Class Test II, MTH201

DATE: November 24th, 2023

TIME ALLOWED: 80 minutes

Name ________________ Student ID ________________

INSTRUCTIONS TO CANDIDATES

1. There are a total of 5 pages numbered 1 through 5. Please ensure that your copy of the test paper is complete.
2. There are a total of 3 questions called Q1 through Q3 in this test paper.
3. The total marks available are 100 and the marks available for each question are indicated in the exam paper.
4. There are no penalties for incorrect answers. Attempt all problems.
5. Do not use pencil or red-ink pen to write your solutions in the booklet(s).
6. Only the university approved calculator (e.g. Casio FS82ES/83ES) can be used.

END OF INSTRUCTIONS TO CANDIDATES

Q 1. *(36 marks in total)*

(a) *(5 marks) What is the name of the equation*

$$u_{tt} = 4u_{xx}, \quad -\infty < x, t < \infty \ ? \tag{1}$$

Solution. It is called the wave equation. □

(b) *(5 marks) Find a nonzero value of constant c so that function*

$$u(x,t) \stackrel{\text{def}}{=} \sin\frac{2\pi x}{3}\cos(ct), \quad -\infty < x, t < \infty \tag{2}$$

satisfies the condition (1).

Solution. $c = 4\pi/3$ □

(c) *(8 marks) Over the interval $0 \le x \le 3$, sketch a graph of u against x when $t = 0$, and a graph of u against x when $t = 1$. In other words, over the interval $0 \le x \le 3$, draw approximately a graph of the function $u(x, t = 0)$, and a graph of the function $u(x, t = 1)$.*

Solution.
$u(x, t = 0) = \sin(2\pi x/3), 0 \le x \le 3$ and is sketched in Figure

$u(x, t = 1) = \sin(2\pi x/3)\cos(4\pi/3) = -\sin(2\pi x/3)/2, 0 \le x \le 3$ and is sketched in Figure □

(d) *(8 marks) Use the trigonometric identity*

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right) \tag{3}$$

to write the function (2) *as the sum of two functions. Each of the two functions represents a travelling wave. What is the speed of the waves ?*

Solution. According to (3),

$$\sin\frac{2\pi x}{3}\cos(ct) = \frac{1}{2}\left(\sin\left(\frac{2\pi x}{3} + ct\right) + \sin\left(\frac{2\pi x}{3} - ct\right)\right) \tag{4}$$

Note that

$$\frac{1}{2}\sin\left(\frac{2\pi x}{3} + ct\right) = F(x + vt) \tag{5}$$

$$\frac{1}{2}\sin\left(\frac{2\pi x}{3} - ct\right) = F(x - vt) \tag{6}$$

with

$$F(u) = \frac{1}{2}\sin\frac{2\pi u}{3} \tag{7}$$

and $v = c/(2\pi/3)$. Therefore, the wave speed is $c/(2\pi/3)$. □

(e) *(10 marks) Fill in the two blanks so that the following initial boundary value problem (IBVP) has a solution which is function in* (2).

$$u_{tt} = 4u_{xx}, \qquad 0 < x < 3, t > 0 \tag{8a}$$

$$u(x,0) = \sin\frac{2\pi x}{3} \qquad 0 \le x \le 3 \tag{8b}$$

$$u_t(x,0) = 0 \qquad 0 \le x \le 3 \tag{8c}$$

$$u(0,t) = 0 \qquad t \le 0 \tag{8d}$$

$$u(3,t) = 0 \qquad t \ge 0 \tag{8e}$$

Q 2. *(34 marks in total) A periodic signal $f = f(t)$ is sketched in Figure 1, and defined over one period as*

$$f(t) = \begin{cases} 1 - |t|/T_1 & \text{when } -T_1 \le t \le T_1 \\ 0 & \text{when } T_1 < t \le T/2 \text{ and } -T/2 \le t < -T_1 \end{cases} \tag{9}$$

(a) *(24 marks) Find the Fourier sereis representation of this periodic signal, by filling in the following eight blanks. Because the signal is periodic with period T, we use functions*

$$1, \cos\frac{\pi x}{T/2}, \sin\frac{\pi x}{T/2}, \cos\frac{2\pi x}{T/2}, \sin\frac{2\pi x}{T/2}, \ldots, \cos\frac{n\pi x}{T/2}, \sin\frac{n\pi x}{T/2}, \ldots \tag{10}$$

to form the Fourier series. According to the Appendix, the Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{T/2} + b_n \sin\frac{n\pi t}{T/2}\right) \tag{11}$$

with

$$\begin{aligned} a_n &= \frac{1}{T/2}\int_{-T/2}^{T/2} f(t)\cos\frac{n\pi t}{T/2}\,\mathrm{d}t = \frac{1}{T/2}\int_{-T_1}^{T_1}(1-|t|/T_1)\cos\frac{n\pi t}{T/2}\,\mathrm{d}t \\ &= \frac{2}{T/2}\int_0^{T_1}(1-t/T_1)\cos\frac{n\pi t}{T/2}\,\mathrm{d}t, \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{12}$$

So

$$a_0 = \frac{2}{T/2}\int_0^{T_1}(1-t/T_1)\,\mathrm{d}t = 2T_1/T \tag{13}$$

and

$$\begin{aligned} a_n &= \frac{2}{T/2}\int_0^{T_1}\cos\frac{n\pi t}{T/2}\,\mathrm{d}t - \frac{2}{T/2}\int_0^{T_1}\frac{t}{T_1}\cos\frac{n\pi t}{T/2}\,\mathrm{d}t \\ &= \frac{2}{n\pi}\sin\frac{n\pi T_1}{T/2} - \frac{2}{n\pi T_1}\int_0^{T_1} t\,\mathrm{d}\sin\frac{n\pi t}{T/2} \\ &= \frac{2}{n\pi}\sin\frac{n\pi T_1}{T/2} - \frac{2}{n\pi}\sin\frac{n\pi T_1}{T/2} + \frac{2}{n\pi T_1}\int_0^{T_1}\sin\frac{n\pi t}{T/2}\,\mathrm{d}t \\ &= \frac{T}{(n\pi)^2 T_1}\left(1 - \cos\frac{n\pi T_1}{T/2}\right), \qquad n = 1, 2, \ldots \end{aligned} \tag{14}$$

$$b_n = \frac{1}{T/2}\int_{-T/2}^{T/2} f(t)\sin\frac{n\pi t}{T/2}\,\mathrm{d}t = 0 \tag{15}$$

So the Fourier series is

$$\frac{T_1}{T} + \frac{T}{\pi^2 T_1}\left(1 - \cos\frac{\pi T_1}{T/2}\right)\cos\frac{\pi t}{T/2} + \frac{T}{4\pi^2 T_1}\left(1 - \cos\frac{2\pi T_1}{T/2}\right)\cos\frac{2\pi t}{T/2} + \cdots \tag{16}$$

(The three blanks are to be filled with the first three nonzero terms in the series)

(b) *(5 marks) Discuss how the Fourier series coefficients change as T_1 increases and approaches $T/2$.*

Solution.

$$\lim_{T_1 \to T/2} a_0 = \lim_{T_1 \to T/2} 2T_1/T = 1 \tag{17}$$

$$\begin{aligned}\lim_{T_1 \to T/2} a_n &= \lim_{T_1 \to T/2} \frac{T}{(n\pi)^2 T_1}\left(1 - \cos\frac{n\pi T_1}{T/2}\right) = \frac{2}{(n\pi)^2}(1 - \cos(n\pi)) \\ &= \frac{2}{(n\pi)^2}(1 - (-1)^n), \qquad n = 1, 2, ...\end{aligned} \tag{18}$$

□

(c) *(5 marks) Will the Gibbs phenomenon occur when the partial sum of the Fourier series is used to approximate this signal ?*

Solution. No. Gibbs phenoemnon would not occur because the function $f = f(t)$ is continuous everywhere. □

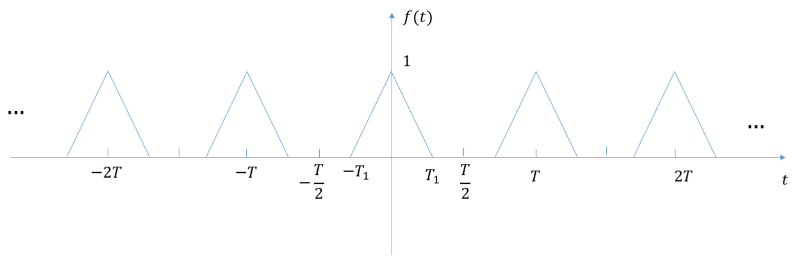

Figure 1: A periodic signal

Q 3. *(30 marks in total) Consider the temperature distribution along a rod which is shown in Figure 2. The rod lies along the x-axis. It is in the shape of a right circular cylinder and its length is 2. Each cross-section of the rod is perpendicular to the x-axis and the temperature is assumed to be the same at every point on each cross-section. Therefore, the temperature distribution along the rod could be described with a function*

$$u = u(x,t), \qquad 0 \le x \le 2, t \ge 0. \tag{19}$$

Here $u(x,t)$ denotes the temperature at cross-section x and at time t.

(a) With regard to function $u = u(x,t)$, a student constructed the following initial boundary value problem (IBVP):

$$\begin{aligned} &u_t = 5u_{xx} && 0 < x < 2, t > 0 && \text{(20a)} \\ &u(x,0) = x && 0 \le x \le 2 && \text{(20b)} \\ &u_t(x,0) = 0 && 0 \le x \le 2 && \text{(20c)} \\ &u(0,t) = 0 && t \ge 0 && \text{(20d)} \\ &u(2,t) = 0 && t \ge 0 && \text{(20e)} \end{aligned}$$

(a.1) (5 marks) What is the name of the equation in (20a) *? In other words, what is it often called ?*

Solution. It is called the heat (or diffusion) equation. □

(a.2) (12 marks) Describe in simple words the meaning of condition (20b)*,* (20c)*,* (20d)*,*(20e)*, respectively.*

Solution.
Condition (20b) means that initially the temperature of the rod at cross-section x is x.
Condition (20c) means that initially the temperature of the rod does not change at anywhere.
Condition (20d) means that the temperature of the rod at the end $x = 0$ is maintained to be zero all the time.
Condition (20e) means that the temperature of the rod at the end $x = 2$ is maintained to be zero all the time. □

(a.3) (8 marks) Some conditions in (20) *contradict with each other. These make the probelm has no solution. Find and explain these errors in simple words.*

(b) (5 marks) Assume the temperature distribution within the rod satisfies the condition (20a)*. The temperature at the end $x = 0$ is maintained at 0, all the time. The temperature at the end $x = 2$ is maintained at 3 all the time. The initial temperrature distribution along the rod is shown in Figure 3. Does the limit*

$$\lim_{t\to\infty} u(1,t) \tag{21}$$

exist ? If so, what is the value of this limit. If not, explain with your reasons.

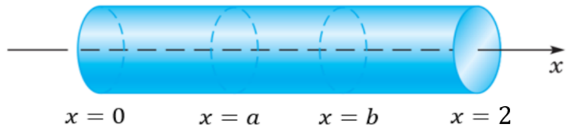

Figure 2: A rod lies along the x-axis

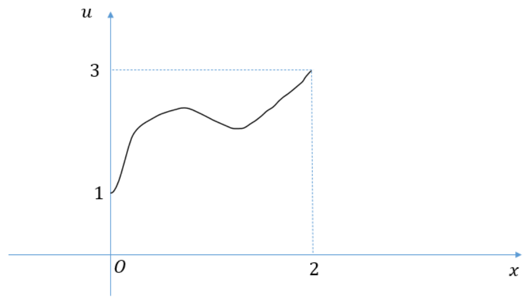

Figure 3: A graph of u as a function of x. $u(x = 0) = 1$ and $u(x = 2) = 3$.

Appendix

The Fourier series for a period function $f = f(x)$ with period $2L$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{22}$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, \mathrm{d}x \quad (n = 0, 1, 2, ...) \tag{23}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x \quad (n = 1, 2, ...) \tag{24}$$