CAN207 Continuous and Discrete Time Signals and Systems

Lecture-7

Continuous-Time Fourier Series

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Content

- 1. Fundamentals of Fourier Analysis
  - Time and frequency analysis
  - Eigen-functions and eigen-values

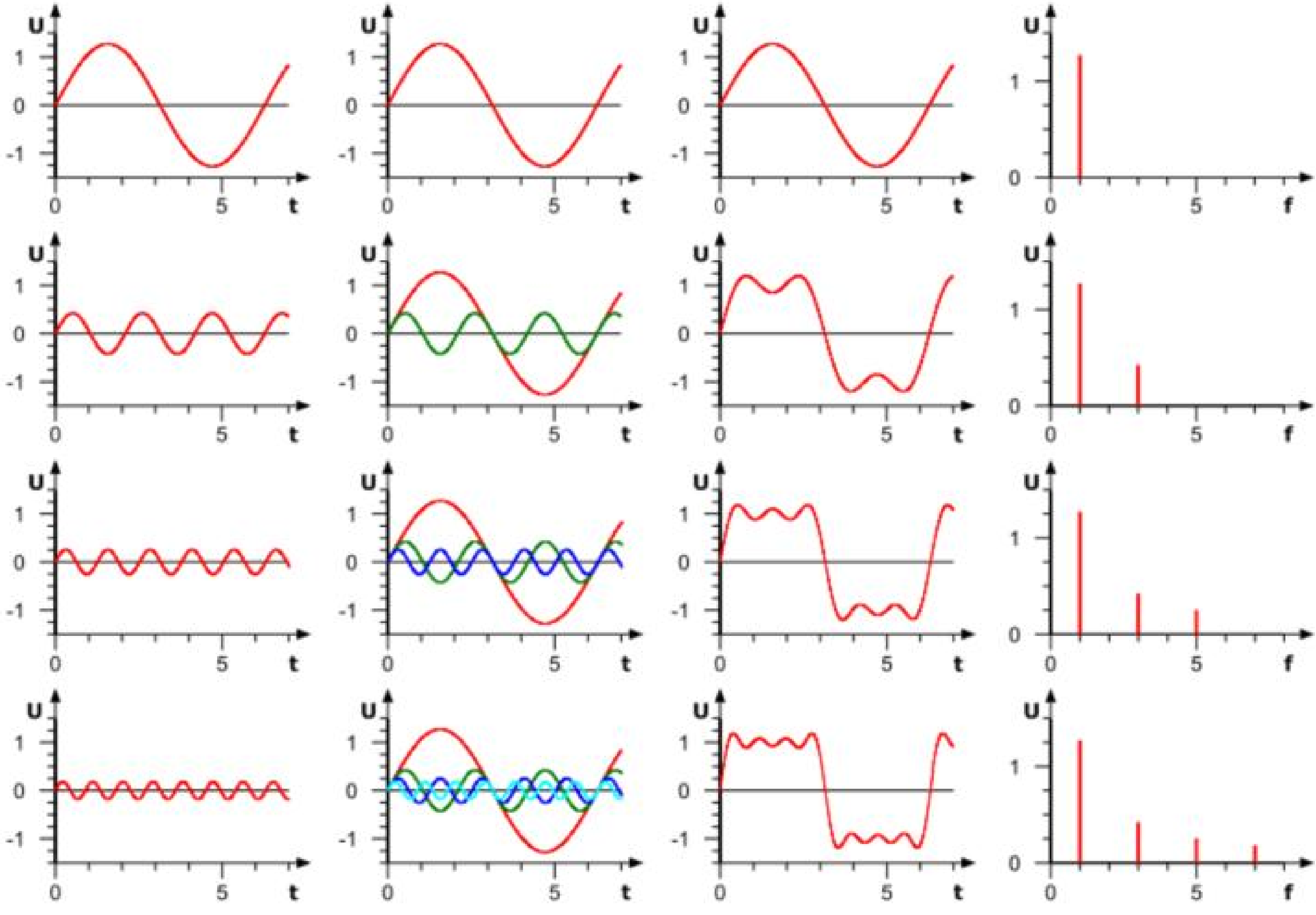
- 2. CTFS for periodic signals
  - synthesis and analysis equations of CTFS
  - FS of real signals
  - Existence - Dirichlet condition
  - Signal spectrum

Fourier Analysis

Jean Baptiste Joseph Fourier(1768-1830)

First published in 1807

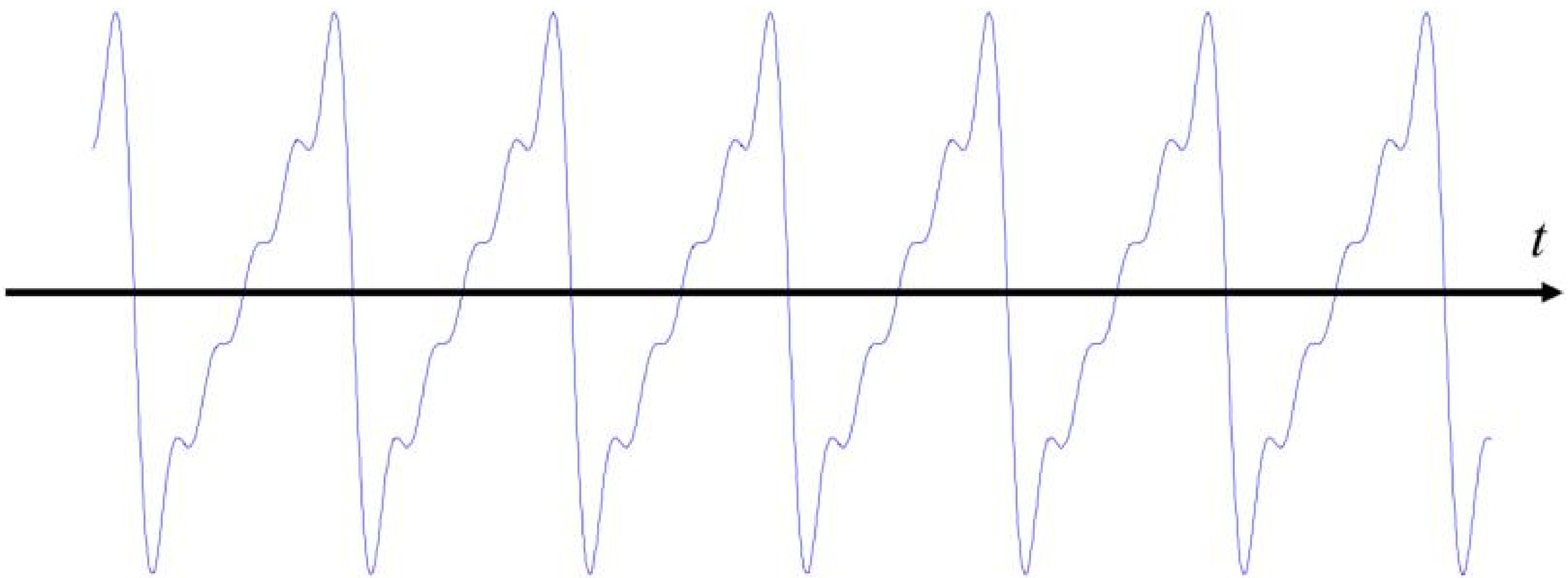
Synthesis of square wave



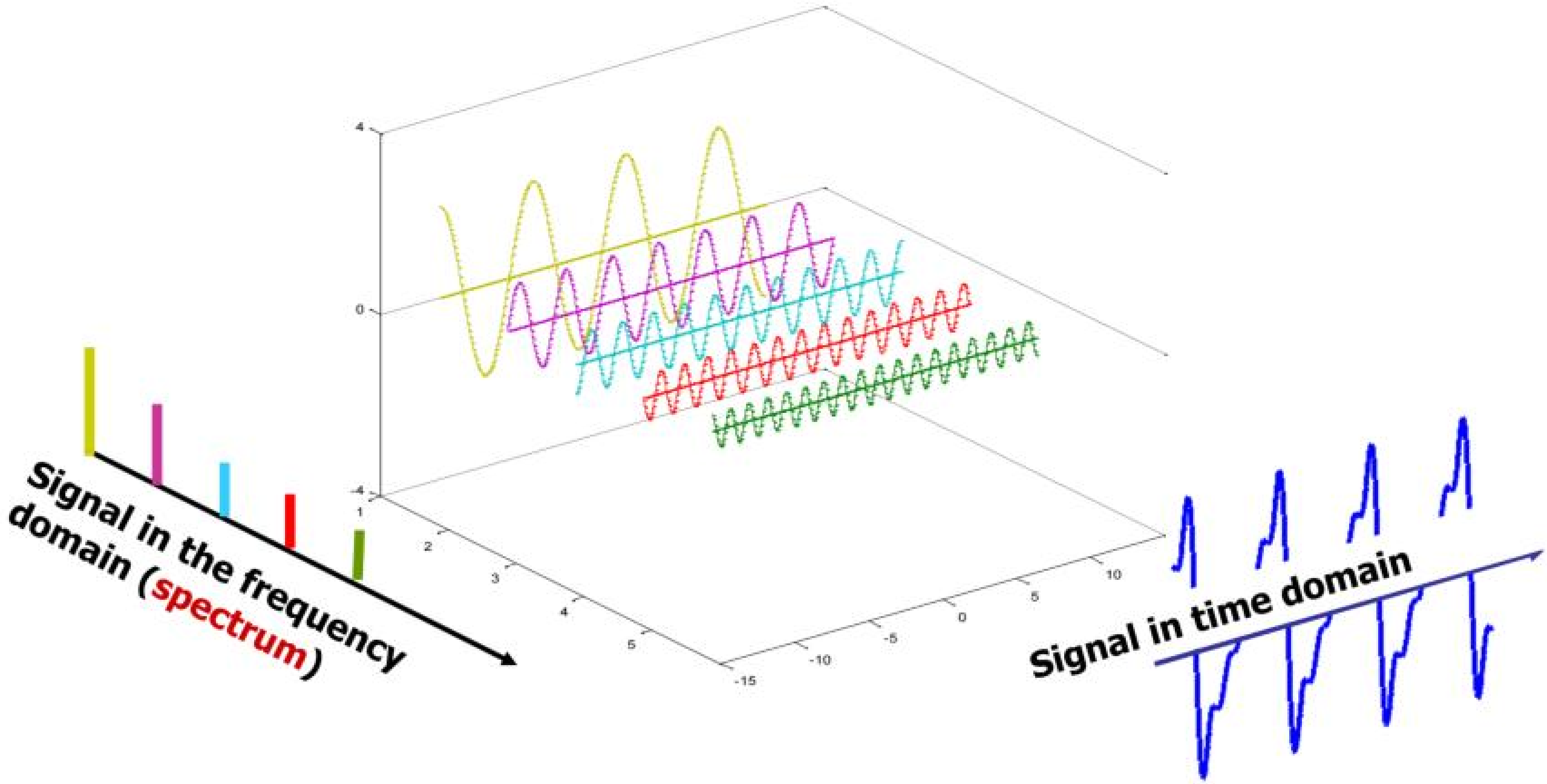
http://zh.wikipedia.org/wiki/%E5%82%85%E9%87%8C%E5%8F%B6%E7%BA%A7%E6%95%B0

Time and frequency

- What is the Fourier series of the following periodic signal ("nearly sawtooth" signal)?

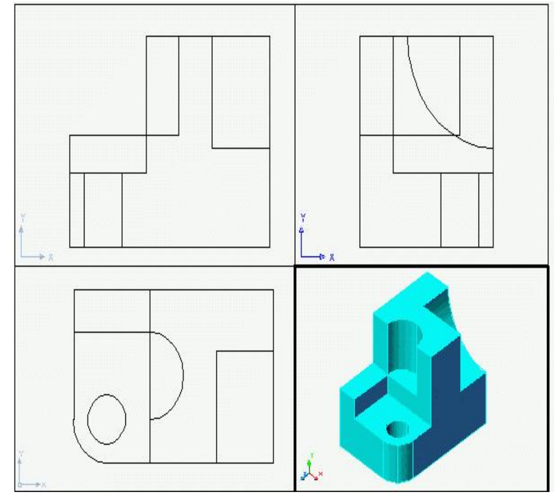


Relationship between time and frequency

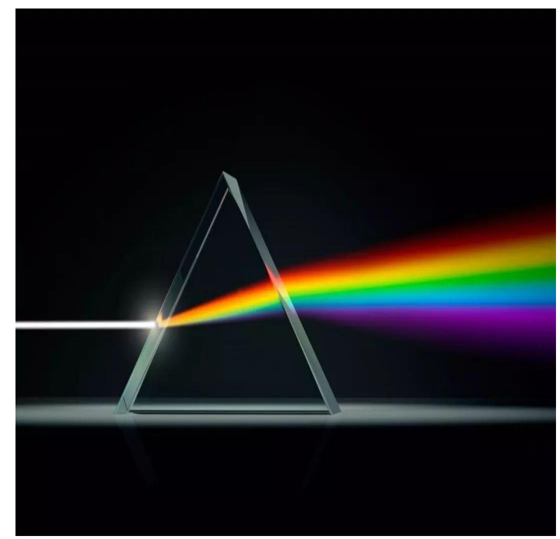


Relationship between time and frequency

- The Fourier series may be thought of as a tool for looking at a signal from a different perspective.



- Fourier series is also called “Mathematical prism”, cause it can decompose a signal down to fundamental building blocks, like what prism does for lights.



Recall Lect.6_p.4

- LTI systems possess the ***superposition property***.
  - Input (linearly combined) → Output (linearly combined)
- Strategy:
  - Decompose input signal into a linear combination of *basic signals*;
  - Choose basic signals so that responses are easy to compute.
- Basic signals?

delayed impulses ⟷ convolution in Time Domain

complex exponentials ⟷ Fourier analyses in Frequency Domain

1.1 Basic signals

- Criteria for choosing a set of *basic signals* in terms of which to decompose the input to a linear system:

$$\text{if:}\quad x = a_1\phi_1 + a_2\phi_2 + \ldots$$

$$\text{then:}\ \ y = a_1\psi_1 + a_2\psi_2 + \ldots$$

  – choose the *basic signal* $\phi_k(t)$ or $\phi_k[n]$ so that:
    - a broad class of signals can be constructed as a linear combination of ϕ_ks
    - responses to ϕ_ks are easy to compute
  – choose ***complex exponentials*** as a set of *basic signals*:
    - CT: $\phi_k(t) = e^{s_k t} => \phi_k(t) = e^{j\Omega_k t}$ when $s_k = j\Omega_k$ is pure imaginary.
      - s_k is complex => Laplace transform
    - DT: $\phi_k[n] = {z_k}^n => \phi_k[n] = e^{j\omega_k n}$ when $z_k = e^{j\omega_k}$ is on unit circle.
      - z_k is complex => Z-transform

1.2 Eigen-function and eigen-values

- The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude

$$\phi_k(t) = e^{j\Omega_k t} \rightarrow H(\Omega_k)e^{j\Omega_k t}$$

  - where the complex amplitude factor $H(\Omega_k)$ is a function of the frequency Ω_k.

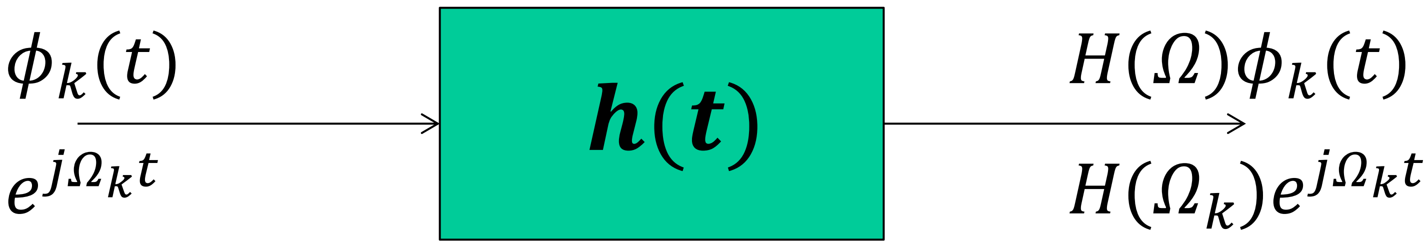
  - Proof:

$$e^{j\Omega_k t} \rightarrow \int_{-\infty}^{\infty} h(\tau)e^{j\Omega_k(t-\tau)}d\tau = \underbrace{\boxed{e^{j\Omega_k t}}}_{\text{eigen-function}}\underbrace{\boxed{\int_{-\infty}^{\infty} h(\tau)e^{-j\Omega_k \tau}d\tau}}_{H(\Omega_k)\ \text{eigen-value}}$$

- A signal for which the system output is a (possibly complex) constant times the input is referred to as an ***eigen-function*** of the system, and the amplitude factor is referred to as the system's ***eigen-value***.

1.2 Eigen-function and eigen-values

- An ***eigenfunction*** of a system (or mathematical equation) is a function which you put it through the system, comes out looking exactly the same, except for its change in amplitude.
- The changing amplitude is the ***eigenvalue***.



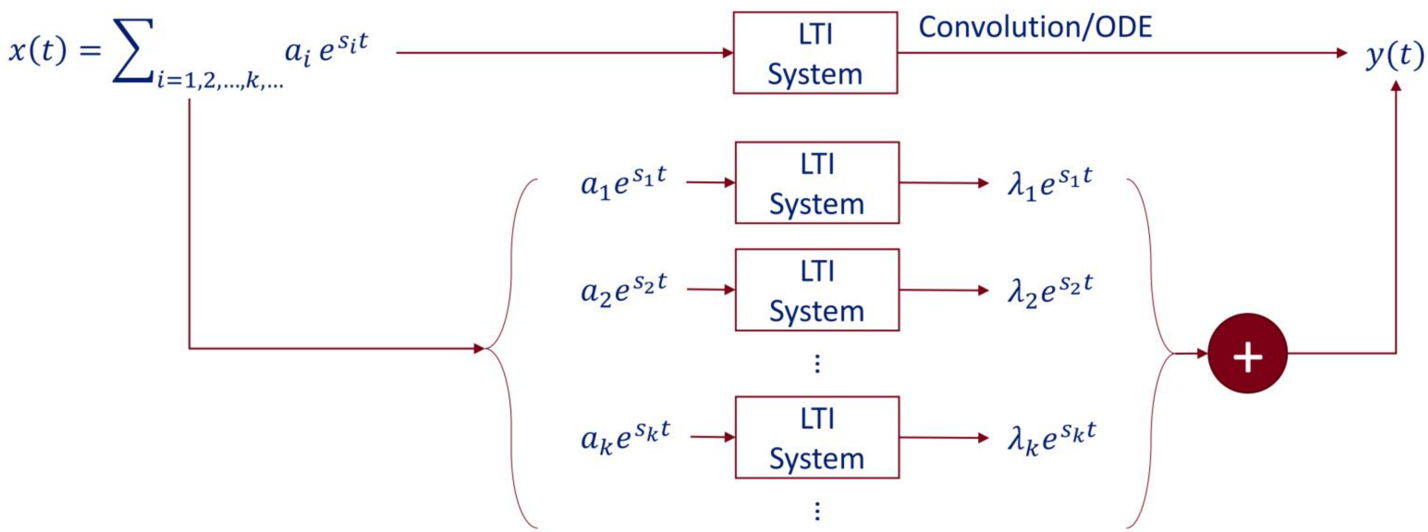
- $e^{j\Omega_k t}$ is a set of conveneint building blocks

  → going through the system, get the same $e^{j\Omega_k t}$ but scaled

1.2 Eigen-function and eigen-values

- $e^{j\Omega_k t}$ (or e^{st}) is an eigenfunction of LTI systems, such that

- If the input signal of an LTI system can be written as a linear combination of complex exponential functions, then:



- the output is also a linear combination of the weighted complex exponential functions.

1.2 Eigen-function and eigen-values

- Let the input signal $x(t)$ correspond to a linear combination of several complex exponentials:

$$x(t) = \sum_k a_k e^{j\Omega_k t}$$

- From the eigen-function property, the response to each separately is

$$a_k e^{j\Omega_k t} \rightarrow a_k H(j\Omega_k) e^{j\Omega_k t}$$

- and from the superposition property, the response is:

$$y(t) = \sum_k a_k H(j\Omega_k) e^{j\Omega_k t}$$

- This can be extended to the complex exponentials $e^{s_k t}$ for CT and ${z_k}^n$ for DT:
  - CT: $\sum_k a_k e^{s_k t} \rightarrow \sum_k a_k H(s_k) e^{s_k t}$
  - DT: $\sum_k a_k z_k^n \rightarrow \sum_k a_k H(z_k) z_k^n$

1.2 Orthogonality for Exponential functions

- Exponential functions
  - The set of complex periodic basis functions is expressed as: $\phi_k = e^{jk\Omega_0 t}$, k is integer
    - with Ω_0 as the fundamental frequency
    - $T_0 = \frac{2\pi}{\Omega_0}$ as the corresponding period.
  - The basis function set is orthogonal in the sense

$$\int_0^{T_0} \phi_m(t)\, \phi_n^*(t) dt = \begin{cases} 0, & n \neq m \\ T_0, & n = m \end{cases}$$

$$\Rightarrow \int_0^{T_0} e^{jm\Omega_0 t}\, e^{-jn\Omega_0 t} dt = \begin{cases} 0, & n \neq m \\ T_0, & n = m \end{cases}$$

Example 1

- Consider an LTI system for which the input x(t) and output y(t) are related by a time shift of 3:

$$y(t) = x(t-3)$$

- If the input to this system is the complex exponential signal $x(t) = e^{j2t}$, determine the eigen-function and corresponding eigen-value of this system.

2.1 CT periodic signals

- To express signals as a linear combination of complex exponentials, we have 2 questions:
  – Does this expression always exist?
  – How broad can this expression be used?

Complex exponentials are periodic

Linear combination of them should also be periodic?
=> Let's start with PERIODIC signals, in CT.

2.1 CT periodic signals

- $x(t)$ is ***periodic*** if, for some positive value of T:

$$x(t) = x(t+T) \quad \text{for all } t$$

  - for all *t*, no exception;
  - minimum positive, nonzero value of T is the ***fundamental period***;
  - $\Omega_0 = \frac{2\pi}{T}$ is referred to as the ***fundamental frequency***.
  - For example:

$$\text{sinusoidal signal:} \quad x(t) = \cos \Omega_0 t$$

$$\text{complex exponential:} \quad x(t) = e^{j\Omega_0 t}$$

    - fundamental frequencies are both Ω_0
    - fundamental periods are $T = \frac{2\pi}{\Omega_0}$

2.1 Synthesis equation

- Harmonically related complex exponential:

$$\phi_k(t) = e^{jk\Omega_0 t} = e^{j\frac{2\pi k}{T}t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- Each of these signals has a fundamental frequency $k\Omega_0$ that is a multiple of Ω_0, and therefore, each is periodic with period T
  – although for $|k| \geq 2$, the fundamental period is a fraction of T.
- The linear combination of harmonically related complex exponentials:

$$a_0 + a_1 e^{j\Omega_0 t} + a_2 e^{j2\Omega_0 t} + \ldots \boxed{= \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} = x(t)}$$

  – is also periodic with period T;
  – the components for k = + N and k = - N are referred to as the N^{th} harmonic components.
- This representation is referred to as the ***synthesis equation*** of ***Fourier Series***.

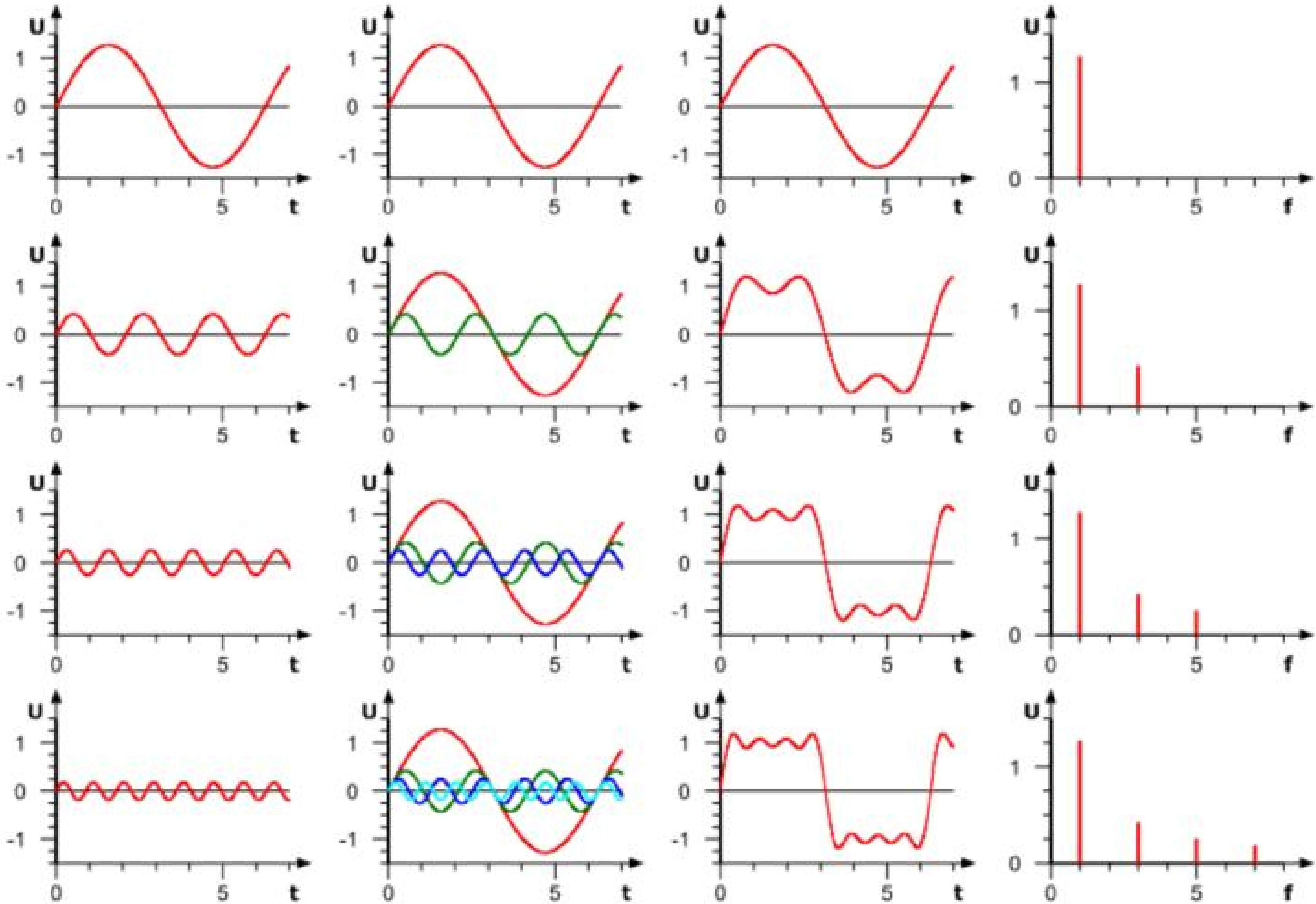
(P5 again) Synthesis of square wave

Fundamental

3rd harmonic

5th harmonic

7th harmonic



2.2 Fourier series coefficients

- Review the property of complex exponentials:

$$\int_T e^{jm\Omega_0 t}\,dt = \begin{cases} T & m = 0 \\ 0 & m \neq 0 \end{cases}$$

- Modify the original Fourier series by:

$$\int_T x(t)e^{-jn\Omega_0 t}dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}\; e^{-jn\Omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_T e^{jk\Omega_0 t}e^{-jn\Omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j(k-n)\Omega_0 t}dt$$

$$\Rightarrow \int_T x(t)e^{-jn\Omega_0 t}dt = a_n T \Rightarrow \boxed{a_n = \frac{1}{T}\int_T x(t)e^{-jn\Omega_0 t}dt}$$

the ***analysis equation*** of ***Fourier Series***.

Example 2

- Consider the signal

$$x(t) = \sin(\omega_0 t)$$

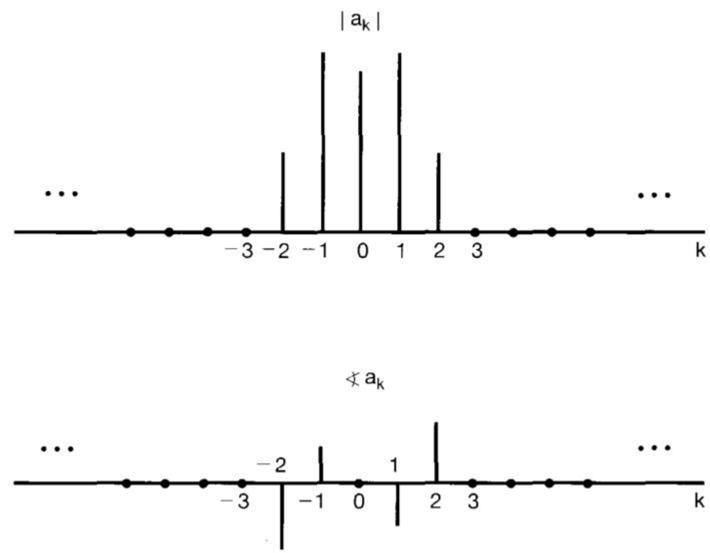
- Determine the Fourier series coefficients a_k for this signal.

Quiz 1

- Consider the signal

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4})$$

- Determine the Fourier series coefficients a_k for this signal.

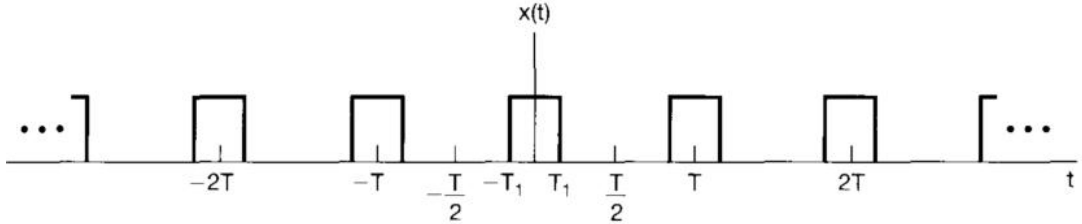


Example 3

- Consider the signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$x(t)$ is periodic with fundamental period T.



- Determine the Fourier series coefficients a_k for this signal.

Example 3

- Solution:
  - For $k = 0$:

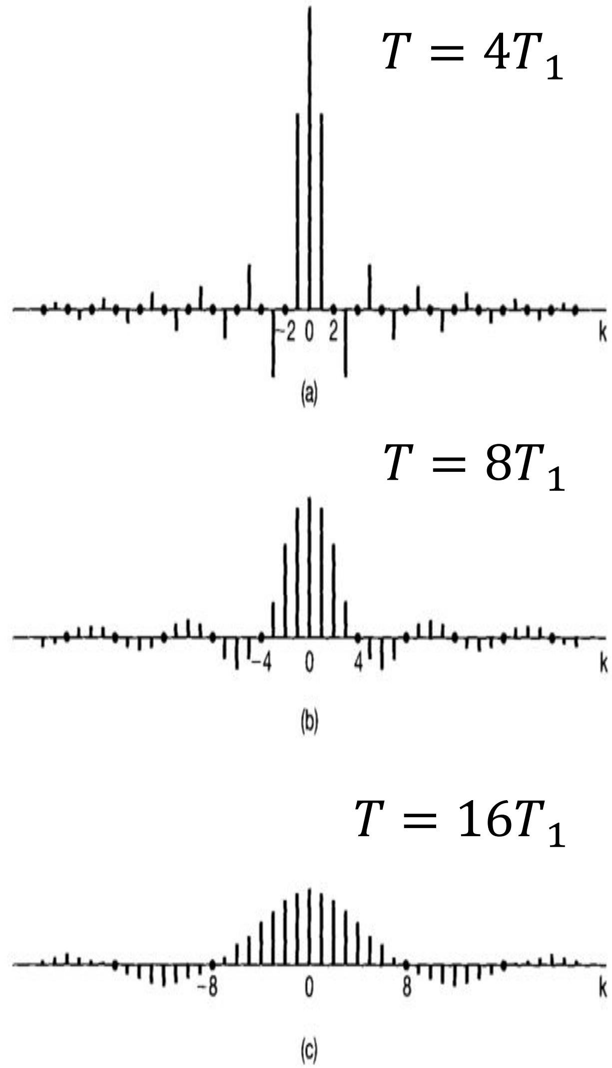
$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

  - For $k \neq 0$:

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\Omega_0 t} dt = -\frac{1}{jk\Omega_0 T} e^{-jk\Omega_0 t}\Big|_{-T_1}^{T_1}$$

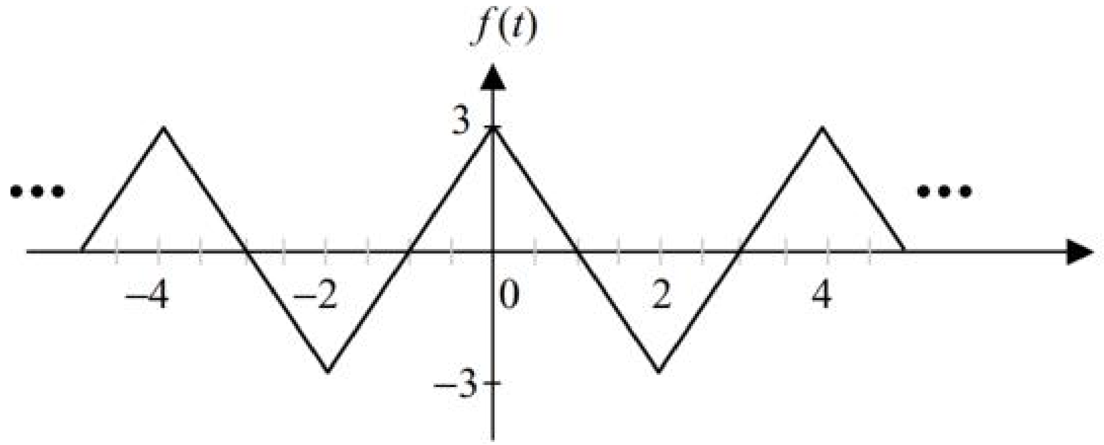
$$= \frac{2}{k\Omega_0 T}\left[\frac{e^{jk\Omega_0 T_1} - e^{-jk\Omega_0 T_1}}{2j}\right] = \frac{2\sin(k\Omega_0 T_1)}{k\Omega_0 T}$$

$$= \frac{\sin(k\Omega_0 T_1)}{k\pi} = \frac{2T_1}{T} sinc(k\Omega_0 T_1)$$



Quiz 2

- Consider a signal as shown, determine the Fourier series of it.



2.2 Fourier Series Properties

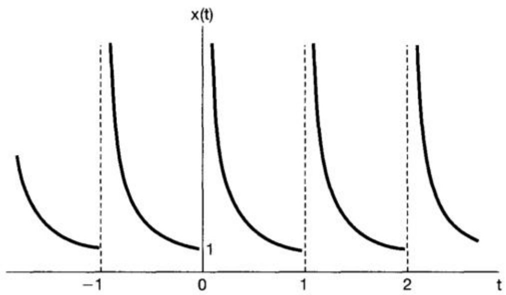
- TD: Complex signal
  - $\mathcal{Re}\{x(t)\}$ is odd, an $\mathcal{Im}\{x(t)\}$ is even
    => FD: Imaginary only
  - $\mathcal{Re}\{x(t)\}$ is even, an $\mathcal{Im}\{x(t)\}$ is odd
    => FD: Real only
- TD: Real signal
  - $x(t)$ is odd
  => FD: Sine only
  - $x(t)$ is even
  => FD: Cosine only

2.3 Existence of the Fourier Series

- Not every periodic signal has a Fourier series
- The Dirichlet conditions are:
  - x(t) must be absolutely integrable;
  - x(t) has a finite number of maxima and minima in any single period;
  - x(t) has a finite number of discontinuities during any time interval.
- The Dirichlet conditions, if met, guarantees that x(t) equals its Fourier series representation except at isolated values of t for which x(t) is discontinuous.
- At a mismatch point, the infinite series converges to the average of the values on either side of the discontinuity.
- Because the isolated points of discontinuity has no effect on an integration operation, the Fourier series representation is as good as the original signal for LTI system analysis.

2.3 Existence of the Fourier Series

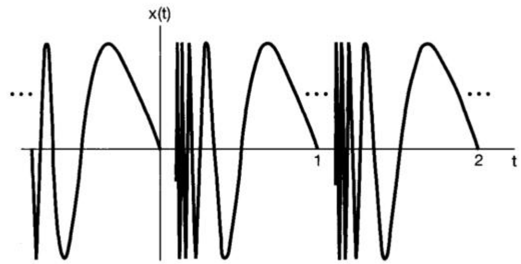
- Some examples (violates the Dirichlet conditions:



(a)

(a) the signal x(t) = 1/t for 0 <t<=1, a periodic signal with period 1;

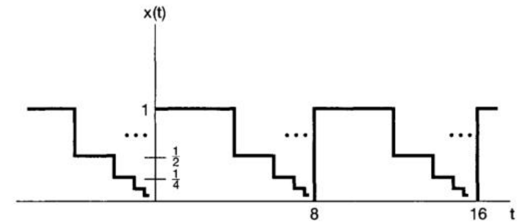
this signal violates the first Dirichlet condition



(b)

(b) the periodic signal of x(t)=sin(2*pi/t);

It violates the second Dirichlet condition



(c)

(c) a signal periodic with period 8 [for 0 s t < 8, the value of x(t) decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2];

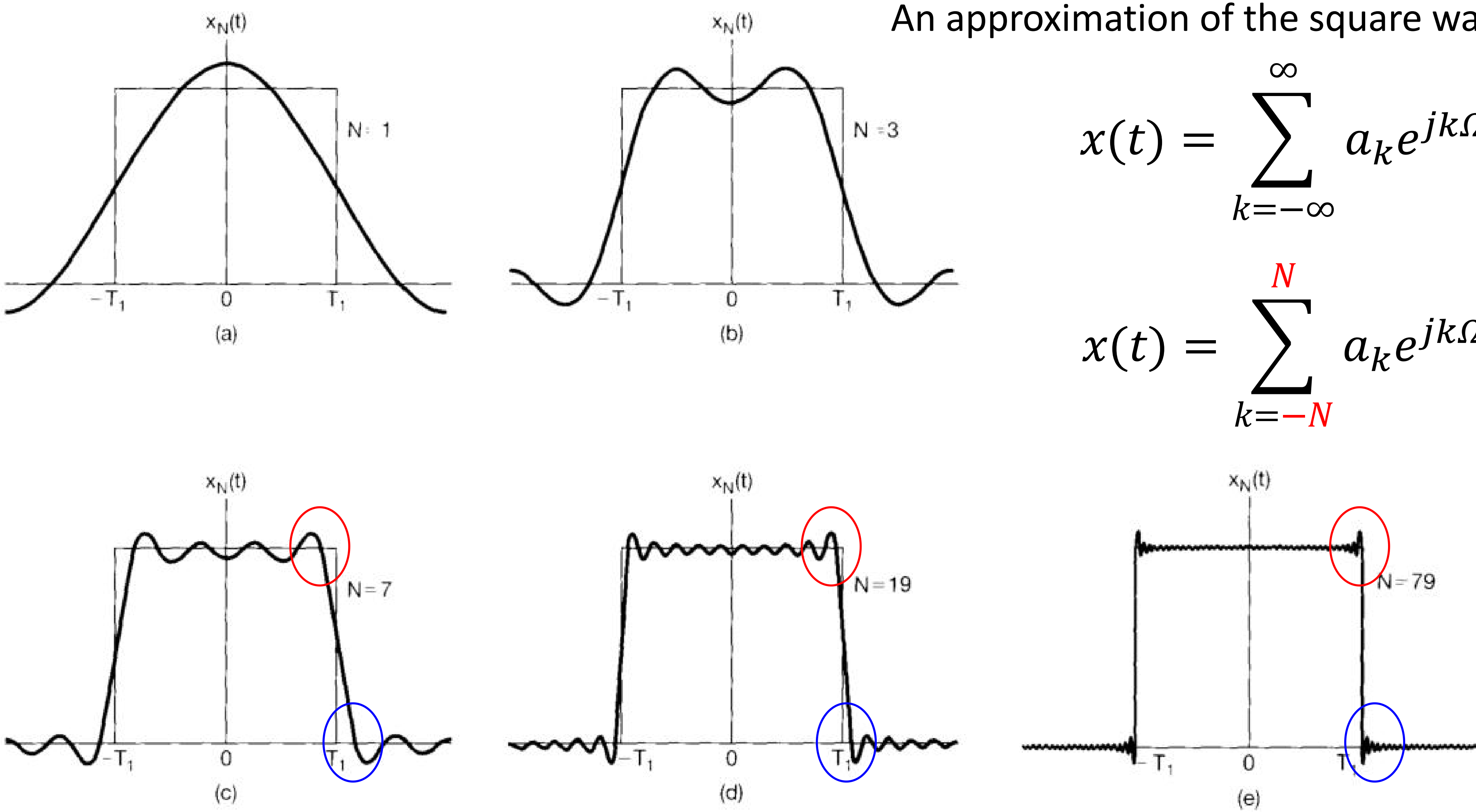
It violates the third Dirichlet condition

2.3 Gibbs phenomenon

An approximation of the square wave

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

$$x(t) = \sum_{k=-N}^{N} a_k e^{jk\Omega_0 t}$$



*These ripples near discontinuity never go away and has a maximum value of **1.09**.*

2.4 Signal spectrum

- Signal spectrum is the graphical representation of a_k

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\frac{2\pi kt}{T}}dt$$

where a_k is a complex function of the variable k.

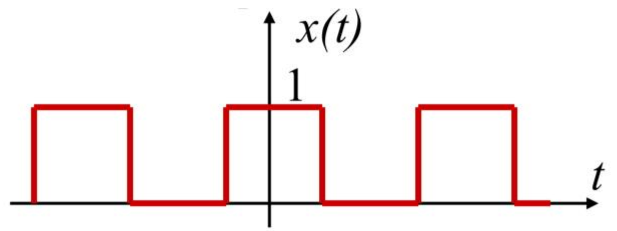
➢ How could we draw it (complex function)?

$$a_k = |a_k|e^{j\angle a_k}$$

- Magnitude $|a_k|$
- Phase $\angle a_k$

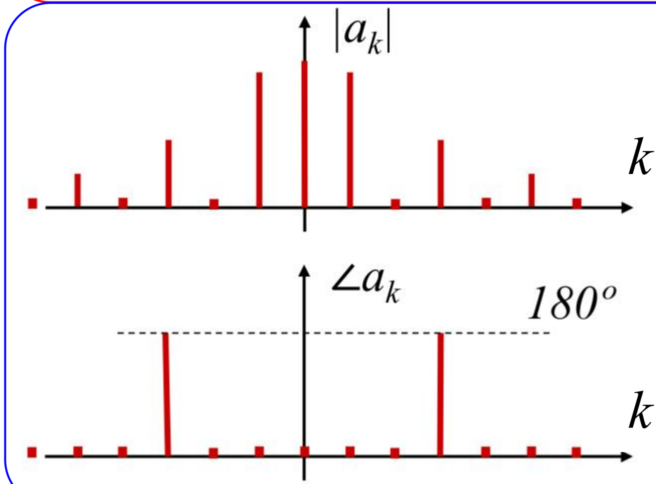
Example

- For the square wave



$$x(t) = \sum_{k=-\infty}^{\infty} rect\left(\frac{t-kT}{T/2}\right)$$

Time Domain

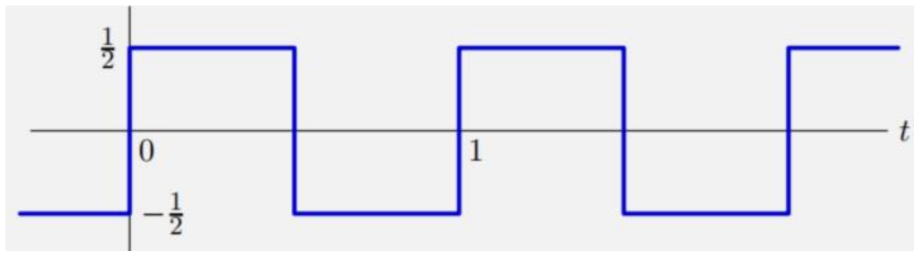


$$a_k = \frac{1}{2} sinc\left(\frac{k}{2}\right)$$

Frequency Domain

Quiz 3

- Let a_k represent the Fourier series coefficients of the following square wave:



- How many of the following statements are true?
  1. $a_k = 0$ if k is even;
  2. a_k is real-valued;
  3. $|a_k|$ decreases with k^2;
  4. there are an infinite number of non-zero a_k;
  5. all of the above.

Next ...

- Continous Time Fourier Transform (CTFT)
  - Fundamentals of CTFT
    - From CTFS to CTFT
  - Fourier transform pairs
    - $e^{j\omega_0 t}$, $\delta(t)$, etc.
  - Fourier transform properties
  - Frequency reponse of a system