Xi’an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

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Quiz 9.2

The state-space model of a system is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

(1) Find the state-transition matrix;

(2) If initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the time response of the state variables when $u(t) = 0$.

(1)

$$\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{s+5}{s^2+5s+6} & \dfrac{6}{s^2+5s+6} \\ \dfrac{-1}{s^2+5s+6} & \dfrac{s}{s^2+5s+6} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 6e^{-2t} - 6e^{-3t} \\ -e^{-2t} + e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

Quiz 9.2

The state-space model of a system is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

(1) Find the state-transition matrix;

(2) If initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the time response of the state variables when $u(t) = 0$.

(2)

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) = \begin{bmatrix} 15e^{-2t} - 14e^{-3t} \\ -5e^{-2t} + 7e^{-3t} \end{bmatrix}$$

Covert Transfer Function to State-space Model

How to obtain the state space model from the transfer function directly without a clear knowledge of the physical system?

Method 1: to develop graphic model of the system and use this model to determine state variables.

$$\boxed{G(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}\qquad n\geq m}$$

Recall **Mason's Signal-flow Gain Formula**:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\sum_k P_k\Delta_k}{\Delta}.$$

Divided by s^n

$$\begin{aligned}\Delta = 1 &- (\text{sum of all different loop gains})\\ &+ (\text{sum of the gain products of all combinations of two nontouching loops})\\ &- (\text{sum of the gain products of all combinations of three nontouching loops})\\ &+ \cdots.\end{aligned}$$

$$G(s)=\frac{b_m s^{-(n-m)}+b_{m-1}s^{-(n-m+1)}+\cdots+b_1 s^{-(n-1)}+b_0 s^{-n}}{1+a_{n-1}s^{-1}+\cdots+a_1 s^{-(n-1)}+a_0 s^{-n}}.$$

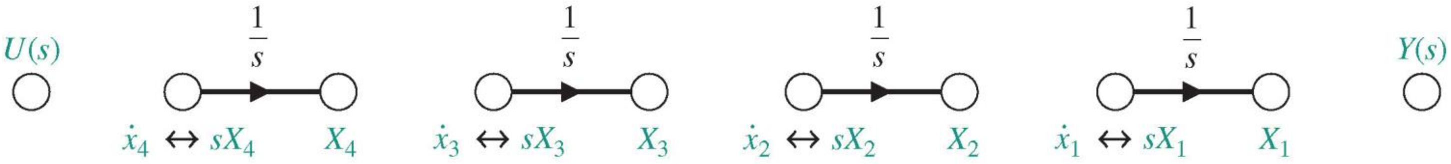
Simple Case

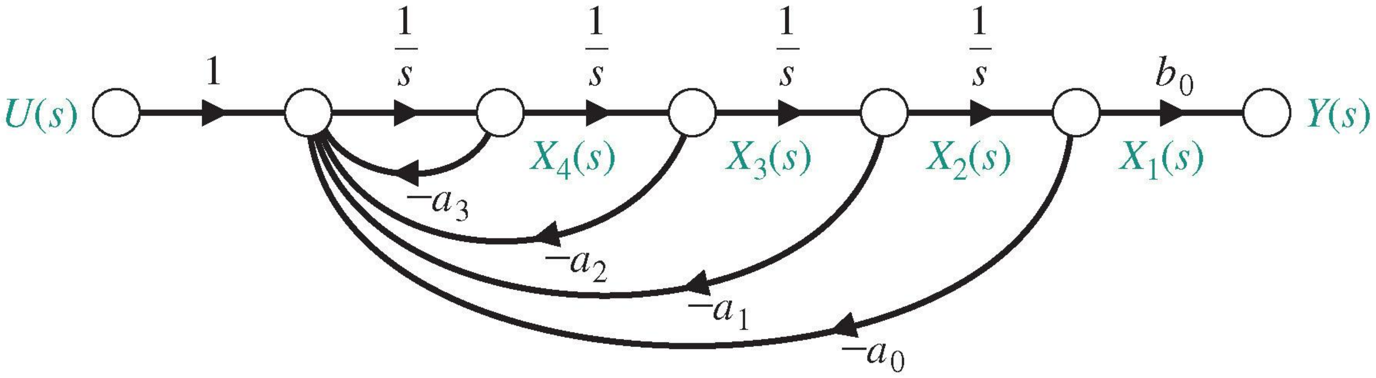
To illustrate the derivation of signal-flow graph from transfer function, let's consider a simple case, when $n = 4$, and $b_m \ldots b_2, b_1 = 0$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$

The system is fourth order, hence we need to identify four state variables: $x_1(t), x_2(t), x_3(t), x_4(t)$

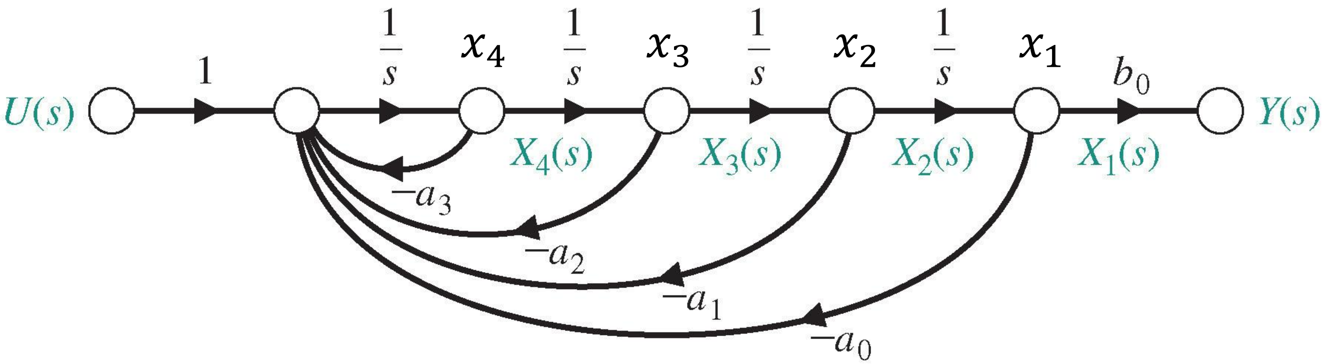




Simple Case

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u$$
$$y = b_0 x_1$$

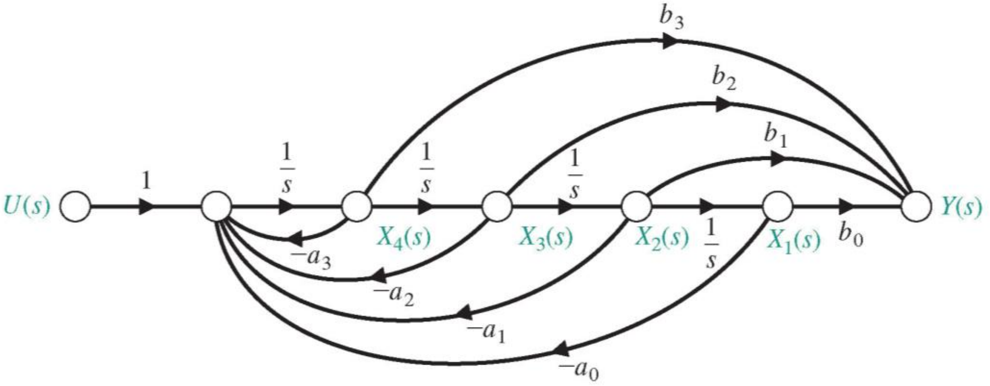
Simple Case

Now consider the numerator is a polynomial in s:

$$G(s) = \frac{\sum_k P_k}{1 - \sum_{q=1}^{N} L_q}$$

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4,$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u.$$

In this equation, $x_1, x_2, \ldots x_n$ are the n **phase variables**.

$$y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3 + b_3 x_4$$

Simple Case

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4,$$

$$\dot{x}_4 = -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + u.$$

$$y = b_0x_1 + b_1x_2 + b_2\,x_3 + b_3\,x_4$$

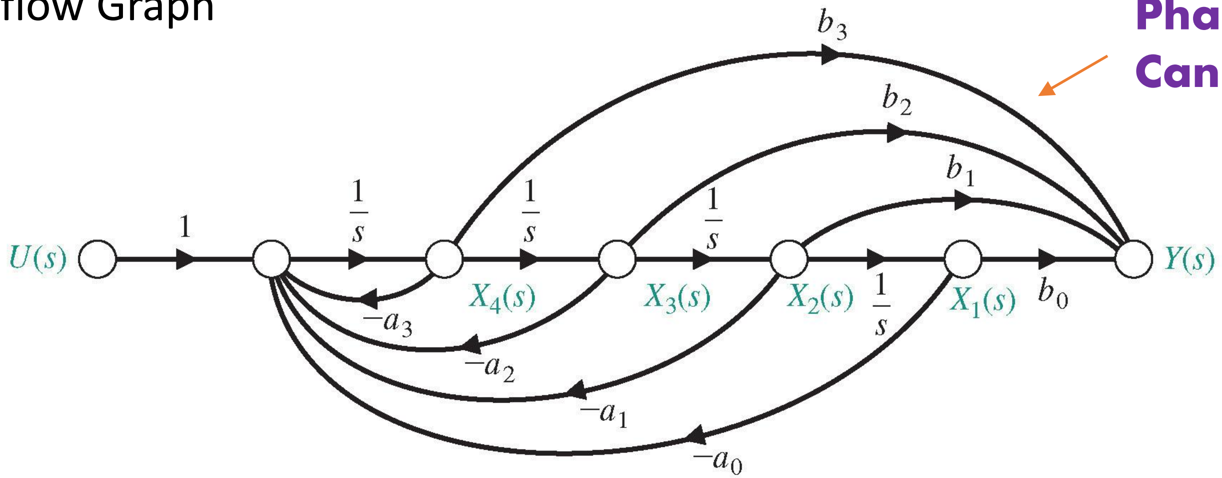
A, B, C, D?

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t). \qquad y(t) = \mathbf{Cx} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

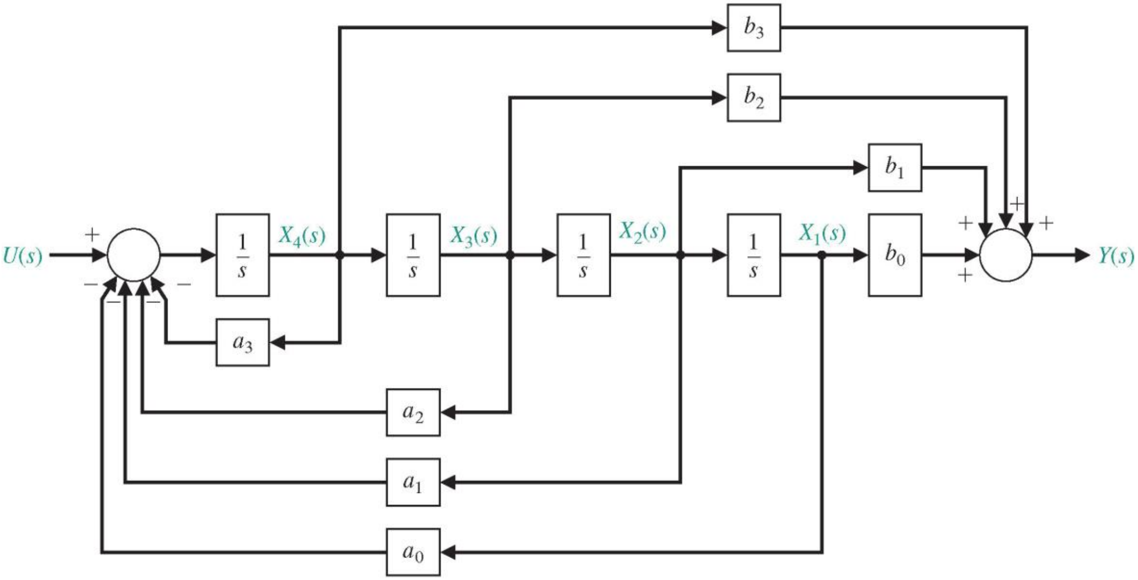
General form of State-space Model

- Signal-flow Graph



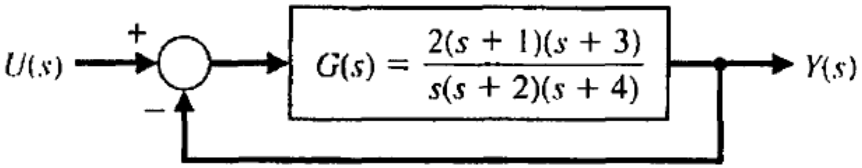
Phase Variable Canonical Form

- Equivalent Block Diagram



Example 10.2

$$T(s)=\frac{Y(s)}{U(s)}=\frac{2s^2+8s+6}{s^3+8s^2+16s+6}$$



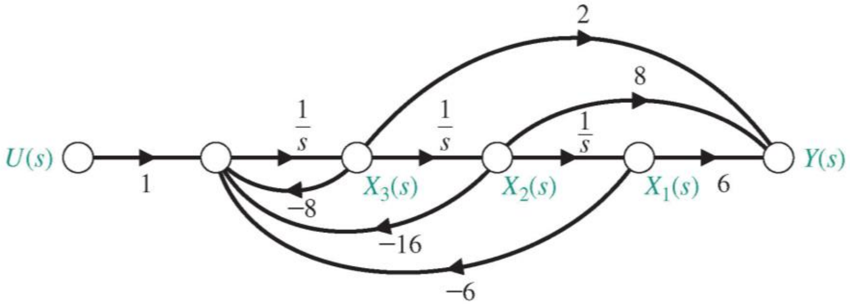
Applying the Phase variable state model:

Multiplying the numerator and denominator by s^{-3}, we have

$$T(s)=\frac{Y(s)}{U(s)}=\frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -16 & -8\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t)$$

$$y(t)=[6\quad 8\quad 2]\mathbf{x}(t)+[0]u(t)$$



(a)

Covert Transfer Function to State-space Model

Method 2: State-space Model can be also obtained by introducing an intermediate variable $Z(s)$.

For simplicity, assume $n = 4$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3s^3 + b_2s^2 + b_1s + b_0}{s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}\frac{Z(s)}{Z(s)}$$

$$Y(s) = (b_3s^3 + b_2s^2 + b_1s + b_0)Z(s)$$

$$U(s) = (s^4 + a_3s^3 + a_2s^2 + a_1s + a_0)Z(s)$$

Then taking inverse Laplace transform of both equations:

$$y = b_3\frac{d^3z}{dt^3} + b_2\frac{d^2z}{dt^2} + b_1\frac{dz}{dt} + b_0z$$

$$u = \frac{d^4z}{dt^4} + a_3\frac{d^3z}{dt^3} + a_2\frac{d^2z}{dt^2} + a_1\frac{dz}{dt} + a_0z.$$

Covert Transfer Function to State-space Model

$$y = b_3\frac{d^3z}{dt^3} + b_2\frac{d^2z}{dt^2} + b_1\frac{dz}{dt} + b_0z \qquad u = \frac{d^4z}{dt^4} + a_3\frac{d^3z}{dt^3} + a_2\frac{d^2z}{dt^2} + a_1\frac{dz}{dt} + a_0z.$$

Define the four state variables as follows:

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{x}_1 = \dot{z} \\ x_3 &= \dot{x}_2 = \ddot{z} \\ x_4 &= \dot{x}_3 = \dddot{z}. \end{aligned}$$

Then the differential equation can be written equivalently as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4, \end{aligned}$$

and

$$\dot{x}_4 = -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + u,$$

and the corresponding output equation is

$$y = b_0x_1 + b_1x_2 + b_2x_3 + b_3x_4.$$

Covert Transfer Function to State-space Model

Method 3: Select state variable with physical meanings.

- Often the control system designer studies an actual control system block diagram that represents physical devices and variables.
- In practice, we wish to select the physical variables as the state variables.
  - While the state-space model could be derived in many forms in math, in practice, we always select physical variables as states such that the **model will be useful**.

Example

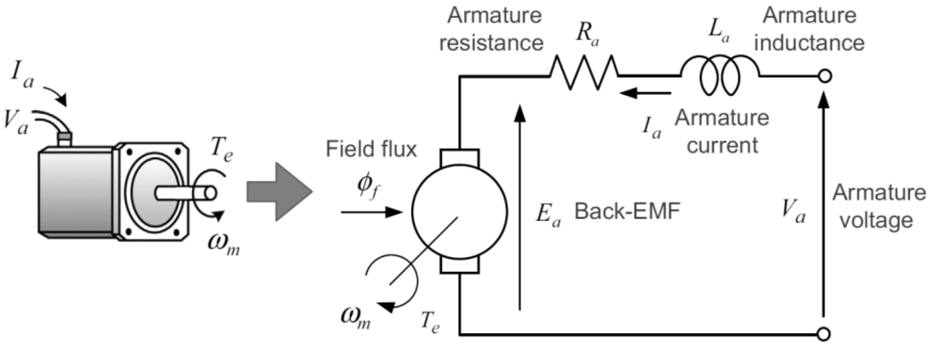
- Recall the model of a DC motor

Voltage equation:

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + k_e \phi_f \omega_m$$

Motion equation

$$k_T \phi_f i_a = J \frac{d\omega_m}{dt} + B\omega_m + T_L$$



Assume $k_e \phi_f = k_t, k_T \phi_f = k_c, B = 0, T_L = 0$

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + k_t \omega_m$$

$$k_c i_a = J \frac{d\omega_m}{dt}$$

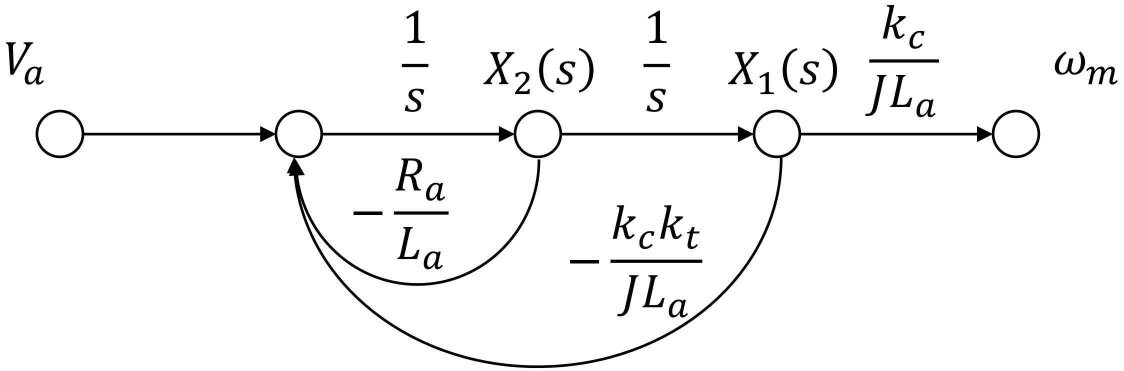
Example

If use method 1 to build state space model:

- The transfer function is

$$\frac{\omega_m}{V_a}=\frac{k_c}{JL_a s^2+JR_a s+k_c k_t}=\frac{\frac{k_c}{JL_a}s^{-2}}{1+\frac{R_a}{L_a}s^{-1}+\frac{k_c k_t}{JL_a}s^{-2}}$$

- Signal flow graph:



Example

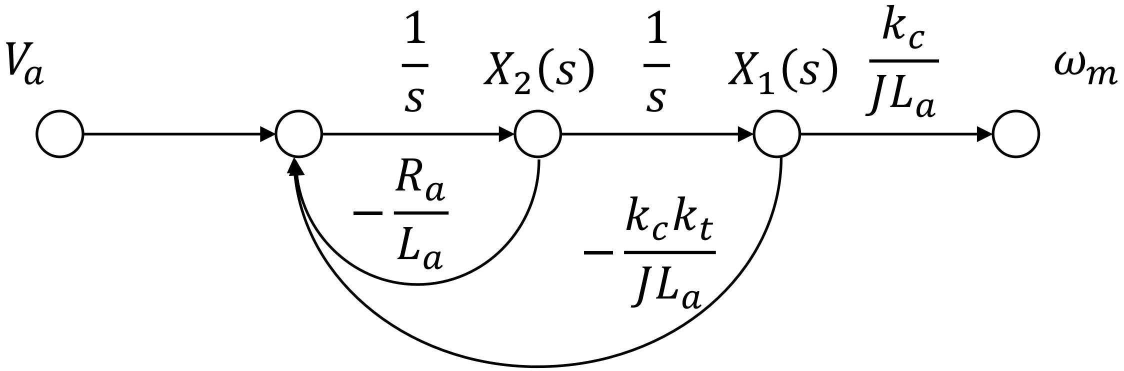
- State space model:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k_c k_t}{J L_a} x_1 - \frac{R_a}{L_a} x_2 + V_a$$

$$y = \omega_m = \frac{k_c}{J L_a} x_1$$

Here the state space variables do not have specific physical meaning



Example

If select physical variable as states:

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + k_c \omega_m$$

$$k_c i_a = J \frac{d\omega_m}{dt}$$

The state variables are useful as they could be measured.

- Select state as $x_1 = \omega_m, x_2 = i_a, y = \omega_m, u = V_a$, then

$$\frac{d\omega_m}{dt} = \dot{x}_1 = \frac{k_c}{J} i_a = \frac{k_c}{J} x_2$$

$$\frac{di_a}{dt} = \dot{x}_2 = -\frac{R_a}{L_a} i_a - \frac{k_c}{L_a} \omega_m + \frac{V_a}{L_a} = -\frac{R_a}{L_a} x_2 - \frac{k_c}{L_a} x_1 + \frac{1}{L_a} u$$

$$y = x_1$$

Quiz 10.1

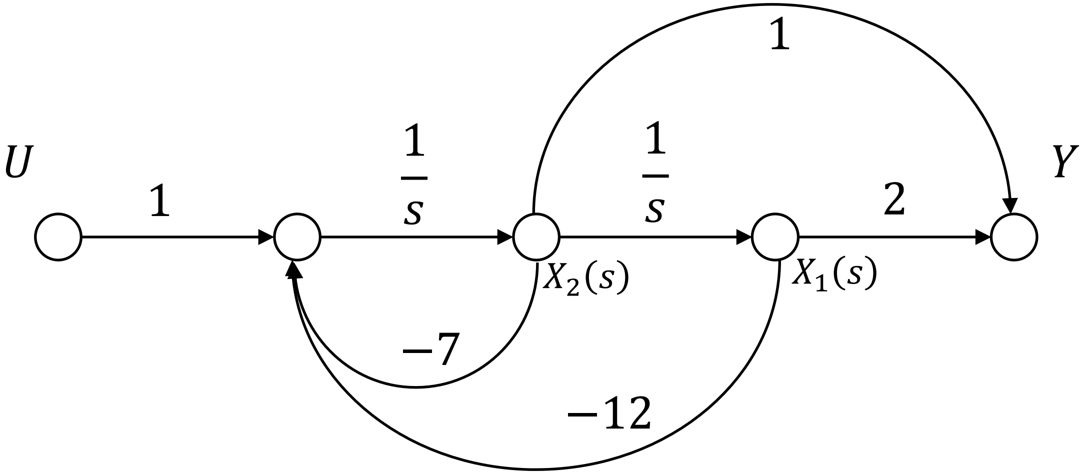
Obtain a state-space model & block diagram for the system with the following transfer equation:

$$\mathbf{G}(s)=\frac{Y(s)}{U(s)}=\frac{s+2}{s^2+7s+12}$$

First transform the function into following form:

$$\mathbf{G}(s)=\frac{Y(s)}{U(s)}=\frac{s^{-1}+2s^{-2}}{1+7s^{-1}+12s^{-2}}$$

Then plot following signal flow graph

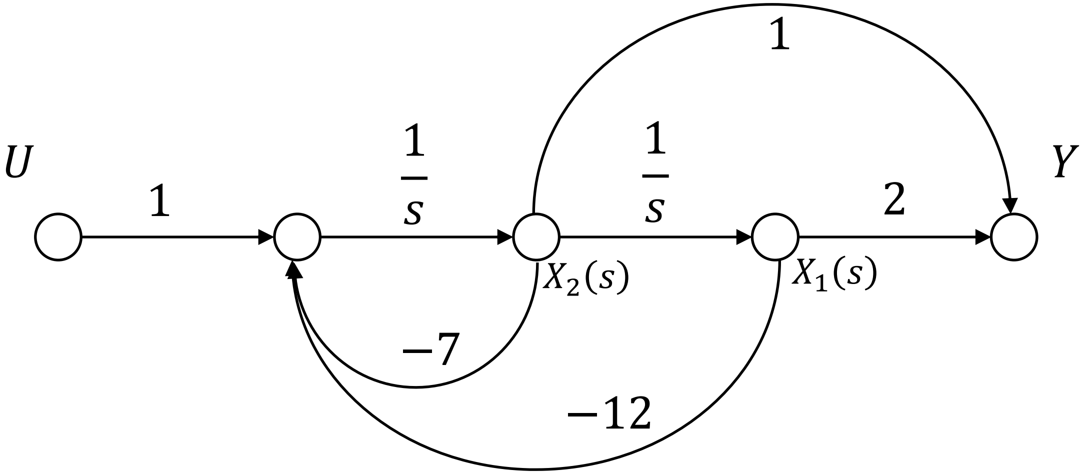


Quiz 10.1

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -12x_1 - 7x_2 + u$$
$$y = 2x_1 + x_2$$

State space model:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \boldsymbol{x} + [0] u$$



Quiz 10.2

Consider following state space model

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

What is the characteristic equation of the system and what are the poles of the system?

Characteristic equation is

$$|sI - A| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 3s + 2$$

Poles are

$$s^2 + 3s + 2 = 0 \Rightarrow s = -1, s = -2$$

Lecture 11

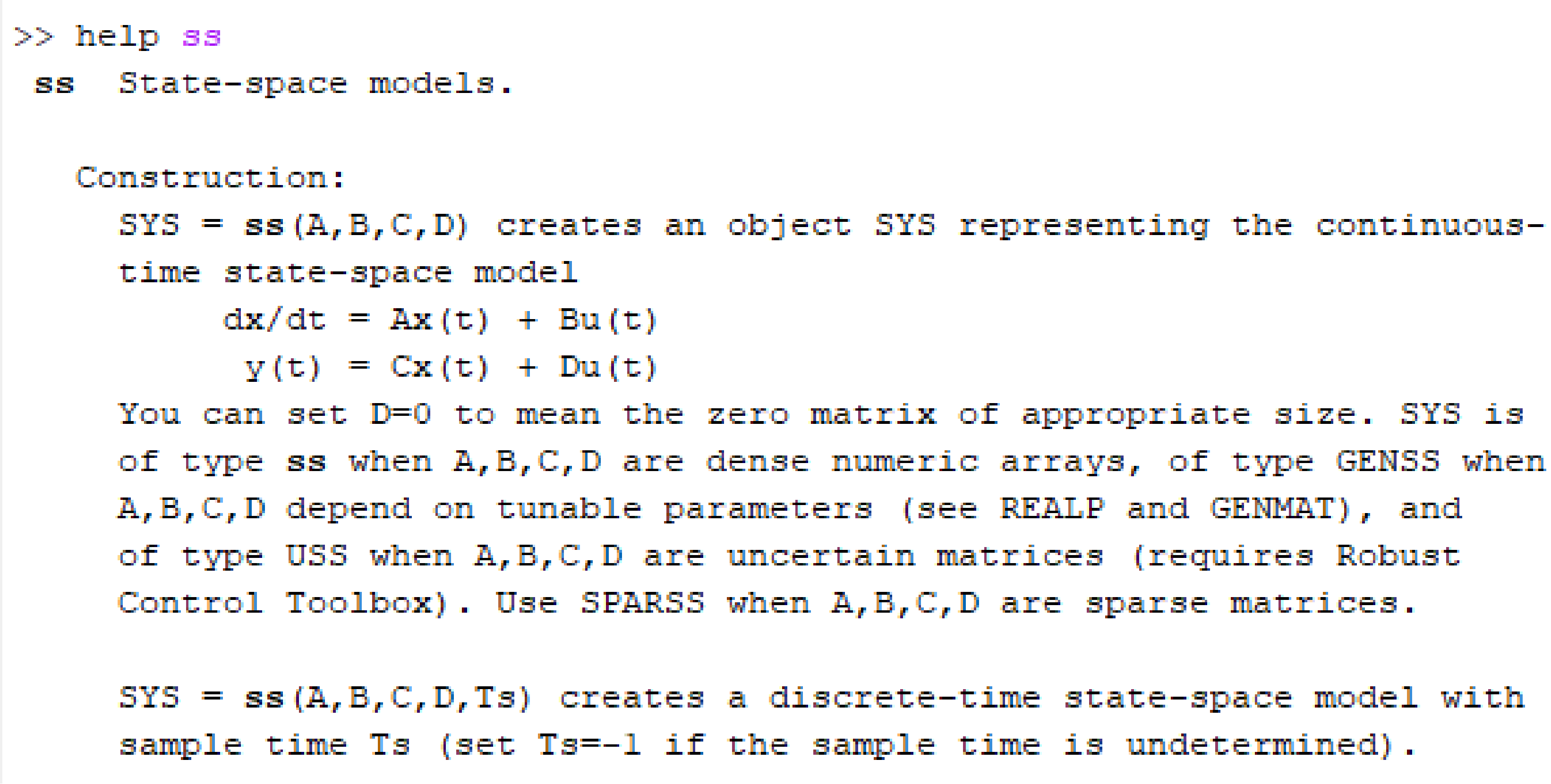
Outline

State Variable Models

- ☐ **Introduction**
- ☐ **State Variables**
- ☐ **State-space Modeling**
- ☐ **State Space Representation in Matrix Form**
- ☐ **Time-domain response (Solution of State-space Models)**
- ☐ **Conversion between State-space Model and Transfer Function**
- ☐ ***Analysis of the State-space Models using Matlab***

Transfer Function to State Space

- Build state-space model with function *ss*



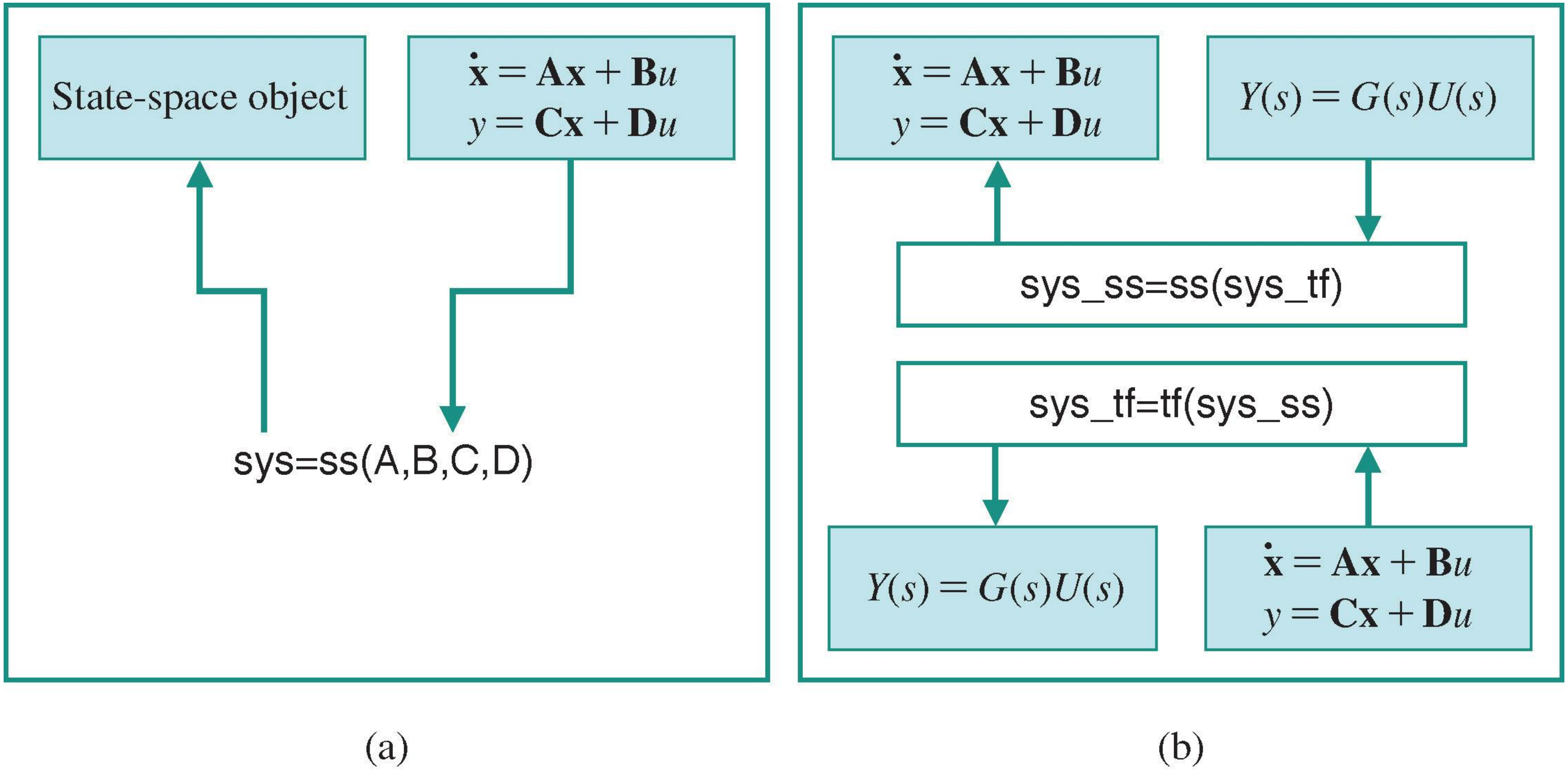
```
>> help ss
 ss  State-space models.

   Construction:
     SYS = ss(A,B,C,D) creates an object SYS representing the continuous-
     time state-space model
          dx/dt = Ax(t) + Bu(t)
           y(t) = Cx(t) + Du(t)
     You can set D=0 to mean the zero matrix of appropriate size. SYS is
     of type ss when A,B,C,D are dense numeric arrays, of type GENSS when
     A,B,C,D depend on tunable parameters (see REALP and GENMAT), and
     of type USS when A,B,C,D are uncertain matrices (requires Robust
     Control Toolbox). Use SPARSS when A,B,C,D are sparse matrices.

     SYS = ss(A,B,C,D,Ts) creates a discrete-time state-space model with
     sample time Ts (set Ts=-1 if the sample time is undetermined).
```

Transfer Function to State Space

- Covert between state space model and transfer function (ss, tf)



(a) (b)

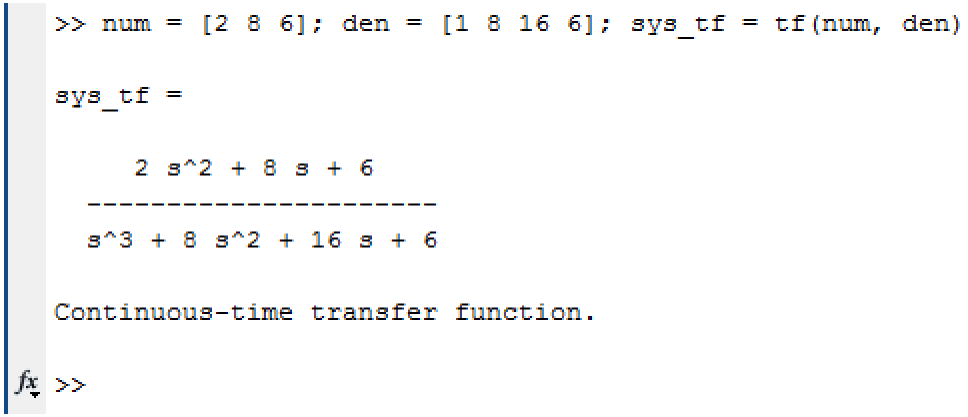
Example

- Consider the following third order system

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$

Compute the state space model using Matlab.

Example



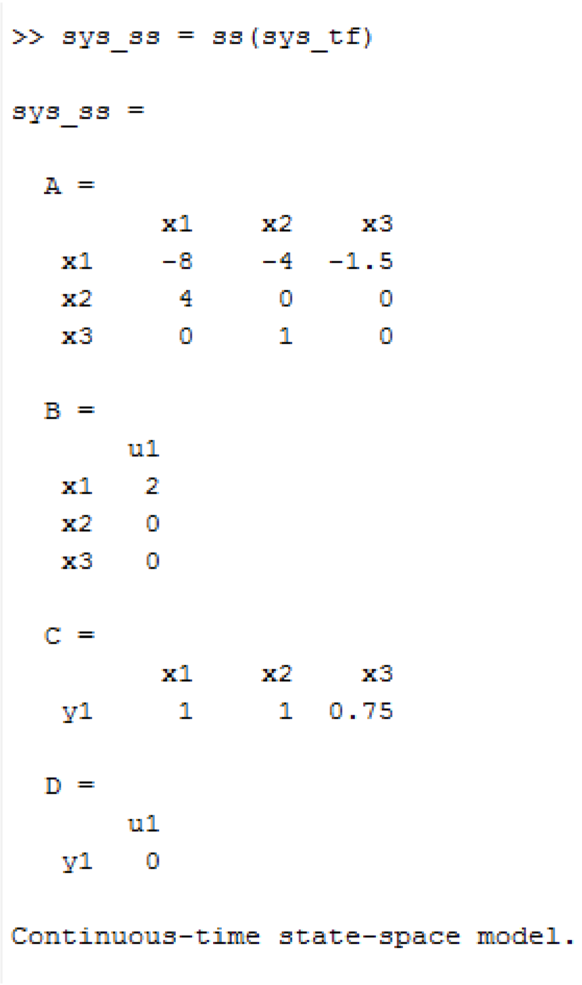
```
>> num = [2 8 6]; den = [1 8 16 6]; sys_tf = tf(num, den)

sys_tf =

     2 s^2 + 8 s + 6
  ----------------------
  s^3 + 8 s^2 + 16 s + 6

Continuous-time transfer function.

>>
```



```
>> sys_ss = ss(sys_tf)

sys_ss =

  A =
          x1    x2    x3
   x1     -8    -4  -1.5
   x2      4     0     0
   x3      0     1     0

  B =
        u1
   x1    2
   x2    0
   x3    0

  C =
          x1    x2    x3
   y1      1     1  0.75

  D =
        u1
   y1    0

Continuous-time state-space model.
```

Please note: a transfer function can be converted to various state space models by choosing different sets of state variables; therefore, it is possible that when using the ss function, the state space model generated will be different, depending on the specific software and version.

Matrix Exponential Function

- Recall solution of state-space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$\Downarrow$ s domain solution

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

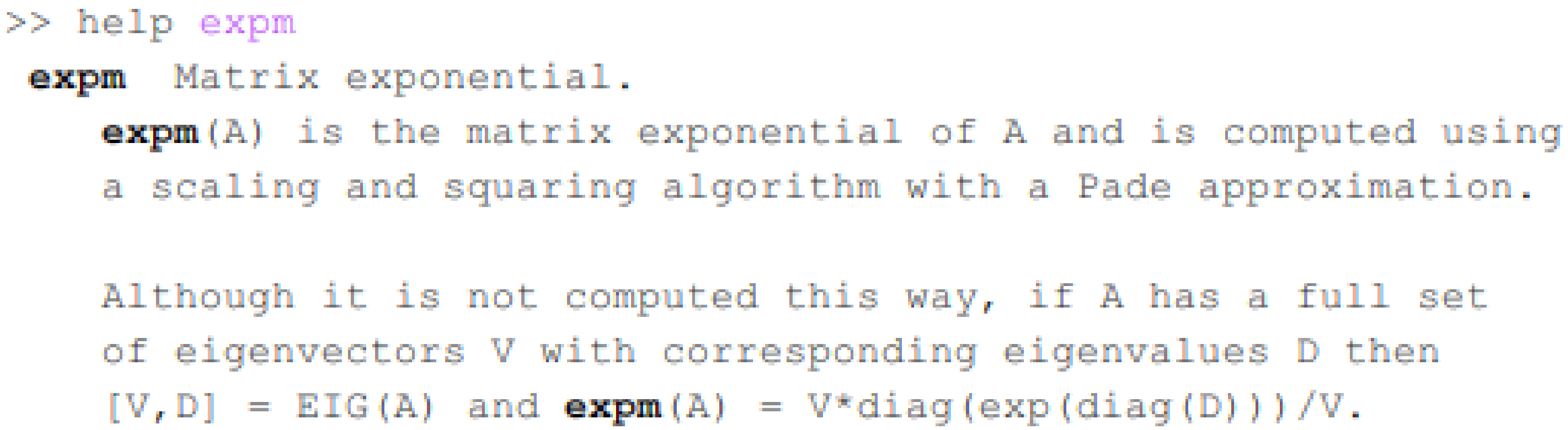
$\Downarrow$ t domain solution

$$\mathbf{x}(t) = \boxed{\boldsymbol{\Phi}(t)}\mathbf{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$\boldsymbol{\Phi}(t) = \exp(\mathbf{A}t)$ State transition matrix

Matrix Exponential Function

- Compute state transition matrix using function *expm*



```
>> help expm
 expm  Matrix exponential.
    expm(A) is the matrix exponential of A and is computed using
    a scaling and squaring algorithm with a Pade approximation.

    Although it is not computed this way, if A has a full set
    of eigenvectors V with corresponding eigenvalues D then
    [V,D] = EIG(A) and expm(A) = V*diag(exp(diag(D)))/V.
```

Compute Time Response

- Method 1: Calculate the output and state response with expm

Compute the state trajectory first and then compute the output.

- The function expm could compute matrix in time sequence.
  - Input time series, e.g., *t=0:0.01:10*

Example

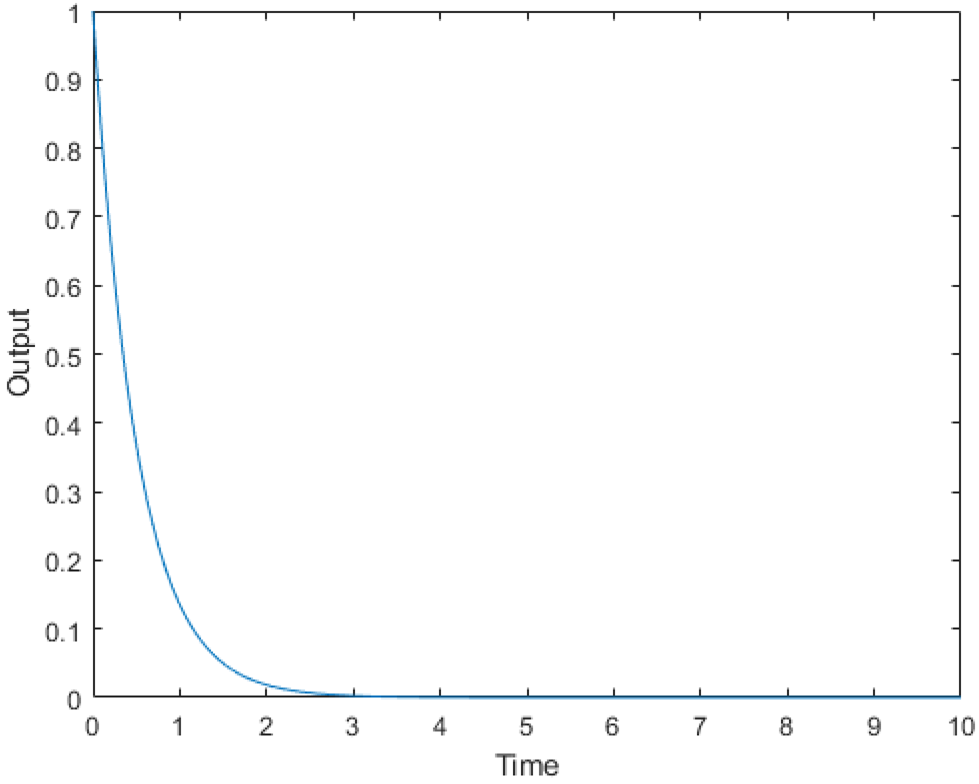
- Consider the RLC network with following state-space model

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

Set $u(t) = 0$, $x_1(0) = x_2(0) = 1$, compute its time response when $t \in [0,10]s$.

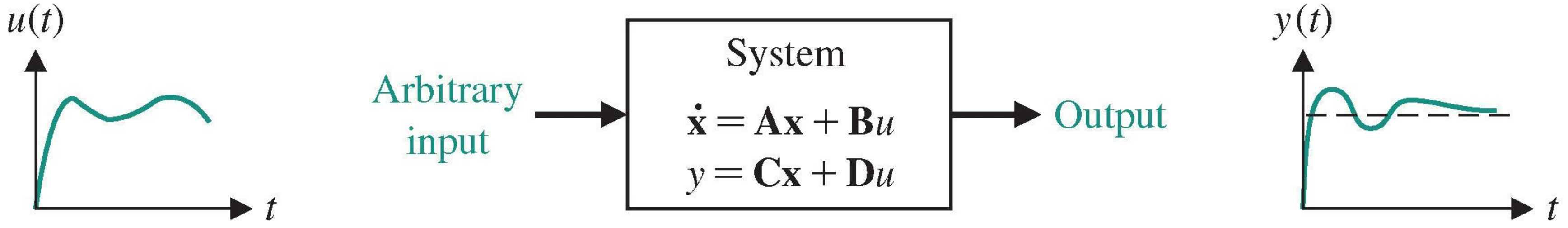
$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$
$$y(t) = C\mathbf{x}(t)$$

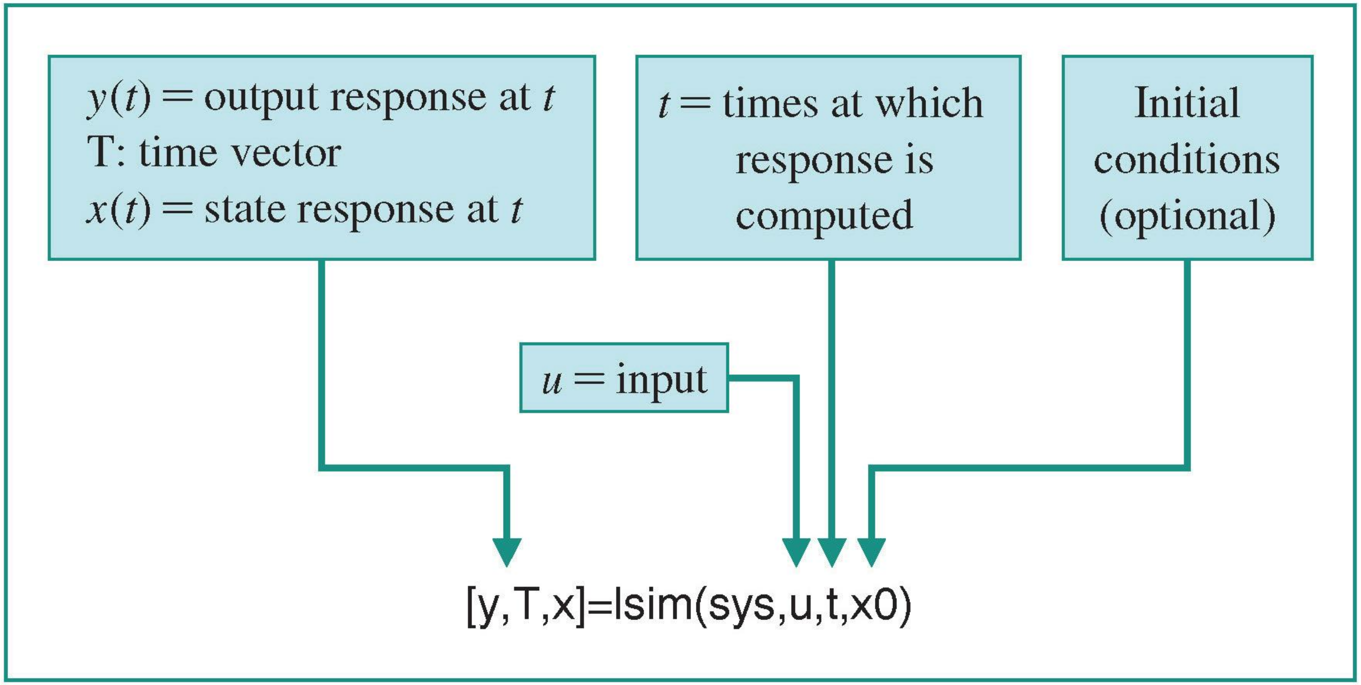
Example



Compute Time Response

Method 2: Calculate the output and state response (lsim)




$y(t)$ = output response at t
T: time vector
$x(t)$ = state response at t

t = times at which response is computed

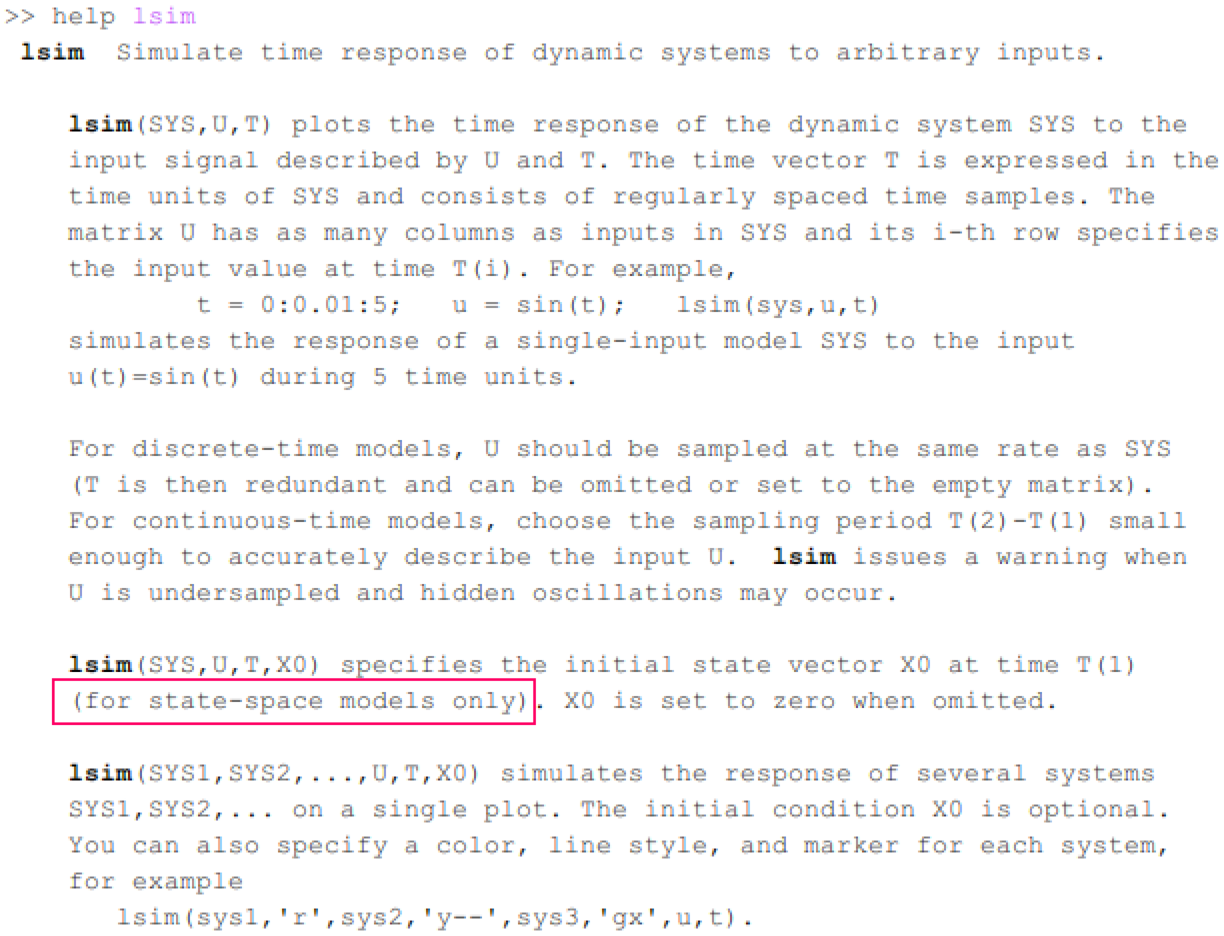
Initial conditions (optional)

u = input

[y,T,x]=lsim(sys,u,t,x0)

Compute Time Response

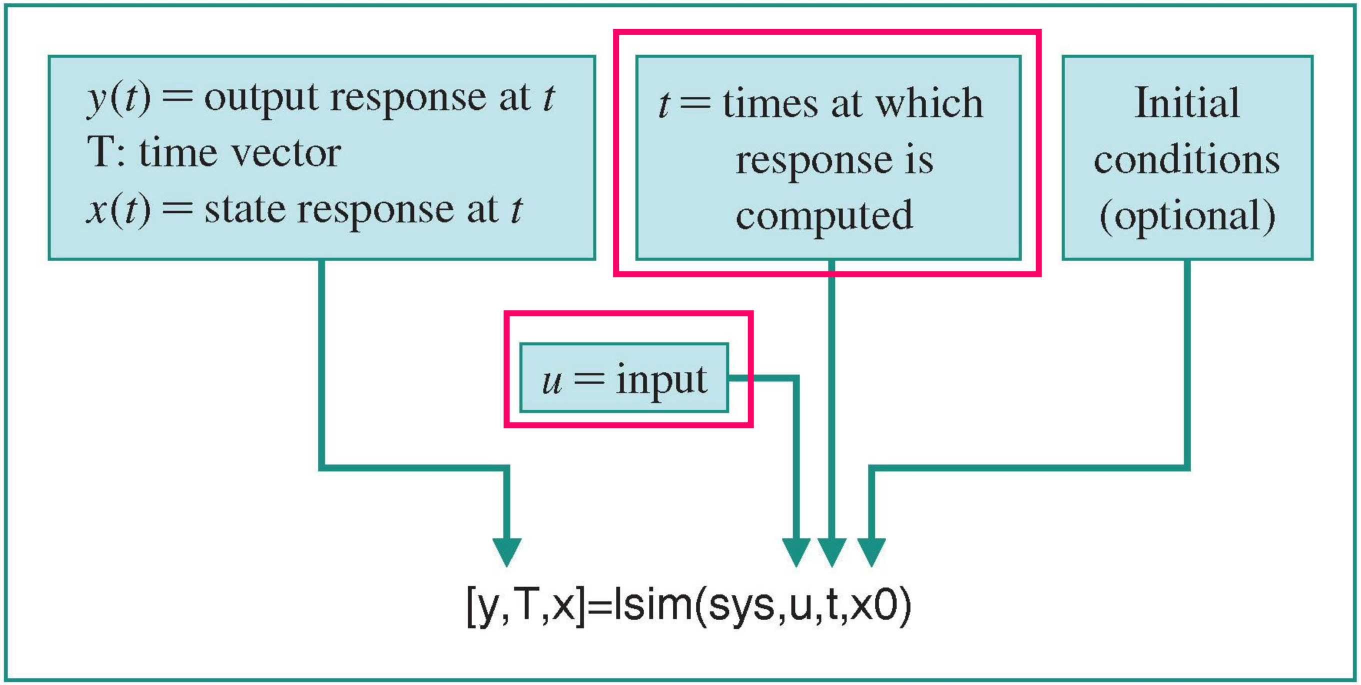
Method 2: Calculate the output and state response (lsim)



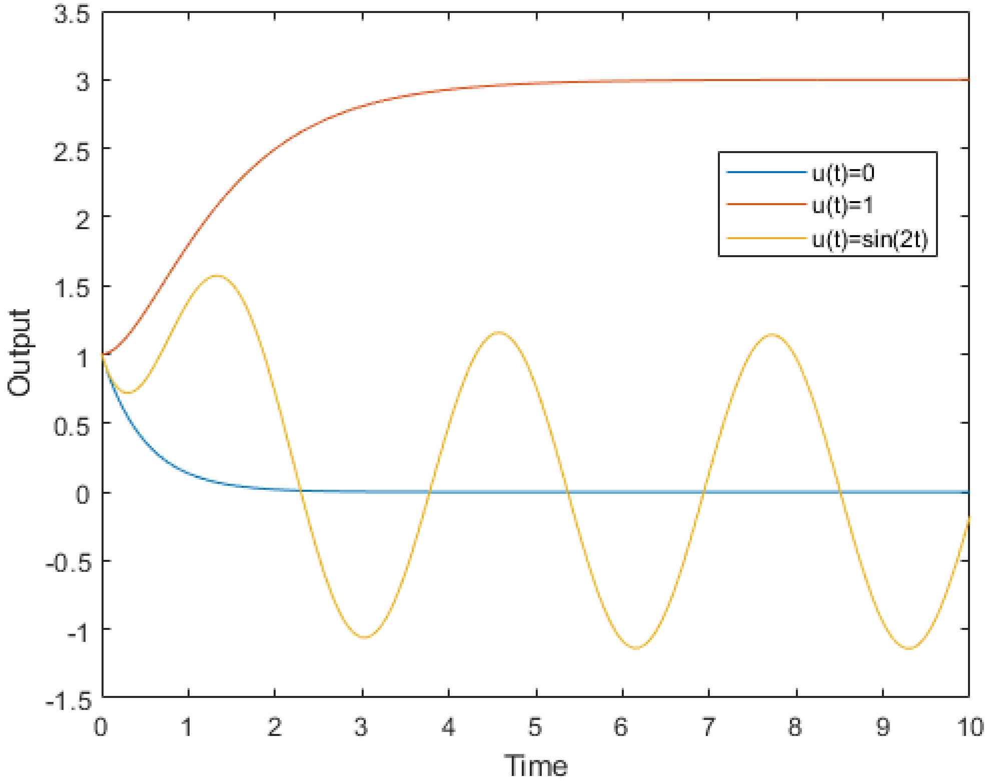
```
>> help lsim
 lsim  Simulate time response of dynamic systems to arbitrary inputs.
 
    lsim(SYS,U,T) plots the time response of the dynamic system SYS to the
    input signal described by U and T. The time vector T is expressed in the
    time units of SYS and consists of regularly spaced time samples. The
    matrix U has as many columns as inputs in SYS and its i-th row specifies
    the input value at time T(i). For example,
            t = 0:0.01:5;   u = sin(t);   lsim(sys,u,t)
    simulates the response of a single-input model SYS to the input
    u(t)=sin(t) during 5 time units.
 
    For discrete-time models, U should be sampled at the same rate as SYS
    (T is then redundant and can be omitted or set to the empty matrix).
    For continuous-time models, choose the sampling period T(2)-T(1) small
    enough to accurately describe the input U.  lsim issues a warning when
    U is undersampled and hidden oscillations may occur.
 
    lsim(SYS,U,T,X0) specifies the initial state vector X0 at time T(1)
    (for state-space models only). X0 is set to zero when omitted.
 
    lsim(SYS1,SYS2,...,U,T,X0) simulates the response of several systems
    SYS1,SYS2,... on a single plot. The initial condition X0 is optional.
    You can also specify a color, line style, and marker for each system,
    for example
       lsim(sys1,'r',sys2,'y--',sys3,'gx',u,t).
```

Example

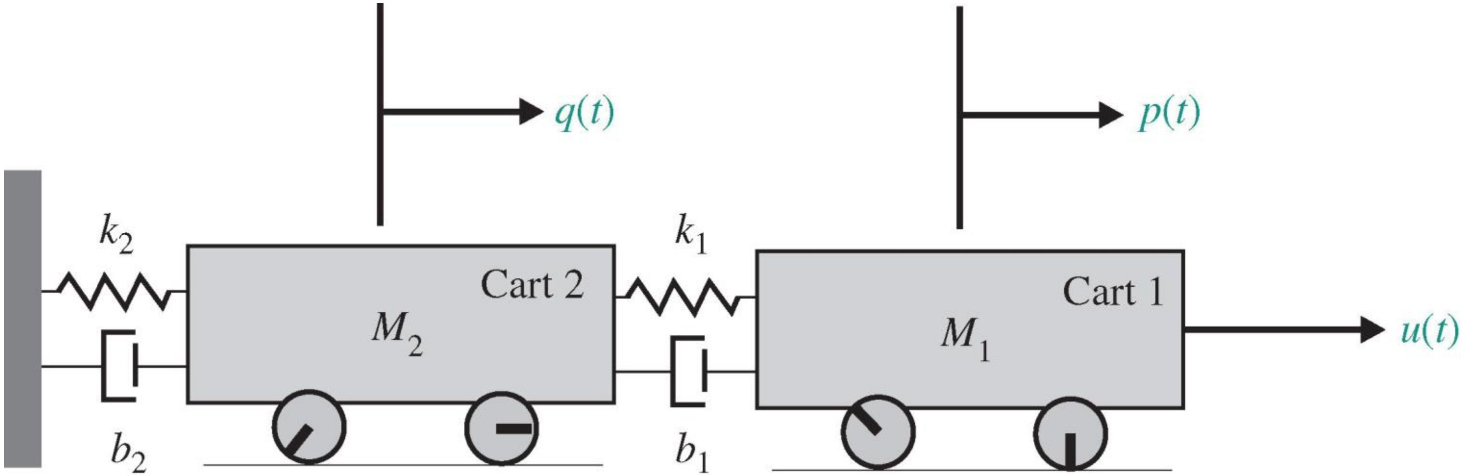
- Consider the same problem in previous example. Compute the time response when control is defined as (1) $u(t) = 0$, (2) $u(t) = 1\ (t)$, and (3) $u(t) = \sin 2t$. The time is set within $t \in [0,10]s$.

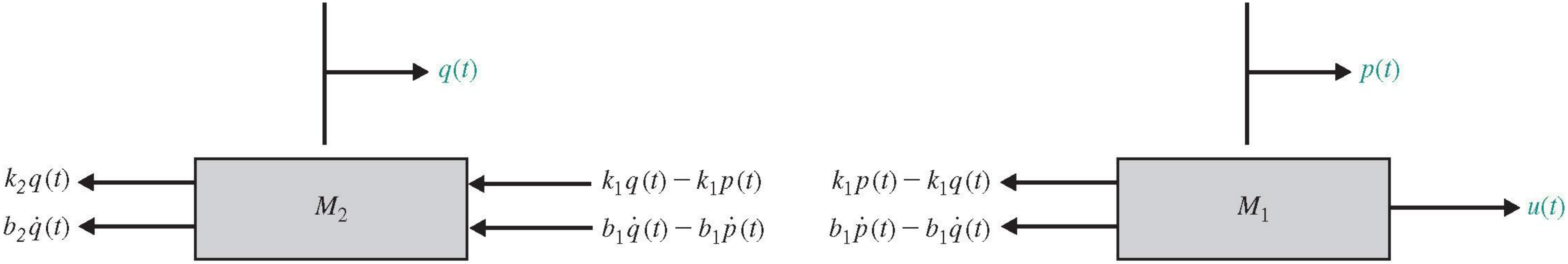


Example


3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
Output
0
1
2
3
4
5
6
7
8
9
10
Time
u(t)=0
u(t)=1
u(t)=sin(2t)

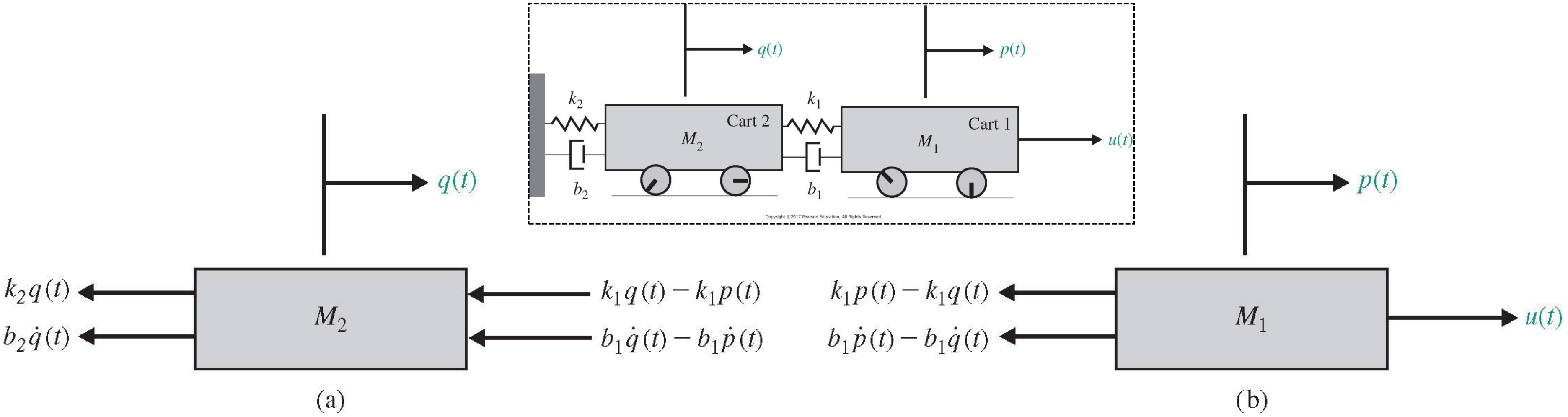
Example 9.4: Two Rolling Carts





(a) (b)

Example 9.4: Two Rolling Carts



where p, q are position of carts; M1 and M2 are mass of carts;
k1, k2: spring coefficients; b1, b2: damper coefficients. u: external force.

For Mass 1: $M_1\ddot{p} = F_1 = u - k_1(p - q) - b_1(\dot{p} - \dot{q})$

For Mass 2: $M_2\ddot{q} = F_2 = k_1(p - q) + b_1(\dot{p} - \dot{q}) - k_2 q - b_2\dot{q}$

Transfer Function

- How to derive the transfer function?

Define $p(t)$ as output, $u(t)$ as input.

$$M_1\ddot{p} = F_1 = u - k_1(p - q) - b_1(\dot{p} - \dot{q})$$

$\Downarrow$ Laplace transform

$$M_1 s^2 P(s) = U(s) - k_1[P(s) - Q(s)] - b_1 s[P(s) - Q(s)]$$

$$M_2\ddot{q} = F_2 = k_1(p - q) + b_1(\dot{p} - \dot{q}) - k_2 q - b_2\dot{q}$$

$\Downarrow$ Laplace transform

$$M_2 s^2 Q(s) = k_1[P(s) - Q(s)] + b_1 s[P(s) - Q(s)] - k_2 Q(s) - b_2 s Q(s)$$

Transfer Function

$$M_2s^2Q(s) = k_1[P(s) - Q(s)] + b_1s[P(s) - Q(s)] - k_2Q(s) - b_2sQ(s)$$

$$\Downarrow$$

$$Q(s) = \frac{k_1 + b_1s}{k_1 + k_2 + b_1s + b_2s + M_2s^2}P(s)$$

$$M_1s^2P(s) = U(s) - k_1[P(s) - Q(s)] - b_1s[P(s) - Q(s)]$$

$$\Downarrow$$

$$G(s) = \frac{P(s)}{U(s)}$$

Block Diagram

- In the system there are mainly three parameters, output $P(s)$, input $U(s)$, and an intermediate term $Q(s)$.
- The key step here is to get the relationship of $Q(s)$ to $P(s)$ and $U(s)$.

$$M_1 s^2 P(s) = U(s) - k_1[P(s) - Q(s)] - b_1 s[P(s) - Q(s)]$$

$$Q(s) = -\frac{1}{k_1 + b_1 s} U(s) + \frac{M_1 s^2 + b_1 s + k_1}{k_1 + b_1 s} P(s)$$

Block Diagram

$$Q(s) = \frac{k_1 + b_1 s}{k_1 + k_2 + b_1 s + b_2 s + M_2 s^2} P(s)$$

$$\Downarrow$$

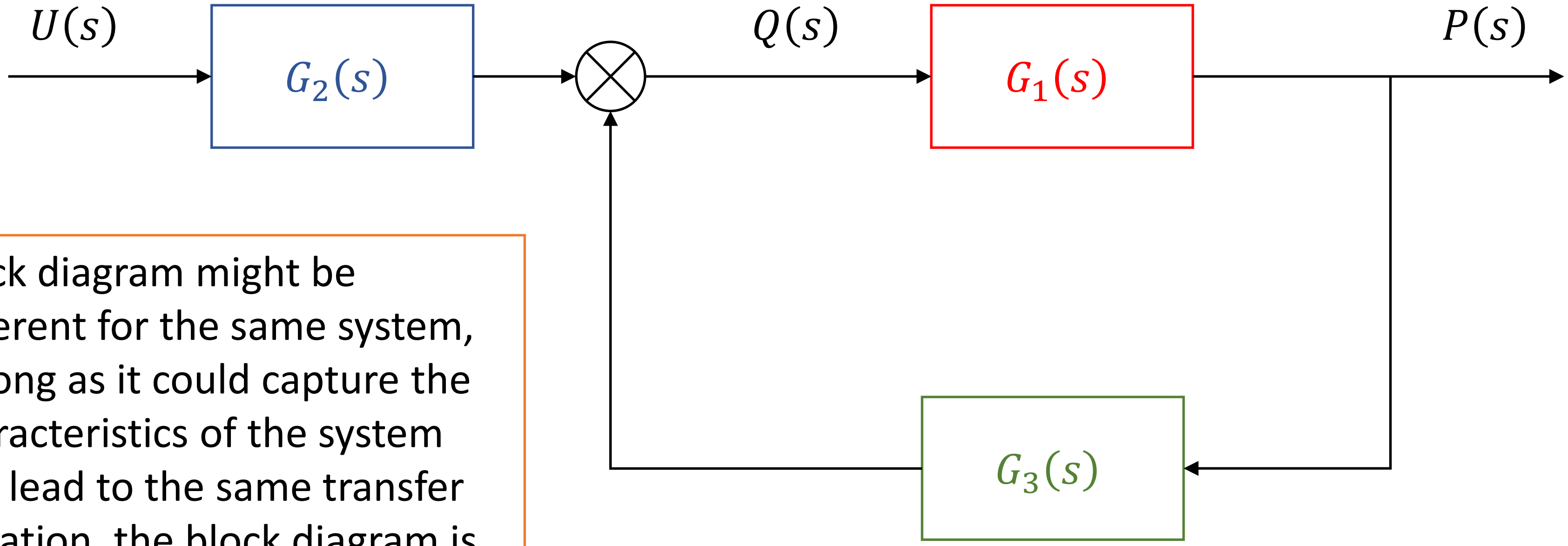
$$P(s) = \frac{k_1 + k_2 + b_1 s + b_2 s + M_2 s^2}{k_1 + b_1 s} Q(s)$$

$$Q(s) = -\frac{1}{k_1 + b_1 s} U(s) + \frac{M_1 s^2 + b_1 s + k_1}{k_1 + b_1 s} P(s)$$

$$P(s) = G_1(s) Q(s), Q(s) = G_2(s) U(s) + G_3(s) P(s)$$

Block Diagram

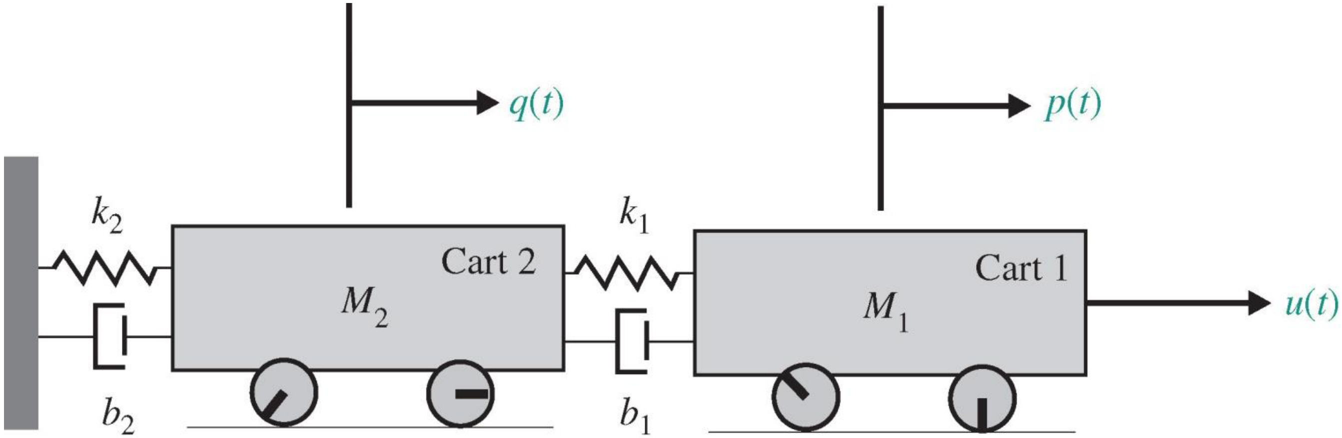
$$P(s) = G_1(s)Q(s), Q(s) = G_2(s)U(s) + G_3(s)P(s)$$



Block diagram might be different for the same system, as long as it could capture the characteristics of the system and lead to the same transfer equation, the block diagram is correct.

State Space Model

- How to derive the state space model?

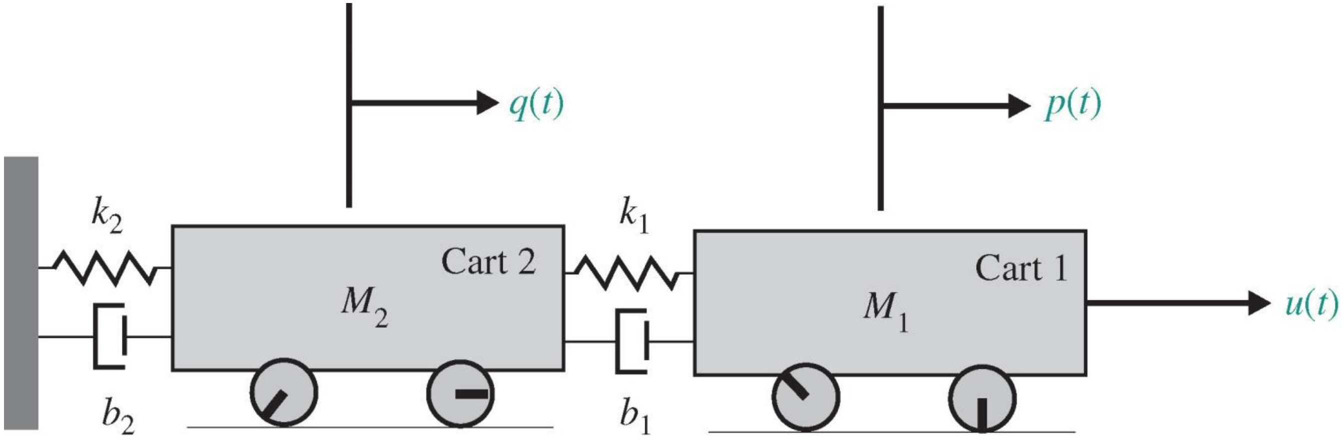


Select state parameter $x_1 = p, x_2 = q, x_3 = \dot{p}, x_4 = \dot{q}, y = x_1$ (they could be measured)

$$\dot{x}_3 = \ddot{p} = -\frac{b_1}{M_1}\dot{p} - \frac{k_1}{M_1}p + \frac{1}{M_1}u + \frac{k_1}{M_1}q + \frac{b_1}{M_1}\dot{q},$$

$$\dot{x}_4 = \ddot{q} = -\frac{k_1 + k_2}{M_2}q - \frac{b_1 + b_2}{M_2}\dot{q} + \frac{k_1}{M_2}p + \frac{b_1}{M_2}\dot{p},$$

State Space Model

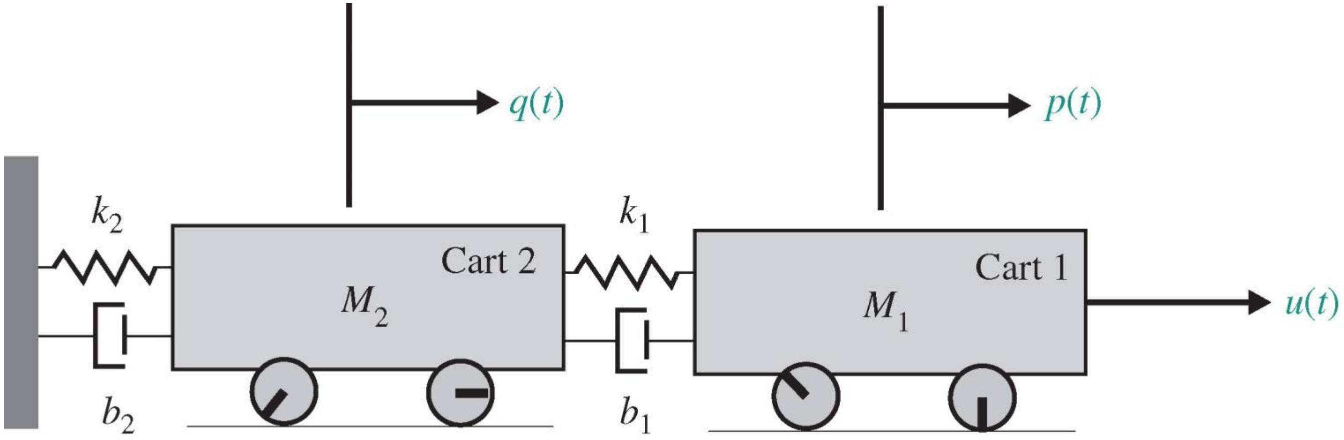


Select state parameter $x_1 = p, x_2 = q, x_3 = \dot{p}, x_4 = \dot{q}, y = x_1$ (they could be measured)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{k_1 + k_2}{M_2} & \frac{b_1}{M_2} & -\frac{b_1 + b_2}{M_2} \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix},$$

$$y = [1 \quad 0 \quad 0 \quad 0]\mathbf{x} = \mathbf{C}\mathbf{x}. \qquad \mathbf{D} = [0]$$

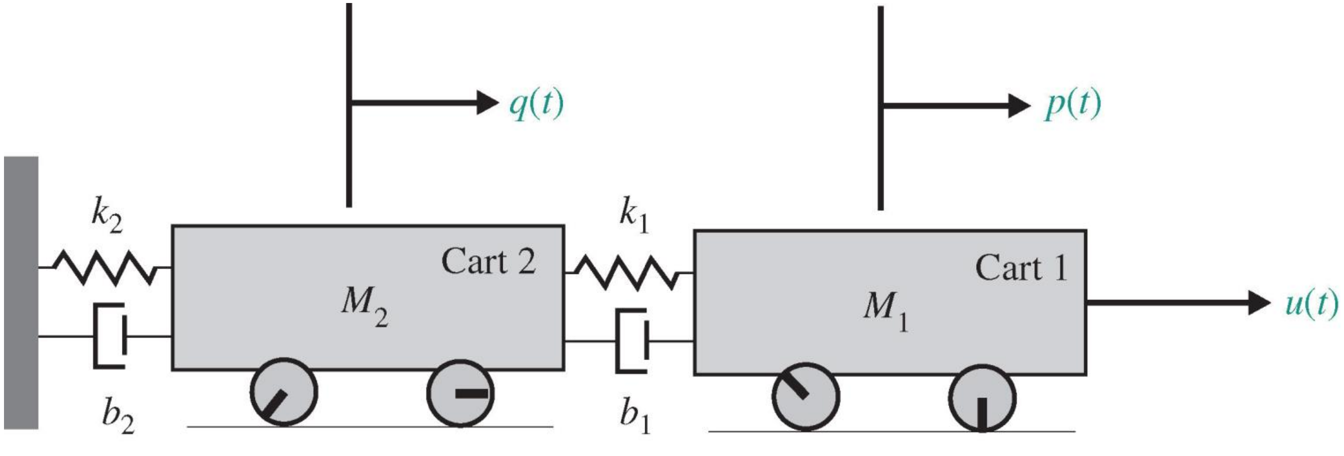
Simulate the Two Rolling Carts System



Suppose that the two rolling carts have the following parameter values: $k_1 = 150$ N/m; $k_2 = 700$ N/m; $b_1 = 15$ N s/m; $b_2 = 30$ N s/m; $M_1 = 5$ kg; and $M_2 = 20$ kg. The

Initial conditions: $p(0) = 10, q(0) = 0, \dot{p}(0) = 0, \dot{q}(0) = 0.$

Simulate the Two Rolling Carts System



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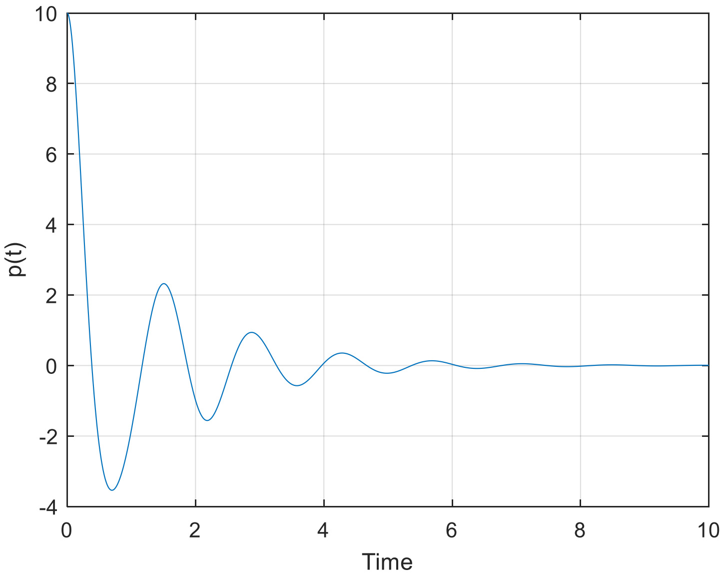
With the given parameters, the state space matrix of the system could be written as

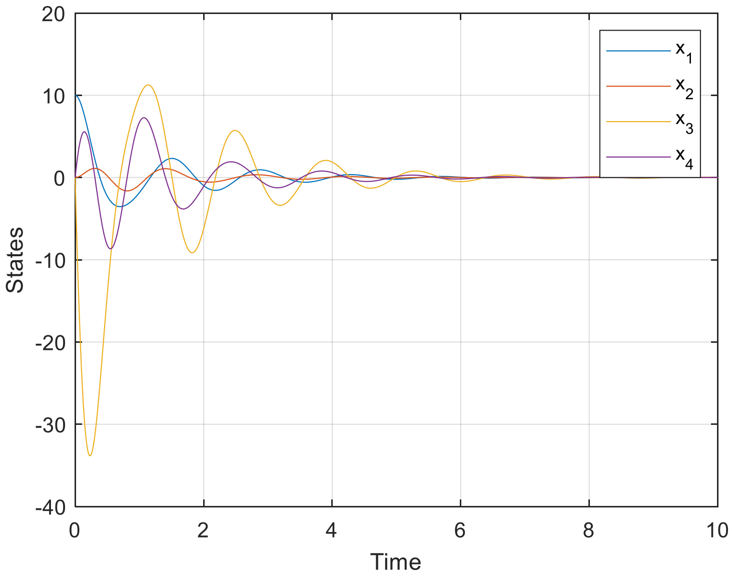
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -30 & 30 & -3 & 3 \\ 7.5 & -42.5 & 0.75 & -2.25 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0 \end{bmatrix}, C = [1 \quad 0 \quad 0 \quad 0], D = [0]$$

Simulate the Two Rolling Carts System

Simulate unforced output of the system

- $u(t) = 0, t \in [0,10]s$

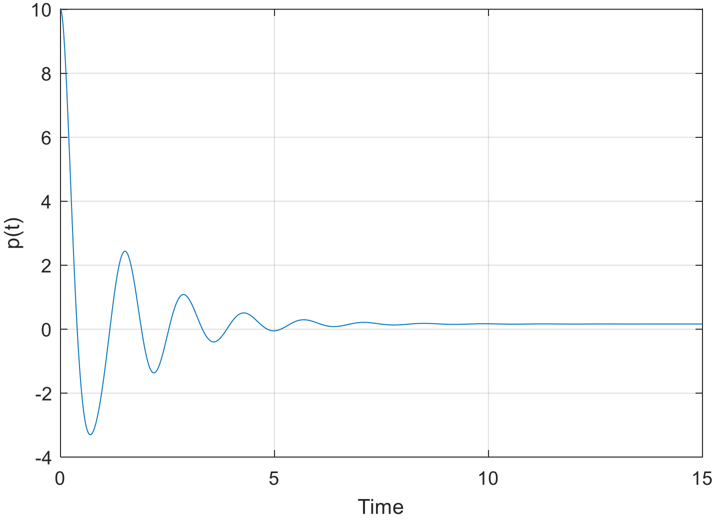


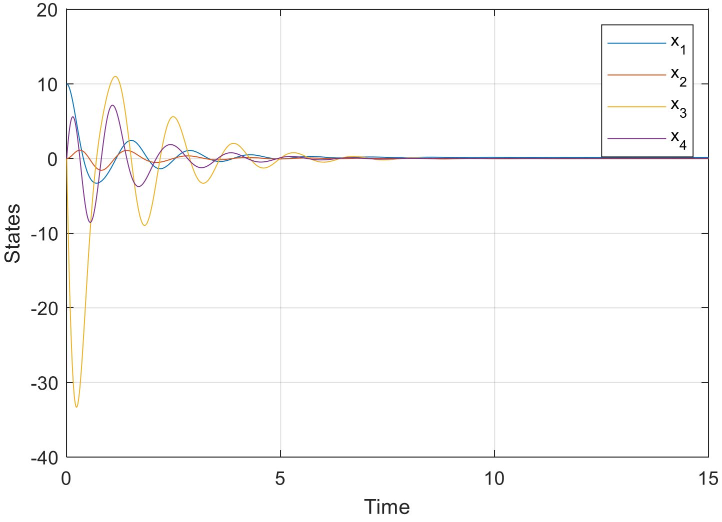


Simulate the Two Rolling Carts System

Simulate step output of the system

- $u(t) = 20(t), t \in [0,15]s$

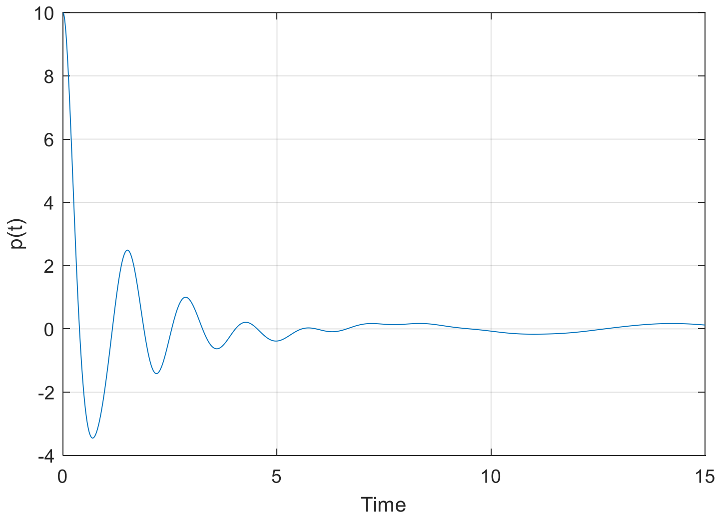


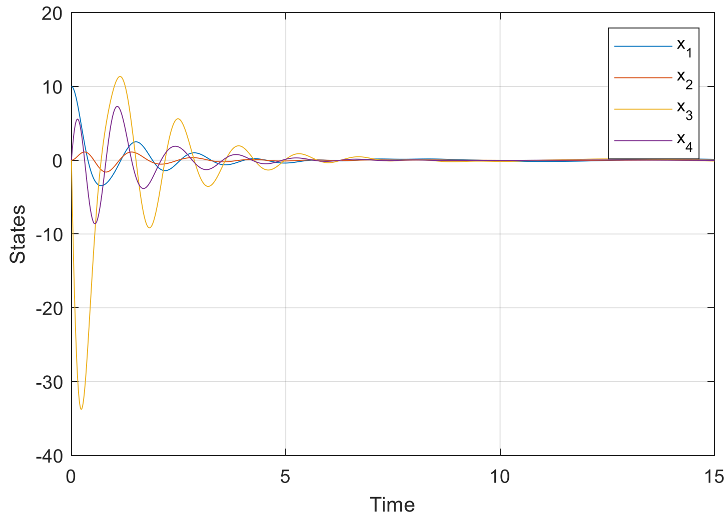


Simulate the Two Rolling Carts System

Simulate sin output of the system

- $u(t) = 20\sin(t), t \in [0,15]s$





Simulate the Two Rolling Carts System

- Compute transfer function with *tf*

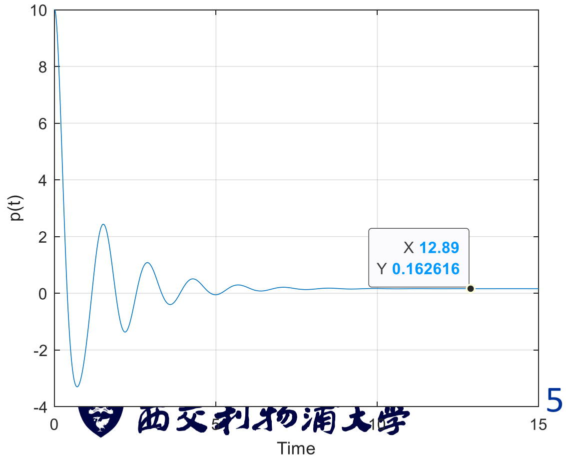
$$G(s) = \frac{P(s)}{U(s)} = \frac{0.2s^2 + 0.45s + 8.5}{s^4 + 5.25s^3 + 77s^2 + 150s + 1050}$$

- Steady value with step input $u(t) = 20(t)$

$$P(s) = \frac{20G(s)}{s} = \frac{20(0.2s^2 + 0.45s + 8.5)}{s(s^4 + 5.25s^3 + 77s^2 + 150s + 1050)}$$

$$\lim_{t\to\infty} p(t) = \lim_{s\to 0} sP(s) = \lim_{s\to 0} \frac{20(0.2s^2 + 0.45s + 8.5)}{s^4 + 5.25s^3 + 77s^2 + 150s + 1050}$$

$$= \frac{20 * 8.5}{1050} = 0.162$$



Simulate the Two Rolling Carts System

If change the output to $q(t)$, i.e., $x_2(t)$, then

- Use state space model, only need to reset C as

$$C = [0 \quad 1 \quad 0 \quad 0],$$

More convenient

and then simulate again

- If use transfer function, we need to rederive transfer function

$$M_2 s^2 P(s) = k_1[P(s) - Q(s)] + b_1 s[P(s) - Q(s)] - k_2 Q(s) - b_2 s Q(s)$$

$$P(s) = \frac{k_1 + k_2 + b_1 s + b_2 s}{k_1 + b_1 s - M_2 s^2} Q(s)$$

$$M_1 s^2 P(s) = U(s) - k_1[P(s) - Q(s)] - b_1 s[P(s) - Q(s)]$$

$$G(s) = \frac{Q(s)}{U(s)}$$

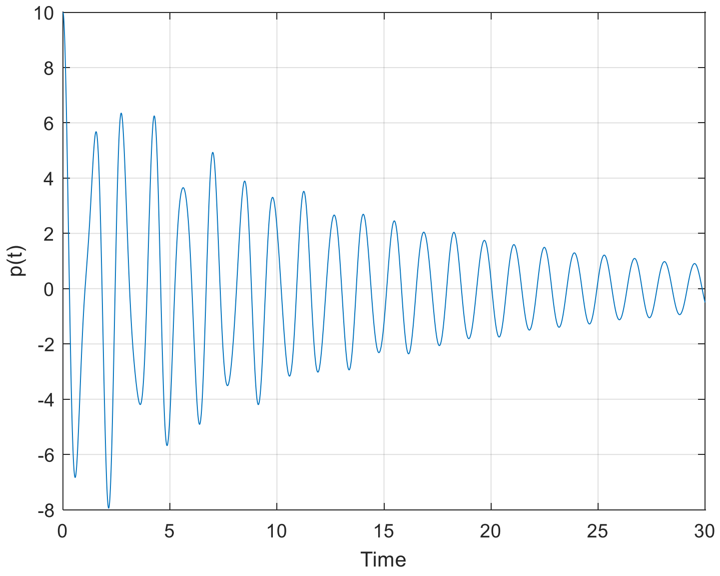
Simulate the Two Rolling Carts System

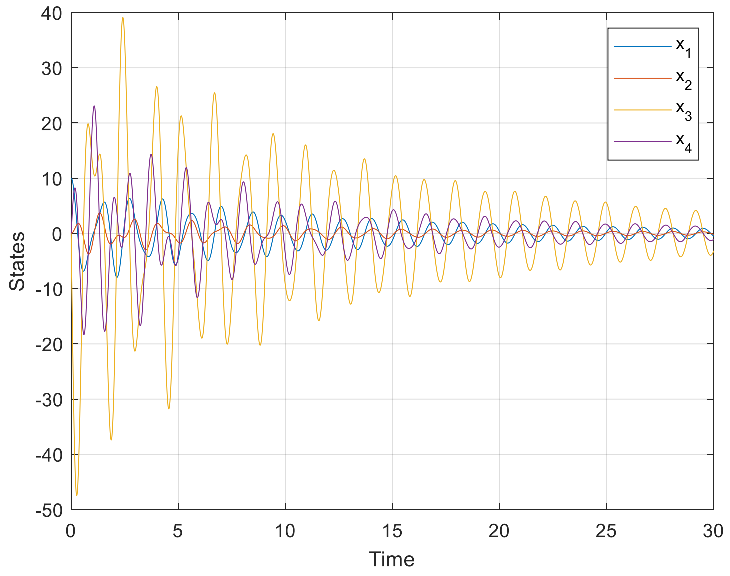
- It is easy to observe that the system is currently in high damp mode
  - Even if the input is large as 20, the system output quickly go to zero
- This is because $b_1 = 15$, $b_2 = 30$.
- What if we reduce the damping?
  - Try $b_1 = 1.5$, $b_2 = 3.0$

Simulate the Two Rolling Carts System

Simulate unforced output of the system with small damping

- $u(t) = 0, t \in [0,30]s$





Thank You !