Class Test I, MTH201

DATE: October 16th, 2024

TIME ALLOWED: 60 minutes

Assume

$$\boldsymbol{F}(P) = y^2\hat{\boldsymbol{x}} \tag{1}$$

for any point P with Cartesian coordinates (x, y, z). Here $\hat{\boldsymbol{x}}$ is a unit vector in the positive direction of x-axis.

Q 1. *(10 Marks)*
Is $\boldsymbol{F}$ a scalar field or a vector field ? Give your reasons.

Solution. It is a vector field. Because the value of $\boldsymbol{F}$ at any point P is a vector. □

Q 2. *(15 Marks)*
Sketch the field $\boldsymbol{F}$ in the xy-plane.

Solution. See Figure

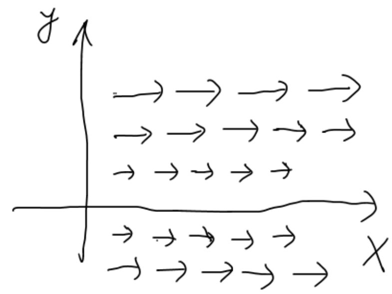

Figure 1: A sketch of vector field $\boldsymbol{F}$ defined in (1)

□

Q 3. *(10 Marks)*
What is the divergence of $\boldsymbol{F}$ at the point with Cartesian coordinates $(x = 1, y = 1, z = 0)$?

Solution.

$$\boldsymbol{F}(x, y, z) = F_1(x, y, z)\hat{\boldsymbol{x}} + F_2(x, y, z)\hat{\boldsymbol{y}} + F_3(x, y, z)\hat{\boldsymbol{z}} \tag{2}$$

with

$$F_1(x, y, z) = y^2, \quad F_2(x, y, z) = 0, \quad F_3(x, y, z) = 0. \tag{3}$$

for any (x, y, z). So

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x} y^2 = 0 \tag{4}$$

and the divergence of $\boldsymbol{F}$ at the point is 0. □

Q 4. *(10 Marks)*
What is the curl of $\boldsymbol{F}$ at the point with Cartesian coordinates $(x=1, y=1, z=0)$?

Solution.

$$\begin{aligned} \nabla \times \boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\boldsymbol{z}} \\ &= -2y\hat{\boldsymbol{z}} \end{aligned} \tag{5}$$

So the curl of $\boldsymbol{F}$ at the point is $-2\hat{\boldsymbol{z}}$. □

Q 5. *(15 Marks)*
Assume S is a surface of

$$\{(x,y,z)|x^2 + (y-1)^2 = 1, x \geq 0\}. \tag{6}$$

The projection of S on the xy-plane is shown in Figure 2.

(i) (5 Marks) Find a surface integral that represents the flux of $\boldsymbol{F}$ through S from its left to its right. Draw a sketch to indicate the unit normal vector $\boldsymbol{n}$ in this surface integral.

(ii) (5 Marks) Is the value of this surface integral positive, negative or zero ?

(iii) (5 Marks) Explain briefly your answer in (ii)

Solution. (i) The surface integral is

$$\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, \mathrm{d}S \tag{7}$$

where $\boldsymbol{n}$ is unit normal vector indicated in Figure 3.
(ii). The value of surface integral is positive.
(iii). At any point P on S (except for the ones on y-axis),

$$\boldsymbol{F}(P) \cdot \boldsymbol{n}(P) = (y(P))^2 \hat{\boldsymbol{x}} \cdot \boldsymbol{n}(P) > 0 \tag{8}$$

So

$$\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, \mathrm{d}S \approx \sum_i \boldsymbol{F}(P_i) \cdot \boldsymbol{n}_i(P_i) \Delta S_i > 0 \tag{9}$$

□

Q 6. *(15 Marks)*
Let Γ be a line of

$$\{(x,y,z)|x^2 + (y-1)^2 = 1, z = 0, x \geq 0\} \tag{10}$$

and is shown in Figure 4. Consider the line integral of $\boldsymbol{F}$ along Γ in the direction as indicated in the Figure 4. Is it positive, negative, or zero (5 Marks) ? Explain your reasons (10 Marks).

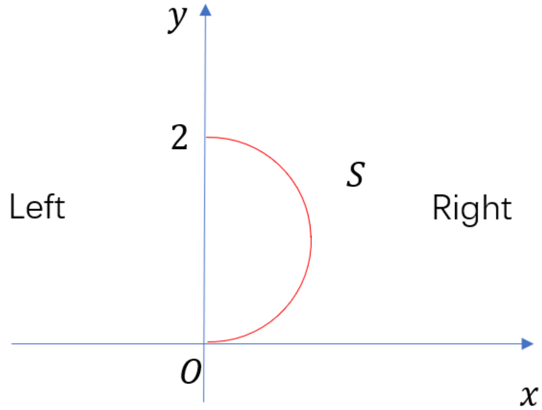

Figure 2: The projection of surface S in the xy-plane.

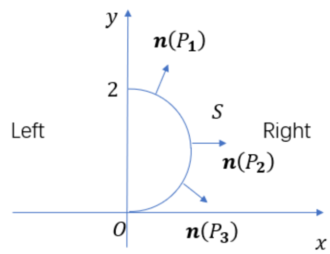

Figure 3: The unit normal vector $\boldsymbol{n}$ at several different points on the surface S.

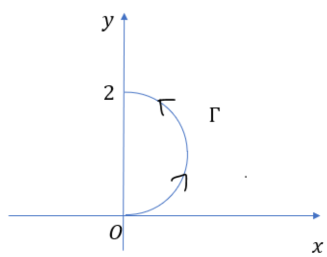

Figure 4: The directed curve Γ on the xy-plane.

Solution. It is negative.

(Explanation I). We divide Γ into two parts Γ_1 and Γ_2 that are symmetric to the dashed line as shown in Figure 5. Then

$$\int_{\Gamma} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s = \int_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s + \int_{\Gamma_2} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s \tag{11}$$

To compare $\int_{\Gamma} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s$ with zero, is equivalent to compare $-\int_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s$ with $\int_{\Gamma_2} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s$.

Let P_1 be a point on the top part Γ_1 while P_2 is the corresponding point on the bottom part Γ_2. We compare $-\int_{\Gamma_1} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s$ with $\int_{\Gamma_2} \boldsymbol{F} \cdot \boldsymbol{T} \, \mathrm{d}s$, by comparing

$$-\boldsymbol{F}(P_1) \cdot \boldsymbol{n}(P_1) \tag{12}$$

with

$$\boldsymbol{F}(P_2) \cdot \boldsymbol{n}(P_2). \tag{13}$$

If $\theta > 0$ is the angle between $\boldsymbol{F}(P_2)$ and $\boldsymbol{n}(P_2)$, then $\pi - \theta$ is the angle between $\boldsymbol{F}(P_1)$ and $\boldsymbol{n}(P_1)$. So

$$\frac{\boldsymbol{F}(P_2)\cdot\boldsymbol{n}(P_2)}{-\boldsymbol{F}(P_1)\cdot\boldsymbol{n}(P_1)} = \frac{|\boldsymbol{F}(P_2)|\cdot|\boldsymbol{n}(P_2)|\cos\theta}{-|\boldsymbol{F}(P_1)|\cdot|\boldsymbol{n}(P_1)|\cos(\pi-\theta)} = \frac{|\boldsymbol{F}(P_2)|}{|\boldsymbol{F}(P_1)|} < 1 \tag{14}$$

So we have

$$-\boldsymbol{F}(P_1)\cdot\boldsymbol{n}(P_1) > \boldsymbol{F}(P_2)\cdot\boldsymbol{n}(P_2). \tag{15}$$

Because $-\int_{\Gamma_1}\boldsymbol{F}\cdot\boldsymbol{T}\,\mathrm{d}s$ is a sum of many terms like $-\boldsymbol{F}(P_1)\cdot\boldsymbol{n}(P_1)\Delta s$; $\int_{\Gamma_2}\boldsymbol{F}\cdot\boldsymbol{T}\,\mathrm{d}s$ is a sum of many terms like $\boldsymbol{F}(P_2)\cdot\boldsymbol{n}(P_2)\Delta s$. The (15) implies that

$$-\int_{\Gamma_1}\boldsymbol{F}\cdot\boldsymbol{T}\,\mathrm{d}s < \int_{\Gamma_2}\boldsymbol{F}\cdot\boldsymbol{T}\,\mathrm{d}s \tag{16}$$

and

$$\int_{\Gamma}\boldsymbol{F}\cdot\boldsymbol{T}\,\mathrm{d}s < 0 \tag{17}$$

□

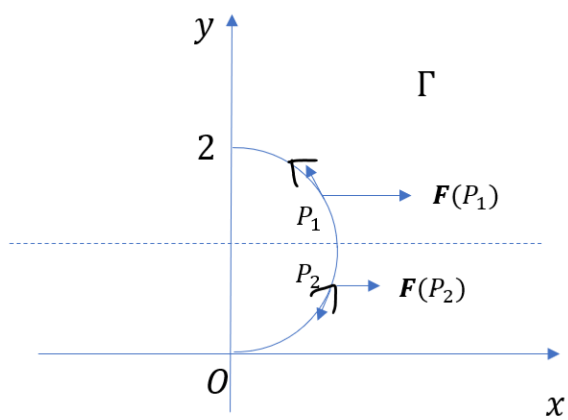

Figure 5: The directed curve Γ on the xy-plane.

Q 7. *(15 Marks)*
Is it possible for $\boldsymbol{F}$ to be the gradient of a scalar field ? If so, give an example of this scalar field. If not, give your reasons.

Solution. No, it is impossible. Because $\boldsymbol{F}$ is not curl-free while the gradient of any scalar field is curl-free. □

Q 8. *(10 Marks)*
Is it possible for $\boldsymbol{F}$ to be the curl of a vector field ? If so, give an example of this vector field. If not, give your reasons.

Solution. Yes. It is possible. Assume

$$\boldsymbol{A} = A_1\hat{\boldsymbol{x}} + A_2\hat{\boldsymbol{y}} + A_3\hat{\boldsymbol{z}} \tag{18}$$

and

$$\nabla\times\boldsymbol{A} = \boldsymbol{F} = y^2\hat{\boldsymbol{x}}. \tag{19}$$

Then

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = y^2. \tag{20}$$

Therefore, if $A_3 = y^3/3, A_1 = A_2 = 0$ and

$$\boldsymbol{A}(P) = y^3/3\hat{\boldsymbol{z}} \tag{21}$$

for any P with Cartesian coordinates (x, y, z), then the curl of $\boldsymbol{A}$ is $\boldsymbol{F}$. □

Some Formula

$$u = u(x, y, z), \quad \boldsymbol{F}(x, y, z) = F_1(x, y, z)\hat{\boldsymbol{x}} + F_2(x, y, z)\hat{\boldsymbol{y}} + F_3(x, y, z)\hat{\boldsymbol{z}} \tag{22}$$

$$\begin{array}{ll}
\text{Gradient} & \nabla u = \dfrac{\partial u}{\partial x}\hat{\boldsymbol{x}} + \dfrac{\partial u}{\partial y}\hat{\boldsymbol{y}} + \dfrac{\partial u}{\partial z}\hat{\mathbf{z}} \\
\text{Divergence} & \nabla \cdot \boldsymbol{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z} \\
\text{Curl} & \nabla \times \boldsymbol{F} = \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right)\hat{\boldsymbol{z}}
\end{array}$$

$$\begin{array}{ll}
\text{Gradient Theorem} & \displaystyle\int_{P_1}^{P_2} (\nabla u) \cdot \mathrm{d}\boldsymbol{s} = u(P_2) - u(P_1) \\
\text{Divergence Theorem} & \displaystyle\int_{\Omega} \nabla \cdot \boldsymbol{F} \,\mathrm{d}\Omega = \oint_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{n} \,\mathrm{d}S \\
\text{Curl Theorem} & \displaystyle\int_{S} (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n} \,\mathrm{d}S = \oint_{\partial S} \boldsymbol{F} \cdot \boldsymbol{T} \,\mathrm{d}s
\end{array}$$

Infinitesimal displacement vector:

$$\mathrm{d}\boldsymbol{s} = \mathrm{d}x\hat{\boldsymbol{x}} + \mathrm{d}y\hat{\boldsymbol{y}} + \mathrm{d}z\hat{\boldsymbol{z}}$$