Xi'an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

2024-25 Semester 2

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Lecture 13

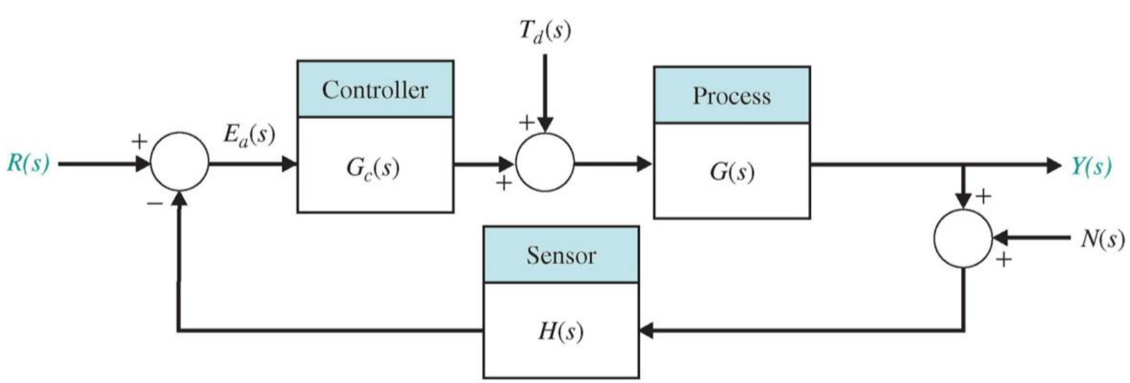
Outline

Time-Domain Performance of Feedback Control

- ❑ Test Input Signals
- ❑ Performance of First-Order and Second-Order System
- ❑ Effects of a Third Pole and a Zero on the Second-Order System Response
- ❑ Pole Location on the *s*-plane and the Transient Response
- ❑ Steady-State Error of Feedback Control Systems
- ❑ System Simulation Using Matlab

Overview

- Ability to control and adjust **transient** and **steady-state** responses of a controlled system is a distinctive advantage of feedback control systems;
- To analyze and design a control system, we often define and measure its performance. The controller parameters can be adjusted to provide the desired response defined by **design specifications**.
- Control systems often deal with the dynamical system. The performance is usually specified in terms of transient and steady-state responses:
  - **Transient response** is the response that disappears with time;
  - **Steady-state response** is the response that exists for a long time following an input signal initiation.



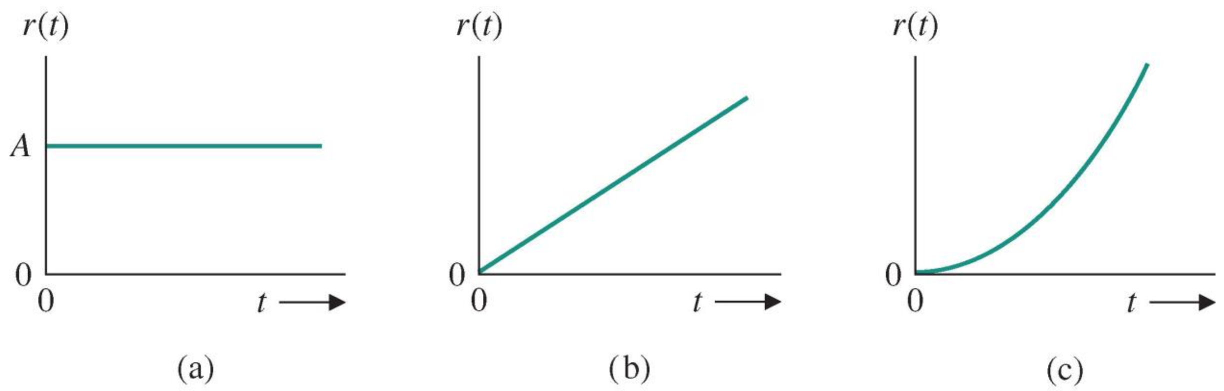
Closed-loop System

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_c G}{1 + G_c G H}$$

Test Input Signals

Control system operates physically in time domain systems, so the transient response is often the response of prime interest in time-critical systems:

- Is the system stable?
- If stable, how to measure and compare the performance of several competing designs?
  - Test signal provides some **measures of performance** (response time, percent overshoot, etc.)
  - Three types of standard test input signals:



Three Standard Test Input Signals

Standard Performance Measures

- To quantify the time performance/behavior subject to a unit step input:
  - Delay time (T_d) is the time required for the response to reach half the final value the very first time.
  - Rise time (T_r) is the time required for the system's output step response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
  - Peak time (T_p) is the time required for the response to reach the first peak of the overshoot.
  - Percent/Maximum overshoot (M_p%) is the maximum peak value of the response curve measured from unity.

$$\frac{x(T_p) - x(\infty)}{x(\infty)} \times 100\%$$

  - Settling time (T_s) is the time required for a system's output step response to settle within some specified band of values, usually ±2% or ±5% about its final value.
- To find the above parameters, one needs to know the inputs. The standard way is to use unit step input.

Test Input Signal in Time- and *s*-Domain (1)

- There is a reasonable correlation between the response of a system to standard test input and the system's ability to perform under normal operating conditions.
- Many control systems experience/take in input signals that are very similar to the standard test signals.

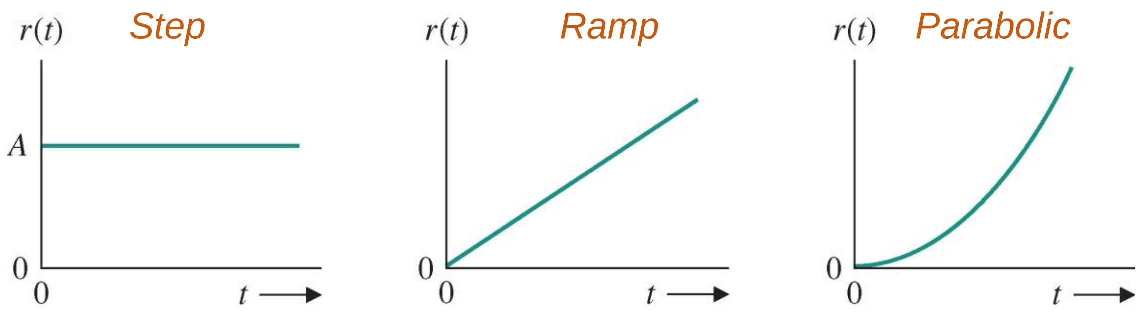


Table 5.1 Test Signal Inputs

Test Signal	$r(t)$	$R(s)$
Step	$r(t) = A, t > 0$ $= 0, t < 0$	$R(s) = A/s$
Ramp	$r(t) = At, t > 0$ $= 0, t < 0$	$R(s) = A/s^2$
Parabolic	$r(t) = At^2, t > 0$ $= 0, t < 0$	$R(s) = 2A/s^3$

Recall from the Laplace Transform table:

$$\boldsymbol{r(t) = t^n}$$

$$\boldsymbol{R(s) = \frac{n!}{s^{n+1}}}$$

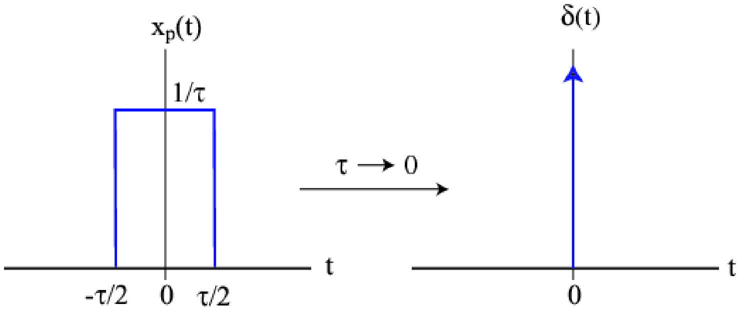
Note: this is a continuous-time impulse.

Test Input Signal in Time- and *s*-Domain (2)

- Some of the real-life inputs should be more precisely approximated as impulse signals, e.g., a hammering impact on a nail, a short voltage pulse into a circuit, etc.

- Unit-impulse signal is another standard test input used rather frequently in the control system study:

Unit impulse



Unit-impulse input in the time-domain:

$$r(t) = \delta(t)$$

where

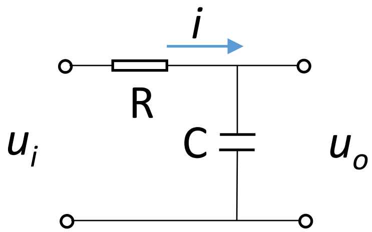
$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases} \qquad \int_{-\infty}^{+\infty} \delta(t).dt = 1$$

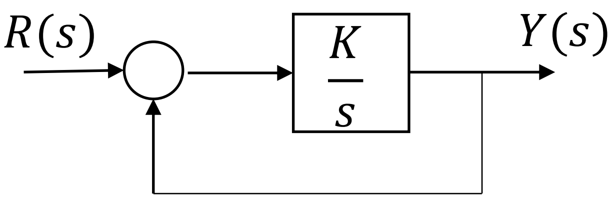
Unit-impulse input in the s-domain:

$$R(s) = 1$$

First-Order system

- Consider a system





$$u_i = Ri + u_C$$

$$i = C\frac{du_C}{dt}$$

$$\Rightarrow \quad U_o = U_C = \frac{1}{RCs+1}U_i$$

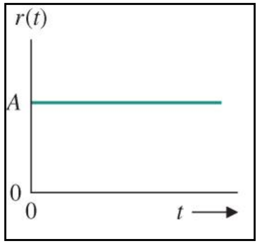
$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s}}{1+\frac{K}{s}} = \frac{K}{s+K}$$

- The **normalized form** is :

$$\frac{Y(s)}{R(s)} = \frac{1}{Ts+1} \qquad T\text{– Time constant}$$

Time Response to Unit-Step Input

$$R(s) = \frac{1}{s}$$



A=1

$$Y(s) = \frac{1}{Ts+1}\frac{1}{s} = \frac{1}{s} - \frac{T}{Ts+1}$$

$$y(t) = 1 - e^{-t/T}$$

$$y(0) = 0$$

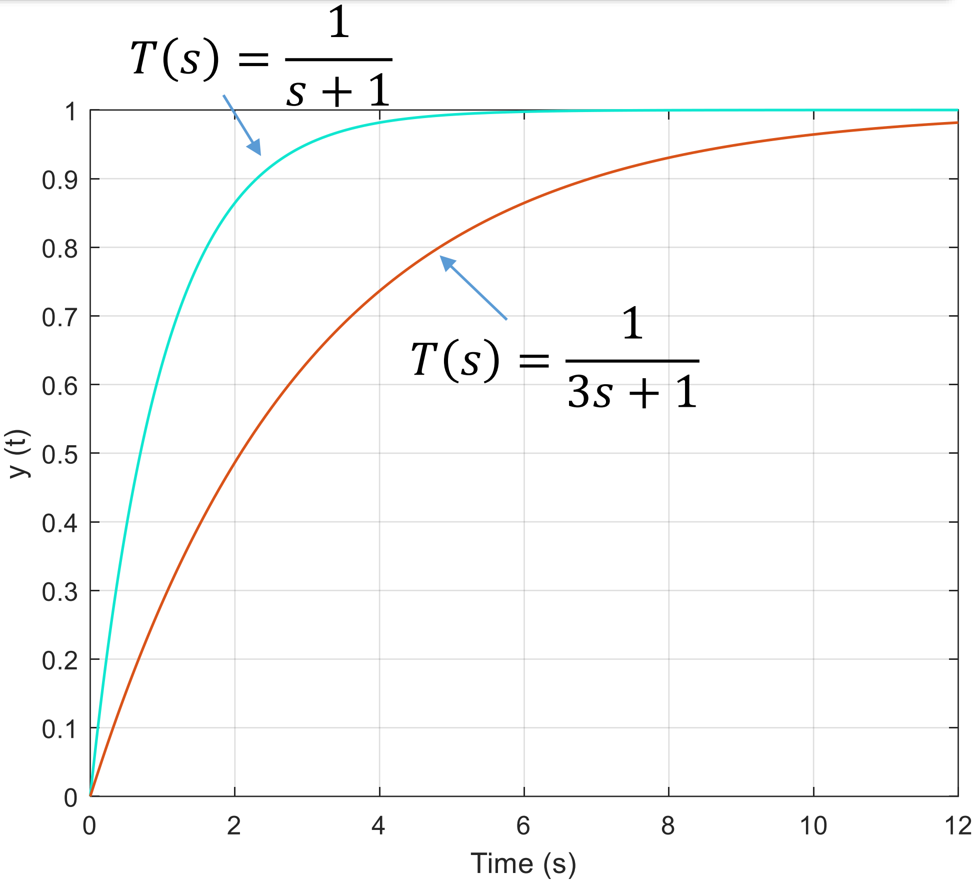
$$y(\infty) = 1$$

$$\dot{y}(0) = 1/T$$

$$y(T) = 0.632$$

$$y(3T) = 0.95$$

$$y(4T) = 0.982$$



Second-Order system

- All linear, time-invariant second-order (and single-loop) systems can be represented by:

$$r(t) = \frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t)$$

$$R(s) = s^2Y(s) + 2\zeta\omega_n sY(s) + \omega_n^2 Y(s)$$

$$\frac{Y(s)}{R(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n – Natural frequency

ζ – Damping ratio/coefficient (zeta)

Q – Quality factor, $\frac{1}{2\zeta}$

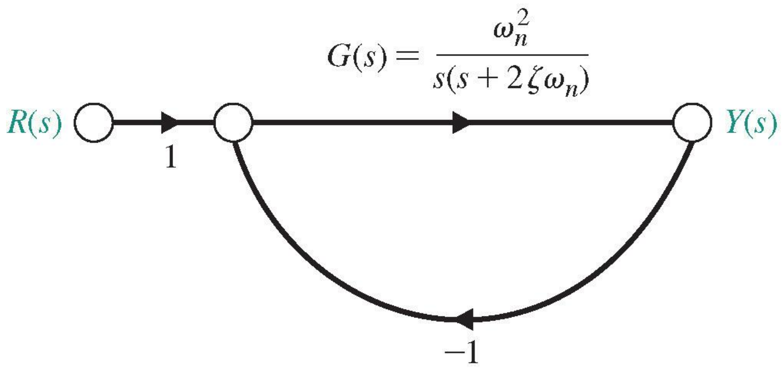
- The **normalized form** is :

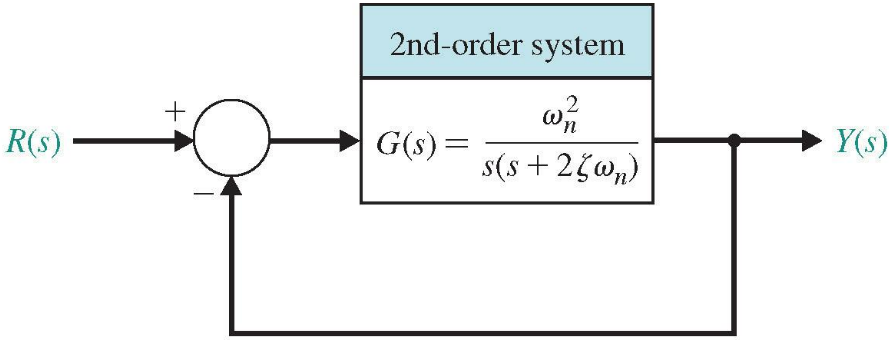
$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- For the normalized form, a simple check using FVT will show that a unit-step input $R(s)$ will result in one unit of output $Y(s)$.

An Alternative Way to View 2nd Order System

$$Y(s) = \frac{G(s)}{1+G(s)}R(s)$$



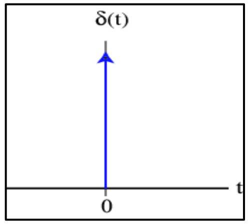


(Just another way of drawing a normalized second-order system but with an inner unity negative feedback path)

$$\boxed{Y(s) = \frac{{\omega_n}^2}{s^2+2\zeta\omega_n s+{\omega_n}^2}R(s)}$$

Time Response to Unit-Impulse Input

$$R(s) = 1$$



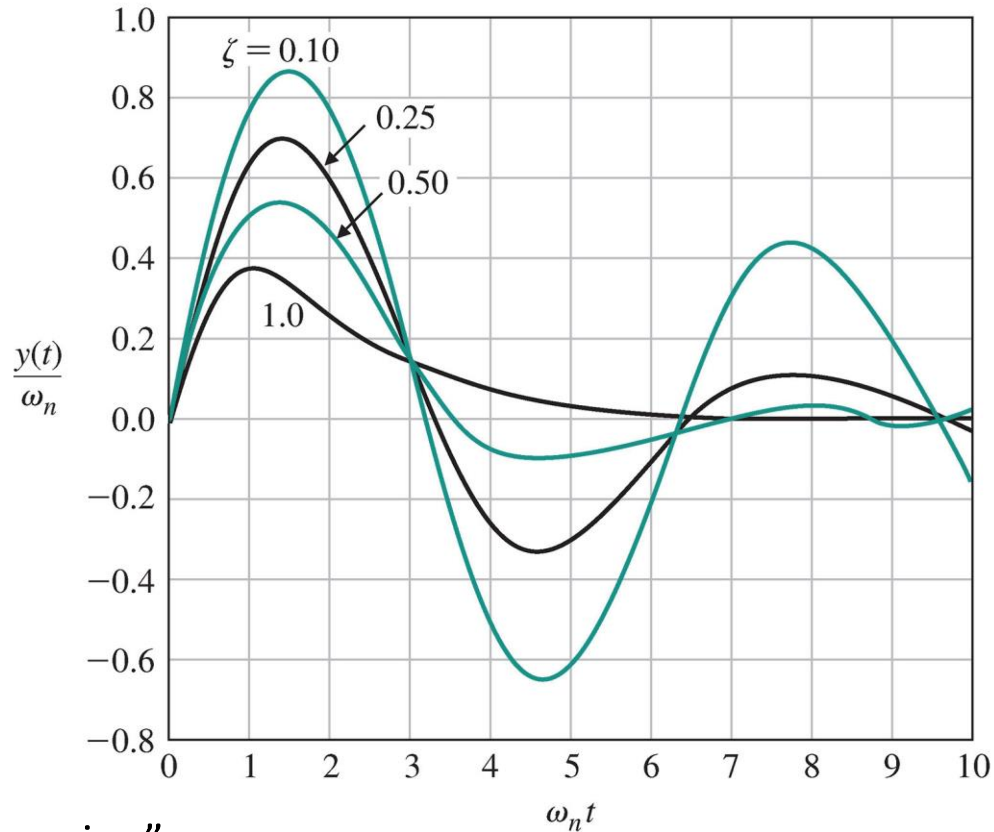
$$Y(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$$\boxed{y(t) = \frac{\omega_n}{\beta} e^{-\zeta\omega_n t} sin(\omega_n \beta t)}$$

where $\beta = \sqrt{1-\zeta^2}$, $0 < \zeta < 1.$

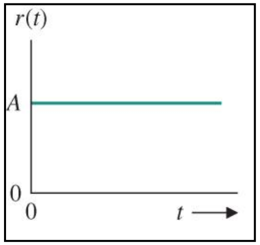
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 1 \end{cases}$$

"formal expression"



Time Response to Unit-Step Input (1)

$$R(s) = \frac{1}{s}$$



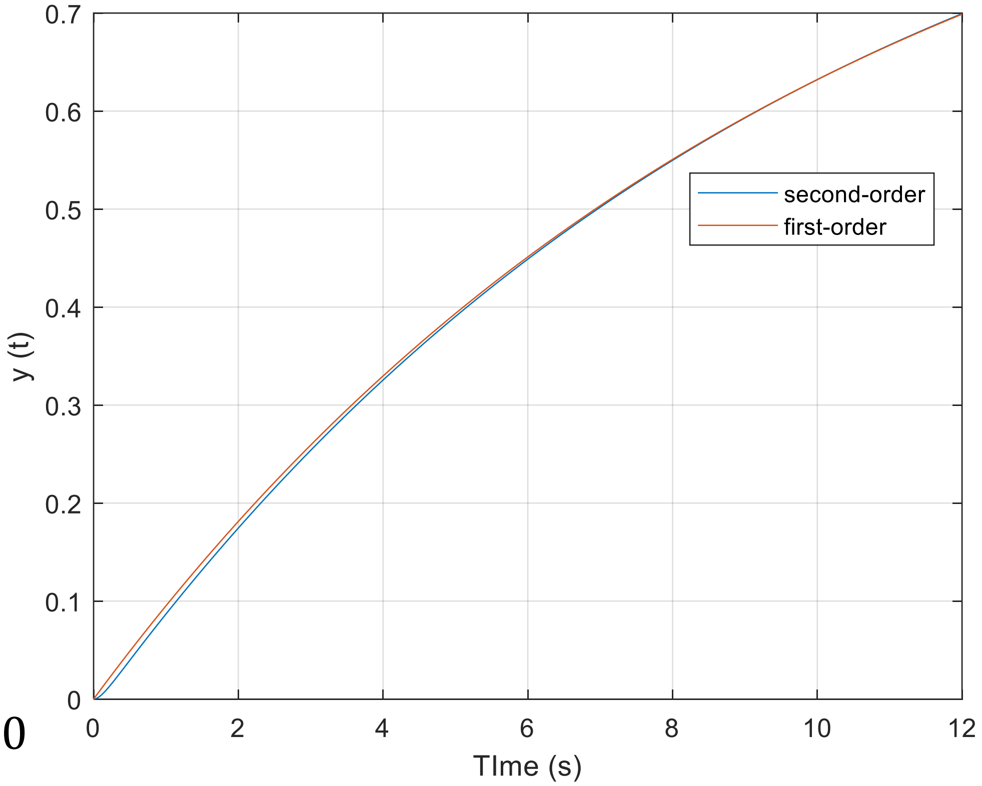
A=1

$$Y(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\frac{1}{s}$$

$$\zeta \geq 1 \qquad overdamped\ (critical\ damped)$$

$$Y(s) = \frac{{\omega_n}^2}{(s + \frac{1}{T_1})(s + \frac{1}{T_2})}\frac{1}{s}$$

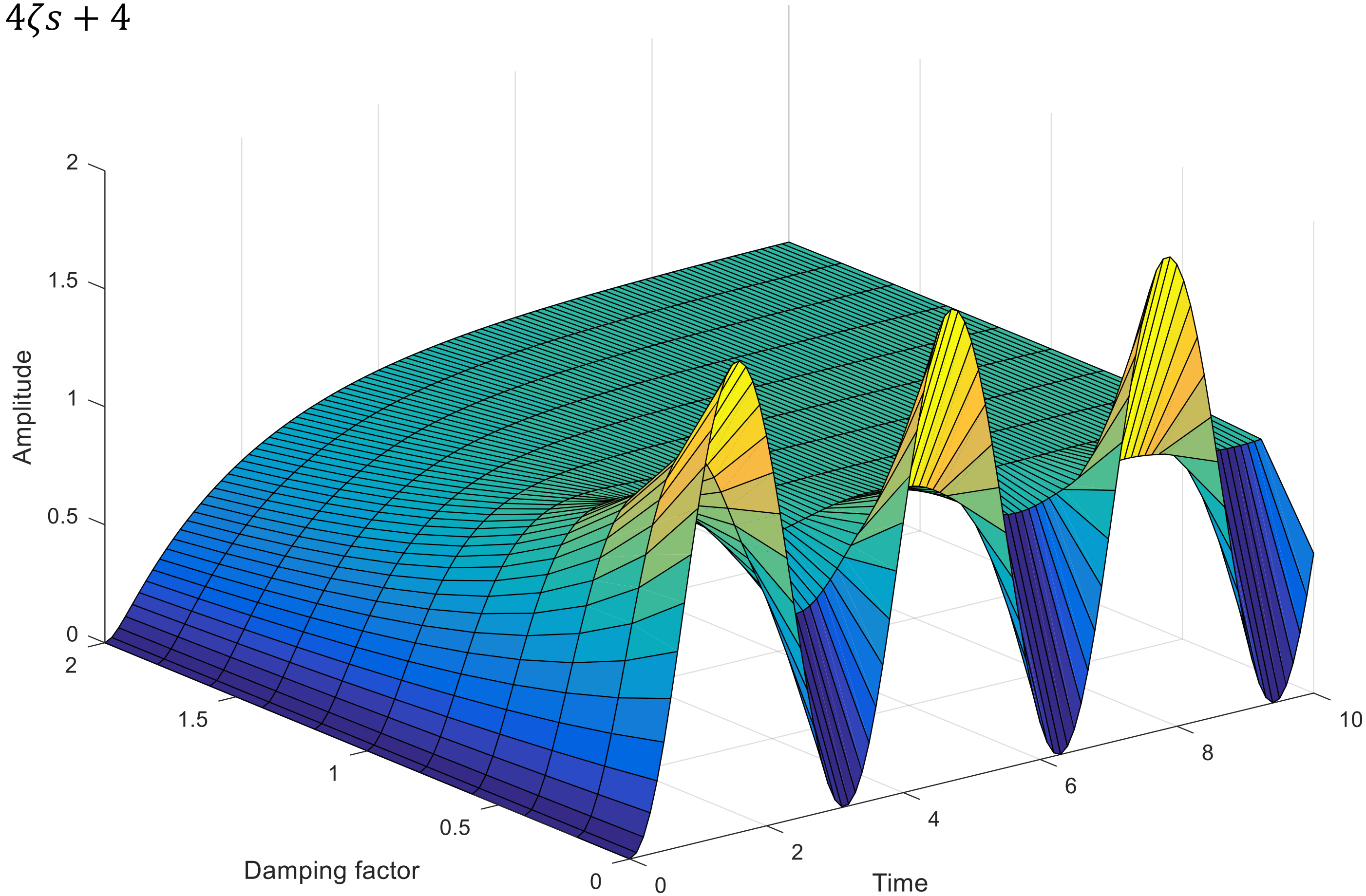
$$\zeta = 5, \omega_n = 1 \Rightarrow s_1 = -0.10, s_2 = -9.90$$



First-order system: $Y(s) = \frac{1}{10s + 1}\frac{1}{s}$

Numerical Example: $\omega_n = 2$ rad/s (for different ζ)

$$\frac{X(s)}{Y(s)} = \frac{4}{s^2 + 4\zeta s + 4}$$



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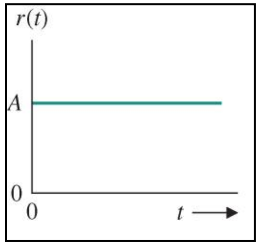
Time Response to Unit-Step Input (2)

$$R(s) = \frac{1}{s}$$



A=1

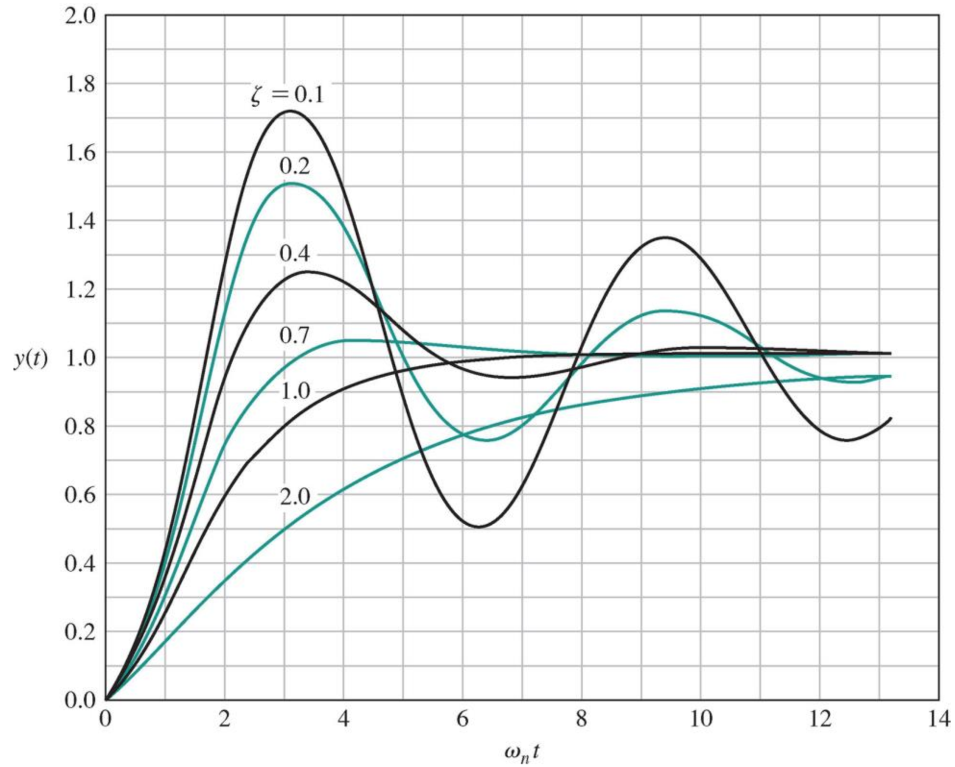
$$Y(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\frac{1}{s}$$

$$0 < \zeta < 1 \quad \text{underdamped}$$

$$\boxed{y(t) = \left[1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)\right]}$$

where $\beta = \sqrt{1 - \zeta^2}$

$$\theta = \cos^{-1}\zeta$$



Alternatively, $y(t)$ can be re-expressed as:

$$\boxed{y(t) = \left[1 - e^{-\zeta\omega_n t}\left(\cos\beta\,\omega_n t + \frac{\zeta}{\beta}\sin\beta\,\omega_n t\right)\right]}$$

Standard Performance Measures (1)

Standard performance measures are often defined in terms of the **unit-step response** of the closed-loop system.

Peak Value: M_{pt}

Final Value: $f_v = 1$

Time Constant: $\tau = \dfrac{1}{\zeta\omega_n}$

Peak Time:

$$\frac{dy(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\beta\,\omega_n t + \frac{\zeta}{\beta}\sin\beta\,\omega_n t\right) + e^{-\zeta\omega_n t}(\beta\omega_n \sin\beta\,\omega_n t - \zeta\omega_n\cos\beta\,\omega_n t)$$

$$\frac{dx(T_p)}{dt} = \left(\frac{\zeta^2\omega_n e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin\beta\,\omega_n T_p\right) + e^{-\zeta\omega_n T_p}\sqrt{1-\zeta^2}\,\omega_n \sin\beta\,\omega_n T_p$$

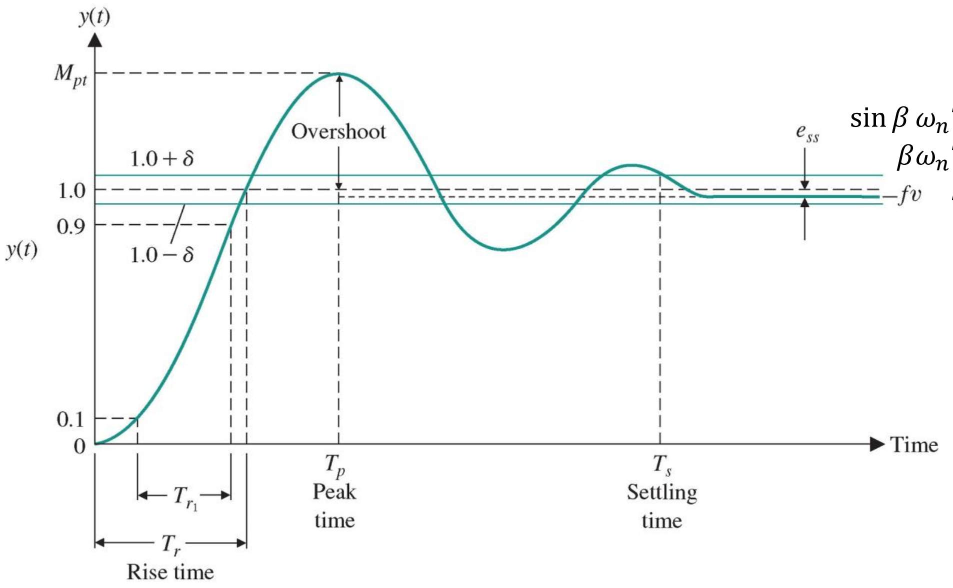
$$0 = \frac{\omega_n e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin\beta\,\omega_n T_p$$

$$\sin\beta\,\omega_n T_p = 0$$

$$\beta\omega_n T_p = 0, \pi, 2\pi, \ldots$$

$$T_p = \frac{\pi}{\beta\omega_n}$$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$



Standard Performance Measures (2)

Standard performance measures are often defined in terms of the **unit-step response** of the closed-loop system.

Rise Time:

$$x(T_r) = 1 = 1 - e^{-\zeta\omega_n T_r}\Big(\cos\beta\,\omega_n T_r$$

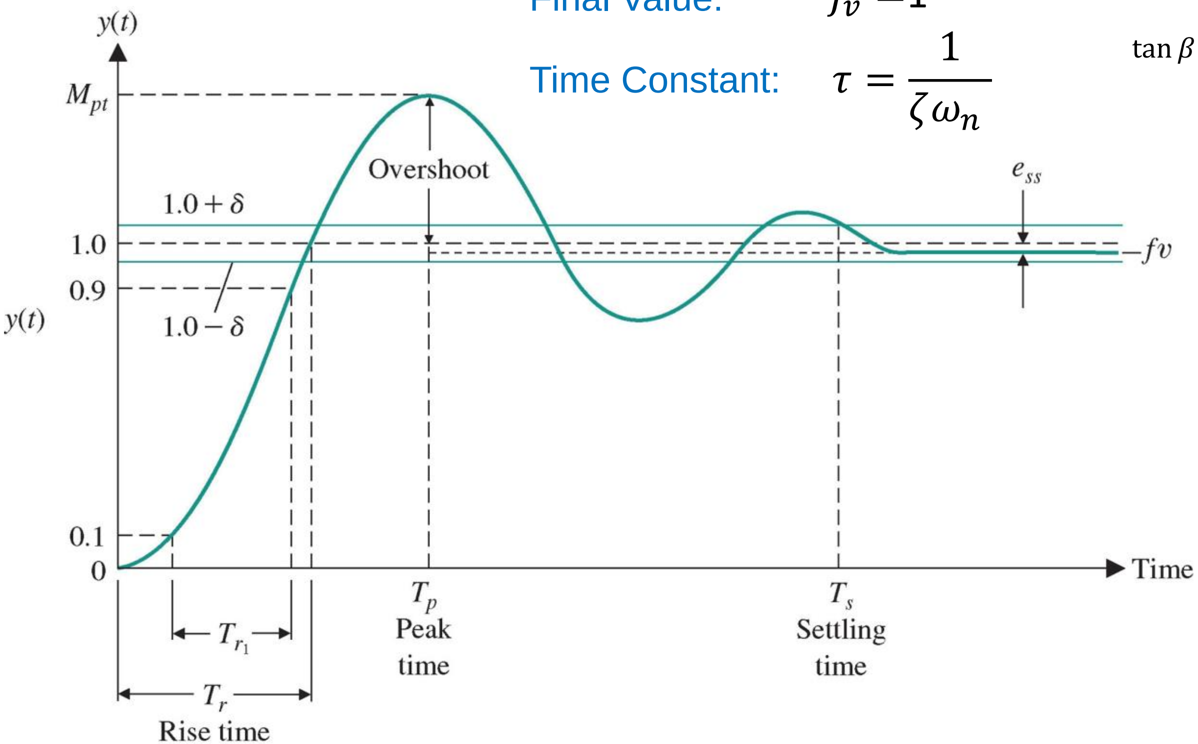
$$0 = \cos\beta\,\omega_n T_r + \frac{\zeta}{\beta}\sin\beta\,\omega_n T_r$$

$$\tan\beta\,\omega_n T_r = -\frac{\beta}{\zeta}$$

Peak Value: M_{pt}

Final Value: $f_v = 1$

Time Constant: $\tau = \dfrac{1}{\zeta\omega_n}$



General

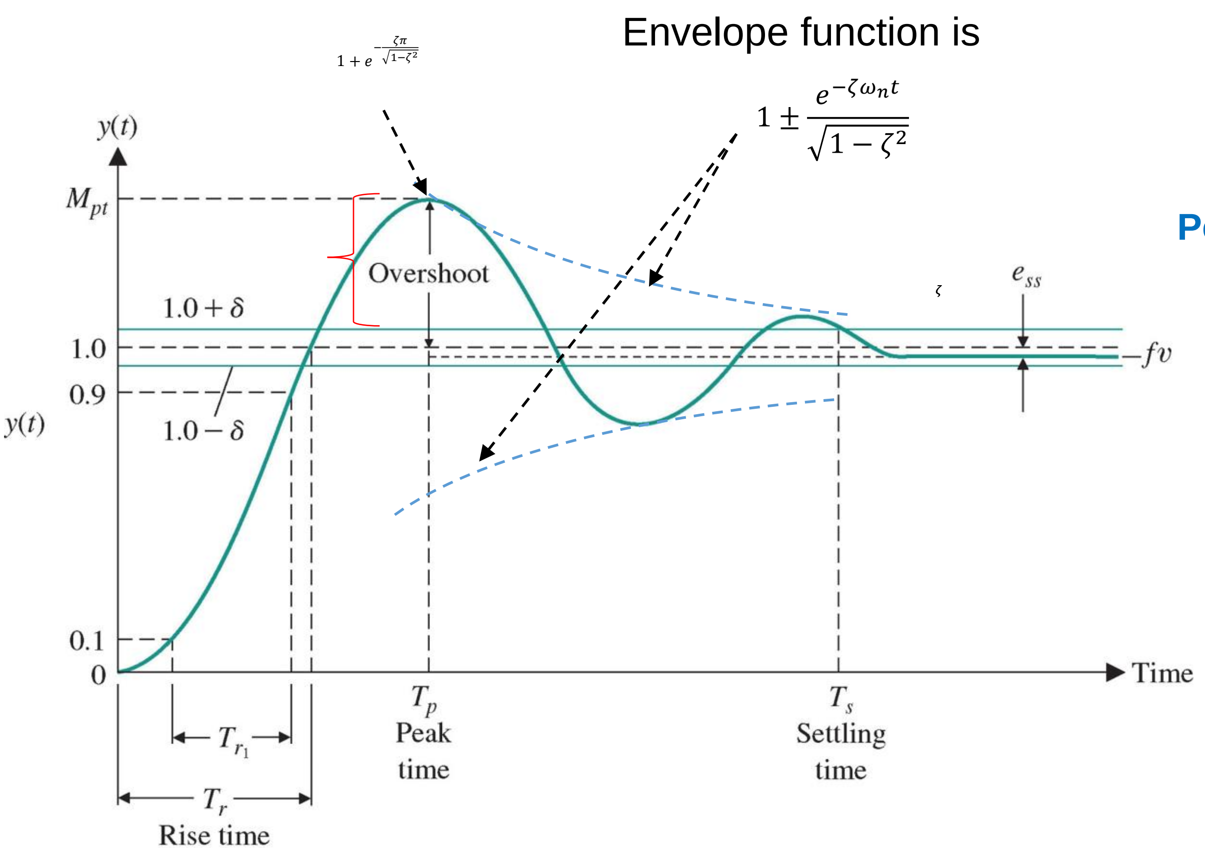
$$\boxed{T_r = \frac{1}{\beta\omega_n}\tan^{-1}\left(-\frac{\beta}{\zeta}\right)}$$

Approximation
$(\mathbf{0.3 < \zeta < 0.8})$

$$\boxed{T_{r1} = \frac{2.16\zeta + 0.60}{\omega_n}}$$

Standard Performance Measures (3)

Standard performance measures are often defined in terms of the **unit-step response** of the closed-loop system.



2% Settling Time (for $0 \leq \zeta \leq 0.9$):

$$\boxed{T_s \cong 4\tau = \frac{4}{\zeta\omega_n}}$$

Percent/Maximum Overshoot (%):

$$M_p = \frac{x(T_p)-1}{1} \times 100\%$$

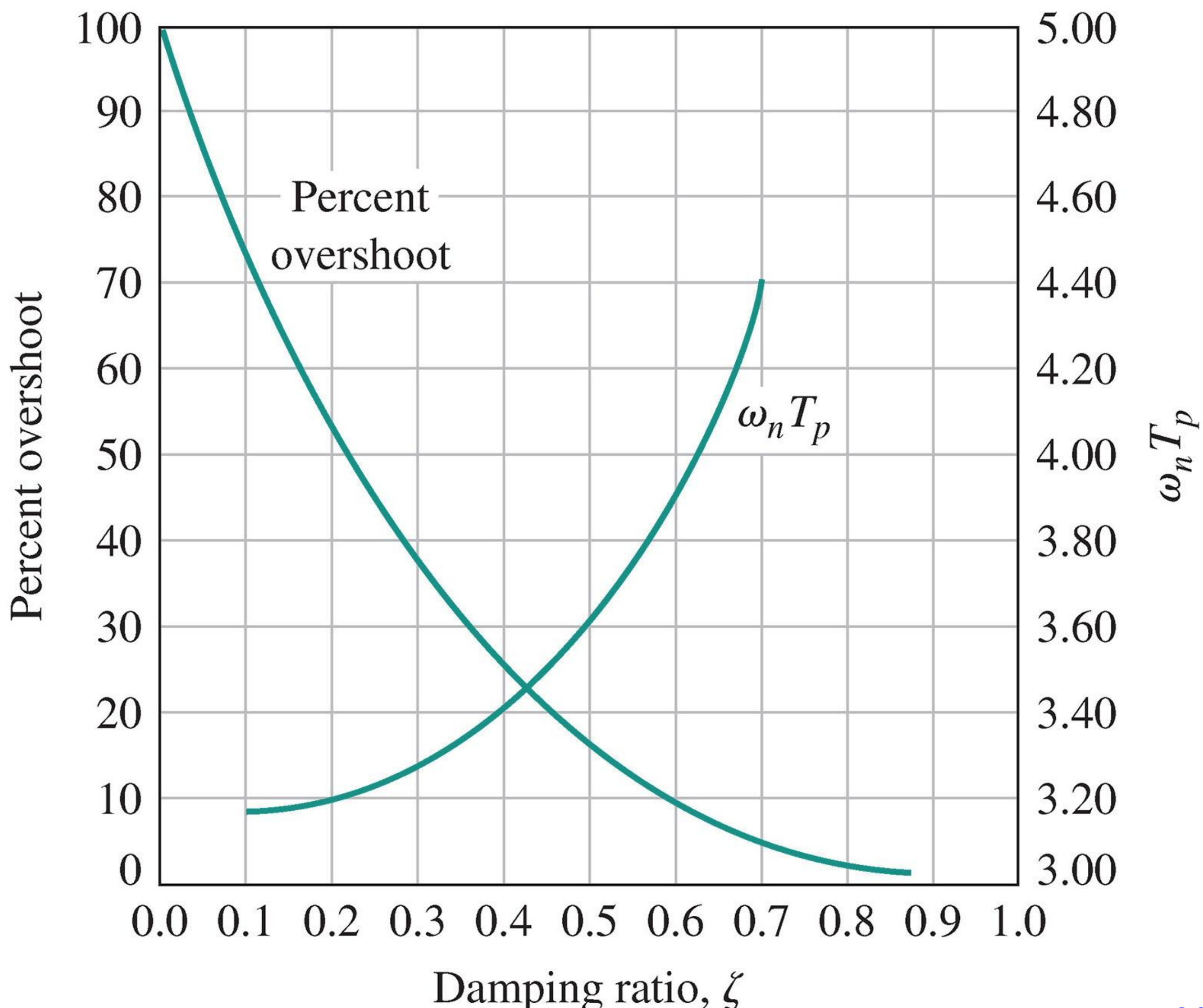
$$M_p = \left[1 - e^{-\zeta\omega_n T_p}\left(\cos\beta\,\omega_n T_p + \frac{\zeta}{\beta}\sin\beta\,\omega_n T_p\right) - 1\right] \times 100\%$$

$$= -e^{-\zeta\omega_n\left(\frac{\pi}{\beta\omega_n}\right)}\left(\cos\pi + \frac{\zeta}{\beta}\sin\pi\right) \times 100\%$$

$$\boxed{P.O.\ or\ M.O. = 100\% \times e^{-\zeta\pi/\sqrt{1-\zeta^2}}}$$

$$M.O.\,or\,P.O. = \frac{M_{pt-fv}}{fv} \times 100\%$$

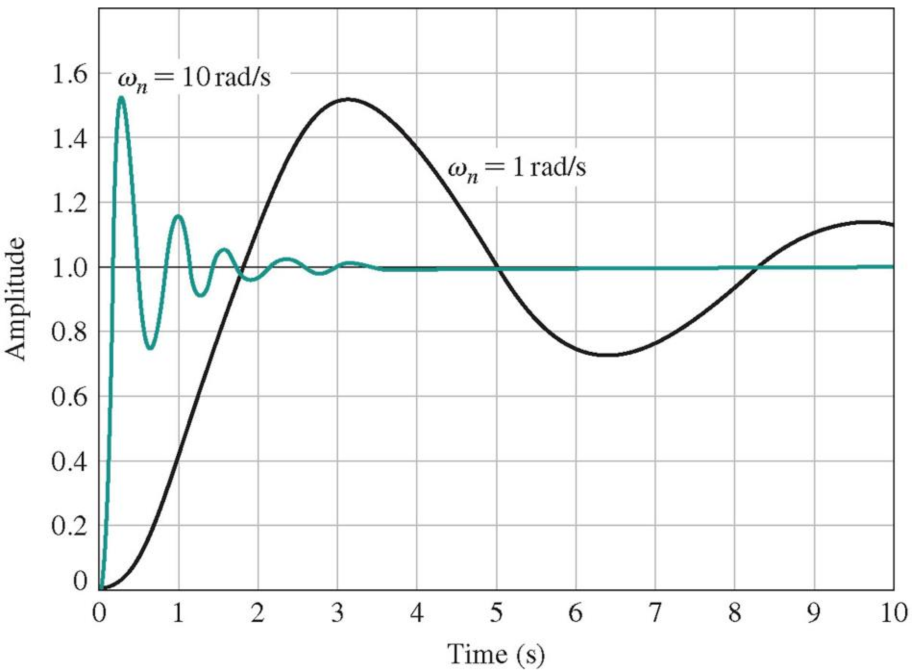
P.O. and Normalized Peak Time vs. ζ



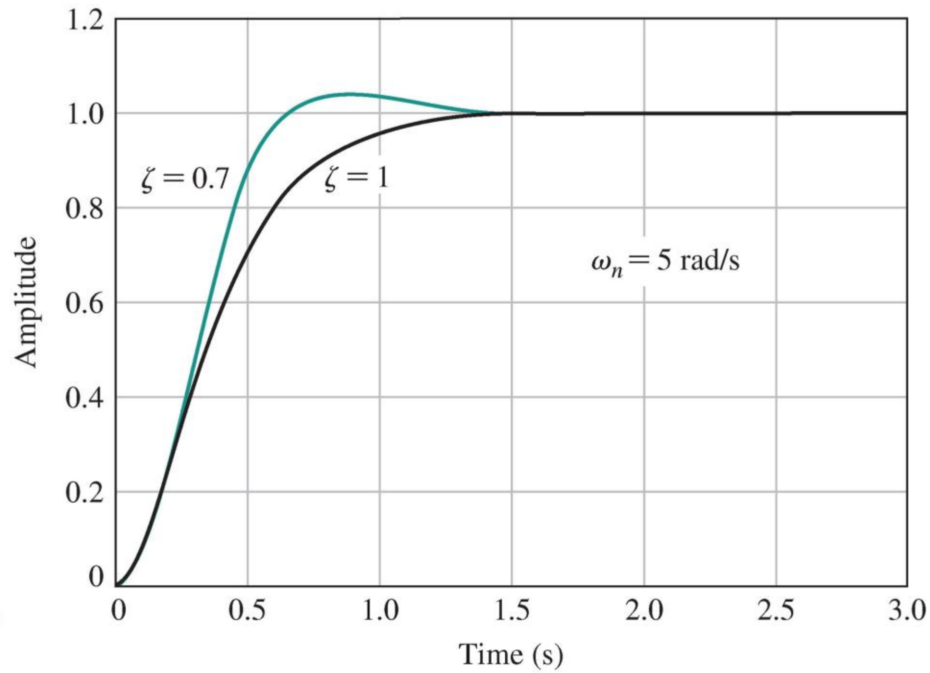
Need to compromise!

Effects of ω_n and ζ on The Step Response

with ζ =0.2,
different ω_n



with $\omega_n = 5$,
different ζ



Peak Time:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Rise Time:

$$T_{r1} = \frac{2.16\zeta + 0.60}{\omega_n} \quad (0.3 < \zeta < 0.8)$$

2% Settling Time:

$$\boxed{T_s = \frac{4}{\zeta\omega_n}}$$

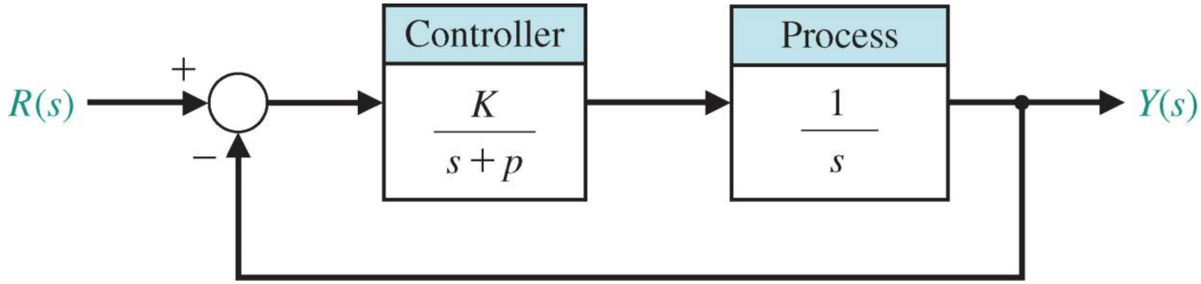
Percent Overshoot:

$$\boxed{P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}}$$

Example 13.1

Consider the following system, select gain K and the parameter p so that the time-domain specifications to a unit step input are satisfied.

- ***Specifications**: 2% settling time $T_s \leq 4$ s; and percent overshoot $P.O. \leq 5\%$.*



Step 1. Transfer function: $T(s) = \dfrac{K}{s^2 + ps + K}$ $\left(= \dfrac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right)$

$$2\zeta\omega_n = p, \qquad {\omega_n}^2 = K$$

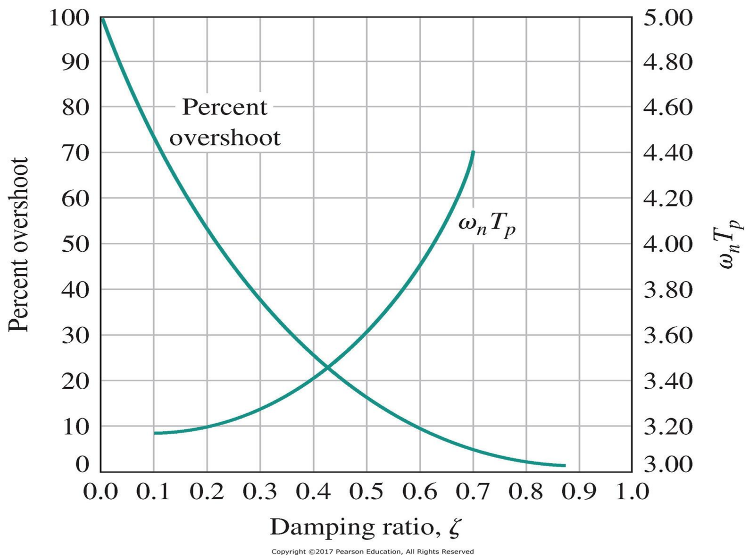
Step 2. To satisfy settling time requirement:

$$\frac{4}{\zeta\omega_n} \leq 4 \longrightarrow \zeta\omega_n \geq 1$$

2% Settling Time:

$$\boxed{T_s = \frac{4}{\zeta\omega_n}}$$

Step 3. To satisfy the P.O. requirement:



Percent Overshoot:

$$\boxed{P.O. = 100e^{-\zeta\pi / \sqrt{1-\zeta^2}}}$$

$$P.O. \leq 5\% \longrightarrow \zeta \geq 0.69$$

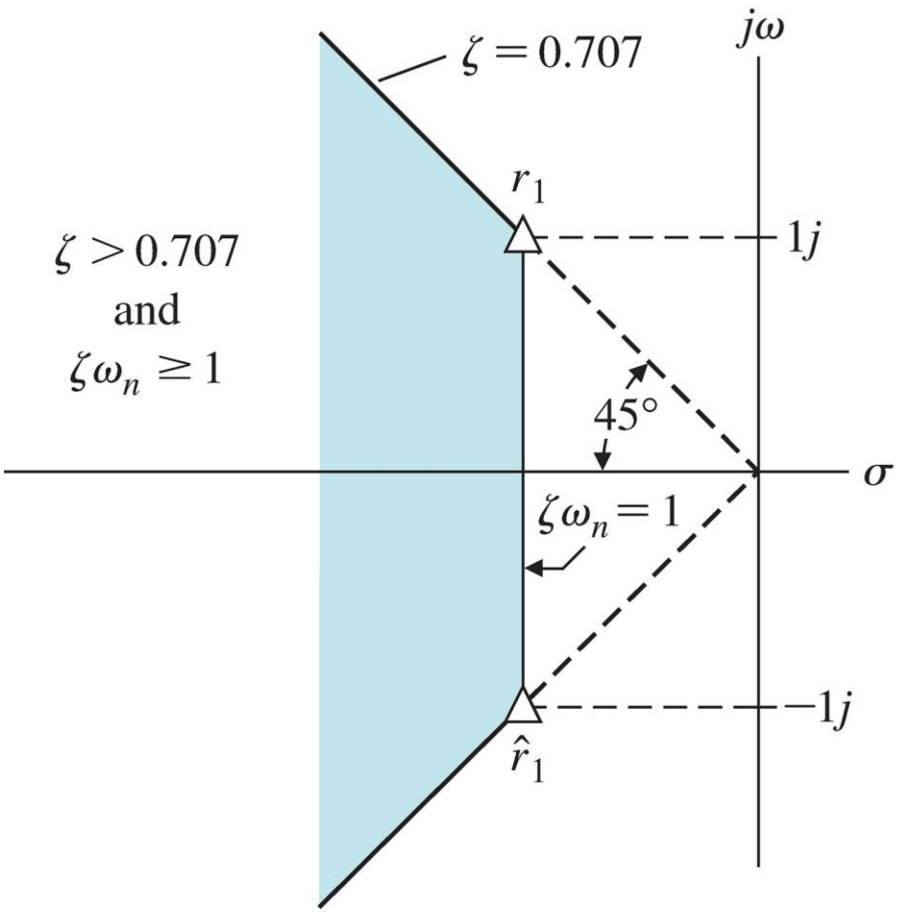
Step 4. Choose suitable values:

Can choose $\zeta\omega_n = 1$

$$\zeta = 0.707 = \frac{1}{\sqrt{2}}$$

$$\longrightarrow \omega_n = \sqrt{2}\, rad/s \longrightarrow \begin{matrix} p = 2 \\ K = 2 \end{matrix}$$

Specifications and Root Locations



$$T(s) = \frac{2}{s^2 + 2s + 2}$$

Poles: $-1 \pm j1$

$$T(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Poles:

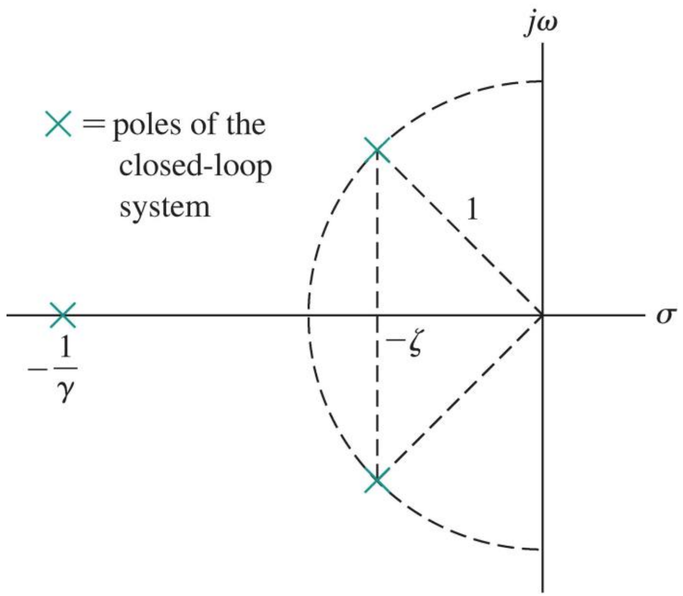
$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Effects of a Third Pole

Assume $\omega_n = 1$, and consider a system with two complex poles and an additional pole:

$$T(s) = \frac{1}{(s^2+2\zeta\omega_n s + 1)(\gamma s + 1)}$$

The time response of a third-order system can be approximated by the **dominant roots** of the second-order system as long as the real part of the dominant roots is less than one tenth of the real part of the third pole.



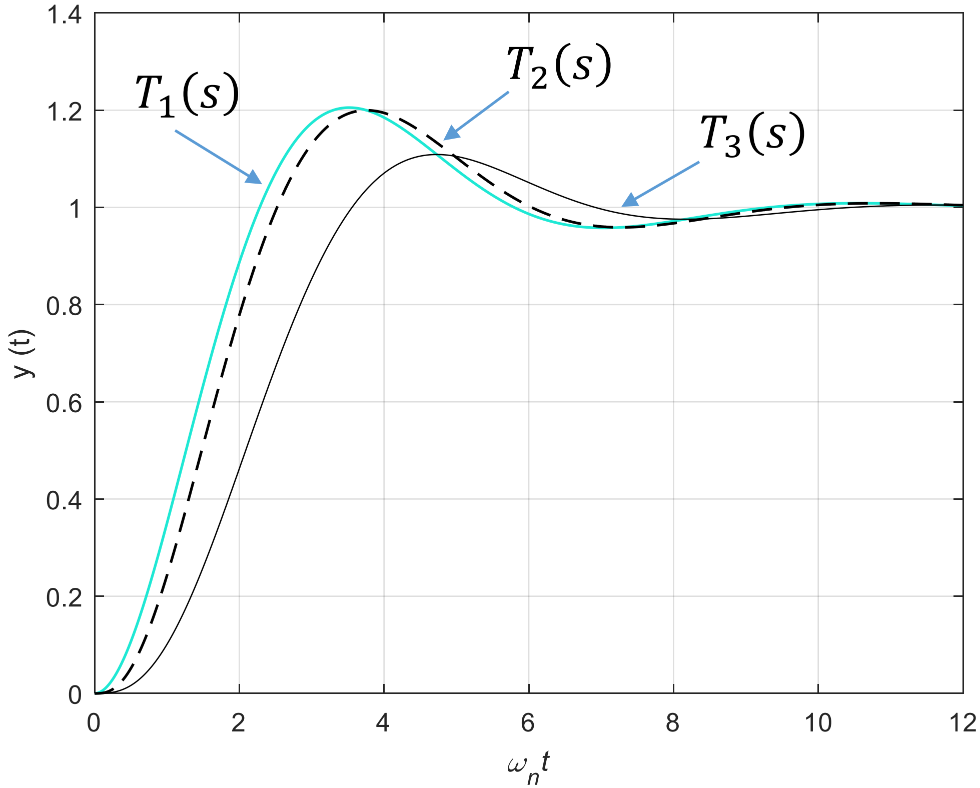
Approximation (rule of thumb):

$$\left|{}^{\mathbf{1}}/_{\boldsymbol{\gamma}}\right| \geq \mathbf{10}|\boldsymbol{\zeta\omega_n}|$$

NOTE: DC gain $T(0)$ should be kept the same after approximation.

Example 13.2

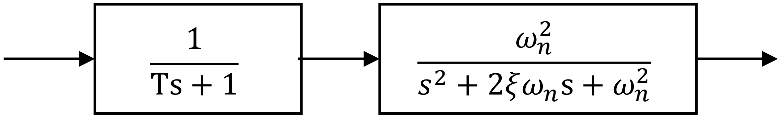
A plant has the transfer function of $T_1(s) = \frac{1}{s^2+0.9s+1}$. Determine whether it is acceptable to approximate it using more-simplified functions $T_2(s) = \frac{1}{(s^2+0.9s+1)(0.22s+1)}$ or $T_3(s) = \frac{1}{(s^2+0.9s+1)(s+1)}$.



System 1 poles: $s = -0.45 \pm j0.893$

System 2 poles: $s = -0.45 \pm j0.893, -4.55$

System 3 poles: $s = -0.45 \pm j0.893, -1$

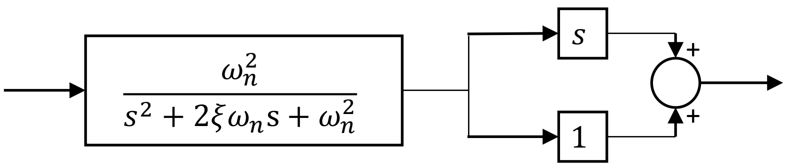


System 1 can be approximated by system 2, but not system 3.

Effects of a Finite Zero

Consider a system (normalized):

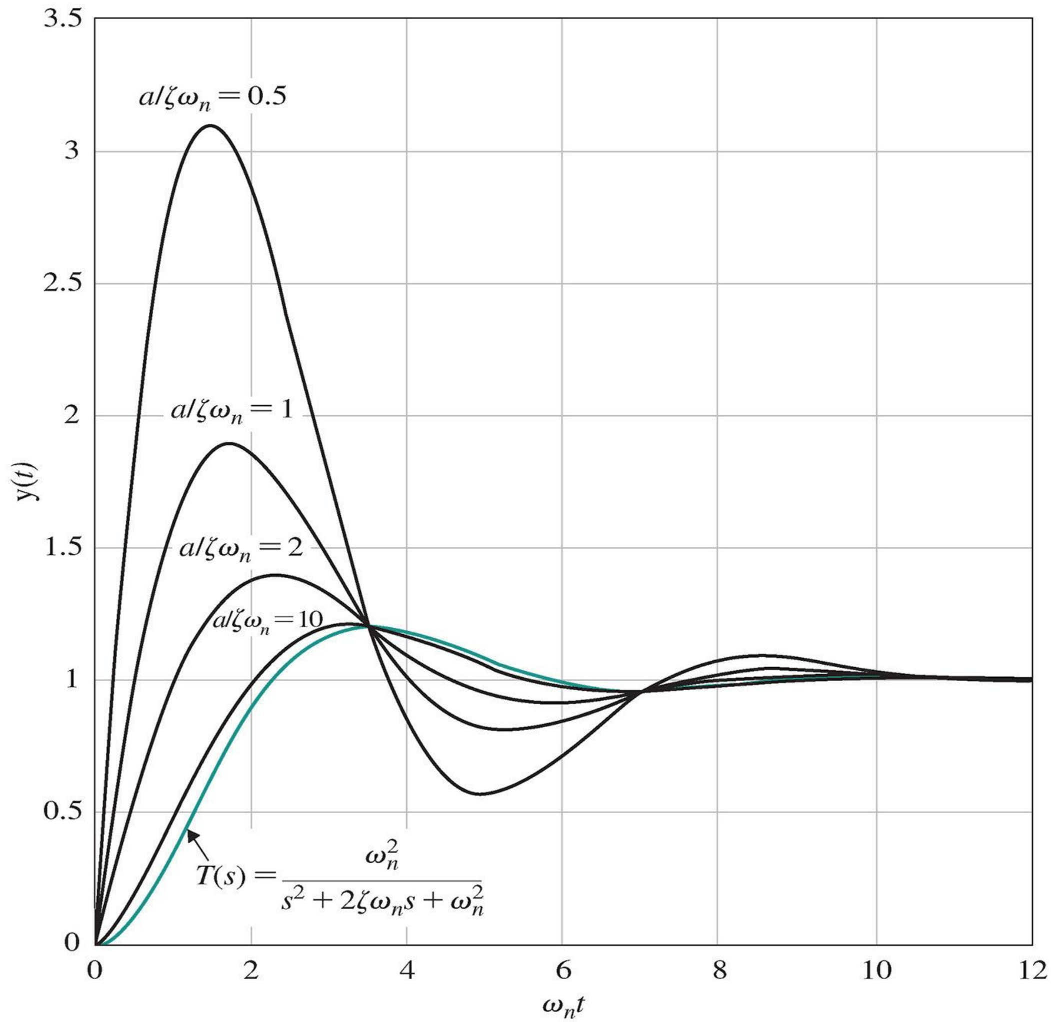
$$T(s) = \frac{\frac{\omega_n{}^2}{a}(s+a)}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$



If $a \gg \zeta\omega_n$ (e.g., 10 times):

the system can be simplified as:

$$T(s) \approx \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$



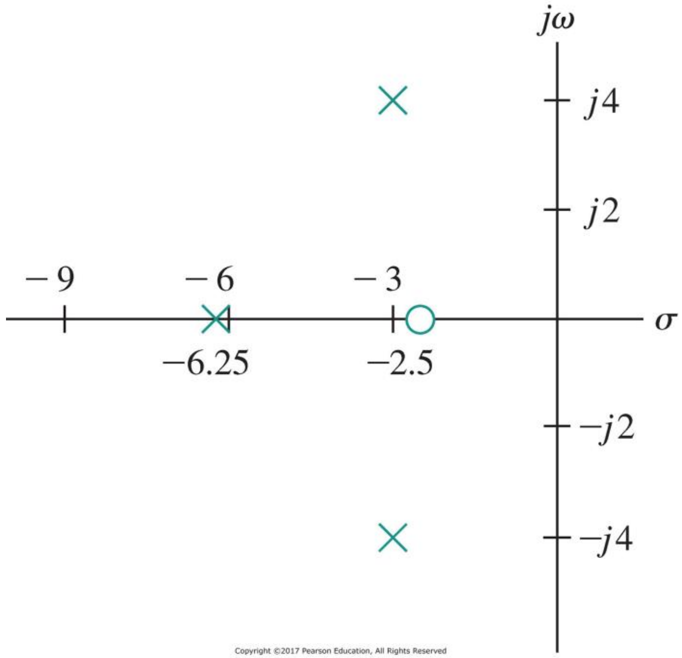
Example 13.3

Consider the following system, analyze the effect of the additional pole and zero

$$T(s) = \frac{1.6(s+2.5)}{(s^2+6s+25)(0.16s+1)}$$

$$T(s) = \frac{\frac{\omega_n{}^2}{a}(s+a)}{(s^2+2\zeta\omega_n s+\omega_n{}^2)(1+\tau s)}$$

- **Tips:** Construct the pole-zero plot on the argand/complex-axis diagram



$$\zeta\omega_n = 3, a = 2.5, \tau = 0.16$$

For this system, based on the rules we just learnt or based on the simply analysis of the s-plane, the zero and third pole can **NOT** be neglected!

For the actual third-order system:

$$T_s = 1.6\ s, P.O. = 38\ \%$$

For a normalized second-order system:

$$T(s) = \frac{25}{s^2+6s+25}$$

The characteristic behavior are:

$$T_s = 1.33\ s, P.O. = 9.5\ \%$$

Example 13.4

Consider the following system, can we neglect the effects of third pole? If yes, obtain approximated transfer function. Estimate P.O.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2000}{(s^2+10s+100)(s+50)}$$

Thought process: (1) Calculate the poles and compare their real parts, or calculate the poles (and zeros) and insert them into the s-plane; (2) if the third pole (and zero) is "far" (rule of thumb used here is 10 times the real part) from the dominant complex poles, it can be neglected, as in this example; (3) Obtain new simplified TF. Obtain ζ and calculate P.O.

(Reminder: calculation steps must be shown/included in the exam)

Final answer: $T = \frac{40}{s^2+10s+100}$, $\text{P.O.} = 16.3\%$

Example 13.5

For a second order system, determine the root locations in *s*-plane which satisfies:

1. 10 % < P.O. < 20 %
2. 0.3 s < 2% Settling time < 0.6 s

Thought process: (1) From the two P.O. limits, calculate the ζ limits; ζ limits determine the slanted lines at $\cos^{-1}\zeta$ angles, which form the boundaries on the LHS of the s-plane. Take note of which side of the lines, left or right, should the root be located.
(2) From the settling time limits, calculate the $\zeta\omega_n$ values. Negated values are the vertical lines' location. Take note on which side of the lines, left or right, should the root be located; (3) estimate poles that comply with conditions 1 and 2.

(Reminder: calculation steps and drawings must be shown/included in the exam)
Final answer: $\text{e.g. } -10 \pm j17.32$

Concluding Remarks

- **Test signal input**
- **First-order system**
  - Step input response
  - Time constant
  - Features: no overshoot, settling time (5%): 3T
- **Second-order system**
  - Step input response
  - Natural frequency, damping ratio
  - Features: peak time, overshoot, settling time, rising time
- **Effect of additional zeros and poles**

Office hour: 2-4 pm Thursday