# MTH102 Solution to Tutorial 04
# Discrete random variables

**Question 1**

Let $X$ be the number of accidents per week in a factory. Let the pmf of $X$ be

$$p(x) = \frac{c}{(x+1)(x+2)}, \; x = 0, 1, 2, \ldots.$$

(a) Determine the constant $c$.

(b) Find the conditional probability of $X \geq 4$, given that $X \geq 1$.

**Answer:**

(a)

$$1 = \sum_{x=0}^{\infty} \frac{c}{(x+1)(x+2)} = c \sum_{x=0}^{\infty} \left( \frac{1}{x+1} - \frac{1}{x+2} \right) = c.$$

(b)

$$P(X \geq 4) = \sum_{x=4}^{\infty} \frac{1}{(x+1)(x+2)} = \sum_{x=4}^{\infty} \left( \frac{1}{x+1} - \frac{1}{x+2} \right) = \frac{1}{5}.$$

$$P(X \geq 1) = \sum_{x=4}^{\infty} \frac{1}{(x+1)(x+2)} = \sum_{x=1}^{\infty} \left( \frac{1}{x+1} - \frac{1}{x+2} \right) = \frac{1}{2}.$$

Hence,

$$P(X \geq 4 | X \geq 1) = \frac{P(X \geq 4)}{P(X \geq 1)} = \frac{2}{5}.$$

**Question 2**

Let $X$ equal the number of flips of a fair coin that are required to observe the same face on consecutive flips.

(a) Find the pmf of $X$.

(b) Find the cdf of $X$.

(c) Find $P(X \geq 2021)$.

**Answer:**

(a) For $k = 2, 3, \ldots$,

$$P(X = k) = 2 \cdot \left(\frac{1}{2}\right)^k = \frac{1}{2^{k-1}}.$$

(b) For $k = 2, 3, \ldots$,

$$P(X \leq k) = \sum_{i=2}^{k} \frac{1}{2^{i-1}} = 1 - \frac{1}{2^{k-1}}.$$

Therefore, the cdf is

$$F(x) = \begin{cases} 0 & x < 2, \\ 1 - \frac{1}{2^{k-1}} & k \leq x < k+1, \ k = 2, 3, \ldots. \end{cases}$$

(c)

$$P(X \geq 2021) = 1 - P(X \leq 2020) = 1 - F(2020) = \frac{1}{2^{2019}}.$$

**Question 3**

Put 2020 balls into 2021 boxes. Find the expected number of empty boxes.

**Answer:**

For $i = 1, 2, \ldots, 2021$, let $X_i$ be defined as

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th box is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P(X_i = 1) = \frac{2020^{2020}}{2021^{2020}}, \ E(X_i) = \frac{2020^{2020}}{2021^{2020}}.$$

Let $X$ be the number of empty boxes, then

$$X = X_1 + X_2 + \cdots + X_{2021}.$$

Hence

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_{2021}) = \frac{2020^{2020}}{2021^{2019}}.$$

**Question 4**

Suppose that the percentage of drivers who are multitaskers (e.g., talk on cell phones, eat a snack, or text message at the same time they are driving) is approximately 80%. In a random sample of 20 drivers, let $X$ be the number of multitaskers.

(a) How is $X$ distributed?

(b) Give the values of the mean, variance, and the standard deviation of $X$.

**Answer:**

(a) $X$ has a binomial distribution with parameters $(20, 0.8)$.

(b)

$$E(X) = 20 \cdot 0.8 = 16, \ Var(X) = 20 \cdot 0.8 \cdot 0.2 = 3.2, \ \sqrt{Var(X)} = \frac{4}{5}\sqrt{5}.$$

**Question 5**

An excellent free-throw shooter attempts several free throws until she misses. If $p = 0.9$ is her probability of making a free throw, what is the probability of having the first miss on the 13th attempt or later?

**Answer:**

Let $X$ be the number of attempts. Then $X$ has a geometric distribution with parameter $1 - p$. Then

$$P(X \geq 13) = 1 - P(X \leq 12) = 0.9^{12}.$$

**Question 6**

A hospital obtains 40% of its flu vaccine from Company A, 50% from Company B, and 10% from Company C. From past experience, it is known that 3% of the vaccines from A are ineffective, 2% of the vaccines from B are ineffective, and 5% of the vaccines from C are ineffective. A shipment from one company arrives at the hospital, and the hospital tests five vaccines from this shipment. If at least one of the five is ineffective, find the conditional probability that this shipment is from Company C.

**Answer:**

Let $A$ be the event that at least of the five vaccines is ineffective, $B_1$ be the event that the shipment is from Company A, $B_2$ be the event that the shipment is from Company B, and $B_3$ be the event that the shipment is from Company C. Hence

$$P(B_1) = 0.4,\ P(B_2) = 0.5,\ P(B_3) = 0.1,$$

$$P(A|B_1) = 1-(1-0.03)^5,\ P(A|B_2) = 1-(1-0.02)^5,\ P(A|B_3) = 1-(1-0.05)^5.$$

By Bayes' rule,

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = 0.1779.$$

**Question 7**

Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume a Poisson distribution, find the probability of at most one flaw appearing in 225 feet.

**Answer:**

Let $X$ be the number of flaws in 225 feet. Then $X$ has a Poisson distribution with parameter

$$\lambda = E(X) = \frac{225}{150} = 1.5.$$

Therefore,

$$P(X \leq 1) = P(X = 0) + P(X = 1) = (1 + \lambda)e^{-\lambda} = 2.5e^{-1.5}.$$

**Question 8**

The number of students visiting the library per day follows a Poisson distribution with mean $\lambda$. The probability that each student borrows books is $p$, and the students borrow books independently.

(a) If there are $n$ students having visited the library on one day, find the conditional probability that there are $k$ of them who have borrowed books.

(b) Determine the distribution of the number of students borrowing books per day.

**Answer:**

For $n = 0, 1, 2, \ldots$, let $B_n$ be the event that there are $n$ students having visited the library per day. Let $X$ be the number of students who have borrowed books per day.

(a) For $k = 0, \ldots, n$

$$P(X = k | B_n) = \binom{n}{k} p^k (1-p)^{n-k}.$$

(b) By the law of total probability, for $k = 0, 1, \ldots$

$$\begin{aligned}
P(X = k) &= \sum_{n=k}^{\infty} P(X = k | B_n) P(B_n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \frac{p^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \lambda^n}{(n-k)!} \\
&= \frac{(\lambda p)^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{(n-k)!} \\
&= \frac{(\lambda p)^k e^{-\lambda}}{k!} e^{\lambda(1-p)} \\
&= \frac{(\lambda p)^k}{k!} e^{-\lambda p}.
\end{aligned}$$

Therefore $X$ has a Poisson distribution with parameter $\lambda p$.