CAN207 Continuous and Discrete Time Signals and Systems

Lecture-13

Unilateral Laplace Transform

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  - Initial- and final-value theorems
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  - System behavior VS system transfer function
- 3. System function algebra and block diagram representations
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1.1 Definition of Unilateral Laplace Transform

- Recall:

$$\boxed{X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt}$$ bilateral LT

  - the bilateral Laplace transform is used to analyze both causal and non-causal LTIC systems;
  - In signal processing, most physical systems and signals are causal.

- Applying the causality condition, the bilateral LT reduces to

$$\boxed{\mathcal{X}(s) = \mathcal{UL}\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}dt}$$ unilateral LT

  - Important in analyzing ***causal*** systems and, particularly, systems specified by LCCDE with ***nonzero initial conditions***.
  - denoted as ULT pair: $x(t) \xleftrightarrow{\mathcal{UL}} \mathcal{X}(s) = \mathcal{UL}\{x(t)\}$

1.1 Some properties

- The lower limit of the integration is set to be 0^{-}, which is to include functions like $\delta(t)$ that is concentrated at t = 0.
- $\mathcal{UL}\{x(t)\}$ and $\mathcal{L}\{x(t)\}$ are the same if $x(t) = 0$ for $t < 0$.
- The ROC of $\mathcal{UL}\{x(t)\}$ is always a right-half plane.
- The evaluation of the inverse unilateral Laplace transforms is the same as for bilateral transforms.

1.1 Examples of ULT

- 1. Calculate the BLT and ULT for

$$x_1(t) = e^{-a(t+1)}u(t+1)$$

$$x_2(t) = \delta(t) + 2u(t) + e^{t}u(t)$$

- 2. Consider the ULT

$$X(s) = \frac{1}{(s+1)(s+2)}$$

Find its inverse transform x(t).

1.2 Differentiation Property

- Recall (in BLT), the pair $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ gives:

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$$

- In ULT, consider the pair $x(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s)$, the ULT of $\frac{dx(t)}{dt}$ is

$$\frac{dx(t)}{dt} \overset{\mathcal{UL}}{\longleftrightarrow} s\mathcal{X}(s) - x(0^-)$$

- Similarly, the second order derivative leads to:

$$\frac{d^2x(t)}{dt^2} \overset{\mathcal{UL}}{\longleftrightarrow} s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

  – $x'(0^-)$ denotes the derivative of $x(t)$ evaluated at $\mathrm{t} = 0^-$.

1.3 Initial- and final- value theorem

- For CAUSAL signal $x(t)$:
- The initial-value $x(0^+)$ of $x(t)$ can be found using the Laplace Transform as follows:

$$x(0^+) = \lim_{s\to\infty} s\mathcal{X}(s)$$

  – x(t) contains no impulses or higher order singularities at the origin;
  – $X(s)$ should be a proper rational function of s.

- The steady-state value $x(\infty)$ can be found by:

$$x(\infty) = \lim_{s\to 0} s\mathcal{X}(s)$$

  – all poles on left-side of s-plane.

Example 1

- Calculate the initial and final values of the functions $x_1(t)$, $x_2(t)$, and $x_3(t)$, whose Laplace transforms are specified below:

$$\text{(i)}\quad X_1(s) = \frac{s+3}{s(s+1)(s+2)} \text{ with ROC } R_1\text{: } \mathrm{Re}\{s\} > 0;$$

$$\text{(ii)}\quad X_2(s) = \frac{s+5}{s^3+5s^2+17s+13} \text{ with ROC } R_2\text{: } \mathrm{Re}\{s\} > -1;$$

$$\text{(iii)}\quad X_3(s) = \frac{5}{s^2+25} \text{ with ROC } R_3\text{: } \mathrm{Re}\{s\} > 0.$$

1.4 Solution of Differential equations

- In Lecture 12, we used a time-domain approach to obtain the zero-input, zero-state, and overall solution of differential equations.
- In this section, we discuss an alternative approach based on the Laplace transform.

- Lecture 12, Example 1

$$\frac{dy}{dt} + 4y(t) = \frac{dx}{dt}$$

  - initial condition $y(0^-) = 2\,V$;
  - a sinusoidal voltage $x(t) = sin(2t)u(t)$ is applied as the input.
- Find the zero-input, zero-state and overall responses.

1.4 Solution of Differential equations

- Solution 1: zero-input response
  - Assume the input $x(t) = 0$, i.e.

$$\frac{dy}{dt} + 4y(t) = 0$$

  - Taking the ULT of the above equation and substituting:

$$s\mathcal{Y}_{zi}(s) - y(0^-) + 4\mathcal{Y}_{zi}(s) = 0$$

  - which reduces to

$$\mathcal{Y}_{zi}(s) = \frac{2}{s+4}$$

  - Performing the inverse ULT results:

$$y_{zi}(t) = 2e^{-4t}u(t)$$

1.4 Solution of Differential equations

- Solution 1: zero-state response
  - Assume the initial condition $y(0^-) = 0$.
  - Taking the ULT of the above equation and substituting:

$$s\mathcal{Y}_{zs}(s) - y(0^-) + 4\mathcal{Y}_{zs}(s) = s\mathcal{X}(s) - x(0^-)$$

  - which reduces to

$$\mathcal{Y}_{zs}(s) = \frac{2s}{(s+4)(s^2+4)}$$

  - Using PFE to perform the inverset ULT, get:

$$y_{zs}(t) = [-0.4e^{-4t} + 0.4\cos 2t + 0.2\sin 2t]u(t)$$

- Overall response

$$y(t) = y_{zi}(t) + y_{zs}(t) = [1.6e^{-4t} + 0.4\cos 2t + 0.2\sin 2t]u(t)$$

1.4 Solution of Differential equations

- Solution 2: find overall response directly
  - Apply the ULT to both sides of the DE, get:

$$s\mathcal{Y}(s) - \underbrace{y(0^-)}_{=2} + 4\mathcal{Y}(s) = s\underbrace{\mathcal{X}(s)}_{=\frac{2}{s^2+4}} - \underbrace{x(0^-)}_{=0}$$

  - Rearranging it get:

$$\mathcal{Y}(s) = \frac{2s^2 + 2s + 8}{(s+4)(s^2+4)}$$

  - Using PFE to perform the inverset ULT, get:

$$y(t) = [1.6e^{-4t} + 0.4\cos 2t + 0.2\sin 2t]u(t)$$

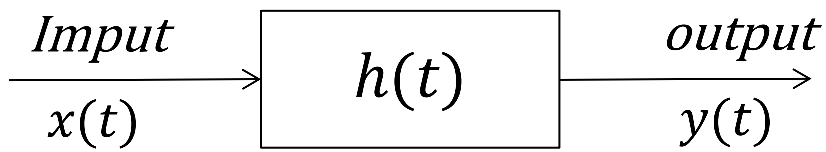
Quiz 1

- The following differential equation was used to model a RLC series circuit.

$$\frac{d^2w}{dt^2} + 7\frac{dw}{dt} + 12w(t) = 12x(t)$$

- Determine the zero-input, zero-state, and overall response of the system produced by the input $x(t) = 2e^{-t}u(t)$ given the initial conditions, $w(0^-) = 5\,V$ and $w'(0^-) = 0$.

2.1 System's representation



- Recall: in time domain, the input-output relationship can be expressed in two ways:
  - Impulse response $h(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

    - when input $x(t) = \delta(t)$, output $y(t) = \delta(t) * h(t) = h(t)$
  - LCCDE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

2.1 System's representation

- Apply LT to both expressions:
  - Impulse response $h(t)$

$$y(t) = x(t) * h(t)$$

$$Y(s) = X(s) \cdot H(s) \quad \Rightarrow \boxed{H(s)} = \frac{Y(s)}{X(s)}$$

    - verify: $\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1$, so $Y(s) = 1 \cdot H(s) = H(s)$

transfer function of
the system
or "system function"

  - LCCDE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s) \quad \Rightarrow \boxed{H(s)} = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Example 2

- Suppose we know that if the input to an LTI system is

$$x(t) = e^{-3t}u(t)$$

- then the output is

$$y(t) = [e^{-t} - e^{-2t}]\,u(t)$$

- Find the impulse response and LCCDE defining this system.

Example 3

- Suppose that we are given the following information about an LTI system:
  1. The system is causal;
  2. The system function is rational and has only two poles, at s = -2 and s = 4;
  3. If x(t) = 1, then y(t) = 0;
  4. The value of the impulse response at $t = 0^+$ is 4.
- Find the transfer function of the system.

Quiz 2

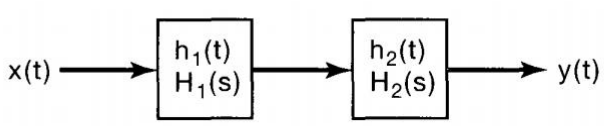
- Consider a stable and causal system with impulse response h(t) and system function H(s). Suppose H(s) is rational, contains a pole at s = -2, and does not have a zero at the origin. The location of all other poles and zeros is unknown.
- Determine whether the following statements are true, false or insufficient information to determine:
  1. $\mathcal{F}\{h(t)e^{3t}\}$ converges;
  2. $\int_{-\infty}^{\infty} h(t)dt = 0$;
  3. $t\,h(t)$ is the impulse response of a causal and stable system;
  4. $\frac{dh(t)}{dt}$ contains at least one pole in its Laplace transform;
  5. $h(t)$ has finite duration;
  6. $H(s) = H(-s)$;
  7. $\lim_{s \to \infty} H(s) = 2$.

3.1 System Functions for Interconnections

- TD differential equation $\overset{\mathcal{L}}{\longleftrightarrow}$ algebraic equation
  - It's convinient for analyzing LTIC system
  - Also important in analyzing interconnections of LTI systems and synthesizing systems as interconnections of elementary system building blocks

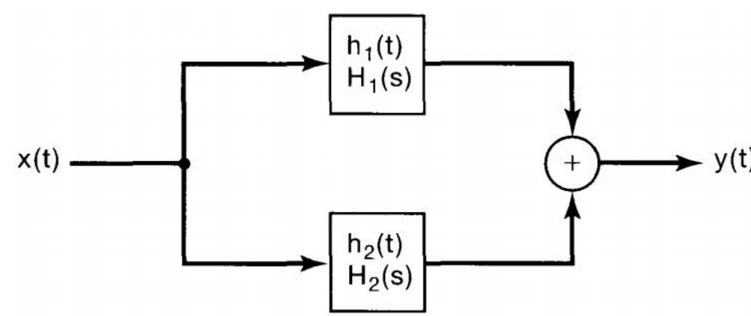
1. Series connection

$$H(s) = H_1(s) \cdot H_2(s)$$



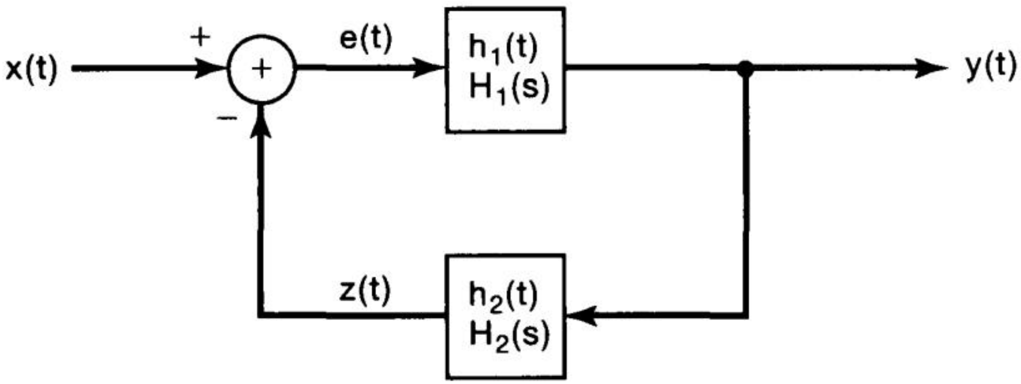
2. Parallel connection

$$H(s) = H_1(s) + H_2(s)$$



3.1 System Functions for Interconnections

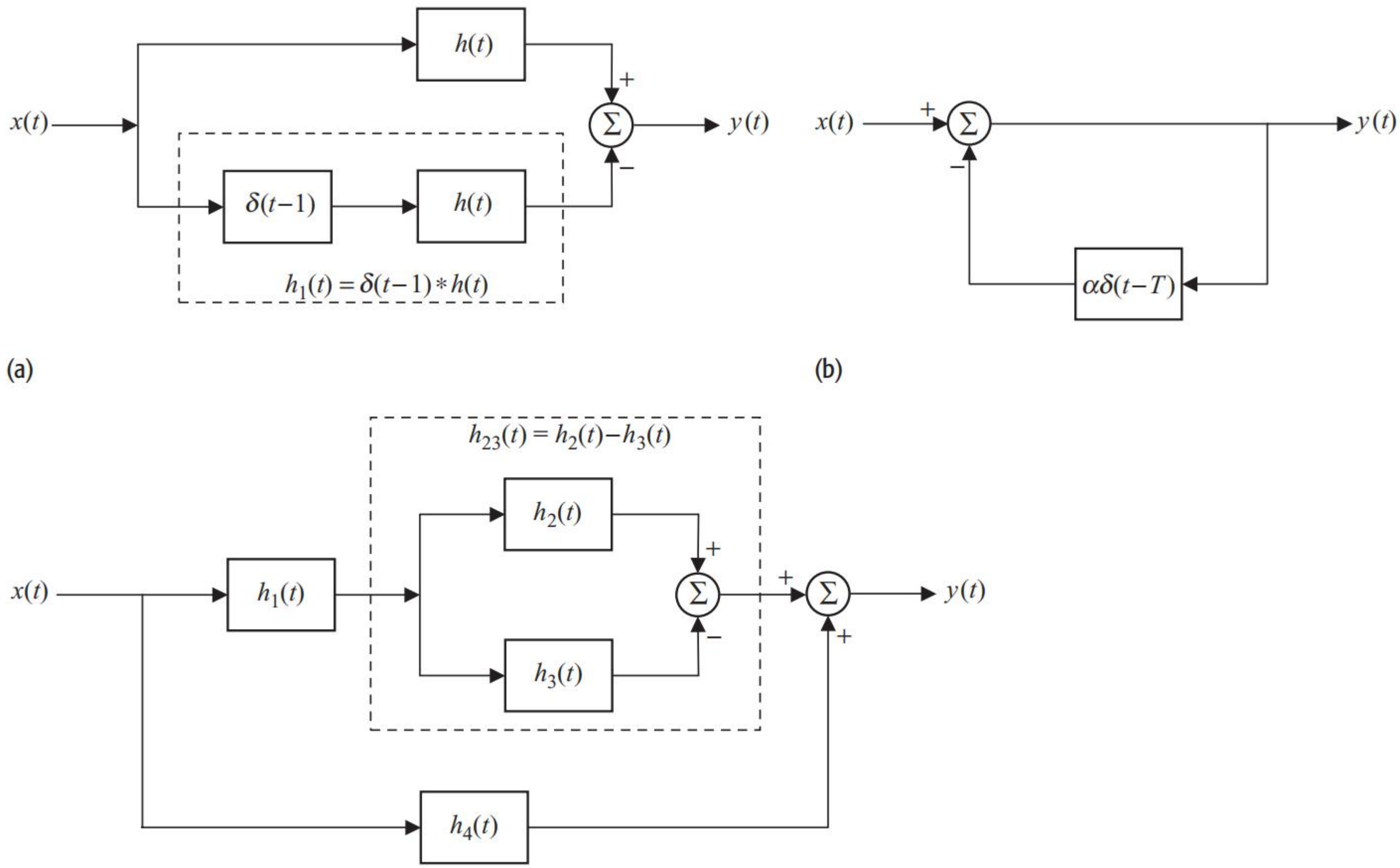
- 3. Feedback connection



$$\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}.$$

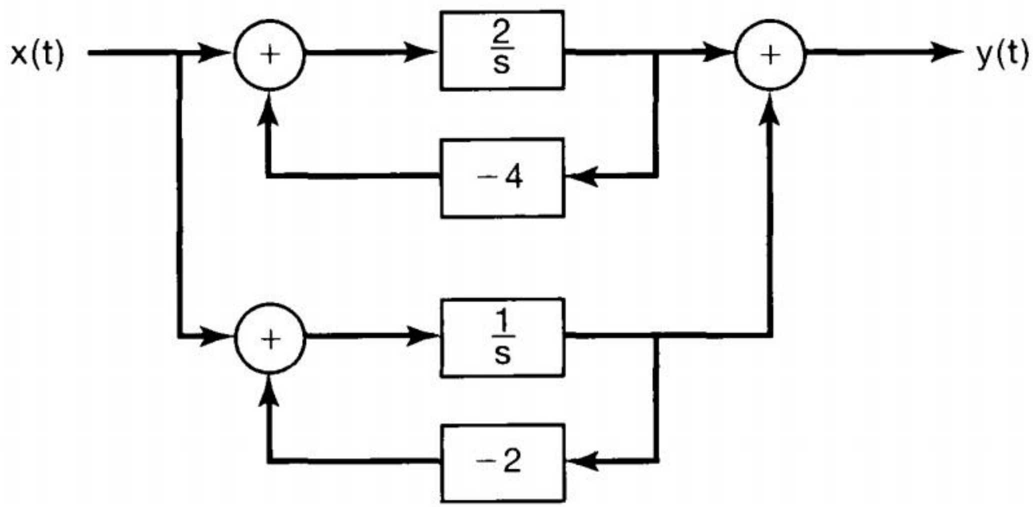
3.2 Examples

- Determine the transfer function of the interconnected systems.



Quiz 3

- A causal LTI system S has the block diagram representation shown below. Determine a differential equation relating the input x(t) to the output y(t) of this system.

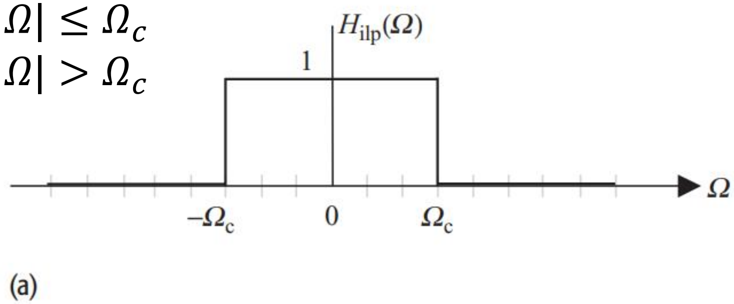


For Assignment 1: Filters

- An ideal frequency-selective filter is a system that passes a prespecified range of frequency components without any attenuation but completely rejects the remaining frequency components.
- Four types of CT filters

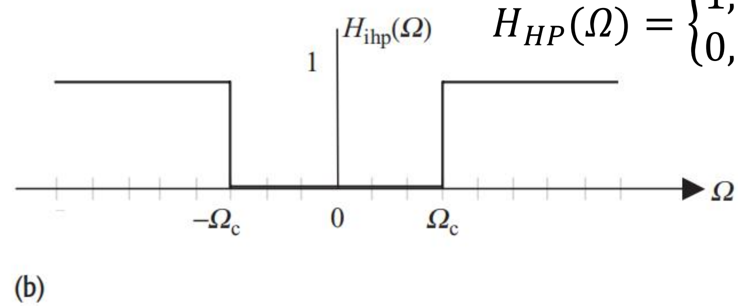
Low-pass

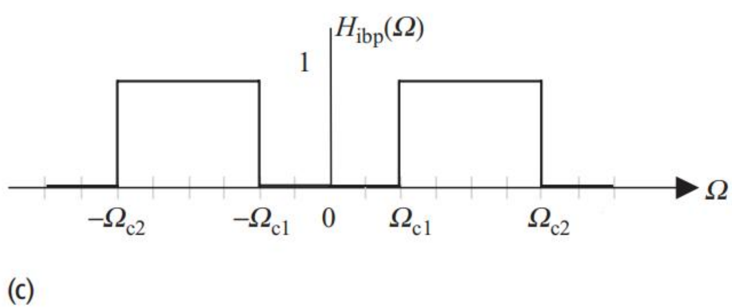
$$H_{LP}(\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

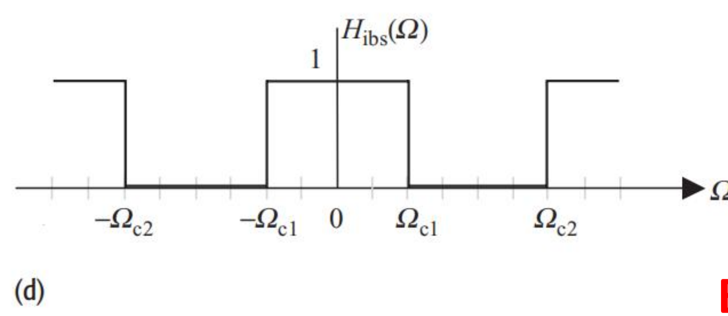


High-pass

$$H_{HP}(\Omega) = \begin{cases} 1, & |\Omega| \ge \Omega_c \\ 0, & |\Omega| < \Omega_c \end{cases}$$







Band-pass

$$H_{LP}(\Omega) = \begin{cases} 1, & \Omega_{c1} \le |\Omega| \le \Omega_{c2} \\ 0, & \text{others} \end{cases}$$

Band-stop

$$H_{LP}(\Omega) = \begin{cases} 1, & \text{others} \\ 0, & \Omega_{c1} \le |\Omega| \le \Omega_{c2} \end{cases}$$

For Assignment 1: Characteristic equation

- ***Characteristic equation*** of a system

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{j=0}^{M} b_j s^j}{\sum_{i=0}^{N} a_i s^i} \Longrightarrow Y(s)\sum_{j=0}^{N} a_j s^j = X(s)\sum_{i=0}^{M} b_i s^i$$

$$\Longrightarrow \sum_{i=0}^{N} a_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^{M} b_j \frac{d^j x(t)}{dt^j}$$

  – Time Domain: to solve the DE, considering the zero-input and $y(t) = Ae^{st}$, the equation about *s* is the ***characteristic equation***:

$$\sum_{i=0}^{N} a_i \frac{d^i Ae^{st}}{dt^i} = Ae^{st}\sum_{i=0}^{N} a_i s^i = 0 \Longrightarrow \sum_{i=0}^{N} a_i s^i = 0$$

    - which is the same as the denominator polynomial of H(s).

  – Frequency Domain: the denominator polynomial of *s* decides the pole locations, i.e. the stability of the system.

Next ...

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