Solution/Hints of Exercises

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Contents

Chapter 1

Scalar and Vector Fields

1.1. *Electric charge, energy associate with the electric field, electrical conductivity, capacitance are scalar quantities. The rest are vector quantities.*

1.2. *Assume point P has Cartesian Coordinates (x, y, z), Spherical Cooordinates (r, ϕ, θ) and Cylindrical coordinates (ρ, ϕ, z). Then the relation between these coordinates are*

$$x = r \sin\theta \cos\phi \tag{1.1a}$$
$$y = r \sin\theta \sin\phi \tag{1.1b}$$
$$z = r \cos\theta \tag{1.1c}$$

$$x = \rho \cos\phi \tag{1.2a}$$
$$y = \rho \sin\phi \tag{1.2b}$$
$$z = z \tag{1.2c}$$

To evaluate the products of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, it is better to express both vectors in terms of Cartesian coordinates. It is not a good idea to express them in terms of spherical coordinates. This is because the unit basis vector $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}$ are different at P and at Q.

1.3. *Pay attention to the minus sign in the third one:*

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \tag{1.3}$$

The last two about the products of three vectors. You could find information in the following webpage

https://ocw.mit.edu/ans7870/18/18.013a/textbook/HTML/chapter05/section06.html

1.4. *Similar to Example 1.2*

1.5. *There can be many possible answers. For example,*

P_1 *with Spherical coordinates* $r = 1, \phi = 0, \theta = 0$

P_2 *with Spherical coordinates* $r = 1, \phi = \pi, \theta = 0$

P_3 *with Spherical coordinates* $r = \sqrt{2}, \phi = 0, \theta = 0$

1.6. *See Figure 1.1, 1.2, and 1.3.*

You could learn how to do it by studying the 'sketch the vector field' under the topic "1. Scalar and Vector Fields" in the module page of LMO.

1.7. *a – (D), b – (C), c – (B), d – (A)*

1.8. *There can many possible answers. For example,*

1.
$$\boldsymbol{E}(P) = -\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} \tag{1.4}$$
for any point P *with Cartesian coordinates* (x, y)

2.
$$\boldsymbol{E}(P) = \hat{\boldsymbol{x}} \tag{1.5}$$
for any point P *with Cartesian coordinates* (x, y)

3.
$$\boldsymbol{E}(P) = \frac{1}{\sqrt{x^2 + y^2}} \left(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}\right) \tag{1.6}$$
for any point P *with Cartesian coordinates* (x, y)

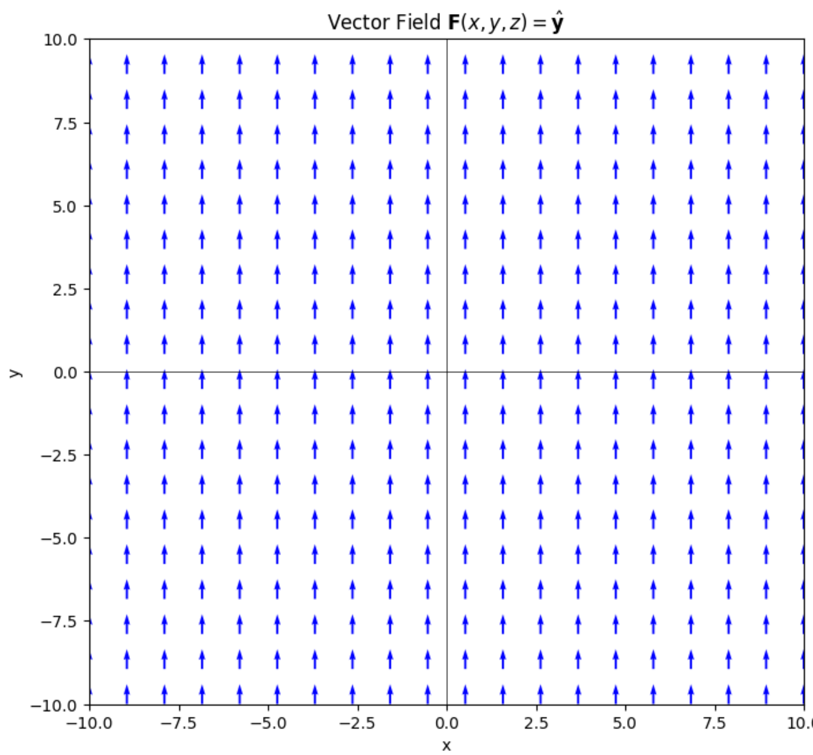

Vector Field $\mathbf{F}(x, y, z) = \hat{\mathbf{y}}$
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x

Figure 1.1

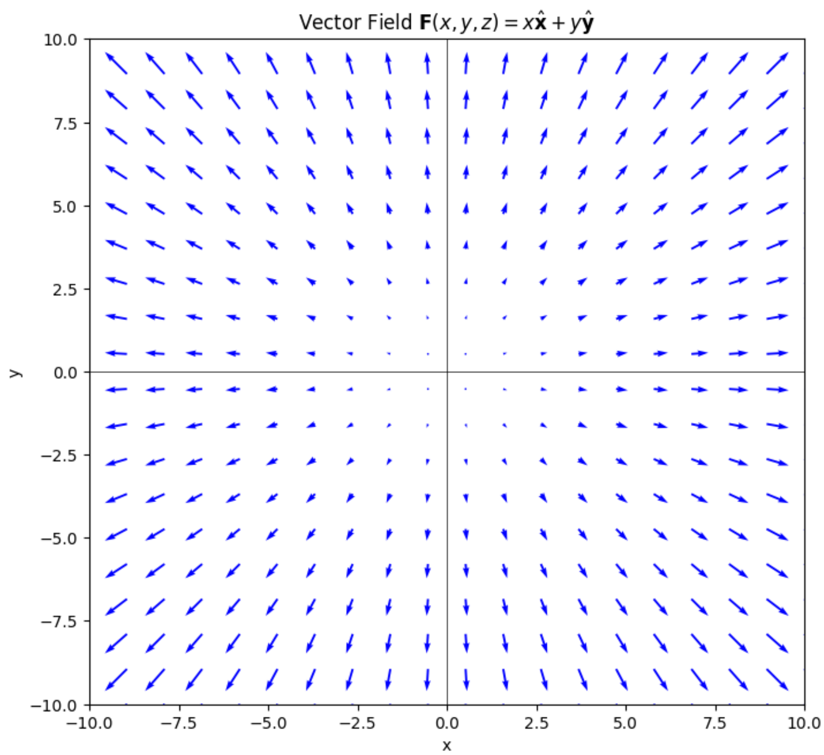
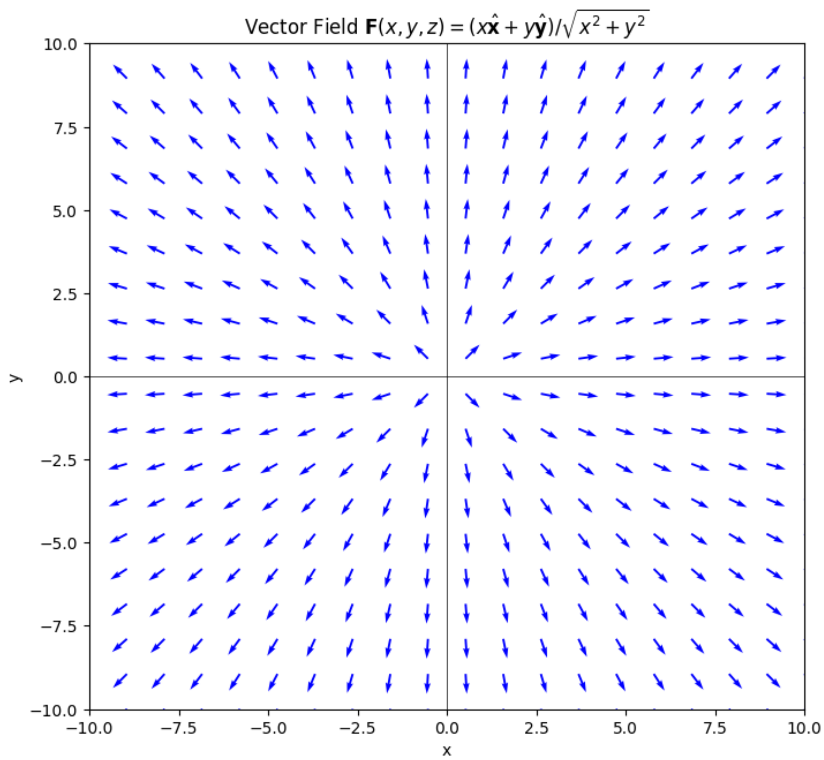

Vector Field $\mathbf{F}(x, y, z) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$
y
x
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

Figure 1.2



Vector Field $\mathbf{F}(x,y,z) = (x\hat{\mathbf{x}} + y\hat{\mathbf{y}})/\sqrt{x^2+y^2}$
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x

Figure 1.3

Chapter 2

Line Integral and Surface Integrals

2.1. *See Figure 2.1*

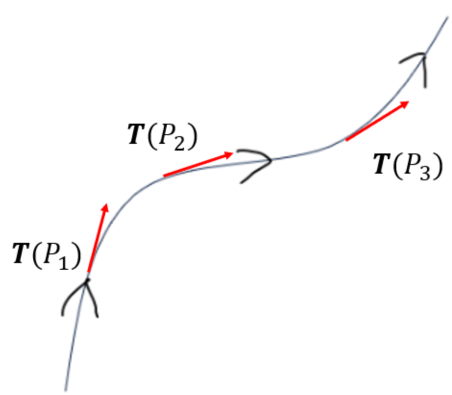


Figure 2.1

2.2. *For any point $P \in \Gamma$, we have*

$$\boldsymbol{B}(P) = \frac{\mu_0 I}{2\pi\rho}\boldsymbol{\phi}(\hat{P}) = \frac{\mu_0 I}{6\pi}\boldsymbol{\phi}(\hat{P}) \tag{2.1}$$

and

$$\boldsymbol{T}(P) = -\hat{\boldsymbol{\phi}}(P) \tag{2.2}$$

So

$$\begin{aligned}\oint_\Gamma \boldsymbol{B}\cdot\boldsymbol{T}\,\mathrm{d}s &= \oint_\Gamma \frac{\mu_0 I}{6\pi}\hat{\boldsymbol{\phi}}(P)\cdot\left(-\hat{\boldsymbol{\phi}}(P)\right)\,\mathrm{d}s \\ &= -\oint_\Gamma \frac{\mu_0 I}{6\pi}\,\mathrm{d}s = -\frac{\mu_0 I}{6\pi}\oint_\Gamma \mathrm{d}s = -\frac{\mu_0 I}{6\pi}\cdot(2\pi\cdot 3) = -\mu_0 I \qquad (2.3)\end{aligned}$$

2.3. *Figure 2.2 shows the relation between* $\boldsymbol{B}$ *and the unit tangent vector* $\boldsymbol{T}$ *in the plane* $z = 1$*. In (a), the horizontal straight line is* Γ *and circles give the direction of* $\boldsymbol{B}$ *at different places. According to (a), at each point* $P \in \Gamma$*, the angle between* $\boldsymbol{B}(P)$ *and* $\boldsymbol{T}(P)$ *is within the range* $[\pi/2, \pi]$*. Therefore*

$$\boldsymbol{B}(P)\cdot\boldsymbol{T}(P) < 0 \qquad (2.4)$$

for each point $P \in \Gamma$*. As a result,*

$$\int_\Gamma \boldsymbol{B}\cdot\boldsymbol{T}\,\mathrm{d}s < 0 \qquad (2.5)$$

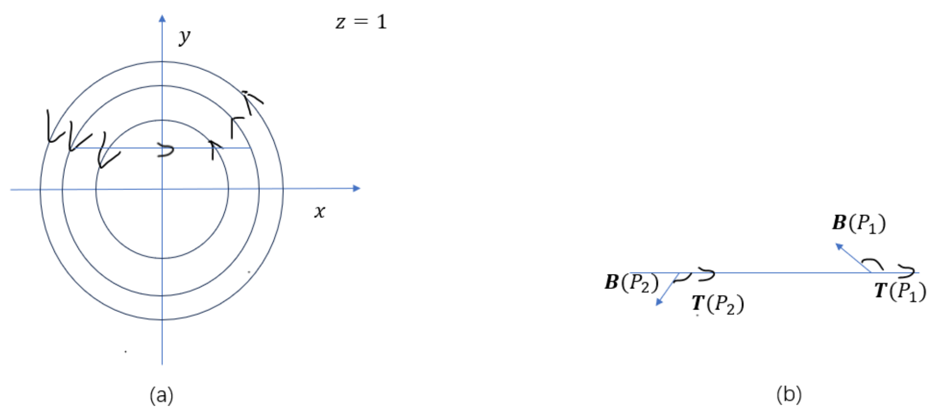


Figure 2.2

2.4.
($P_1 \to P_2$)
Let Γ_1 *be the straight line from* P_1 *to* P_2*. According to formula (2.26)*

$$\begin{aligned}\int_{\Gamma_1} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s} &= \int_{\Gamma_1} F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z \\ &= \int_{\Gamma_1} y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z \qquad (2.6)\end{aligned}$$

For any point $P \in \Gamma_1$, its x and y coordinates are related as

$$x(P) + \frac{y(P)}{2} = 1 \tag{2.7}$$

So

$$\int_{\Gamma_1} y \mathrm{d}x = \int_{\Gamma_1} 2(1-x)\mathrm{d}x. \tag{2.8}$$

As a point moves along Γ_1 from P_1 to P_2, its x-coordinate changes from $x = 1$ to $x = 0$. Therefore,

$$\int_{\Gamma_1} 2(1-x)\mathrm{d}x = \int_1^0 2(1-x)\mathrm{d}x \tag{2.9}$$

Overall,

$$\int_{\Gamma_1} y\mathrm{d}x = \int_{\Gamma_1} 2(1-x)\mathrm{d}x = \int_1^0 2(1-x)\mathrm{d}x = -1 \tag{2.10}$$

For any point $P \in \Gamma_1$, its z-coordinate is 0, therefore

$$\int_{\Gamma_1} z\mathrm{d}y = \int_{\Gamma_1} 0\,\mathrm{d}y = 0 \tag{2.11}$$

As a point moves along Γ_1, its z-coordinate does not change, therefore

$$\int_{\Gamma_1} x\mathrm{d}z = 0. \tag{2.12}$$

Overall,

$$\int_{\Gamma_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_{\Gamma_1} y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = -1 + 0 + 0 = -1 \tag{2.13}$$

($P_2 \to P_3$)

Let Γ_2 be the straight line segment from P_2 to P_3.

As a point moves along Γ_2, its x-coordinate does not change. Therefore

$$\int_{\Gamma_2} y\mathrm{d}x = 0. \tag{2.14}$$

For any point $P \in \Gamma_2$, its y and z coordinates are related as

$$\frac{y(P)}{2} + \frac{z(P)}{3} = 1. \tag{2.15}$$

$$\int_{\Gamma_2} z\mathrm{d}y = \int_{\Gamma_2} \left(3 - \frac{3}{2}y\right) \mathrm{d}y = \int_2^0 \left(3 - \frac{3}{2}y\right) \mathrm{d}y = -3 \tag{2.16}$$

For any point $P \in \Gamma_2$, its x-coordinate is always zero. Therefore

$$\int_{\Gamma_2} x\mathrm{d}z = 0. \tag{2.17}$$

Overall,

$$\int_{\Gamma_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_{\Gamma_2} y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = 0 + (-3) + 0 = -3 \tag{2.18}$$

($P_3 \to P_1$)

Let Γ_3 be the straight line segment from P_3 to P_1.

For any point on Γ_3, its y-coordinate is alway zero. Therefore

$$\int_{\Gamma_2} y\mathrm{d}x = 0. \tag{2.19}$$

As a point moves along Γ_3, its y does not change, therefore

$$\int_{\Gamma_3} z\mathrm{d}y = 0 \tag{2.20}$$

For any point $P \in \Gamma_2$, its x and z coordinates are related as

$$x(P) + \frac{z(P)}{3} = 1 \tag{2.21}$$

Therefore,

$$\int_{\Gamma_3} x\mathrm{d}z = \int_{\Gamma_3} \left(1 - \frac{z}{3}\right) \mathrm{d}z = \int_3^0 \left(1 - \frac{z}{3}\right) \mathrm{d}z = -3/2 \tag{2.22}$$

In summary,

$$\begin{aligned}\int_{\Gamma_1+\Gamma_2+\Gamma_3} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} &= \int_{\Gamma_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} + \int_{\Gamma_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} + \int_{\Gamma_3} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} \\ &= -1 + (-3) + (-3/2) = -11/2 \end{aligned} \tag{2.23}$$

2.5. *The force on the charge* q^* *is* $\boldsymbol{F} = q^*\boldsymbol{E}$.

According to the Example 2.9, the line integral of $\boldsymbol{F}$ *along a curve* Γ *from point* P_1 *to* P_2 *is*

$$\int_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \frac{q^* q}{4\pi\epsilon_0} \int_{r(P_1)}^{r(P_2)} \frac{1}{r^2}\,\mathrm{d}r = \frac{q^* q}{4\pi\epsilon_0}\left(\frac{1}{r(P_1)} - \frac{1}{r(P_2)}\right) \tag{2.24}$$

where

$$r(P_i) = |\overrightarrow{OP_i}|, i = 1, 2. \tag{2.25}$$

In this case,

$$r(P_1) = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \tag{2.26}$$

$$r(P_2) = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \tag{2.27}$$

Therefore, the work done is

$$\begin{aligned} \int_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} &= \frac{q^* q}{4\pi\epsilon_0}\left(\frac{1}{r(P_1)} - \frac{1}{r(P_2)}\right) \\ &= \frac{q^* q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{29}}\right) \end{aligned} \tag{2.28}$$

2.6.

(i) $\boldsymbol{n} = \hat{\boldsymbol{y}}$ *if the surface integral represents the current flowing through* S *from the side* $y < 2$ *to the side* $y > 2$. $\boldsymbol{n} = -\hat{\boldsymbol{y}}$ *if the surface integral represents the currnt flowing from the side* $y > 2$ *to the side* $y < 2$.

(ii) *They are the same as in (i).*

2.7. *Figure 2.3 shows the unit normal vector* $\boldsymbol{n}$ *at two different points on the surface* S. *In addition to be perpendicular to* S, *the* $\boldsymbol{n}$ *at each point also approximately points to the right.*

2.8. *Let*

S_1 *be the triangular surface with vertices* O, P_1, P_2.

S_2 *be the triangular surface with vertices* O, P_1, P_3

S_3 *be the triangular surface with vertices* O, P_2, P_3

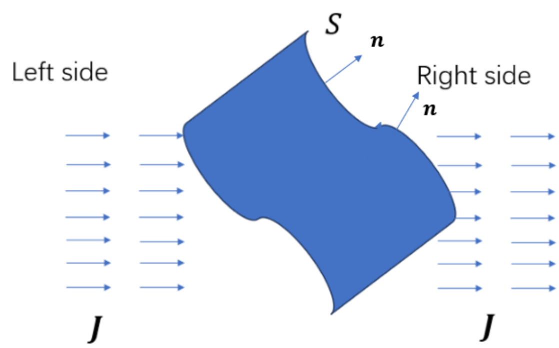


Figure 2.3

S_4 be the triangular surface with vertices P_1, P_2, P_3

(i) The flux of $\boldsymbol{E}$ through S_1 and into the tetrahedron is

$$\int_{S_1} \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_{S_1} \boldsymbol{E} \cdot \hat{\boldsymbol{z}} \, \mathrm{d}S = \int_{S_1} z \, \mathrm{d}S = \int_{S_1} 0 \, \mathrm{d}S = 0 \tag{2.29}$$

The third '=' is because the z-coordinate of any point on S_1 is zero
(ii) The flux of $\boldsymbol{E}$ through S_2 and into the tetrahedron is

$$\int_{S_2} \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_{S_2} \boldsymbol{E} \cdot \hat{\boldsymbol{y}} \, \mathrm{d}S = \int_{S_2} x \, \mathrm{d}S \tag{2.30}$$

Because S_2 lies on the xz-plane,

$$\int_{S_2} x \, \mathrm{d}S = \int_0^1 \left(\int_0^{3(1-x)} \mathrm{d}z \right) x \mathrm{d}x = \int_0^1 (3x - 3x^2) \, \mathrm{d}x = 1/2 \tag{2.31}$$

(iii) The flux of $\boldsymbol{E}$ through S_3 and into the tetrahedron is

$$\int_{S_3} \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_{S_3} \boldsymbol{E} \cdot \hat{\boldsymbol{x}} \, \mathrm{d}S = \int_{S_3} y \, \mathrm{d}S \tag{2.32}$$

Because S_3 lies on the yz-plane,

$$\int_{S_3} y \, \mathrm{d}S = \int_0^2 \left(\int_0^{3(1-y/2)} \mathrm{d}z \right) y \mathrm{d}y = \int_0^2 (3y - 3y^2/2) \, \mathrm{d}x = 2 \tag{2.33}$$

(iv) The flux of $\boldsymbol{E}$ through S_4 and into the tetrahedron is

$$\begin{aligned}\int_{S_4} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S &= \int_{S_4} \boldsymbol{E}\cdot\left((\boldsymbol{n}\cdot\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} + (\boldsymbol{n}\cdot\hat{\boldsymbol{y}})\hat{\boldsymbol{y}} + (\boldsymbol{n}\cdot\hat{\boldsymbol{z}})\hat{\boldsymbol{z}}\right)\,\mathrm{d}S \\ &= \int_{S_4} y\boldsymbol{n}\cdot\hat{\boldsymbol{x}}\,\mathrm{d}S + \int_{S_4} x\boldsymbol{n}\cdot\hat{\boldsymbol{y}}\,\mathrm{d}S + \int_{S_4} z\boldsymbol{n}\cdot\hat{\boldsymbol{z}}\,\mathrm{d}S \qquad (2.34)\end{aligned}$$

Since $\boldsymbol{n}$ is perpendicular to S_4 and points towards the origin,

$$\boldsymbol{n}\cdot\hat{\boldsymbol{x}}\,\mathrm{d}S = -\cos\theta\,\mathrm{d}S = -\mathrm{d}S_3 \qquad (2.35)$$

Here θ the angle between the surface S_4 and S_3. $\mathrm{d}S_3$ is the area of a small piece on S_3. This small piece is the projection of another small piece that is on S_4 and has area $\mathrm{d}S$. So

$$\int_{S_4} y\boldsymbol{n}\cdot\hat{\boldsymbol{x}}\,\mathrm{d}S = -\int_{S_3} y\mathrm{d}S_3 = -1 \qquad (2.36)$$

Similarly,

$$\int_{S_4} x\boldsymbol{n}\cdot\hat{\boldsymbol{y}}\,\mathrm{d}S = -\int_{S_2} x\mathrm{d}S_2 = -1/2 \qquad (2.37)$$

For any point $P \in S_4$, its Cartesian coordinates are related as

$$\frac{x(P)}{1} + \frac{y(P)}{2} + \frac{z(P)}{3} = 1 \qquad (2.38)$$

So

$$\begin{aligned}\int_{S_4} z\boldsymbol{n}\cdot\hat{\boldsymbol{z}}\,\mathrm{d}S &= \int_{S_4}\left(3\left(1-x-\frac{y}{2}\right)\right)\boldsymbol{n}\cdot\hat{\boldsymbol{z}}\,\mathrm{d}S \\ &= -\int_{S_1}\left(3\left(1-x-\frac{y}{2}\right)\right)\,\mathrm{d}S_1 = -\int_0^1\int_0^{2-2x}\left(3\left(1-x-\frac{y}{2}\right)\right)\,\mathrm{d}y\mathrm{d}x \\ &= -1 \qquad (2.39)\end{aligned}$$

Overall,

$$\int_{S_4} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S = -2 + (-1/2) + (-1) = -7/2 \qquad (2.40)$$

2.9.

(i) There can be many different answers. For example, let S be a region on the plane $z = 1$. If the area of S is $20\epsilon_0/\sigma$, then the flux of $\boldsymbol{E}$ through S in the direction of $\hat{\boldsymbol{z}}$ is

$$\int_S \boldsymbol{E} \cdot \hat{\boldsymbol{z}}\, \mathrm{d}S = \int_S \frac{\sigma}{2\epsilon_0} \hat{\boldsymbol{z}} \cdot \boldsymbol{z}\, \mathrm{d}S = \frac{\sigma}{2\epsilon_0} \int_S \mathrm{d}S = \frac{\sigma}{2\epsilon_0} \frac{20\epsilon_0}{\sigma} = 10 \tag{2.41}$$

(ii) According to Figure 2.4 , for any point $P \in S$, $\boldsymbol{E}(P)$ and $\boldsymbol{n}(P)$ forms an acute angle. This implies that

$$\boldsymbol{E}(P) \cdot \boldsymbol{n}(P) > 0 \tag{2.42}$$

for any $P \in S$. As a result,

$$\int_S \boldsymbol{E} \cdot \boldsymbol{n}\, \mathrm{d}S > 0 \tag{2.43}$$

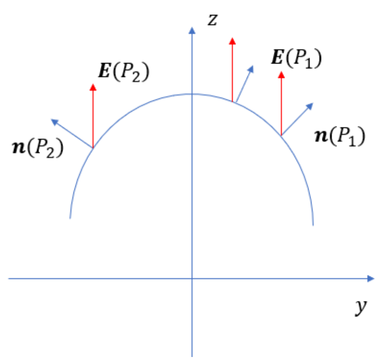


Figure 2.4

Chapter 3

Derivatives of Fields and Fundamental Theorems

3.1. *If P has Cartesian coordinates (x, y, z), then*

$$u(P) = -\frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{x^2 + y^2}. \tag{3.1}$$

Then

$$\frac{\partial u}{\partial z} = 0 \tag{3.2}$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{\lambda}{2\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2}} \cdot 2x \\ &= -\frac{\lambda}{2\pi\epsilon_0} \frac{x}{x^2 + y^2}\end{aligned} \tag{3.3}$$

and similarly,

$$\frac{\partial u}{\partial y} = -\frac{\lambda}{2\pi\epsilon_0} \frac{y}{x^2 + y^2}. \tag{3.4}$$

Overall,

$$\begin{aligned}\boldsymbol{E}(P) = -\nabla u &= -\left(\frac{\partial u}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial u}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial u}{\partial z}\hat{\boldsymbol{z}}\right) \\ &= \frac{\lambda}{2\pi\epsilon_0} \frac{1}{x^2 + y^2} \left(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}\right)\end{aligned} \tag{3.5}$$

3.2. *When $\boldsymbol{T} = \hat{\boldsymbol{r}}$, the directional derivative of u at P and in the direction of $\boldsymbol{T}$ is*

$$\frac{u(r+\mathrm{d}r,\theta,\phi)-u(r,\theta,\phi)}{\mathrm{d}r} = \frac{\partial u}{\partial r} \tag{3.6}$$

Therefore,

$$\mathit{grad}\, u \cdot \hat{\boldsymbol{r}} = \frac{\partial u}{\partial r} \tag{3.7}$$

When $\boldsymbol{T} = \hat{\boldsymbol{\theta}}$, the directional derivative of u at P and in the direction of $\boldsymbol{T}$ is

$$\frac{u(r,\theta+\mathrm{d}\theta,\phi)-u(r,\theta,\phi)}{r\mathrm{d}\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} \tag{3.8}$$

Therefore,

$$\mathit{grad}\, u \cdot \hat{\boldsymbol{\theta}} = \frac{1}{r}\frac{\partial u}{\partial \theta} \tag{3.9}$$

When $\boldsymbol{T} = \hat{\boldsymbol{\phi}}$, the directional derivative of u at P and in the direction of $\boldsymbol{T}$ is

$$\frac{u(r,\theta,\phi+\mathrm{d}\phi)-u(r,\theta,\phi)}{r\sin\theta\,\mathrm{d}\phi} = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} \tag{3.10}$$

Therefore,

$$\mathit{grad}\, u \cdot \hat{\boldsymbol{\phi}} = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} \tag{3.11}$$

Overall,

$$\begin{aligned} \mathit{grad}\, u &= (\mathit{grad}\, u \cdot \hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{r}} + (\mathit{grad}\, u \cdot \hat{\boldsymbol{\theta}}) \cdot \hat{\boldsymbol{\theta}} + (\mathit{grad}\, u \cdot \hat{\boldsymbol{\phi}}) \cdot \hat{\boldsymbol{\phi}} \\ &= \frac{\partial u}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\hat{\boldsymbol{\phi}} \end{aligned} \tag{3.12}$$

3.3. *Yes. According to the definition of gradient or the formula (3.5), if*

$$u(P) = v(P) + C \qquad \textit{for any } P, \tag{3.13}$$

then the scalar fields u and v have the same gradient.

3.4. *This scalar field must be a constant field. In other words, if this scalar field is denoted by u, then*

$$u(P) = c \qquad \textit{for any } P \tag{3.14}$$

where c is a constant.

3.5.

$$\begin{aligned}\nabla \times \nabla u &= \nabla \times \left(\frac{\partial u}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial u}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial u}{\partial z}\hat{\boldsymbol{z}}\right) \\ &= \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial z} - \frac{\partial}{\partial z}\frac{\partial u}{\partial y}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial}{\partial z}\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\frac{\partial u}{\partial z}\right)\hat{\boldsymbol{y}} \\ &+ \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial x}\frac{\partial u}{\partial y}\right)\hat{\boldsymbol{z}} \\ &= 0\hat{\boldsymbol{x}} + 0\hat{\boldsymbol{y}} + 0\hat{\boldsymbol{z}} = \mathbf{0} \end{aligned} \tag{3.15}$$

3.6. *Because*

$$r = \sqrt{x^2 + y^2 + z^2} \tag{3.16}$$

and

$$\hat{\boldsymbol{r}} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\sqrt{x^2 + y^2 + z^2}} \tag{3.17}$$

If P has Cartesian coordinates (x, y, z), then

$$\begin{aligned}\boldsymbol{E}(P) = kr^3\hat{\boldsymbol{r}} &= k(x^2 + y^2 + z^2)^{3/2}\frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\sqrt{x^2 + y^2 + z^2}} \\ &= k\left(x^2 + y^2 + z^2\right)(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) \\ &= kr^2(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) \end{aligned} \tag{3.18}$$

According to Gauss's law in differential form, the charge density at P is

$$\begin{aligned}\rho(P) = \epsilon_0\nabla \cdot \boldsymbol{E} &= \epsilon_0\left(\frac{\partial}{\partial x}(kr^2x) + \frac{\partial}{\partial y}(kr^2y) + \frac{\partial}{\partial z}(kr^2z)\right) \\ &= \epsilon_0 k\left(\left(2r\frac{\partial r}{\partial x}x + r^2\right) + \left(2r\frac{\partial r}{\partial y}y + r^2\right) + \left(2r\frac{\partial r}{\partial z}z + r^2\right)\right) \\ &= 5\epsilon_0 kr^2 = 5\epsilon_0 k\left(x^2 + y^2 + z^2\right) \end{aligned} \tag{3.19}$$

3.7. *There can be many answers. For example, if*

$$\boldsymbol{F}(P) = y\hat{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{G}(P) = x\hat{\boldsymbol{x}} \tag{3.20}$$

for any P with Cartesian coordinates (x, y, z), then

$$div\,\boldsymbol{F}(P) = 0 \quad \text{and} \quad div\,\boldsymbol{G}(P) = 1 \tag{3.21}$$

for any P.

3.8. *To ensure*

$$\oint \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_\Omega div\,\boldsymbol{E}\,\mathrm{d}\Omega, \tag{3.22}$$

the $\boldsymbol{E}$ *has to be continuously differentiable* **everywhere** *in* Ω.

The $\boldsymbol{E}$ *in Example 2.11 is not defined at the origin. The magnitude of vector* $\boldsymbol{E}(P)$ *approaches infinity, as* P *approaches the origin. These make* $\boldsymbol{E}$ *be not differentiable at the origin. So the divergence theorem does not hold for* $\boldsymbol{E}$ *over* Ω *which contains the origin.*

3.9.

$$\begin{aligned}\int_\Omega \nabla\cdot\boldsymbol{F}\,\mathrm{d}\Omega = \int_\Omega (2x+2y)\,\mathrm{d}\Omega = \int_0^1\int_0^1\int_0^1 2(x+y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ = \int_0^1\int_0^1 (2y+1)\mathrm{d}y\mathrm{d}z = \int_0^1 2\mathrm{d}z = 2\end{aligned} \tag{3.23}$$

According to Figure 3.1, the flux of $\boldsymbol{F}$ *through each of the six faces of the* *cube is*

$$\int_{S_{(i)}} \boldsymbol{F}\cdot\boldsymbol{n}_{(i)}\,\mathrm{d}S = \int_{S_{(i)}} (y^2\hat{\boldsymbol{x}} + (2xy+z^2)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}})\cdot\hat{\boldsymbol{y}}\,\mathrm{d}S$$
$$= \int_{S_{(i)}} (2xy+z^2)\,\mathrm{d}S = \int_0^1\int_0^1 (2x+z^2)\mathrm{d}x\mathrm{d}z = 4/3 \tag{3.24}$$

$$\int_{S_{(ii)}} \boldsymbol{F}\cdot\boldsymbol{n}_{(ii)}\,\mathrm{d}S = \int_{S_{(ii)}} (y^2\hat{\boldsymbol{x}} + (2xy+z^2)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}})\cdot(-\hat{\boldsymbol{x}})\,\mathrm{d}S$$
$$= -\int_{S_{(ii)}} y^2\,\mathrm{d}S = -\int_0^1\int_0^1 y^2\mathrm{d}y\mathrm{d}z = -1/3 \tag{3.25}$$

$$\int_{S_{(iii)}} \boldsymbol{F}\cdot\boldsymbol{n}_{(iii)}\,\mathrm{d}S = \int_{S_{(iii)}} (y^2\hat{\boldsymbol{x}} + (2xy+z^2)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}})\cdot\hat{\boldsymbol{z}}\,\mathrm{d}S$$
$$= \int_{S_{(iii)}} 2yz\,\mathrm{d}S = \int_0^1\int_0^1 2y\mathrm{d}y\mathrm{d}x = 1 \tag{3.26}$$

$$\int_{S_{(iv)}} \boldsymbol{F}\cdot\boldsymbol{n}_{(iv)}\,\mathrm{d}S = \int_{S_{(iv)}} (y^2\hat{\boldsymbol{x}} + (2xy+z^2)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}})\cdot(-\hat{\boldsymbol{z}})\,\mathrm{d}S$$
$$= \int_{S_{(iv)}} 2yz\,\mathrm{d}S = 0 \tag{3.27}$$

$$\int_{S_{(v)}} \boldsymbol{F}\cdot\boldsymbol{n}_{(v)}\,\mathrm{d}S = \int_{S_{(v)}} (y^2\hat{\boldsymbol{x}} + (2xy+z^2)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}})\cdot\hat{\boldsymbol{x}}\,\mathrm{d}S$$
$$= \int_{S_{(v)}} y^2\,\mathrm{d}S = \int_0^1\int_0^1 y^2\mathrm{d}y\mathrm{d}z = 1/3 \tag{3.28}$$

$$\int_{S_{(vi)}} \boldsymbol{F}\cdot\boldsymbol{n}_{(vi)}\,\mathrm{d}S = \int_{S_{(vi)}} (y^2\hat{\boldsymbol{x}} + (2xy+z^2)\hat{\boldsymbol{y}} + (2yz)\hat{\boldsymbol{z}})\cdot(-\hat{\boldsymbol{y}})\,\mathrm{d}S$$
$$= -\int_{S_{(vi)}} (2xy+z^2)\,\mathrm{d}S = \int_0^1\int_0^1 z^2\mathrm{d}x\mathrm{d}z = -1/3 \tag{3.29}$$

Overall,

$$\oint_{\partial\Omega} \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S = \frac{4}{3} + (-\frac{1}{3}) + 1 + 0 + \frac{1}{3} + (-\frac{1}{3}) = 2$$
$$= \int_\Omega \nabla\cdot\boldsymbol{F}\,\mathrm{d}\Omega \tag{3.30}$$

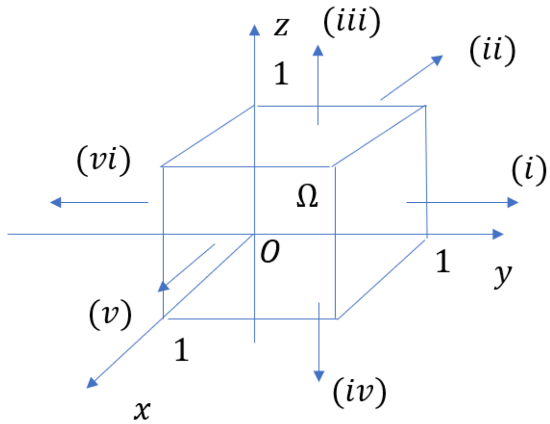


Figure 3.1: The six faces of the cube with the direction of outward normal vector.

3.10. *They are the same. Let S_{12} be the surface that is shared by Ω_1 and Ω_2 as shown in Figure 3.2. Then*

$$\partial\Omega = \partial\Omega_1 \backslash S_{12} + \partial\Omega_2 \backslash S_{12} \tag{3.31}$$

Here $\partial\Omega_i \backslash S_{12}$ denotes the set of points which are on $\partial\Omega_i$ but not on $S_{12}, i = 1, 2$.

Therefore,

$$\begin{aligned}
\int_{\partial\Omega_1} \boldsymbol{F}\cdot\boldsymbol{n}_1 \,\mathrm{d}S + \int_{\partial\Omega_2} \boldsymbol{F}\cdot\boldsymbol{n}_2 \,\mathrm{d}S &= \int_{\partial\Omega_1\backslash S_{12}} \boldsymbol{F}\cdot\boldsymbol{n}_1 \,\mathrm{d}S + \int_{S_{12}} \boldsymbol{F}\cdot\boldsymbol{n}_{12} \,\mathrm{d}S \\
&\quad + \int_{\partial\Omega_2\backslash S_{12}} \boldsymbol{F}\cdot\boldsymbol{n}_2 \,\mathrm{d}S + \int_{S_{12}} \boldsymbol{F}\cdot(-\boldsymbol{n}_{12}) \,\mathrm{d}S \\
&= \int_{\partial\Omega_1\backslash S_{12}} \boldsymbol{F}\cdot\boldsymbol{n}_1 \,\mathrm{d}S + \int_{\partial\Omega_2\backslash S_{12}} \boldsymbol{F}\cdot\boldsymbol{n}_2 \,\mathrm{d}S \\
&= \int_{\partial\Omega} \boldsymbol{F}\cdot\boldsymbol{n} \,\mathrm{d}S
\end{aligned} \tag{3.32}$$

3.11. *Check if the vector field is curl-free. If so, then its line integral is path-independent. Otherwise, its line integral is path-dependent.*

3.12. *Assume*

$$\boldsymbol{F} = F_1\hat{\boldsymbol{x}} + F_2\hat{\boldsymbol{y}} + F_3\hat{\boldsymbol{z}}. \tag{3.33}$$

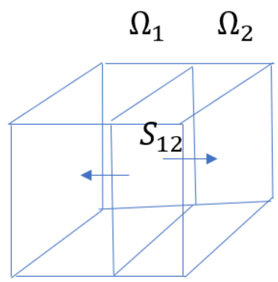


Figure 3.2

Then

$$\begin{aligned}\nabla \cdot (\nabla \times \boldsymbol{F}) &= \nabla \cdot \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{\boldsymbol{y}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\boldsymbol{z}} \right) \\ &= \left(\frac{\partial^2 F_3}{\partial y \partial x} - \frac{\partial^2 F_2}{\partial z \partial x} \right) + \left(\frac{\partial^2 F_1}{\partial z \partial y} - \frac{\partial^2 F_3}{\partial x \partial y} \right) + \left(\frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y \partial z} \right) = 0\end{aligned} \tag{3.34}$$

3.13. *There can be many answers. If*

$$\boldsymbol{F}(P) = x\hat{\boldsymbol{x}} \quad \textit{and} \quad \boldsymbol{G}(P) = y\hat{\boldsymbol{z}} \tag{3.35}$$

for any P with Cartesian coordinates (x, y, z), then

$$\nabla \times \boldsymbol{F} = \mathbf{0} \quad \textit{and} \quad \nabla \times \boldsymbol{G} = \hat{\boldsymbol{x}} \tag{3.36}$$

3.14. *There can be many answers. For example, if*

$$\boldsymbol{F}(P) = yz\hat{\boldsymbol{x}} + xz\hat{\boldsymbol{y}} + xy\hat{\boldsymbol{z}} \tag{3.37}$$

for any P with Cartesian coordiantes (x, y, z), then it can be shown that

$$\nabla \cdot \boldsymbol{F} = 0 \quad \textit{and} \quad \nabla \times \boldsymbol{F} = \mathbf{0} \tag{3.38}$$

3.15.

$$\int_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s = \int_{\Gamma_1} \big((2xz + 3y^2)\hat{\boldsymbol{y}} + (4yz)\hat{\boldsymbol{z}}\big) \cdot \hat{\boldsymbol{z}} \, \mathrm{d}s$$

$$= \int_{\Gamma_1} 4yz\mathrm{d}s = \int_0^1 4z\mathrm{d}z = 2 \tag{3.39}$$

$$\int_{\Gamma_2} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s = \int_{\Gamma_2} \big((2xz + 3y^2)\hat{\boldsymbol{y}} + (4yz)\hat{\boldsymbol{z}}\big) \cdot (-\hat{\boldsymbol{y}}) \, \mathrm{d}s$$

$$= -\int_{\Gamma_2} (2xz + 3y^2) \, \mathrm{d}s = \int_0^1 3y^2 \, \mathrm{d}y = -1 \tag{3.40}$$

$$\int_{\Gamma_3} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s = \int_{\Gamma_3} \big((2xz + 3y^2)\hat{\boldsymbol{y}} + (4yz)\hat{\boldsymbol{z}}\big) \cdot (-\hat{\boldsymbol{z}}) \, \mathrm{d}s$$

$$= -\int_{\Gamma_3} 4yz \, \mathrm{d}s = 0$$

$$\int_{\Gamma_4} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s = \int_{\Gamma_4} \big((2xz + 3y^2)\hat{\boldsymbol{y}} + (4yz)\hat{\boldsymbol{z}}\big) \cdot \hat{\boldsymbol{y}} \, \mathrm{d}s$$

$$= \int_{\Gamma_4} (2xz + 3y^2) \, \mathrm{d}s = \int_0^1 3y^2 \, \mathrm{d}y = 1 \tag{3.41}$$

Overall,

$$\oint_{\Gamma} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s = 2 + (-1) + 0 + 1 = 2 = \tag{3.42}$$

On the other hand,

$$\nabla \times \boldsymbol{F} = \left(\frac{\partial}{\partial y}(2xz + 3y^2) - \frac{\partial}{\partial z}(4yz)\right) \hat{\boldsymbol{x}} + \left(\frac{\partial}{\partial x}(2xz + 3y^2) - 0\right) \hat{\boldsymbol{z}}$$

$$= 2y\hat{\boldsymbol{x}} + 2z\hat{\boldsymbol{z}} \tag{3.43}$$

Then

$$\int_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n} \, \mathrm{d}S = \int_S (2y\hat{\boldsymbol{x}} + 2z\hat{\boldsymbol{z}}) \cdot \hat{\boldsymbol{x}} \, \mathrm{d}S = \int_S 2y \, \mathrm{d}S$$

$$= \int_0^1 \int_0^1 2y\mathrm{d}z\mathrm{d}y = 2 = \oint_{\Gamma} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s \tag{3.44}$$

3.16. *They are the same. Let*

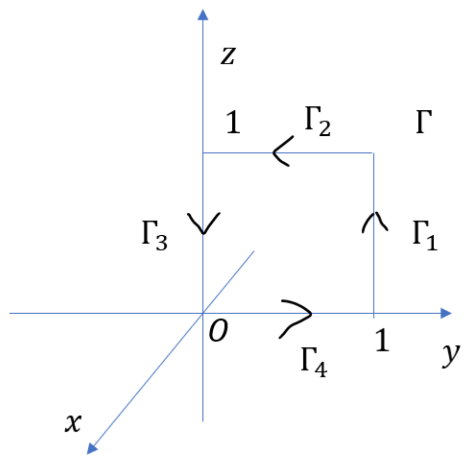


Figure 3.3

$\hat{\Gamma}_1$ be the path $D \to C \to O \to E$.

$\hat{\Gamma}_2$ be the path $E \to A \to B \to D$.

Γ_{12} be the path $E \to D$

Then

$$\oint_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{T}_1 \, \mathrm{d}s = \oint_{\hat{\Gamma}_1} \boldsymbol{F} \cdot \boldsymbol{T}_1 \, \mathrm{d}s + \int_{\Gamma_{12}} \boldsymbol{F} \cdot \boldsymbol{T}_1 \mathrm{d}s \tag{3.45}$$

$$\oint_{\Gamma_2} \boldsymbol{F} \cdot \boldsymbol{T}_2 \, \mathrm{d}s = \oint_{\hat{\Gamma}_2} \boldsymbol{F} \cdot \boldsymbol{T}_2 \, \mathrm{d}s + \int_{\Gamma_{12}} \boldsymbol{F} \cdot \boldsymbol{T}_2 \mathrm{d}s \tag{3.46}$$

Because

$$\int_{\Gamma_{12}} \boldsymbol{F} \cdot \boldsymbol{T}_1 \mathrm{d}s = -\int_{\Gamma_{12}} \boldsymbol{F} \cdot \boldsymbol{T}_2 \mathrm{d}s, \tag{3.47}$$

$$\begin{aligned}\oint_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{T}_1 \, \mathrm{d}s + \oint_{\Gamma_2} \boldsymbol{F} \cdot \boldsymbol{T}_2 \, \mathrm{d}s &= \oint_{\hat{\Gamma}_1} \boldsymbol{F} \cdot \boldsymbol{T}_1 \, \mathrm{d}s + \oint_{\hat{\Gamma}_1} \boldsymbol{F} \cdot \boldsymbol{T}_1 \, \mathrm{d}s \\ &= \oint_{\Gamma} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s \end{aligned} \tag{3.48}$$

3.17. *The vector fileds in (A) and (B) are curl-free. The vector fields in (C) and (D) are divergence free.*

These vector fields are actually the ones in Exercise 1.7.

Another Exercise on the Applications of Divergence Theorem. Use the divergence theorem to find the surface integral of the vector field (2.68) through the face $\overline{P_1P_2P_3}$ in Exercise 2.8.

Solution. Let Ω be the tetrahedron $\overline{OP_1P_2P_3}$. Then its boundary is

$$\partial\Omega = S_1 \cup S_2 \cup S_3 \cup S_4 \tag{3.49}$$

with

S_1 be the triangular surface with vertices O, P_1, P_2.

S_2 be the triangular surface with vertices O, P_1, P_3

S_3 be the triangular surface with vertices O, P_2, P_3

S_4 be the triangular surface with vertices P_1, P_2, P_3

So the surface integral to be computed is

$$\int_{S_4} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{\partial\Omega} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_{S_1} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_{S_2} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_{S_3} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S \tag{3.50}$$

Here $\boldsymbol{n}$ is the unit normal vector pointing to the interior of the tetrahedron.

$$\int_{S_1} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{S_1} \boldsymbol{E}\cdot\hat{\boldsymbol{z}}\,\mathrm{d}S = \int_{S_1} z\,\mathrm{d}S = \int_{S_1} 0\mathrm{d}S = 0 \tag{3.51}$$

$$\int_{S_2} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{S_2} \boldsymbol{E}\cdot\hat{\boldsymbol{y}}\,\mathrm{d}S = \int_{S_2} x\,\mathrm{d}S = \int_0^1 \left(\int_0^{3(1-x)} \mathrm{d}z\right) x\,\mathrm{d}x = 1/2 \tag{3.52}$$

$$\int_{S_3} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{S_3} \boldsymbol{E}\cdot\hat{\boldsymbol{x}}\,\mathrm{d}S = \int_{S_3} y\,\mathrm{d}S = \int_0^2 \left(\int_0^{3(1-y/2)} \mathrm{d}z\right) y\mathrm{d}y = 2 \tag{3.53}$$

According to the divergence theorem,

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S &= -\int_{\partial\Omega} \boldsymbol{E}\cdot(-\boldsymbol{n})\,\mathrm{d}S = -\int_{\Omega} \nabla\cdot\boldsymbol{E}\,\mathrm{d}\Omega \\ &= -\int_{\Omega} \left(\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial z}{\partial z}\right) \mathrm{d}\Omega = -\int_{\Omega} 1\,\mathrm{d}\Omega = -|\Omega| = -1\end{aligned} \tag{3.54}$$

Overall,

$$\int_{S_4} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S = -1 - (0 + 1/2 + 2) = -7/2 \tag{3.55}$$

□

Chapter 4

Introduction to PDE

4.1. *(1)*

$$u(x,t) = t, \quad 0 \le x \le 1, t \ge 0 \tag{4.1}$$

is a solution of PDE (4.12).

The general solutin of PDE (4.12) is

$$u(x,t) = \phi(t) \quad 0 \le x \le 1, t \ge 0, \tag{4.2}$$

Here $\phi = \phi(t)$ *is some arbitrary function of* t *that is defined on* $t \ge 0$.

(2)

$$u(x,t) = x + t \qquad 0 \le x \le 1, t \ge 0 \tag{4.3}$$

is a solution of PDE (4.13). The general solution of PDE (4.13) is

$$u(x,t) = \phi(x) + t \qquad 0 \le x \le 1, t \ge 0 \tag{4.4}$$

Here $\phi = \phi(x)$ *is an arbitray function of* x *and defined on* $0 \le x \le 1$.

(3)

$$u(x,t) = tx + 1 \qquad 0 \le x \le 1, t \ge 0 \tag{4.5}$$

is a solution of PDE (4.14). The general solution of PDE (4.14) is

$$u(x,t) = tx + \phi(x) \qquad 0 \le x \le 1, t \ge 0 \tag{4.6}$$

Here $\phi = \phi(x)$ *is an arbitrary function of* x *and defined on* $0 \le x \le 1$.

(4)

$$u(x,t) = \frac{t^2}{2} + x \qquad 0 \le x \le 1, t \ge 0 \tag{4.7}$$

is a solution of PDE (4.15). The general solution of PDE (4.15) is

$$u(x,t) = \frac{t^2}{2} + \phi(x) \qquad 0 \le x \le 1, t \ge 0. \tag{4.8}$$

$$u(x,t) = xe^t \qquad 0 \le x \le 1, t \ge 0 \tag{4.9}$$

is a solution of PDE (4.16). The general solution of PDE (4.16) is

$$u(x,t) = \phi(x)e^t \qquad 0 \le x \le 1, t \ge 0 \tag{4.10}$$

The PDE (4.12) means that at any instant, the temperature is the same everyhwere on the rod.

(Followings are the solutions of 4.1 with more details, **Optional***)*
Let

$$v(t) \equiv u(x_0, t) \quad t \ge 0 \tag{4.11}$$

where x_0 is any constant within $(-\infty, \infty)$.
(1) With (4.11)*,*

$$u_x = 0 \quad -\infty < x < \infty, t > 0 \tag{4.12}$$

implies

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \quad t > 0 \tag{4.13}$$

whose solution is

$$v(t) = v(0) \quad t \ge 0 \tag{4.14}$$

Given (4.11)*, the* (4.14) *means*

$$u(x_0, t) = u(x_0, 0) \qquad t \ge 0 \tag{4.15}$$

Because x_0 can be any constant within $(-\infty, \infty)$, (4.15) *means*

$$u(x,t) = u(x,0) \qquad -\infty < x < \infty, t \ge 0 \tag{4.16}$$

The (4.16) *is the solution of PDE* (4.12)*.*

(2) With (4.11),

$$u_t = 1 \quad -\infty < x < \infty, t > 0 \tag{4.17}$$

implies

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 1 \quad t > 0 \tag{4.18}$$

whose solution is

$$v(t) = t + v(0) \quad t \geq 0 \tag{4.19}$$

Given (4.11), *the* (4.19) *means*

$$u(x_0, t) = t + u(x_0, 0) \qquad t \geq 0 \tag{4.20}$$

Because x_0 *can be any constant within* $(-\infty, \infty)$, (4.20) *means*

$$u(x, t) = t + u(x, 0) \qquad -\infty < x < \infty, t \geq 0 \tag{4.21}$$

The (4.21) *is the solution of* (4.17).

(3)

$$u_t = x \quad -\infty < x < \infty, t > 0 \tag{4.22}$$

implies

$$u_t(x_0, t) = x_0 \quad t > 0. \tag{4.23}$$

With (4.11), *the* (4.23) *means*

$$\frac{\mathrm{d}v}{\mathrm{d}t} = x_0 \quad t > 0 \tag{4.24}$$

whose solution is

$$v(t) = x_0 t + v(0) \quad t \geq 0 \tag{4.25}$$

Given (4.11), *the* (4.25) *means*

$$u(x_0, t) = x_0 t + u(x_0, 0) \qquad t \geq 0 \tag{4.26}$$

Because x_0 *can be any constant within* $(-\infty, \infty)$, (4.26) *means*

$$u(x, t) = xt + u(x, 0) \qquad -\infty < x < \infty, t \geq 0 \tag{4.27}$$

The (4.27) *is the solution of* (4.22).

(4) With (4.11),

$$u_t = t \qquad 0 < x < 1, t > 0 \tag{4.28}$$

implies

$$\frac{\mathrm{d}v}{\mathrm{d}t} = t \qquad t > 0 \tag{4.29}$$

whose solution is

$$v(t) = \frac{t^2}{2} + v(0) \qquad t > 0 \tag{4.30}$$

The (4.30) *means*

$$u(x_0, t) = \frac{t^2}{2} + u(x_0, 0) \qquad t > 0 \tag{4.31}$$

Since x_0 *can be any constant within* $(-\infty, \infty)$*, the* (4.31) *means*

$$u(x, t) = \frac{t^2}{2} + u(x, 0) \qquad -\infty < x < \infty, t > 0 \tag{4.32}$$

The (4.32) *is the solution of PDE* (4.28)*.*

(5)

$$u_t = u \quad -\infty < x < \infty, t > 0 \tag{4.33}$$

implies

$$u_t(x_0, t) = u(x_0, t) \quad t > 0. \tag{4.34}$$

With (4.11)*, the* (4.34) *means*

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v \quad t > 0 \tag{4.35}$$

whose solution is

$$v(t) = v(0)e^t \quad t \geq 0 \tag{4.36}$$

Given (4.11)*, the* (4.36) *means*

$$u(x_0, t) = u(x_0, 0)e^t \qquad t \geq 0 \tag{4.37}$$

Because x_0 *can be any constant within* $(-\infty, \infty)$*,* (4.37) *means*

$$u(x, t) = u(x, 0)e^t \qquad -\infty < x < \infty, t \geq 0 \tag{4.38}$$

The (4.38) *is the solution of PDE* (4.33)

4.2. *(1)*
$$u(x,y,z,t) = y + z + t, \quad 0 \le x,y,z \le 1, t \ge 0 \tag{4.39}$$
is a solution of PDE (4.17).

The general solutin of PDE (4.17) is
$$u(x,y,z,t) = \phi(y,z,t) \quad 0 \le x,y,z \le 1, t \ge 0, \tag{4.40}$$
Here $\phi = \phi(y,z,t)$ *is some arbitrary function of* y,z,t *that is defined on* $0 \le y,z \le 1, t \ge 0.$

(2)
$$u(x,y,z,t) = x + y + z + t \qquad 0 \le x,y,z \le 1, t \ge 0 \tag{4.41}$$
is a solution of PDE (4.18). The general solution of PDE (4.18) is
$$u(x,y,z,t) = \phi(x,y,z) + t \qquad 0 \le x,y,z \le 1, t \ge 0 \tag{4.42}$$
Here $\phi = \phi(x,y,z)$ *is an arbitray function of* x,y,z *and defined on* $0 \le x,y,z \le 1.$

(3)
$$u(x,y,z,t) = tx + xyz \qquad 0 \le x,y,z \le 1, t \ge 0 \tag{4.43}$$
is a solution of PDE (4.19). The general solution of PDE (4.19) is
$$u(x,y,z,t) = tx + \phi(x,y,z) \qquad 0 \le x,y,z \le 1, t \ge 0 \tag{4.44}$$
Here $\phi = \phi(x,y,z)$ *is an arbitrary function of* x,y,z *and defined on* $0 \le x,y,z \le 1.$

(4)
$$u(x,y,z,t) = \frac{t^2}{2} + xyz \qquad 0 \le x,y,z \le 1, t \ge 0 \tag{4.45}$$
is a solution of PDE (4.20). The general solution of PDE (4.20) is
$$u(x,y,z,t) = \frac{t^2}{2} + \phi(x,y,z) \qquad 0 \le x,y,z \le 1, t \ge 0. \tag{4.46}$$
Here $\phi = \phi(x,y,z)$ *is an arbitrary function of* x,y,z *and defined on* $0 \le x,y,z \le 1.$

(5)
$$u(x,y,z,t) = xyze^t \qquad 0 \le x,y,z \le 1, t \ge 0 \tag{4.47}$$
is a solution of PDE (4.21). The general solution of PDE (4.21) is
$$u(x,y,z,t) = \phi(x,y,z)e^t \qquad 0 \le x,y,z \le 1, t \ge 0 \tag{4.48}$$
Here $\phi = \phi(x,y,z)$ *is an arbitrary function of* x,y,z *and defined on* $0 \le x,y,z \le 1.$

Chapter 5

1D Advection Equation

5.1. *The conservation of chemical means that the increase in the amount of the chemical in each portion of the tube is always equal to the amount of chemical that flows into this portion, minus the amount that flows out of this portion.*

Assume the tube is a right cylinder and its cross-section has area A. Consider a short portion of the tube between $x = x_0$ and $x = x_0 + \Delta x, \Delta x > 0$. In terms of ϕ, the increase of chemical in this portion during a short time period from $t = t_0$ to $t = t_0 + \Delta t, \Delta t > 0$ is approximately

$$(\phi(x_0, t_0) - \phi(x_0 + \Delta x, t_0))A\Delta t \tag{5.1}$$

In terms of u, the increase is approximately

$$(u(x_0, t_0 + \Delta t) - u(x_0, t_0))A\Delta x \tag{5.2}$$

Since both (5.1) *and* (5.2) *represent the same amount, they should be equal to each other. So we have*

$$(\phi(x_0, t_0) - \phi(x_0 + \Delta x, t_0))A\Delta t = (u(x_0, t_0 + \Delta t) - u(x_0, t_0))A\Delta x \tag{5.3}$$

or equivalently

$$\frac{\phi(x_0, t_0) - \phi(x_0 + \Delta x, t_0)}{\Delta x} = \frac{u(x_0, t_0 + \Delta t) - u(x_0, t_0)}{\Delta t} \tag{5.4}$$

As $\Delta t \to 0, \Delta x \to 0$, (5.4) *becomes*

$$-\phi_x(x_0, t_0) = u_t(x_0, t_0) \tag{5.5}$$

Since (5.5) *holds for any x_0 and t_0, we obtain 5.3.*

5.2. *The following functions are*

$$u(x,t) = x + ct \qquad -\infty < x < \infty, t \geq 0 \tag{5.6}$$
$$u(x,t) = (x + ct)^2 \qquad -\infty < x < \infty, t \geq 0 \tag{5.7}$$
$$u(x,t) = \sin(x + ct) \qquad -\infty < x < \infty, t \geq 0 \tag{5.8}$$
$$u(x,t) = e^{x+ct} \qquad -\infty < x < \infty, t \geq 0 \tag{5.9}$$

some solutions of PDE (5.2).

5.3. *If u and $\hat{u}$ are solutions of PDE (5.2), then*

$$u_t = cu_x \qquad -\infty < x < \infty, t > 0 \tag{5.10}$$
$$\hat{u}_t = c\hat{u}_x \qquad -\infty < x < \infty, t > 0 \tag{5.11}$$

So

$$(c_1 u + c_2 \hat{u})_t = c_1 u_t + c_2 \hat{u}_t = c_1 c u_x + c_2 c \hat{u}_x = c(c_1 u_x + c_2 \hat{u}_x) \tag{5.12}$$

That is to say, $c_1 u + c_2 \hat{u}$ satisfies PDE (5.2).

5.4. *If*

$$u(x,t) \equiv f(x + ct) \qquad -\infty < x < \infty, t \geq 0, \tag{5.13}$$

then

$$u_x = f'(x + ct) \qquad u_t = f'(x + ct) \cdot c \quad -\infty < x < \infty, t > 0 \tag{5.14}$$

So

$$u_t = cu_x \qquad -\infty < x < \infty, t > 0 \tag{5.15}$$

i.e. u defined in (5.13) *is a solution of PDE (5.2)*

5.5. *The three profiles of u against x at three diferent time are sketched in Figure* **??**.

5.6. *(i) The equation*

$$u_t = -2tu_x \qquad -\infty < x < \infty, t > 0 \tag{5.16}$$

is a result of

$$u_t = -\phi_x \qquad -\infty < x < \infty, t > 0 \tag{5.17}$$

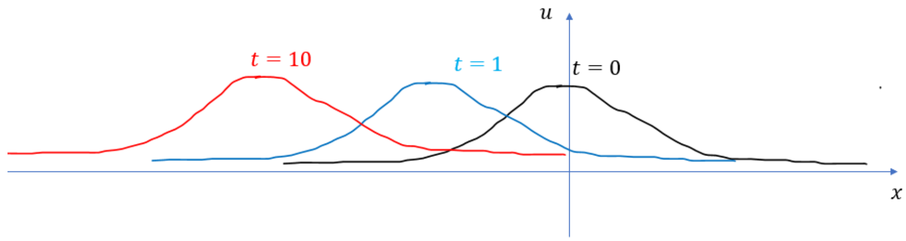


Figure 5.1

and

$$\phi(x,t) = 2tu(x,t) \qquad -\infty < x < \infty, t > 0 \tag{5.18}$$

The equation (5.18) means that the pollutant moves to the right. The flux of the pollutant is 2 times the product of time t and the value of mass density at that time and place.

(ii) The solution curve of differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \qquad t > 0 \tag{5.19}$$

is

$$x(t) = t^2 + x(0) \qquad t \geq 0. \tag{5.20}$$

When a point (x,t) moves along this curve, the rate of change of u with respect to time t is

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = u_x\frac{\mathrm{d}x}{\mathrm{d}t} + u_t = 2tu_x + u_t = 0 \qquad t > 0, \tag{5.21}$$

i.e. u does not change its value.

For any point (x^*,t^*), it is on the curve (5.20) with

$$x(0) = x^* - (t^*)^2 \tag{5.22}$$

Therefore,

$$u(x^*,t^*) = u(x(0),0) = u\left(x^* - (t^*)^2, 0\right) = \left(x^* - (t^*)^2\right)^2 \tag{5.23}$$

In other words, the solution of this IVP is

$$u(x,t) = \left(x - t^2\right)^2 \qquad -\infty < x < \infty, t \geq 0 \tag{5.24}$$

5.7. *When*

$$\phi(x,t) = -c(x,t)u(x,t) \qquad -\infty < x < \infty, t > 0 \tag{5.25}$$

the

$$u_t = -\phi_x \qquad -\infty < x < \infty, t > 0 \tag{5.26}$$

becomes

$$u_t = (c(x,t)u(x,t))_x = c_x u + c u_x \qquad -\infty < x < \infty, t > 0 \tag{5.27}$$

This is the PDE about $u = u(x,t)$ in this case.

Chapter 6

1D Diffusion Equation

6.1. *First, from*

$$q_t = -\phi_x \qquad 0 < x < L, t > 0 \tag{6.1}$$

and

$$\phi = -kq_x \qquad 0 < x < L, t > 0, \tag{6.2}$$

we obtain

$$q_t = kq_{xx} \qquad 0 < x < L, t > 0 \tag{6.3}$$

Second, the

$$\mathrm{d}q \propto \mathrm{d}u, \tag{6.4}$$

implies that

$$q_t = c'u_t, \quad q_x = c'u_x, \quad q_{xx} = c'u_{xx} \tag{6.5}$$

where c' is some constant.

With (6.5), *we can write* (6.3) *in terms of u as follows:*

$$u_t = ku_{xx} \qquad 0 < x < L, t > 0 \tag{6.6}$$

This equation is the 1D diffusion equation (6.2) where $c = k$.

6.2. *The solution is*

$$u(x,t) = 2x + 1, \qquad 0 \leq x \leq 3, t \geq 0 \tag{6.7}$$

You could solve this IBVP with its physical meaning in the context of heat conduction along the rod.

6.3. *Since both u and $\hat{u}$ are solutions of the diffusion equation (6.2), we have*

$$u_t = cu_{xx} \qquad 0 < x < L, t > 0 \tag{6.8}$$
$$\hat{u}_t = c\hat{u}_{xx} \qquad 0 < x < L, t > 0 \tag{6.9}$$

$(6.8) \times c_1 + (6.8) \times c_2$ *gives*

$$c_1 u_t + c_2 \hat{u}_t = c_1 c u_{xx} + c_2 c \hat{u}_{xx} \qquad 0 < x < L, t > 0 \tag{6.10}$$

or equivalently

$$(c_1 u + c_2 \hat{u})_t = c(c_1 u + c_2 \hat{u})_{xx} \qquad 0 < x < L, t > 0 \tag{6.11}$$

i.e. $c_1 u + c_2 \hat{u}$ *is also a solution of the diffusion equation (6.2)*

6.4. *If*

$$u(x,t) = e^{at}\cos(bx) \qquad 0 \leq x \leq L, t \geq 0, \tag{6.12}$$

then

$$u_t = ae^{at}\cos(bx) = au \quad 0 < x < L, t > 0 \tag{6.13}$$

and

$$u_x = -be^{at}\sin(bx) \tag{6.14}$$
$$u_{xx} = (-b)\cdot be^{at}\cos(bx) = -b^2 u \quad 0 < x < L, t > 0 \tag{6.15}$$

Comparing (6.13) *with* (6.15)*, we find that constants a, b and c should be related as*

$$a = -cb^2 \tag{6.16}$$

in order for (6.12) *to be a solution of 1D diffusion equation (6.2).*

6.5. *The condition*

$$u(x, t=0) = x \qquad 0 \leq x \leq L \tag{6.17}$$

implies that

$$u(x=L, t=0) = L \tag{6.18}$$

The condition

$$u(x=L, t) = 0 \qquad t \geq 0 \tag{6.19}$$

implies that

$$u(x=L, t=0) = 0 \tag{6.20}$$

Since L is a positive constant, condition (6.18) *contradicts condition* (6.20)*. Therefore, condition* (6.17) *contradicts condition* (6.19).

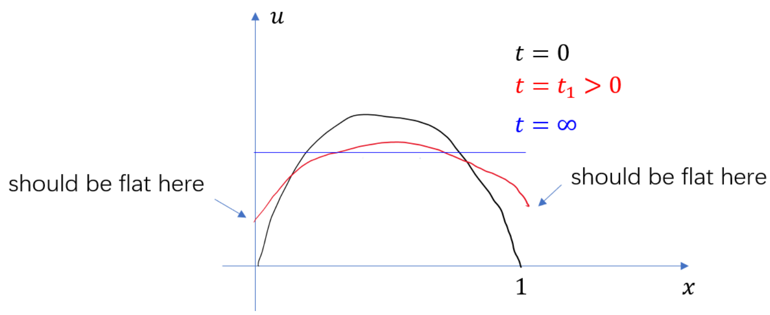


Figure 6.1: The graph of u against x at three typical times.

6.6. *The behavior of solution at three typical times is shown in Figure 6.1. The blue curve is a horizontal straight line.*
J*ustification: We consider this IBVP as a heat conduction problem. The* $u(x,t), 0 \leq x \leq L, t \geq 0$ *represents the temperature of the rod at cross-section* x *and at time* t.

According to the initial condition (6.14b), temperature is highest in the middle of the rod and lowers towards both ends. Because of this, heat would flow from the middle to the two ends of the rod. This will decreases the temperature in the middle.

In the meantime, the boundary conditions (6.14c and d) mean that no heat flows happens at any end of the rod. ($u_x = 0$ *at the end means that temperature is approximately the same around this end). Therefore, the temperature at the two ends would increase because they are receiving the heat from the middle.*

After some time, the heat flow along the rod would get weaker as the difference of the temperature between the middle and the end of the rod decreases. In the end, heat would stop flowing when the temperature is the same everywhere on the rod.

6.7. *The behavior of solution is shown in Figure 6.2. The blue curve is the graph of equation*

$$u(x) = -2x^2 + 3x \qquad 0 \leq x \leq 1 \tag{6.21}$$

J*ustification Because of* $h(x) \equiv 1, 0 < x < 1$*, the rod is heated and heat is flowing into the rod through its side surface. Therefore, temperature will increases everywhere on the rod except for its two ends.*

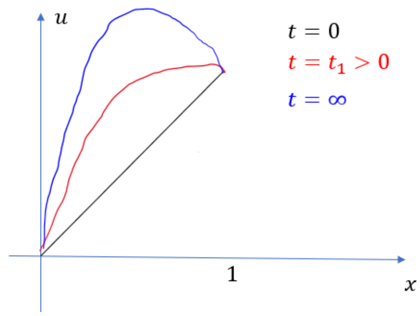


Figure 6.2: The graph of u against x at three typical times.

The temperature at the two ends are maintained to be zero for all the time because of the boundary conditions. To keep the temperature at the two ends to be zero when the rod is heated, heat has to flow out of the rod through its two ends. Because of this, temperature increases most rapidly in the middle of the rod and slowly towards the ends of the rod. As a result, the graph of temperature u against x after some time may look like the red curve in the Figure.

When the temperature is distributed as the blue curve of equation (6.21), *we have*

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = u_{xx} = -4 \qquad 0 < x < 1. \tag{6.22}$$

The (6.22) *implies*

$$u_t = 0 \qquad 0 < x < 1 \tag{6.23}$$

according to (6.16a). That is to say, the temperature does not change anywhere on the rod. Therefore, once the temperature of the rod reaches this distribution, it will not change afterwards.

Chapter 7

1D Laplace/Poisson Equation

7.1.

$$u(x) = 1 \qquad 0 \le x \le L \tag{7.1}$$

and

$$u(x) = x \qquad 0 \le x \le L \tag{7.2}$$

are two functions that satisfy the Laplace equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0 \qquad 0 < x < L \tag{7.3}$$

7.2. *The general solution of 1D Laplace equation* (7.3) *is*

$$u(x) = c_1 x + c_2 \qquad 0 \le x \le L \tag{7.4}$$

where c_1 and c_2 are constants.

7.3. *From* (7.4), *we have*

$$u(x = 0) = c_1 \cdot 0 + c_2 = 1 \tag{7.5}$$

$$u(x = 1) = c_1 \cdot 1 + c_2 = 0 \tag{7.6}$$

which are equivalent to

$$c_1 = -c_2 = -1 \tag{7.7}$$

Therefore, function

$$u(x) = -x + 1 \qquad 0 \le x \le 1 \tag{7.8}$$

is the solution of the BVP (7.9) of the 1D Laplace equation.

7.4. *From (7.8a), we have*

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \int x \mathrm{d}x = \frac{x^2}{2} + c_1 \tag{7.9}$$

From (7.9)*, we have*

$$u(x) = \int \left(\frac{x^2}{2} + c_1\right) \mathrm{d}x = \frac{x^3}{6} + c_1 x + c_2 \tag{7.10}$$

If $u = u(x)$ *in* (7.10) *satisfies the boundary condition (7.8b) and (7.8c), then*

$$u(x=0) = \frac{0^3}{6} + c_1 \cdot 0 + c_2 = 1 \tag{7.11}$$

$$u(x=1) = \frac{1^3}{6} + c_1 \cdot 1 + c_2 = 0 \tag{7.12}$$

In other words,

$$c_1 = -7/6, \quad c_2 = 1 \tag{7.13}$$

Overall, the solution of the BVP (7.8) is

$$u(x) = \frac{x^3}{6} - \frac{7x}{6} + 1 \tag{7.14}$$

7.5. *According to the solution to question 7.4,*

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = x \quad 0 < x < 1 \tag{7.15}$$

implies that

$$u(x) = \frac{x^3}{6} + c_1 x + c_2 \tag{7.16}$$

where c_1 *and* c_2 *are constants.*

From (7.16)*, we have*

$$u'(x) = \frac{x^2}{2} + c_1 \tag{7.17}$$

In order to meet both conditions (7.12b) and (7.12c), we require

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = \frac{0^2}{2} + c_1 = c_1 = 1 \tag{7.18}$$

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=1} = \frac{1^2}{2} + c_1 = c_1 = 0 \tag{7.19}$$

simultaneously. This however can not be acheived. Therefore, there is NO solution of this BVP (7.12).

7.6. *The condition*

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) = 5 \qquad 0 < x < 3 \tag{7.20}$$

is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) = 5x^2 \qquad 0 < x < 3 \tag{7.21}$$

The condition (7.21) *means that*

$$x^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \int 5x^2\,\mathrm{d}x = \frac{5x^3}{3} + c_1 \qquad 0 < x < 3 \tag{7.22}$$

where c_1 *is a constant. The condition* (7.22) *is equivalent to*

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{5x}{3} + \frac{c_1}{x^2} \qquad 0 < x < 3 \tag{7.23}$$

The condition (7.23) *means that*

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \int\left(\frac{5x}{3} + \frac{c_1}{x^2}\right)\mathrm{d}x = \frac{5x^2}{6} - \frac{c_1}{x} + c_2 \qquad 0 < x < 3 \tag{7.24}$$

where c_2 *is a constant. The condition* (7.24) *means that*

$$\begin{aligned} u(x) &= \int\left(\frac{5x^2}{6} - \frac{c_1}{x} + c_2\right)\mathrm{d}x \\ &= \frac{5x^3}{18} - c_1\ln x + c_2x + c_3 \qquad 0 < x \leq 3 \end{aligned} \tag{7.25}$$

where c_3 *is also a constant.*

If $c_1 \neq 0$*, then* (7.25) *implies that*

$$\lim_{x\to0^+} u(x) = \infty \tag{7.26}$$

and function $u = u(x)$ *is not continuous at* $x = 0$*. Because of this,* c_1 *has to be zero and*

$$u(x) = \frac{5x^3}{18} + c_2x + c_3 \qquad 0 \leq x \leq 3 \tag{7.27}$$

Chapter 8

1D Wave Equation

8.1.

In (8.51a), the v should be the same as the v in the 1D wave equation (8.1)

In (8.51b),

$$u(x,t) = A\sin\left(2\pi\left(\frac{x}{\lambda} \pm \frac{t}{\tau}\right)\right) = A\sin\left(\frac{2\pi}{\lambda}\left(x \pm \frac{\lambda}{\tau}t\right)\right) \tag{8.1}$$

So λ/τ must be equal to the v in the 1D wave equation.

In (8.51c),

$$u(x,t) = A\sin\left(2\pi(\kappa x \pm vt)\right) = A\sin\left(2\pi\kappa\left(x \pm \frac{v}{\kappa}t\right)\right) \tag{8.2}$$

So v/κ must be equal to the v in the 1D wave equation.

In (8.51d),

$$u(x,t) = A\sin(kx \pm wt) = A\sin\left(k\left(x \pm \frac{w}{k}t\right)\right) \tag{8.3}$$

So w/k must be equal to the v in the 1D wave equation.

In (8.51e),

$$u(x,t) = A\sin\left(2\pi\nu\left(\frac{x}{v} \pm t\right)\right) = A\sin\left(\frac{2\pi\nu}{v}\left(x \pm vt\right)\right) \tag{8.4}$$

So the v here should be the same as the v in the 1D wave equation.

8.2. *Since both u and $\hat{u}$ are the solution of the 1D wave equation (8.1), we have*

$$
\begin{aligned}
u_{tt} &= v^2 u_{xx} && -\infty < x < \infty,\ t > 0 && (8.5)\\
\hat{u}_{tt} &= v^2 \hat{u}_{xx} && -\infty < x < \infty,\ t > 0 && (8.6)
\end{aligned}
$$

$c_1 \times (8.5)\ + c_2 \times (8.6)$ *gives*

$$c_1 u_{tt} + c_2 \hat{u}_{tt} = v^2 c_1 u_{xx} + v^2 c_2 \hat{u}_{xx}, \quad -\infty < x < \infty, t > 0 \tag{8.7}$$

or equivalently

$$\left(c_1 u + c_2 \hat{u}\right)_{tt} = v^2 \left(c_1 u + c_2 \hat{u}\right)_{xx}, \quad -\infty < x < \infty, t > 0 \tag{8.8}$$

The equation (8.8) *means that* $c_1 u + c_2 \hat{u}$ *is a solution of the 1D wave equation (8.1).*

8.3. *Function*

$$u(x,t) = F(x - vt) + G(x + vt) \qquad -\infty < x < \infty, t > 0 \tag{8.9}$$

satisfies the 1D wave equation (8.52a) for any function F and G. So we look for F and G so that the function (8.9) *also satisfies initial conditions (8.52b) and (8.52c).*

From (8.9)*, we have*

$$
\begin{aligned}
u(x,0) &= F(x) + G(x) && -\infty < x < \infty && (8.10)\\
u_t(x,t) &= (-v)F'(x - vt) + vG'(x + vt) && -\infty < x < \infty && (8.11)\\
u_t(x,0) &= (-v)F'(x) + vG'(x) && -\infty < x < \infty && (8.12)
\end{aligned}
$$

So condition (8.52b) becomes

$$F(x) + G(x) = f(x) \qquad -\infty < x < \infty \tag{8.13}$$

and condition (8.53c) becomes

$$(-v)F'(x) + vG'(x) = 0 \qquad -\infty < x < \infty \tag{8.14}$$

The (8.14) *means that*

$$F'(x) = G'(x) \qquad -\infty < x < \infty \tag{8.15}$$

or

$$G(x) = F(x) + C \qquad -\infty < x < \infty \tag{8.16}$$

for some constant C*. From* (8.13) *and* (8.16)*, we have*

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2}C \quad -\infty < x < \infty \tag{8.17}$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2}C \quad -\infty < x < \infty \tag{8.18}$$

So (8.9) *becomes*

$$u(x,t) = \frac{1}{2}(f(x-vt) + f(x+vt)) \qquad -\infty < x < \infty, t \geq 0 \tag{8.19}$$

This is the solution of the IVP.

2. When $v = 2$*, the solution* (8.19) *becomes*

$$u(x,t) = \frac{1}{2}(f(x-2t) + f(x+2t)) \qquad -\infty < x < \infty, t \geq 0 \tag{8.20}$$

So

$$u(x,t=1) = \frac{1}{2}(f(x-2) + f(x+2)) \qquad -\infty < x < \infty \tag{8.21}$$

$$u(x,t=10) = \frac{1}{2}(f(x-20) + f(x+20)) \qquad -\infty < x < \infty \tag{8.22}$$

The graph of (8.21) *is shown in Figure* (8.1)*. The graph of* (8.22) *is shown in Figure* (8.2)*.*

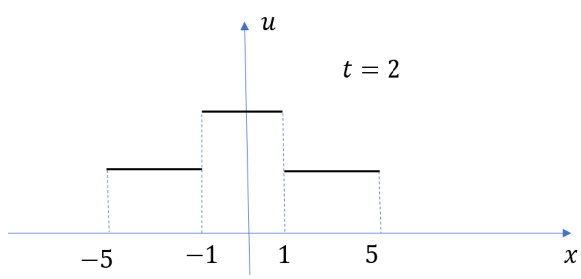


Figure 8.1

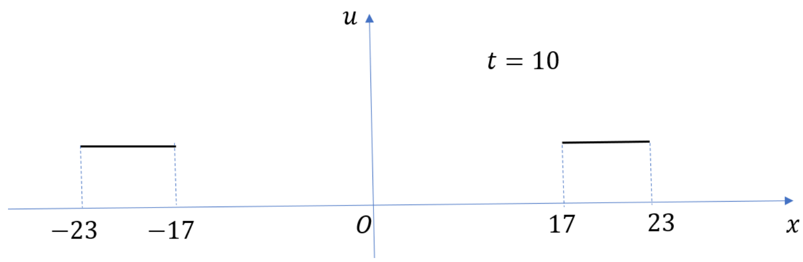


Figure 8.2

8.4. *Function* (8.9) *satisfies the 1D wave equation (8.54a) for any function F and G. So we look for F and G so that the function* (8.9) *also satisfies the initial condition (8.54b) and (8.54c).*

From (8.9), *we have*

$$u(x,0) = F(x) + G(x) \qquad -\infty < x < \infty \tag{8.23}$$

$$u_t(x,t) = (-v)F'(x-vt) + vG'(x+vt) \qquad -\infty < x < \infty \tag{8.24}$$

$$u_t(x,0) = (-v)F'(x) + vG'(x) \qquad -\infty < x < \infty \tag{8.25}$$

So condition (8.54b) becomes

$$F(x) + G(x) = 0 \qquad -\infty < x < \infty \tag{8.26}$$

and condition (8.54c) becomes

$$(-v)F'(x) + vG'(x) = g(x) \qquad -\infty < x < \infty \tag{8.27}$$

From (8.26) *and* (8.27), *we have*

$$F'(x) = -\frac{1}{2v}g(x) \qquad -\infty < x < \infty \tag{8.28}$$

which means

$$F(x) = -\frac{1}{2v}\int_0^x g(t)\,\mathrm{d}t + C \qquad -\infty < x < \infty \tag{8.29}$$

where C is some constant. So

$$G(x) = \frac{1}{2v}\int_0^x g(t)\,\mathrm{d}t - C \qquad -\infty < x < \infty \tag{8.30}$$

Given (8.29) *and* (8.30), *the function* (8.9) *becomes*

$$\begin{aligned} u(x,t) &= -\frac{1}{2v}\int_0^{x-vt} g(t)\,\mathrm{d}t + \frac{1}{2v}\int_0^{x+vt} g(t)\,\mathrm{d}t \\ &= \frac{1}{2v}\int_{x-vt}^{x+vt} g(t)\,\mathrm{d}t \qquad -\infty < x < \infty, t \geq 0 \end{aligned} \tag{8.31}$$

This is the solution of the IVP.

8.5. *The solution of this IVP is the sum of the solution of IVP in Example 8.3 and the solution of IVP in Example 8.4, i.e.*

$$u(x,t) = \frac{1}{2}\left(f(x-vt) + f(x+vt)\right) + \frac{1}{2v}\int_{x-vt}^{x+vt} g(t)\,\mathrm{d}t \qquad -\infty < x < \infty, t \geq 0 \tag{8.32}$$

8.6. *(OPTIONAL)*
(1) The condition

$$w_t + v w_x = 0 \qquad -\infty < x < \infty, t > 0 \tag{8.33}$$

means

$$w(x,t) = f(x-vt) \qquad -\infty < x < \infty, t > 0 \tag{8.34}$$

for some function f. *From* (8.34), *we have*

$$w(x,t=0) = f(x) \qquad -\infty < x < \infty, t > 0. \tag{8.35}$$

In order to satisfy the condition (8.57), we must require

$$f(x) = 2vxe^{-x^2} \qquad -\infty < x < \infty. \tag{8.36}$$

Given (8.36), *the* (8.34) *becomes*

$$w(x,t) = 2v(x-vt)e^{-(x-vt)^2} \qquad -\infty < x < \infty, t > 0 \tag{8.37}$$

(2) Given that we know w *which is* (8.37), *we then solve the following IVP for function* $u = u(x,t)$*:*

$$u_t - vu_x = 2v(x-vt)e^{-(x-vt)^2} \qquad -\infty < x < \infty, t > 0 \tag{8.38a}$$

$$u(x,t=0) = e^{-x^2} \qquad -\infty < x < \infty \tag{8.38b}$$

For point (x^*, t^*), it is on the curve defined by

$$x(t) - x^* = -v(t - t^*) \qquad t \geq 0 \tag{8.39}$$

On this cuve, we have

$$\begin{aligned}\frac{\mathrm{d}u(x(t), t)}{\mathrm{d}t} &= u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_t = u_t - vu_x \\ &= 2v(x(t) - vt)e^{-(x(t)-vt)^2} \\ &= \frac{1}{2}\frac{\mathrm{d}e^{-(x(t)-vt)^2}}{\mathrm{d}t} \qquad t > 0\end{aligned} \tag{8.40}$$

Therefore

$$\begin{aligned}u(x(t), t) &= \frac{1}{2}e^{-(x(t)-vt)^2} + u(x(0), 0) - \frac{1}{2}e^{-x^2(0)} \\ &= \frac{1}{2}e^{-(x(t)-vt)^2} + u(x^* + vt^*, 0) - \frac{1}{2}e^{-(x^*+vt^*)^2} \\ &= \frac{1}{2}e^{-(x(t)-vt)^2} + e^{-(x^*+vt^*)^2} - \frac{1}{2}e^{-(x^*+vt^*)^2} \\ &= \frac{1}{2}e^{-(x(t)-vt)^2} + \frac{1}{2}e^{-(x^*+vt^*)^2} \qquad t \geq 0\end{aligned} \tag{8.41}$$

and

$$\begin{aligned}u(x^*, t^*) = u(x(t^*), t^*) &= \frac{1}{2}e^{-(x(t^*)-vt^*)^2} + \frac{1}{2}e^{-(x^*+vt^*)^2} \\ &= \frac{1}{2}e^{-(x^*-vt)^2} + \frac{1}{2}e^{-(x^*+vt^*)^2}\end{aligned} \tag{8.42}$$

That is to say, the solution of IVP is

$$u(x, t) = \frac{1}{2}e^{-(x-vt)^2} + \frac{1}{2}e^{-(x+vt)^2} \qquad -\infty < x < \infty, t \geq 0 \tag{8.43}$$

8.7. *The solution of IBVP (8.59) is*

$$u(x, t) = \frac{L}{\pi v}\sin\frac{\pi x}{L}\sin\frac{\pi vt}{L} \qquad 0 \leq x \leq L, t \geq 0 \tag{8.44}$$

The solution of IBVP (8.60) is

$$u(x, t) = \frac{L}{k\pi v}\sin\frac{k\pi x}{L}\sin\frac{k\pi vt}{L} \qquad 0 \leq x \leq L, t \geq 0 \tag{8.45}$$

The solution of IBVP (8.61) is

$$u(x,t) = \sum_{k=1}^{\infty} \frac{c_k L}{k\pi v} \sin\frac{k\pi x}{L} \sin\frac{k\pi v t}{L} \qquad 0 \le x \le L, t \ge 0 \tag{8.46}$$

The solution of IBVP (8.62) is the sum of the solution of IBVP (8.61) and IBVP

$$\begin{aligned}
u_{tt} &= v^2 u_{xx} && 0 < x < L, t > 0 && (8.47a)\\
u(x,0) &= \sum_{k=1}^{\infty} d_k \sin\frac{k\pi x}{L} && 0 \le x \le L && (8.47b)\\
u_t(x,0) &= 0 && 0 \le x \le L && (8.47c)\\
u(0,t) &= 0 && t \ge 0 && (8.47d)\\
u(L,t) &= 0 && t \ge 0 && (8.47e)
\end{aligned}$$

According to Example 8.7, the solution of IBVP (8.47) *is*

$$u(x,t) = \sum_{k=1}^{\infty} d_k \sin\frac{k\pi x}{L} \cos\frac{k\pi v t}{L} \qquad 0 \le x \le L, t \ge 0. \tag{8.48}$$

Therefore, the solution of IBVP (8.62) is

$$\sum_{k=1}^{\infty} \frac{c_k L}{k\pi v} \sin\frac{k\pi x}{L} \sin\frac{k\pi v t}{L} + \sum_{k=1}^{\infty} d_k \sin\frac{k\pi x}{L} \cos\frac{k\pi v t}{L} \qquad 0 \le x \le L, t \ge 0 \tag{8.49}$$

8.8. *When* $v = 2, L = 3$,

$$u(x,t) = \frac{L}{\pi v} \sin\frac{\pi x}{L} \sin\frac{\pi v t}{L} = \frac{3}{2\pi} \sin\frac{\pi x}{3} \sin\frac{2\pi t}{3} \quad 0 \le x \le 3, t \ge 0 \tag{8.50}$$

So

$$\begin{aligned}
u(x,t=0) &= 0 && 0 \le x \le 3 && (8.51a)\\
u(x,t=1) &= \frac{3}{2\pi} \sin\frac{\pi x}{3} \sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{4\pi} \sin\frac{\pi x}{3} && 0 \le x \le 3 && (8.51b)\\
u(x,t=2) &= \frac{3}{2\pi} \sin\frac{\pi x}{3} \sin\frac{4\pi}{3} = -\frac{3\sqrt{3}}{4\pi} \sin\frac{\pi x}{3} && 0 \le x \le 3 && (8.51c)\\
u(x,t=3) &= \frac{3}{2\pi} \sin\frac{\pi x}{3} \sin(2\pi) = 0 && 0 \le x \le 3 && (8.51d)
\end{aligned}$$

A sketch of the graphs of u *against* x *at various times are shown in Figure*

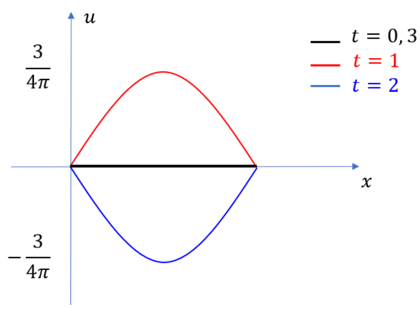


Figure 8.3: A sketch of the graph of u against x at time $t = 0, 1, 2, 3$, respectively

8.9. *The function (8.63) is the solution of IBVP*

$$u_{tt} = v^2 u_{xx} \qquad 0 < x < L, t > 0 \tag{8.52a}$$

$$u(x, 0) = \cos\frac{\pi x}{L} \qquad 0 \leq x \leq L \tag{8.52b}$$

$$u_t(x, 0) = 0 \qquad 0 \leq x \leq L \tag{8.52c}$$

$$u(0, t) = \cos\frac{v\pi t}{L} \qquad t \geq 0 \tag{8.52d}$$

$$u(L, t) = -\cos\frac{v\pi t}{L} \qquad t \geq 0 \tag{8.52e}$$

The function (8.64) is the solution of IBVP

$$u_{tt} = v^2 u_{xx} \qquad 0 < x < L, t > 0 \tag{8.53a}$$

$$u(x, 0) = 0 \qquad 0 \leq x \leq L \tag{8.53b}$$

$$u_t(x, 0) = \frac{v\pi}{L}\cos\frac{\pi x}{L} \qquad 0 \leq x \leq L \tag{8.53c}$$

$$u(0, t) = \sin\frac{v\pi t}{L} \qquad t \geq 0 \tag{8.53d}$$

$$u(L, t) = -\sin\frac{v\pi t}{L} \qquad t \geq 0 \tag{8.53e}$$

8.10. *The solution of this IBVP is the sum of the solution of IBVP*

$$u_{tt} = v^2 u_{xx} \qquad 0 < x < L, t > 0 \tag{8.54a}$$
$$u(x,0) = \sin\frac{\pi x}{L} \qquad 0 \le x \le L \tag{8.54b}$$
$$u_t(x,0) = 0 \qquad 0 \le x \le L \tag{8.54c}$$
$$u(0,t) = 0 \qquad t \ge 0 \tag{8.54d}$$
$$u(L,t) = 0 \qquad t \ge 0 \tag{8.54e}$$

and IBVP

$$u_{tt} = v^2 u_{xx} \qquad 0 < x < L, t > 0 \tag{8.55a}$$
$$u(x,0) = \frac{x}{L} + 1 \qquad 0 \le x \le L \tag{8.55b}$$
$$u_t(x,0) = 0 \qquad 0 \le x \le L \tag{8.55c}$$
$$u(0,t) = 1 \qquad t \ge 0 \tag{8.55d}$$
$$u(L,t) = 2 \qquad t \ge 0 \tag{8.55e}$$

The solution of IBVP (8.54) *is*

$$u(x,t) = \sin\frac{\pi x}{L}\cos\frac{v\pi x}{L} \qquad 0 \le x \le L, t \ge 0 \tag{8.56}$$

and the solution of IBVP (8.54) *is*

$$u(x,t) = \frac{x}{L} + 1 \qquad 0 \le x \le L, t \ge 0 \tag{8.57}$$

Therefore, the solution of this problem is

$$u(x,t) = \frac{x}{L} + 1 + \sin\frac{\pi x}{L}\cos\frac{v\pi x}{L} \qquad 0 \le x \le L, t \ge 0 \tag{8.58}$$

Chapter 9

Fourier Series

9.1.
(i) When $m = n = 0$,

$$f_m(x)f_n(x) = 1 \qquad \textit{for any } x \tag{9.1}$$

and

$$\int_{-L}^{L} f_m(x)f_n(x)\,\mathrm{d}x = \int_{-L}^{L} 1\mathrm{d}x = 2L. \tag{9.2}$$

(ii) When $m = n \neq 0$ and

$$f_m(x)f_n(x) = \sin^2\frac{m^*\pi x}{L} = \frac{1}{2}\left(1 - \cos\frac{2m^*\pi x}{L}\right) \tag{9.3}$$

for some positive integer m^*,

$$\begin{aligned}\int_{-L}^{L} f_m(x)f_n(x)\,\mathrm{d}x &= \int_{-L}^{L} \frac{1}{2}\,\mathrm{d}x - \int_{-L}^{L} \cos\frac{2m^*\pi x}{L}\,\mathrm{d}x \\ &= L - \left[\frac{L}{2m^*\pi}\sin\frac{2m^*\pi x}{L}\right]_{x=-L}^{x=L} = L \end{aligned} \tag{9.4}$$

(iii) When $m = n \neq 0$ and

$$f_m(x)f_n(x) = \cos^2\frac{m^*\pi x}{L} = \frac{1}{2}\left(1 + \cos\frac{2m^*\pi x}{L}\right) \tag{9.5}$$

for some positive integer m^*,

$$\int_{-L}^{L} f_m(x) f_n(x)\,\mathrm{d}x = \int_{-L}^{L} \frac{1}{2}\,\mathrm{d}x + \int_{-L}^{L} \cos\frac{2m^*\pi x}{L}\,\mathrm{d}x$$
$$= L + \left[\frac{L}{2m^*\pi}\sin\frac{2m^*\pi x}{L}\right]_{x=-L}^{x=L} = L \tag{9.6}$$

(iv) When $m \neq n$ and

$$f_m(x) f_n(x) = \sin\frac{m^*\pi x}{L}\cos\frac{n^*\pi x}{L} = \frac{1}{2}\left(\sin\frac{(m^*+n^*)\pi x}{L} + \sin\frac{(m^*-n^*)\pi x}{L}\right) \tag{9.7}$$

for some nonnegative integers m^* and n^*,

$$\int_{-L}^{L} f_m(x) f_n(x)\,\mathrm{d}x = \int_{-L}^{L} \frac{1}{2}\sin\frac{(m^*+n^*)\pi x}{L}\,\mathrm{d}x + \int_{-L}^{L} \frac{1}{2}\sin\frac{(m^*-n^*)\pi x}{L}\,\mathrm{d}x$$
$$= \left[\frac{-L}{2(m^*+n^*)\pi}\cos\frac{(m^*+n^*)\pi x}{L}\right]_{x=-L}^{x=L}$$
$$+ \left[\frac{-L}{2(m^*-n^*)\pi}\cos\frac{(m^*-n^*)\pi x}{L}\right]_{x=-L}^{x=L} = 0 \tag{9.8}$$

9.2. *The proof is already shown in the notes.*

9.3. *Since $L = 3$, we use functions*

$$1,\quad \sin\frac{\pi x}{3}, \cos\frac{\pi x}{3},\quad \sin\frac{2\pi x}{3}, \cos\frac{2\pi x}{3},\quad \sin\frac{3\pi x}{3}, \cos\frac{3\pi x}{3}, \ldots \tag{9.9}$$

to construct the series. The series should be in the form

$$\frac{a_0}{2} + b_1\sin\frac{\pi x}{3} + a_1\cos\frac{\pi x}{3} + b_2\sin\frac{2\pi x}{3} + a_2\cos\frac{2\pi x}{3} + b_3\sin\frac{3\pi x}{3} + a_3\cos\frac{3\pi x}{3} + \cdots \tag{9.10}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{3}\int_{-3}^{3}\cos\frac{n\pi x}{3}\,\mathrm{d}x = \begin{cases} 2 & n = 0 \\ 0 & n = 1, 2, \ldots \end{cases} \tag{9.11}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{3}\int_{-3}^{3}\sin\frac{n\pi x}{3}\,\mathrm{d}x = 0, n = 1, 2, 3 \ldots \tag{9.12}$$

Therefore, the Fourier series is 1 (i.e. itself) over the interval $(-3, 3)$.

9.4. *In this case, $L = \pi$. So the Fourier series should be*

$$\frac{a_0}{2} + b_1 \sin x + a_1 \cos x + b_2 \sin(2x) + a_2 \cos(2x) + b_3 \sin(3x) + a_3 \cos(3x) + \cdots \tag{9.13}$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_{-2}^{2} \mathrm{d}x = \frac{4}{\pi} \tag{9.14}$$

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\int_{-2}^{2}\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{2}\cos(nx)\,\mathrm{d}x \\ &= \frac{2}{n\pi}\int_{0}^{2}\mathrm{d}\sin(nx) = \frac{2}{n\pi}\sin(2n),\ \ n = 1, 2, ... \end{aligned} \tag{9.15}$$

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{1}{\pi}\int_{-2}^{2}\sin(nx)\,\mathrm{d}x = 0,\ \ n = 1, 2, ... \end{aligned} \tag{9.16}$$

In summary, we write

$$f(x) \sim \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{2\sin(2n)}{n\pi}\cos(nx) \qquad (-\pi < x < \pi) \tag{9.17}$$

or

$$f(x) \sim \frac{2}{\pi} + \frac{2\sin 2}{\pi}\cos x + \frac{\sin(4)}{\pi}\cos(2x) + \frac{2\sin(6)}{3\pi}\cos(3x) + \cdots \qquad (-\pi < x < \pi) \tag{9.18}$$

9.5. *Let f_o be the odd extension of $f(x)$, i.e.*

$$f_o(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0. \end{cases} \tag{9.19}$$

The Fourier sine series of function $f(x)$ over the interval $(0, \pi)$ is the Fourier series of f_o over the interval $(-\pi, \pi)$. It is the series (9.23) with $L = \pi$ and

Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f_o(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\cos(nx)\,\mathrm{d}x, \quad n = 0, 1, 2, ... \tag{9.20}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_o(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin(nx)\,\mathrm{d}x, \quad n = 1, 2, ... \tag{9.21}$$

Since f_o is an odd function, the integral in (9.20) *is zero and hence* $a_n = 0, n = 0, 1, 2, \ldots$. *Then the series becomes*

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin(nx) \tag{9.22}$$

with

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\,\mathrm{d}x \\ &= \left[\frac{-2}{n\pi}\cos(nx)\right]_{x=0}^{x=\pi} = \frac{2}{n\pi}(1-\cos(n\pi)) = \frac{2}{n\pi}\left(1-(-1)^n\right),\ n = 1, 2, ... \end{aligned} \tag{9.23}$$

In summary, we write

$$1 \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right)\,\sin(nx) \qquad (0 < x < \pi) \tag{9.24}$$

or

$$1 \sim \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x) + \cdots \qquad (0 < x < \pi) \tag{9.25}$$

9.6. *Let f_e be the even extension of f, i.e.*

$$f_e(x) = 1 \qquad -\pi < x < \pi. \tag{9.26}$$

The Fourier cosine series of function f over the interval $(0, \pi)$ is the Fourier series of f_e over the interval $(-\pi, \pi)$. It is the series (9.23) with $L = \pi$ and Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos(nx)\,\mathrm{d}x, \quad n = 0, 1, 2, ... \tag{9.27}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\sin(nx)\,\mathrm{d}x, \quad n = 1, 2, ... \tag{9.28}$$

Since f_e is even, the integral in (9.28) *is zero and hence* $b_n = 0, 1, 2, \ldots.$ *Then the series becomes*

$$\sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} = \sum_{n=0}^{\infty} a_n \cos(nx) \tag{9.29}$$

with

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\mathrm{d}x = 2 \tag{9.30}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\cos(nx)\,\mathrm{d}x \\ &= \left[\frac{2}{n\pi}\sin(nx)\right]_{x=0}^{x=\pi} = 0, \qquad n = 1, 2, \ldots \end{aligned} \tag{9.31}$$

In summary, the required Fourier series is 1 (i.e. the function that takes constant value 1).

9.7. *According to Euler's formula*

$$e^{ia} = \cos a + i\sin a, \tag{9.32}$$

we have

$$\cos a = \frac{1}{2}\left(e^{ia} + e^{-ia}\right) \tag{9.33}$$

$$\sin a = \frac{1}{2i}\left(e^{ia} - e^{-ia}\right) = \frac{i}{2}\left(e^{-ia} - e^{ia}\right) \tag{9.34}$$

Then

$$\cos\frac{n\pi x}{L} = \frac{1}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right) \tag{9.35a}$$

$$\sin\frac{n\pi x}{L} = \frac{i}{2}\left(e^{-in\pi x/L} - e^{in\pi x/L}\right), \quad n = 0, 1, 2, \ldots \tag{9.35b}$$

With (9.35)*, we can rewrite the Fourier series (9.23) in terms of complex exponential functions as follows:*

$$\begin{aligned} &\frac{a_0}{2}\cdot e^{0\pi x/L} + b_1\cdot\frac{i}{2}\left(e^{-i\pi x/L} - e^{i\pi x/L}\right) + a_1\cdot\frac{1}{2}\left(e^{i\pi x/L} + e^{-i\pi x/L}\right) \\ &\qquad + b_2\cdot\frac{i}{2}\left(e^{-i2\pi x/L} - e^{i2\pi x/L}\right) + a_2\cdot\frac{1}{2}\left(e^{i2\pi x/L} + e^{-i2\pi x/L}\right) \\ &\qquad + b_3\cdot\frac{i}{2}\left(e^{-i3\pi x/L} - e^{i3\pi x/L}\right) + a_3\cdot\frac{1}{2}\left(e^{i3\pi x/L} + e^{-i3\pi x/L}\right) \\ &\qquad + \cdots \end{aligned} \tag{9.36}$$

After combining similar terms in (9.36), *we get*

$$
\begin{aligned}
\cdots &+ \left(\frac{a_3 + b_3 i}{2}\right) e^{-3\pi x/L} + \left(\frac{a_2 + b_2 i}{2}\right) e^{-2\pi x/L} + \left(\frac{a_1 + b_1 i}{2}\right) e^{-\pi x/L} \\
&+ \left(\frac{a_0}{2}\right) e^{0\pi x/L} \\
&+ \left(\frac{a_1 - b_1 i}{2}\right) e^{\pi x/L} + \left(\frac{a_2 - b_2 i}{2}\right) e^{2\pi x/L} + \left(\frac{a_3 - b_3 i}{2}\right) e^{3\pi x/L} + \cdots
\end{aligned}
\tag{9.37}
$$

The series (9.37) *becomes*

$$
\begin{aligned}
\cdots &+ c_{-3} e^{-3\pi x/L} + c_{-2} e^{-2\pi x/L} + c_{-1} e^{-\pi x/L} \\
&+ c_0 e^{0\pi x/L} \\
&+ c_1 e^{\pi x/L} + c_2 e^{2\pi x/L} + c_3 e^{3\pi x/L} + \cdots
\end{aligned}
\tag{9.38}
$$

or equivalently

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \tag{9.39}$$

if we can let

$$\cdots$$

$$c_{-3} = \frac{a_3 + b_3 i}{2} \tag{9.40a}$$

$$c_{-2} = \frac{a_2 + b_2 i}{2} \tag{9.40b}$$

$$c_{-1} = \frac{a_1 + b_1 i}{2} \tag{9.40c}$$

$$c_0 = \frac{a_0}{2} \tag{9.40d}$$

$$c_1 = \frac{a_1 - b_1 i}{2} \tag{9.40e}$$

$$c_2 = \frac{a_2 - b_2 i}{2} \tag{9.40f}$$

$$c_3 = \frac{a_3 - b_3 i}{2} \tag{9.40g}$$

$$\cdots$$

The conditions (9.40) *are equivalent to (9.63) because*

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,\mathrm{d}x$$
$$= \frac{1}{2}\cdot\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x - \frac{i}{2}\cdot\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x,\ \ n = 0, \pm1, \pm2, \cdots \tag{9.41}$$

From (9.41)*, we know that when* $n = 0$*,*

$$c_0 = \frac{a_0}{2}, \tag{9.42}$$

when $n > 0$*,*

$$c_n = \frac{a_n - ib_n}{2} \qquad n = 1, 2, ... \tag{9.43}$$

when $n < 0$*,*

$$c_n = \frac{a_{-n} + ib_{-n}}{2} \qquad n = -1, -2, ... \tag{9.44}$$

9.8. *The IBVP to be solved is the IBVP (8.45) in Example 8.7 of Chapter 8, with* $v = 3, L = 1$ *and* $f(x) = x(1-x), 0 \le x \le 1$*. According to the discussion in Example 8.7, if*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \qquad 0 \le x \le 1, \tag{9.45}$$

where $b_1, b_2, ...$ *are constants, then the solution of this IBVP is*

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{vn\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\cos(3n\pi x) \qquad 0 \le x \le 1, t \ge 0. \tag{9.46}$$

In order for (9.45)*, we use the Fourier sine series of* $f(x) = x(1-x)$ *over the interval* $(0,1)$*. It is*

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) \tag{9.47}$$

with

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x = 2\int_0^1 x(1-x)\sin(n\pi x)\,\mathrm{d}x$$
$$= 2\int_0^1 x\sin(n\pi x)\,\mathrm{d}x - 2\int_0^1 x^2\sin(n\pi x)\,\mathrm{d}x, \quad n = 1, 2, ... \tag{9.48}$$

Since

$$\int_0^1 x\sin(n\pi x)\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}\frac{-\cos(n\pi x)}{n\pi}$$
$$= \left[x\frac{-\cos(n\pi x)}{n\pi}\right]_{x=0}^{x=1} - \int_0^1 \frac{-\cos(n\pi x)}{n\pi}\,\mathrm{d}x = \frac{(-1)^{n+1}}{n\pi} \tag{9.49}$$

$$\int_0^1 x^2\sin(n\pi x)\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}\frac{-\cos(n\pi x)}{n\pi}$$
$$= \left[x^2\frac{-\cos(n\pi x)}{n\pi}\right]_{x=0}^{x=1} - \int_0^1 2x\frac{-\cos(n\pi x)}{n\pi}\,\mathrm{d}x$$
$$= \frac{(-1)^{n+1}}{n\pi} + \frac{1}{(n\pi)^2}\int_0^1 2x\,\mathrm{d}\sin(n\pi x)$$
$$= \frac{(-1)^{n+1}}{n\pi} - \frac{2}{(n\pi)^2}\int_0^1 \sin(n\pi x)\,\mathrm{d}x$$
$$= \frac{(-1)^{n+1}}{n\pi} - \frac{2}{(n\pi)^3}\left(1-(-1)^n\right), \qquad n = 1, 2, ... \tag{9.50}$$

Therefore,

$$b_n = \frac{4}{(n\pi)^3}\left(1-(-1)^n\right), \quad n = 1, 2, ... \tag{9.51}$$

and the solution of this IBVP is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}\left(1-(-1)^n\right)\sin(n\pi x)\cos(3n\pi x) \qquad 0 \le x \le 1, t \ge 0 \tag{9.52}$$

Chapter 10

3D PDEs

10.1. *Consider the amount $M(\Omega, t_1, t_2)$ of chemicals that are moving out of Ω between time $t = t_1 > 0$ and time $t = t_2 > t_1$. In terms of flux density $\boldsymbol{\phi}$,*

$$M(\Omega, t_1, t_2) = \int_{t_1}^{t_2} \oint_{\partial\Omega} \boldsymbol{\phi}(t) \cdot \boldsymbol{n} \, \mathrm{d}S \, \mathrm{d}t = \int_{t_1}^{t_2} \int_{\Omega} \nabla \cdot \boldsymbol{\phi} \, \mathrm{d}\Omega \, \mathrm{d}t \tag{10.1}$$

This is because

$$\left(\oint_{\partial\Omega} \boldsymbol{\phi}(t) \cdot \boldsymbol{n} \, \mathrm{d}S \right) \Delta t \tag{10.2}$$

is approximately the amount of chemcials moving out of Ω between $t = t$ and $t = t + \Delta t$.

In terms of concentration u,

$$\begin{aligned} M(Q, t_1, t_2) &= \int_{\Omega} u(P, t_1) \, \mathrm{d}\Omega - \int_{\Omega} u(P, t_2) \, \mathrm{d}\Omega = \int_{\Omega} (u(P, t_1) - u(P, t_2)) \, \mathrm{d}\Omega \\ &= \int_{\Omega} \int_{t_2}^{t_1} u_t(P, t) \, \mathrm{d}t \, \mathrm{d}\Omega = -\int_{t_1}^{t_2} \int_{\Omega} u_t(P, t) \, \mathrm{d}\Omega \mathrm{d}t \end{aligned} \tag{10.3}$$

This is the total amount of chemcials is conserved. The amount moving out of Ω must be equal to the amount that is decreased within Ω during the same time period.

Equating (10.1) *with* (10.3), *we get*

$$\int_{t_1}^{t_2} \int_{\Omega} \nabla \cdot \boldsymbol{\phi} \, \mathrm{d}\Omega \, \mathrm{d}t = -\int_{t_1}^{t_2} \int_{\Omega} u_t(P, t) \, \mathrm{d}\Omega \, \mathrm{d}t \tag{10.4}$$

Since (10.4) *has to hold for any region* Ω *and any time* t_1 *and* t_2*, we must require the integrand of the integrals on both sides be equal to each other, i.e.*

$$\nabla \cdot \boldsymbol{\phi} = -u_t \qquad \text{in } \mathbb{R}^3, t > 0 \tag{10.5}$$

10.2. *(i) If chemcial moves in the direction of* x*-axis i.e.*

$$\boldsymbol{d} = \pm \hat{\boldsymbol{x}}, \tag{10.6}$$

then u *is independent of* y *and* z*. So*

$$\nabla u = u_x \hat{\boldsymbol{x}} + u_y \hat{\boldsymbol{y}} + u_z \hat{\boldsymbol{z}} = u_x \hat{\boldsymbol{x}} \tag{10.7}$$

Then the PDE (10.3) becomes

$$u_t = -v \nabla u \cdot \boldsymbol{d} = -v(u_x \hat{\boldsymbol{x}}) \cdot (\pm \hat{\boldsymbol{x}}) = -v u_x \tag{10.8}$$

which is PDE (5.2) with $c = \pm v$*.*

(ii) When u *is independent of* y *and* z*, then*

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = u_{xx}. \tag{10.9}$$

With (10.9)*, the 3D diffusion equation (10.14) becomes the 1D diffusion equation (6.2), when it is restricted within the interval* $(0, L)$*; the 3D Poisson equation (10.19) becomes the 1D poisson equation (7.1); the 3D Poisson equation (10.40) becomes the 1D poisson equation (8.1).*

10.3. *The PDE could be rewritten as*

$$u_t = -v \nabla u \cdot \boldsymbol{d} \qquad \text{in } \mathbb{R}^3, t > 0 \tag{10.10}$$

with

$$\boldsymbol{d} = -\frac{1}{\sqrt{3}} \left(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}} \right) \tag{10.11}$$

and $v = \sqrt{3}$*. Therefore, this PDE could model some advection process where the chemical moves in the direction of vector* $\boldsymbol{d}$ *in* (10.11)*, with speed* $\sqrt{3}$

10.4. *Let* $u(P)$ *be the potential at point* P*. Then* $u(P) = 0$ *if* P *is on the boundary of region* Ω*, i.e.*

$$u = 3 \qquad \text{on } \partial\Omega \tag{10.12}$$

On the other hand, since there is no charge within Ω, we have

$$\nabla^2 u = 0 \qquad \textit{in } \Omega \tag{10.13}$$

according to the discussion in Section 10.3.

To find the potential distribution over the reigon, is to determine the function u that satisfies both the bounary condition (10.12) *and the Laplace Equation* (10.13). *In other words, we need to solve the boundary value problem*

$$\begin{aligned} \nabla^2 u &= 0 && \textit{in } \Omega && (10.14a) \\ u &= 3 && \textit{on } \partial\Omega && (10.14b) \end{aligned}$$

for the unknown function u.

Apparently,

$$u = 3 \qquad \textit{on } \Omega \tag{10.15}$$

is a solution of BVP (10.14). *Because the BVP* (10.14) *has at most one solution, the function* (10.15) *must be the* **unique** *solution. In other words, the conditions* (10.14) *implies* (10.15).

In summary, the voltage is 3 units everywhere within Ω, regardless of the shape of Ω.

10.5.

First, we set up a Cartesian coordinate system where the z-axis aligns with the axis of the cylinder.

Second, let

- *the annular cylinder be denoted as Ω, i.e.*

$$\Omega = \{(x, y, z) | 1 \le x^2 + y^2 \le 2^2\} \tag{10.16}$$

- *the inner surface of annular cylinder be denoted as $\partial\Omega_1$, i.e.*

$$\partial\Omega_1 = \{(x, y, z) | x^2 + y^2 = 1\} \tag{10.17}$$

- *the outer surface of annular cylinder be denoted as $\partial\Omega_2$*

$$\partial\Omega_2 = \{(x, y, z) | x^2 + y^2 = 2^2\} \tag{10.18}$$

- *the electrical potential at any point $P \in \Omega$ be denoted as $u(P)$*

Third, the potential u should satisfy all the following conditions simulateniously.

$$\nabla^2 u = -\frac{\rho^*}{\epsilon_0} \qquad \text{in } \Omega \tag{10.19a}$$

$$u = 1 \qquad \text{on } \partial\Omega_1 \tag{10.19b}$$

$$u = 2 \qquad \text{on } \partial\Omega_2 \tag{10.19c}$$

The Poisson's equation (10.19a) *is a mathematical description of the condition that the electric charge is uniformly distributed within Ω. ρ^* is the constant charge density within Ω, ϵ_0 is the physical constant in the Gauss's Law (10.17).*

Fourth, we claim that for any point $P \in \Omega$, $u(P)$ is independent of the cylindrical coordinates z and ϕ. This is because the axis of the annular cylinders aligns with the z axis of the Coordinate system, and the electrical charge is uniformly distributed within Ω.

Fifth, we formulate the BVP (10.19) *in terms of the cylindrical coordiantes ρ, ϕ, z and get*

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{\rho^*}{\epsilon_0} \qquad \text{in } \Omega \tag{10.20a}$$

$$u(\rho = 1) = 0 \tag{10.20b}$$

$$u(\rho = 2) = 2 \tag{10.20c}$$

Given that u is independent of ϕ and z coordinates, the BVP (10.20) *is simplified to be*

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}u}{\mathrm{d}\rho}\right) = -\frac{\rho^*}{\epsilon_0} \qquad 1 < \rho < 2 \tag{10.21a}$$

$$u(\rho = 1) = 0 \tag{10.21b}$$

$$u(\rho = 2) = 2 \tag{10.21c}$$

Sixth, we solve the BVP (10.21) *for u. From* (10.21a)*, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}u}{\mathrm{d}\rho}\right) = -\frac{\rho\rho^*}{\epsilon_0} \qquad 1 < \rho < 2 \tag{10.22}$$

which means

$$\rho\frac{\mathrm{d}u}{\mathrm{d}\rho} = -\int \frac{\rho\rho^*}{\epsilon_0}\,\mathrm{d}\rho = -\frac{\rho^2\rho^*}{2\epsilon_0} + C_1 \qquad 1 < \rho < 2 \tag{10.23}$$

where C_1 *is some constant. From* (10.23)*, we have*

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = -\frac{\rho\rho^*}{2\epsilon_0} + \frac{C_1}{\rho} \qquad 1 < \rho < 2 \tag{10.24}$$

The solution of (10.24) *is*

$$u(\rho) = -\frac{\rho^2\rho^*}{4\epsilon_0} + C_1 \ln|\rho| + C_2 \qquad 1 \le \rho \le 2 \tag{10.25}$$

In order for the function (10.25) *to meet the conditions* (10.21b) *and* (10.21c)*, we require*

$$u(\rho = 1) = -\frac{1^2\rho^*}{4\epsilon_0} + C_2 = 0 \tag{10.26a}$$

$$u(\rho = 2) = -\frac{2^2\rho^*}{4\epsilon_0} + C_1 \ln 2 + C_2 = 2 \tag{10.26b}$$

The system of equations (10.26) *is equivalent to*

$$C_1 = \frac{1}{\ln 2}\left(2 + \frac{3\rho^*}{4\epsilon_0}\right) \tag{10.27a}$$

$$C_2 = \frac{\rho^*}{4\epsilon_0} \tag{10.27b}$$

Overall, the solution of BVP (10.21) *is*

$$u(\rho) = -\frac{\rho^2\rho^*}{4\epsilon_0} + \frac{1}{\ln 2}\left(2 + \frac{3\rho^*}{4\epsilon_0}\right)\ln\rho + \frac{\rho^*}{4\epsilon_0} \quad 1 \le \rho \le 2 \tag{10.28}$$

Seventh, in summary, the potential distribution over the annular sylinder could be described by (10.28)*. It means that the electrical potential at a point* P *within the cylinder is* (10.28)*, provided that the distance from* P *to the axis of annular cylinder is* ρ*.*

10.6.

(i) If both $\boldsymbol{E}$ *and* $\boldsymbol{B}$ *are constant and time-independent vector fields, then they are both divergence free and curl free. So they will satisfy the Maxwell's equations (10.34) in the notes.*

(ii) Yes, it is possible. For example

$$\boldsymbol{E}(x,y,z) = y\hat{\boldsymbol{z}} \qquad \textit{for any } x,y,z \tag{10.29}$$

$$\boldsymbol{B}(x,y,z) = -t\hat{\boldsymbol{x}} \qquad \textit{for any } x,y,z \tag{10.30}$$

Then they satisfy Maxwell's equations (10.34) in the notes.

(iii) If

$$\boldsymbol{E}(P,t) = \hat{\boldsymbol{z}}E_0 \sin(kx - vt) \tag{10.31}$$

then

$$\boldsymbol{B}(P,t) = -\hat{\boldsymbol{y}}\frac{kE_0}{v}\sin(kx - vt) \tag{10.32}$$

(other $\boldsymbol{B}$ *are possible) and the pair* $(\boldsymbol{E},\boldsymbol{B})$ *satisfies the Maxwell's equation (10.34) in the notes.*

If

$$\boldsymbol{E}(P,t) = \hat{\boldsymbol{z}}E_0 \sin(kz - vt) \tag{10.33}$$

then $\boldsymbol{E}$ *is not divergence free all the time. So we cannot find a pair* $(\boldsymbol{E},\boldsymbol{B})$ *that satisfies the Maxwell's equation (10.34) in the notes.*

If

$$\boldsymbol{E}(P,t) = \hat{\boldsymbol{z}}E_0 \cos(ky - vt) \tag{10.34}$$

then

$$\boldsymbol{B}(P,t) = \hat{\boldsymbol{x}}\frac{kE_0}{v}\cos(ky - vt) \tag{10.35}$$

(other $\boldsymbol{B}$ *are possible) and the pair* $(\boldsymbol{E},\boldsymbol{B})$ *satisfies the Maxwell's equation (10.34) in the notes.*