MTH102 Solution to Tutorial 03 Conditional probability & independence

Question 1

Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

Answer:

Let A be the event that at least one lands on 6 and B be the event that the dice land on different numbers. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{10/36}{30/36} = \frac{1}{3}.$$

Question 2

52 percent of the students at a certain college are females. 5 percent of the students in this college are majoring in computer science. 2 percent of the students are women majoring in computer science. If a student is selected at random, find the conditional probability that

(a) the student is female given that the student is majoring in computer science;

(b) this student is majoring in computer science given that the student is female.

Answer:

Let F be the event that the student is female and C be the event that the student is majoring in computer science. Then

$$P(F) = 0.52,\ P(C) = 0.05,\ P(F \cap C) = 0.02.$$

(a)

$$P(F|C) = \frac{P(F \cap C)}{P(C)} = \frac{2}{5}.$$

(b)

$$P(C|F) = \frac{P(F \cap C)}{P(F)} = \frac{1}{26}.$$

Question 3

Let A_1 and A_2 be the events that a person is left-eye dominant or right-eye dominant respectively. When a person folds his or her hands, let B_1 and B_2 be the events that the right thumb and the left thumb, respectively, are on top. A survey in one statistics class yielded the following table:

	B_1	B_2	Total
A_1	10	15	25
A_2	25	10	35
Total	35	25	60

If a student is selected randomly, find the following probabilities:

(a) $P(A_2|B_1)$ and $P(B_2|A_1)$.

(b) If the students had their hands folded and you hoped to select a left-eye dominant student, would you select a "right thumb on top" or a "left thumb on top" student? Why?

Answer:

(a)

$$P(A_2|B_1) = \frac{P(A_2 \cap B_1)}{P(B_1)} = \frac{25}{35} = \frac{5}{7},$$
$$P(B_2|A_1) = \frac{P(B_2 \cap A_1)}{P(A_1)} = \frac{15}{25} = \frac{3}{5}.$$

(b)

$$P(A_1|B_1) = \frac{P(A_1 \cap B_1)}{P(B_1)} = \frac{10}{35} = \frac{2}{7},$$
$$P(A_1|B_2) = \frac{P(A_1 \cap B_2)}{P(B_2)} = \frac{15}{25} = \frac{3}{5}.$$

Since $P(A_1|B_1) < P(A_1|B_2)$, it is better to choose a "left thumb on top" student.

Question 4

In each game, the player Wu Lei scores a goal with probability 0.4 if he receives an assist, with probability 0.1 if he takes a free kick and with probability 0.9 if he takes a penalty kick. Suppose that he receives an assist with probability 0.6, he is awarded a free kick with probability 0.3 and a penalty kick with probability 0.1.

(a) What is the probability that Lei scores a goal in one shooting?

(b) If Lei has scored a goal in one shooting, what is the conditional probability that it is from a free kick?

Answer:

Let A be the event that Lei scores a goal in one shooting, B_1 be the event that Lei receives an assist, B_2 be the event that Lei takes a free kick and B_3 be the event that Lei takes a penalty kick. Thus,

$$P(B_1) = 0.6,\ P(B_2) = 0.3,\ P(B_3) = 0.1,$$

$$P(A|B_1) = 0.4,\ P(A|B_2) = 0.1,\ P(A|B_3) = 0.9.$$

(a) By the law of total probability

$$P(A) = \sum_{i=1}^{3} P(B_i)P(A|B_i) = 0.24 + 0.03 + 0.09 = 0.36.$$

(b) By Bayes' rule

$$P(B_2|A) = \frac{P(B_2)P(A|B_2)}{P(A)} = \frac{0.03}{0.36} = \frac{1}{12}.$$

Question 5

A boy has five blue and four white marbles in his left pocket and four blue and five white marbles in his right pocket. He transfers one marble at random from his left to his right pocket, and then draw one marble from his right pocket.

(a) What is the probability that it is a blue marble?

(b) If it is a blue marble, what is the conditional probability that he has transferred one blue marble from his left to his right pocket?

Answer:

Let A be the event that it is a blue marble, B_1 be the event that he has transferred one blue marble from his left to his right pocket, and B_2 be the event that he has transferred one white marble from his left to his right pocket. Then

$$P(B_1) = \frac{5}{9},\ P(B_2) = \frac{4}{9},\ P(A|B_1) = \frac{5}{10},\ P(A|B_2) = \frac{4}{10}.$$

(a) By the law of total probability,

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) = \frac{41}{90}.$$

(b) By Bayes' rule,

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(A)} = \frac{25}{41}.$$

Question 6

On rainy days, Joe is late to work with probability 0.3; on nonrainy days, he is late with probability 0.1. With probability 0.7, it will rain tomorrow.

(a) Find the probability that Joe is early tomorrow.

(b) Given that Joe was early, what is the conditional probability that it rained?

Answer:

Let A be the event that Joe is early, B_1 be the event that it is a rainy day, and B_2 be the event that it is a nonrainy day. Therefore,

$$P(B_1) = 0.7,\ P(B_2) = 0.3,\ P(A|B_1) = 1 - 0.3 = 0.7,\ P(A|B_2) = 1 - 0.1 = 0.9.$$

(a) By the law of total probability,

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) = 0.76.$$

(b) By Bayes' rule,

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(A)} = \frac{49}{76}.$$

Question 7

Three independent reviewers are reviewing a book. Let A_i denote the event that a favorable review is submitted by reviewer i, $i = 1, 2, 3$. Assume that A_1, A_2, A_3 are mutually independent and that $P(A_1) = 0.6$, $P(A_2) = 0.57$ and $P(A_3) = 0.4$.

(a) Compute the probability that at least one of the reviewers submits a favorable review.

(b) Compute the probability that exactly two reviewers submit favorable reviews.

Answer:

(a) The desired event is $A_1 \cup A_2 \cup A_3$. Therefore,

$$P(A_1 \cup A_2 \cup A_3) = 1 - P(A_1^c \cap A_2^c \cap A_3^c).$$

Since A_1, A_2, A_3 are mutually independent, then

$$P(A_1 \cup A_2 \cup A_3) = 1 - P(A_1^c)P(A_2^c)P(A_3^c) = 1-(1-0.6)(1-0.57)(1-0.4) = 0.8968.$$

(b) The desired event is $A_1A_2A_3^c \cup A_1A_2^cA_3 \cup A_1^cA_2A_3$ and it is a union of mutually exclusive events. Therefore,

$$\begin{aligned}
& P(A_1A_2A_3^c \cup A_1A_2^cA_3 \cup A_1^cA_2A_3) \\
= \; & P(A_1A_2A_3^c) + P(A_1A_2^cA_3) + P(A_1^cA_2A_3) \\
= \; & P(A_1)P(A_2)P(A_3^c) + P(A_1)P(A_2^c)P(A_3) + P(A_1^c)P(A_2)P(A_3) \\
= \; & 0.6 \times 0.57 \times (1-0.4) + 0.6 \times (1-0.57) \times 0.4 + (1-0.6) \times 0.57 \times 0.4 \\
= \; & 0.3996.
\end{aligned}$$

Question 8

Two hunters shoot at a target with probabilities of p_1 and p_2, respectively. Assuming independence, can p_1 and p_2 be selected so that P(zero hits)=P(one hit)=P(two hits)?

Answer: We have

$$P(\text{zero hits}) = (1-p_1)(1-p_2),$$

$$P(\text{one hit}) = p_1(1-p_2) + p_2(1-p_1),$$

$$P(\text{two hits}) = p_1p_2.$$

The condition P(zero hits)=P(one hit)=P(two hits) implies that

$$p_1 + p_2 = 1 \text{ and } p_1p_2 = \frac{1}{3},$$

which have no real solutions.