CAN102 Electromagnetism and Electromechanics

Lecture-2 Mathematics Background

3D Coordinate Systems & Vector Analysis

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Outline

- 3D Coordinate Systems
  - Key concepts about a Coordinate System
  - Rectangular, Cylindrical, Spherical CSs

- Vector Analysis
  - Integrals
    - Line/Surface/Volume Integrals
    - Differential Elements in Three CSs
  - Differentials
    - Gradient, Divergence, Curl and Laplacian
  - Theorems
    - Gauss's and Stokes' Theorems

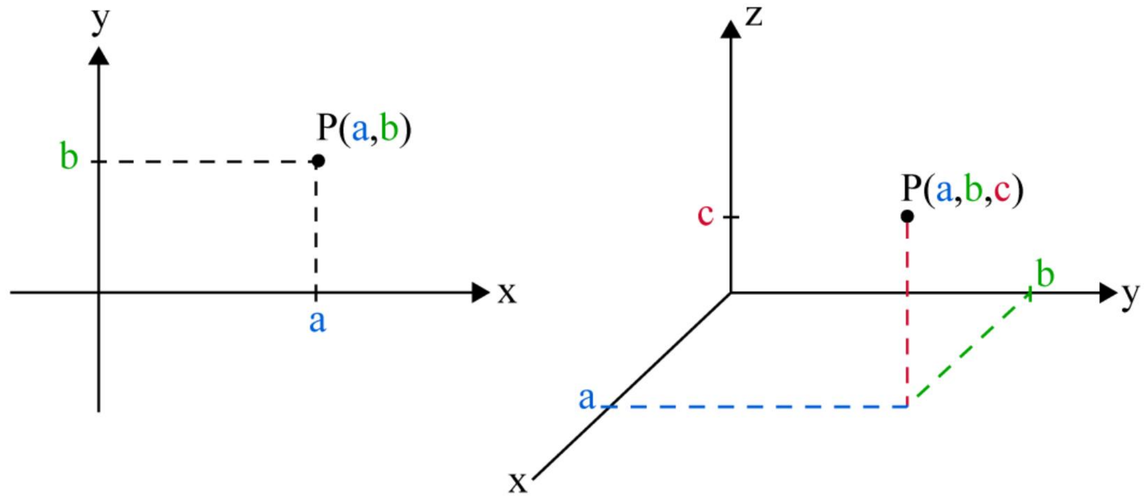
1.1 Key Points

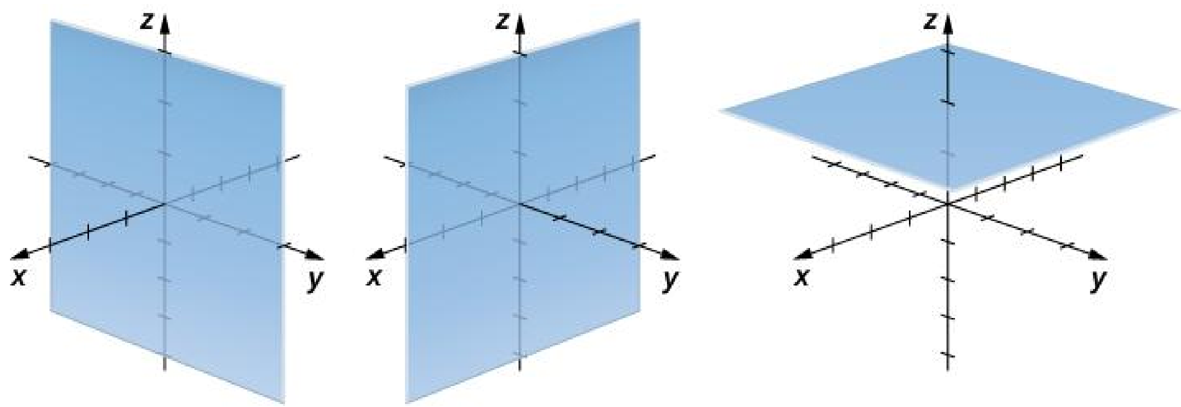
The 3D coordinate system allows us to represent a quantity in a space that contains three mutually perpendicular axes. Through the 3D coordinate system, we can now visualize points and surfaces with respect to three axes.

- Variables
- Axes and Origin
- Unit Vectors
- Position Vector
- Range of variables
- Cutting planes
- Dot and Cross products
- Conversion with other CS

1.2 Rectangular - from 2D to 3D

- Variables:
  - x, y, z
- Planes:
  - fix x => y-z plane;
    fix y => x-z plane;
    fix z => x-y plane.
- Unit vectors:
  - $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ or $\hat{\boldsymbol{a}}_x, \hat{\boldsymbol{a}}_y, \hat{\boldsymbol{a}}_z$
- Position vector:
  - $\vec{\boldsymbol{P}} = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}} + c\hat{\boldsymbol{z}}$





(a) (b) (c)

1.2 Rectangular - from 2D to 3D

- Range of variables:
  - $x, y, z \in \mathbb{R}$
- Dot product:
  - $\boldsymbol{a_x} \cdot \boldsymbol{a_x} = \boldsymbol{a_y} \cdot \boldsymbol{a_y} = \boldsymbol{a_z} \cdot \boldsymbol{a_z} = 1$
  - $\boldsymbol{a_x} \cdot \boldsymbol{a_y} = \boldsymbol{a_y} \cdot \boldsymbol{a_z} = \boldsymbol{a_z} \cdot \boldsymbol{a_x} = 0$

$$\boldsymbol{a_i} \cdot \boldsymbol{a_j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- Cross product:
  - $\boldsymbol{a_i} \times \boldsymbol{a_j} = \boldsymbol{a_k}$, sequence $\boldsymbol{a_x} \rightarrow \boldsymbol{a_y} \rightarrow \boldsymbol{a_z} \rightarrow \boldsymbol{a_x}$
- $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ are ***constant*** unit vectors
  - They are not changing according to values x, y and z.

1.3 From 2D Polar to 3D Cylindrical

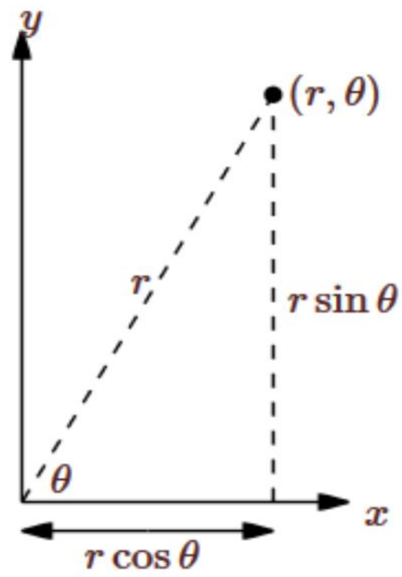
- Variables:
  - ρ, φ, z or r, φ, z
- Planes:

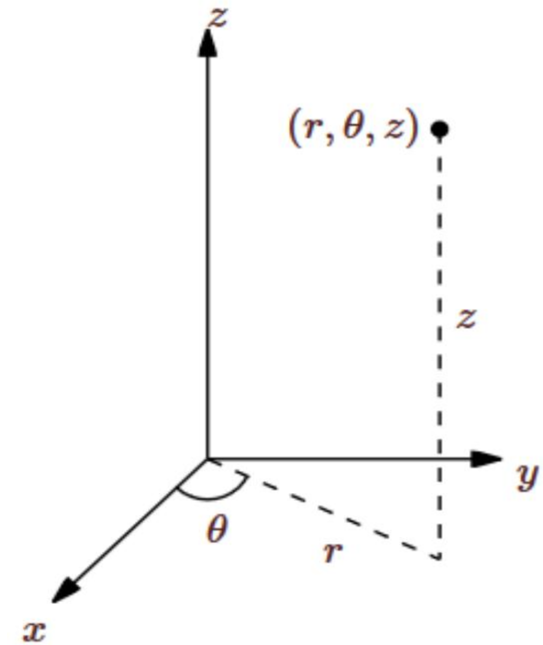
– fix ρ =>

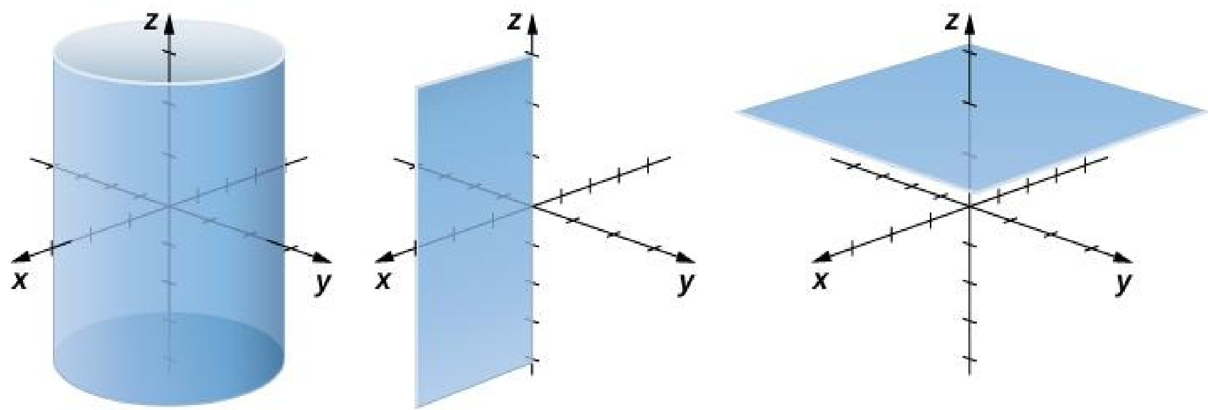
  fix φ =>

  fix z =>

- Unit vectors:
  - $\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{\varphi}}, \hat{\mathbf{z}}$
- Position vector
  - $\vec{\boldsymbol{P}} = A_\rho \widehat{\boldsymbol{\rho}} + A_z \hat{\mathbf{z}}$







(a) (b) (c)

1.3 From 2D Polar to 3D Cylindrical

- Range of variables:
  - $\rho \in [0, +\infty), \varphi \in [0, 2\pi), z \in (-\infty, +\infty)$
- Dot product:
  - $\boldsymbol{a}_\rho \cdot \boldsymbol{a}_\rho = \boldsymbol{a}_\varphi \cdot \boldsymbol{a}_\varphi = \boldsymbol{a}_z \cdot \boldsymbol{a}_z = 1$
  - $\boldsymbol{a}_\rho \cdot \boldsymbol{a}_\varphi = \boldsymbol{a}_\varphi \cdot \boldsymbol{a}_z = \boldsymbol{a}_z \cdot \boldsymbol{a}_\rho = 0$

$$\boldsymbol{a}_i \cdot \boldsymbol{a}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- Cross product:
  - $\boldsymbol{a}_i \times \boldsymbol{a}_j = \boldsymbol{a}_k$, sequence $\boldsymbol{a}_\rho \rightarrow \boldsymbol{a}_\varphi \rightarrow \boldsymbol{a}_z \rightarrow \boldsymbol{a}_\rho$

1.3 3D Rectangular vs. Cylindrical

- Conversion between the two CSs:
  - Values: $x = \rho\cos\varphi$
    $y = \rho\sin\varphi$
    $z = z$

$$\boxed{\begin{bmatrix} \boldsymbol{a_x} \\ \boldsymbol{a_y} \\ \boldsymbol{a_z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{a_\rho} \\ \boldsymbol{a_\varphi} \\ \boldsymbol{a_z} \end{bmatrix}}$$

  - Unit vectors:

$$\begin{aligned} \hat{\boldsymbol{x}} &= \hat{\rho}\cos\varphi - \hat{\varphi}\sin\varphi \\ \hat{\boldsymbol{y}} &= \hat{\rho}\sin\varphi + \hat{\varphi}\cos\varphi \\ \hat{\boldsymbol{z}} &= \hat{\boldsymbol{z}} \end{aligned}$$

and

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \hat{x}\cos\varphi + \hat{y}\sin\varphi \\ \hat{\boldsymbol{\varphi}} &= -\hat{x}\sin\varphi + \hat{y}\cos\varphi \\ \hat{\boldsymbol{z}} &= \hat{\boldsymbol{z}} \end{aligned}$$

- In Rectangular CS: $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are ***constant*** unit vectors whose magnitudes are 1 and directions unchanged
- In Cylindrical CS: $\hat{\boldsymbol{z}}$ is ***constant*** unit vector. But $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varphi}}$ are not, their directions changed according to the value φ.

Quiz 1

- Prove

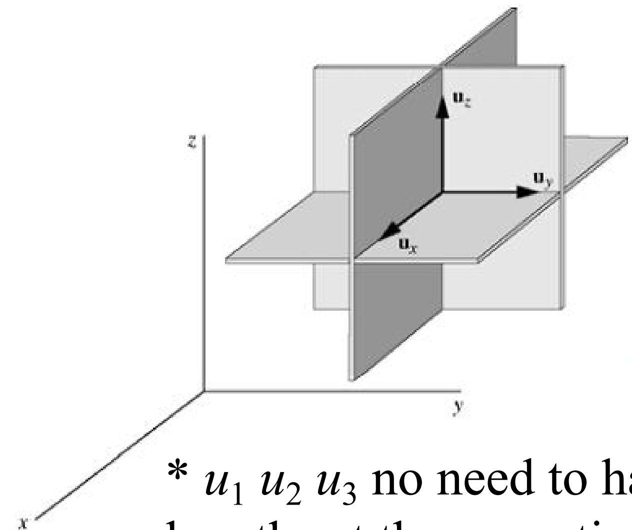
$$\frac{\partial \boldsymbol{a_\rho}}{\partial \varphi} = \boldsymbol{a_\varphi}, \qquad \frac{\partial \boldsymbol{a_\varphi}}{\partial \varphi} = -\ \boldsymbol{a_\rho}$$

- Answer:

Orthogonal Coordinate Systems

- Assume that the three families of surfaces are described by

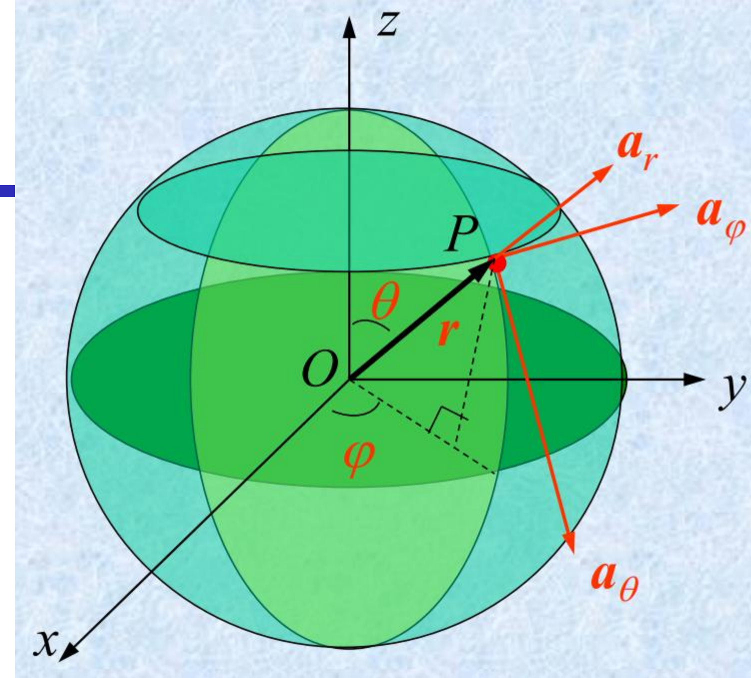
$$u_1 = constant$$
$$u_2 = constant$$
$$u_3 = constant$$



* u_1 u_2 u_3 no need to have lengths at the same time

- When these three surfaces are mutually **perpendicular** to one another, we have an ***orthogonal coordinate system***.
  - Non-orthogonal CSs are not used because they complicate the given problems.

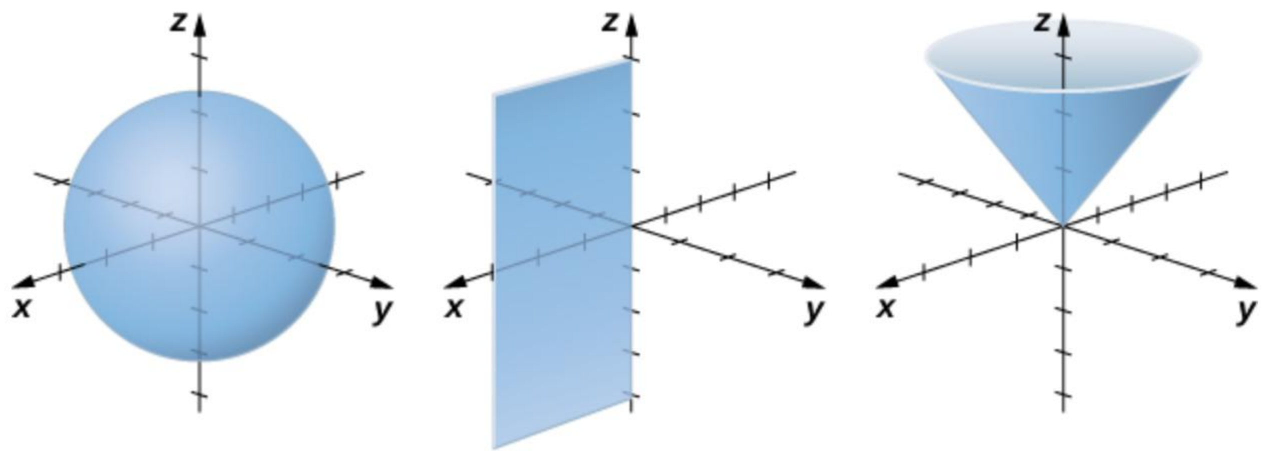
1.4 Spherical Coordinates



- Variables and range:
  - $r \in [0, +\infty)$: distance to origin
  - $\theta \in [0, \pi)$: angle between $\boldsymbol{r}$ and $+\boldsymbol{z}$ axis
  - $\varphi \in [0, 2\pi)$: angle between the projection of $\boldsymbol{r}$ in x-y plane and $+\boldsymbol{x}$ axis

- Planes:

– fix r =>
fix θ =>
fix φ =>



(a) (b) (c)

1.4 Spherical Coordinates

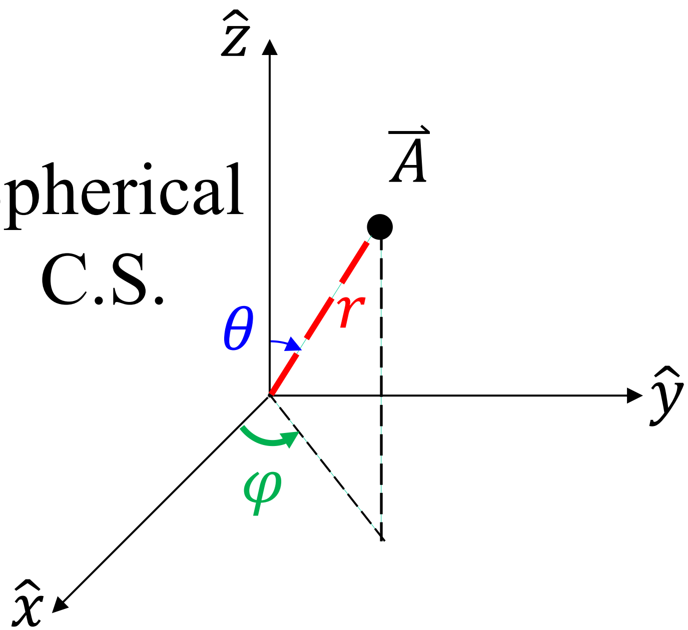
- Unit vectors:
  - $\hat{\boldsymbol{r}}, \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\varphi}}$ or $\boldsymbol{a_r}, \boldsymbol{a_\theta}, \boldsymbol{a_\varphi}$ (sometimes $\hat{\boldsymbol{r}} \rightarrow \widehat{\boldsymbol{R}}$)
- Position vector
  - $\overrightarrow{\boldsymbol{P}} = A_r \hat{\boldsymbol{r}}$
- Dot product: $\boldsymbol{a_i} \cdot \boldsymbol{a_j} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$
- Cross product: $\boldsymbol{a_i} \times \boldsymbol{a_j} = \boldsymbol{a_k}$,

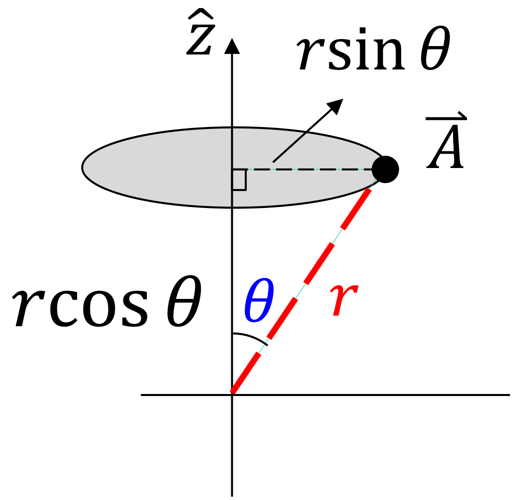
  sequence $\boldsymbol{a_r} \rightarrow \boldsymbol{a_\theta} \rightarrow \boldsymbol{a_\varphi} \rightarrow \boldsymbol{a_r}$

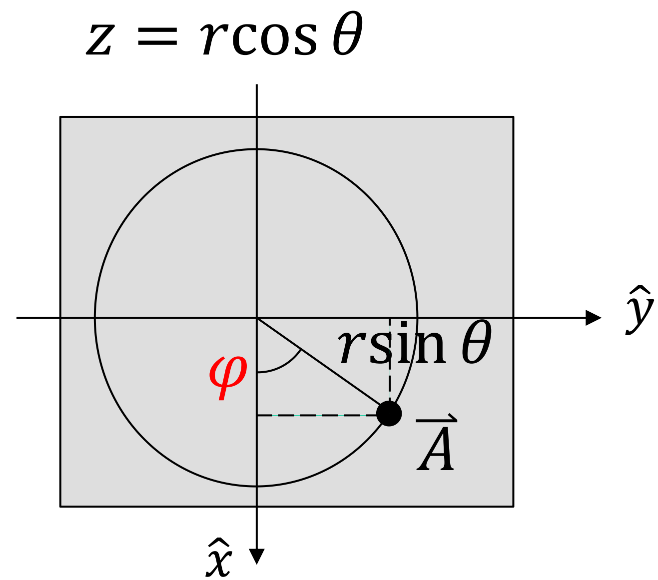
1.4 Spherical Coordinates

- Conversion between Rectangular and Spherical:

Spherical C.S.







$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arctan\frac{\sqrt{x^2 + y^2}}{z}$$

$$\varphi = \arctan\frac{y}{x}$$

1.4 Spherical Coordinates

- In Spherical CS: $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ are all changing according to the values θ and φ.

$$\begin{cases} \boldsymbol{a}_r = \boldsymbol{a}_x \sin\theta\cos\varphi + \boldsymbol{a}_y \sin\theta\sin\varphi + \boldsymbol{a}_z \cos\theta \\ \boldsymbol{a}_\theta = \boldsymbol{a}_x \cos\theta\cos\varphi + \boldsymbol{a}_y \cos\theta\sin\varphi - \boldsymbol{a}_z \sin\theta \\ \boldsymbol{a}_\varphi = -\boldsymbol{a}_x \sin\varphi + \boldsymbol{a}_y \cos\varphi \end{cases}$$

$$\begin{cases} \boldsymbol{a}_x = \boldsymbol{a}_r \sin\theta\cos\varphi + \boldsymbol{a}_\theta \cos\theta\cos\varphi - \boldsymbol{a}_\varphi \sin\varphi \\ \boldsymbol{a}_y = \boldsymbol{a}_r \sin\theta\sin\varphi + \boldsymbol{a}_\theta \cos\theta\sin\varphi + \boldsymbol{a}_\varphi \cos\varphi \\ \boldsymbol{a}_z = \boldsymbol{a}_r \cos\theta - \boldsymbol{a}_\theta \sin\theta \end{cases}$$

$$\begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_\varphi \end{bmatrix}$$

1. Summary of the Coordinate Systems

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi$ $+\hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi$ $+\hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi$ $+A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi$ $+A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi$ $+\hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi$ $+\hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi$ $+A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi$ $+A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

1. Summary of the Coordinate Systems

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A	$\mathbf{a}_x A_x + A_y\,\mathbf{a}_y + A_z\,\mathbf{a}_z$	$\mathbf{a}_r A_r + A_\phi\,\mathbf{a}_\phi + A_z\,\mathbf{a}_z$	$\mathbf{a}_R A_R + A_\theta\,\mathbf{a}_\theta + A_\phi\,\mathbf{a}_\phi$
Magnitude of A, $\lvert A\rvert$	$+\sqrt{A_x^2 + A_y^2 + A_z^2}$	$+\sqrt{A_r^2 + A_\phi^2 + A_z^2}$	$+\sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$
Base vectors properties	$\mathbf{a}_x \bullet \mathbf{a}_x = \mathbf{a}_y \bullet \mathbf{a}_y = \mathbf{a}_z \bullet \mathbf{a}_z = 1$ $\mathbf{a}_x \bullet \mathbf{a}_y = \mathbf{a}_y \bullet \mathbf{a}_z = \mathbf{a}_z \bullet \mathbf{a}_x = 0$ $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z, \quad \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$ $\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$	$\mathbf{a}_r \bullet \mathbf{a}_r = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = \mathbf{a}_z \bullet \mathbf{a}_z = 1$ $\mathbf{a}_r \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_z = \mathbf{a}_z \bullet \mathbf{a}_r = 0$ $\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z, \quad \mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r$ $\mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi$	$\mathbf{a}_R \bullet \mathbf{a}_R = \mathbf{a}_\theta \bullet \mathbf{a}_\theta = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = 1$ $\mathbf{a}_R \bullet \mathbf{a}_\theta = \mathbf{a}_\theta \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_R = 0$ $\mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi, \quad \mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R$ $\mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta$
Dot product, $\mathbf{A} \cdot \mathbf{B}$	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\phi B_\phi + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\mathbf{A} \times \mathbf{B}$	$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_\phi & \mathbf{a}_z \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta & \mathbf{a}_\phi \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$

Quiz 2

- The geometry defined by $R = 5$ and $\theta = 90^\circ$ is a __________, and the geometry defined by $\theta = 60^\circ$ and $\varphi = 60^\circ$ is a __________.
  - (a) A circle and a ray;
  - (b) A straight line and a spherical surface;
  - (c) A circle and a circle;
  - (d) A planar surface and a spherical surface

Quiz 3

- Express the vector in the rectangular coordinate system.

$$\boldsymbol{A} = \frac{1}{\rho}\hat{\rho} + 5\sin 2\varphi\,\hat{z}$$

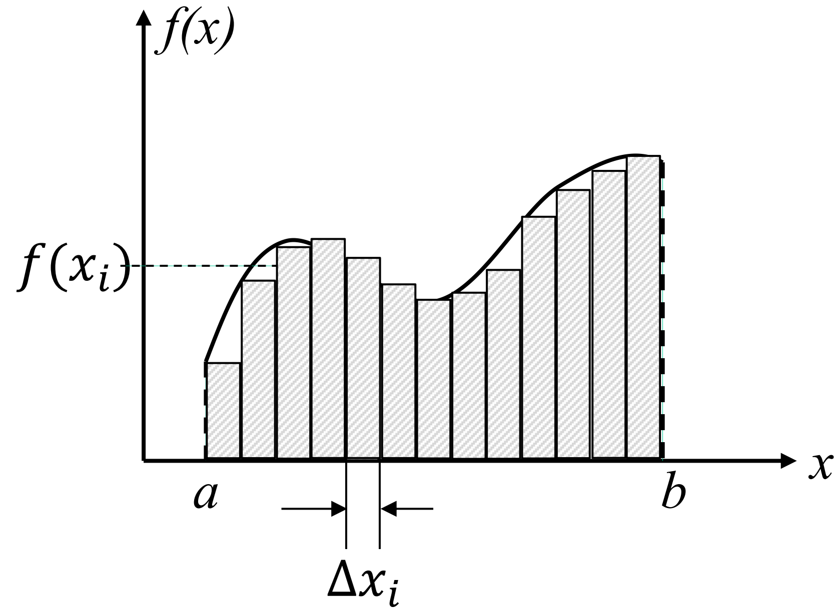
2.1 Vector Analysis - Integrals

- Vector Analysis - Integrals
  - Line Integral
  - Surface Integral
  - Volume Integral

- Differential Elements
  - Rectangular CS
  - Cylindrical CS
  - Spherical CS

2.1.1 Line Integral

- 1D Scalar function *f(x)* of single variable *x*:
  - Continuous, single-valued;
  - Variable x;
  - Limits: $a \leq x \leq b$.



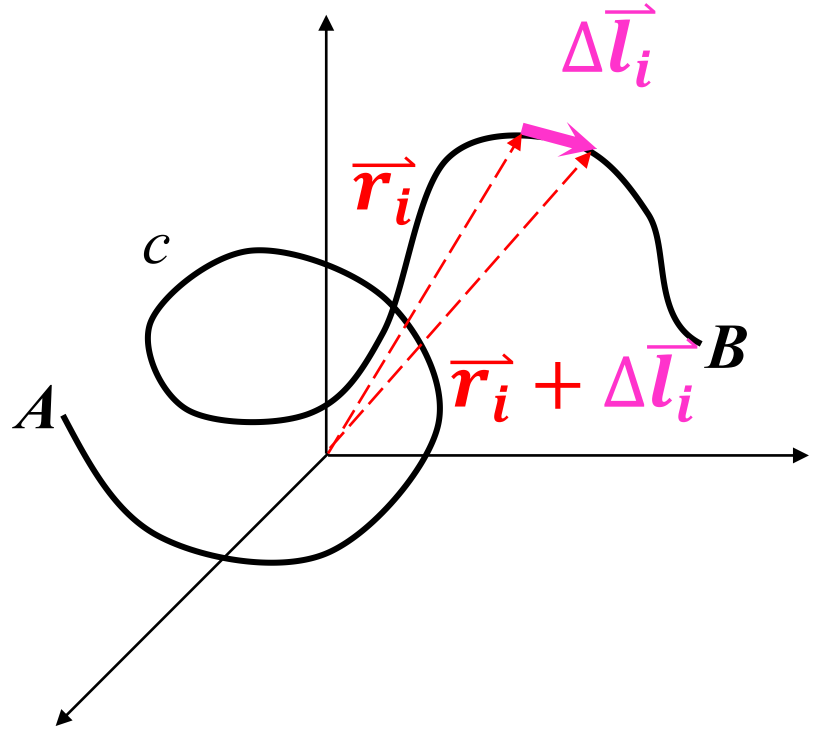
Integral of a continuous, single-valued function (scalar function)

- The integral is defined as:

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

2.1.1 Line Integral

- 3D Scalar function $f(\vec{r})$:
  - Continuous, single-valued;
  - Variable $\vec{r}$;
  - Limits: point A to B.



- Integral is defined as

$$\int_c f(\vec{r})d\vec{l} = \lim_{\substack{n\to\infty \\ \Delta\vec{l}_i \to 0}} \sum_{i=1}^{n} f(\vec{r}_i)\Delta\vec{l}_i$$

- Example:
  - Total electric field of a charge-carrying line with uneven distribution

- $\Delta\vec{l}_i$ - length vectors
- $\vec{r}_i$ - position vectors
- $f(\vec{r}_i)$ - scalar function value

2.1.1 Line Integral

- Scalar line integral for a **vector** field $\vec{F}(\vec{r})$:

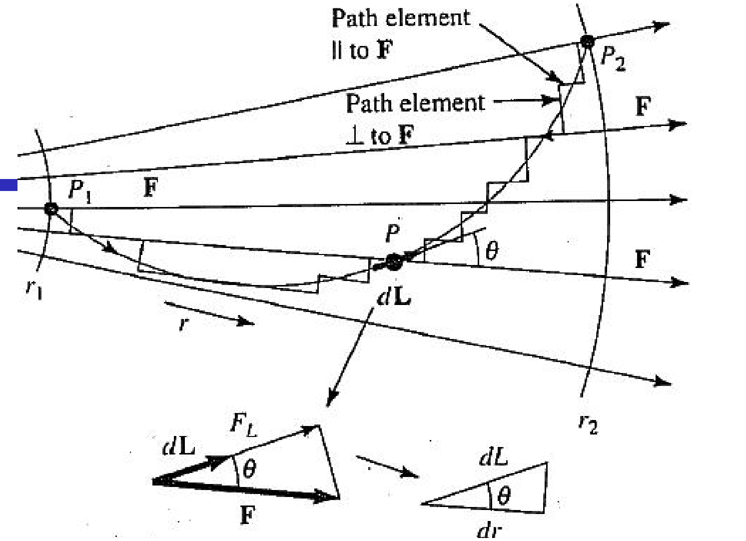
$$\int_c \vec{F}(\vec{r}) \cdot d\vec{l} = \lim_{\substack{n\to\infty \\ \Delta\vec{l}_i \to 0}} \sum_{i=1}^{n} \vec{F}(\vec{r}_i)\Delta\vec{l}_i$$

  – Example: Electric potential along a line

- Vector line integral for a **vector** field $\vec{F}(\vec{r})$:

$$\int_c \vec{F}(\vec{r}) \times d\vec{l} = \lim_{\substack{n\to\infty \\ \Delta\vec{l}_i \to 0}} \sum_{i=1}^{n} \vec{F}(\vec{r}_i) \times \Delta\vec{l}_i$$

  – Example: Total magnetic field of a current-carrying wire



Example: Work done by moving from point P_1 to P_2 along a curved path $\boldsymbol{l}$.

- Force $\vec{F}$ along dL segment:

$$\vec{F} \cdot d\vec{L} = F cos\theta dL$$

- So, the work dW is:

$$dW = \vec{F} \cdot d\vec{L} = F cos\theta dL$$

- Sum dW as P moves from P_1 to P_2:

$$W = \int_{P_1}^{P_2} dW = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{L}$$

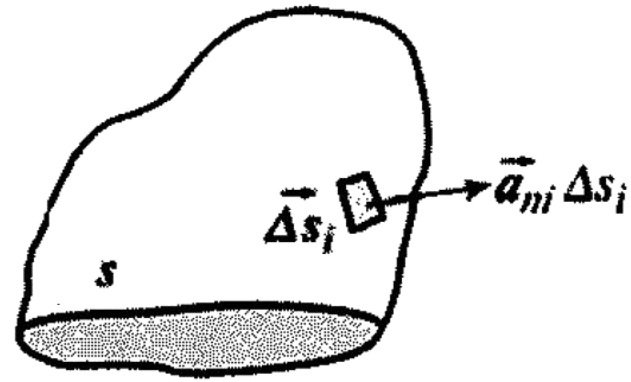
2.1.1 Line Integral

- Close path integral: the path of integration can be around a closed curve.
  - The starting and ending points *a* and *b* coincide, so no need to give starting and ending points;
  - The path still has directions: cw or ccw;
  - All line integrals can be performed on closed curves.

$$\oint_c f(\vec{r})d\vec{l} \qquad \oint_c \vec{F}(\vec{r})\cdot d\vec{l} \qquad \oint_c \vec{F}(\vec{r})\times d\vec{l}$$

2.1.2 Surface Integral

- Divide the given surface s into a large number of n small surfaces:
  - All $\Delta s_i \to 0$ in the limit;
  - Vector surface $\Delta \vec{s}_i$
    - Area Δs_i;
    - Direction $\vec{a}_{ni} \perp \Delta s_i$.



- Surface integral of a scalar function:

$$\iint_s f(\vec{r})d\vec{s} = \lim_{\substack{n\to\infty \\ \Delta\vec{s}_i \to 0}} \sum_{i=1}^{n} f(\vec{r}_i)\Delta\vec{s}_i$$

- Scalar surface integral of a vector function:

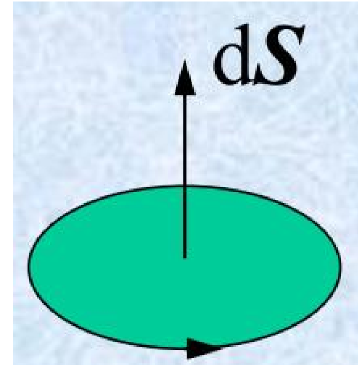
$$\iint_s \vec{F}(\vec{r}) \cdot d\vec{s} = \lim_{\substack{n\to\infty \\ \Delta\vec{s}_i \to 0}} \sum_{i=1}^{n} \vec{F}(\vec{r}_i) \cdot \Delta\vec{s}_i$$

- Vector surface integral of a vector function:

$$\iint_s \vec{F}(\vec{r}) \times d\vec{s} = \lim_{\substack{n\to\infty \\ \Delta\vec{s}_i \to 0}} \sum_{i=1}^{n} \vec{F}(\vec{r}_i) \times \Delta\vec{s}_i$$

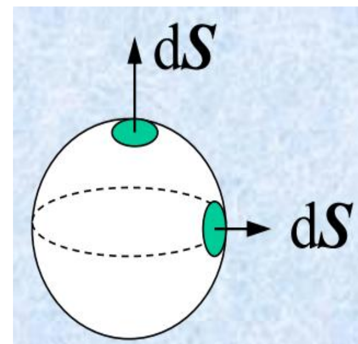
2.1.2 Surface Integral

- Open surface
  - Bounded by a closed line (with direction)
  - Direction of the surface: **Right-hand rule**



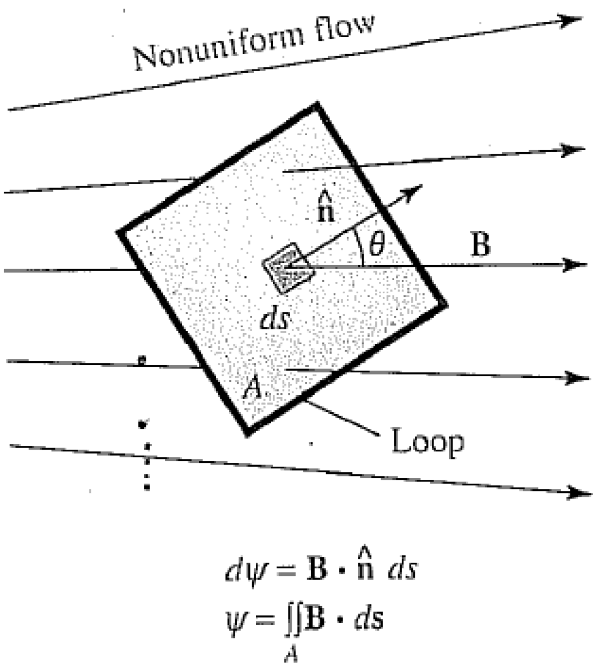
- Closed surface
  - No boundary
  - Seperate the space as *interior* and *exterior*
  - Direction of the surface:

Pointing outwards, normal direction



2.1.2 Surface Integral

- Example: In a vector field $\vec{B}$ (*e.g.*, rate of the water flow), find the flux flowing through a surface **A.**



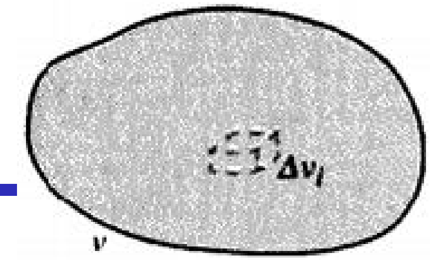
- $d\vec{s}$ – magnitude A; direction $\hat{n} \perp$ surface;
- $\vec{B}$ – magnitude B and direction of the flow;
- θ – the angle between $\vec{B}$ and $\hat{n}$;
- $d\psi$ – the incremental flux through a surface area element ds (with normal $\hat{n}$):

$$d\psi = \vec{B} \cdot d\vec{s} = \vec{B} \cdot \hat{n} ds$$

- Integrate the increment flux $d\psi$ over the whole area A gives:

$$\psi = \iint_A \vec{B} \cdot \hat{n} ds$$

2.1.3 Volume Integral



- Divide a given volume *v* into *n* small volume elements:

$$\Delta v \to 0 \text{ as } n \to \infty$$

– Scalar volume integral:

$$\iiint_v f(\vec{r})dv = \lim_{\substack{n\to\infty \\ \Delta v_i \to 0}} \sum_{i=1}^{n} f(\vec{r}_i)\Delta v_i$$

- Example: Overall E-field potential

– Vector volume integral:

$$\iiint_v \vec{F}(\vec{r})dv = \lim_{\substack{n\to\infty \\ \Delta v_i \to 0}} \sum_{i=1}^{n} \vec{F}(\vec{r}_i)\Delta v_i$$

- Example: Overall E-field intensity

Example 1:

Total charge over a region with the charge density $\rho(\vec{r})$:

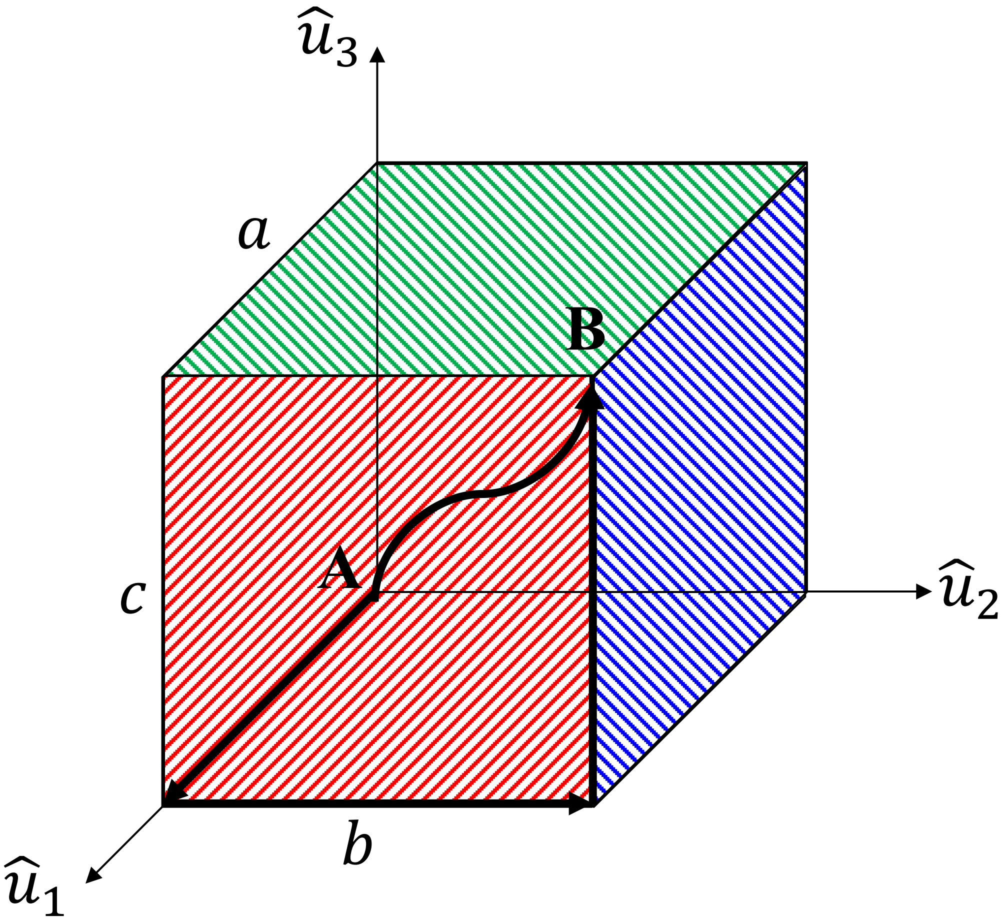
$$Q = \iiint_v \rho(\vec{r})dv$$

Example 2:

Total electric field intensity at point $\vec{P}(\vec{r})$ generated by the charged region with charge density of $\rho(\vec{r}')$:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_v \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^2}\hat{n}_{rr'}dv'$$

2.2 Differential Elements



- Line vector

  $\vec{l}_{AB} = a\hat{u}_1 + b\hat{u}_2 + c\hat{u}_3$

- Surfaces Vectors:

  $\hat{u}_1$ direction: $\vec{s}_{u_1} = b \times c\,\hat{u}_1$

  $\hat{u}_2$ direction: $\vec{s}_{u_2} = a \times c\,\hat{u}_2$

  $\hat{u}_3$ direction: $\vec{s}_{u_3} = a \times b\,\hat{u}_3$

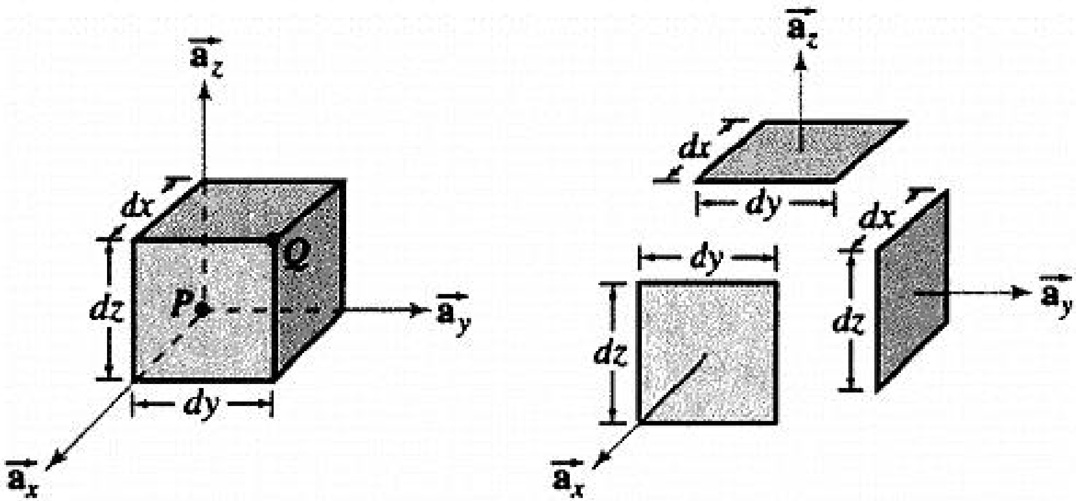
- Volume (scalar):

  $v = a \times b \times c$

2.2.1 In Cartesian CS

Cartesian (rectangular) CS

- $\hat{u}_1 \leftrightarrow \hat{x},\ \ \hat{u}_2 \leftrightarrow \hat{y},\ \ \hat{u}_3 \leftrightarrow \hat{z}$
- $a \leftrightarrow dx,\ \ b \leftrightarrow dy,\ \ c \leftrightarrow dz$



Differential line element:

$$d\boldsymbol{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$

3 differential surface element:

$$d\boldsymbol{s_x} = \hat{x}dydz$$

$$d\boldsymbol{s_y} = \hat{y}dzdx$$

$$d\boldsymbol{s_z} = \hat{z}dxdy$$

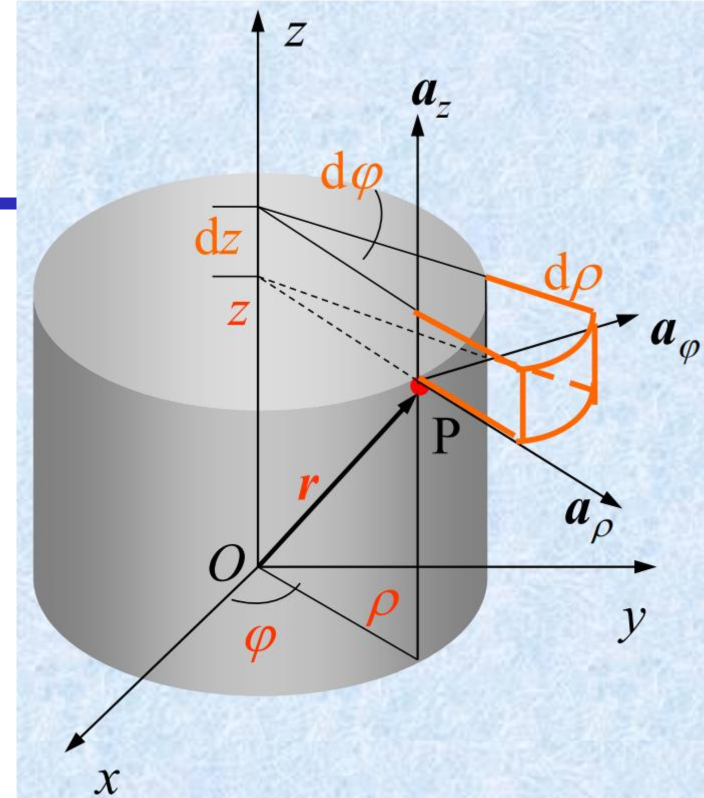
Differential volume element:

$$dv = dxdydz$$

2.2.2 In Cylindrical CS

Cylindrical CS

- $\hat{\boldsymbol{u}}_1 \leftrightarrow \hat{\boldsymbol{\rho}},\ \hat{\boldsymbol{u}}_2 \leftrightarrow \hat{\boldsymbol{\phi}},\ \hat{\boldsymbol{u}}_3 \leftrightarrow \hat{\mathbf{z}}$
- $a \leftrightarrow d\rho,\ b \leftrightarrow \rho d\phi,\ c \leftrightarrow dz$



Differential line element:

$$d\boldsymbol{l} = \hat{\boldsymbol{\rho}} d\rho + \hat{\boldsymbol{\phi}} \rho d\phi + \hat{\mathbf{z}} dz$$

3 differential surface element:

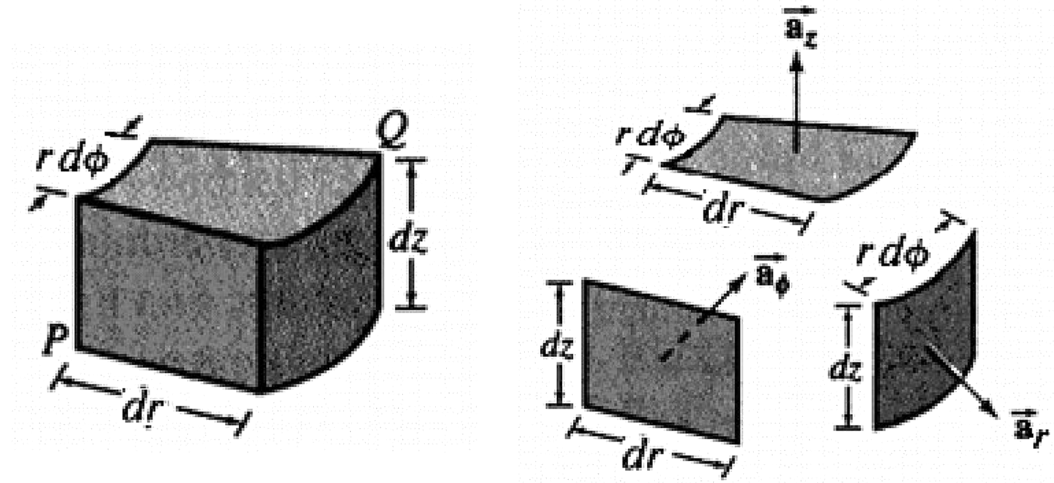
$$d\boldsymbol{s}_{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}} \rho d\phi dz$$

$$d\boldsymbol{s}_{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} d\rho dz$$

$$d\boldsymbol{s}_{\boldsymbol{z}} = \hat{\mathbf{z}} \rho d\rho d\phi$$

Differential volume element:

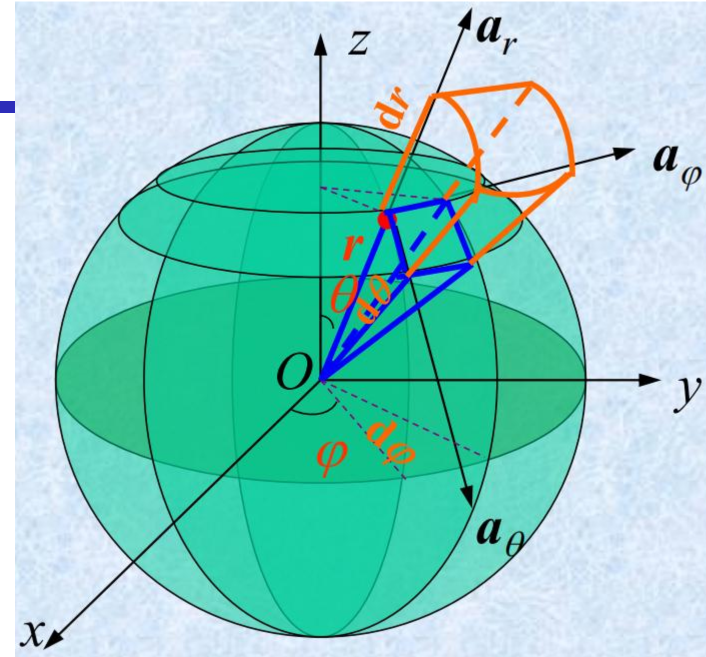
$$dv = \rho d\rho d\phi dz$$



2.2.3 In Spherical CS

Spherical CS

- $\hat{\boldsymbol{u}}_1 \leftrightarrow \hat{\boldsymbol{r}},\ \hat{\boldsymbol{u}}_2 \leftrightarrow \hat{\boldsymbol{\theta}},\ \hat{\boldsymbol{u}}_3 \leftrightarrow \hat{\boldsymbol{\phi}}$
- $a \leftrightarrow dr$
- $b \leftrightarrow rd\theta$
- $c \leftrightarrow r\sin\theta\, d\phi$



Differential line element:

$$d\boldsymbol{l} = \hat{\boldsymbol{r}}dr + \hat{\boldsymbol{\theta}}rd\theta + \hat{\boldsymbol{\phi}}r\sin\theta\, d\phi$$

3 differential surface element:

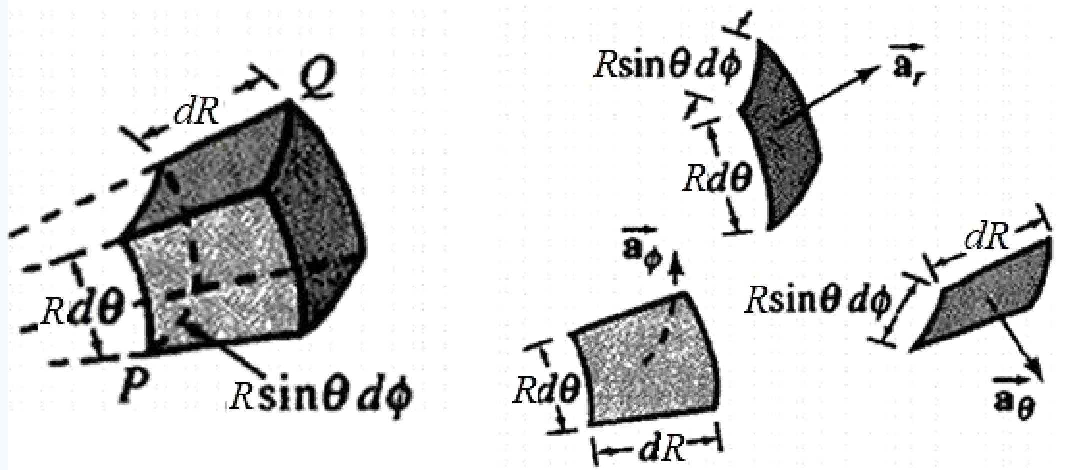
$$d\boldsymbol{s}_{\boldsymbol{r}} = \hat{\boldsymbol{r}}r^2\sin\theta\, d\theta d\phi$$

$$d\boldsymbol{s}_{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}r\sin\theta\, d\phi dr$$

$$d\boldsymbol{s}_{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}rd\theta dr$$

Differential volume element:

$$dv = r^2\sin\theta\, drd\theta d\phi$$



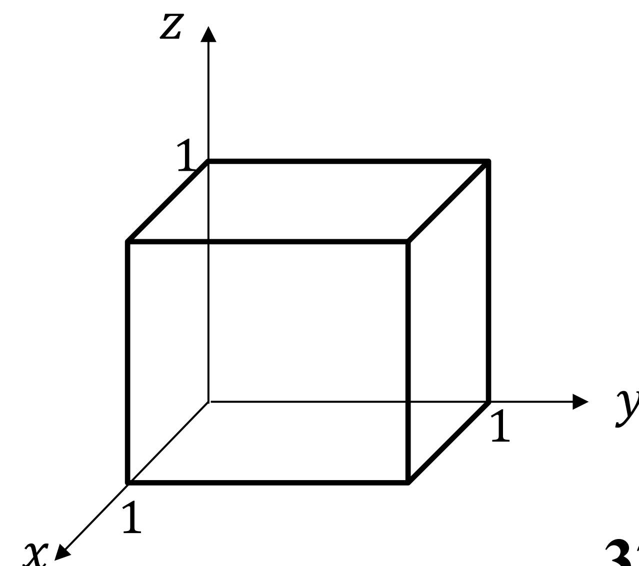
Quiz 4

1. True or False:

   – Over a closed curve c, $\oint_c \, d\vec{l} = 0$.

2. Evaluate $\oiint \vec{r} \cdot d\vec{s}$ over the closed surface of the cube bounded by $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$, where $\vec{r}$ is the position vector of any point **on** the surface of the cube.

   A) 0 B) -6 C) 6 D) 3

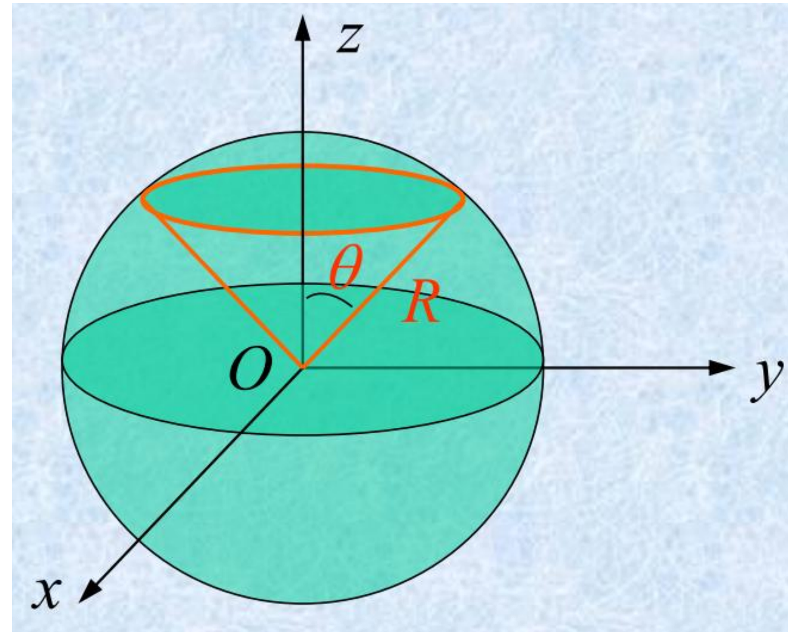


Quiz 5

- Calculate the surface integral

$$\int_S \boldsymbol{a_r} \cdot d\boldsymbol{S}$$

  – where S is the area cutted from a sphere by a cone surface with the angle θ_0, as shown on the right.



Next ...

- Static Electric Fields
  - Coulomb's Law
  - Visualisation of Electric-field
  - Electric-fields produced by continuous charge distributions
    - using line integral, surface integral and volume integral
  - Electric flux and Flux density
  - Gauss's Law and Divergence