# MTH 101 Engineering Model Report
# Low-pass series RLC filter

## *Group ICAC*

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# I. Introduction

RLC circuits are widely used in production and life. For example, the television can effectively attenuate the original image distortion caused by horizontal driven impulse by tuning the RLC circuit as a filter circuit, which is also the reason why we potential engineers take this circuit into research [1].

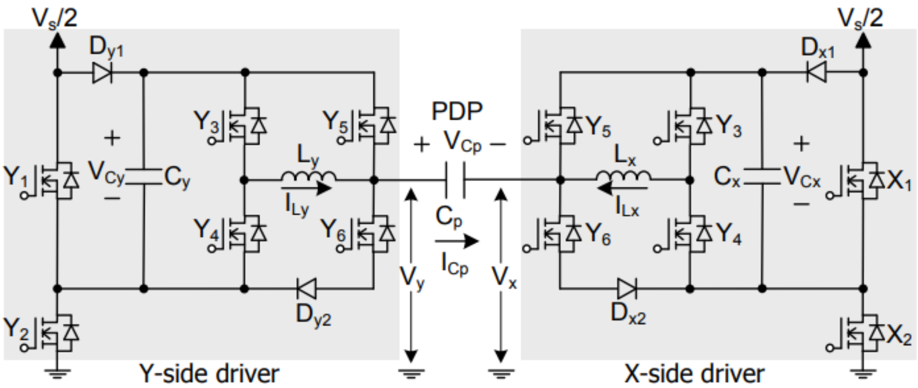



Fig.1. Practical circuit of Television. [2]

This report will cover 5 sections. Following the introduction, we will describe and solve the ODE by approaches like Laplace transform and complex analysis of our model in section 2 and 3. After that, Bode plot, an integration with our model and engineering will be provided. Ultimately, we would like to make a summary of our research.

# II. Description of the model

What we dig into is an active series RLC circuit. We will use the mathematics we have learned in combination with physics to research RLC circuits. The diagram of the circuit is shown as below.

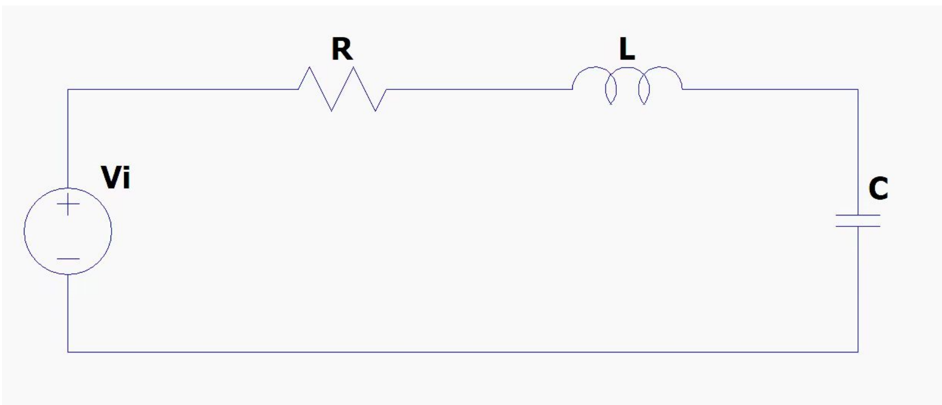



Fig.2.

## III. Problem-solving technique by Mathematics

### 1) Establishment of ODE

Trying to use Laplace transform to find the voltage of capacitor, we assume the voltages of input source and capacitor in the form of (1), and capacitor with zero initial voltage (2).

$$V_i = V_m cos(wt + \emptyset), \;\; V_C = y(t) \tag{1}$$

$$y(0) = 0 \qquad y'(0) = 0 \tag{2}$$

$$\mathcal{L}[V_i] = V(s), \;\; \mathcal{L}[y(t)] = Y(s)$$

To find the relationship of these components, we apply the KVL (Kirchhoff Voltage Law) to the loop [3].

$$V_i = V_R + V_L + V_C$$

Since all the components in the circuit are in series, the current flow through the capacitor and inductor is the same. According to Ohm's low and the formulas of capacitor and inductor (3) [3], we can get the ODE as (4).

$$i_c = C\frac{dV_c}{dt} \;,\; V_L = L\frac{di_c(t)}{dt} \;,\; V_R = i_C * R \tag{3}$$

$$V_i = RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) \tag{4}$$

### 2) An attempt to solve this ODE by Laplace Transform

Take the Laplace transformation of equation (4) on left side [4].

$$\begin{aligned} V(s) &= \mathcal{L}[V_i] \\ &= \mathcal{L}[V_m \cos(\omega t + \emptyset)] \\ &= \mathcal{L}[V_m \cos(\emptyset)\cos(\omega t)] - \mathcal{L}[V_m \sin(\emptyset)\sin(\omega t)] \\ &= V_m cos(\emptyset)\frac{s}{s^2+\omega^2} - V_m sin(\emptyset)\frac{\omega}{s^2+\omega^2} \end{aligned} \tag{5}$$

For the right side, we consider the transform of y(t).

$$Y(s) = \mathcal{L}[y(t)]$$

$$= \frac{V(s)}{1 + RCS + LCS^2}$$

$$= \frac{V_m cos(\emptyset)}{LC} * \frac{\frac{s}{s^2+\omega^2}}{\frac{1}{LC}+\frac{R}{L}S+S^2} - \frac{V_m sin(\emptyset)}{LC} * \frac{\frac{\omega}{s^2+\omega^2}}{\frac{1}{LC}+\frac{R}{L}S+S^2} \tag{6}$$

We adopt that the value of R, L, C are equal to $1k\Omega, 0.5mH, 2\mu F$ respectively. To make it clear, we just concentrate on part of $Y(S)$

$$\frac{\frac{s}{s^2+\omega^2}}{\frac{1}{LC}+\frac{R}{L}S+S^2} = \frac{\frac{s}{s^2+\omega^2}}{(S+1000)^2}$$

Obviously, if we only focus on only nominator or denominator separately, they can be easily calculated. But if they appear as a ratio, we may use convolution. Here, we define and find the Laplace transform [4].

$$Q(S) = \frac{1}{(S+1000)^2} \qquad R(S) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}[Q(s)] = q(t) = t * e^{-1000t} \qquad \mathcal{L}^{-1}[R(S)] = r(t) = cos\,\omega t$$

By the definition of convolution [4]

$$\mathcal{L}^{-1}[R(S)Q(S)] = q(t) * r(t)$$

$$= \int_0^t r(t-\tau)q(\tau)d\tau$$

$$= \int_0^t cos(t-\tau) * \tau * e^{-1000\tau}d\tau$$

[* means convolution, * means multiplication]

However, we cannot solve this definite integral, because there are 3 functions in the integral, and that is beyond our capability.

## 3) Explore another way to solve this ODE by complex analysis

$$V_i = RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) \quad (4)$$

It fails if we apply Laplace transform to this ODE. However, we may solve it if we transfer all of these functions into the complex plane with the tool called phasor. According to Euler's Formula (7) [5]

$$e^{i\emptyset} = cos\emptyset + isin\emptyset \quad (7)$$

$$cos\emptyset = Re\left[e^{i\emptyset}\right]$$

Therefore, we can transform the source voltage in equation (1) into equation (8).

$$V_i = V_m cos(wt + \emptyset)$$

$$V_i = Re\left[V_m e^{i(\omega t+\emptyset)}\right] \quad (8)$$

This equation realizes a transform from time-domain to complex-domain. Thus，we define the phasor as the constant part of complex expression (9).

$$V_m e^{i(\omega t+\emptyset)} = V_m e^{i\emptyset} * e^{i\omega t}$$

$$= \boldsymbol{V_i} * e^{i\omega t}$$

$$\boldsymbol{V_i} = V_m e^{j\emptyset} \quad (9)$$

*The complex unit $i$ is equivalent to $j$

We can also write the exponential part as (10) for simplify.

$$V_m e^{j\emptyset} = V_m \angle \emptyset \quad (10)$$

Notice that we assume the $V_C$ as expression (11) , as all the components in this circuit are linearly dependent, which means V can be represent in sinusoidal form. Additionally, we set $V_{Cmax}$ as the maximum voltage of the capacitor, and the phase angle $\theta$ between $V_C$ and $V_i$.

$$V_C = y(t) = V_{Cmax}\, cos(\omega t + \emptyset - \theta) \quad (11)$$

Now we can do the transform of the capacitor by using phasor.

$$y(t) = V_{Cmax}\, cos(\omega t + \emptyset - \theta)$$

$$= Re\left[V_{Cmax} e^{j(\emptyset - \theta)} * e^{j(\omega t)}\right]$$

$$= Re\left[\boldsymbol{y(t)} * e^{(j\omega t)}\right]$$

$$\boldsymbol{y(t)} = V_{Cmax} e^{j\emptyset} e^{j(-\theta)} \tag{12}$$

Now we try to solve the derivative, notice that $\boldsymbol{y'(t)}$ represent the phasor of the first derivative of $y(t)$.

$$\frac{dy(t)}{dt} = \omega V_{Cmax}\, cos\left(\omega t + \emptyset - \theta + \frac{\pi}{2}\right)$$

$$= Re\left[\omega V_{Cmax} e^{j(\emptyset - \theta + \frac{\pi}{2})} * e^{j(\omega t)}\right]$$

$$= Re\left[j\omega V_1 e^{j\emptyset} e^{j(-\theta)} * e^{j(\omega t)}\right]$$

$$= Re\left[\boldsymbol{y'(t)} * e^{j(\omega t)}\right]$$

$$\boldsymbol{y'(t)} = j\omega V_{Cmax} e^{j\emptyset} e^{j(-\theta)} \tag{13}$$

By the same way, we can get $\boldsymbol{y''(t)}$ of the second derivative of $y(t)$.

$$\boldsymbol{y''(t)} = -\omega^2 V_{Cmax} e^{j\emptyset} e^{j(-\theta)} \tag{14}$$

Above all we have calculated the phasor form of $y(t)$ with its first and second derivatives. By replace (9) (12) (13) (14) into equation (4), we get (15).

$$\boldsymbol{V_i} = RC * \boldsymbol{y'(t)} + LC * \boldsymbol{y''(t)} + \boldsymbol{y(t)}$$

$$V_m e^{j\emptyset} = (j\omega RC - \omega^2 LC + 1) V_{Cmax} e^{j\emptyset} e^{j(-\theta)}$$

$$V_{Cmax} = \frac{V_m}{(1 - \omega^2 LC) + j\omega RC} e^{j(\theta)} \tag{15}$$

Replace (15) into with (12), we get the expression of $\boldsymbol{y(t)}$ as (16).

$$\boldsymbol{y(t)} = \frac{V_m}{(1 - \omega^2 LC) + j\omega RC} e^{j\emptyset} \tag{16}$$

From the inverse transform of phasor, we get the expression (17) in time domain.

$$y(t) = \frac{V_m}{(1-\omega^2 LC) + j\omega RC} cos(\omega t + \emptyset) \tag{17}$$

Take the modulus of the complex number in the denominator and modify the angle in cosine, we get the final expression.

$$y(t) = \frac{V_m}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}}\ cos\ (\omega t + \emptyset - tan^{-1}(\frac{\omega RC}{1-\omega^2 LC}))$$

## IV. Integration with Engineering Field

To analyze the characteristic of this circuit as a filter, we need to calculate the transfer function T(s), which is the ratio of the Laplace ransform of the output and the input which are Y(s) and V(s) [3].

In our previous analysis, we have got V(s) as expression (5) and Y(s) as expression (6), so T(s) can be calculated as follow.

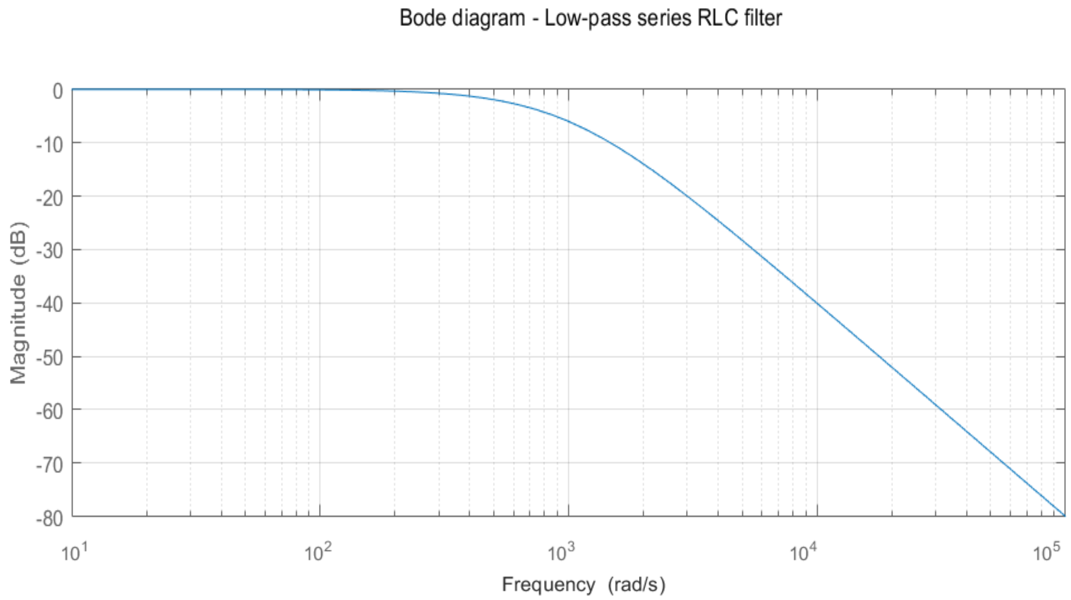
$$T(S) = \frac{Y(S)}{V(S)}$$

$$= \frac{\frac{1}{LC}}{\frac{1}{LC} + \frac{R}{L}S + S^2}$$

$$= \frac{1}{1 + RCS + LCS^2}$$

$$= \frac{1}{S^2 + 2000S + 10^6} * 10^6$$

Next, we need to use MATLAB for further interpretation. We choose bode plot and compile the relevant parameters.




The Bode diagram verifies the low-pass filter trait of this circuit. From the graph, we can find that the corner frequency $f_0$ is about 600HZ [6], which manifests that the signal will get dampened when the input frequency is larger than $f_0$ , and vice versa.

## V. Conclusion

In this report, we take television as our model and set up ODE initially. Then, we attempt to solve this ODE by Laplace transform but it fails. Subsequently, we convert our method to complex analysis, and derive the answer with the tool of phasor. Ultimately, we plot the Bode diagram to verify the functionality of our model as a low-pass filter.

However, the establishment of our model has some limitations as well. On the one hand, we have to consider the interaction of components like mutual induction between inductors, and coupling between capacitors in real applications. On the other hand, the assumption of our model is linear circuits, in real production, components like transistors can reverse linear circuit to non-linear status, which cannot be analyzed by the methods above [2].

Despite the limitation of our model, our model is still valid for deriving the corner frequency of this low-pass filter which marks the characteristic of this filter. More importantly, we have a better understanding about Laplace transform and complex analysis through this coursework.

In further study, we will contact more intricate RLC filter circuits, and we may capitalize on Matrices to replace components for simpler calculation [7], Fourier Transform to transfer from time domain to frequency domain [8]. Equipped with these tools, we are able to continue in-depth study about RLC circuits.

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