MTH102 Engineering Mathematics II

Lesson 1: Permutations and combinations

Term: 2024

Outline

- Reference textbook: 1) *Miller & Freund's Probability and Statistics for Engineers*, A. R. Johnson
  2) *Introduction to Statistics and Data Analysis*, R. Peck, T. Short, C. Olsen
- Assessment
  - Coursework 1 (15%): online quiz, 7 days, due date: Week 7-8
  - Coursework 2 (15%): online quiz, 7 days, due date: Week 11-12
  - Final exam (70%): onsite exam, 2 hours, Final Exam Week
- Teaching material accessibility:
  1) The lecture slides and tutorial sheets will be uploaded on Learning Mall 1 week in advance usually.
  2) The solutions to tutorial sheets will be uploaded after tutorial class weekly
- Office hour: to be announced by your teacher

Outline

The basic principle of counting

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Example: There are 3 types of size for a pizza: small, medium and large, and 8 toppings. How many varieties of pizza can be made if we can choose one size with one topping?

Sol: $3 \times 8 = 24$

The generalized basic principle of counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiments; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if ..., then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Example: A combo meal consists of a starter, a main course and a dessert. A restaurant offers 4 starters, 5 main courses and 3 desserts. In how many ways a combo meal can be offered?

Sol : $4 \times 5 \times 3 = 60$

The generalized basic principle of counting

Example 1: For three distinct letters a,b,c how many ways are there to form a three-letter word? We distinguish between choosing

- with repetition: the same element can be chosen again.
- without repetition: only different elements can be chosen.

Example 2: How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers? And how many license plates would be possible if repetition among letters or numbers were prohibited?

1. $26^3 \times 10^4$
2. $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$

Permutations

- A **permutation** of objects is an ordered arrangement of these objects in a row.
- **Example**: given 3 letters a, b and c, there are 6 possible permutations by direct enumeration:

$$abc, \ acb, \ bac, \ bca, \ cab, \ cba.$$

Or by the basic principle, the first place in the permutation can be any of the 3, the second place in the permutation can then be chosen from any of the remaining 2, and the third place in the permutation is then the remaining 1. Thus, there are $3 \cdot 2 \cdot 1 = 6$ permutations.

- In general, given n objects, there are

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

different permutations of the n objects.

Permutations: example 1

Jim has 10 books to put on the bookshelf.

1. How many different arrangements are possible? $P_{10} = 10!$
2. Of these books, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. If all the books dealing with the same subject are together on the shelf, how many different arrangements are possible?

Sol: ① 10!

② There are 4 cathegories of books. There are 4! ways to permute the 4 cathegories and in each cathegory, there are resp. 4!, 3!, 2! and 1! ways to permute books. So in total, there are 4!4! 3! 2! 1! ways to arrange the books.

Permutations: exercise

Four married couples are sitting in a row. In how many ways can the 8 people be seated if each couple must sit together?

Sol: Similar to the last example, classify the 8 people into 4 groups (couples), then starting from arranging the 4 groups in $4!$ ways and in each group, there are $2!$ ways to arrange the couple. Hence, in total there are $4!(2!)^4$ ways to arrange the couples.

Permutations: example 2

Consider a four-digit password.

1. It is known that the four-digit password contains the four integers $6, 7, 8, 9$, which can only appear once (we call this *without replacement*). Then how many different passwords would be possible?
2. It is known that the four-digit password contains exactly the four integers $6, 7, 8, 8$ but the orders are unknown. Then how many different passwords would be possible?

$$P_4^{(1,1,2)} = \frac{4!}{2!} = \frac{4\times3\times2\times1}{2\times1} = 12$$

Permutations

In general, given a set of n objects of which certain are indistinguishable from each other. There are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots$, n_r are alike.

Example: If a box contains 3 red, 2 blue and 5 white marbles, in how many ways can we arrange them?

Solution. There are $n = 10$ marbles in total, of which $n_1 = 3$ are red, $n_2 = 2$ are blue and $n_3 = 5$ are white. Therefore the number of permutations is

$$\frac{10!}{3!2!5!} = 2520.$$

Permutations: example

- How many different letter arrangements can be made from the following letters
  1. XJTLU?
  2. Engineer?
  3. Mississippi?

① $P_5 = 5! = 5\times4\times3\times2\times1 = 120$

② $P_8^{(3,2,1,1)} = \dfrac{8!}{3!\times2!} = 3360$

③ $P_{11}^{(1,4,4,2)} = \dfrac{11!}{1!\,4!\,4!\,2!} = 34650$

Outline

Combinations

- By taking 3 objects at a time from 5 different objects, there are
$$5 \cdot 4 \cdot 3 = \frac{5!}{(5-3)!}$$
possible (ordered) arrangements.
- By taking 3 objects at a time from 5 different objects, there are
$$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{(5-3)!3!}$$
possible unordered combinations.

In general, $n(n-1)\cdots(n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant. And as each group of r items will be counted $r!$ times in this count, it follows that **the number of different groups of r items that could be formed from a set of n items** is
$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}.$$

Combinations

- For positive integers n and r with $r \leq n$, we define

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.

- **The binomial theorem**:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

- **Properties**:

$$\binom{n}{0} = 1, \ \binom{n}{1} = n, \ \binom{n}{k} = \binom{n}{n-k}, \ k = 0, \ldots, n, \ \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Combinations

Example: A committee of 4 is to be formed from a group of 10 people.

1. How many different committees are possible? $\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4!} = 210$
2. If there are 4 mathematicians and 6 physicists in the group, how many different committees consisting of 2 mathematicians and 2 physicists can be formed? $\binom{4}{2}\binom{6}{2} = \frac{4 \times 3}{2!} \times \frac{6 \times 5}{2!} = 90$
3. If there are 4 mathematicians and 6 physicists in the group, but 2 of the mathematicians are feuding and refuse to serve on the committee together, how many different committees consisting of 2 mathematicians and 2 physicists can be formed?

$$\left[\binom{4}{2} - 1\right] \times \binom{6}{2} = \left(\frac{4 \times 3}{2} - 1\right) \times \frac{6 \times 5}{2} = 75$$

Combinations

Exercise: A person has 8 friends, of whom 5 will be invited to a party.

1. How many choices are there if 2 of the friends are feuding and will not attend together? $\binom{8}{5} - \binom{6}{3} = \binom{8}{3} - \binom{6}{3} = \frac{8\times7\times6 - 6\times5\times4}{3!} = 36$
2. How many choices if 2 of the friends will only attend together?

$\binom{6}{3} = 20$

Outline

Probability of counting: example 1

Permutations and combinations are quite commonly used in finding probabilities in counting problems.

Problem: Compute the probability of obtaining exactly three '6's after rolling a fair ~~dice~~ 4 times.
die

Probability of counting: example 1

Solution 1:

There are 4 trials and 6 possible results for each. Then the total number of results is

$$6^4 = 1296.$$

The number of ways of arranging the three '6's in a row of 4 positions is $\binom{4}{3}$ (666x,66x6,6x66,x666). For the non '6' position, there are 5 possible result. Therefore there are $\binom{4}{3} * 5 = 20$ arrangements in total.

So the required probability is $\frac{20}{1296}$.

Probability of counting: example 1

Solution 2:

There are 4 trials and 6 possible results for each. Then the total number of results is

$$6^4 = 1296.$$

The number of ways of arranging the non '6' number in a row of 4 positions is $\binom{4}{1}$ (666x,66x6,6x66,x666), there are 5 possible result for this number. Therefore there are $\binom{4}{1} * 5 = 20$ arrangements in total.

So the required probability is $\frac{20}{1296}$.

Probability of counting: example 1

Solution 3:

There are four types of arrangements: 666x,66x6,6x66,x666. For each arrangement, 666x for example, the probability to get a '6' in a single rolling is $\frac{1}{6}$ and the probability to get a non '6' in a single rolling is $\frac{5}{6}$. Therefore each arrangement has a probability

$$\left(\frac{1}{6}\right)^3 \times \frac{5}{6} = \frac{5}{1296}.$$

So the required probability is $4 \times \frac{5}{1296} = \frac{20}{1296}$.

Probability of counting: example 2

Given a class of 6 girls and 5 boys.

(a) What is the probability that a committee of five, chosen at random from the class, consists of 3 girls and 2 boys?

(b) What is the probability that a committee of five, chosen at random from the class, consists of at least one boy?

Sol: Sample space $S = \{$all combinations of 5 objects chosen among 11$\}$

So $|S| = \binom{11}{5} = 462$

(a) $\mathbb{P}$(Committee of 5 people with 3 girls and 2 boys) $= \dfrac{\binom{6}{3} \times \binom{5}{2}}{\binom{11}{5}} = \dfrac{100}{231}$

(b) $\mathbb{P}$(At least 1 boy in the committee) $= \dfrac{\binom{11}{5} - \#\{\text{No boy}\}}{\binom{11}{5}} = 1 - \dfrac{\binom{6}{5}}{\binom{11}{5}} = \dfrac{228}{231}$

Open question

How many ways are there to distribute n sweets between person A and person B?

Sol: * If the n sweets are all different, then each sweet will have 2 choices, so in total there are 2^n ways to distribut sweets to A and B.

* If the n sweets are all the same, then we only need to count number of sweets distributed to A and B. That is the no. of non negative integer solutions (x, y) to the equation $x+y=n$. There are $n+1$ solutions.