Xi'an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

2024-25 Semester 2

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Lecture 16

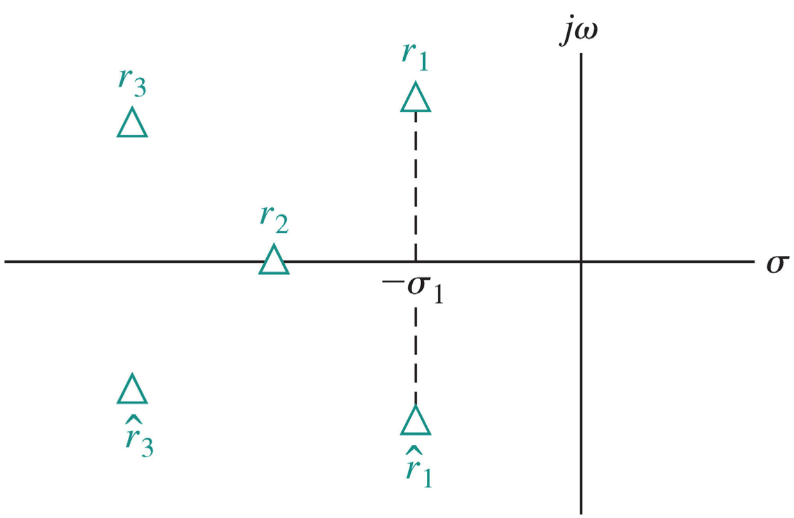
Outline

Stability of Linear Feedback Systems

- ❑ The Concept of Stability
- ❑ Routh-Hurwitz Stability Criterion
- ❑ **Relative Stability of Feedback Control Systems**
- ❑ **Stability of State Space/Variable Systems**
- ❑ **System Stability Using Matlab**

Relative Stability (through RHC)

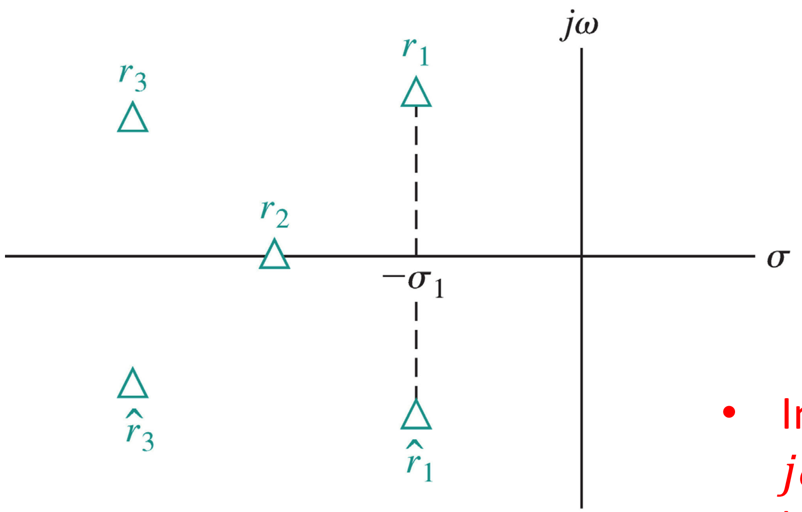
- The Routh-Hurwitz criterion ascertain the **absolute stability** of a system by determining whether any of the roots of the characteristic equation lie in the right half of the s-plane;
- However, if a system satisfies the Routh-Hurwitz criterion and is stable, it is desirable to determine the **relative stability** (the degree of stability, or how close the system is to instability);
- The relative stability can be determined by as the property that is measured by the relative real part of each root or pair of roots.



In this figure, root r_2 is relatively more stable than the roots $r_1, \widehat{r_1}$.

For Examining Relative Stability: Axis Shift

- This approach is the extension of Routh-Hurwitz criterion to ascertain relative stability;
- The approach can be accomplished by utilizing a change of variable, which shifts the $j\omega$-axis in the s-plane in order to utilize the Routh-Hurwitz criterion.



In this figure, it can be noticed that a shift of the $j\omega$-axis in the s-plane to $-\sigma_1$ will result in the roots appearing on the newly shifted axis (-marginally stability).

- In practice, the correct magnitude to shift the $j\omega$-axis must be obtained on a trial-and-error basis. Then, without solving $q(s)$ (5^{th} order in this case), we may determine the real-part of the dominant roots.

Example 16.1

Consider the third-order system with the following characteristic equation

$$q(s) = s^3 + 4s^2 + 6s + 4$$

To determine relative stability:

1. Apply Routh-Hurwitz criterion on this characteristic equation, the system is stable (absolute stability);
2. As a first try, we can shift the $j\omega$-axis by $\frac{1}{2}$, in other words, let us assume $s_n = s + \frac{1}{2}$, then the new characteristic equation can be obtained. Applying Routh-Hurwitz criterion, we'll find that the system is still stable after shifting the $j\omega$-axis by $\frac{1}{2}$;
3. Then we try shifting the $j\omega$-axis by 1, i.e., we assume $s_n = s + 1$, new characteristic equation now is

$$(s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n^3 + s_n^2 + s_n + 1$$

$$(s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n^3 + s_n^2 + s_n + 1$$

The Routh array is:

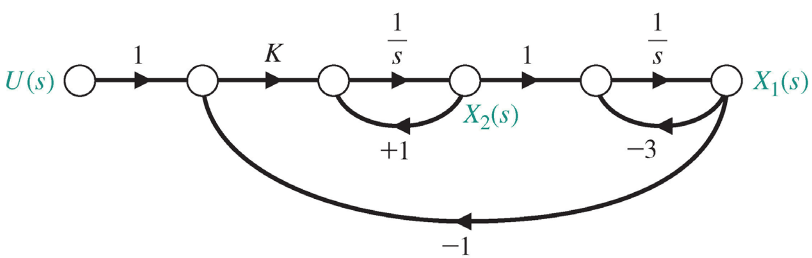
$$\begin{array}{c|cc} s_n^3 & 1 & 1 \\ s_n^2 & 1 & 1 \\ s_n^1 & 0 & 0 \\ s_n^0 & 1 & 0 \end{array}$$

There is a row with all zeros in the Routh array, indicating that there are a pair of roots on the shifted imaginary axis. These two roots can be obtained from the auxiliary polynomial

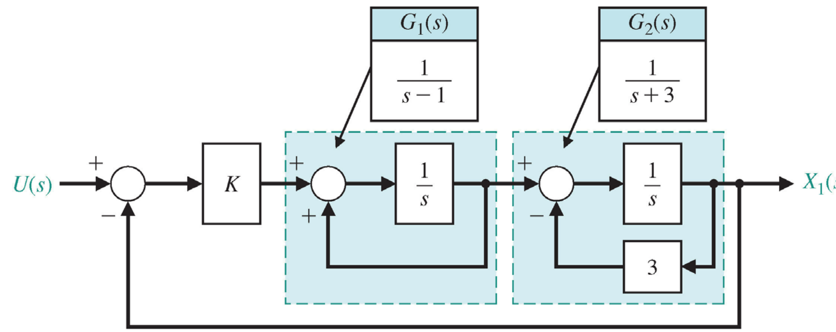
$$U(s_n) = s_n^2 + 1 = (s_n + j)(s_n - j)$$

Stability of State Variable System

- If the system is represented by signal-flow graph (a) or block diagram (b), stability can be assessed by firstly obtaining the transfer function of the system, then applying Routh-Hurwitz criterion to the characteristic equation.



(a)



(b)

- How about the system represented by state-space model?

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(\mathrm{t}) + \mathbf{B}\mathbf{u}(\mathrm{t})$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(\mathrm{t}) + \mathbf{D}\mathbf{u}(\mathrm{t})$$

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

Characteristic Equation from State-space Model

Transfer function $\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$

$$\mathbf{G}(s) = \mathbf{C}\frac{adj(s\mathbf{I} - \mathbf{A})}{|s\mathbf{I} - \mathbf{A}|}\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}[adj(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|}$$

Note: for a 2×2 matrix M=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, assume its inverse matrix is M^{-1}, (i.e., $\mathrm{M}^{-1}\mathrm{M} = \mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)
Its adjugate is adj(M)=$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, its determinant is det(M)=$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
Then $\mathrm{M}^{-1} = \frac{adj(\mathrm{M})}{\det(\mathrm{M})}$.

Setting the denominator of the transfer function matrix **G**(s) to be zero, we get the **characteristic equation:**

$$\Delta(\boldsymbol{s}) = |s\mathbf{I} - \mathbf{A}| = \mathbf{0}$$

$\boldsymbol{n}$**: order of the system**
$\boldsymbol{A}$**:** $\boldsymbol{n} \times \boldsymbol{n}$ **matrix**
$s\mathbf{I} - \mathbf{A}$**:** $\boldsymbol{n} \times \boldsymbol{n}$ **matrix**
$|s\mathbf{I} - \mathbf{A}|$**:** $\boldsymbol{n}$**-th order polynomial**

IMPORTANT OBSERVATION:

- *It means one does not need to obtain the full transfer function to proceed with stability analysis (which requires determination of an inverse matrix).*
- *One only needs to extract the characteristic function from the determinant* $|s\boldsymbol{I} - \boldsymbol{A}|$.

Supplementary on Determinant of 2x2 and 3x3 Matrices (recall)

- For a 2 x 2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- The determinant is **A**, |**A**|, is:

$$= ad - cd$$

- Example:

$$\mathbf{A} = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$|\mathbf{A}| = 4 \times 2 - 6 \times 3 = 14$$

- For a 3 x 3 matrix

$$\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- The determinant is **B**, |**B**|, is:

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

- Example:

$$\mathbf{B} = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$|\mathbf{B}|$$
$$= 6(-14 - 40) - (1)(28 - 10)$$
$$+ (1)(32 + 4) = -306$$

Example 16.2

A system is described by the following model, determine its stability.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix} \mathbf{x}$$

Step 1: Form the characteristic equation

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix},$$

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix} \right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 3 & s+1 & 0 \\ 2 & 1 & s+2 \end{vmatrix} = s^3 + 3s^2 + 5s + 6$$

Step 2: Apply Routh Hurwitz Criterion

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 3 & 6 \\ s^1 & 3 & 0 \\ s^0 & 6 & \end{array}$$

No sign change in the first column, the system is stable.

Example 16.3

For the following system, choose values of k to make it stable.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & -1 \\ k & 0 & 2 \\ -k & -2 & -k \end{bmatrix} \mathbf{x}$$

Solutions:

$$[s\boldsymbol{I} - \boldsymbol{A}] = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ k & 0 & 2 \\ -k & -2 & -k \end{bmatrix} = \begin{bmatrix} s & 0 & 1 \\ -k & s & -2 \\ k & 2 & s+k \end{bmatrix}$$

$$\Delta(s) = \det\left(\begin{bmatrix} s & 0 & 1 \\ -k & s & -2 \\ k & 2 & s+k \end{bmatrix}\right) = \mathrm{s}\begin{vmatrix} s & -2 \\ 2 & s+k \end{vmatrix} - 0\begin{vmatrix} -k & -2 \\ k & s+k \end{vmatrix} + 1\begin{vmatrix} -k & s \\ k & 2 \end{vmatrix}$$

$$= s(s^2 + ks + 4) - 2k - ks = s^3 + ks^2 + (4-k)s - 2k$$

$$\Delta(s) = s^3 + ks^2 + (4-k)s - 2k$$

The Routh array is

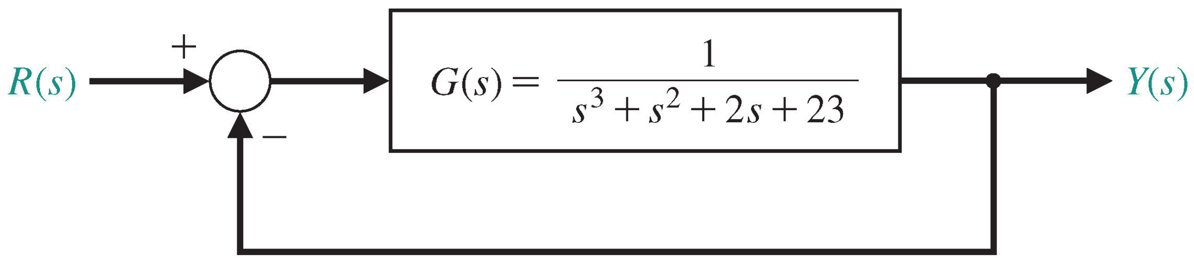
$$\begin{array}{c|cc} s^3 & 1 & 4-k \\ s^2 & k & -2k \\ s^1 & \dfrac{\begin{vmatrix} 1 & 4-k \\ k & -2k \end{vmatrix}}{-k} = 6-k & 0 \\ s^0 & -2k & \end{array}$$

Conditions to fulfill: $6-k>0, -2k>0$ (impossible to achieve)

Therefore, no value of k can make the system stable.

Stability Analysis Using Matlab

pole function: compute poles of the closed-loop control system.



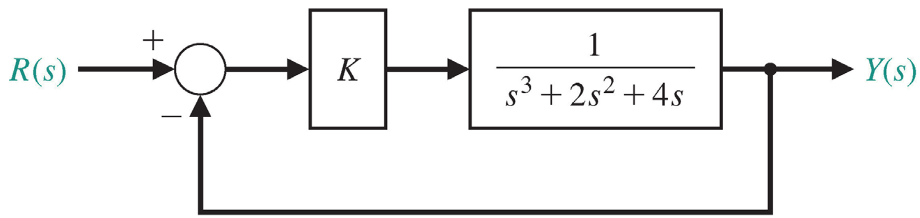

```
>>numg=[1]; deng=[1 1 2 23]; sysg=tf(numg,deng);
>>sys=feedback(sysg,[1]);
>>pole(sys)


ans =


      -3.0000
       1.0000 + 2.6458i
       1.0000 - 2.6458i
```

Unstable poles

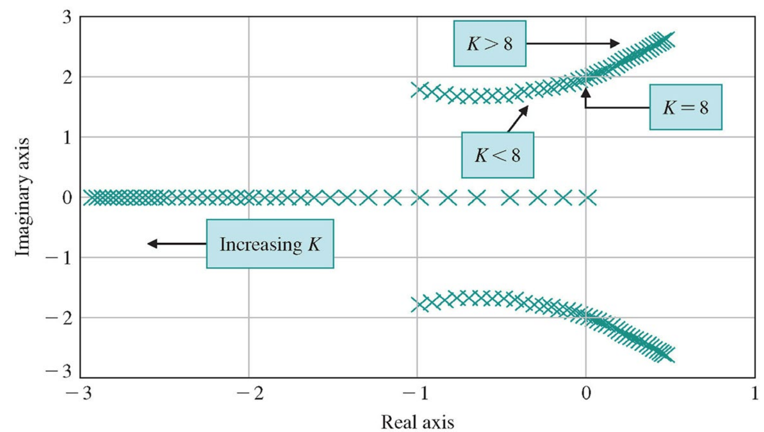
Use **roots** function to calculate root locations of $q(s) = s^3 + 2s^2 + 4s + K$ for $0 \le K \le 20$




```
% This script computes the roots of the characteristic
% equation q(s) = s^3 + 2 s^2 + 4 s + K for 0<K<20
%
K=[0:0.5:20];
for i=1:length(K)
    q=[1 2 4 K(i)];
    p(:,i)=roots(q);
end
plot(real(p),imag(p),'x'), grid
xlabel('Real axis'), ylabel('Imaginary axis')
```

Loop for roots as a function of K

(b)



(a)

Computing the characteristic polynomial of **A** using the **poly** function.

Coefficients of characteristic polynomial in descending order

$n \times n$ matrix

p=poly(A)

Characteristic polynomial

```
>>A=[-8 -16 -6;1 0 0;0 1 0];
>>p=poly(A)

p =
     1.0000     8.0000     16.0000     6.0000

>>roots(p)

ans =
   -5.0861
   -2.4280
   -0.4859
```

Stable

$$\mathbf{A} = \begin{bmatrix} -8 & -16 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^3 + 8s^2 + 16s + 6$$

Example 16.4 (in-class)

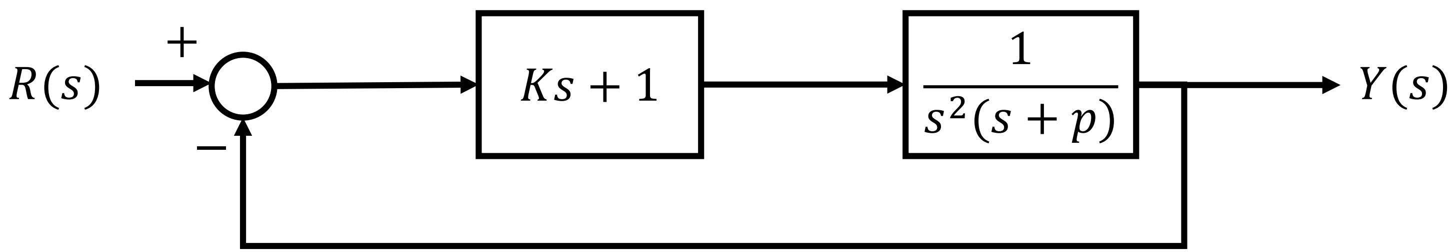
For the following system, find the value of *k* for which the system is stable.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 1 \\ -k & -1 & -2 \end{bmatrix} \mathbf{x}$$

Flow of thoughts: (1) Recognize the fact that the system's characteristic equation can be obtained through $|s\mathbf{I} - \mathbf{A}| = 0$; (2) Apply RHC to the characteristic equation and concludes about *k* in terms of system stability.

Answer: $s^3 - s^2 - 5s + k = 0$; no *k* value can stabilize the system.

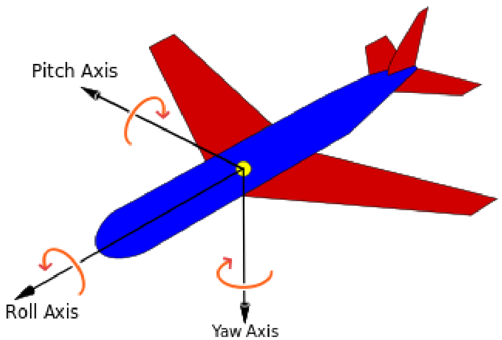
Example 16.5: RHC can deal with two unknown rather easily.

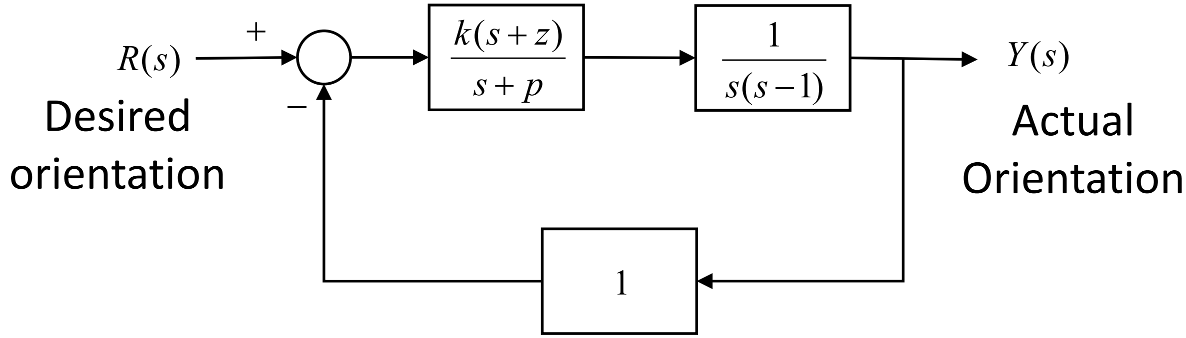


- A closed-loop feedback system is shown in Figure above. Cvalues of *K* and *p* is the system stable?

Answer: Since the stability conditions to be met are $p > 0$ $K > \frac{1}{p}$, we can choose, for example, $p = 1$ and $K = 2$.

Example 16.6 (Design example - Single-dimensional orientation control of a jet)

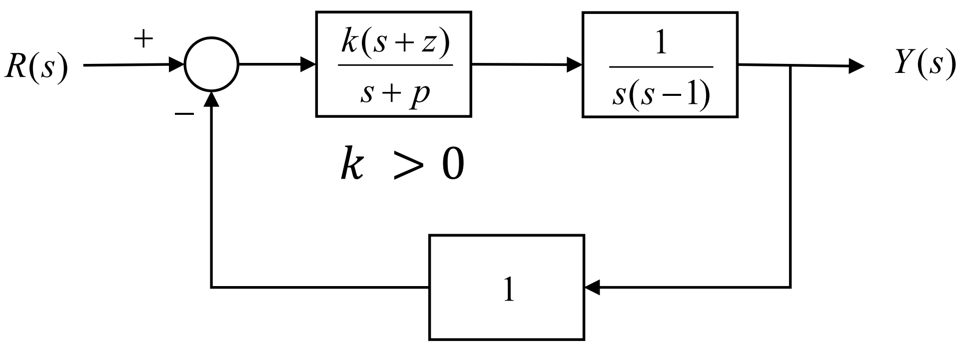


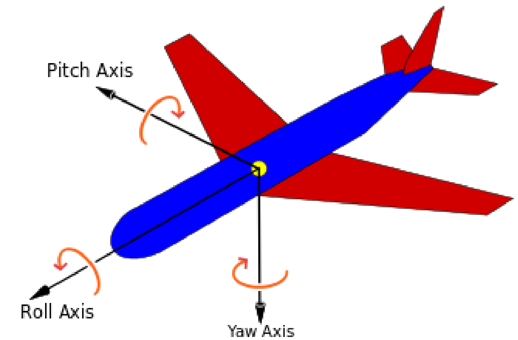


The above is the orientation control system of a jumbo-jet aircraft, where the controller gain *k* is greater than 0, and the *z* and *p* are the controller's zero and pole location.

- Is the system open-loop stable?
- Can the system be stabilized through the closed-loop control? If yes, what are the ranges of *k*, *p*, and *z* values that can ensure stability?

Solution





1) CL system's characteristic equation:

$$1+\frac{k(s+z)}{s+p}\frac{1}{s(s-1)}=0 \quad \Rightarrow \quad s^3+(p-1)s^2+(k-p)s^1+kz=0$$

2) Apply RHC:

Routh array

s^3	1	(*k*-*p*)
s^2	(*p*-1)	*kz*
s^1	a_1	*0*
s^0	*kz*	

$$a_1=-\frac{1}{(p-1)}\begin{vmatrix} 1 & (k-p) \\ (p-1) & kz \end{vmatrix}$$

To ensure overall stability:

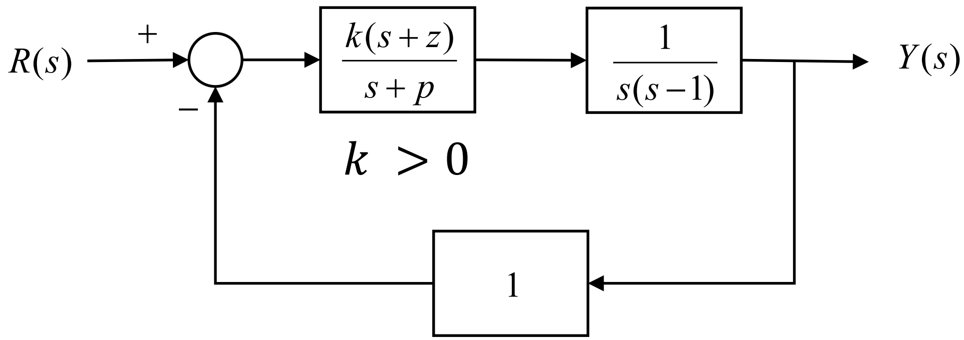
Condition 1: $p-1>0$

Condition 2: $a_1>0$

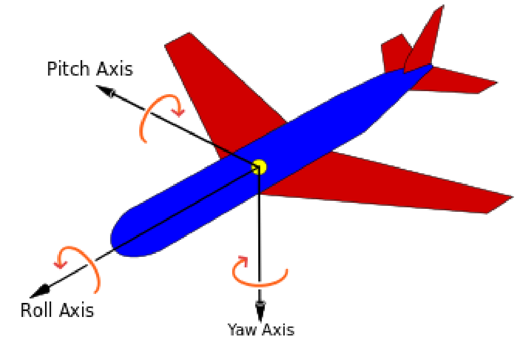
Condition 3: $kz\ >0$

$$\Rightarrow \quad \begin{gathered} p>1 \\ k[z-(p-1)]+p(p-1)<0 \\ kz\ >0 \rightarrow z\ >0 \end{gathered}$$

Solution



$k > 0$

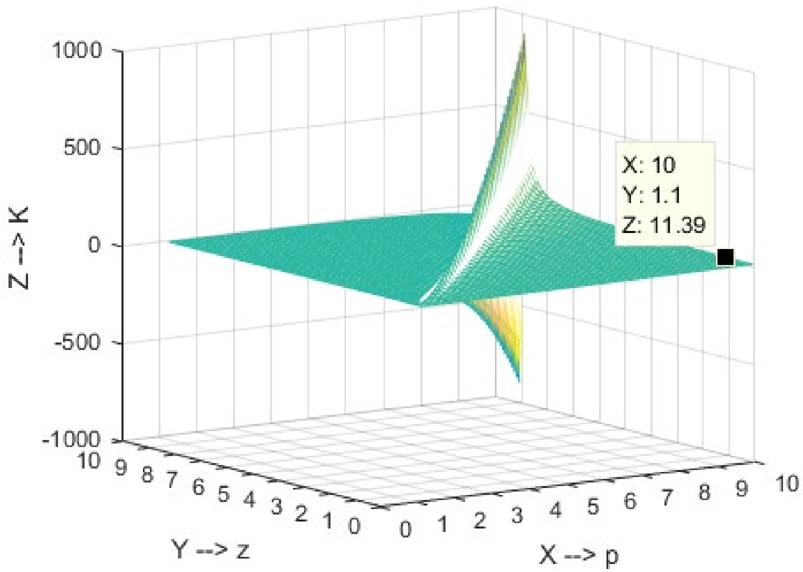


3) Obtain and analyse the conditions for stability

Analysing Condition 2 further:

- If $(p-1) > z$, $k > \frac{p(p-1)}{(p-1)-z}$
- If $(p-1) < z$, $k < \frac{p(p-1)}{(p-1)-z}$ (Not consider further, because the denominator is always negative, leading to the case of *k* less than a negative value)

Only the first condition is feasible. To visualize the result, we plot the surface of $Z_1 = p(p-1)/[(p-1)-z]$, then we can choose a value of *k* above the surface Z_1, fulfilling the first condition, together with the conditions of "*p*>1", "*z*>0" and "(*p*-1)>*z*", to ensure the overall stability of the CL system.



$p > 1 \qquad z < p-1 \qquad k > \frac{p(p-1)}{(p-1)-z}$

Additional Exercises (self-check)

Textbook ("Modern Control Systems" by R. C. Dorf & R. H. Bishop, 14th edition), Chapter 6:

- Skills Check - Can be found from Textbook pg. 428-430 (answer in pg. 445) or LMO MEC280's "Quizzes/Tutorials" section.

Additional:

E6.1, E6.4, E6.7, E6.9, E6.15, E6.17, E6.26

Partial answers to E6.25 and E6.26 (other answers are provided in the textbook):

6.26(a): $2s^2 + (K - 20)s + 10 - 10K = 0$

6.26(b): No value of *K* can make the system stable.

Important Announcement (now is Week 9)

- Coursework 2 (Lab 2) is now released, on **17/18th April** of **Week 9**.
  - Computer **Lab support/workshop sessions** are scheduled in Timetable on **28th April (Mon., all groups) and 30th April (Wed., all groups)** of **Week 11**.
  - The due date is **11th May, Sunday of Week 12**.

  Please refer to your own timetable in *e-bridge*. For the exact lab locations (there are 4 lab groups, assigned to 4 locations).

- In this lab exercise, you will analyze and solve some control problems, only within the scope of Lectures 12-19 (until the end of RLM), through the use of MATLAB, specifically the control system toolbox.
- You are encouraged to start preparing and doing the coursework at your own effort from now onwards. You can do the coursework during the workshop session (only 1 session/student), and you may bring along the questions that you have doubt with.
- You are encouraged to bring your own laptop as backup.

Concluding Remarks

- **What have been covered:** “Stability of Linear Feedback Systems”
  - Concept of stability
  - Routh Hurwitz Stability Criterion
  - Stability of state-space/variable system
  - Stability analysis using Matlab

- **In our next three lectures:** we will learn about “Root Locus Method (and analysis)”
- **What you can do from now till the next lecture:** revise the material, further reading, group study, and skill checks (LMO tutorial, or Textbook chapter 6).
- **How to get in touch**: through LMO Module’s *“General question and answer forum”* section or during my weekly consultation hour(s).