MTH102 Engineering Mathematics II

Lesson 2: Probability theory

Term: 2024

Outline

A brief history of probability

- Gambling questions on profitable strategies, 1650s
- Equally likely models: Blaise Pascal & Pierre de Fermat.
- Law of large numbers: Bernoulli 1713 & de Moivre 1718.
- Applications on scientific and practical problems other than games of chance, Laplace 1812.
- Set theory, Cantor 1870s.
- Probability theory on an axiomatic basis, Kolmogorov 1933.
- A branch of measure theory, nowadays.

Outline

1. Set theory
2. Axioms of probability
3. Equally likely models

Sample space and events

- Random experiment: the outcome of an experiment is not predictable with certainty.
- Sample space: the set of all possible outcomes of an experiment.
- Event: any subset of the sample space, i.e. a set consisting of possible outcomes of the experiment.

Example:

- If the experiment consists of flipping a fair coin, then the sample space is

$$S = \{H, T\},$$

  where the outcome H means head and T means tail. If $E = \{H\}$, then E is the event that the coin is head.
- If the experiment consists of flipping two fair coins, then the sample space

$$S = \{HH, HT, TH, TT\}.$$

  If $E = \{HH, HT\}$, then E is the event that a head appears on the first coin.

Sample space and events

Examples:

- If the experiment consists of tossing one die, then the sample space is
$$S = \{1, 2, 3, 4, 5, 6\}. \quad |S| = 6$$
If E is the event that the number is less than 3, then $E = \{1, 2\}$.
- If the experiment consists of tossing two dice, then the sample space consists of 36 outcomes
$$S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}. \quad |S| = 6 \times 6 = 36$$
If E is the event that the sum of the two dice is 6, then $E = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$.
- In an experiment, a die is rolled continually until a 6 appears, at which point the experiment stops. Consider the number of rolls, then
$$S = \{1, 2, 3, \ldots\}. \quad |S| = \infty$$
- If the experiment consists of the waiting time for a bus, then
$$S = \{x : 0 \leq x < \infty\} = [0, \infty). \quad |S| = \infty$$

Sample space: exercise

In an training, an archer keeps shooting for one target until he successfully shoots the target for the first time. What is the sample space of this experiment?

Sol: The sample space S in the experiment can be in many different forms, such as

$S = \{0, 1, \cdots\} = \mathbb{N}$, the no. of shoots

$S = \{S, FS, FFS, FFFS, \cdots\}$, the possible results of shoots.

....

S Sample space is NOT unique for the same random experiment!

Events: logical relations

Let S be the sample space, and E, F are two events (two subsets of S).

- If the outcome of an experiment is contained in E, then we say that E has occurred.
- $E \cup F$: the **union** of E and F, i.e. either E or F occur.
- EF ($E \cap F$): the **intersection** of E and F, i.e. both E and F occur.
- E^c: the **complement** of E, i.e. E does not occur.
- $E \subset F$: E is contained in F, i.e. if E occurs, then F occurs.

Example: the experiment consists of flipping two coins and the sample space

$$S = \{HH, HT, TH, TT\}.$$

Let E be the event that the first coin is heads, F be the event that the outcomes of the two coins are different. Then

$$E = \{HH, HT\}, \ F = \{HT, TH\}.$$

Moreover

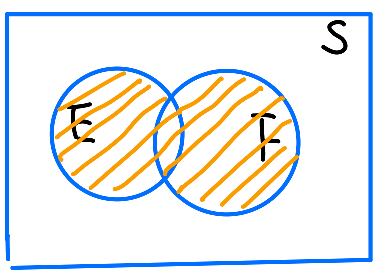
$$E \cup F = \{HH, HT, TH\}, \ EF = \{HT\}, \ E^c = \{TH, TT\}.$$

Events: logical relation

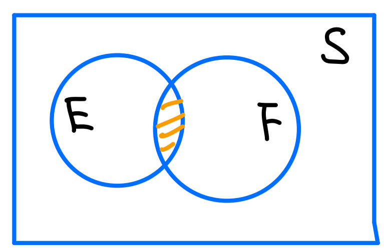
- The null event $\emptyset$: the event consisting no outcomes.
- $S^c = \emptyset$, and $\emptyset^c = S$.
- If $EF = \emptyset$, then E and F are said to be **mutually exclusive**.
- E and E^c are mutually exclusive.
- If $E_1, E_2, \ldots$ are events, then $\bigcup\limits_{n=1}^{\infty} E_n$ denotes the union of these events, i.e. at least one of these events occurs.
- If $E_1, E_2, \ldots$ are events, then $\bigcap\limits_{n=1}^{\infty} E_n$ denotes the intersection of these events, i.e. all these events occur.

Venn diagrams

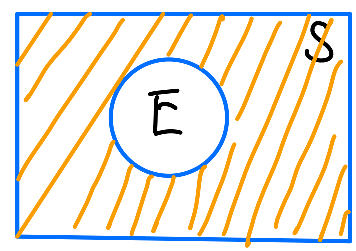
The Venn diagram is a graphical representation of logical relations among events.



$E \cup F$



$E \cap F$



E^c

Rules from the set theory

Let E, F, G be the subsets of S, then the following are satisfied.

- Commutative laws:

$$E \cup F = F \cup E, \;\; EF = FE.$$

- Associative laws:

$$(E \cup F) \cup G = E \cup (F \cup G), \;\; (EF)G = E(FG).$$

- Distributive laws:

$$(E \cup F)G = EG \cup FG, \;\; EF \cup G = (E \cup G)(F \cup G).$$

- DeMorgan's laws:

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c, \;\; \left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c.$$

Outline

Axiomatic approach

- **Axiomatic approach**, in logic, a procedure by which an entire system is generated in accordance with specified rules by logical deductions from certain basic axioms, which in turn are constructed from a few terms taken as primitive.
- The oldest example: Euclid's geometry.

- Early in the 20th century, Russel & Whitehead attempted to formalize all of mathematics in an axiomatic manner.
- In 1933, Kolmogorov outlined the axiomatic basis for the modern probability theory.

Axioms of probability

Consider an experiment whose sample space is S. For each event E of S, we assume that a number $P(E)$ is defined and satisfies the following three axioms:

1. $$0 \leq P(E) \leq 1.$$
2. $$P(S) = 1.$$
3. For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_i E_j = \emptyset$ when $i \neq j$),
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

We refer to $P(E)$ as the probability of the event E.

Axioms of probability

If an experiment consists of tossing a fair coin, then a head is as likely to appear as a tail. Therefore,

$$P(\{H\}) = \frac{1}{2}, \; P(\{T\}) = \frac{1}{2}.$$

On the other hand, if the coin is biased and we feel that a head is twice as likely to appear as a tail, then we have

$$P(\{H\}) = \frac{2}{3}, \; P(\{T\}) = \frac{1}{3}.$$

Axioms of probability

If a die is rolled and we suppose that all six sides are equally likely to appear, then we would have

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6}.$$

From Axiom 3, it would thus follow that the probability of rolling an even number would equal

$$P(\{2,4,6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{2}.$$

Basic propositions of probability

The complementation rule

Proposition

For any event E,

$$P(E^c) = 1 - P(E).$$

Proof.

E and E^c are mutually exclusive and $E \cup E^c = S$, we have, by Axioms 2 and 3,

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c),$$

where the desired result is deduced. □

The complementation rule: example

Five fair coins are flipped simultaneously. Find the probability of the event A that at least one head turns up.

Solution:

$$P(A^c) = P(\text{"no heads"}) = P(\text{"5 tails"}) = P(\{TTTTT\}) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Therefore,

$$P(A) = 1 - P(A^c) = 1 - \frac{1}{32} = \frac{31}{32}.$$

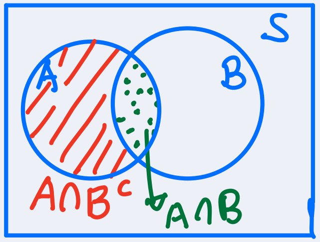
Basic propositions of probability

Law of total probability

Proposition

For any event A and B, it holds that

$$P(A) = P(A \cap B) + P(A \cap B^c).$$



$A = (A \cap B^c) \cup (A \cap B)$ union of disjoint events

Proof.

Since $S = B \cup B^c$,

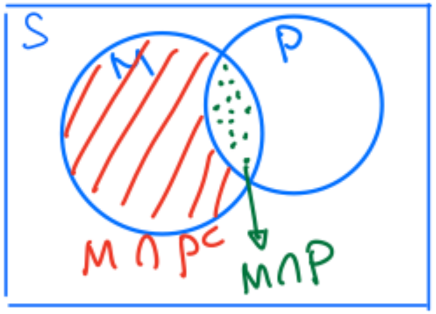
$$A = A \cap S = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c).$$

Note that $B \cap B^c = \emptyset$, we have thus $(A \cap B) \cap (A \cap B^c) = \emptyset$. Therefore, by Axiom 3,

$$P(A) = P(A \cap B) + P(A \cap B^c).$$

□

Law of total probability: example



From Law of total probability,

$$\mathbb{P}(M \cap P^c) = \mathbb{P}(M) - \mathbb{P}(M \cap P)$$

Out of 40 students, it is observed that 10 take Maths, 15 take Physics and 5 take both. What is the probability of randomly selecting a student who takes Maths but not Physics?

Sol: $S = \{\text{all 40 students}\}$, then $|S| = 40$.

Let $M = \{\text{the student taking Maths}\}$, $P = \{\text{the student taking Physics}\}$, then $|M \cap P| = 5$. Therefore, by Law of total probability,

$$\mathbb{P}(\text{a student selects Maths but not Physics}) = \mathbb{P}(M \cap P^c)$$

$$= \mathbb{P}(M) - \mathbb{P}(M \cap P) = \frac{|M|}{|S|} - \frac{|M \cap P|}{|S|} = \frac{10}{40} - \frac{5}{40} = \frac{5}{40} = \frac{1}{8}.$$

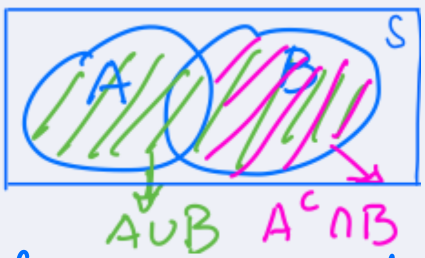
Basic propositions of probability

Addition rule

Proposition

For any event A and B, it holds that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



$B = (A \cap B) \cup (A^c \cap B)$ and $A \cup B = A \cup (A^c \cap B)$ union of disjoint events

Proof.

Note that $A \cup B$ can be written as the union of the two disjoint events A and $A^c \cap B$. Thus, from Axiom 3, we obtain

by Axiom 3

$$P(A \cup B) = P(A) + P(A^c \cap B).$$

By the law of total probability, we have $P(B) = P(A \cap B) + P(A^c \cap B)$. Therefore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Addition rule: example

Out of 40 students, it is observed that 10 take Maths, 15 take Physics and 5 take both. What is the probability of randomly selecting a student

(a) who takes Maths or Physics?

(b) who takes Maths or Physics but not both?

Sol: Recall $|S| = 40$, $|M| = 10$, $|P| = 15$, $|M \cap P| = 5$.

(a) $\mathbb{P}(M \cup P) = \mathbb{P}(M) + \mathbb{P}(P) - \mathbb{P}(M \cap P) = \frac{|M|}{|S|} + \frac{|P|}{|S|} - \frac{|M \cap P|}{|S|}$

$= \frac{1}{40}(10 + 15 - 5) = \frac{20}{40} = \frac{1}{2}$, by Additional rule.

(b) $\mathbb{P}(\underbrace{(M \cup P)}_{=A} \cap \underbrace{(M \cap P)^c}_{=B^c}) = \mathbb{P}(M \cup P) - \mathbb{P}((M \cup P) \cap (M \cap P))$ by LTP

$= \mathbb{P}(M \cup P) - \mathbb{P}(M \cap P)$, since $M \cap P \subset M \cup P$

$= \frac{1}{2} - \frac{|M \cap P|}{|S|} = \frac{1}{2} - \frac{5}{40} = \frac{3}{8}$

Exercise

Consider the distribution of pass/fail in a course by students' gender.

	Pass	Fail	Total
Male	60	30	90
Female	9	1	10
Total	69	31	100

Find $P(\text{Male} \cap \text{Pass})$ and $P(\text{Male} \cup \text{Pass})$.

Sol: $\mathbb{P}(\text{Male} \cap \text{Pass}) = \frac{|\text{Male} \cap \text{Pass}|}{|\text{students}|} = \frac{60}{100} = \frac{3}{5}$

$\mathbb{P}(\text{Male} \cup \text{Pass}) = \frac{|\text{Male}| + |\text{Pass}| - |\text{Male} \cap \text{Pass}|}{|\text{students}|} = \frac{90+69-60}{100} = \frac{99}{100}$

Outline

Equally likely models

- The sample space S consists of finite outcomes:

$$S = \{x_1, x_2, \ldots, x_n\}.$$

- All the outcomes are equally likely to occur, i.e.

$$P(\{x_1\}) = P(\{x_2\}) = \cdots = P(\{x_n\}) = \frac{1}{n}.$$

- The probability of an event A is

$$P(A) = \frac{\text{number of outcomes in A}}{\text{number of outcomes in S}}, = \frac{|A|}{|S|}$$

i.e. $P(A)$ equals the proportion of outcomes in S that are contained in A.

Equally likely models: example 1

In rolling a fair die once, what is the probability of

(a) the event A of obtaining a 5 or 6?

(b) the event B of obtaining an even number?

Equally likely models: example 1

Solution

$S = \{1, 2, 3, 4, 5, 6\}$ and the outcomes are equally likely.

(a) The event $A = \{5, 6\}$, therefore,

$$P(A) = \frac{2}{6} = \frac{1}{3}.$$

(b) The event $B = \{2, 4, 6\}$, therefore,

$$P(B) = \frac{3}{6} = \frac{1}{2}.$$

Equally likely models: example 2

In rolling a fair die twice, what is the probability that the same number appears twice?

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$S = \{(i,j), i,j \in \{1,2,\cdots,6\}\} = \{1,\cdots,6\} \times \{1,\cdots,6\}$, $|S| = 6 \times 6 = 36$.

Solution. The desired event

$$A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}.$$

Therefore, $P(A) = \frac{6}{36} = \frac{1}{6}$.

Exercise

In rolling a fair die twice, what is the probability that

(a) the sum of the two numbers is even?

(b) the product of the two numbers is even?

	1	2	3	4	5	6	
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	3
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	6
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	3
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	6
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	3
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	6

Sol: (a) $\mathbb{P}(\text{the sum of the 2 no. is even}) = \frac{|\{(i,j)\in S: i+j=2k, k=1,\cdots,6\}|}{36} = \frac{6\times 3}{36} = \frac{1}{2}$

(b) $\mathbb{P}(\text{Product of the 2 no. is even}) = \frac{|\{(i,j)\in S: ij=2k, k=1,\cdots,18\}|}{36} = \frac{3\times 3+3\times 6}{36} = \frac{3}{4}$