Xi'an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

2024-25 Semester 2

Dr. Chee Shen LIM (Room SC469)

MEC208 office hour: Thursday, 2-4pm

Email: cheeshen.lim@xjtlu.edu.cn

Lecture 20

Outline

Frequency Response Methods – Bode Plot and Nyquist Plot

- ❑ **Introduction of Bode Plot**
- ❑ **Frequency Response Plot – Bode Plot**
- ❑ **Frequency Response Measurements**
- ❑ Performance Specifications, Gain Margin, and Phase Margin
  Compensator Structure and Design
- ❑ Frequency Response Methods Using Matlab
- ❑ Another Frequency Response Plot –Nyquist/Polar Plot

Learning Aims

- To interpret and describe the concept of frequency response and its role in control system
- To sketch Bode plot and also how to obtain a computer-generated Bode plot
- To interpret Bode plot for performance specification and relative stability
- To design a controller to meet desired specification using frequency response methods
- To describe the main features of Nyquist plots and explain its relevance to stability and margin designs.

Introduction

In this chapter, we consider the steady-state response of a system to a **sinusoidal** input signal.

$$r(t) = A \sin \omega t$$

$$R(s) = \frac{A\omega}{s^2 + \omega^2}$$

- The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal.
- The sinusoid is a unique input signal, and the resulting output signal for a linear constant coefficient system is sinusoidal in the steady state;
- The output signal differs from the input only in **amplitude** and **phase angle**.

Advantages & Disadvantages of Frequency Response Method

Advantages:

- Sinusoid test signals for various range of frequency and amplitudes are readily available. Thus, the **experimental determination of the system frequency** response is easily accomplished;
- The **unknown transfer function** of a system can often be deduced from the experimentally determined frequency response of the system;
- The design of the system in the frequency domain allows the designers to **direct control the bandwidth** properties, as well as some to measure the response towards undesired noise and disturbances.
- The transfer function describing the sinusoidal steady-state behavior of a system can be easily obtained by replacing s with $j\omega$ in the system transfer function $T(s)$.

Basic Disadvantage:

- The relationship between frequency and time domains may not be obvious.

Frequency Response

Consider the system $Y(s) = T(s)R(s)$, with $r(t) = A\sin\omega t$. We have

$$R(s) = \frac{A\omega}{s^2 + \omega^2} \qquad T(s) = \frac{m(s)}{q(s)} = \frac{m(s)}{\prod_{i=1}^{n}(s + p_i)}$$

where $-p_i$ are assumed to be distinct poles. Then in partial forms, we have

$$Y(s) = \frac{k_1}{s + p_1} + \cdots + \frac{k_n}{s + p_n} + \frac{\alpha s + \beta}{s^2 + \omega^2}$$

Taking the inverse Laplace transform yields

$$y(t) \; = k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t} + \mathcal{L}^{-1}\{\frac{\alpha s + \beta}{s^2 + \omega^2}\}$$

where α and β are constants which are problem dependent. If the system is stable, then all p_i should have positive real parts, the steady-state output is

$$\lim_{t\to\infty} y(t) = \mathcal{L}^{-1}\{\frac{\alpha s + \beta}{s^2 + \omega^2}\}$$

15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$

$Y(s) = T(s)R(s)$

So, it can be shown that, for $t \to \infty$ (i.e., the steady state),

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega^2}\right\} = A|T(j\omega)|\sin(\omega t + \phi)$$

where $\emptyset = \angle T(j\omega)$.

The steady-state response described above is true <u>only for stable systems</u>.

Therefore, the steady-state output signal depends only on the magnitude ($|T(j\omega)|$) and phase ($\emptyset$) of $T(j\omega)$.

Example 20.1: Exact mag. and phase plot upon frequency sweeping

$$T(s) = \frac{s}{s+2} \xrightarrow{\text{Letting } s = j\omega} T(j\omega) = \frac{j\omega}{j\omega + 2}$$

- Manipulate the result further:

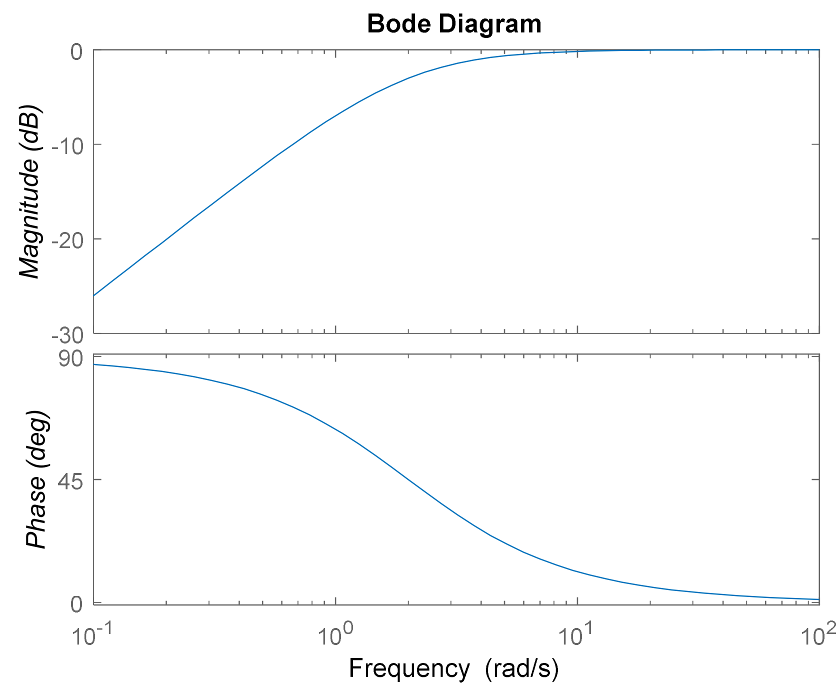
$$T(j\omega) = \frac{\omega \angle \frac{\pi}{2}}{\sqrt{4+\omega^2} \angle tan^{-1}\left(\frac{\omega}{2}\right)}$$

$$= \left| \frac{\omega}{\sqrt{4+\omega^2}} \right| \angle \frac{\pi}{2} - tan^{-1}\left(\frac{\omega}{2}\right)$$

- In decibel unit (for magnitude):

$$T(j\omega)[dB] = 20\log\left(\frac{\omega}{\sqrt{4+\omega^2}}\right) \angle \frac{\pi}{2} - tan^{-1}\left(\frac{\omega}{2}\right)$$

$$= \left(20\log\omega - 10\log(4+\omega^2)\right) \angle \frac{\pi}{2} - tan^{-1}\left(\frac{\omega}{2}\right)$$



Bode Plots

- Hendrik Wade Bode (1940) proposed the use of a semi-log analysis using the asymptotic behaviour to plot the frequency response of a system. This approach has a unique advantage where the effect of adding control elements on the gain and phase can be readily determined.
- Given $G(j\omega) = |G(j\omega)|e^{j\phi(\omega)}$, taking Natural Logs gives

$$\ln G(j\omega) = ln|G(j\omega)| + j\phi(\omega)$$

where the units of $ln|G(j\omega)|$ being Neper (rarely used).

- It is more understandable to use dB – express gain as $20log_{10}|G(j\omega)|\ (dB)$. The use of a log scale allows straight line approximations to be applicable.
- The Bode plot consists of two plots:
  - A gain plot of the magnitude expressed in dB (linear scale) plotted against frequency (Hz or rad/s, log scale)
  - A phase plot (radians or degrees, linear scale) plotted against frequency (Hz or rad/s, log scale).

Example 20.2: Approximation of Curves

Consider the transfer function of an RC filter

A system with a first-order pole

$$G(j\omega) = \frac{1}{j\omega(RC)+1} = \frac{1}{j\omega\tau+1} \qquad \text{where } \tau = RC$$

The logarithm gain is

$$20log|G(j\omega)| = 20log\left(\frac{1}{1+(\omega\tau)^2}\right)^{1/2} = -10log\left(1+(\omega\tau)^2\right)$$

- For small frequencies – that is, $\omega \ll {}^{1}/_{\tau}$, the logarithm gain is

$$20log|G(j\omega)| = -10log(1) = 0\ dB,\ \omega \ll \frac{1}{\tau}$$

An horizontal line at 0db

- For large frequencies – that is, $\omega \gg {}^{1}/_{\tau}$, the logarithm gain is

A straight line function of magnitude vs log(ω)

$$20log|G(j\omega)| = -20log(\omega\tau) =-20log(\tau) - 20log(\omega),\ \omega \gg \frac{1}{\tau}$$

- And, at $\boldsymbol{\omega = {}^{1}/_{\tau}}$ (**break frequency** or **corner frequency**), we have

$$20log|G(j\omega)| = -10log(2) = -3.01\ dB,\ \omega = \frac{1}{\tau}$$

Phase angle of the transfer function is

$$\emptyset|\omega| = -tan^{-1}(\omega\tau)$$

- Logarithmic scale of frequency is preferred over linear scale, for straight line slope. Horizontal axis is **log $\boldsymbol{\omega}$.** An interval of two frequencies with a ratio equal to 10 is called a **decade**. E.g., if from ω_1 is increased to 10 times (becoming ω_2), the new frequency is called a decade from ω_1 (+1 in log scale).
- The logarithmic gains, for $\omega \gg {}^{1}/_{\tau}$, over a decade increase of frequency is

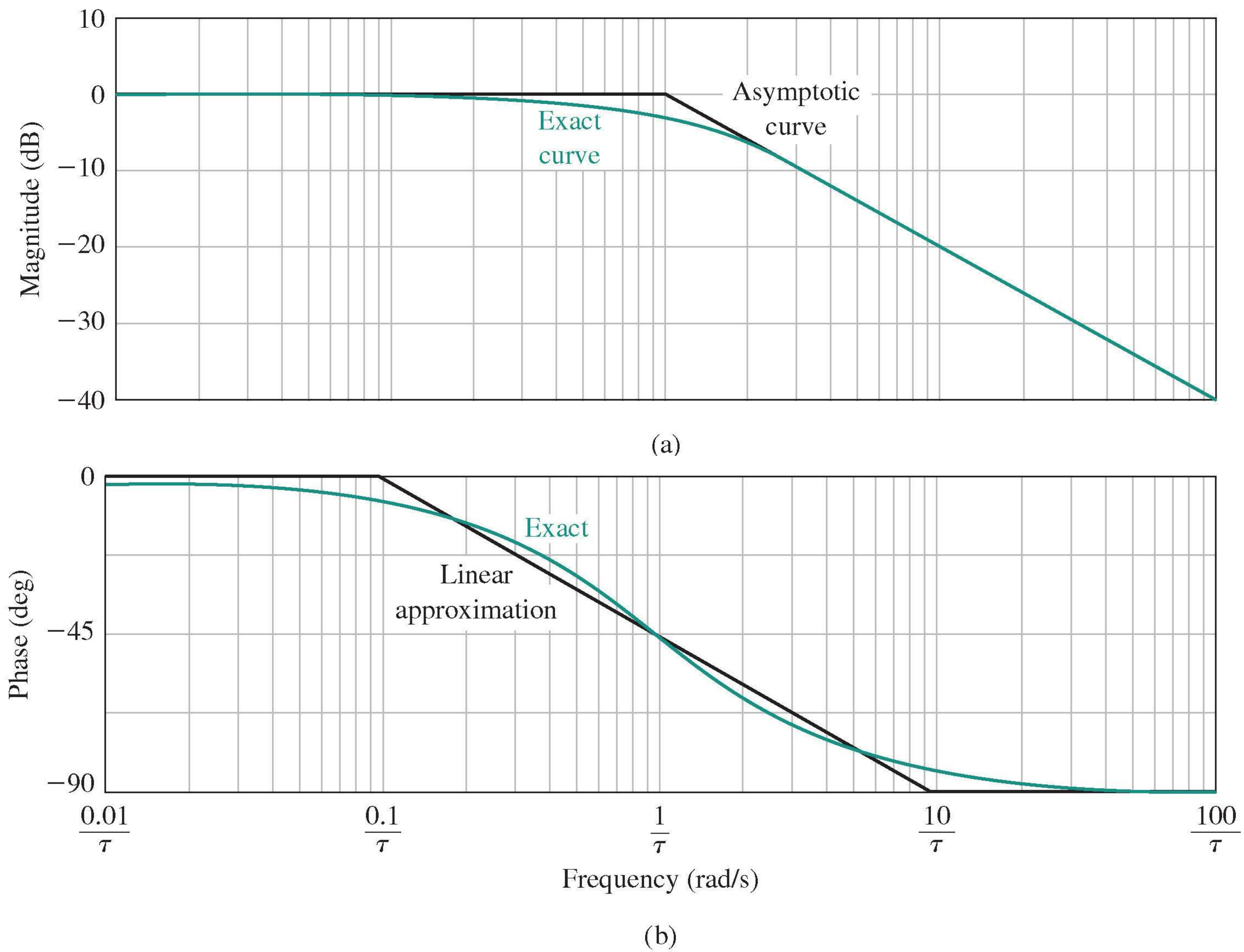
$$20log|G(j\omega_1)| - 20log|G(j\omega_2)| = -20log(\tau\omega_1) - \big(-20log(\tau\omega_2)\big)$$
$$= -20log\left(\frac{\tau\omega_2}{\tau\omega_1}\right) = -20log(10)$$
$$= -20\ dB$$

That is, the slope of the asymptotic line of this first-order transfer function for $\omega \gg {}^{1}/_{\tau}$, usually beyond 10 times the corner frequency, is -20 dB/decade.

Concept of Asymptotes or Asymptotic Plots

Bode diagram for $(1 + j\omega\tau)^{-1}$.

e.g., poles (or zeroes) on the real axis.



Generalisation of Bode Plot

- Consider the following generalised transfer function:

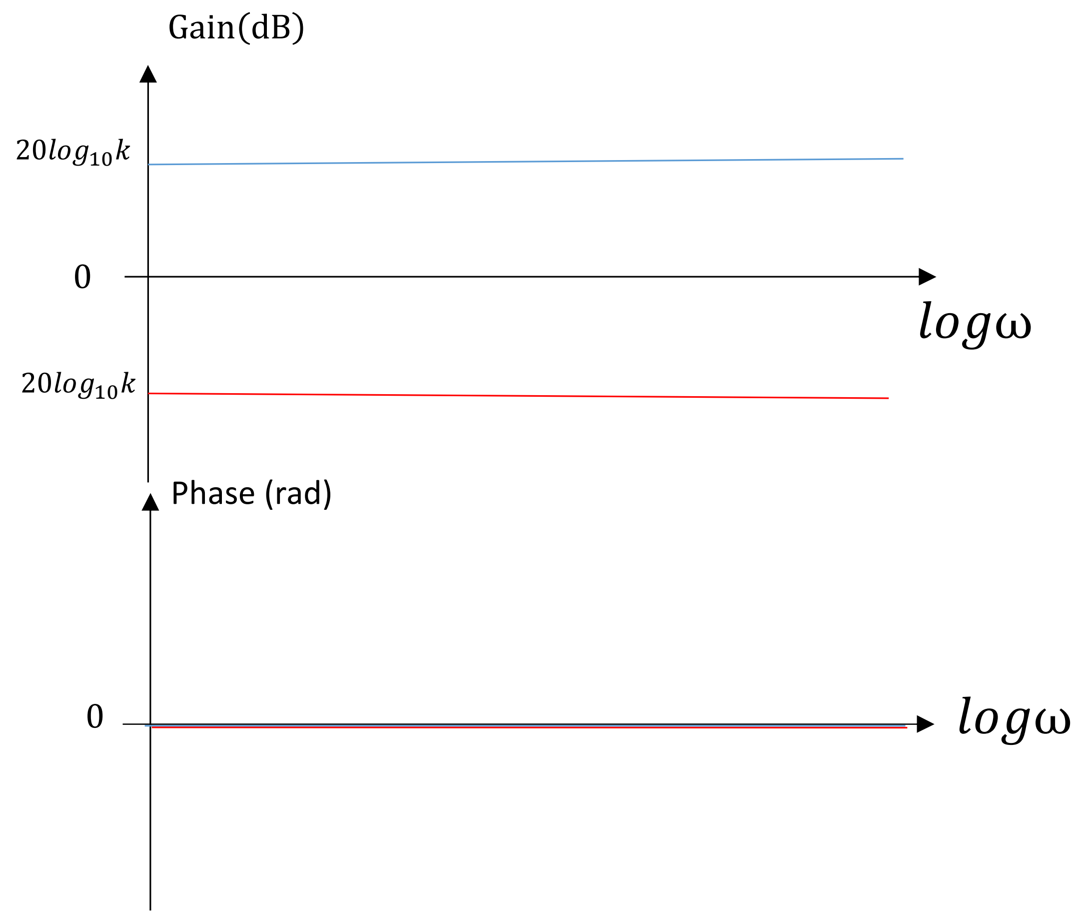
$$G(j\omega) = \frac{k\prod(1 + j\omega\tau_z)}{(j\omega)^N\prod(1 + j\omega\tau_p)\prod\left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

- 5 terms are to be considered:
  - Scalar gain
  - Negative real zeroes
  - Poles (and zeroes) at origin
  - Negative real poles
  - Complex conjugate poles
- Use the general forms of asymptotic behavior to obtain the overall frequency response for any system by summing the curves that result for each element.
- **Important**: the transfer function must be in the form of the product of the terms $(j\omega\tau)$, $(1 + j\omega\tau)$, $\left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\omega\left(\frac{2\zeta}{\omega_n}\right)\right)$, and/or *k*.

Asymptotic Plot – Constant Gain

$$G(j\omega) = \frac{k\prod(1+j\omega\tau_z)}{(j\omega)^N\prod(1+j\omega\tau_p)\prod\left(1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

- Element 1: Constant gain *k*
  - Gain(dB) $= 20log_{10}k$
  - Phase$(rad) = 0\ for\ all\ \omega$

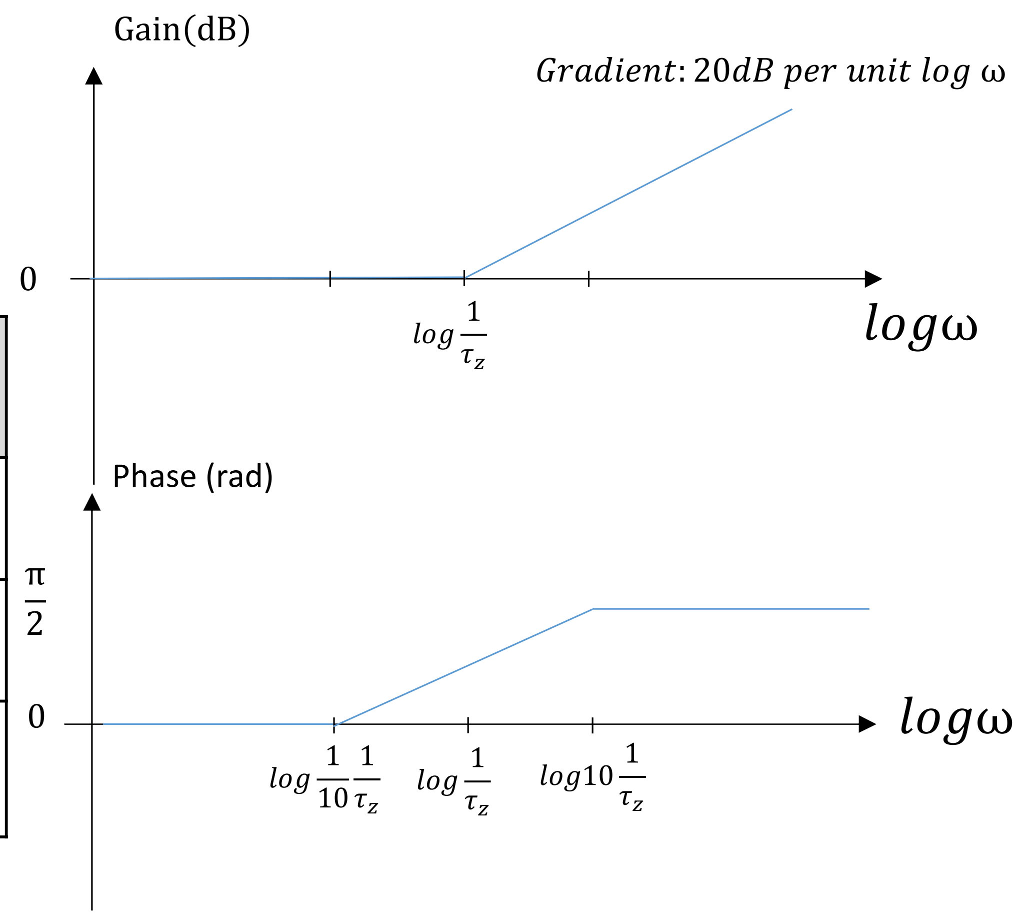


Asymptotic Plot – Negative Real Zeroes

$$G(j\omega) = \frac{k\prod(1+j\omega\tau_z)}{(j\omega)^N\prod(1+j\omega\tau_p)\prod\left(1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

- Element 2: Negative real zeroes
  - $\text{Gain(dB)} = 20log_{10}|1+j\omega\tau_z| = 10log_{10}\left(1+\omega^2\tau_z^2\right)$
  - $\text{Phase}(rad) = tan^{-1}(\omega\tau_z)\ for\ all\ \omega$

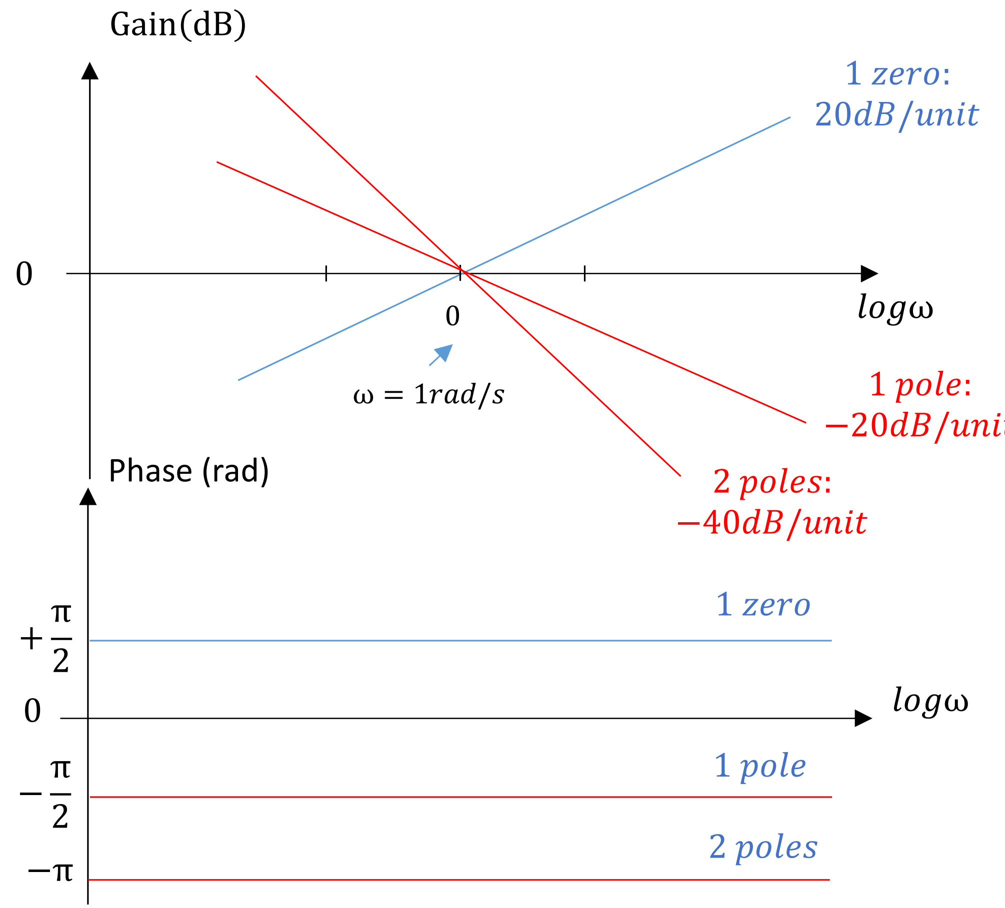
Range of ω	**Gain**	**Phase** (use points $1/10\tau_z$, $1/\tau_z$, and $10/\tau_z$)
$\omega \ll \frac{1}{\tau_z}$	$10log_{10}(1) = 0dB$	0
$\omega = \frac{1}{\tau_z}$	$10log_{10}(2) = 3.01dB$	45° or $\frac{\pi}{4}$
$\omega \gg \frac{1}{\tau_z}$	$20log_{10}(\omega\tau_z) = 20log_{10}(\omega) + 20log_{10}(\tau_z)$	90° or $\frac{\pi}{2}$



Asymptotic Plot – Poles at the Origin

$$G(j\omega) = \frac{k\prod(1 + j\omega\tau_z)}{(j\omega)^N \prod(1 + j\omega\tau_p)\prod\left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

- Element 3: N poles at the origin (extendable to zeroes)
  - $\text{Gain(dB)} = 20log_{10}\left|\frac{1}{j\omega^N}\right| = -20Nlog_{10}\omega$
  - $\text{Phase}(rad) = -\frac{N\pi}{2}\ for\ all\ \omega$
  - For zeroes: $20Nlog_{10}\omega,\ \frac{N\pi}{2}$

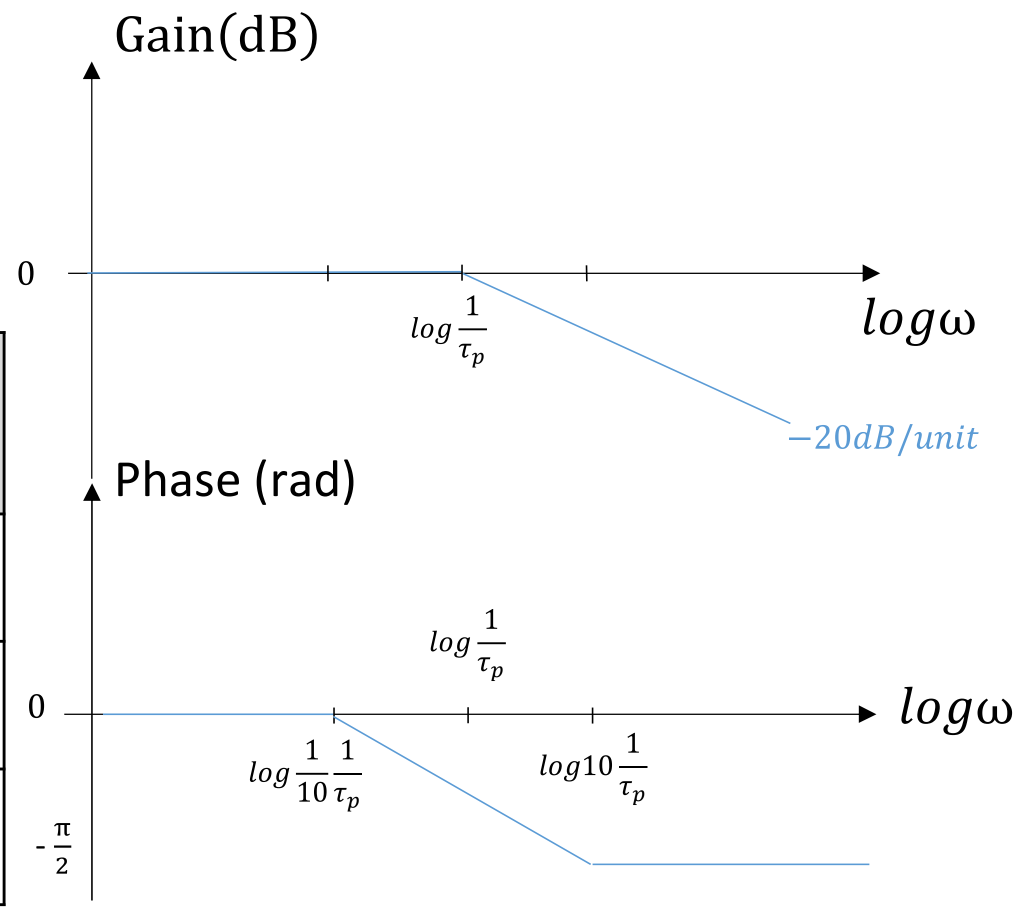


Asymptotic Plot – Negative Real Poles

$$G(j\omega) = \frac{k\prod(1 + j\omega\tau_z)}{(j\omega)^N \prod(1 + j\omega\tau_p) \prod\left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

- Element 4: Negative real poles
  - $\text{Gain(dB)} = 20log_{10}\left|\frac{1}{1+j\omega\tau_p}\right| = -10log_{10}\left(1+\omega^2\tau_p^2\right)$
  - $\text{Phase}(rad) = -tan^{-1}(\omega\tau_p)\ for\ all\ \omega$

Range of ω	**Gain**	**Phase** (use points $1/10\tau_p$, $1/\tau_p$, and $10/\tau_p$)
$\omega \ll \frac{1}{\tau_p}$	$-10log_{10}(1) = 0dB$	0
$\omega = \frac{1}{\tau_p}$	$-10log_{10}(2) = -3.01dB$	-45° or $-\frac{\pi}{4}$
$\omega \gg \frac{1}{\tau_p}$	$-20log_{10}(\omega\tau_p)$ $= -20log_{10}(\omega) - 20log_{10}(\tau_p)$	-90° or $-\frac{\pi}{2}$

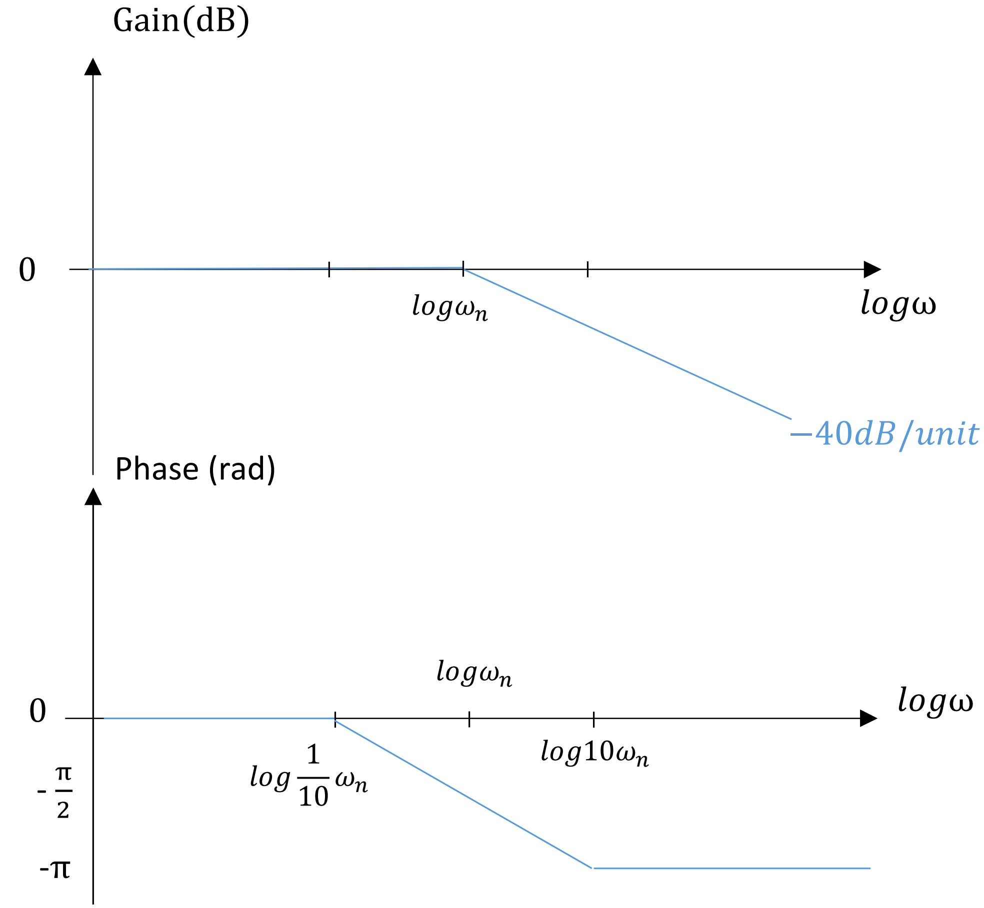


Asymptotic Plot – Complex Conjugate Poles

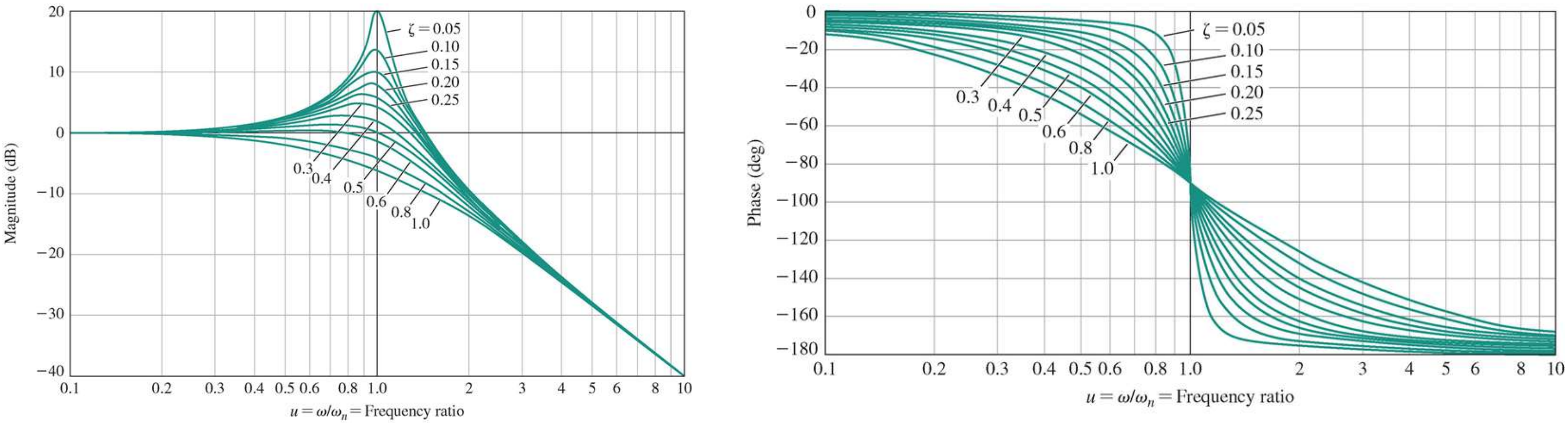
$$G(j\omega) = \frac{k\prod(1 + j\omega\tau_z)}{(j\omega)^N \prod(1 + j\omega\tau_p) \prod\left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

- Element 5: Complex conjugate poles
  - $\text{Gain(dB)} = -10log_{10}\left(\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right)$
  - $\text{Phase}(rad) = -tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right) for\ all\ \omega$

Range of ω	Gain	Phase (use points $\omega_n/10$, ω_n, and $10\omega_n$)
$\omega \ll \omega_n$	$-10log_{10}(1) = 0dB$	0
$\omega = \omega_n$	$-20log_{10}(2\zeta)$	-90° or $-\frac{\pi}{2}$
$\omega \gg \omega_n$	$-40log_{10}\left(\frac{\omega}{\omega_n}\right) = -40log_{10}(\omega) + 40log_{10}(\omega_n)$	-180° or $-\pi$



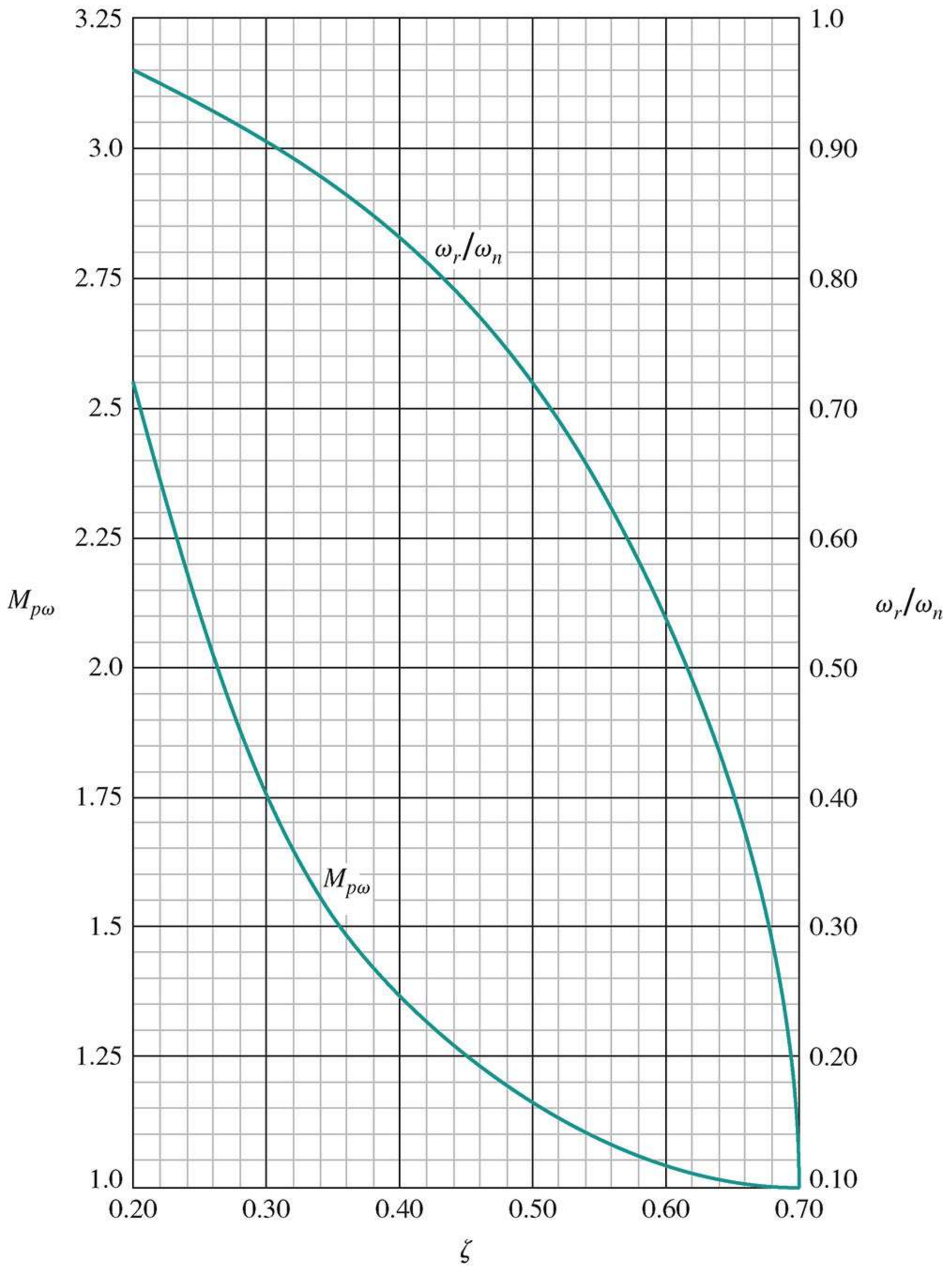
Complex Conjugate Poles - Different Damping Factors



(a) (b)

Range of $\boldsymbol{\omega}$	Gain	Phase (use points $\omega_n/10$, ω_n, and $10\omega_n$)
$\omega \ll \omega_n$	$-10log_{10}(1) = 0dB$	0
$\omega = \omega_n$	$-20log_{10}(2\zeta)$	-90° or $-\frac{\pi}{2}$
$\omega \gg \omega_n$	$-40log_{10}\left(\frac{\omega}{\omega_n}\right) = -40log_{10}(\omega) + 40log_{10}(\omega_n)$	-180° or $-\pi$

Complex Conjugate Poles – Resonant Frequency and $M_{p\omega}$



The maximum $M_{p\omega}$ of the frequency response and the resonant frequency ω_r versus ζ for a pair of conjugate poles.

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}, \zeta < 0.707$$

$$M_{p\omega} = |G(j\omega_r)| = \left(2\zeta\sqrt{1-\zeta^2}\right)^{-1}, \zeta < 0.707$$

Note:

- These formula are valid only for $\zeta < 0.707$.
- $M_{p\omega}$ is derived from finding the optimum(max.) point of the gain against varying ζ.

$$\frac{d|G(j\omega)|}{d\omega} = 0$$

Sketching a Bode Plot

$$G(j\omega) = \frac{k\prod(1+j\omega\tau_z)}{(j\omega)^N\prod(1+j\omega\tau_p)\prod\left(1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

Consider the system with the following transfer function

$$G(j\omega) = \frac{5(1+j0.1\omega)}{j\omega(1+j0.5\omega)\left(1+j0.6\left(\frac{\omega}{50}\right)+\left(j\frac{\omega}{50}\right)^2\right)}$$

Solutions:

There are five terms in the transfer function:

1. A constant gain $K = 5$
2. A pole at the origin
3. A pole at $\omega = 2$
4. A zero at $\omega = 10$
5. A pair of complex poles at $\omega_n = 50, 2\zeta = 0.6 \rightarrow \zeta = 0.3$

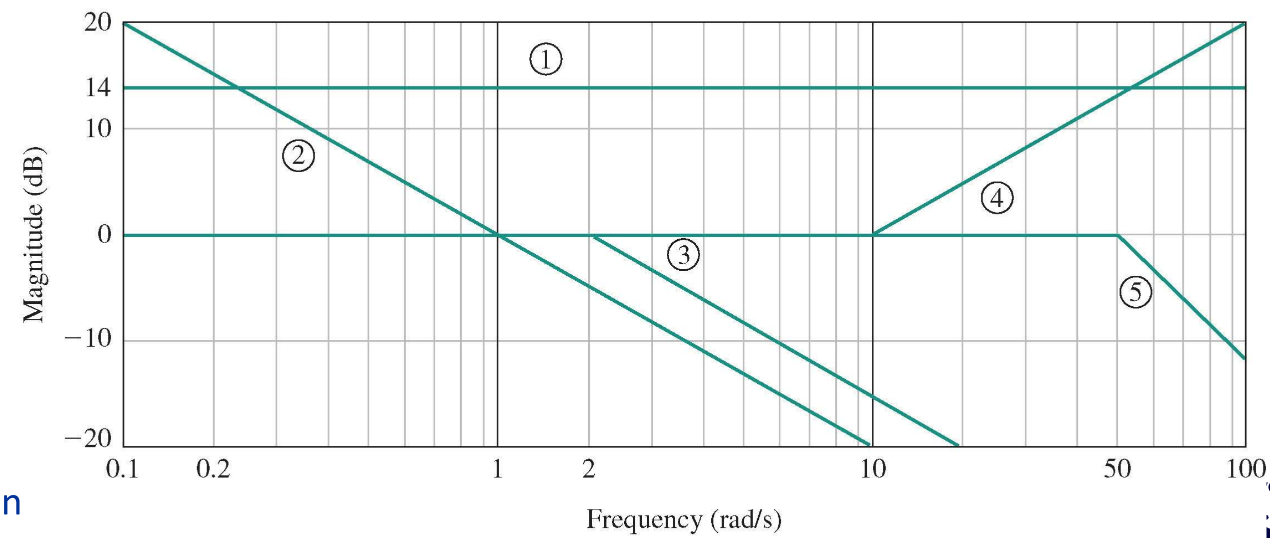
Magnitude Plot

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)[1 + j0.6(\omega/50) + (j\omega/50)^2]}$$

We plot the magnitude characteristic for each individual pole and zero factor and the constant gain:

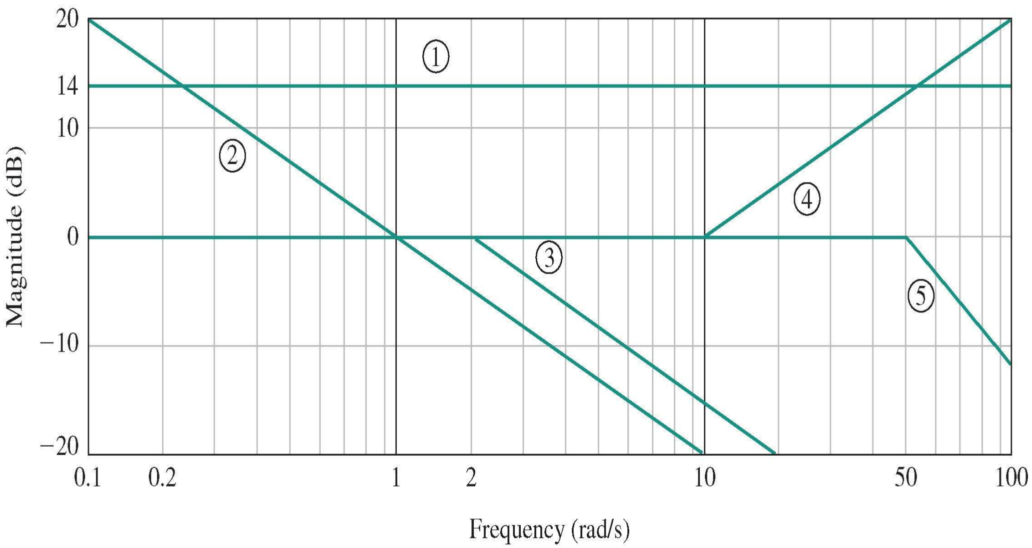
1. The constant gain is $20\log 5 = 14\text{dB}$.
2. The magnitude of the pole at the origin extends from zero to infinite frequencies, and has a slope of -20dB/dec intersecting the 0dB line at $\omega = 1\ rad/s$.
3. The asymptotic (linear) approximation of the magnitude of the pole at $\omega = 2\ rad/s$ has a slope of -20dB/dec beyond the corner frequency of $\omega = 2\ rad/s$. The asymptotic magnitude below the corner frequency is 0dB.
4. The asymptotic (linear) approximation of the magnitude of the zero at $\omega = 10\ rad/s$ has a slope of $+20\text{dB/dec}$ beyond the corner frequency at $\omega = 10\ rad/s$. The asymptotic magnitude below the corner frequency is also 0dB.
5. The magnitude slope for the complex poles after the corner frequency, $\omega = \omega_n = 50\,rad/s$, is -40dB/dec. The linear approximation around the corner frequency ω_n must be corrected based on the damping ratio ζ, in the example being 0.3 (approximately, ζ lower than $1/\sqrt{2}$ has a peak; higher than $1/\sqrt{2}$ has none).

Magnitude's asymptotic lines



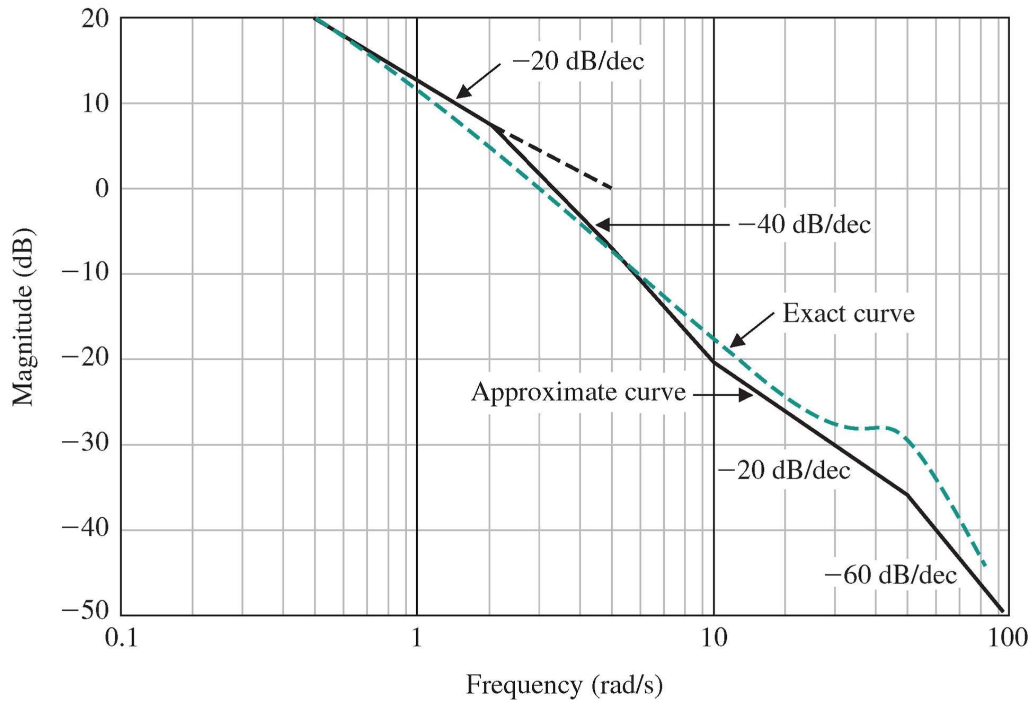
Mag. Plot

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)[1 + j0.6(\omega/50) + (j\omega/50)^2]}$$



Magnitude's asymptotic lines

FIGURE 8.20 Magnitude characteristic.



Sketch of the magnitude characteristic

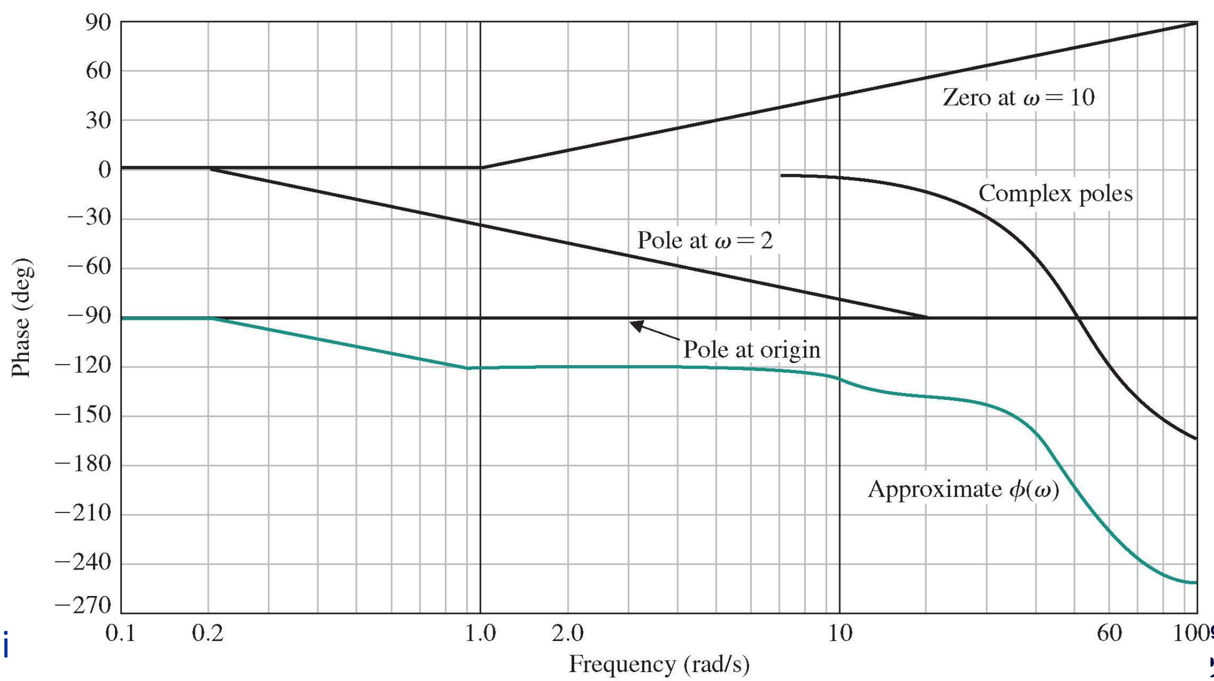
Phase Plot

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)[1 + j0.6(\omega/50) \ + (j\omega/50)^2]}$$

Then, we plot the phase characteristic for each individual pole and zero factor and the constant gain:

1. The phase of the constant gain is 0^o.
2. The phase of the pole at the origin is a constant -90^o.
3. The linear approximation of the phase characteristic for the pole at $\omega = 2\ rad/s$ is three-section straight lines (as shown below) with -45^o at $\omega = 2\ rad/s$.
4. The linear approximation of the phase characteristic for the zero at $\omega = 10\ rad/s$ is three-section straight lines (as shown below) with $+45^o$ at $\omega = 10\ rad/s$.
5. The phase characteristic approximation for the complex poles with the corner frequency $\omega_n = 50\ rad/s$ is the corrected curve based on the damping factorbased on the damping ratio ζ, in the example being 0.3 (reminder: ζ near 1 approximates to the three-section straight lines; ζ near 0 approximates to two-section step without the slope).

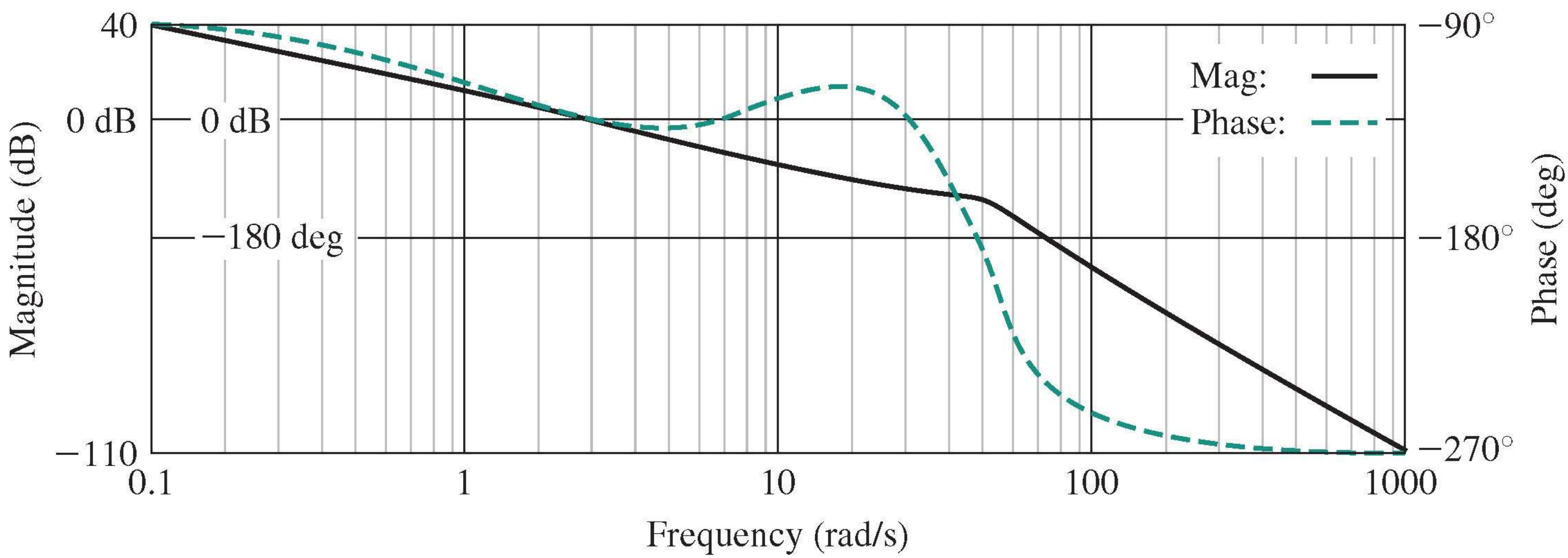
Phase's asymptotic lines



Phase Plot

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)[1 + j0.6(\omega/50) + (j\omega/50)^2]}$$

Sketch of the phase characteristic



Example 20.3 (full hand sketch)

For the following transfer function

$$L(s) = G_c(s)G(s) = \frac{300(s + 100)}{s(s + 10)(s + 40)}$$

1. Determine the phase angle and logarithmic gain of $L(j\omega)$ of when $\omega = 28.3$ rad/s;
2. Sketch the Bode Plot (means mag. and phase plots) for this transfer function.

Solution (pg. 1)

$$L(s) = G_c(s)G(s) = \frac{300(s+100)}{s(s+10)(s+40)}$$

- Convert the transfer functions into the standard forms:

$$L(j\omega) = \frac{300(j\omega+100)}{j\omega(j\omega+10)(j\omega+40)}$$

$$= \frac{\frac{300\cdot 100}{10\cdot 40}\left(1+j\frac{\omega}{100}\right)}{j\omega\left(1+j\frac{\omega}{10}\right)\left(1+j\frac{\omega}{40}\right)} = \frac{75\left(1+j\frac{\omega}{100}\right)}{j\omega\left(1+j\frac{\omega}{10}\right)\left(1+j\frac{\omega}{40}\right)}$$

$$= \frac{300\sqrt{\omega^2+100^2}}{\omega\sqrt{(\omega^2+10^2)(\omega^2+40^2)}}\angle \tan^{-1}\left(\frac{\omega}{100}\right) - \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{40}\right)$$

- Solutions for parts (1):

$$\measuredangle L(j28.3) = \tan^{-1}\left(\frac{28.3}{100}\right) - \frac{\pi}{2} - \tan^{-1}\left(\frac{28.3}{10}\right) - \tan^{-1}\left(\frac{28.3}{40}\right)$$

$$= -3.1419rad = -180^o$$

$$|L(j28.3)| = \frac{300\sqrt{28.3^2+100^2}}{28.3\sqrt{(28.3^2+10^2)(28.3^2+40^2)}} = 0.749 = -2.5dB$$

Solution (pg. 2, mag. plot)

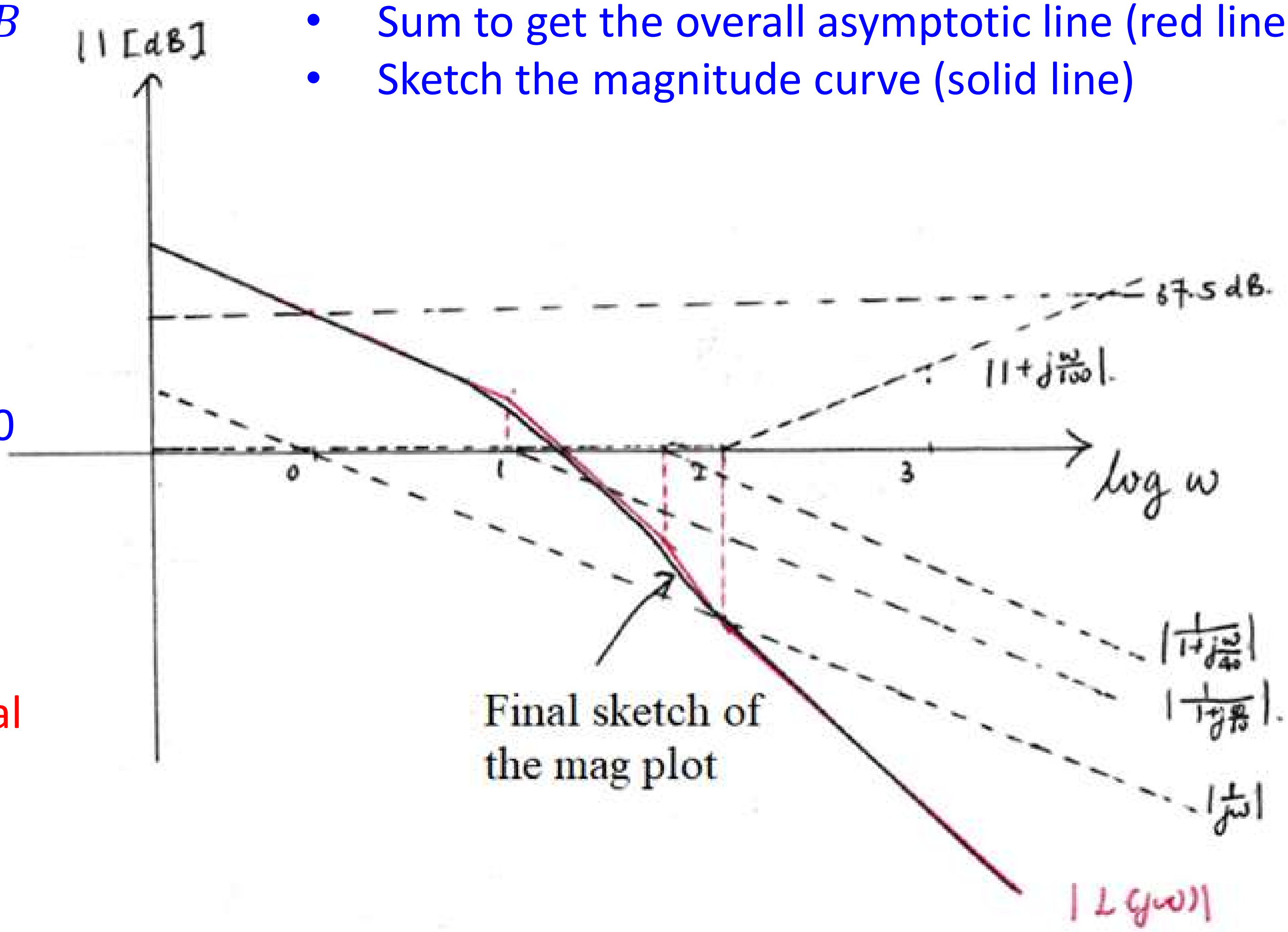
$$L(j\omega) = \frac{75\left(1 + j\frac{\omega}{100}\right)}{j\omega\left(1 + j\frac{\omega}{10}\right)\left(1 + j\frac{\omega}{40}\right)}$$
$$= \frac{300\sqrt{\omega^2 + 100^2}}{\omega\sqrt{(\omega^2 + 10^2)(\omega^2 + 40^2)}} \angle \tan^{-1}\left(\frac{\omega}{10}\right) - \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{40}\right)$$

Identify all the terms of gain, poles & zeros:

- Gain: $75 = 37.5dB$
- A pole @ origin
- A real first-order zero: $\omega = 100$
- A real first-order pole: $\omega = 10$
- Another real first-order pole: $\omega = 40$

Steps:

- Sketch all the asymptotic lines
- Sum to get the overall asymptotic line (red line)
- Sketch the magnitude curve (solid line)



Sketch all individual gain asymptotes, and sum all to get the final asymptote. Then use it to draw the final sketch.

Solution (pg. 3, phase plot)

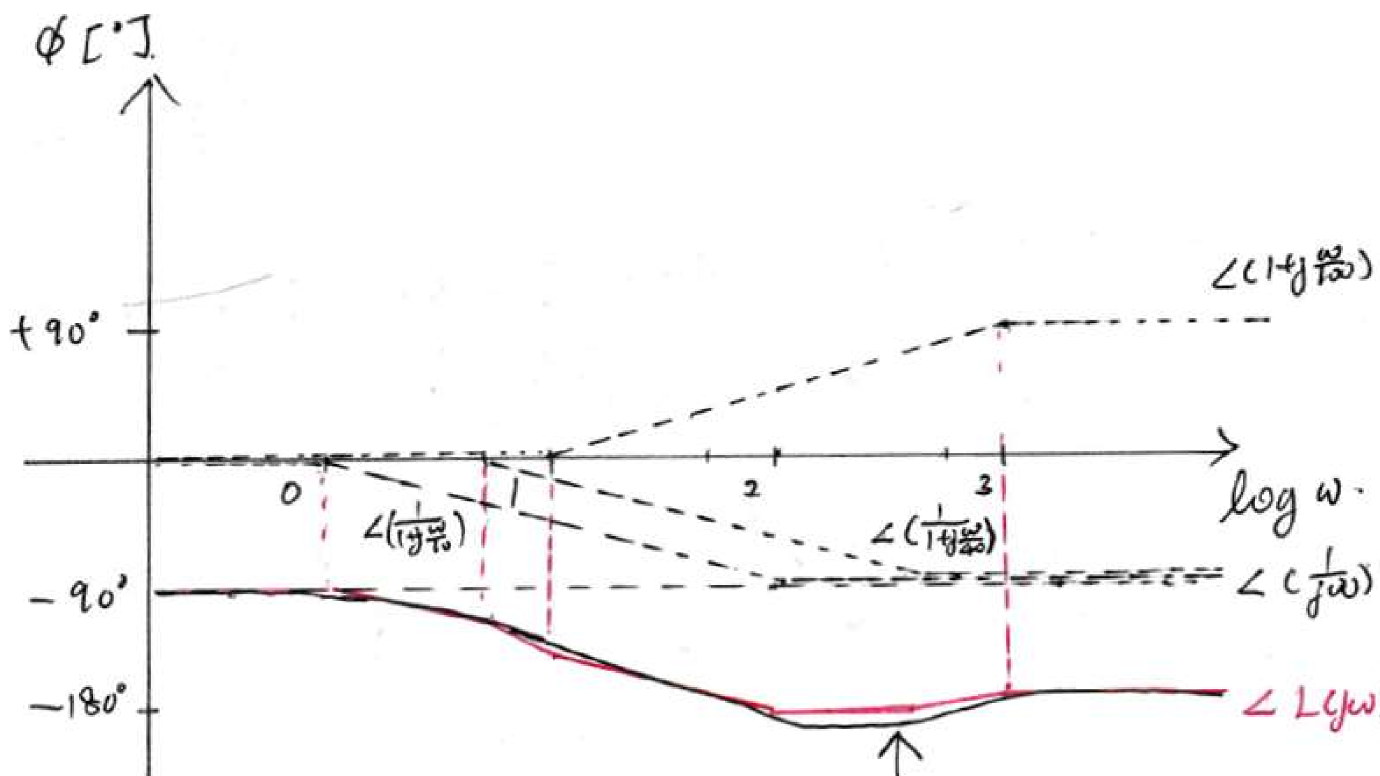
Identify all the terms of gain, poles & zeros:

- Gain: $75 = 37.5dB$
- A pole @ origin
- A real first-order zero: $\omega = 100$
- A real first-order pole: $\omega = 10$
- Another real first-order pole: $\omega = 40$

Sketch all individual phase asymptotes, and sum all to get the final asymptote. Then use it to draw the final sketch.

Steps:

- Sketch all the asymptotic lines
- Sum to get the overall asymptotic lines (red line)
- Sketch the magnitude curve (solid line)



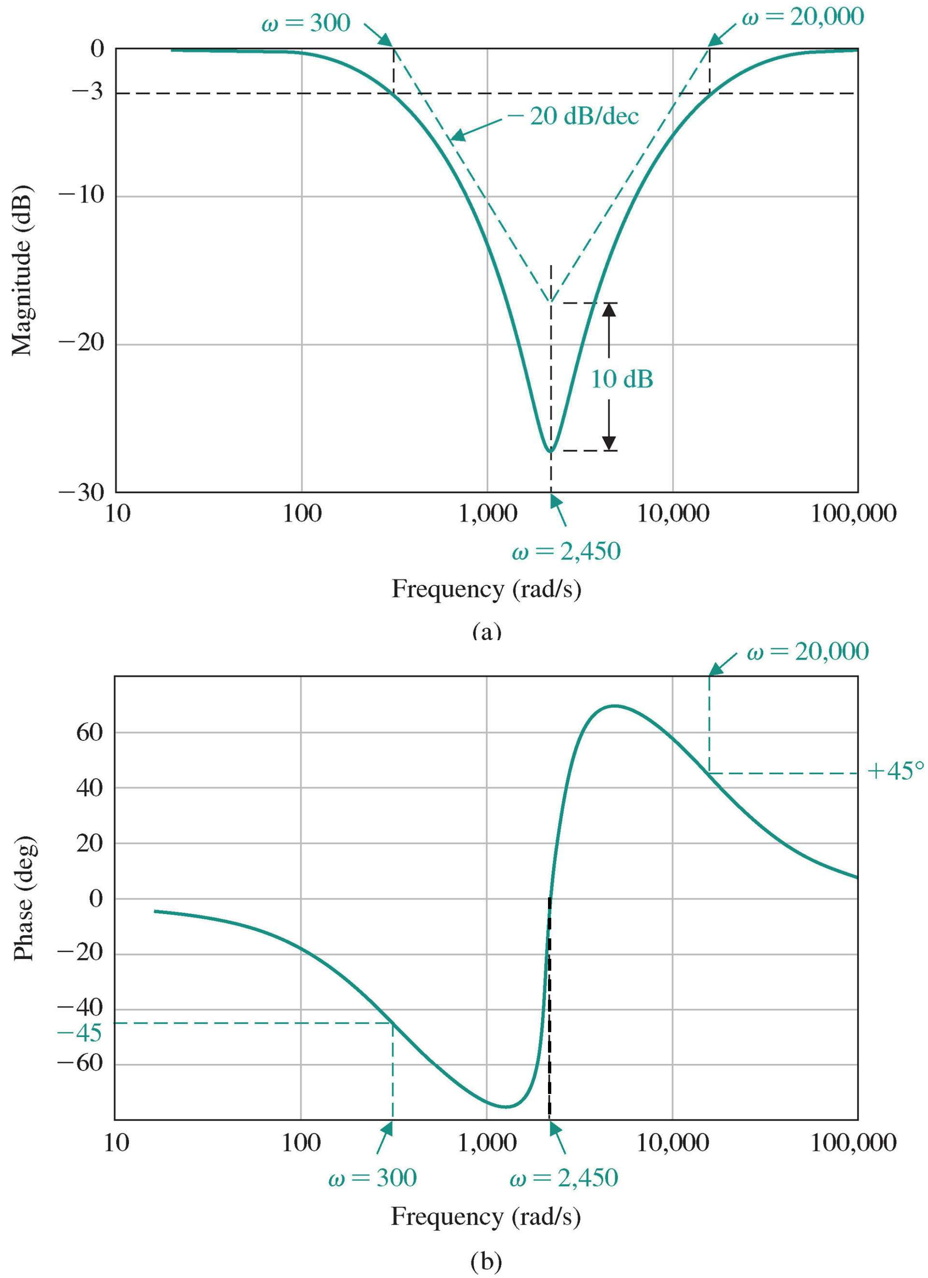
Final sketch of the phase plot

Frequency Response - Features

- A sine wave can be used to measure the open-loop frequency response of a system. In practice, a plot of amplitude versus frequency and phase versus frequency will be obtained. From these two plots, the transfer function (including both the loop TF and the closed-loop TF) of the system can be deduced.

- **Control bandwidth** of the system under control can be easily deduced from the plots. Behavior/response of the system towards measurement noise and disturbances can also be understood (recall the materials from Lecture 9).

Example 20.4: TF Deduction

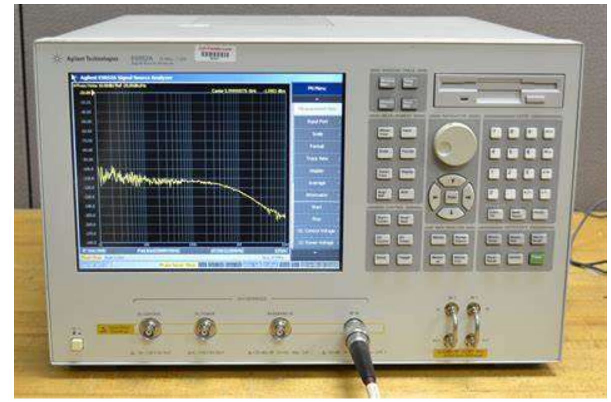


- Magnitude is 0db for low ω. No pole at the origin.
- The magnitude declines at about -20 dB/decade as ω increase between 100 and 1000; and because the phase is -45° and the magnitude is -3dB at 300 rad/s → there must be a pole at $p_1 = 300$;
- Phase abruptly increases by nearly 180° and passing 0° at 2450 rad/s; also, the slope of the magnitude changes from -20 dB/decade to 20 dB/decade at 2450 rad/s → there must be a pair of quadratic zeros existing at $\omega_n = 2450$;
- Difference in the corner frequency of the asymptotes to the minimal response is 10 dB → $\zeta = 0.16$.
- Slope of magnitude returns to 0 dB/decade as ω exceeds 50000; specifically, magnitude is -3 dB (and phase is 45°) at 20000 rad/s → there must be a second pole at $p_2 = 20000$.

$$T(s) = \frac{(s/2450)^2 + (0.32/2450)s + 1}{(s/300 + 1)(s/20000 + 1)}$$

Frequency Response Measurement (Practical)

- A device called a **wave analyzer** can be used to measure the amplitude and phase variations as the frequency of the input sine wave is altered. Also, a device called a **transfer function analyzer** can be used to measure the loop transfer function and closed-loop transfer function.



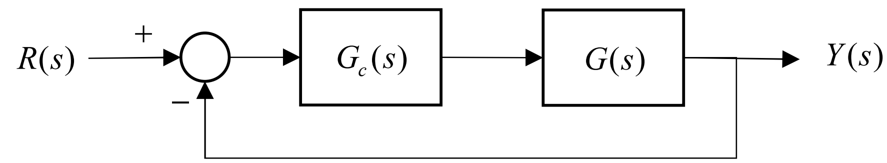
- Nowadays, signal analyzer instrument can perform frequency response measurement different ranges, depending on application. For example,
  - In power electronic circuit analyzer, response in the range of DC to 1MHz range is relevant.
  - In communication, frequency response in the range of several MHz to GHz is relevant.
  - High-end analyzer can perform a very wide range of frequency sweeping.

- Built-in analysis and modeling capabilities can derive poles and zeros from measured frequency responses or construct phase and magnitude responses from user-supplied models. This device can also synthesize the frequency response of a model of a system, allowing a comparison with an actual response.

Example 20.5 (in-class)

A space shuttle was used to repair satellites. The closed-loop control system of the space robotic arm has the following loop transfer function:

$$L(s) = G_c(s)G(s) = \frac{87}{s^2 + 15.9s}$$



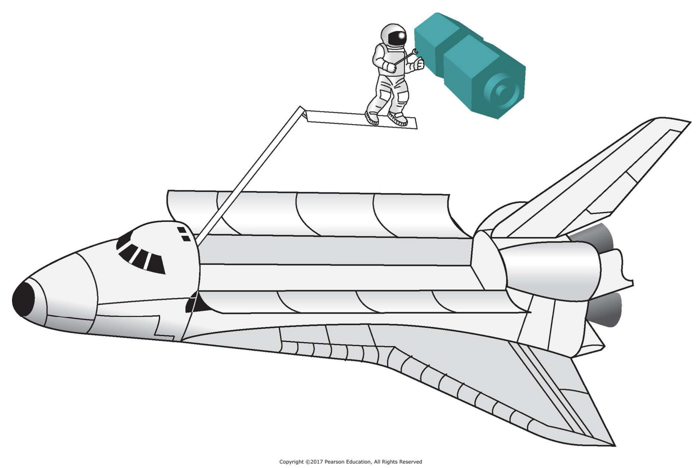
1. Sketch the mag. and phase plots of the loop transfer function. Clearly indicate the details of the asymptotes and the final sketch.

2. Determine the gain crossover frequency and the phase margin of the system.

3. Determine the phase crossover frequency and the gain margin of the system.

4. Sketch the mag. and phase plots of the closed-loop system.

5. Determine the bandwidth of the closed-loop system.



Bode asymptotic plotting (LTF)

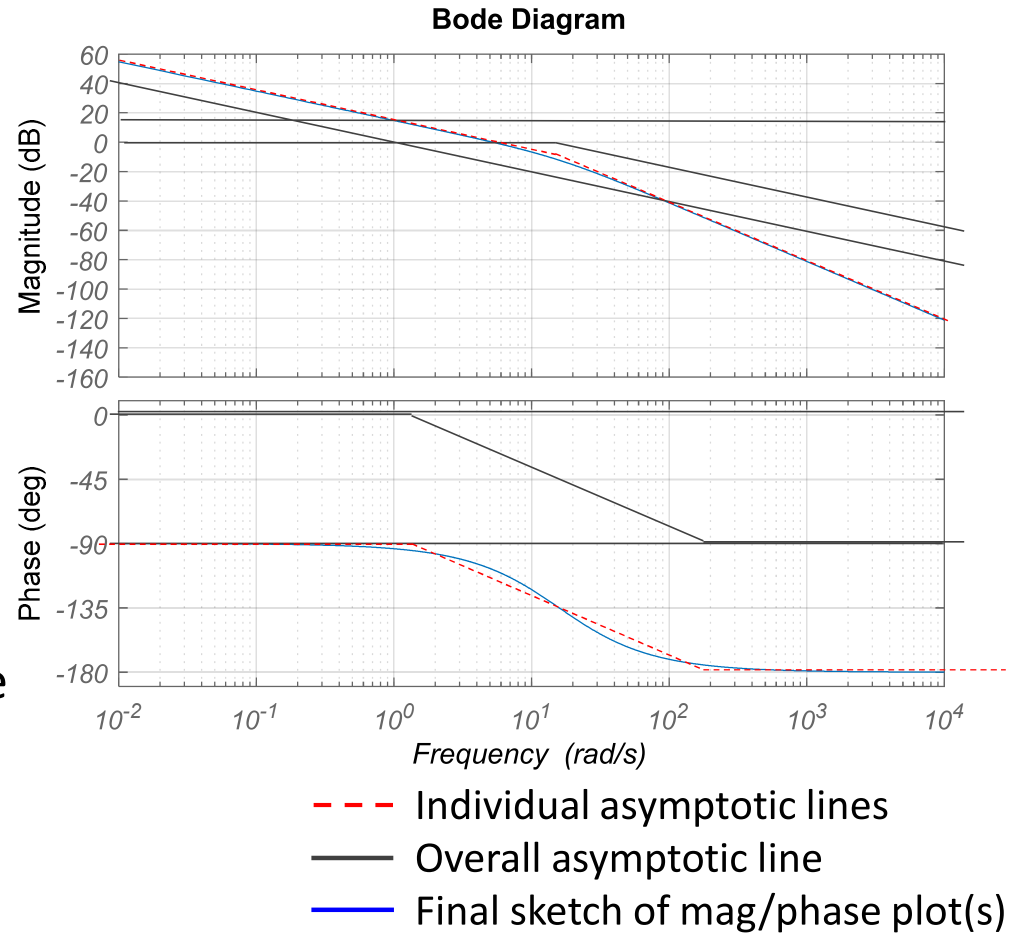
$$L(s) = \frac{87}{s^2 + 15.9s} = \frac{87}{s(s+15.9)}$$

$$L(j\omega) = \frac{5.47}{j\omega(1 + j\frac{\omega}{15.9})} = \left|\frac{87}{\omega\sqrt{\omega^2 + 15.9^2}}\right| \angle -90^o - \tan^{-1}\frac{\omega}{15.9}$$

1. Substitute $s = j\omega$, rearrange to the standard forms (preferably both).

2. Identify all the terms of gain, poles & zeros:
   - Gain: 5.47 = 14.6dB
   - Origin pole: $j\omega$
   - Real pole: $(1 + j\frac{\omega}{15.9})$

3. Sketch all individual asymptotes (in the gain and phase plots), and sum all to get the final asymptote. Then use it to draw the final sketch.



Gain crossover frequency and Phase Margin (PM) calculation

$$L(j\omega) = \frac{5.47}{j\omega(1 + j\frac{\omega}{15.9})} = \left|\frac{87}{\omega\sqrt{\omega^2 + 15.9^2}}\right| \angle -90^o - \tan^{-1}\frac{\omega}{15.9}$$

4. If relevant, calculate the gain crossover frequency and PM by setting gain to 0 db:

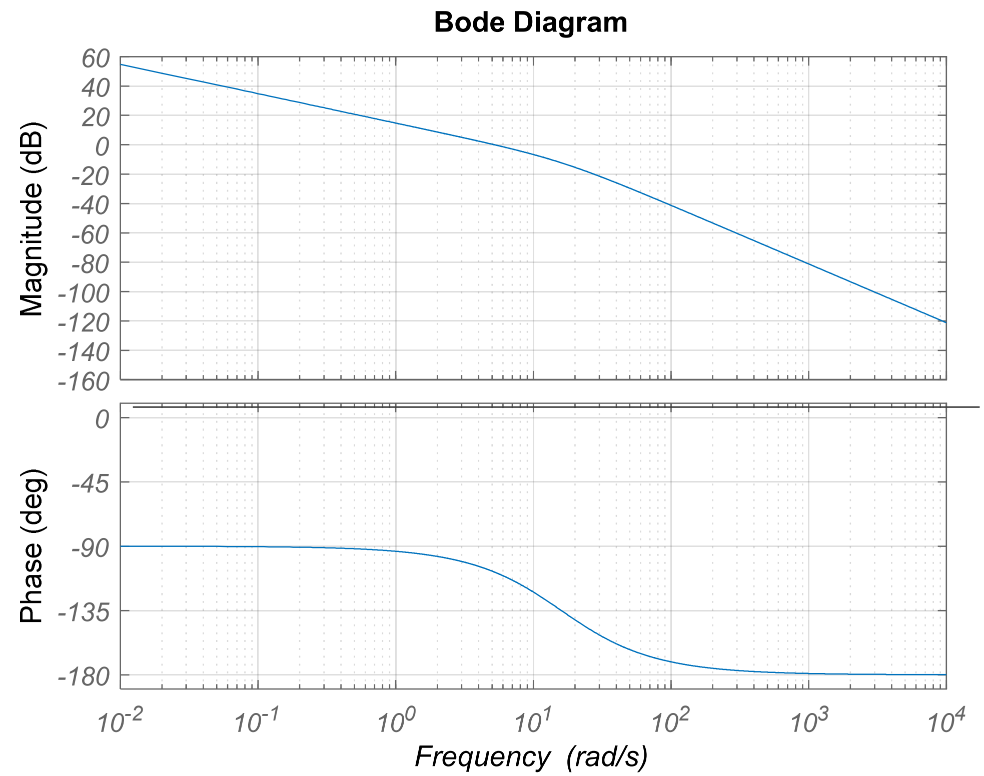
$$\left|\frac{87}{\omega\sqrt{\omega^2 + 15.9^2}}\right| = 1$$

Gain crossover frequency $\omega_{PM} = 5.2rad \cdot s^{-1}$

(the other answer, $\sqrt{-279.86}$, is rejected)

$$\angle L(j\omega) = -90^o - \tan^{-1}\frac{5.2}{15.9} = -108.1^o$$

Phase Margin = $-108.1^o - (-180^o) = 71.9^o$.



Phase crossover frequency and Gain Margin (GM) calculation

$$L(j\omega) = \frac{5.47}{j\omega(1 + j\frac{\omega}{15.9})} = \left|\frac{87}{\omega\sqrt{\omega^2 + 15.9^2}}\right| \angle -90^o - \tan^{-1}\frac{\omega}{15.9}$$

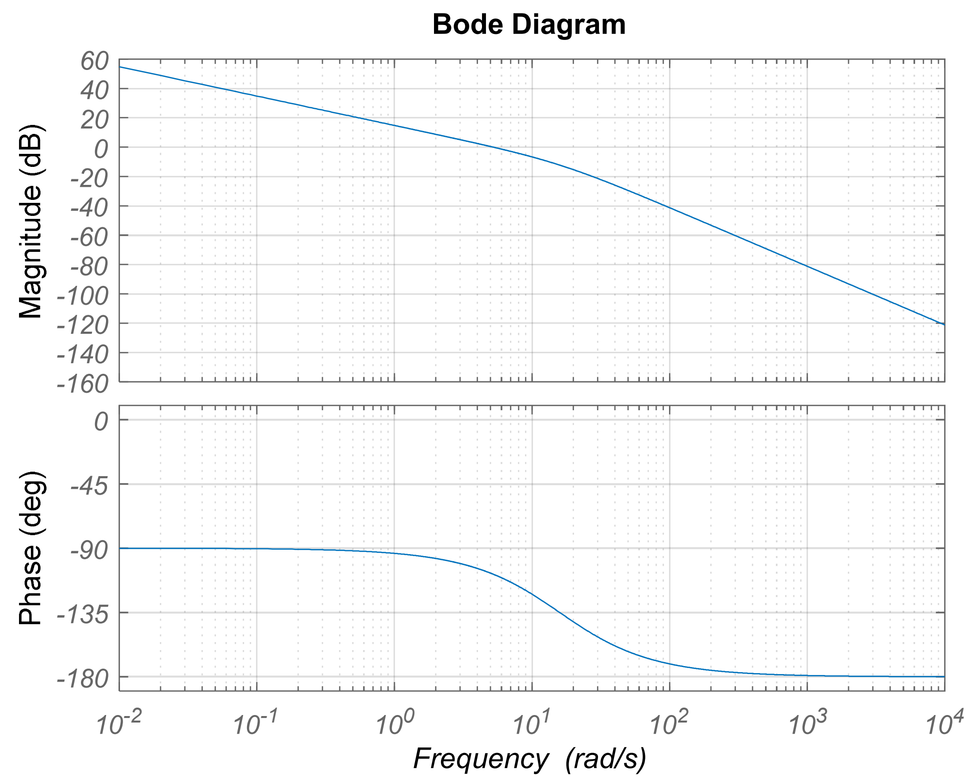
4. If relevant, calculate the phase crossover frequency and GM by setting phase to -180º:

$$-90^o - \tan^{-1}\frac{\omega}{15.9} = -180^o$$

Phase crossover frequency $\omega_{GM} = \infty\, rad \cdot s^{-1}$

$$|L(j\omega)| = 0 = -\infty db$$

Gain Margin = $0 - (-\infty) = \infty\ db$



Bode asymptotic plotting (CLTF)

$$G(j\omega) = \frac{k}{1-\left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)}$$

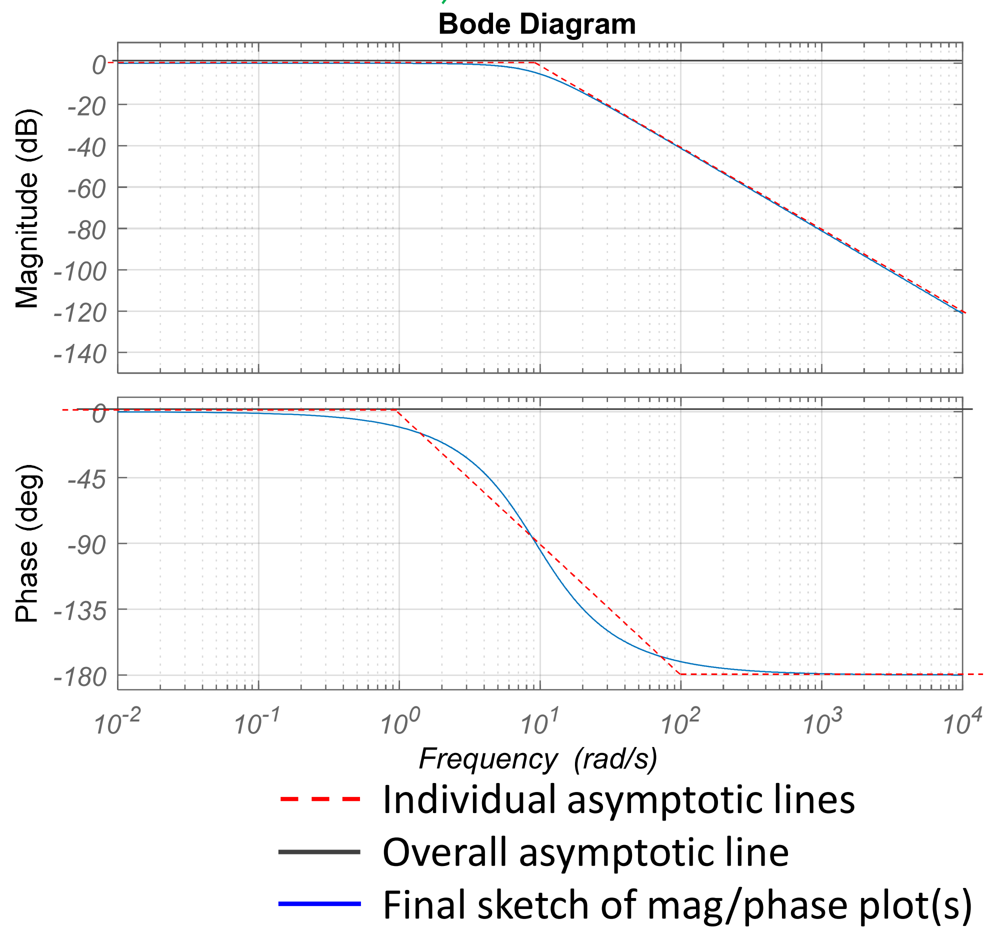
$$\text{Gain}(\textit{complex pole}) = -10log_{10}\left(\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right)$$

$$\text{Phase}(\textit{complex pole}) = -tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right)$$

$$L(s) = \frac{G_cG}{1+G_cG} = \frac{87}{s^2+15.9s+87}$$

$$L(j\omega) = \frac{1}{1-\left(\frac{\omega}{\sqrt{87}}\right)^2 + j\left(\frac{15.9}{\sqrt{87}}\right)\left(\frac{\omega}{\sqrt{87}}\right)} = \left(\left\{1-\left(\frac{\omega}{\sqrt{87}}\right)^2\right\}^2 + \left(\frac{15.9}{\sqrt{87}}\right)^2\left(\frac{\omega}{\sqrt{87}}\right)^2\right)^{-1/2} \angle tan^{-1}\left(\frac{15.9\omega}{87-\omega^2}\right)$$

1. Substitute $s = j\omega$, rearrange to the standard forms (preferably both).
2. Identify all the terms of gain, poles & zeros:
   - Gain: $1 = 0dB$
   - Complex poles: $\omega_n = \sqrt{87} = 9.3rad\cdot s^{-1}, \zeta = \frac{15.9}{2\sqrt{87}} = 0.85$
3. Sketch all individual asymptotes (in the gain and phase plots), and sum all to get the final asymptote. Then use it to draw the final sketch.

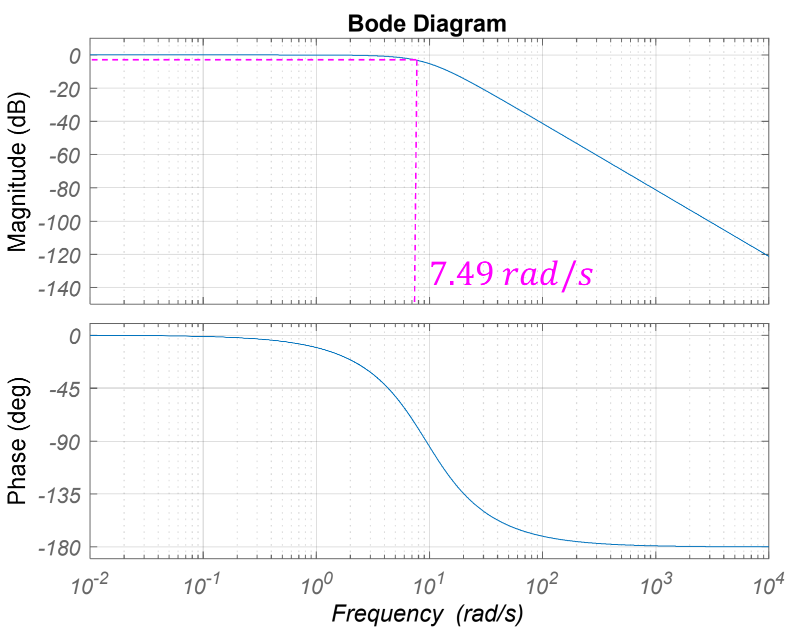


Bandwidth (BW) calculation

$$L(j\omega) = \frac{1}{1-\left(\frac{\omega}{\sqrt{87}}\right)^2 + j\left(\frac{15.9}{\sqrt{87}}\right)\left(\frac{\omega}{\sqrt{87}}\right)} = \left(\left\{1-\left(\frac{\omega}{\sqrt{87}}\right)^2\right\}^2 + \left(\frac{15.9}{\sqrt{87}}\right)^2\left(\frac{\omega}{\sqrt{87}}\right)^2\right)^{-1/2} \angle \tan^{-1}\left(\frac{15.9\omega}{87-\omega^2}\right)$$

- Bandwidth is defined as the frequency at which the frequency response has declined 3dB from its low-frequency gain value (which happens to be 1 or 0dB, hence just need to find ω at -3dB).

- Bandwidth = 7.49 rad/s



$$20db \cdot log\left(\left\{1-\left(\frac{\omega}{\sqrt{87}}\right)^2\right\}^2 + \left(\frac{15.9}{\sqrt{87}}\right)^2\left(\frac{\omega}{\sqrt{87}}\right)^2\right)^{-1/2} = -3dB$$

$$\left(\left\{1-\left(\frac{\omega}{\sqrt{87}}\right)^2\right\}^2 + \left(\frac{15.9}{\sqrt{87}}\right)^2\left(\frac{\omega}{\sqrt{87}}\right)^2\right)^{-1/2} = 10^{-\frac{3}{20}} = 0.7071$$

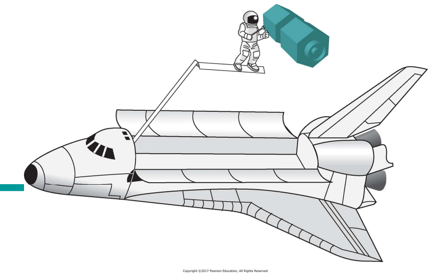
$$\left\{1-\left(\frac{\omega}{\sqrt{87}}\right)^2\right\}^2 + \left(\frac{15.9}{\sqrt{87}}\right)^2\left(\frac{\omega}{\sqrt{87}}\right)^2 = 2$$

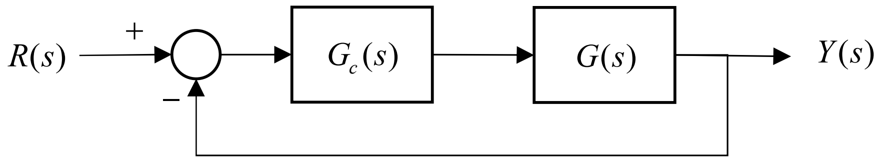
$$\left(\frac{\omega}{\sqrt{87}}\right)^2 = 0.645(accept), -1.55\ (reject)$$

$$\omega = 7.49 rad \cdot s^{-1}$$

Exact plots

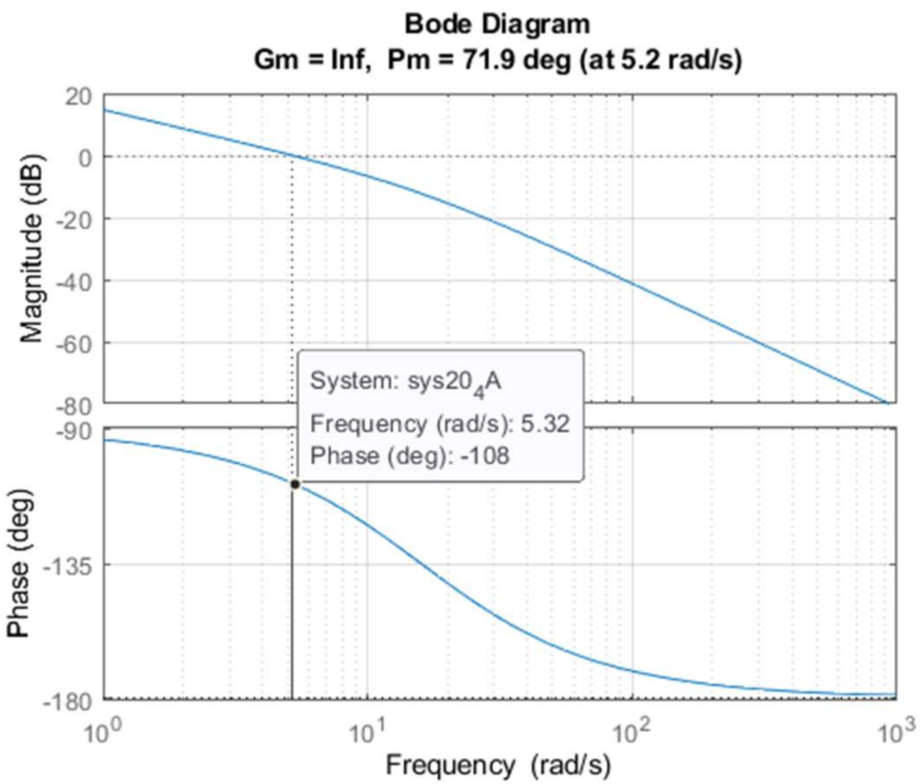
$$L(s) = \frac{87}{s^2 + 15.9s}$$

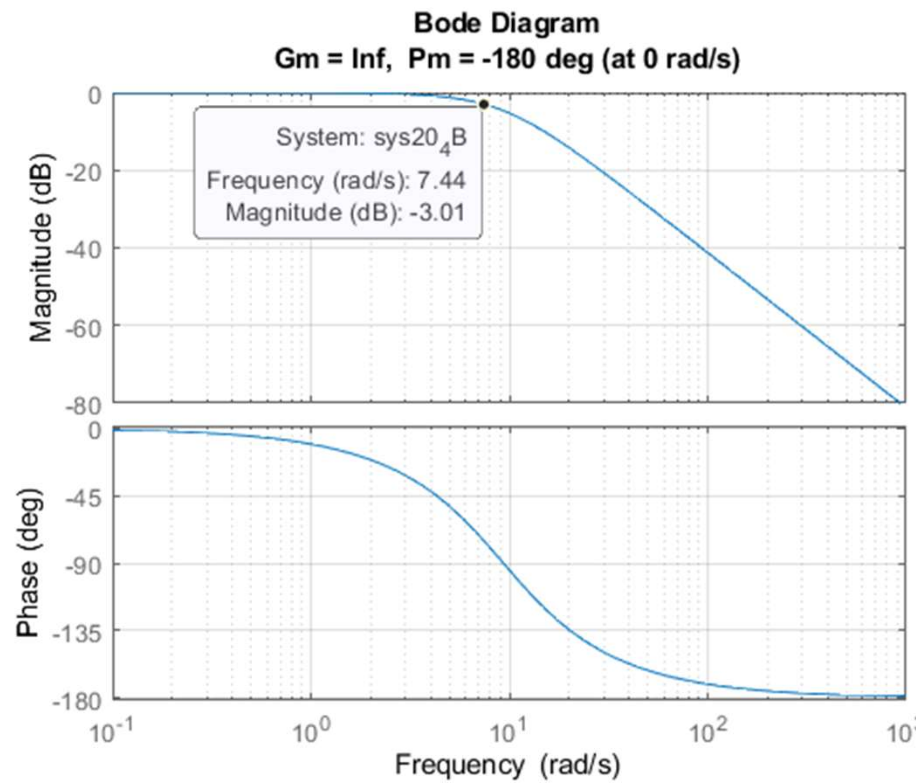




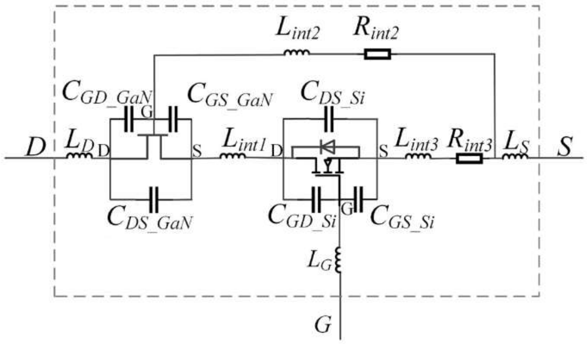
$$L(j\omega) = \frac{5.47}{j\omega(1 + j\frac{\omega}{15.9})}$$

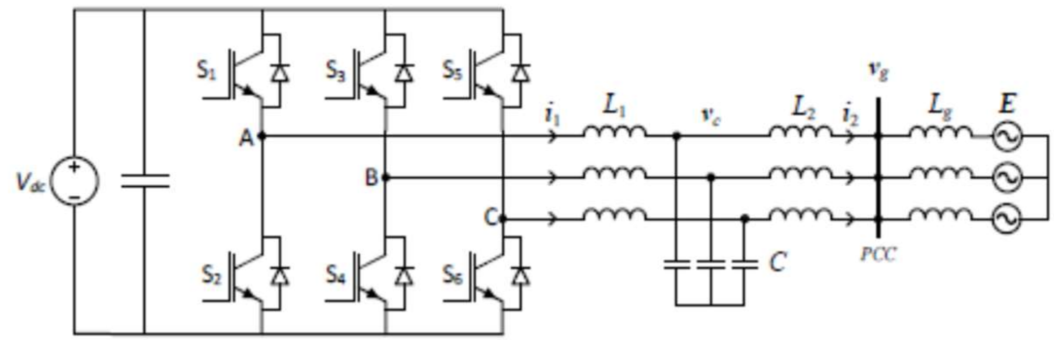
$$T(j\omega) = \frac{1}{1 - \left(\frac{\omega}{\sqrt{87}}\right)^2 + j\left(\frac{15.9}{\sqrt{87}}\right)\left(\frac{\omega}{\sqrt{87}}\right)}$$

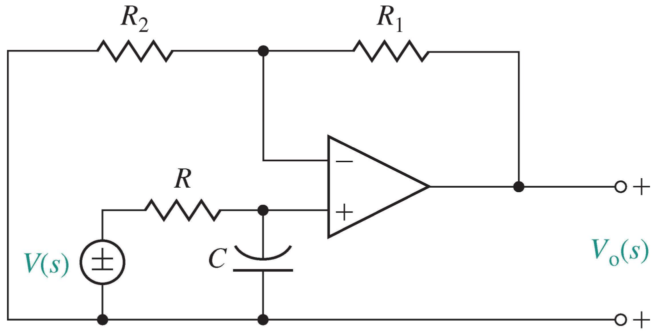


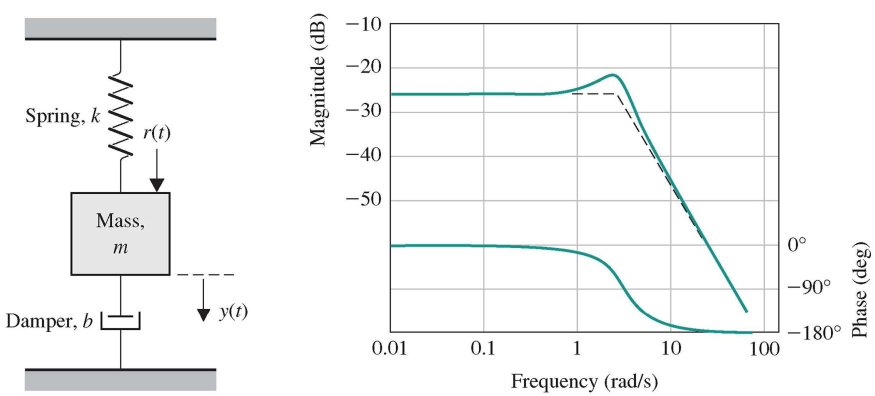


Frequency response (and root locus, and other more advanced control theory and methods) are used in wide-ranging technological applications.

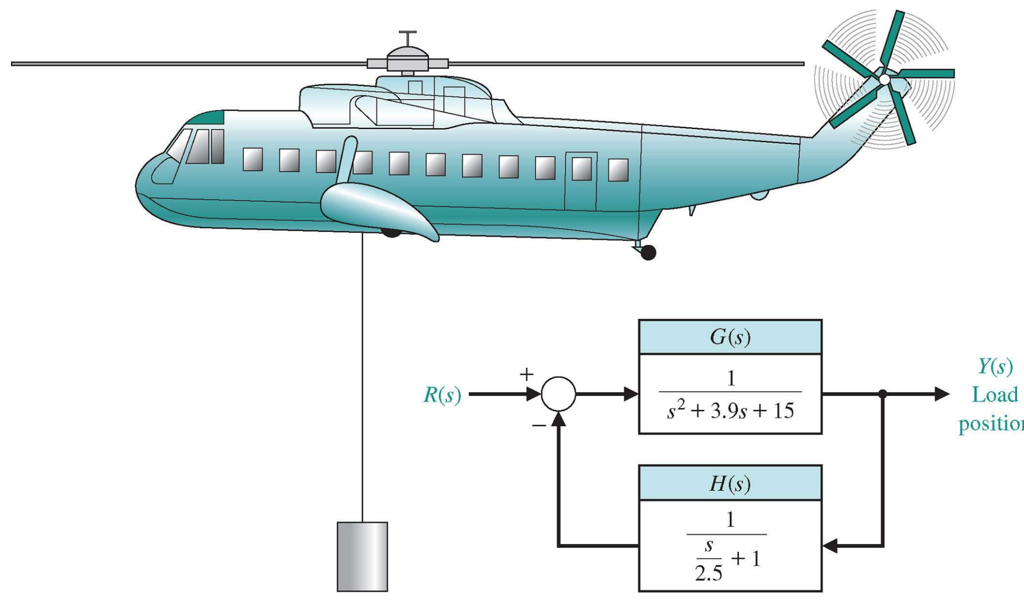








(a) (b)



(a) (b)

Concluding Remarks and Next Lecture

- **What have been covered:** Frequency Response Method "Bode plot and Nyquist Plot"
    - Basic principle of frequency response
    - Bode plotting/sketching method
    - Transfer function deduction using Bode plot
    - Frequency response measurement specification: $M_{p\omega}$, ω_B
- **In our next lecture:** we will finish off the remaining part of Bode plot (stability and margin/transient design) and proceed to "Nyquist plot" and its stability criterion.
- **What you can do from now till the next lecture:** revise the material, further reading, group study, and skill checks (LMO tutorial, or Textbook chapter 8).
- **How to get in touch**: through LMO Module's "*General question and answer forum*" section or during my weekly consultation hour(s).