MTH102 Engineering Mathematics II

Lesson 5: Introduction to discrete random variables

Term: 2024

Outline

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Motivations 1

A sample space S may be difficult to describe mathematically if the elements of S are not numbers. We need to find a way to associate each element s of S with a real number x, which leads to the notion of random variables.

Example. In an training, an archer keeps shooting for one target until he successfully shoots the target for the first time. We have the following options of sample space.

- $$S = \{H,\ FH,\ FFH,\ FFFH, \ldots\},$$
  where F represents that he has missed the target and H represents that he has hit the target.
- Consider the total number of shots, then
  $$S = \{1, 2, 3, \ldots\}.$$
- Consider the number of failure before he hits the target, then
  $$S = \{0, 1, 2, \ldots\}.$$

Motivations 2

Frequently, when an experiment is performed, we are interested mainly in some function of the outcome as opposed to the actual outcome itself.

Example. In an training, an archer keeps shooting for one target for three times. Let F represent that he has missed the target and H represent that he has hit the target. Then the sample space is

$$S = \{FFF, HFF, FHF, FFH, HHF, HFH, FHH, HHH\}.$$

However, the coach only wants to know how many times the archer has hit the target during these 3 trials. Let X be the number of success, then

$$\{X = 0\} = \{FFF\},\ \{X = 1\} = \{HFF, FHF, FFH\},$$

$$\{X = 2\} = \{HHF, HFH, FHH\},\ \{X = 3\} = \{HHH\}.$$

X is actually a function defined on the sample space, known as a *random variable*.

Definition of random variables

Definition

Given a random experiment with a sample space S, a function X that assigns one and only one real number $X(s) = x$ to each element s in S is called a **random variable**. *The* **space** *of X is the set of real numbers*

$$\{x : X(s) = x, \;\; s \in S\},$$

where $s \in S$ means that the element s belongs to the set S.

Example 1

Flip a fair coin and observe the outcome. In this case

$$S = \{\text{heads, tails}\}.$$

Let X be defined as

$$X(s) = \begin{cases} 1, & \text{if } s = \text{heads}, \\ 0, & \text{if } s = \text{tails}. \end{cases}$$

Then X is a random variable.

Example 2

A gambler flips a fair coin and observe the outcome. If it is a head, he will win 5 dollars. Otherwise, he will lose 5 dollars.

$$S = \{\text{heads, tails}\}.$$

Let X be defined as

$$X(s) = \begin{cases} 5, & \text{if } s = \text{heads}, \\ -5, & \text{if } s = \text{tails}. \end{cases}$$

Then X is a random variable.

Example 3

Roll a fair die. In this case

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Let X be defined as

$$X(s) = s, \;\; s = 1, 2, 3, 4, 5, 6.$$

Then X is a random variable.

Example 4

Roll two fair dice. In this case

$$S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}.$$

The definition of random variables depends on our interests in the applications.

- If we consider the sum of the two dice, then

$$X((i, j)) = i + j, \ \forall\, i, j = 1, 2, 3, 4, 5, 6.$$

- If we are interested in if the product of the two numbers is even, then

$$X((i, j)) = \begin{cases} 0 & \text{if } i, j = 1, 3, 5, \\ 1 & \text{otherwise.} \end{cases}$$

Example 5

Suppose that the office hour is from 6:00 pm. to 8:00 pm. each Tuesday. A student is coming to ask questions during the office hour. Consider the arrival time of the student, then

$$S = \{\text{the time between 6:00 pm and 8:00 pm}\}.$$

Let X be the time difference in minutes between 6:00 pm and the arrival time, then

$$X(s) \in [0, 120], \ \forall s \in S.$$

Remark. X can be defined in different ways.

Discrete random variable

- A random variable that can take on at most a countable number of possible values is said to be **discrete random variable**.
- For a discrete random variable X and any value x, the probability $P(X = x)$ is frequently denoted by $p(x)$, called the **probability mass function**. It is hereafter abbreviated **pmf**.

X	x_1	x_2	$\cdots$	x_i	$\cdots$
$P(X = x)$	$p(x_1)$	$p(x_2)$	$\cdots$	$p(x_i)$	$\cdots$

Probability mass function (pmf)

Definition

Let X be a discrete random variable and $x_1, x_2, \ldots$ be the values that X can take on. The pmf $p(x)$ is a function that satisfies the following properties:

(a) $p(x_i) \geq 0, \ \ i = 1, 2, \ldots;$

(b) $\sum_{i=1}^{\infty} p(x_i) = 1;$

(c) $P(X \in A) = \sum_{x_i \in A} p(x_i)$, *for any event* A.

Cumulative distribution function (cdf)

- We call the function defined by

$$F(x) = P(X \le x), \ -\infty < x < \infty,$$

  the **cumulative distribution function** and abbreviate it as **cdf**.

- Let $p(x)$ be the pmf of the random variable X, and $x_1, x_2, \ldots$ be the values that X can take on. Then for any $x \in \mathbb{R}$

$$F(x) = \sum_{x_i \le x} p(x_i).$$

- For any $x \in \mathbb{R}$,

$$0 \le F(x) \le 1, \ \lim_{x \to -\infty} F(x) = 0, \ \lim_{x \to \infty} F(x) = 1.$$

- $F(\cdot)$ is a nondecreasing function, i.e. $F(x) \le F(y)$ for $x \le y$.

Example 6

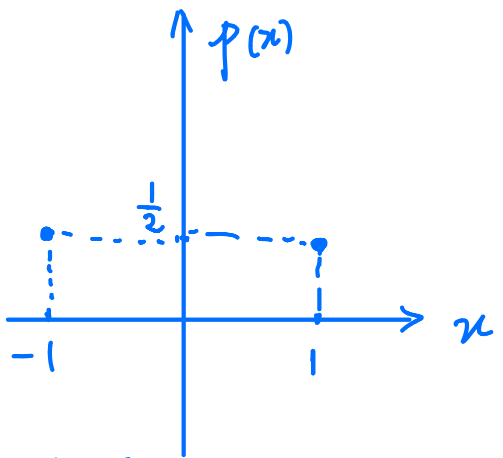
A gambler rolls a fair dice. If the number is larger than 3, he will win 1 dollar. Otherwise, he will lose 1 dollar. Let X be the gambler's gain. Find the pmf and the cdf of X, and sketch the cdf.

Sol: $S=\{1,2,\cdots,6\}$. $X\in\{-1,1\}$. The pmf of X is

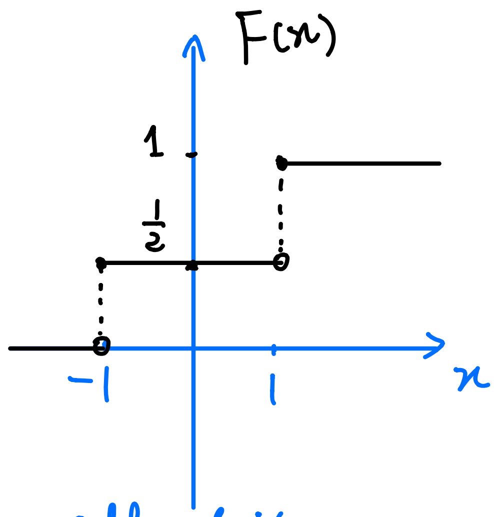
$$p(-1)=\mathbb{P}(X=-1)=\mathbb{P}(\{1,2,3\})=\frac{3}{6}=\frac{1}{2}$$
$$p(1)=\mathbb{P}(X=1)=\mathbb{P}(\{4,5,6\})=\frac{3}{6}=\frac{1}{2}.$$

The cdf of X is

$$F(x)=\mathbb{P}(X\le x)=\begin{cases}0, & \text{if } x<-1\\ p(-1)=\frac{1}{2}, & \text{if } -1\le x<1\\ p(-1)+p(1)=\frac{1}{2}+\frac{1}{2}=1, & \text{if } x\ge 1.\end{cases}$$



pmf of X



cdf of X

Example 7

Roll two fair dice. Let X be the sum of the two dice. Find the pmf and cdf of X, and sketch them.

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$S = \{(i,j), \; i,j = 1, 2, \cdots, 6\}$

$|S| = 6 \times 6 = 36$

<u>Sol</u>: Let $A = \{2, 3, \cdots, 12\}$. then X takes values in A. From the definition of pmf and cdf, we have

$$p(2) = \mathbb{P}(X=2) = \mathbb{P}(\{(1,1)\}) = \frac{1}{36}$$

$$p(3) = \mathbb{P}(X=3) = \mathbb{P}(\{(1,2),(2,1)\}) = \frac{2}{36} = \frac{1}{18}$$

$$\vdots$$

$$p(12) = \mathbb{P}(X=12) = \mathbb{P}(\{(6,6)\}) = \frac{1}{36}$$

In general,

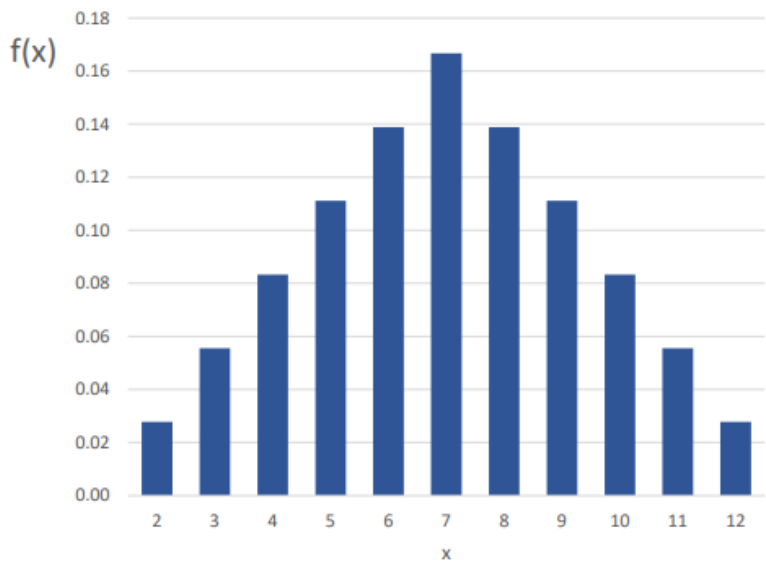
$$p(k) = \mathbb{P}(X=k) = \begin{cases} \frac{k-1}{36}, & k = 2, 3, \cdots, 7 \\ \frac{13-k}{36}, & k = 8, 9, \cdots, 12 \end{cases}$$

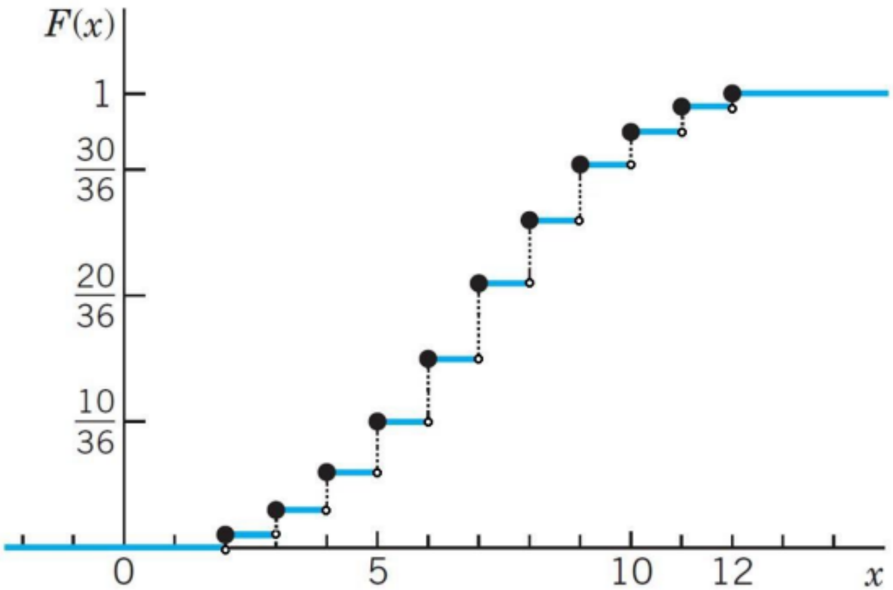
Therefore,

$$F(x) = \begin{cases} 0, & \text{if } x < 2 \\ p(2) = \frac{1}{36}, & \text{if } 2 \le x < 3 \\ p(2)+p(3) = \frac{1+2}{36} = \frac{3}{36}, & \text{if } 3 \le x < 4 \\ \vdots & \vdots \\ p(2)+p(3)+\cdots+p(7) = \frac{1+\cdots+6}{36} = \frac{7}{12}, & \text{if } 7 \le x < 8 \\ p(2)+p(3)+\cdots+p(8) = \frac{7}{12} + \frac{5}{36} = \frac{13}{18}, & \text{if } 8 \le x < 9 \\ \vdots & \vdots \\ p(2)+p(3)+\cdots+p(11) = \frac{35}{36}, & \text{if } 11 \le x < 12 \\ p(2)+\cdots+p(12) = 1, & \text{if } x \ge 12 \end{cases}$$

Example 7

Roll two fair dice. Let X be the sum of the two dice. Find the pmf and cdf of X, and sketch them.





Exercise

Mr. Siegfried hits a target with probability $p = \frac{1}{3}$, and he has made two trials. Let X be the number of hits. Find the pmf and cdf of X, and sketch the cdf.

Sol: $\mathbb{P}(\text{hit the target}) = \frac{1}{3}$

$\mathbb{P}(\text{NOT hit the target}) = 1 - \frac{1}{3} = \frac{2}{3}$

$X \in \{0, 1, 2\}$.

$p(0) = \mathbb{P}(X=0) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$p(1) = \mathbb{P}(X=1) = \binom{2}{1} \cdot \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$

$p(2) = \mathbb{P}(X=2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

And so the cdf of X is

$$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0, & \text{if } x < 0 \\ p(0) = \frac{4}{9}, & \text{if } 0 \leq x < 1 \\ p(0) + p(1) = \frac{4}{9} + \frac{4}{9} = \frac{8}{9}, & \text{if } 1 \leq x < 2 \\ p(0) + p(1) + p(2) = 1, & \text{if } x \geq 2. \end{cases}$$

Properties of cdf

- The cdf $F(\cdot)$ is right continuous, i.e. for any $x \in \mathbb{R}$
$$F(x) = F(x+0),$$
where $F(x+0)$ is the right limit of F at x.
- For any $x \in \mathbb{R}$,
$$P(X = x) = F(x) - F(x-0),$$
$$P(X < x) = P(X \leq x) - P(X = x) = F(x-0),$$
where $F(x-0)$ is the left limit of F at x.
- For $a < b$,
$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a),$$
$$P(a \leq X < b) = P(X < b) - P(X < a) = F(b-0) - F(a-0).$$

Outline

Definition of mean

Definition

If X is a discrete random variable having a probability mass function $p(x)$, then the mean, or the expectation, of X, denoted by $E(X)$, is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x).$$

If the values that X can take on are $x_1, x_2, \ldots$, then

$$E[X] = \sum_{i=1}^{\infty} x_i p(x_i).$$

In words, the mean of X is a weighted average of the possible values that X can take on, each value being weighted by the probability that X assumes it.

Example 8

(a) The diameters of steel spheres can take three values: 0.5mm, 0.52mm and 0.55mm, with probability 0.2, 0.5 and 0.3, respectively. Find the expected diameter of a steel sphere.

(b) With the same setting as above, find the expected volume of a steel sphere.

<u>Sol</u> : Let $A = \{0.5,\ 0.52,\ 0.55\}$

(a) Let X be the diameter, then $X \in A$.

$$\mathbb{E}(X) = \sum_{x \in A} x\, p(x) = 0.5 \times 0.2 + 0.52 \times 0.5 + 0.55 \times 0.3$$
$$= 0.525$$

(b) Let Y be the ~~value~~ volume of the steal sphere, then

$$Y = \frac{1}{6}\pi x^3 .$$

$$\mathbb{E}(Y) = \frac{1}{6}\pi\, 0.5^3 \times 0.2 + \frac{1}{6}\pi\, 0.52^3 \times 0.5 + \frac{1}{6}\pi\, 0.55^3 \times 0.3$$

$$= \frac{1}{6}\pi\left(0.5^3 \times 0.2 + 0.52^3 \times 0.5 + 0.55^3 \times 0.3\right)$$

$$\simeq 0.076\ (mm^3)$$

Example 9

(a) Find $E[X]$, where X is the outcome when we roll a fair die.

(b) A gambler rolls a fair dice. If the outcome is less than 4, he will lose 1 dollar. If the outcome is 6, he will win 2 dollars. Otherwise he won't win or lose. Find the expected gain of the gambler.

$S = \{1, \cdots, 6\}$

(a) $\mathbb{E}(X) = \frac{1}{6}(1+2+\cdots+6) = \frac{1}{6} \times \frac{(1+6)\times 6}{2}$

$= \frac{7}{2}$

(b) Let X be the gain of the gambler.

Then $X \in \{-1, 0, 2\}$.

$\mathbb{P}(X=-1) = \mathbb{P}(\{1,2,3\}) = \frac{3}{6} = \frac{1}{2}$

$\mathbb{P}(X=2) = \mathbb{P}(\{6\}) = \frac{1}{6}$

$\mathbb{P}(X=0) = \mathbb{P}(\{4,5\}) = \frac{2}{6} = \frac{1}{3}$

therefore, the expected gain $\mathbb{E}(X)$ is

$$\mathbb{E}(X) = (-1)\,\mathbb{P}(X=-1) + 2\,\mathbb{P}(X=2) + 0\,\mathbb{P}(X=0)$$

$$= -1 \times \frac{1}{2} + 2 \times \frac{1}{6}$$

$$= -\frac{3}{6} + \frac{2}{6}$$

$$= -\frac{1}{6}.$$

<u>In conclusion</u>, the gambler is expected to lose $\frac{1}{6}$ dollar in this game.

Expectation of a function of a random variable

If X is a discrete random variable that takes on the values $x_1, x_2, \ldots$, with the pmf $p(x)$, then for any real-valued function $g : \mathbb{R} \to \mathbb{R}$,

$$E[g(X)] = \sum_{i=1}^{\infty} g(x_i)p(x_i).$$

Remark. $Y = g(X)$ is actually a new random variable. However, to compute the mean of Y, i.e. $E[Y] = E[g(X)]$, it is not necessary to find the distribution of Y once the distribution of X is given.

Definition of variance

Definition

The variance of a random variable X, denoted by $Var(X)$, is defined as the mean of the function of X with $g(X) = (X - E[X])^2$, i.e.

$$Var(X) = E[(X - E[X])^2].$$

In the practice, it is more convenient to compute the variance via the following equivalent formula

$$Var(X) = E[X^2] - (E[X])^2.$$

- The variance is always nonnegative.
- The square root of $Var(X)$, i.e. $\sqrt{Var(X)}$, is called the *standard deviation* of X.

Variance

The variance provides a measure of spread of X around its mean $E(X)$. For example, consider the following three random variables:

$$X = 0 \text{ with probability } 1.$$

$$Y = \begin{cases} -1 & \text{with probability } \frac{1}{2}, \\ 1 & \text{with probability } \frac{1}{2}. \end{cases}$$

$$Z = \begin{cases} -100 & \text{with probability } \frac{1}{2}, \\ 100 & \text{with probability } \frac{1}{2}. \end{cases}$$

They all have the same expectation 0. But there is a much greater spread in the possible values of Y than in those of X (which is a constant) and in the possible values of Z than in those of Y.

Example 10 (continued from Example 9)

(a) Find $Var(X)$, where X is the outcome when we roll a fair die.

(b) A gambler rolls a fair die. If the outcome is less than 4, he will lose 1 dollar. If the outcome is 6, he will win 2 dollars. Otherwise he won't win or lose. Find the variance of the gain of the gambler.

Sol: (a)

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2.$$

$$\mathbb{E}(X^2) = \sum_{x=1}^{6} x^2\, \mathbb{P}(X = x)$$

$$= 1^2 \times \mathbb{P}(X=1) + 2^2 \times \mathbb{P}(X=2) + \cdots + 6^2 \times \mathbb{P}(X=6)$$

$$= \frac{1}{6}(1^2 + 2^2 + \cdots + 6^2)$$

$$= \frac{91}{6}$$

Hence,

$$\mathrm{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}$$

(b)

$$\mathbb{E}(X^2) = (-1)^2 \times \frac{1}{2} + 2^2 \times \frac{1}{6} + 0^2 \times \frac{1}{3}$$

$$= \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

Then we have

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \frac{7}{6} - \left(-\frac{1}{6}\right)^2$$

$$= \frac{7}{6} - \frac{1}{36}$$

$$= \frac{41}{36}$$

Properties of mean and variance

Let X be a random variable, a, b, and c be constants. Consider $Y = aX + b$ as a linear function of X. Then

- $E(c) = c$, $Var(c) = 0$.
- $$E(Y) = E(aX + b) = aE(X) + b.$$
- $$Var(Y) = Var(aX + b) = a^2 Var(X).$$

The formula $Var(X) = E[X^2] - (E[X])^2$ can be proved using the above properties.

Mean of a sum of random variables

Let $X_1, X_2, \ldots, X_n$ be random variables. The mean of a sum of random variables equals the sum of the mean of each random variable, i.e.

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n).$$

Moreover, let $a_1, a_2, \ldots, a_n$ be constant. Then

$$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \cdots + a_nE(X_n).$$

Example 11

We play a game in two steps:

- We flip a coin. If it is a head we win 1 dollar. Otherwise we lose 1 dollar.
- Then we roll a dice. If it is less than 4, we lose 1 dollar. If it is 6 we win 2 dollars. Otherwise we don't win or lose.

Find

(a) the expected gain for one game.

(b) the expected gain for 5 games.

Sol: Let X be the total gain if we play this game, X_i be the gain in the step i, $i = 1, 2$.

Then $X = X_1 + X_2$.

(a) $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2)$

Since $\mathbb{E}(X_1) = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$.

and $\mathbb{E}(X_2) = (-1) \times \frac{1}{2} + 2 \times \frac{1}{6} + 0 \times \frac{1}{3} = -\frac{1}{6}$

So we have the expected gain for 1 game is

$\mathbb{E}(X) = 0 - \frac{1}{6} = -\frac{1}{6}$.

(b) Since the expected gain for each game is $(-\frac{1}{6})$, the expected gain for 5 games is $5 \times (-\frac{1}{6}) = -\frac{5}{6}$.

Exercise

Put 2020 balls into 2021 boxes. Find the expected number of empty boxes.

Sol: For $i = 1, \cdots, 2021$, define the random variable by

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th box is empty} \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\mathbb{P}(X_i = 1) = \frac{2020^{2020}}{2021^{2020}}$$

$$\mathbb{E}(X_i) = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0)$$

$$= \frac{2020^{2020}}{2021^{2020}}$$

Let X be the no. of empty boxes.

Then

$$X = X_1 + \cdots + X_{2021}$$

Therefore,

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{2021})$$

$$= \frac{2020^{2020}}{2021^{2019}}$$