MTH102 Solution to Tutorial 02
Permutations & combinations, Probability theory

Question 1

(a) How man different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?

(b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.

Answer:

(a) $26^2 \times 10^5 = 6.76 \times 10^7$.

(b) $26 \times 25 \times 10 \times 9 \times 8 \times 7 \times 6 = 1.9656 \times 10^7$.

Question 2

In how many ways can 8 people be seated in a row if

(a) there are no restrictions on the seating arrangement?

(b) person A and B must sit next to each other?

(c) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?

(d) there are 5 men and they must sit next to each other?

(e) there are 3 married couples and each couple must sit together?

Answer:

(a) $8! = 40320$.

(b) $7! \times 2! = 10080$.

(c) $2 \times 4! \times 4! = 1152$.

(d) $4! \times 5! = 2880$.

(e) $5! \times 2^3 = 960$.

Question 3

Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

Answer:

$\binom{20}{2} = 190$.

Question 4

A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

Answer:

$\binom{12}{5} \times \binom{10}{5} \times 5! = 23950080$.

Question 5

A student has to buy 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if

(a) both books are to be on the same subject?

(b) the books are to be on different subjects?

Answer:

(a) $\binom{6}{2} + \binom{7}{2} + \binom{4}{2} = 42$.

(b) $6 \times 7 + 7 \times 4 + 4 \times 6 = 94$.

Question 6

From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if

(a) 2 of the men refuse to serve together?

(b) 2 of the women refuse to serve together?

(c) 1 man and 1 woman refuse to serve together?

Answer:

(a) $\binom{8}{3} \times \left[\binom{6}{3} - \binom{4}{1}\right] = 896.$

(b) $\binom{6}{3} \times \left[\binom{8}{3} - \binom{6}{1}\right] = 1000.$

(c) $\binom{8}{3}\binom{6}{3} - \binom{5}{2}\binom{7}{2} = 910.$

Question 7

Suppose that A and B are mutually exclusive events for which $P(A) = 0.3$ and $P(B) = 0.5$. What is the probability that

(a) both A and B occur?

(b) either A or B occurs?

(c) A occurs but B does not?

Answer:

(a) $P(AB) = 0.$

(b) $P(A \cup B) = P(A) + P(B) - P(AB) = P(A) + P(B) = 0.8.$

(c) $P(AB^c) = P(A) - P(AB) = P(A) = 0.3.$

Question 8

60 percent of the students at a certain school wear neither a ring nor a necklace. 20 percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing

(a) a ring or a necklace?

(b) a ring and a necklace?

Answer:

Let A :=”the student is wearing a ring” and B :=”the student is wearing a necklace”. We have

$$P(A^cB^c) = 0.6,\ P(A) = 0.2,\ P(B) = 0.3.$$

(a) $P(A \cup B) = P((A^cB^c)^c) = 1 - P(A^cB^c) = 0.4$.

(b) $P(AB) = P(A) + P(B) - P(A \cup B) = 0.1$.

Question 9

Two cards are chosen at random from a deck of 52 playing cards. What is the probability that they

(a) are both aces?

(b) have the same value?

Answer:

(a)

$$\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{1}{221}.$$

(b)

$$\frac{\binom{13}{1}\binom{4}{2}}{\binom{52}{2}} = \frac{1}{17}.$$

Question 10

A woman has n keys, of which one will open her door.

(a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try?

(b) What if she does not discard previously tried keys?

Answer:

(a) Consider the sample space

$$S = \{\text{all the permutations of } k \text{ keys selected from } n \text{ keys}\}.$$

Then $|S| = P_n^k$.
Let A be the event that the key on the kth try is the correct one with discards. Then

$$A = \{\text{all the permutations of } (k-1) \text{ wrong keys selected out of } (n-1)\}.$$

Therefore, $|A| = P_{n-1}^{k-1}$ and we can obtain from equally likely assumption,

$$P(A) = \frac{|A|}{|S|} = \frac{P_{n-1}^{k-1}}{P_n^k} = \frac{(n-1)\times(n-2)\cdots\times[(n-1)-(k-1)+1]}{n\times(n-1)\cdots\times(n-k+1)} = \frac{1}{n}.$$

(b) Consider the sample space

$$S = \{(a_1, a_2, \ldots, a_k);\ a_1, a_2, \ldots, a_k \in \{1, \ldots, n\}\},$$

where a_i represents the key number on the ith try. Then $|S| = n^k$.
Let A be the event that the key on the kth try is the correct one without discards. Then $|A| = (n-1)^{k-1}$ and we can obtain from equally likely assumption,

$$P(A) = \frac{|A|}{|S|} = \frac{(n-1)^{k-1}}{n^k} = \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}.$$