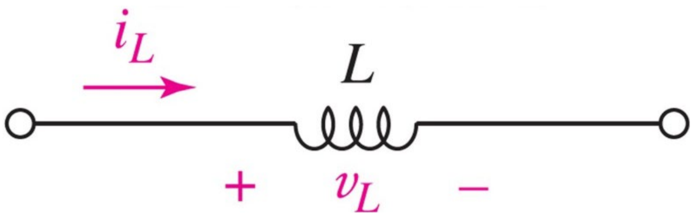




Solution:

$$v(t) = \begin{cases} 0, & t \leq 0 \text{ ms} \\ 4000t, & 0 \leq t \leq 2 \text{ ms} \\ 8 & t > 2 \text{ ms} \end{cases}$$

# The inductor

- The ideal inductor a passive element with circuit symbol



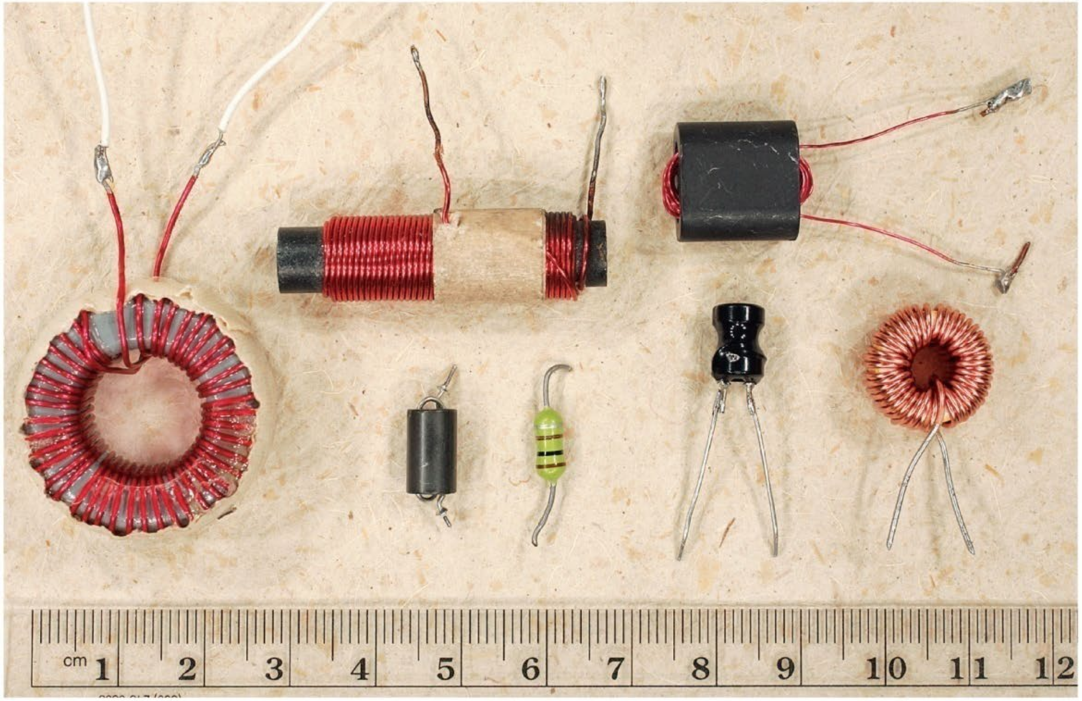

- The current-voltage relation is

$$v = L\frac{di}{dt}$$

- The unit of inductance *L* is volt-second per ampere or henry (H)

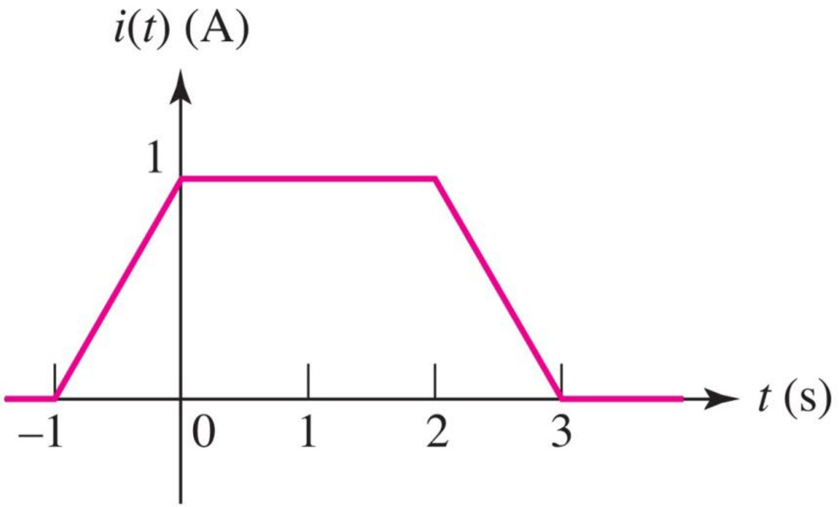
# Example Inductors

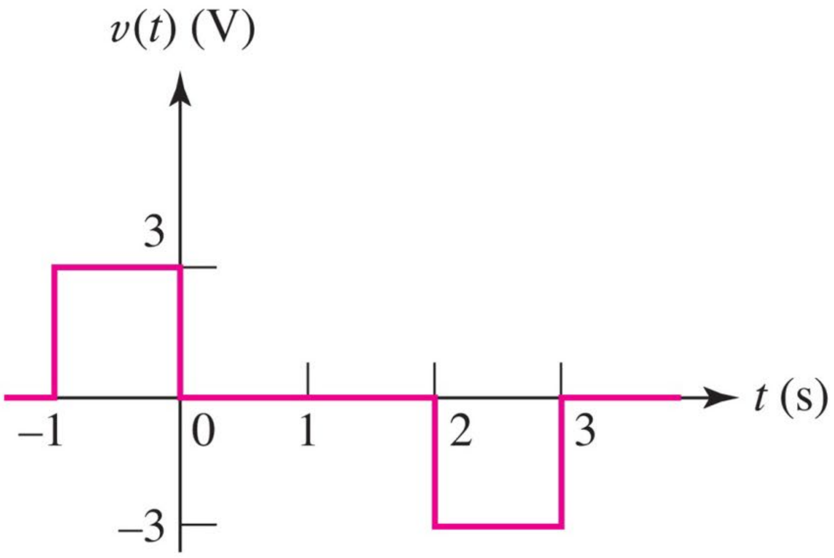
- Inductors can be bulky, with typical values ranging from µH to H




# Example 3

Show that the following graphs are matching voltage and current graphs for an inductor of $L$ = 3 H.



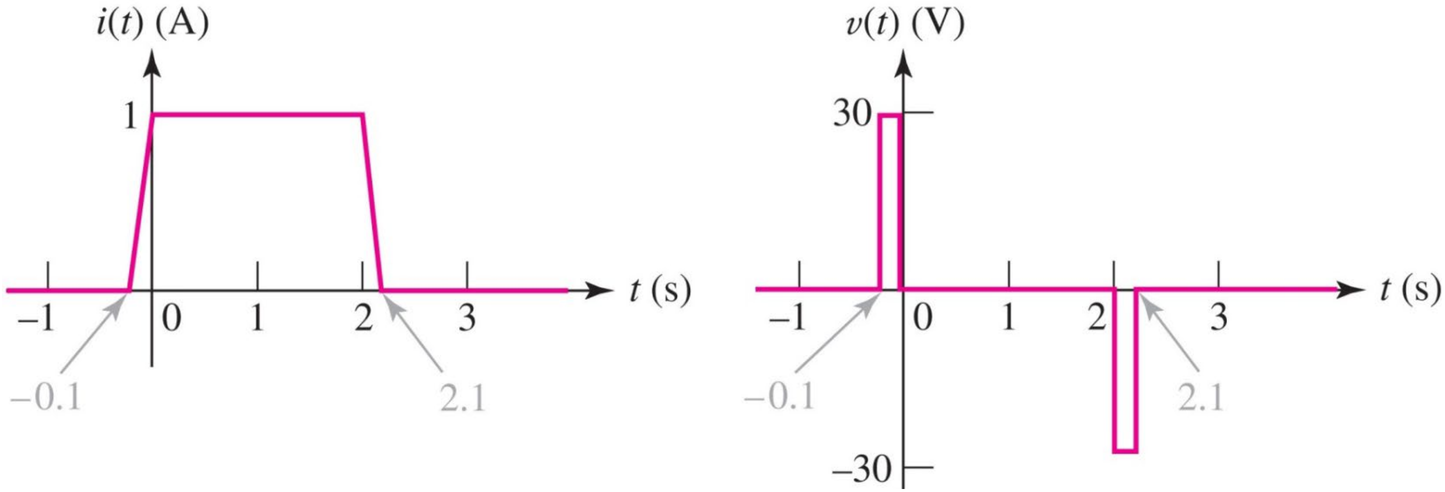

Solution:

$$v(t) = 3\,\frac{di}{dt}$$

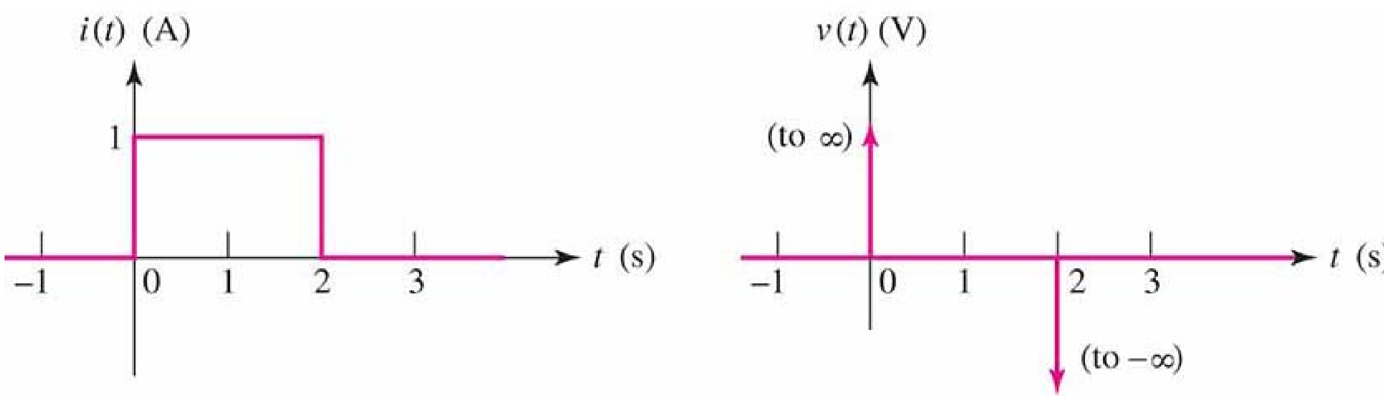
$$v(t) = \begin{cases} 0, & t \le -1 \\ 3 & -1 \le t \le 0 \\ 0, & 0 \le t \le 2 \\ -3, & 2 \le t \le 3 \\ 0 & t > 3 \end{cases}$$

# The inductor behavior

- For the same 3-H inductor, the voltages are 10 times larger when the current is ramped 10 times faster.




- A sudden change in the current will cause the infinite voltage "spikes",



# Key inductor behaviors

- A constant current across a inductor results in zero voltage across it; thus inductors are "short circuits" to constant dc currents.
- A sudden or discontinuous change in the current must be associated with an infinite voltage across the inductor, which is physically impossible. Therefore, the current through an inductor *cannot* jump.

$$v = L\frac{di}{dt}$$

# Integral Current-Voltage relationships

- The inductor current-voltage relationship

$$v = L\frac{di}{dt} \qquad [1]$$

- Integrating Eq.[1], we first obtain

$$di = \frac{1}{L}v(t)dt \qquad [2]$$

- Then integrate between times $t_0$ and $t$ and between corresponding currents $i(t_0)$ and $i(t)$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t')dt' + i(t_0) \qquad [3]$$

- Eq.[2] may also be written as an indefinite integral plus a constant of integration:

$$i(t) = \frac{1}{L}\int v dt + k \qquad [4]$$

# Integral Voltage-Current relationships

- We also may assume that we are solving a realistic problem that no current or energy in the inductor at initial stage. Thus set $t_0 = -\infty$ and $i(-\infty)$ =0， so that

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v\, dt' \qquad [5]$$

# Capacitor energy storage

- Capacitors store energy (*iv* > 0) or deliver energy (*iv* < 0).

- Since

$$p(t) = vi = Cv\frac{dv}{dt} \qquad [1]$$

- Integrate over times between $t_0$ and $t$.

$$\int_{t_0}^{t} p(t')dt' = C\int_{t_0}^{t} v\frac{dv}{dt'}dt' = C\int_{v(t_0)}^{v(t)} v'dv' \qquad [2]$$

- Thus,

$$w_c(t) - w_c(t_0) = \frac{1}{2}C\{[v(t)]^2 - [v(t_0)]^2\} \qquad [3]$$

- Assuming zero energy at $t_0$, then the energy stored in a capacitor is:

$$w_c(t) = \frac{1}{2}Cv^2 \qquad [4]$$

# Inductor energy storage

- Inductors store energy (*iv* > 0) or deliver energy (*iv* < 0).

- Since

$$p(t) = vi = Li\frac{di}{dt} \qquad [1]$$

- Integrate over times between $t_0$ and $t$.

$$\int_{t_0}^{t} p(t')dt' = L\int_{t_0}^{t} i\frac{di}{dt'}dt' = L\int_{i(t_0)}^{i(t)} i'di' \qquad [2]$$

- Thus,

$$w_L(t) - w_L(t_0) = \frac{1}{2}L\{[i(t)]^2 - [i(t_0)]^2\} \qquad [3]$$

- Assuming zero current at $t_0$, then the energy stored in a inductor is:

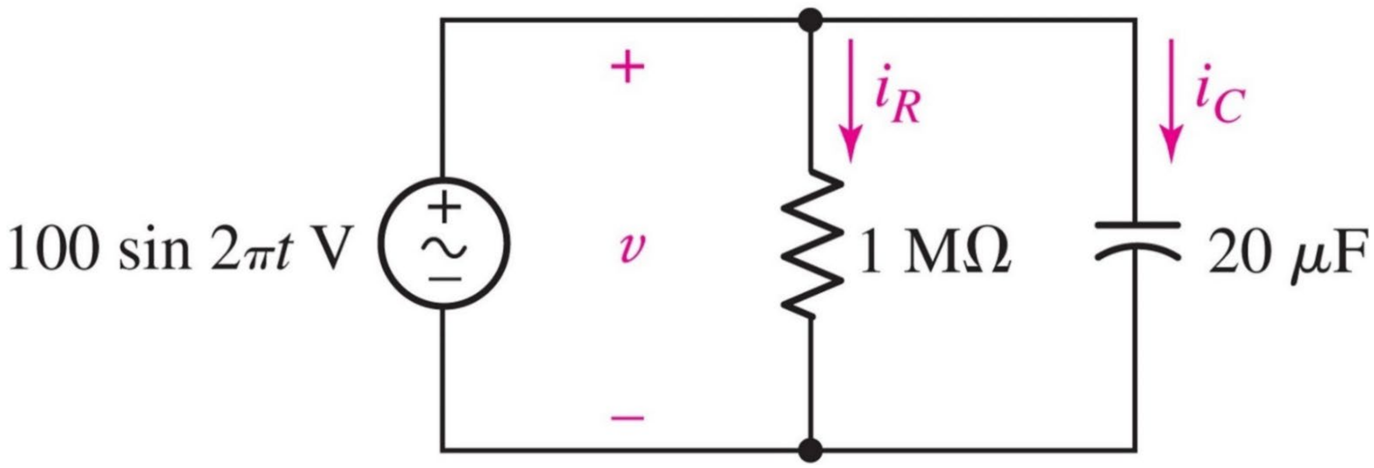
$$w_L(t) = \frac{1}{2}Li^2 \qquad [4]$$

# Important characteristics of the ideal capacitor and inductor

- A finite amount of energy can be stored in a capacitor even if the current through the capacitor is zero, such as when the voltage across it is constant.
- A finite amount of energy can be stored in a inductor even if the voltage across the inductor is zero, such as when the current through it is constant.
- An ideal capacitor and inductor never dissipates energy, but only stores it.

# Example 4

Determine the maximum energy stored in the capacitor, and plot $i_R$ and $i_c$.




Note: The 1 MΩ resistor might represent the finite resistance of the “real” capacitor’s dielectric layer.

# Solution to Example 4

- The energy stored in the capacitor is:

$$w_c(t) = \frac{1}{2} C v^2 = 0.1 \sin^2 2\pi t \text{ J}$$

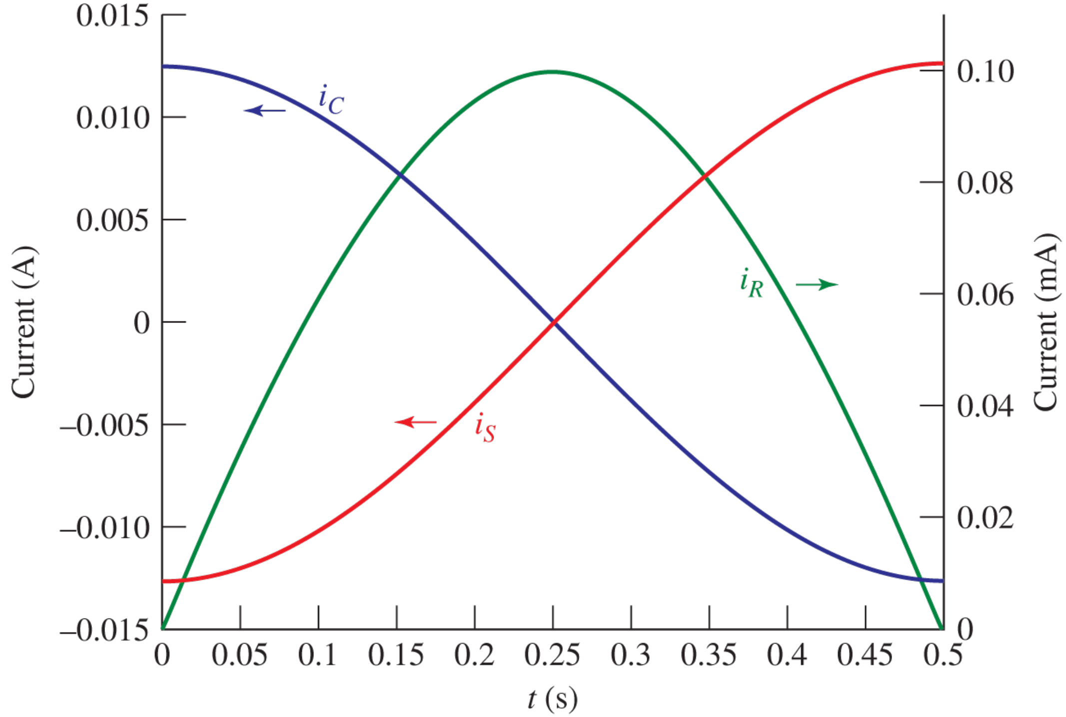
- The resistor current:

$$i_R = \frac{v}{R} = 10^{-4} \sin 2\pi t \text{ A}$$

- The capacitor current:

$$i_C = C \frac{dv}{dt} = 20 \times 10^{-6} \frac{dv}{dt} = 0.04\pi \cos 2\pi t \text{ A}$$

# Solution to Example 4

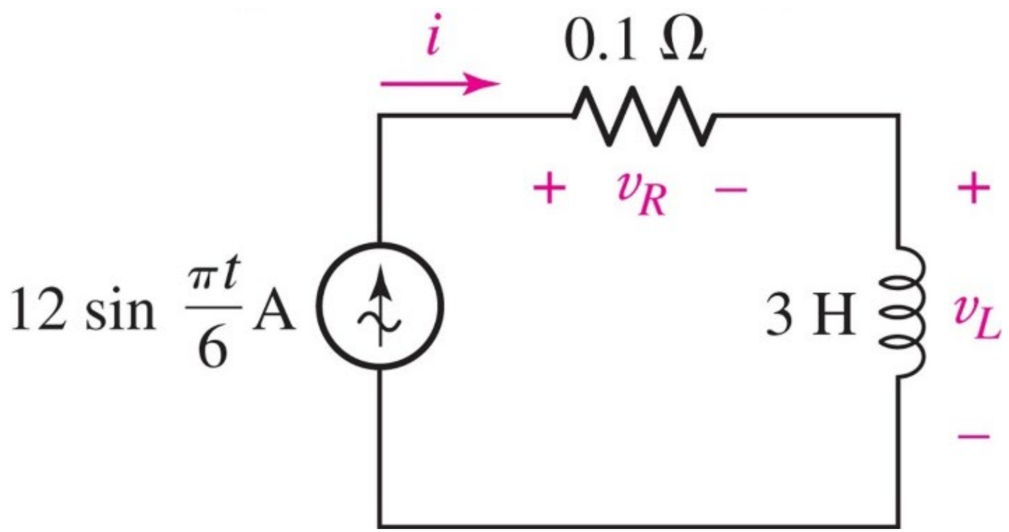



The current $i_s$ is defined as the current flowing into the positive terminal of the source:

$$i_s = -i_c - i_R$$

# Example 5

Determine the maximum energy stored in the inductor, and find the energy lost to resistor from t = 0 to t = 6 s.




Note: The 0.1Ω resistor represents the inherent resistance of the wire from which the inductor is fabricated.

# Solution to Example 5

- The energy stored in the inductor is:

$$w_L(t) = \frac{1}{2}Li^2 = 216\sin^2\frac{\pi t}{6}\ \text{J}$$

Thus the maximum energy stored in the inductor is 216 J at t=3s.

- The power dissipated in the resistor is:

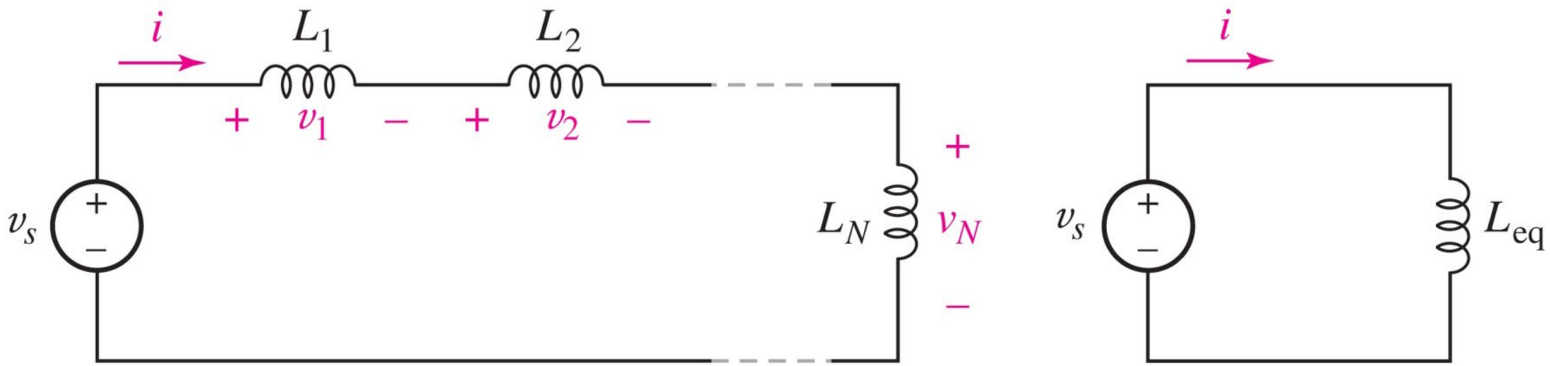
$$p_R = i^2R = 14.4\sin^2\frac{\pi t}{6}\ \text{W}$$

- The energy lost to resistor from t = 0 to t = 6 s:

$$w_R = \int_0^6 p_R dt = \int_0^6 14.4\left(\frac{1}{2}\right)\left(1-\cos\frac{\pi}{3}t\right)dt = 43.2\ \text{J}$$

Note: $\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}, \sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$

# Inductors in series




- Applying KVL to the original circuit:

$$\begin{aligned} v_s &= v_1 + v_2 + \cdots + v_N \\ &= L_1\frac{di}{dt} + L_2\frac{di}{dt} + \cdots + L_N\frac{di}{dt} \qquad [1] \\ &= (L_1 + L_2 + \cdots + L_N)\frac{di}{dt} \end{aligned}$$

- For the equivalent circuit we have:

$$v_s = L_{\text{eq}}\frac{di}{dt} \qquad [2]$$

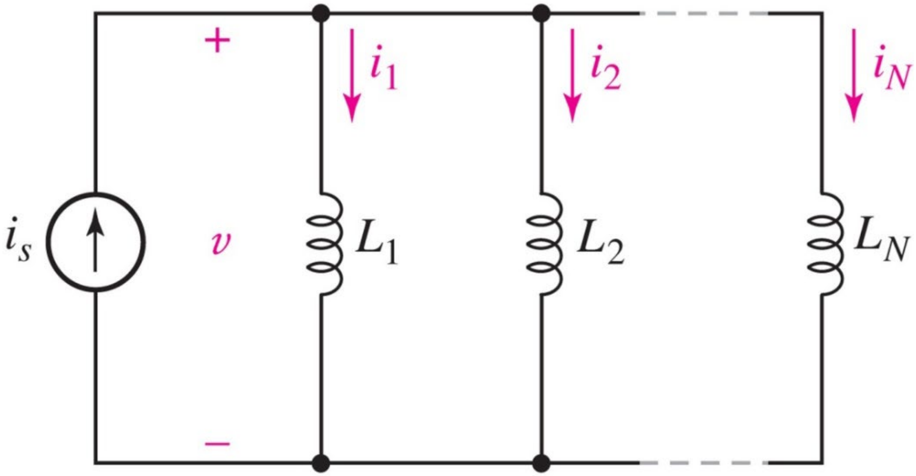
# Inductors in series

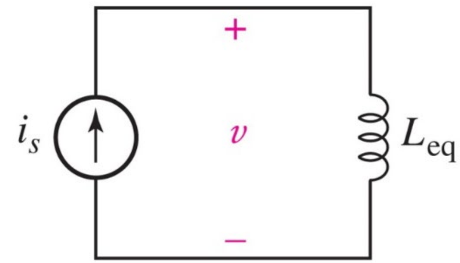
- Thus the equivalent inductance is:

$$L_{eq} = L_1 + L_2 + \cdots + L_N$$ [3]

- The equivalent inductance is the sum of all the inductances connected in series. This is exactly the same result we obtained for the resistor in series.

# Inductors in Parallel




- Applying KCL to the original circuit:

$$i_s = \sum_{n=1}^{N} i_n = \sum_{n=1}^{N} \left[\frac{1}{L_n}\int_{t_0}^{t} v\,dt' + i_n(t_0)\right]$$
$$= \left(\sum_{n=1}^{N} \frac{1}{L_n}\right)\int_{t_0}^{t} v\,dt' + \sum_{n=1}^{N} i_n(t_0) \qquad [1]$$

- For the equivalent circuit we have:

$$i_s = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,dt' + i_s(t_0) \qquad [2]$$

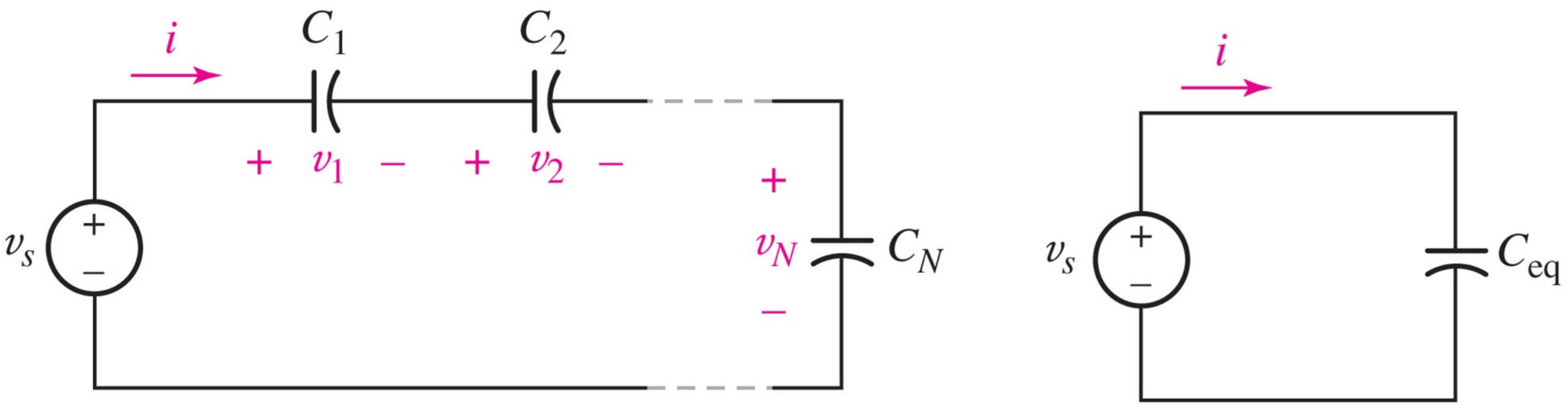
# Inductors in Parallel

- According to KCL, $i_s(t_0)$ is equal to the sum of branch currents at $t_0$, thus,

$$L_{eq} = \frac{1}{1/L_1 + 1/L_2 + \cdots + 1/L_N} \qquad [3]$$

- This is also exactly the same result we obtained for the resistor in parallel.

# Capacitors in series




- Applying KVL to the original circuit:

$$v_s = \sum_{n=1}^{N} v_n = \sum_{n=1}^{N} \left[ \frac{1}{C_n} \int_{t_0}^{t} i\,dt' + v_n(t_0) \right]$$
$$= \left( \sum_{n=1}^{N} \frac{1}{C_n} \right) \int_{t_0}^{t} i\,dt' + \sum_{n=1}^{N} v_n(t_0) \qquad [1]$$

- For the equivalent circuit we have:

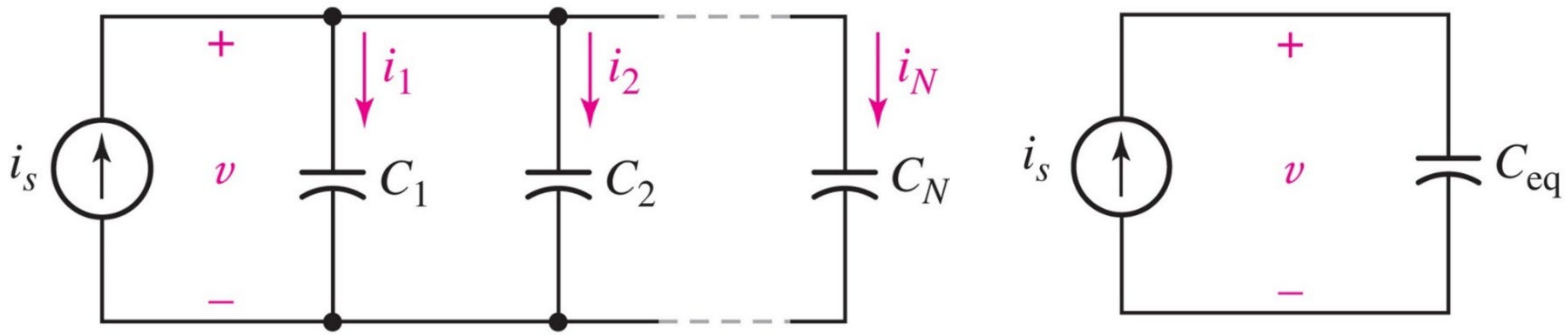
$$v_s = \frac{1}{C_{eq}} \int_{t_0}^{t} i\,dt' + v_s(t_0) \qquad [2]$$

# Capacitors in series

- According to KVL, $v_s(t_0)$ is equal to the sum of capacitor voltages at $t_0$, thus,

$$C_{\text{eq}} = \frac{1}{1/C_1 + 1/C_2 + \cdots + 1/C_N}$$ [3]

# Capacitors in parallel




$$C_{eq} = C_1 + C_2 + \cdots + C_N \qquad [1]$$

- $C_{eq}$ can be found by applying KCL and equation $i = C\frac{dv}{dt}$.

# Two-element Shortcuts

- Two capacitors in series:

$$C_{eq} = \frac{c_1 c_2}{c_1 + c_2}$$

- Two inductors in parallel:

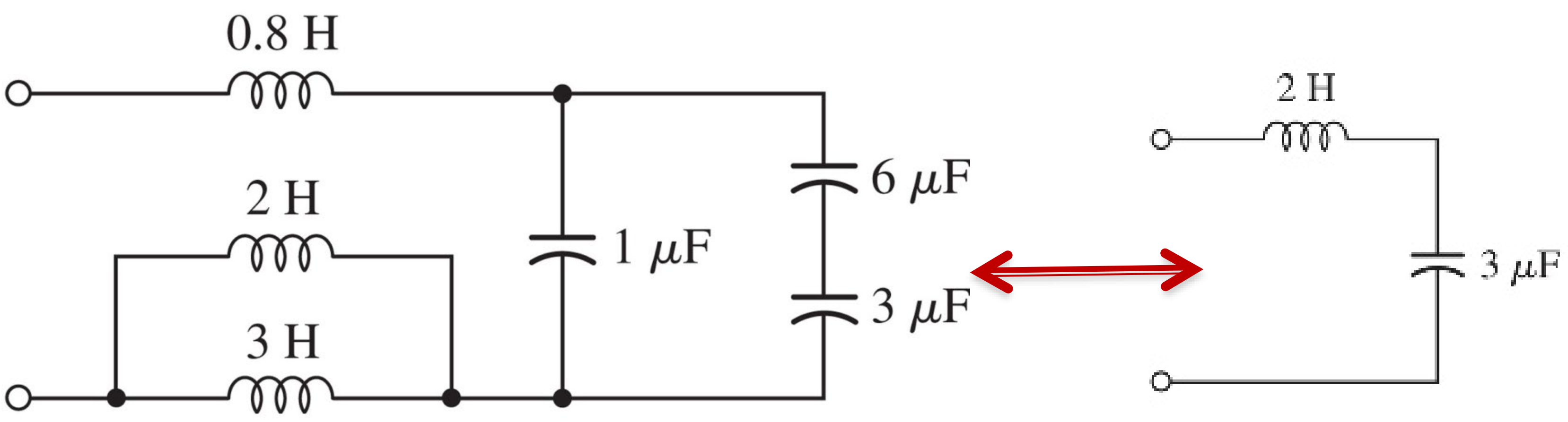
$$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$$

- Two resistors in parallel:

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

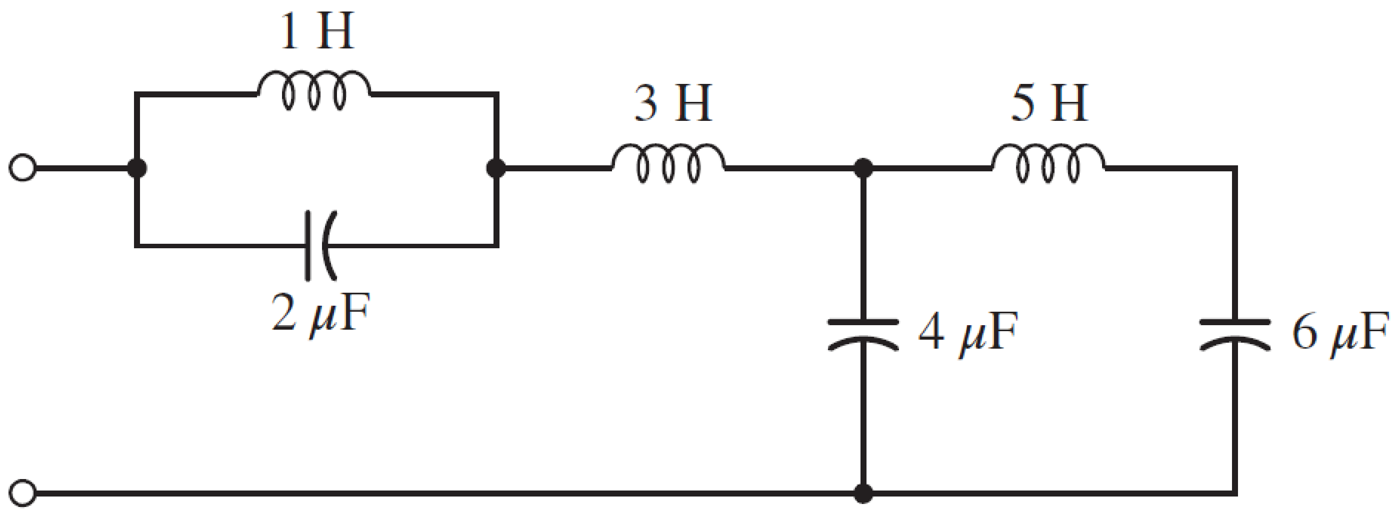
# Example 6

Simplify the network shown below.



# Example 7

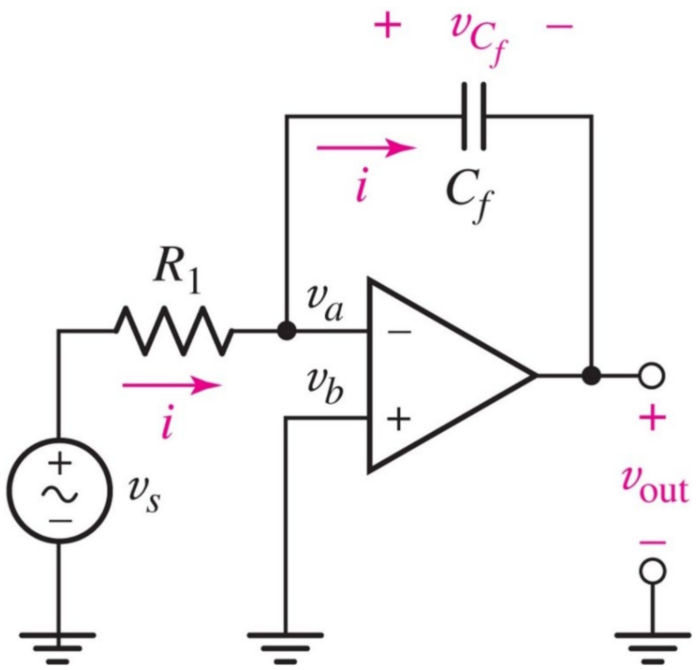
Simplify the network shown below.




Solution: cannot be simplified 😂.

# Op Amp Integrator

- Incorporating energy storage elements (L or C) with op amp circuits can provide important time-varying functions.
- Solve $V_{out}$, use KVL, KCL, and op amp rules.




- According to op amp rule 1:

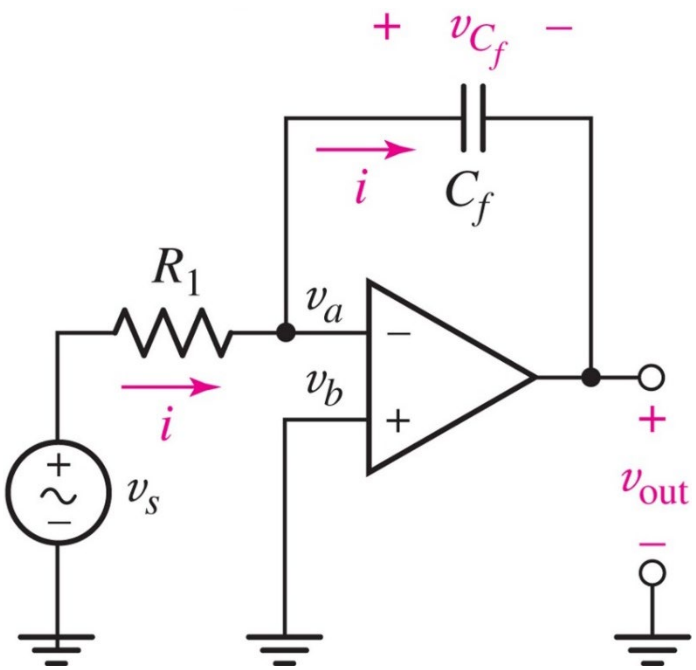
$$i_{c_f}(t) = i = \frac{v_s - v_a}{R_1} \quad [1]$$

- According to op amp rule 2, we know that $v_a = v_b = 0$, so

$$v_{c_f}(t) = -v_{out}(t)$$
$$= \frac{1}{C_f}\int_{t_0}^{t} i_{C_f}(t')dt' + v_{c_f}(t_0) \quad [2]$$

$$i_{c_f}(t) = \frac{v_s - v_a}{R_1} = \frac{v_s}{R_1} \quad [3]$$

# Op Amp Integrator



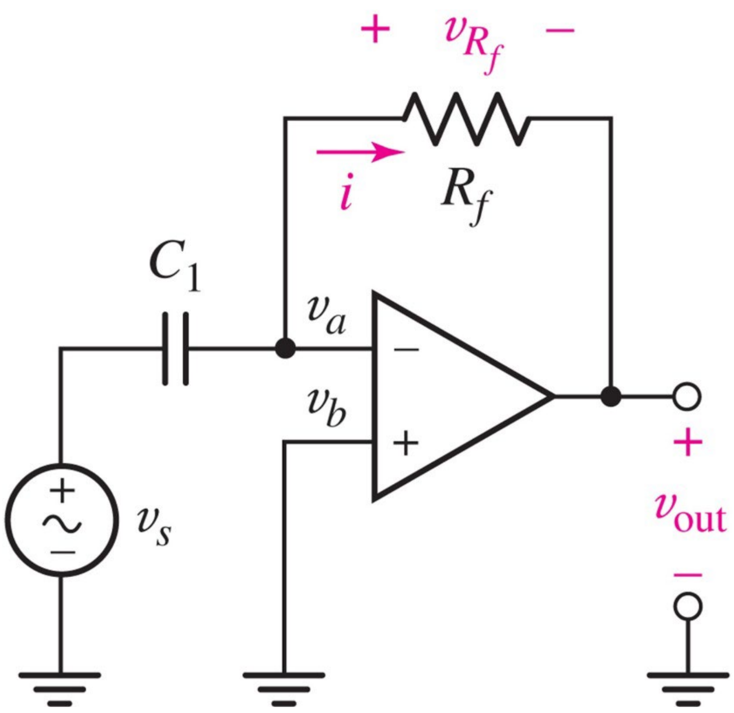

- Substituting Eq.[3] into Eq.[2], and assuming $t_0 = 0$,

$$v_{out}(t) = -\frac{1}{R_1 C_f}\int_0^t v_s(t')dt' - v_{c_f}(0)$$

- We therefore have combined a resistor, a capacitor and an op amp to form an integrator.

# Op Amp Differentiator

- Solve $V_{out}$, use KVL, KCL, and op amp rules.




- According to op amp rule 1:

$$C_1 \frac{dv_{c_1}}{dt} = \frac{v_a - v_{out}}{R_f} \qquad [1]$$

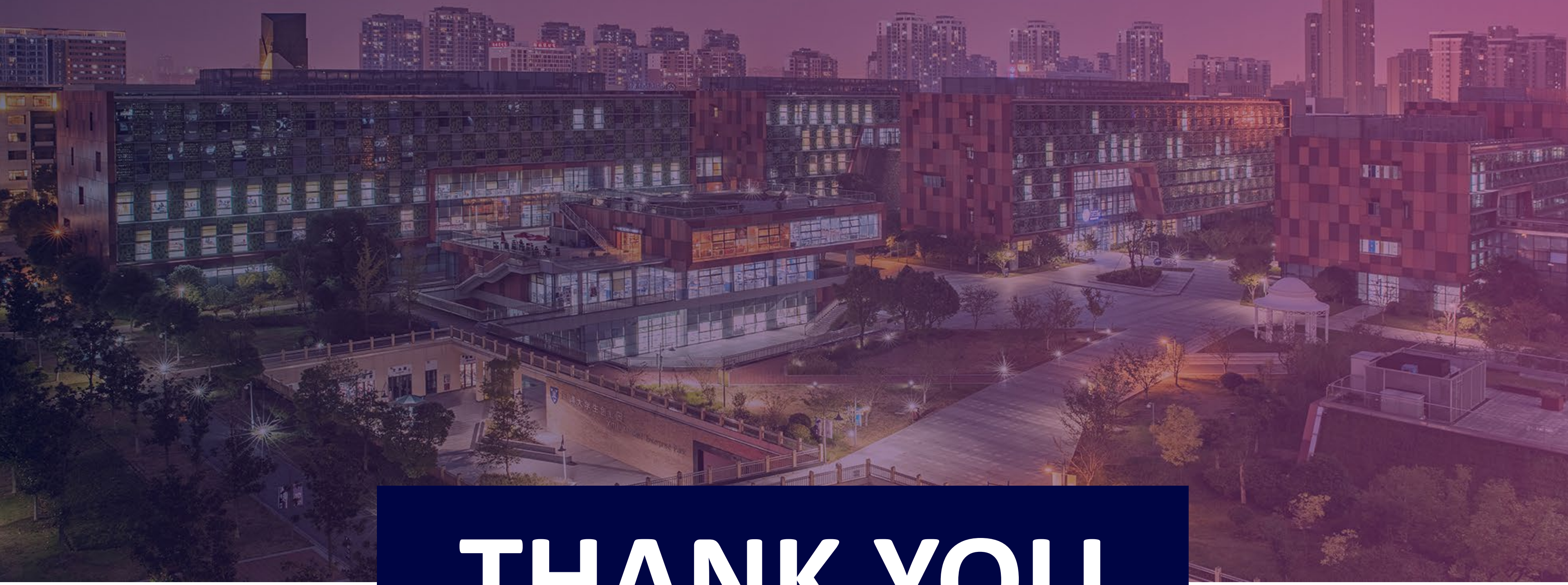
- According to op amp rule 2, $v_a = v_b = 0$, so

$$C_1 \frac{dv_{c_1}}{dt} = \frac{-v_{out}}{R_f} \qquad [2]$$

- Since $v_c = v_s - v_a = v_s$, solving $v_{out}$,

$$v_{out} = -R_f C_1 \frac{dv_s}{dt} \qquad [3]$$

- We therefore have combined a resistor, a capacitor and an op amp to form an differentiator.



# THANK YOU

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