CAN209 Advanced Electrical Circuits and Electromagnetics

Lecture 13 Transient Response of 2nd-Order Circuits (Step Response)

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OUTLINE

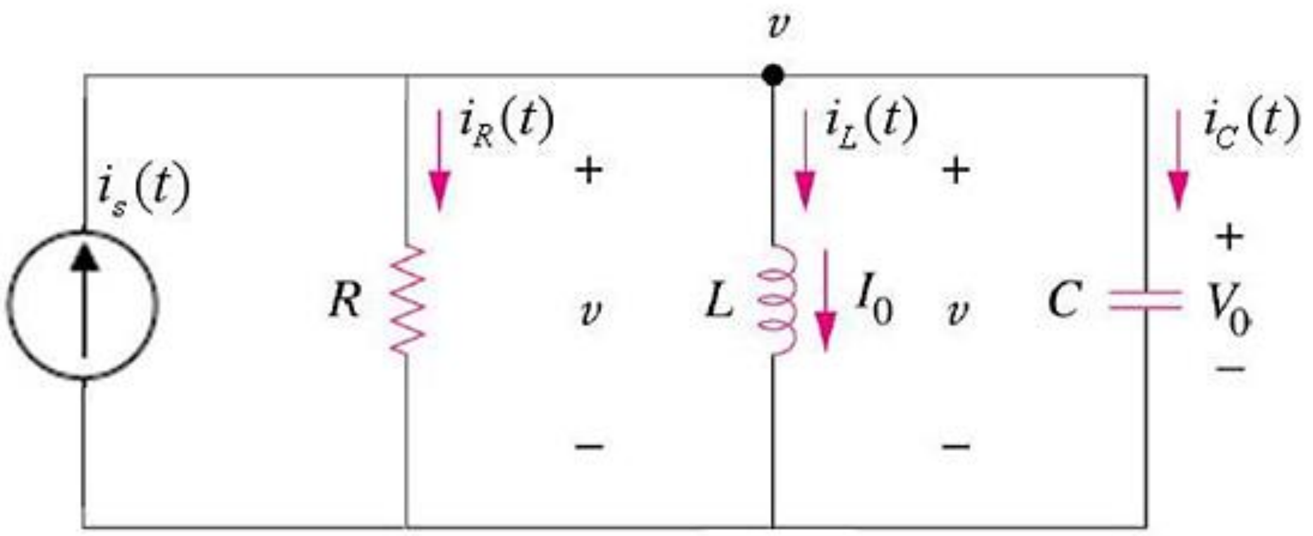
- 2nd Transient Analyses (Step Response)
  - Parallel *RLC* Circuits
    - Response Form of the Circuit (2nd ODE)
    - Solutions to the 2nd ODE
  - Series *RLC* Circuits
    - Response Form of the Circuit (2nd ODE)
    - Solutions to the 2nd ODE
- General 2nd – order Circuit

Inhomogeneous ODE
and its general solution

1 PARALLEL RLC CIRCUIT

Applying KCL:

$$i_R(t) + i_L(t) + i_C(t) = i_s(t)$$



$$\left.\begin{aligned} v &= L\frac{di_L}{dt} \\ i_R(t) &= \frac{v}{R} = \frac{L}{R}\frac{di_L}{dt} \\ i_C(t) &= C\frac{dv}{dt} = LC\frac{d^2 i_L}{dt^2} \end{aligned}\right\} \quad \frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = \frac{1}{LC}i_s$$

Inhomogeneous ODE

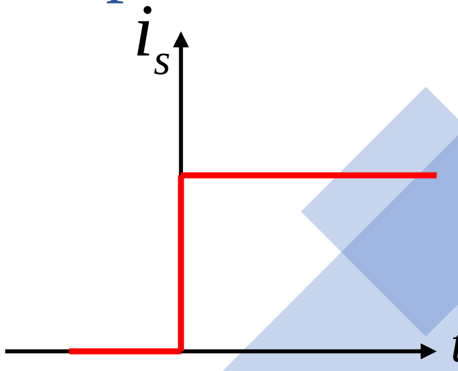
General Solution: the sum of the **final** current and its **natural** response

Complete response

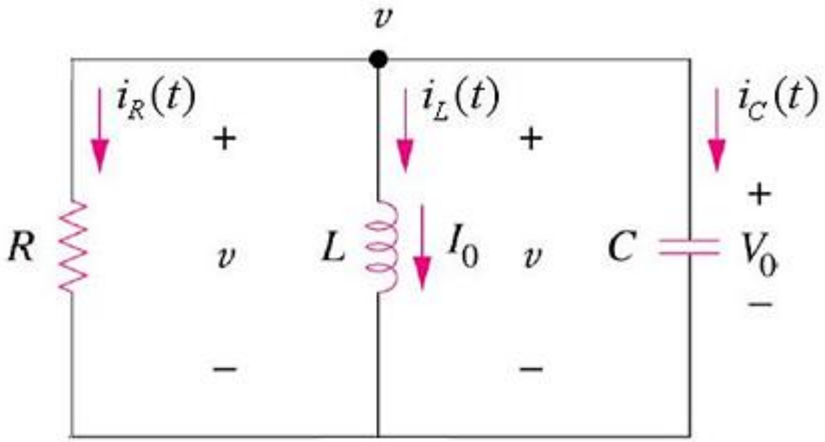
$$\boxed{i_L(t) = i_{L,f}(t) + i_{L,n}(t) = i_{ss}(\infty) + i_{L,n}(t)}$$

The natural response is the same as the source-free case.

The forced response should be the steady-state case ($t = \infty$).



RECALL NATURAL RESPONSE...



$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = 0$$

Characteristic eq.:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\therefore s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$and \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\alpha = \frac{1}{2RC}$$

s_1 and s_2 are called complex frequencies.

- Over Damped → $\alpha > \omega_0$:

  s_1 & s_2 are two unequal real numbers

  Response: $i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

- Critical Damped → $\alpha = \omega_0$:

  s_1 & s_2 are two equal real numbers

  Response: $i_L(t) = e^{-\alpha t}(A_1 t + A_2)$

- Under Damped → $\alpha < \omega_0$:

  s_1 & s_2 are two complex numbers

  $s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2}$ and $s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$

  Response:

  $$i_L(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

  $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

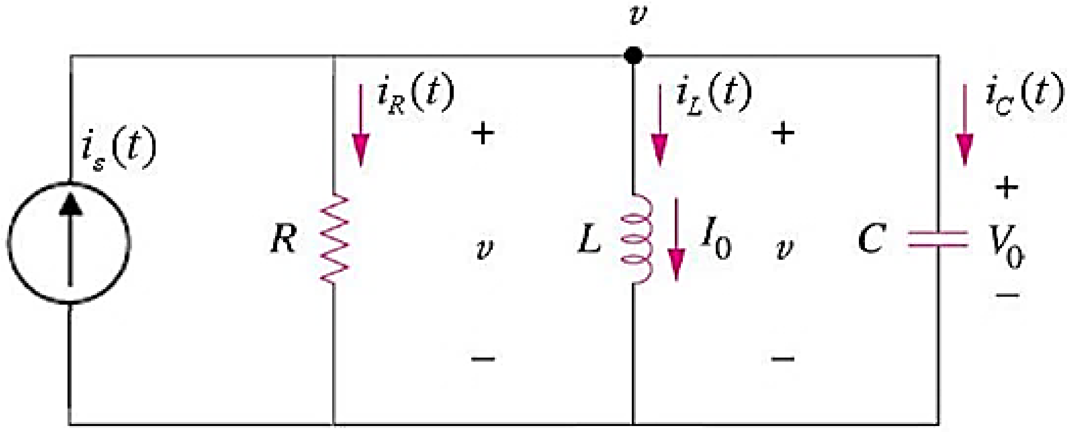
QUIZ 1

For a parallel *RLC* circuit as shown in the figure below, values of the passive components and the initial conditions are as follows:

$$R = 500\Omega, C = 1\mu\text{F}, L = 0.2\text{H}$$

$$i_L(0) = 50\text{mA}, v_c(0) = 0, i_s(t) = 100u(t)\text{mA}$$

Find the response of $i_L(t)$,$v_C(t)$, and $i_R(t)$.



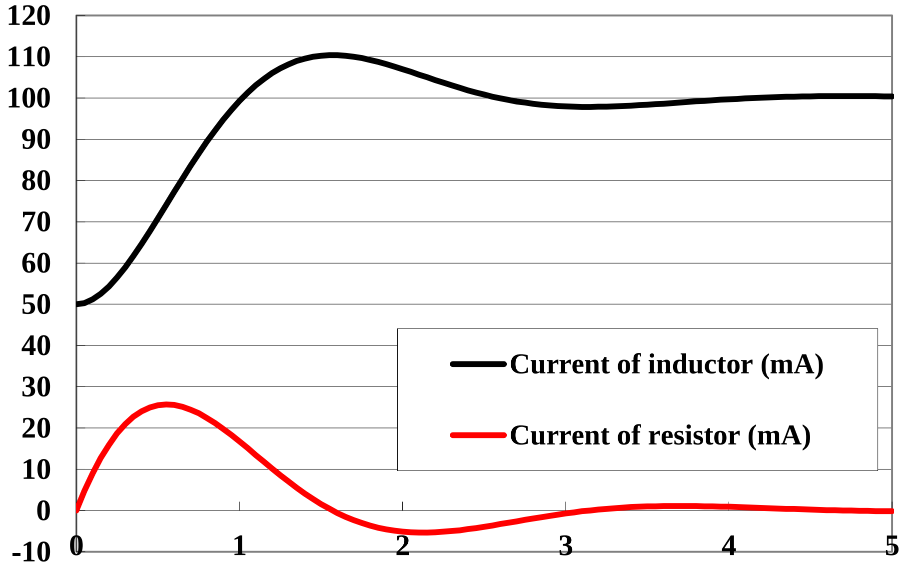
COMPARISON

Complete Response ($t > 0$)

$$i_L(t) = 100 + 25e^{-10^3 t}(-2\cos 2000t - \sin 2000t)\ \text{mA}$$

$$v_C(t) = 25e^{-1000t}\sin 2000t\ \text{V}$$

$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_L(t)}{R} = 50e^{-1000t}\sin 2000t\ \text{mA}$$

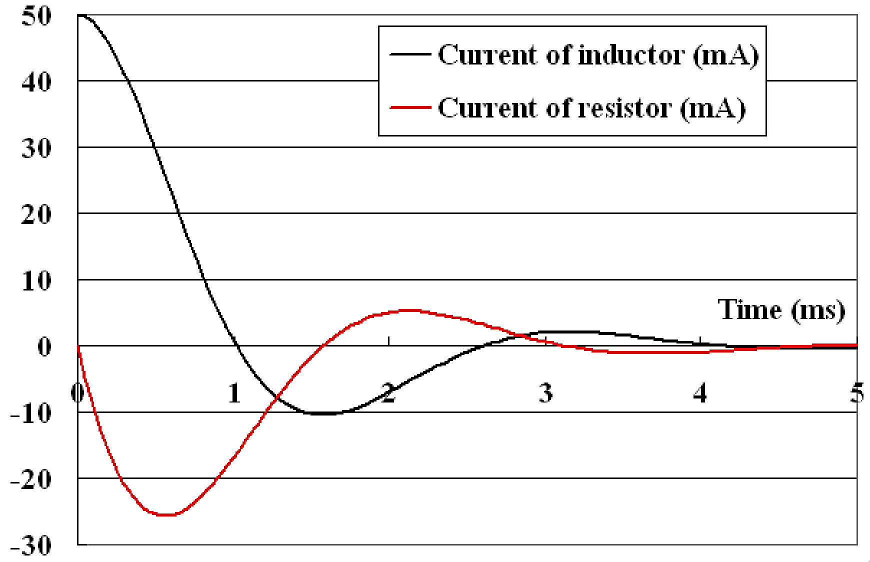


Natural Response ($t > 0$)

$$i_L(t) = 25e^{-10^3 t}(2\cos 2000t + \sin 2000t)\ \text{mA}$$

$$v_C(t) = -25e^{-1000t}\sin 2000t\ \text{V}$$

$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_L(t)}{R} = -50e^{-1000t}\sin 2000t\ \text{mA}$$



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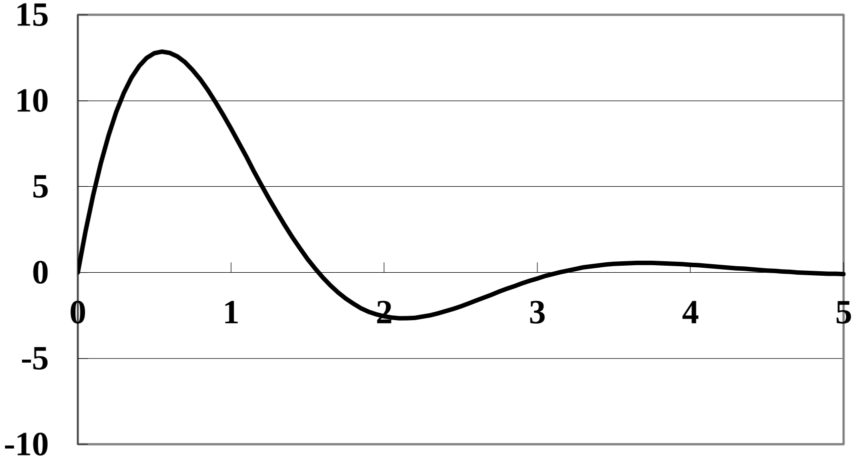
Voltage (V)



Natural Response ($t > 0$)

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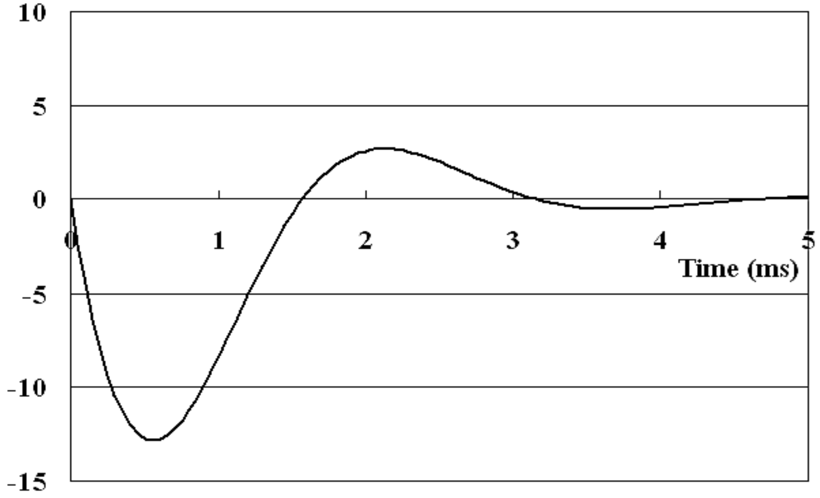

Applications: B**andpass/Bandstop** filters

OUTLINE

- 2nd Transient Analyses (Step Response)
  - ✓ Parallel *RLC* Circuits
    - Response Form of the Circuit (2nd ODE)
    - Solutions to the 2nd ODE
  - ✓ Series *RLC* Circuits
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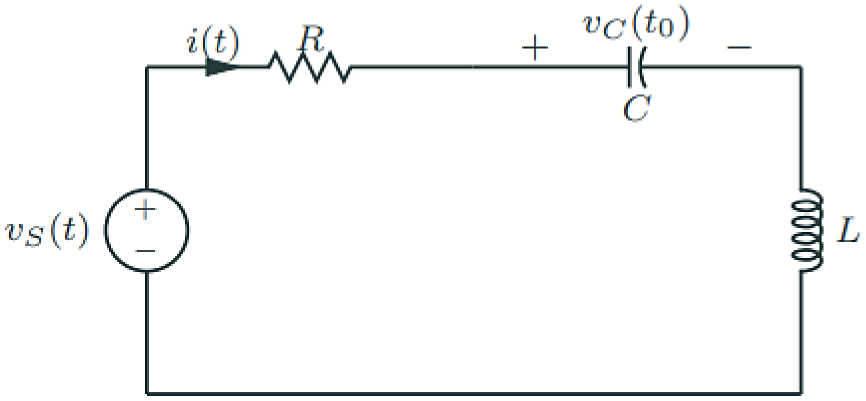
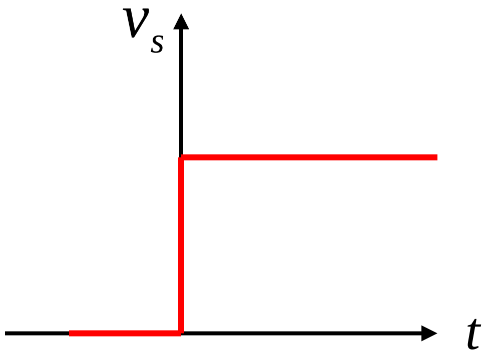
2.1 SERIES RLC CIRCUIT

Apply KVL:





$$v_C(t) + v_R(t) + v_L(t) = v_S(t)$$

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_c = \frac{1}{LC}v_s$$

General Solution:

$$\boxed{v_c(t) = v_{c,f}(t) + v_{c,n}(t) = v_{ss}(\infty) + v_{c,n}(t)}$$

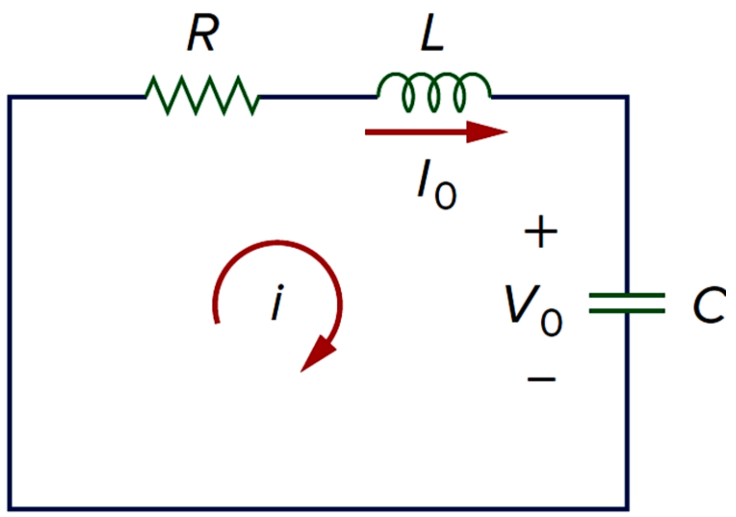
Complete
response

$$\left\{\begin{aligned} & i_R = i_L = i_C(t) = C\frac{dv_C}{dt} \\ & v_R = Ri_R = RC\frac{dv_C}{dt} \\ & v_L(t) = L\frac{di_L}{dt} = LC\frac{d^2 v_C}{dt^2} \end{aligned}\right.$$

The natural response is the same as the source-free case.

The forced response should be the steady-state case ($t = \infty$).

RECALL NATURAL RESPONSE...



Characteristic eq.:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\therefore s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$and \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

s_1 and s_2 are called complex frequencies.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\alpha = \frac{R}{2L}$$

- Over Damped → $\alpha > \omega_0$:

  s_1 & s_2 are two unequal real numbers

  Response: $v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

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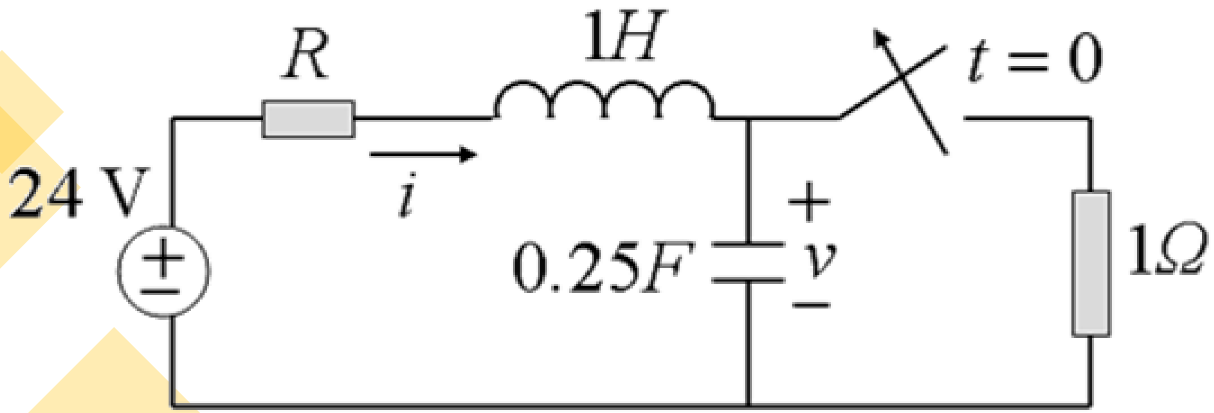
  Response:

  $$v_C(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

  $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

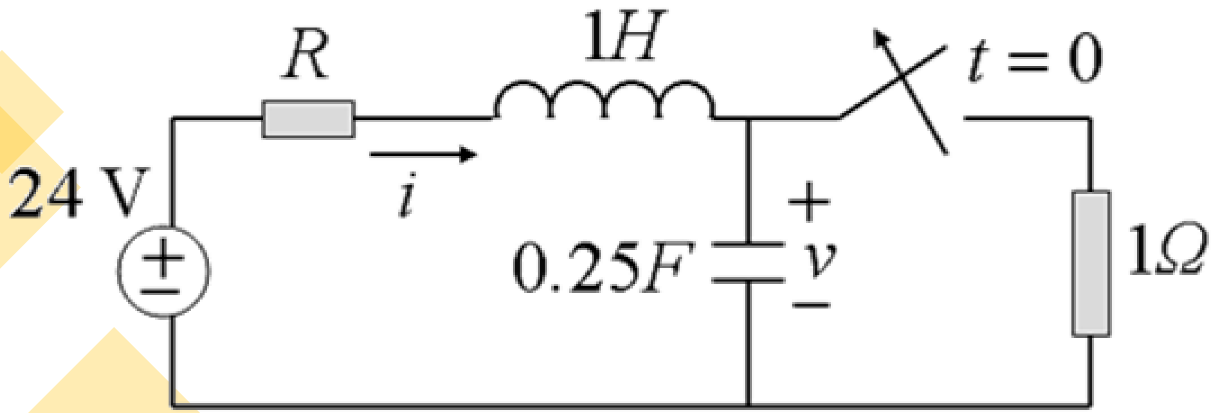
QUIZ 2

The circuit below has $R = 5\ \Omega$, $C = 0.25\ \mu\text{F}$ and $L = 1$ H. The switch has been closed for a long time and is open at $t = 0$. Find the capacitor voltage and inductor current for $t \geq 0$.



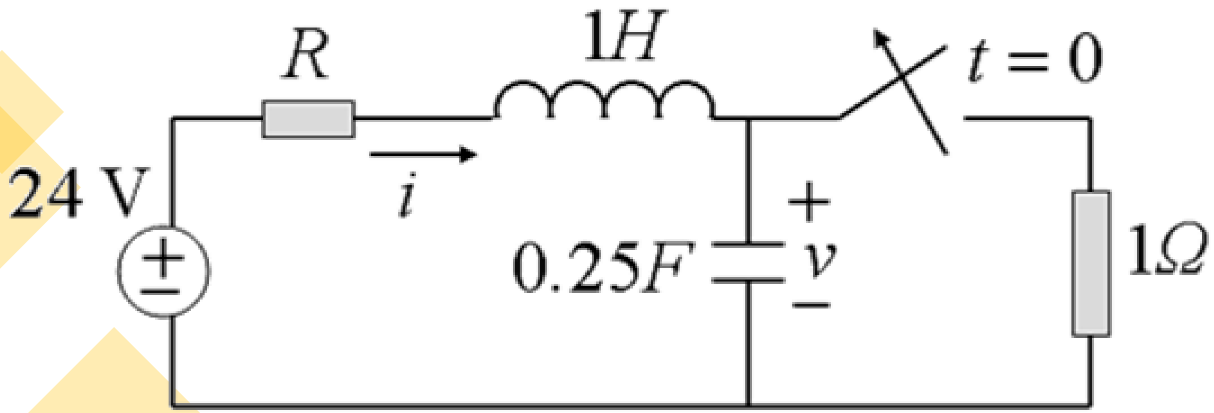
QUIZ 3

The circuit below has $R = 4\ \Omega$, $C = 0.25\ \mu\text{F}$ and $L = 1\ \text{H}$. The switch has been closed for a long time and is open at $t = 0$. Find the capacitor voltage and inductor current for $t \geq 0$.



QUIZ 4

The circuit below has $R = 1\ \Omega$, $C = 0.25\ \mu\text{F}$ and $L = 1$ H. The switch has been closed for a long time and is open at $t = 0$. Find the capacitor voltage and inductor current for $t \geq 0$.



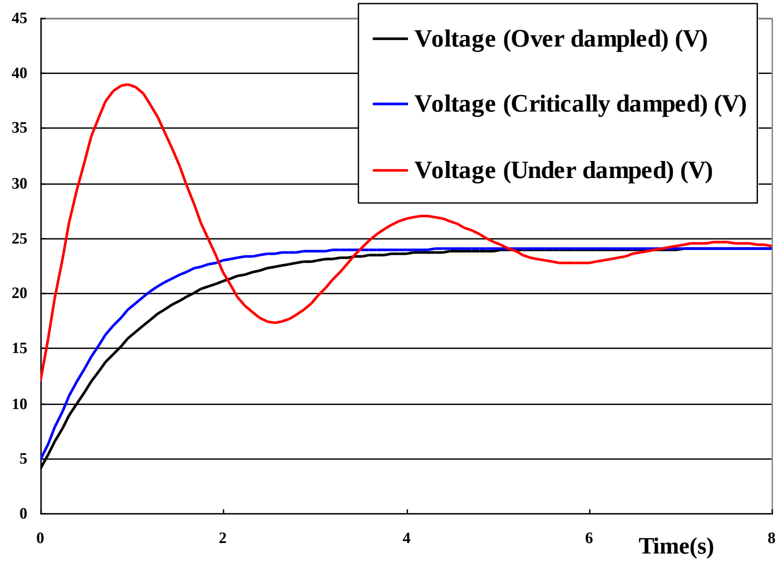
COMPARISON

Complete Response ($t > 0$)

$$v_C(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t})\ \text{V}$$

$$v_C(t) = 24 - 19.2(1 + t)e^{-2t}\ \text{V}$$

$$v_C(t) = 24 + (21.7 \sin 1.936\, t - 12 \cos 1.936\, t)e^{-0.5t}\text{V}$$

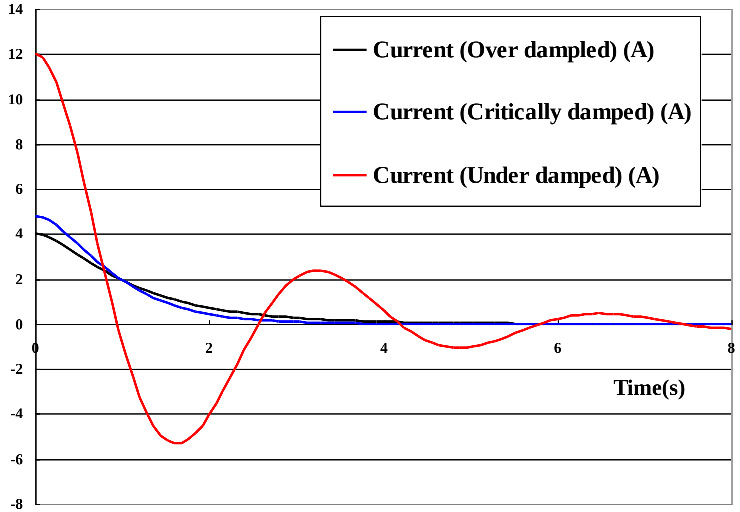


Natural Response ($t > 0$)

$$v_C(t) = 16e^{-500t} - e^{-8000t}\ \text{V}$$

$$v_C(t) = 15e^{-2000t}(2000t + 1)\ \text{V}$$

$$v_C(t) = e^{-500t}\left(15\cos\left(500\sqrt{15}t\right) + \sqrt{15}\sin\left(500\sqrt{15}t\right)\right)\text{V}$$



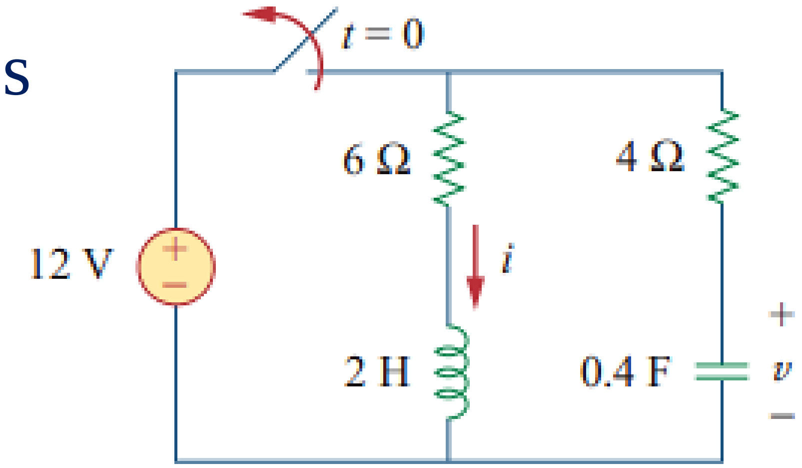
Applications:

A **bandpass** filter such as IF amplifier for the AM radio.

A **lowpass** filter with a **sharper** cutoff than with an RC circuit

QUIZ 5

1. For the given circuit, determine the initial values of $i(0^+)$ and $di(0^+)/dt$:



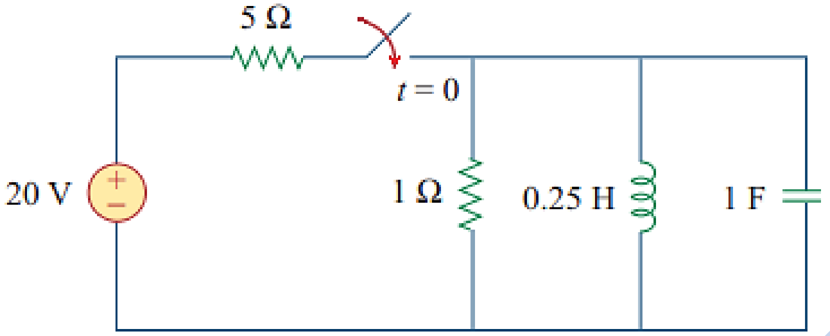
(a) 2 A; - 4 A/s (b) 1.2 A; 2 A/s

(c) 2 A; 0 A/s (d) 1.2 A; 1.2 A/s

2.For the given circuit on the right side, determine its damping case:

(a) under damped (b) over damped

(c) critical damped (d) un-damped



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3.1 GENERAL CASE

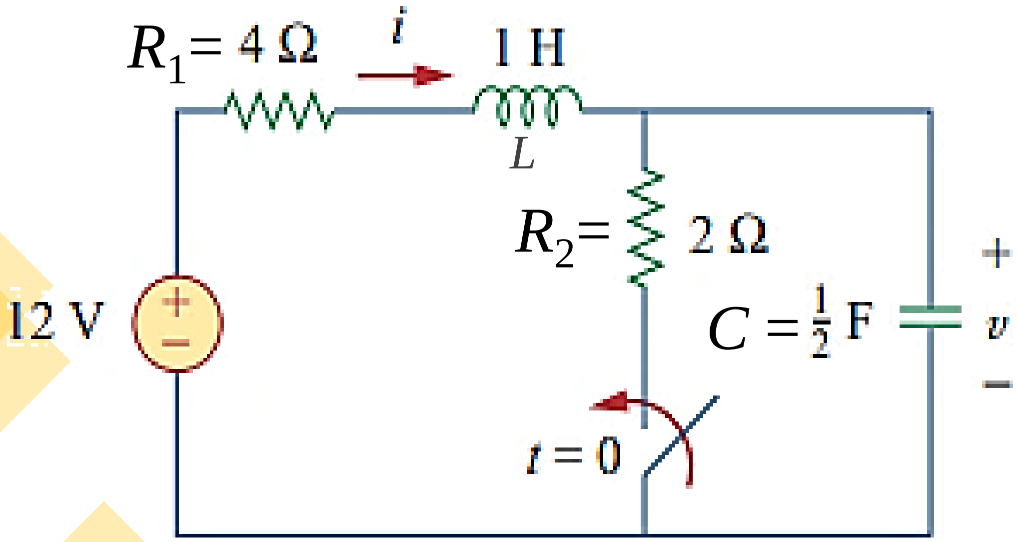
The idea of parallel/series *RLC* circuits can be extended to *any* 2nd order circuits.

Find the step response of a 2nd order system takes 5 steps:

1. Determine initial conditions $x(0)$ and $dx(0)/dt$, and the final value $x(\infty)$.
2. Turn off the independent sources and find the form of the natural response $x_n(t)$ by applying KCL or KVL.
3. Obtain the forced response as $x_f(t) = x(\infty)$.
4. The complete response is the sum of the natural response and forced response $x(t) = x_n(t) + x_f(t)$.
5. Determine the constants associated with the natural response by imposing the initial conditions $x(0)$ and $dx(0)/dt$ found in step 1.

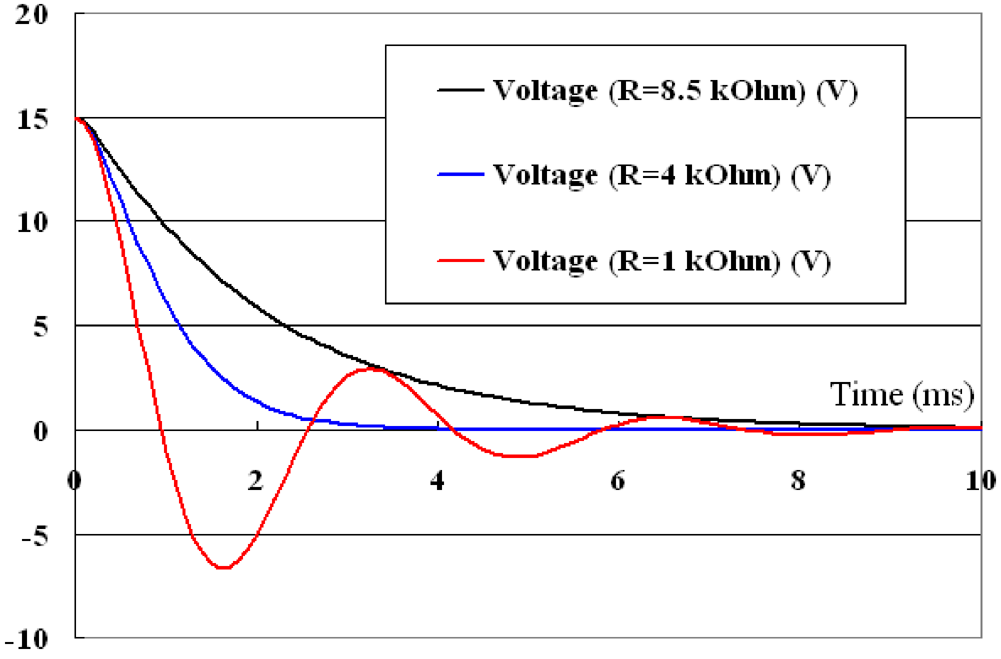
EXAMPLE

Find the complete response $v_c(t)$ and then $i_L(t)$ for $t > 0$.



DAMPING

Under Damped | Critical Damped | Over Damped



Take the series circuit as an example:

Black: R=8.5 kΩ – over

Blue: R=4 kΩ – critical

Red: R=1 kΩ – under

NEXT

- Three-phase Systems
- Tutorial