Xi'an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

2024-25 Semester 2

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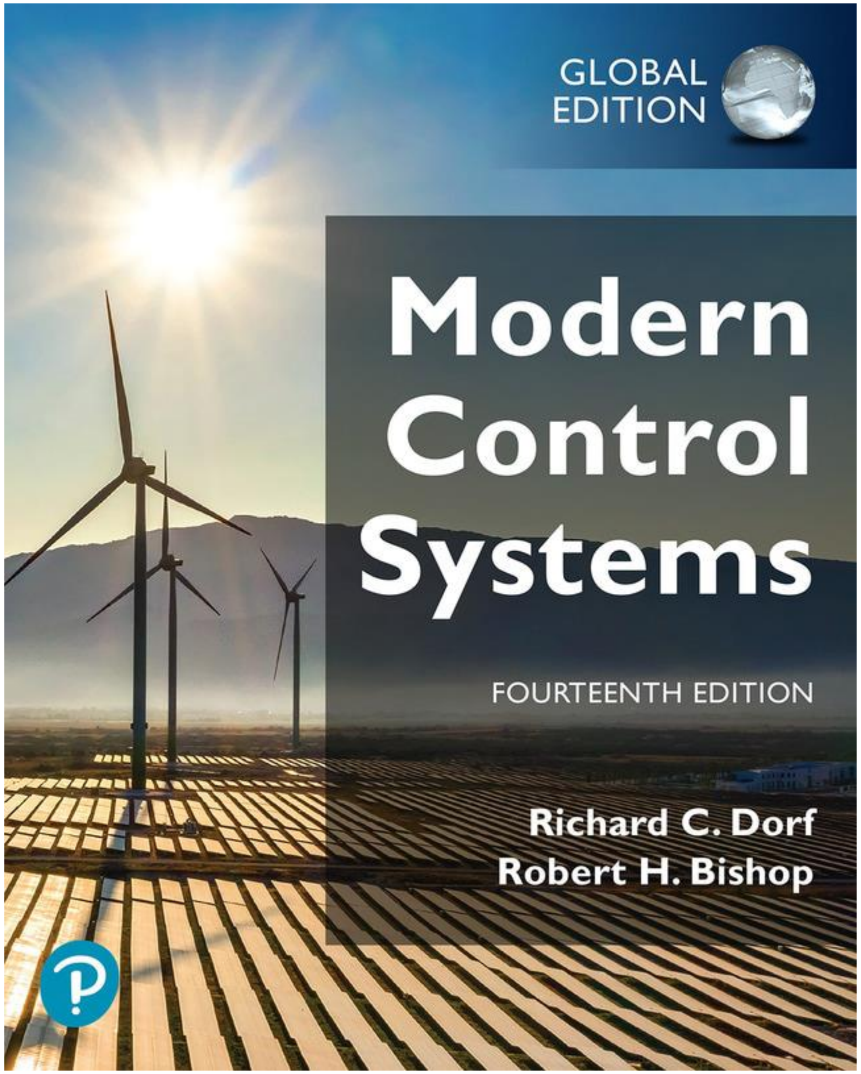
Content for Part 2



- Chapter 3: state variable model

  3 lectures

- Chapter 4: Feedback Control System Characteristics

  1 lecture

- Chapter 5: Time-Domain Performance of Feedback Control

  2 lectures

Lecture 9

Outline

State Variable Models

- Introduction
- State Variables
- State-space Modeling
- State Space Representation in Matrix Form
- Time-domain response (Solution of State-space Models)
- Conversion between State-space Model and Transfer Function
- Analysis of the State-space Models using Matlab

Outline

State Variable Models

- ❑ ***<u>Introduction</u>***
- ❑ **State Variables**
- ❑ **State-space Modeling**
- ❑ **State Space Representation in Matrix Form**
- ❑ **Time-domain response (Solution of State-space Models)**
- ❑ **Conversion between State-space Model and Transfer Function**
- ❑ **Analysis of the State-space Models using Matlab**

Introduction

In the preceding classes, we studied several useful approaches to the analysis of design of control system including (suppose the system order is n):

- Ordinary differential equations (ODEs): nth-order differential equations; time-domain description of system;
- Transfer function: Laplace transform of ODEs; frequency-domain description;
- Block diagram & signal-flow graph: graphic ways derived from transfer functions.

In this chapter, we'll learn

- **State-space model** (or **state variable model**) which
  - is a set of **1st order differential equations** therefore a time-domain description of a system and can be further represented in a convenient matrix-vector form;
  - powerful mathematical & computational tools available for analyzing and designing a system using state-space model;
  - can be extended to nonlinear, time-varying & multiple input-output system.

State-space model is an essential basis for modern control theory.

Introduction

- The transfer function only considers the relation between output and input

$$G(s) = \frac{Y(s)}{R(s)}$$

- But in state space model, it considers the states inside a system, the output is included into the states.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$ Describe the system itself

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$$ Output is only part of states

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- ❑ Introduction
- ❑ *State Variables*
- ❑ State-space Modeling
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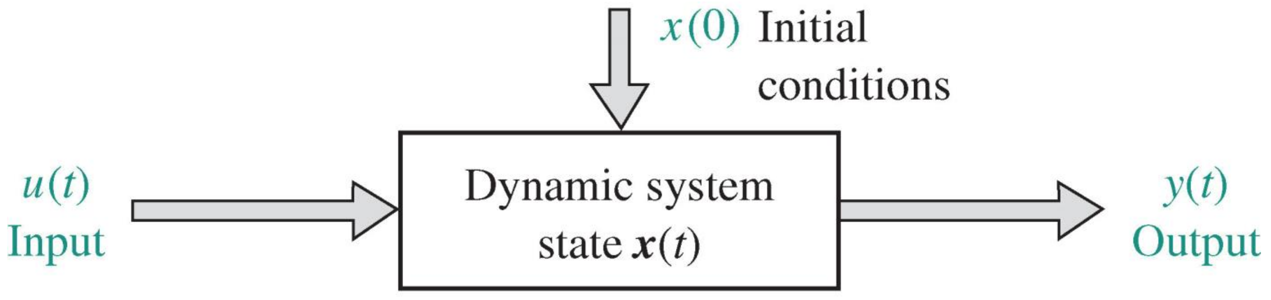
State of a System

The **State** of a system is a set of variables whose values, together with the input signals and the equations describing the dynamics, will provide the future state and output of the system;

It is the minimum information needed about the system in order to determine its future behavior.

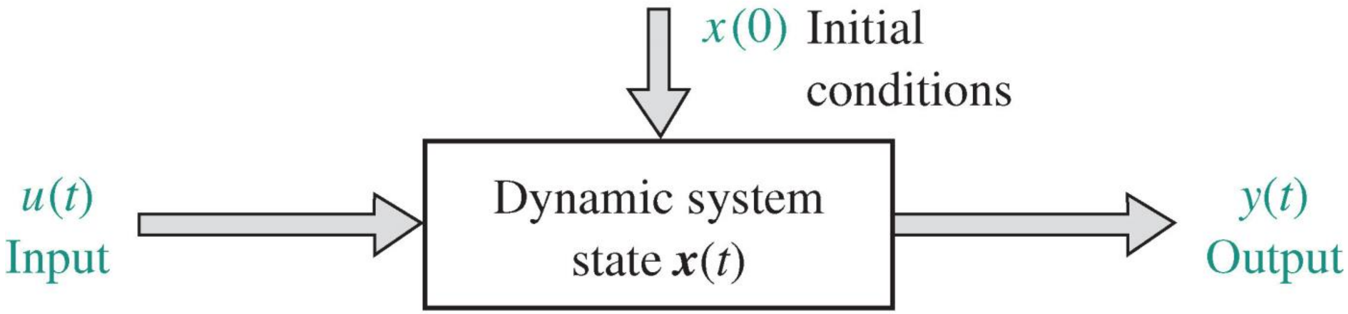


State variables: $[x_1(t), x_2(t), x_3(t), \ldots x_n(t)]$ $\quad \boldsymbol{x}(t)$

Input signals: $[u_1(t), u_2(t), \ldots u_m(t)]$ $\quad \boldsymbol{u}(t)$

Output signals: $[y_1(t), y_2(t), \ldots y_p(t)]$ $\quad \boldsymbol{y}(t)$

State Variables



- The **state variables** of a system are defined as a **minimal set of variables**, which are set of variables of smallest possible size that together with any input to the system is sufficient to determine the future behavior (i.e., state, output) of the system.
- At any initial time $t = t_0$, the state variables $x_1(t_0), x_2(t_0)$, $x_3(t_0)$, … $x_n(t_0)$ defines the initial states of the system;
- Once the inputs of the system for $t \geq t_0$ and the initial states just defined are specified, the state variables should completely define the future behavior of the system.

State Variables

- For differential equations with n^{th} order, there should also be state variables with number n, to fully describe the system.

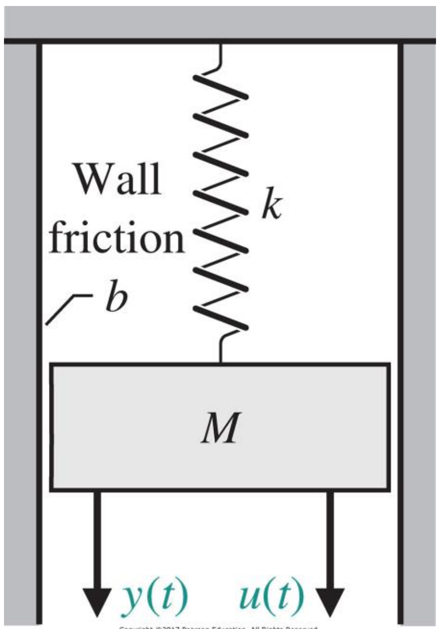
$$x^{(n)} + b_1 x^{(n-1)} + b_2 x^{(n-2)} + \cdots + b_{n-1} x^{(1)} + b_n x = u$$

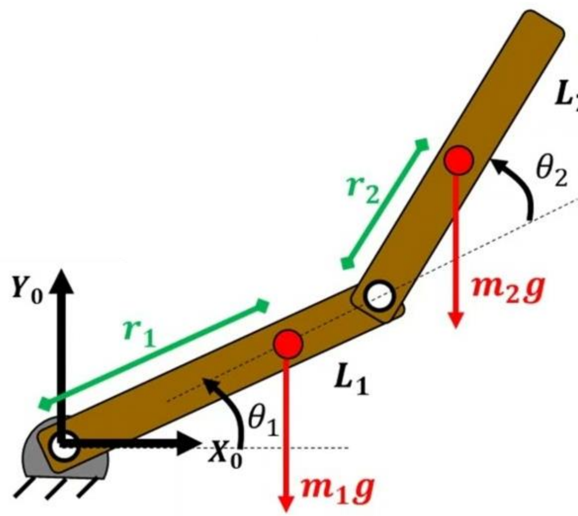
The state variable could be selected as

$$\boldsymbol{x} = \begin{bmatrix} x^{(n-1)} \\ x^{(n-2)} \\ \vdots \\ x^{(1)} \\ x \end{bmatrix}$$

State Variables

- For mechanical systems, a set of state variables which are sufficient to describe the system includes the position and velocity of the object.





$$\boldsymbol{x} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

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- ❑ ***State-space Modeling***
- ❑ State Space Representation in Matrix Form
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State-Space Modeling

- **State differential equations**: first-order differential equations written in terms of the state variables $\boldsymbol{x} = (x_1(t), x_2(t), \ldots x_n(t))$ and the input $\boldsymbol{u} = (u_1(t), u_2(t), \ldots u_m(t))$;

- A **state-space model** represents a system by a series of first-order differential state equations and algebraic output equations; It simplifies analysis of complex systems with multiple inputs and outputs;

- State-space models are numerically efficient to solve, can handle complex systems, allow for a more geometric understanding of systems, and form the basis of modern control theory.

Example 9.1

Consider the following system where $u(t)$ is the input and $\dot{x}(t)$ is the output:

$$\dddot{x} + 5\ddot{x} + 3\dot{x} + 2x = u$$

How to generate state-space model?

- Changing variables, let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$,

$$\dddot{x} = -5\ddot{x} - 3\dot{x} - 2x + u \longrightarrow \left.\begin{aligned} \dot{x}_3 &= -5x_3 - 3x_2 - 2x_1 + \mathrm{u} \\ \dot{x}_2 &= x_3 \\ \dot{x}_1 &= x_2 \end{aligned}\right\}$$

State equations

- Obtaining output:

$$\left. y = \dot{x} = x_2 \right\}$$

Output equation

The system has 1 input ($u(t)$), 1 output ($\mathrm{y}(t)$) and 3 state variables ($x_1(t), x_2(t), x_3(t)$).

State-Space Modeling: General Form

General state-space models have the following form:

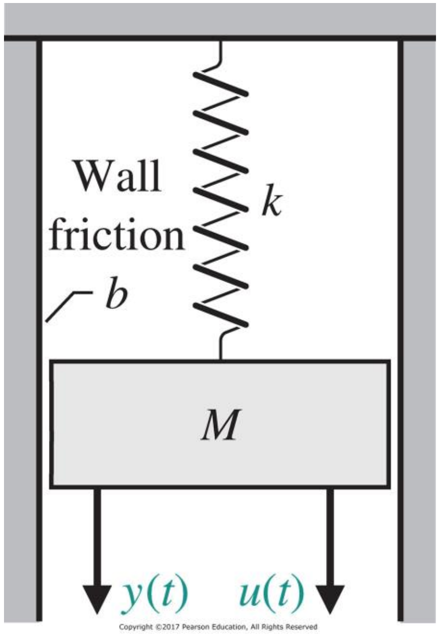
State equations

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m,) \\ \dot{x}_2 = f_2(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m,) \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m,) \end{cases}$$

Output equations

$$\begin{cases} y_1 = h_1(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m,) \\ y_2 = h_2(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m,) \\ \vdots \\ y_p = h_p(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m,) \end{cases}$$

Example 9.2: Spring-Mass-Damper System



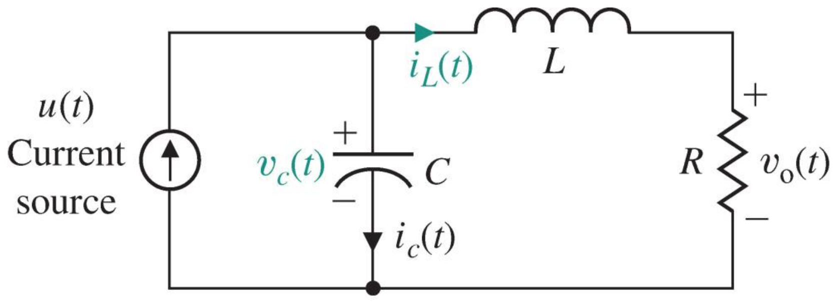
- let $x_1 = y(t)$, $x_2 = \dot{y}(t)$, therefore, x_1 and x_2 will represent displacement and velocity respectively.

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = u(t) \longrightarrow \dot{x}_2 = \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

$$\dot{x}_1 = x_2$$

Output: $y = x_1$

Example 9.3: RLC Circuit



Ordinary Differential Equations: $i_c(t) = C\frac{dv_c}{dt} = u(t) - i_L(t)$

$$L\frac{di_L}{dt} = -Ri_L(t) + v_c(t)$$

Output: $v_0(t) = Ri_L(t)$

- Choose variables, let $x_1(t) = v_c(t)$, $x_2(t) = i_L(t)$

$$\frac{dx_1}{dt} = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t) \qquad \dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u$$

$$\frac{dx_2}{dt} = \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t) \quad \longrightarrow \quad \dot{x}_2 = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

$$y(t) = v_o(t) = Rx_2(t) \qquad y = Rx_2$$

State Variables Selection

State variables are typically has associated with energy storage:

- Each state Variable has memory;
- Each state variable has an "initial condition";

For a system, the number of state variables required is equal to the number of independent energy-storage elements.

- In electric circuits, the energy storage devices are the **capacitors** and **inductors**. They contain all of the state information or "memory" in the system. State variables:
  - Voltage across capacitors
  - Current through inductors

- In mechanical systems, energy is stored in **springs** and **masses**. State variables:
  - Spring displacement
  - Mass position and velocity

- **Resistors** (in electric circuits) and **dampers** (mechanical systems) are energy dissipaters, **they don't store energy**.

State Variables Selection

- The state variables describe a system are not a unique set
  - Several alternative sets of states can be chosen
- The state variables are only required to be any dependent linear combinations.
- For example, in the RLC circuit, we can select
  - $x_1(t) = v_c(t)$, $x_2(t) = i_L(t)$ or
  - $x_1(t) = v_c(t)$, $x_2(t) = v_L(t)$
- In engineering practice, the states are usually required to be physical parameters that are easy to measure
  - Voltage, current, velocity, position, pressure, temperature …

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State-space Representation in Matrix Form

Linear first-order differential equations:

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_{11}u_1(t) + \cdots + b_{1m}u_m(t)$$

$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_{21}u_1(t) + \cdots + b_{2m}u_m(t)$$

$$\vdots$$

$$\dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_{n1}u_1(t) + \cdots + b_{nm}u_m(t)$$

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \boldsymbol{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix} \qquad \boldsymbol{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{pmatrix}$$

**note: boldface small letter: vector; boldface capital letter: matrix.*

State-space Representation in Matrix Form

Linear first-order differential equations:

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_{11}u_1(t) + \cdots + b_{1m}u_m(t)$$

$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_{21}u_1(t) + \cdots + b_{2m}u_m(t)$$

$$\vdots$$

$$\dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_{n1}u_1(t) + \cdots + b_{nm}u_m(t)$$

State Space Equations:

System matrix

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

Input matrix

Output matrix

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$$

Feed-forward matrix

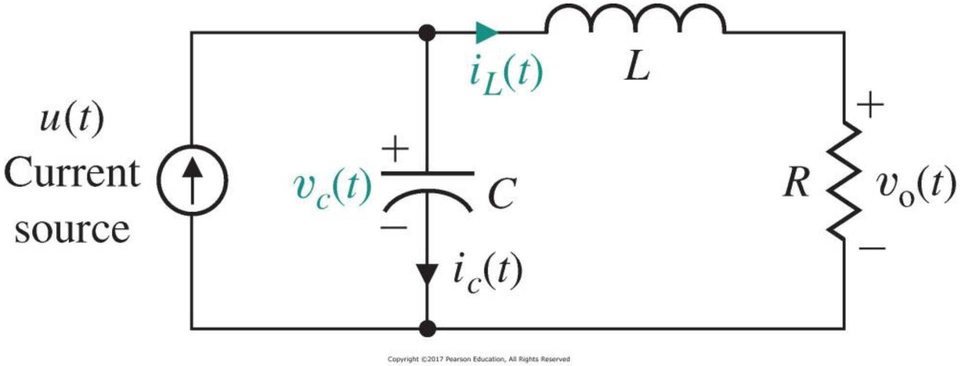
Dimensions: $\mathbf{x}(n \times 1), \mathbf{u}(m \times 1), \mathbf{y}(p \times 1); \mathbf{A}(n \times n), \mathbf{B}(n \times m), \mathbf{C}(p \times n), \mathbf{D}(p \times m).$

**note: boldface small letter: vector; boldface capital letter: matrix.*

Examples

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$$



$$\dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u$$

$$\dot{x}_2 = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

$$y = Rx_2$$

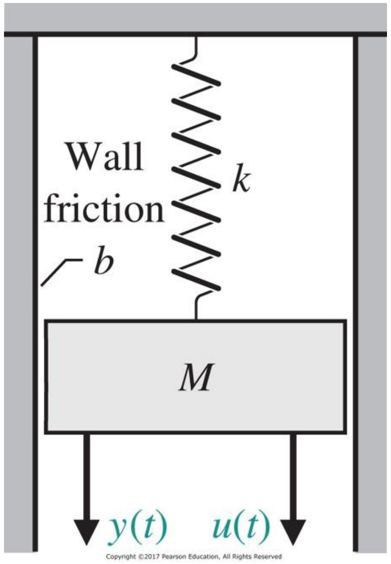
$$A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$C = [0 \quad R], \qquad D = [0]$$

Examples

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$$



$$\dot{x}_1 = x_2$$

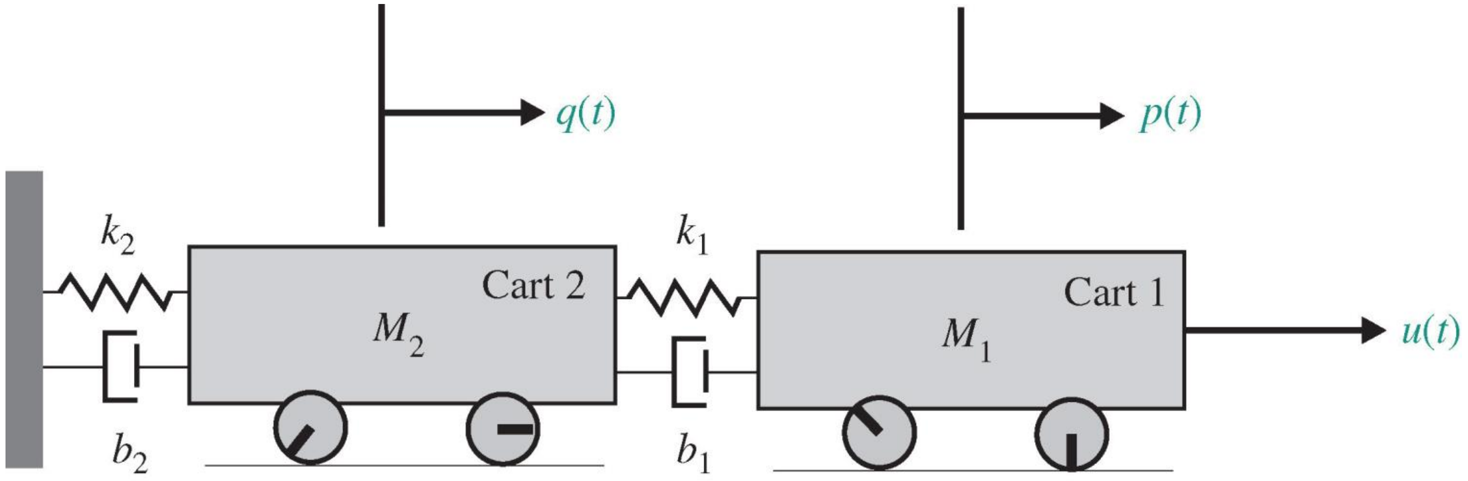
$$\dot{x}_2 = \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

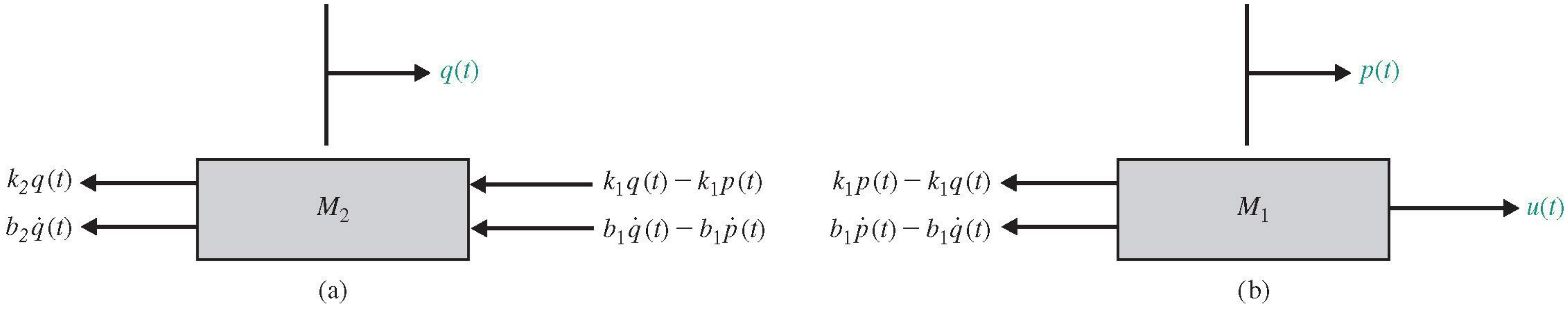
$$y = x_1$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

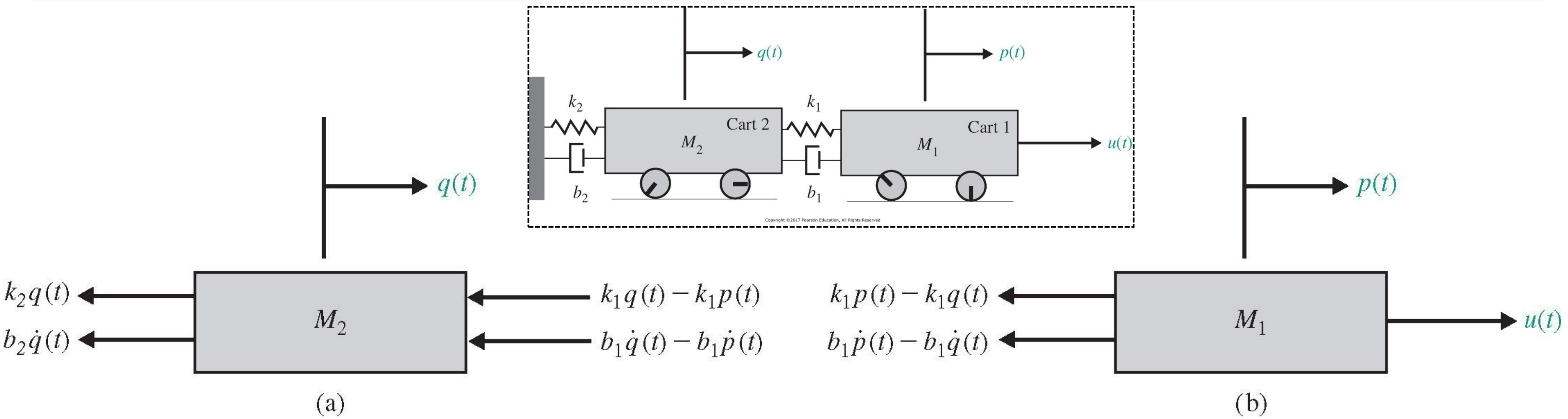
$$C = [1 \quad 0], \qquad D = [0]$$

Example 9.4: Two Rolling Carts





Example 9.4: Two Rolling Carts



where p, q are position of carts; M1 and M2 are mass of carts;
k1, k2: spring coefficients; b1, b2: damper coefficients. u: external force.

For Mass 1: $M_1\ddot{p} = F_1 = u - k_1(p-q) - b_1(\dot{p} - \dot{q})$

For Mass 2: $M_2\ddot{p} = F_2 = k_1(p-q) + b_1(\dot{p} - \dot{q}) - k_2 q - b_2\dot{q}$

How many state variables required?

Example 9.4: Two Rolling Carts

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$x_1 = p, \qquad x_3 = \dot{x}_1 = \dot{p},$$
$$x_2 = q. \qquad x_4 = \dot{x}_2 = \dot{q}.$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} p \\ q \\ \dot{p} \\ \dot{q} \end{pmatrix}$$

$$\dot{x}_3 = \ddot{p} = -\frac{b_1}{M_1}\dot{p} - \frac{k_1}{M_1}p + \frac{1}{M_1}u + \frac{k_1}{M_1}q + \frac{b_1}{M_1}\dot{q},$$

$$\dot{x}_4 = \ddot{q} = -\frac{k_1 + k_2}{M_2}q - \frac{b_1 + b_2}{M_2}\dot{q} + \frac{k_1}{M_2}p + \frac{b_1}{M_2}\dot{p},$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{k_1 + k_2}{M_2} & \frac{b_1}{M_2} & -\frac{b_1 + b_2}{M_2} \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix},$$

$$y = [1 \quad 0 \quad 0 \quad 0]\mathbf{x} = \mathbf{C}\mathbf{x}. \qquad \mathbf{D} = [0]$$

Quiz 9.1

Consider the system with the mathematical model given by the differential equation:

$$5\frac{d^3y(t)}{dt^3} + 10\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = u(t)$$

Obtain a state variable model of this system.

Concluding Remarks

- **What are state variables**
    - How to select state variables
- **How to model a system in a state-space way**
    - First-order ODE
    - State equations, output equations
- **How to express a state-space model in the matrix form**
    - System matrix, input matrix, output matrix and feed-forward matrix

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Solution of State Equations

Consider the first-order differential equation (**one-dimension**):

$$\dot{x} = ax + bu$$

With Laplace transform,

$$sX(s) - x(0) = aX(s) + bU(s)$$

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

Inverse Laplace transform satisfies, note : $\mathcal{L}^{-1}(f \cdot g) = \mathcal{L}^{-1}(f) * \mathcal{L}^{-1}(g)$:

$$x(t) = e^{at}x(0) + be^{at} * u(t)$$

Therefore:

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau$$

Solutions for Matrix Form

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s),$$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

And we let $[s\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{\Phi}(s)$, it is the Laplace form of

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$$

Assume $\mathbf{U}(s) = 0$, then

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0)$$

$\mathbf{\Phi}(s)$ describes the unforced response of the system.

Solutions for Matrix Form

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s),$$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

The matrix exponential function of $e^{\mathbf{A}t}$ is defined as:

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^kt^k}{k!} + \ \ldots$$

Then:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Solution to the matrix:

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\,\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Fundamental/State Transition Matrix

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$$ State Transition Matrix

The solution to the unforced system (when u(t)=0):

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{bmatrix} \emptyset_{11} & \dots & \emptyset_{1n} \\ \emptyset_{12} & \dots & \emptyset_{2n} \\ \vdots & \ddots & \vdots \\ \emptyset_{n1} & \dots & \emptyset_{nn} \end{bmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix}$$

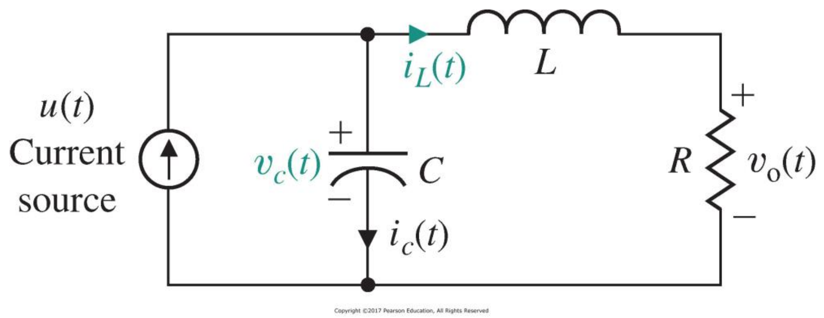
– $\emptyset_{ij}$ is the response of the i^{th} state variable due to an initial condition on the j^{th} state variable when there are zero initial conditions on all the other variables.

Example 9.5: Time Response of Unforced System

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{\Phi}(s)$$

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t) = \mathcal{L}^{-1}(\mathbf{\Phi}(s))$$ --- ***fundamental/state transition matrix***



$$\mathbf{A} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [0 \quad R], \quad \mathbf{D} = [0]$$

Assume C = ½, L = 1, R = 3, then:

$$\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \longrightarrow s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}$$

Inverse of a 2*2 matrix

$$\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{-2}{s^2+3s+2} \\ \frac{1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

Example 9.5: Time Response of Unforced System

$$\boldsymbol{\Phi}(t)=\mathcal{L}^{-1}(\boldsymbol{\Phi}(s)) = \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{-2}{s^2+3s+2} \\ \frac{1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \left(\frac{2}{s+1}+\frac{-1}{s+2}\right) & \left(\frac{-2}{s+1}+\frac{2}{s+2}\right) \\ \left(\frac{1}{s+1}+\frac{-1}{s+2}\right) & \left(\frac{-1}{s+1}+\frac{2}{s+2}\right) \end{bmatrix}$$

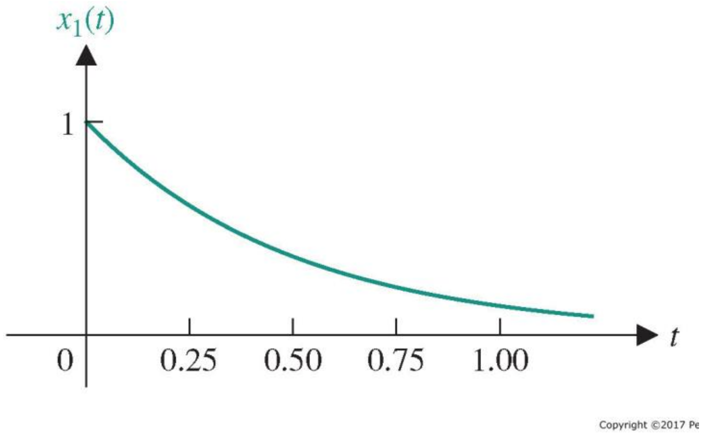
$$= \begin{bmatrix} (2e^{-t}-e^{-2t}) & (-2e^{-t}+2e^{-2t}) \\ (e^{-t}-e^{-2t}) & (-e^{-t}+2e^{-2t}) \end{bmatrix}$$

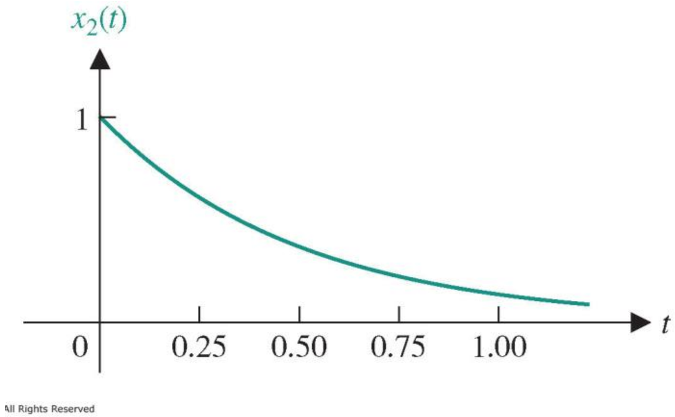
Assume $x_1(0) = x_2(0) = 1$: $\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{x}(0)$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} (2e^{-t}-e^{-2t}) & (-2e^{-t}+2e^{-2t}) \\ (e^{-t}-e^{-2t}) & (-e^{-t}+2e^{-2t}) \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$

Example 9.5: Time Response of Unforced System

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$





How about $y(t)$?

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Example 9.6: Time Response of Forced Systems

Consider the same system in example 9.5, compute the step response.

The response with control is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

If we define $\tau' = t - \tau, \tau \in [0, t]$, then

$$\tau' = t - \tau \in [0, t]$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(\tau')\mathbf{B}\mathbf{u}(t-\tau')\,d\tau'$$

$$= \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(\tau)\mathbf{B}\mathbf{u}(t-\tau)\,d\tau$$

Sometimes this is more convenient

Example 9.6: Time Response of Forced Systems

The state transition matrix is the compute before as

$$\mathbf{\Phi}(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

When $u(t) = 1(t)$, the response is written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(\tau)\mathbf{B}\, d\tau$$

While the first term is the same, the second term is

$$\int_0^t \mathbf{\Phi}(t)\mathbf{B}\, d\tau = \int_0^t \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} d\tau$$

Example 9.6: Time Response of Forced Systems

$$\int_0^t \mathbf{\Phi}(t)\mathbf{B}\, d\tau = \int_0^t \begin{bmatrix} 4e^{-\tau} - 2e^{-2\tau} \\ 2e^{-\tau} - 2e^{-2\tau} \end{bmatrix} d\tau = \begin{bmatrix} -4e^{-\tau} + e^{-2\tau} \\ -2e^{-\tau} + e^{-2\tau} \end{bmatrix}\Bigg|_0^t$$

$$= \begin{bmatrix} -4e^{-t} + e^{-2t} + 3 \\ -2e^{-t} + e^{-2t} + 1 \end{bmatrix}$$

So the time response is

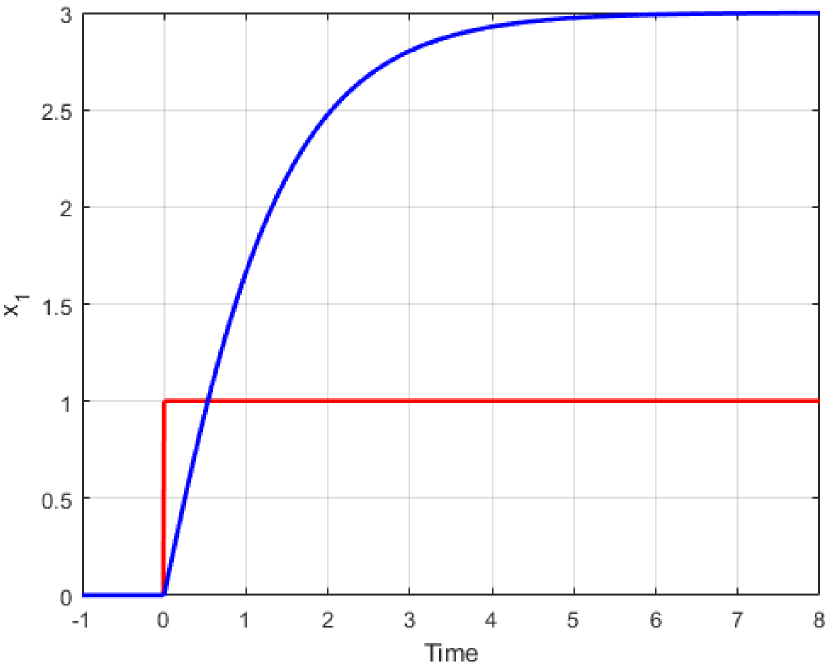
$$\mathbf{x}(t)$$

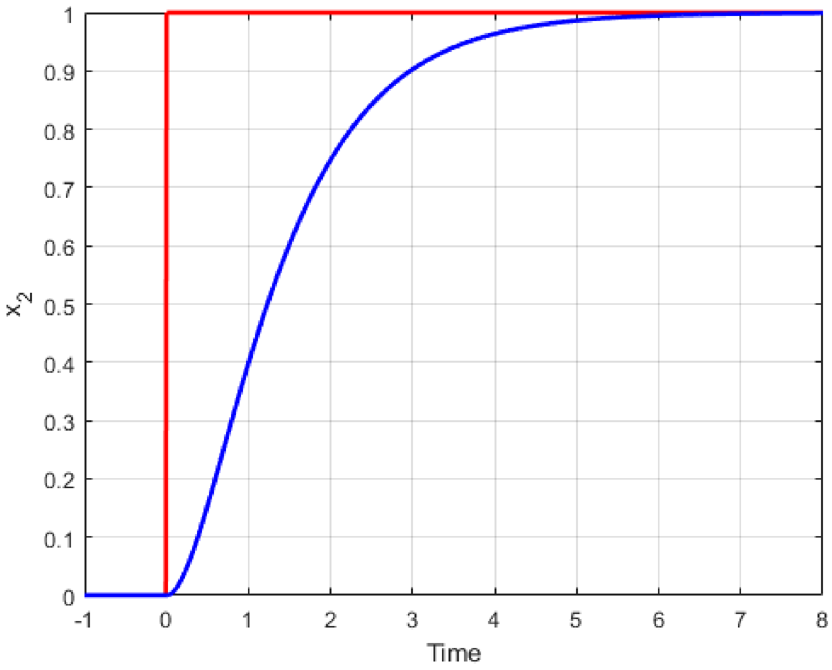
$$= \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} -4e^{-t} + e^{-2t} + 3 \\ -2e^{-t} + e^{-2t} + 1 \end{bmatrix}$$

Example 9.6: Time Response of Forced Systems

If $x(0) = 0$, then

$$\mathbf{x}(t) = \begin{bmatrix} -4e^{-t} + e^{-2t} + 3 \\ -2e^{-t} + e^{-2t} + 1 \end{bmatrix}$$





Quiz 9.2

The state-space model of a system is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

(1) Find the state-transition matrix;

(2) If initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the time response of the state variables when $u(t)$=0.

Thank You！