MTH102 Engineering Mathematics II

Lesson 4: Bayes' rule and independence

Term: 2024

Outline

Outline

Bayes' rule

Let the events $B_1, B_2, \ldots, B_n$ be mutually exclusive with $P(B_i) > 0$ and

$$S = B_1 \cup B_2 \cup \cdots \cup B_n.$$

For any event A, it follows by law of total probability that

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

Therefore for $i = 1, 2, \ldots, n$

$$P(B_i|A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)},$$

which is called *Bayes' Rule*.

$S = \{\text{participant in the seminar}\}$

Example 1

Recall: Let $T = \{\text{Coming by train}\}$, $C = \{\text{Coming by car}\}$, $F = \{\text{Coming by flight}\}$, $L = \{\text{Delay}\}$. Then $\mathbb{P}(T) = 0.5$, $\mathbb{P}(C) = 0.3$, $\mathbb{P}(F) = 0.2$. $\mathbb{P}(L|T) = 0.1$, $\mathbb{P}(L|C) = 0.3$, $\mathbb{P}(L|F) = 0.4$. Since $S = \{T\} \cup \{F\}$ and it's a union of mutually exclusive events, Using Bayes' rule, we have

A visitor is coming for a seminar in XJTLU. The probability that he comes by train, car and flight is respectively 0.5, 0.3 and 0.2. And the probability of delay by these three means of transport is respectively 0.1, 0.3 and 0.4. If the visitor is late for the seminar, what is the probability that

(a) he came by train?

(b) he came by car?

(c) he came by flight?

(a) $$\mathbb{P}(T|L) = \frac{\mathbb{P}(L|T)\mathbb{P}(T)}{\mathbb{P}(L|T)\mathbb{P}(T) + \mathbb{P}(L|C)\mathbb{P}(C) + \mathbb{P}(L|F)\mathbb{P}(F)} = \frac{0.1 \times 0.5}{0.1 \times 0.5 + 0.3 \times 0.3 + 0.4 \times 0.2} = \frac{0.05}{0.22} = \frac{5}{22}$$

(b) Similarly, $$\mathbb{P}(C|L) = \frac{\mathbb{P}(L|C)\mathbb{P}(C)}{\mathbb{P}(L|T)\mathbb{P}(T) + \mathbb{P}(L|C)\mathbb{P}(C) + \mathbb{P}(L|F)\mathbb{P}(F)} = \frac{0.3 \times 0.3}{0.22} = \frac{9}{22}$$

(c) Similarly, $$\mathbb{P}(F|L) = \frac{\mathbb{P}(L|F)\mathbb{P}(F)}{\mathbb{P}(L|T)\mathbb{P}(T) + \mathbb{P}(L|C)\mathbb{P}(C) + \mathbb{P}(L|F)\mathbb{P}(F)} = \frac{0.4 \times 0.2}{0.22} = \frac{8}{22} = \frac{4}{11}$$

Example 2

$S = \{\text{all tested people}\}$

A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability 0.01, the test result will imply that he or she has the disease.) If 0.5 percent of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?

Sol: Let $A = \{\text{sufferer}\}$, $P = \{\text{test shows positive result}\}$, then $A^c = \{\text{not a sufferer}\}$, $P^c = \{\text{test shows negative}\}$, $\mathbb{P}(P|A) = 0.95$
$\mathbb{P}(P|A^c) = 0.01$, $\mathbb{P}(A) = 0.005$, $\mathbb{P}(A^c) = 1 - 0.005 = 0.995$
Since $S = A \cup A^c$, A and A^c are mutually exclusive events;
using Bayes' rule, we have

$$\mathbb{P}(A \mid P) = \frac{\mathbb{P}(P \mid A)\,\mathbb{P}(A)}{\mathbb{P}(P \mid A)\,\mathbb{P}(A) + \mathbb{P}(P \mid A^c)\,\mathbb{P}(A^c)}$$

$$= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} = \frac{4.75}{14.7} \simeq 0.3231$$

In conclusion, the probability that a person has the disease given that the test shows positive result is around 0.3231

Bayes' rule

Let the events $B_1, B_2, \ldots, B_n$ be mutually exclusive with $P(B_i) > 0$ and

$$S = B_1 \cup B_2 \cup \cdots \cup B_n.$$

For any event A, for $i = 1, 2, \ldots, n$

$$P(B_i|A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}.$$

Here $P(B_i)$ is called the **prior probability**, and $P(B_i|A)$ is called the **posterior probability**.

Prior probability represents what is originally believed before new evidence is introduced, and posterior probability is the revised or updated probability after taking the new information into account. Bayes' rule can be used to update a previous belief once new information is obtained.

Aesop's Fables "The boy who cried 'Wolf'"

There was once a young Shepherd Boy who tended his sheep at the foot of a mountain near a dark forest. It was rather lonely for him all day, so he thought upon a plan by which he could get a little company and some excitement. He rushed down towards the village calling out "Wolf, Wolf," and the villagers came out to meet him, and some of them stopped with him for a considerable time. This pleased the boy so much that a few days afterwards he tried the same trick, and again the villagers came to his help. But shortly after this a Wolf actually did come out from the forest, and began to worry the sheep, and the boy of course cried out "Wolf, Wolf," still louder than before. But this time the villagers, who had been fooled twice before, thought the boy was again deceiving them, and nobody stirred to come to his help. So the Wolf made a good meal off the boy's flock, and when the boy complained, the wise man of the village said:

"A liar will not be believed, even when he speaks the truth."

Aesop's Fables "The boy who cried 'wolf'"

Let $A =$"the boy is lying", $B =$"the boy is reliable". Assume that

$$P(B) = 0.8,\ P(A|B) = 0.1,\ P(A|\overline{B}) = 0.5.$$

After the villagers have been fooled for the first time, the reliability of the boy turns to

$$P(B|A) = \frac{P(B)P(A|B)}{P(B)P(A|B) + P(\overline{B})P(A|\overline{B})} = \frac{0.8 \times 0.1}{0.8 \times 0.1 + (1 - 0.8) \times 0.5} = 0.4444.$$

From now on, $P(B) = 0.4444$. After the villagers have been fooled for the second time, the reliability of the boy turns to

$$P(B|A) = \frac{P(B)P(A|B)}{P(B)P(A|B) + P(\overline{B})P(A|\overline{B})} = \frac{0.4444 \times 0.1}{0.4444 \times 0.1 + 0.5556 \times 0.5} = 0.1379.$$

Here $P(B)$ is the prior probability and $P(B|A)$ is the posterior probability.

Bayes' rule: exercise

$S = \{$any 1 ball drawn either from Urn A or Urn B$\}$.

Urn A has 5 white and 7 black balls. Urn B has 3 white and 9 black balls. We flip a fair coin. If the outcome is heads, then a ball from urn A is selected, whereas if the outcome is tails, then a ball from urn B is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails?

Sol: Let $U_1 = \{$Urn A$\}$, $U_2 = \{$Urn B$\}$, $W = \{$white ball$\}$, $B = \{$black ball$\}$. Then we have $\{$Heads$\} = U_1$, $\{$Tails$\} = U_2$ and $\mathbb{P}(U_1) = \mathbb{P}(\{\text{Heads}\}) = 1/2$, $\mathbb{P}(U_2) = \mathbb{P}(\{\text{Tails}\}) = 1/2$,

$$\mathbb{P}(W|U_1) = \frac{5}{5+7} = \frac{5}{12}, \quad \mathbb{P}(W|U_2) = \frac{3}{3+9} = \frac{1}{4}.$$

Since $S = U_1 \cup U_2$, and U_1, U_2 are mutually exclusive events, from Bayes' rule,

$$\mathbb{P}(\{\text{Tails}\} \mid W) = \mathbb{P}(U_2 \mid W)$$

$$= \frac{\mathbb{P}(W \mid U_2)\,\mathbb{P}(U_2)}{\mathbb{P}(W \mid U_1)\,\mathbb{P}(U_1) + \mathbb{P}(W \mid U_2)\,\mathbb{P}(U_2)}$$

$$= \frac{\frac{1}{4} \times \cancel{\frac{1}{2}}}{\frac{5}{12} \times \cancel{\frac{1}{2}} + \frac{1}{4} \times \cancel{\frac{1}{2}}} = \frac{3}{5+3} = \frac{3}{8}$$

In conclusion, if a white ball is selected, the probability that the coin landed tails is 3/8.

Outline

Independent events

Definition

Two events A and B are said to be independent if

$$P(A \cap B) = P(A)P(B).$$

Two events A and B that are not independent are said to be dependent.

Remark. Pay attention, A and B are independent does not mean that $A \cap B = \emptyset$!

Property. If A and B are independent, then

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B).$$

The probability of A does not depend on the occurrence or nonoccurrence of B, and conversely.

Example 3

A card is selected at random from an ordinary deck of 52 playing cards. If A is the event that the selected card is an ace and B is the event that it is a spade. Then

$$P(A) = \frac{4}{52}, \ P(B) = \frac{13}{52}, \ P(A \cap B) = \frac{1}{52}.$$

Therefore A and B are independent since $P(A \cap B) = P(A)P(B)$.

Example 4

Two coins are flipped, and the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Let A be the event that the first coin lands on heads and B be the event that the second lands on tails. Then

$$P(A) = P(\{HH, HT\}) = \frac{1}{2}, \; P(B) = P(\{HT, TT\}) = \frac{1}{2},$$

and

$$P(A \cap B) = P(\{HT\}) = \frac{1}{4} = P(A)P(B).$$

We thus deduce that A and B are independent.

Remarks

- The events A and B are mutually exclusive, i.e. $A \cap B = \emptyset$, does not imply that A and B are independent.
- For any event A, A and S are independent, A and $\emptyset$ are independent.
- In the applications, the independence can be judged by the analysis of the pattern of the experiment instead of computation. For example, we select at random one ball and then another ball from a box containing 2 black balls and 3 white balls. Let A be the event that the first ball is black, and B be the event that the second ball is black.
  - ➢ If the selection is with replacement, then A and B are independent.
  - ➢ If the selection is without replacement, then A and B are not independent.

① With replacement, $\mathbb{P}(A \cap B) = \frac{2}{5} \times \frac{2}{5} = \mathbb{P}(A)\mathbb{P}(B)$

② Without replacement, $\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) = \frac{1}{4} \times \frac{2}{5} \neq \frac{2}{5} \times \frac{2}{5} = \mathbb{P}(A)\mathbb{P}(B)$

Open question

Two fair dice are tossed. Let A be the event that the first die equals m ($m = 1, 2, \ldots, 6$), B be the event that the sum of the two dice is n ($n = 2, 3, \ldots, 12$). Since the computation of the sum of the two dice will depend on the first die, so A and B are not independent. Is that true? (*Hint: discuss the case* $m = 1, n = 2$ *and* $m = 1, n = 7$.) No.

Sol: $S = \{(i,j);\ i,j = 1,2,\cdots,6\}$, $|S| = 6^2 = 36$

① Let $A_1 = \{$1st die is 1$\}$, $B_1 = \{$sum of 2 dice is 2$\}$. Then

$A_1 = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$ and

$B_1 = \{(1,1)\}$. Hence, $\mathbb{P}(A_1) = \frac{|A_1|}{|S|} = \frac{6}{36} = \frac{1}{6}$

and $\mathbb{P}(B_1) = \frac{|B_1|}{|S|} = \frac{1}{36}$. Therefore,

$$\mathbb{P}(A_1 \cap B_1) = \mathbb{P}(\{(1,1)\}) = \frac{1}{36}$$
$$\mathbb{P}(A_1)\mathbb{P}(B_1) = \frac{1}{6} \times \frac{1}{36}$$
$$\Rightarrow \mathbb{P}(A_1 \cap B_1) \neq \mathbb{P}(A_1)\mathbb{P}(B_1)$$

A_1 and B_1 are <u>not independent</u>.

(2) Let $B_2 = \{$sum of the 2 dice is 7$\}$. Then

$$B_2 = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

and $\mathbb{P}(B_2) = \frac{|B_2|}{|S|} = \frac{6}{36} = \frac{1}{6}$. We have

$$\mathbb{P}(A_1 \cap B_2) = \mathbb{P}(\{(1,6)\}) = \frac{1}{36}$$
$$\mathbb{P}(A_1)\mathbb{P}(B_2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$
$$\Rightarrow \mathbb{P}(A_1 \cap B_2) = \mathbb{P}(A_1)\mathbb{P}(B_2)$$

A_1 and B_2 are <u>independent</u>.

<u>In conclusion</u>, sometimes A and B are independent, sometimes not, depending on the value of the sum n.

Property of independent events

If the events A and B are independent, then so are

- A and B^c;
- A^c and B;
- A^c and B^c.

The probability of A does not depend on the occurrence or nonoccurrence of B, and conversely.

Example. Two fair dice are tossed. Let A be the event that the first die equals 1, C be the event that the sum of the two dice is not 7. Are A and C are independent?

Sol : Method 1 : Recall $S = \{(i,j);\ i,j = 1, \cdots, 6\}$. $|S| = 36$.

Let $A = \{\text{1st die is } 1\} = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$.

and $C = \{\text{the sum of 2 dice is NOT } 7\}$

$= \{\text{the sum of 2 dice is } 7\}^c$

Then we have $A \cap C = \{(1,1),(1,2),(1,3),(1,4),(1,5)\}$,

$$\mathbb{P}(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

$\mathbb{P}(C) = 1 - \mathbb{P}(\{\text{the sum of 2 dice is } 7\})$, by complement rule.

$$= 1 - \mathbb{P}(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\})$$

$$= 1 - \frac{6}{36} = 1 - \frac{1}{6} = \frac{5}{6}$$

and $\mathbb{P}(A \cap C) = \dfrac{|A \cap C|}{|S|} = \dfrac{5}{36}$. Therefore, we have

$$\mathbb{P}(A)\mathbb{P}(C) = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36} = \mathbb{P}(A \cap C).$$

In conclusion, A and C are independent events.

Method 2 : Let $B = \{\text{the sum of 2 dice is } 7\}$. Then $B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$A \cap B = \{(1,6)\}$ and we have

$$\left.\begin{array}{l} \mathbb{P}(A \cap B) = \dfrac{1}{36} \\ \mathbb{P}(A)\mathbb{P}(B) = \dfrac{1}{6} \times \dfrac{1}{6} = \dfrac{1}{36} \end{array}\right\} \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

So A and B are independent. From property above, we have A and $B^c = C$ must be also independent.

Independence of three events

Mutually independent events

Definition

Three events A, B and C are said to be independent if

$$P(A \cap B) = P(A)P(B),$$

$$P(B \cap C) = P(B)P(C),$$

$$P(C \cap A) = P(C)P(A),$$

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

Example 5

There are four boxes of T-shirts numbered 1,2,3 and 4. There are only red T-shirts in box 1, only white T-shirts in box 2, only black T-shirts in box 3 and T-shirts of all the three colors in box 4. A box is drawn at random, and let $A = \{\text{we get red T-shirts}\}$, $B = \{\text{we get white T-shirts}\}$ and $C = \{\text{we get black T-shirts}\}$. Then

$$S = \{1, 2, 3, 4\},\ A = \{1, 4\},\ B = \{2, 4\},\ C = \{3, 4\}.$$

Therefore, $P(A) = P(B) = P(C) = 1/2$. Furthermore,

$$P(A\cap B) = \frac{1}{4} = P(A)P(B),\ P(B\cap C) = \frac{1}{4} = P(B)P(C), P(C\cap A) = \frac{1}{4} = P(C)P(A).$$

However,

$$P(A \cap B \cap C) = P(\{4\}) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}.$$

Example 6

There are eight boxes of T-shirts numbered $1, 2, \ldots, 8$. We put red T-shirts in box $1, 2, 3, 4$, white T-shirts in box $1, 2, 3, 5$, and black T-shirts in box $1, 6, 7, 8$. A box is drawn at random, and let let $A = \{\text{we get red T-shirts}\}$, $B = \{\text{we get white T-shirts}\}$ and $C = \{\text{we get black T-shirts}\}$. Then

$$S = \{1, 2, \ldots, 8\}, \;A = \{1, 2, 3, 4\}, \;B = \{1, 2, 3, 5\}, \;C = \{1, 6, 7, 8\}.$$

Therefore, $P(A) = P(B) = P(C) = 1/2$ and

$$P(A \cap B \cap C) = P(\{1\}) = \frac{1}{8} = P(A)P(B)P(C).$$

However,

$$P(A \cap B) = P(\{1, 2, 3\}) = \frac{3}{8} \neq P(A)P(B) = \frac{1}{4}.$$

Example 7

Problem. Suppose that the probability that the missile reaches the target during an experiment is 0.8. What is the probability that the missile reaches the target at least once in three independent experiments?

Solution. For $i = 1, 2, 3$, let

$$A_i = \{\text{the missile reaches the target in the } i\text{-th experiment}\}.$$

Then A_1, A_2, A_3 are mutually independent and $P(A_i) = 0.8$. The desired event is $A_1 \cup A_2 \cup A_3$, and

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= 1 - P((A_1 \cup A_2 \cup A_3)^c) \\ &= 1 - P(A_1^c \cap A_2^c \cap A_3^c) && \text{by De Morgan's law} \\ &= 1 - P(A_1^c)P(A_2^c)P(A_3^c) && \text{by independence} \\ &= 1 - (1 - 0.8)^3 = 0.992. \end{aligned}$$

Experiment and trials

Sometimes, an experiment consists of performing a sequence of subexperiments. For instance, if the experiment consists of tossing a coin several times, we may think of each toss as being a subexperiment. If the subexperiments are independent and each subexperiment has the same set of possible outcomes, then the subexperiments are called **trials**.

Example 8

A person shoots for one target with several independent trials. In each trial, he hits the target successfully with probability p and misses the target with probability $1-p$. What is the probability that

(a) at least 1 success occurs in the first n trials;

(b) exactly k successes occur in the first n trials.

Sol : (a) $\mathbb{P}(\{\text{at least 1 succes occurs in the first } n \text{ trials}\})$

$= 1 - \mathbb{P}(\{\text{no succes occurs in the first } n \text{ trials}\})$

$= 1 - \mathbb{P}(\{\text{failure in the 1st trail}\} \cap \{\text{failure in the 2nd trail}\} \cap \cdots \cap \{\text{failure in the n-th trail}\})$

$= 1 - \underbrace{(1-p)(1-p)\cdots(1-p)}_{n\text{-times}}$, by independence of n trials

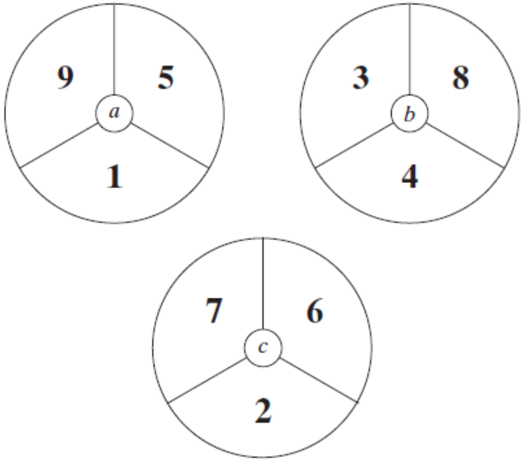
$= 1 - (1-p)^n$

(b) $\mathbb{P}(\{$ Exactly k success occur in the 1st n trials $\})$

$= \binom{n}{k} p^k (1-p)^{n-k}$, by independence of n trails

Exercise

There are three spinners a, b, c as in the following figure.



Three gamblers take one spinner each and spin their spinner. If the outcome on a is larger than b, then we say a beats b. If you are one of them, which spinner will you choose if you want to beat the other two? (*Hint: Find the probability of the three events that a beats b, b beats c and c beats a.*)

Sol :

$$\mathbb{P}(\{a \text{ beats } b\}) = \mathbb{P}(\{(9,8),(9,4),(9,3),(5,4),(5,3)\})$$
$$= 5 \times \frac{1}{3^2} = \frac{5}{9} \Rightarrow \mathbb{P}(\{b \text{ beats } a\}) = 1 - \frac{5}{9} = \frac{4}{9}$$

$$\mathbb{P}(\{a \text{ beats } c\}) = 1 - \mathbb{P}(\{c \text{ beats } a\}) = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\mathbb{P}(\{b \text{ beats } c\}) = \mathbb{P}(\{(8,7),(8,6),(8,2),(4,2),(3,2)\}) = 5 \times \frac{1}{3^2} = \frac{5}{9}$$

$$\mathbb{P}(\{c \text{ beats } a\}) = \mathbb{P}(\{(7,4),(7,3),(6,4),(6,3),(7,1)\}) = \frac{5}{9}$$

$$\mathbb{P}(\{c \text{ beats } b\}) = 1 - \mathbb{P}(\{b \text{ beats } c\}) = 1 - \frac{5}{9} = \frac{4}{9}$$

Therefore, we have

* if choose a, $\mathbb{P}(\{a \text{ beats } b \text{ and } c\}) \overset{\text{by independence}}{=} \mathbb{P}(a \text{ beats } b)\,\mathbb{P}(a \text{ beats } c) = \frac{4}{9} \times \frac{5}{9} = \frac{20}{81}$

* if chose b, $\mathbb{P}(\{b \text{ beats } a \text{ and } c\}) = \mathbb{P}(b \text{ beats } a\})\,\mathbb{P}(\{b \text{ beats } c\})$
$$= \frac{4}{9} \times \frac{5}{9} = \frac{20}{81}$$

* if chose c, $\mathbb{P}(\{c \text{ beats } a \text{ and } b\}) = \mathbb{P}(\{c \text{ beats } a\})\,\mathbb{P}(\{c \text{ beats } b\})$
$$= \frac{5}{9} \times \frac{4}{9} = \frac{20}{81}$$

Combining the results above, we have $\mathbb{P}(\{a \text{ beats } b \text{ and } c\}) = \mathbb{P}(\{b \text{ beats } a \text{ and } c\})$
$$= \mathbb{P}(\{c \text{ beats } a \text{ and } b\})$$

In conclusion, if I am one of the gamblers, I will choose any one spinner among a, b, c, because each spinner beats the other two equally probably.