CAN207 Continuous and Discrete Time Signals and Systems

Lecture 17

Filtering

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Room SC340

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- 1. Continuous-Time (analogue) vs. Discrete-Time (digital) filtering
  - Frequency relationship
- 2. Ideal and Practical filters
  - Four types: lowpass, highpass, bandpass and bandstop
- 3. CT Filters examples
  - Simple electrical circuit filters
  - Butterworth filters
- 4. DT Filters examples
  - Recursive and non-recursive filters
  - FIR vs. IIR filters.

1. Analogue and Digital Filters

- A frequency-selective filter processes the received signal to eliminate (suppress) certain frequency components and pass (keep) other frequency components.
- Continuous-Time (Analogue) filters:

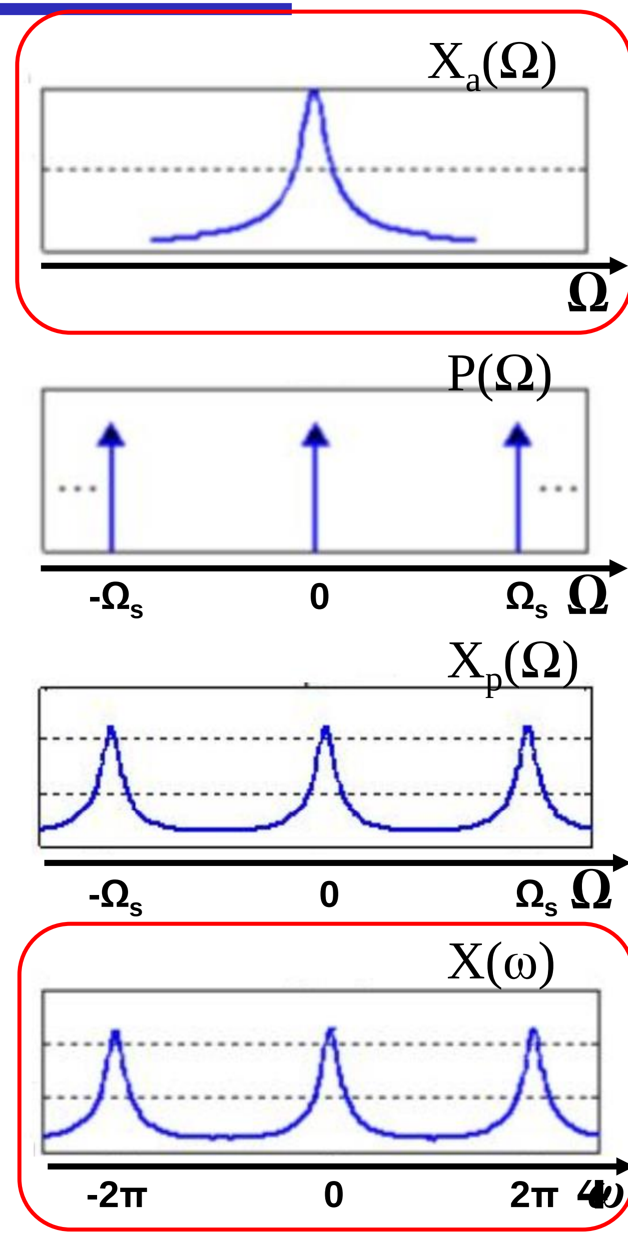
$$x(t) \longrightarrow \boxed{\text{LTIC filter}} \longrightarrow y(t)$$

- Discrete-Time (Digital) filters:

$$x(t) \longrightarrow \boxed{\text{Sampling}} \xrightarrow{x[n]} \boxed{\text{LTID filter}} \xrightarrow{y[n]} \boxed{\text{Recontruction}} \longrightarrow y(t)$$

1. Analogue and Digital Filters

- Frequency mapping
  - For CT signals and systems:
    - frequency range is $0 \leq |\Omega| < \infty$
    - Low frequency: 0
    - High frequency: $\pm\infty$
  - For DT signals and systems:
    - frequency range is $0 \leq |\omega| < \pi$
    - Low frequency: $0 = 2\pi$
    - High frequency: π
  - Relationship: $\frac{\Omega}{\Omega_s} = \frac{\omega}{2\pi}$



2.1 Ideal Filters

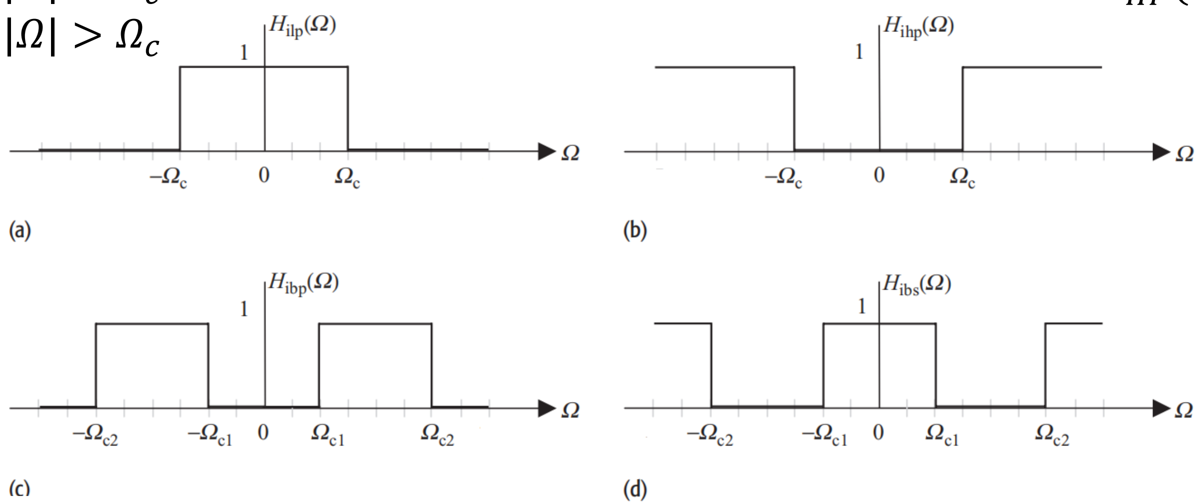
- An ideal frequency-selective filter is a system that passes a prespecified range of frequency components without any attenuation but completely rejects the remaining frequency components.
- Four types of CT filters

Low-pass

$$H_{LP}(\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

High-pass

$$H_{HP}(\Omega) = \begin{cases} 1, & |\Omega| \ge \Omega_c \\ 0, & |\Omega| < \Omega_c \end{cases}$$



Band-pass

$$H_{BP}(\Omega) = \begin{cases} 1, & \Omega_{c1} \le |\Omega| \le \Omega_{c2} \\ 0, & \text{others} \end{cases}$$

Band-stop

$$H_{BS}(\Omega) = \begin{cases} 1, & \text{others} \\ 0, & \Omega_{c1} \le |\Omega| \le \Omega_{c2} \end{cases}$$

2.1 Ideal Filters

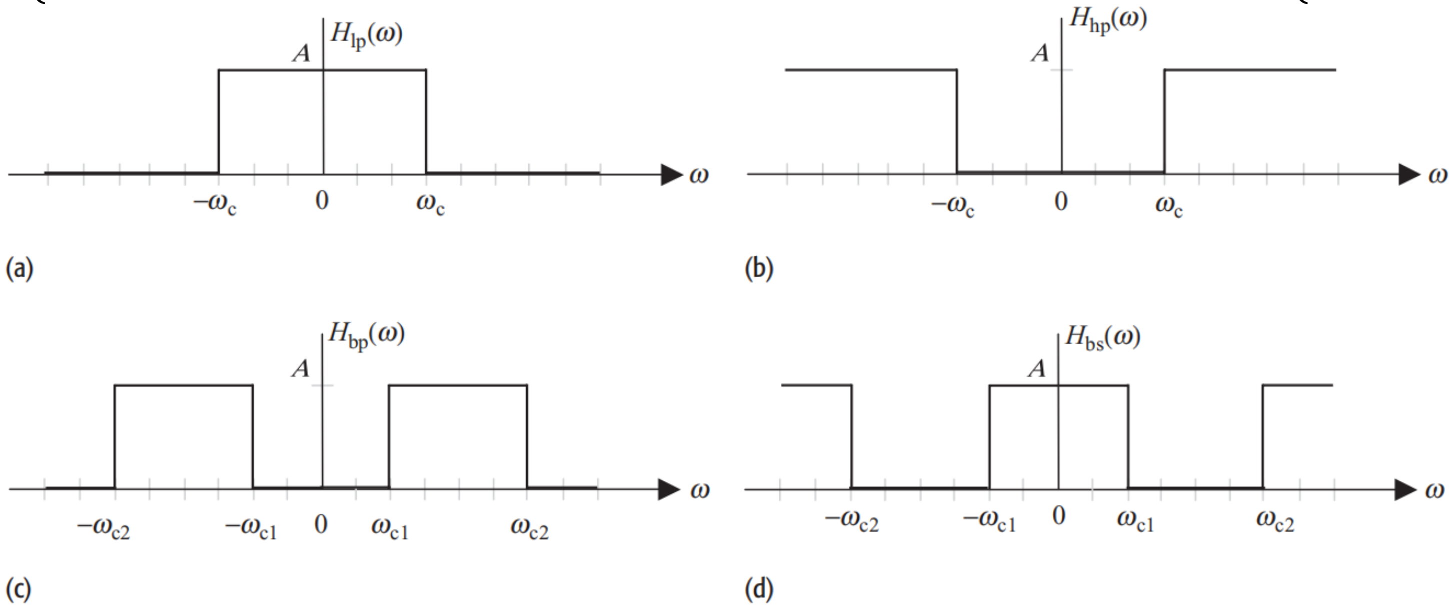
- Four types of DT filters

Low-pass

$$H_{LP}(\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & \omega_c<|\omega|\le\pi\end{cases}$$

High-pass

$$H_{HP}(\omega)=\begin{cases}1, & \omega_c\le|\omega|<\pi\\ 0, & |\omega|<\omega_c\end{cases}$$



Band-pass

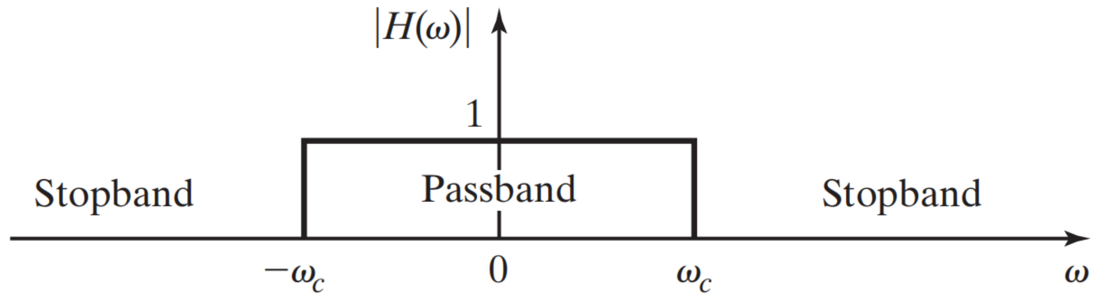
$$H_{BP}(\omega)=\begin{cases}1, & \omega_{c1}\le|\omega|\le\omega_{c2}\\ 0, & \text{others}\end{cases}$$

Band-stop

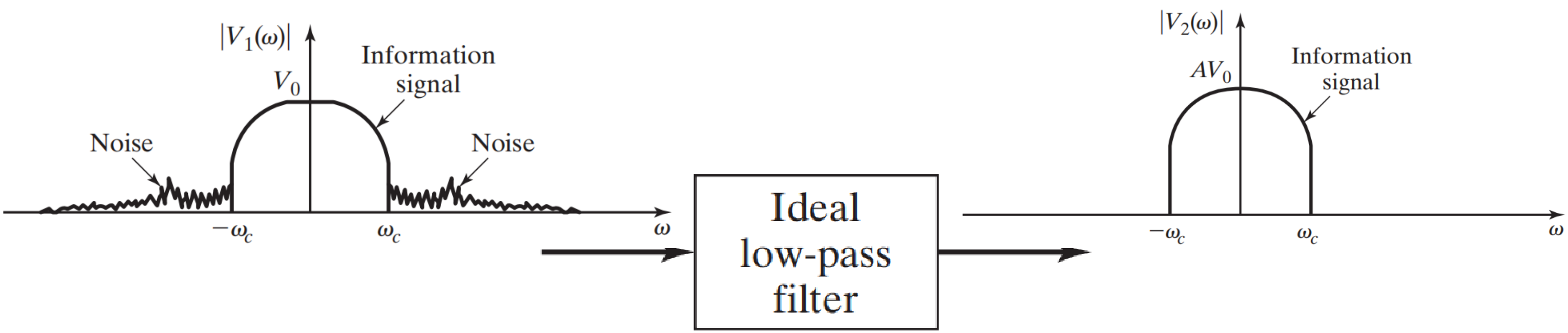
$$H_{BS}(\omega)=\begin{cases}1, & \text{others}\\ 0, & \omega_{c1}\le|\omega|\le\omega_{c2}\end{cases}$$

2.1 Ideal Filters

- Example: a low-pass filter
  - Key parameters:
    - Passband gain
    - Cut-off frequency

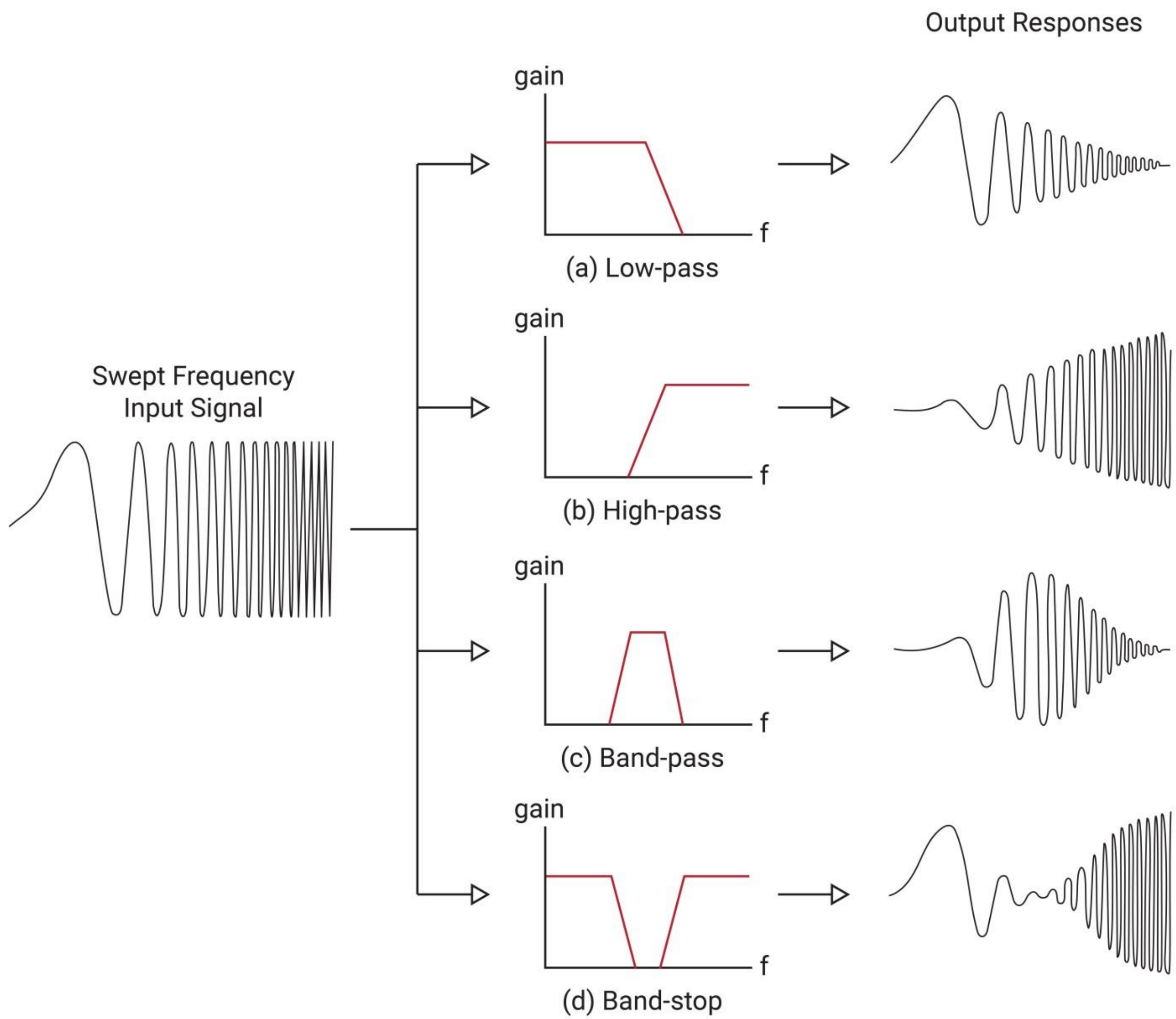


- In such a system:



2.1 Ideal Filters

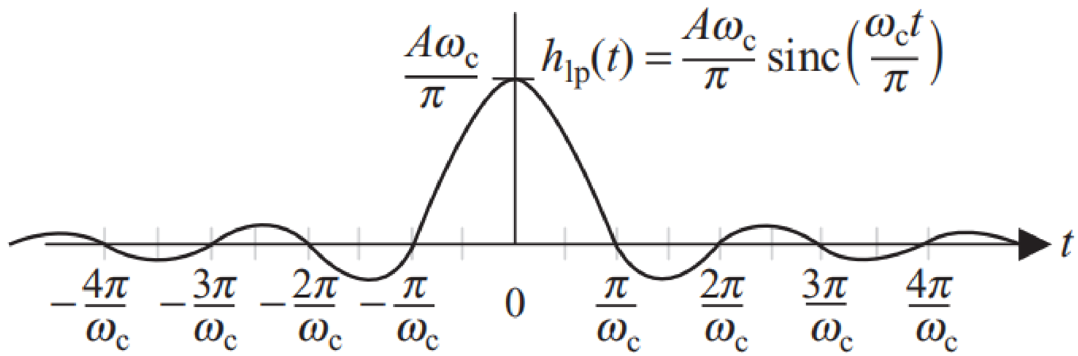
- Another example:



2.1 Ideal Filters

- Impulse response of an ideal filter:
  - LPF: with a gain of A within the passband and a cut-off frequency of Ω_c

$$h_{lp}(t) = \frac{A\Omega_c}{\pi} sinc\left(\frac{\Omega_c t}{\pi}\right)$$

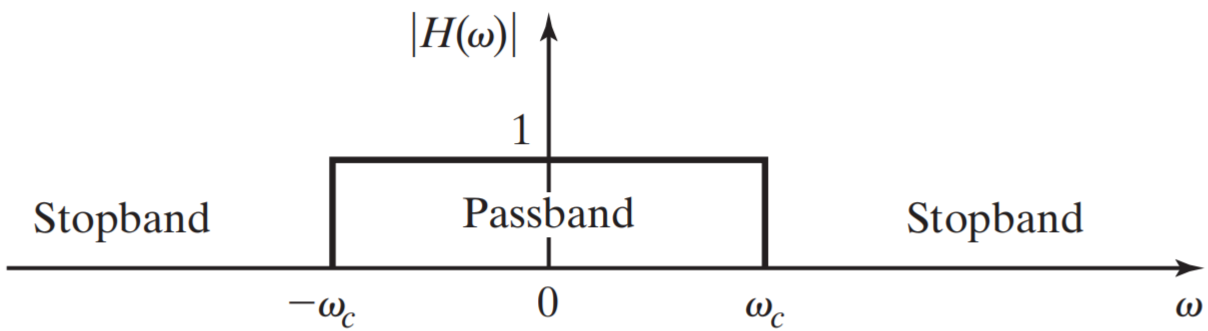


  - the filter has an infinite length in the time domain
  - the filter is non-causal

not physically realizable

2.2 Practical Filters

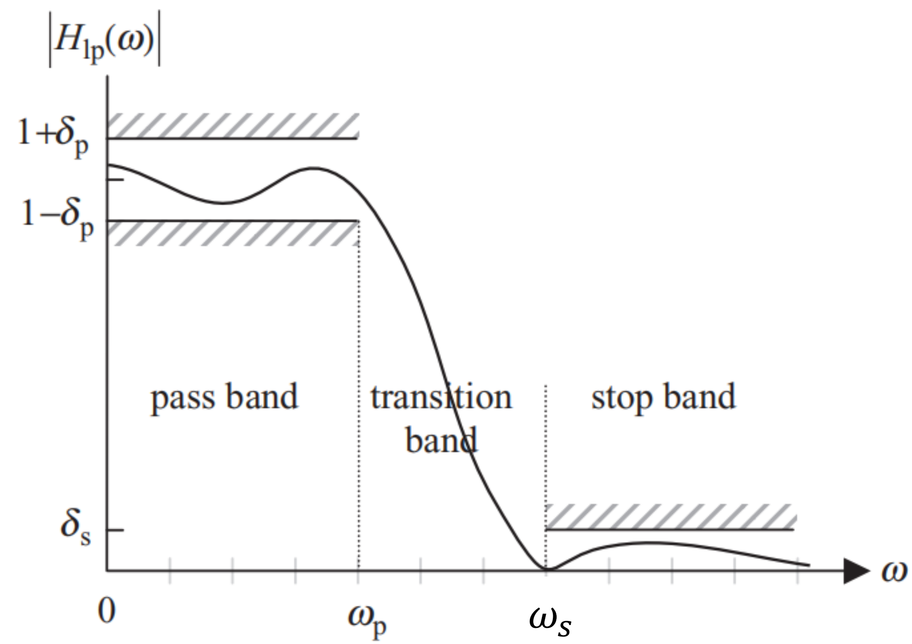
- Problems of the ideal filters:
  - **constant gains** within the pass and stop bands.
  - **abrupt transitions** from passbands to stopbands.



  - To obtain a physically realizable filter, it is necessary to **relax** some of the requirements of the ideal filters.

2.2 Practical Filters

- Relaxed requirements:
  - The gains of the practical filters within the pass and stop bands are not constant but vary within the following limits:
    - Pass bands: $1-\delta_p \leq |H(\omega)| \leq 1+\delta_p$
    - Stop bands: $0 \leq |H(\omega)| \leq \delta_s$
  - Transition bands of non-zero bandwidth are included in between the pass and stop bands of the practical filters.
    - Pass band edge: $H(\omega_p) = 1-\delta_p$
    - Stop band edge: $H(\omega_s) = \delta_s$
    - the discontinuity at the cut-off frequency is eliminated.

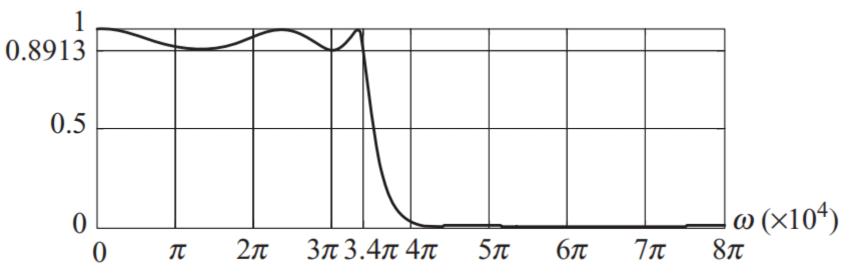


2.2 Practical Filters

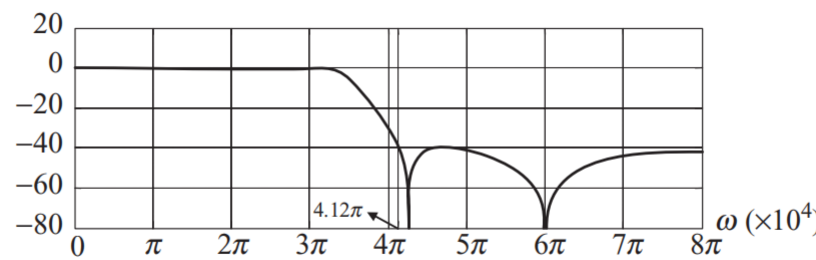
- Example: consider a practical lowpass filter with the following transfer function:

$$H(s)=\frac{5.018\times10^{3}s^{4}+2.682\times10^{14}s^{3}-1.026\times10^{4}s+3.196\times10^{24}}{s^{5}+9.863\times10^{4}s^{4}+2.107\times10^{10}s^{3}+1.376\times10^{15}s^{2}+1.026\times10^{20}s+3.196\times10^{24}}.$$

- Assuming that the ripple δ_p within the pass band is limited to 1 dB and the ripple δ_s within the stop band is limited to 40 dB, determine the pass band, transition band and stop band of the lowpass filter.



(a)



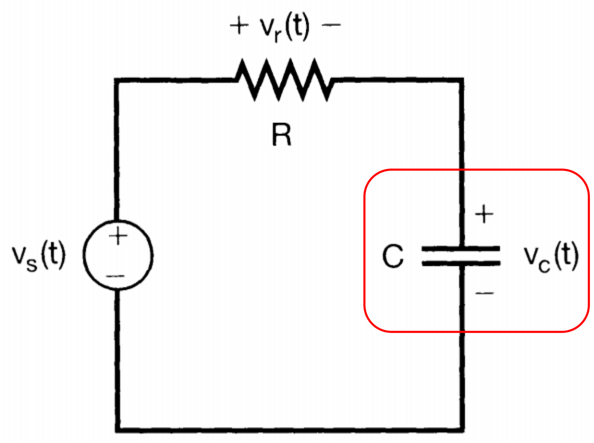
(b)

Quiz 1

- Electro-cardiology is the study of the electric signals produced by the heart. These signals maintain the heart's rhythmic beat, and can be measured by an instrument called an electrocardiograph. This instrument must be capable of detecting periodic signals whose frequency is about **1 Hz** (72 beat/minute), and be able to remove the signals from surrounding electrical environment of **50 Hz** – the frequency at which electric power is supplied.
- Determine the cut off frequency for an analogue filter;
- Determine the cut off frequency for a digital filter (if the signal is sampled every 5 ms).

3.1 Simple electrical circuits

- 1. A simple RC circuit

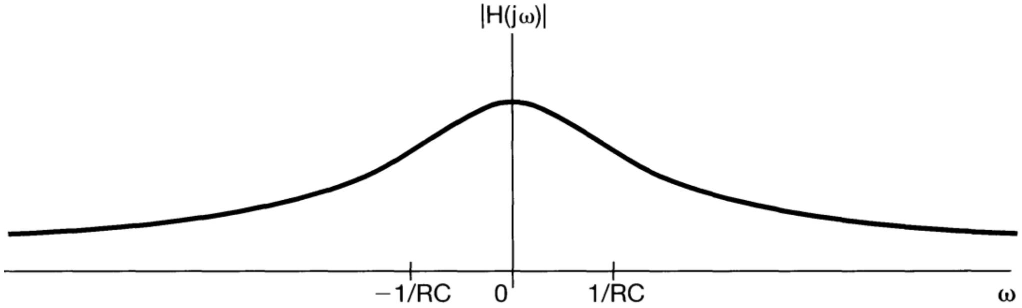


$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t).$$

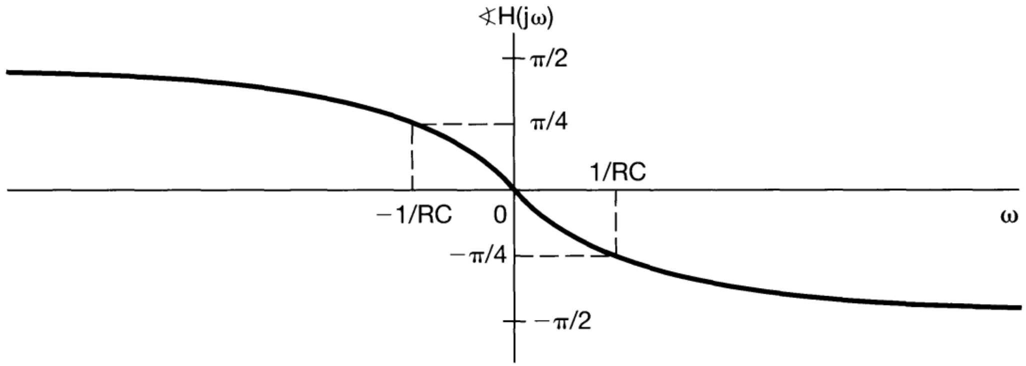
$$\Downarrow$$

$$H(j\omega) = \frac{1}{1 + RCj\omega}.$$

Lowpass



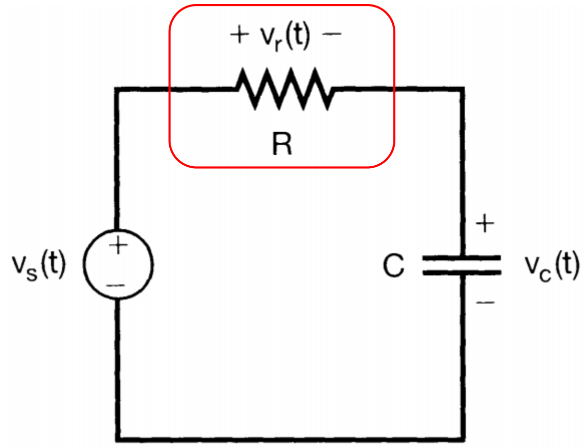
(a)



(b)

3.1 Simple electrical circuits

- 2. Another RC circuit

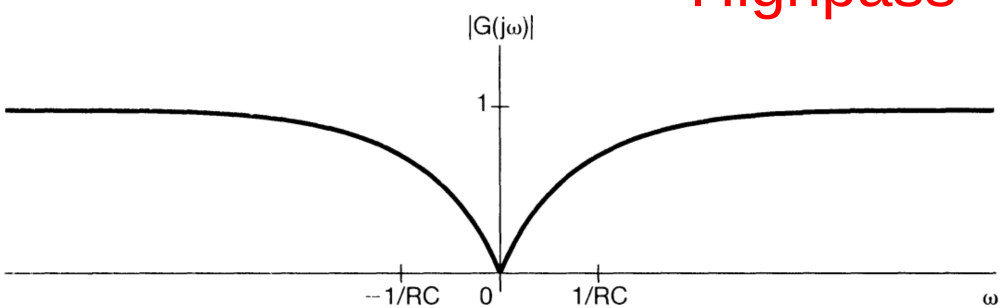


$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}.$$

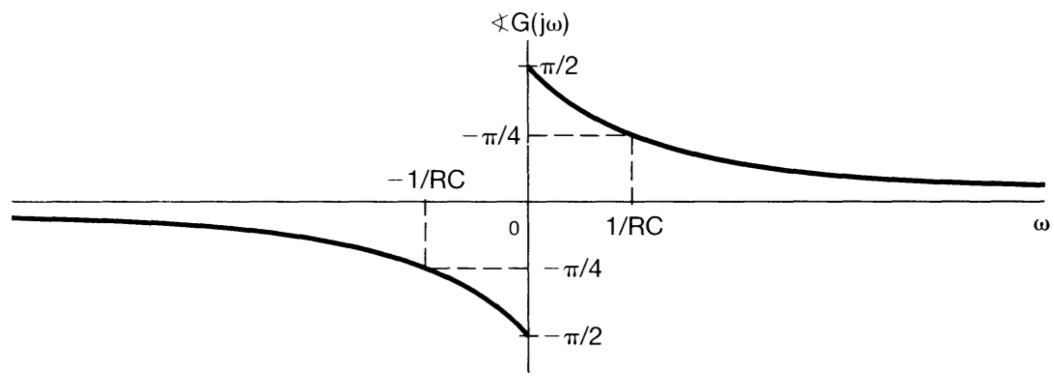
$$\Downarrow$$

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}.$$

Highpass



(a)



(b)

3.1 Simple electrical circuits

- First order circuits:
  - RC & RL
  - Possible types: lowpass or highpass
- Second order circuits:
  - RLC (or RCC, RLL)
  - Possible types: lowpass, highpass, bandpass or bandstop
- Limitations:
  - Selection of R, L and C values can only change cut-off frequency
  - Cannot design based on the requirement of transition bandwidth and passband and stopband ripples

3.2 The Butterworth Filters (Optional)

- The magnitude-square response of an N^th order analogue lowpass Butterworth filter:

$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

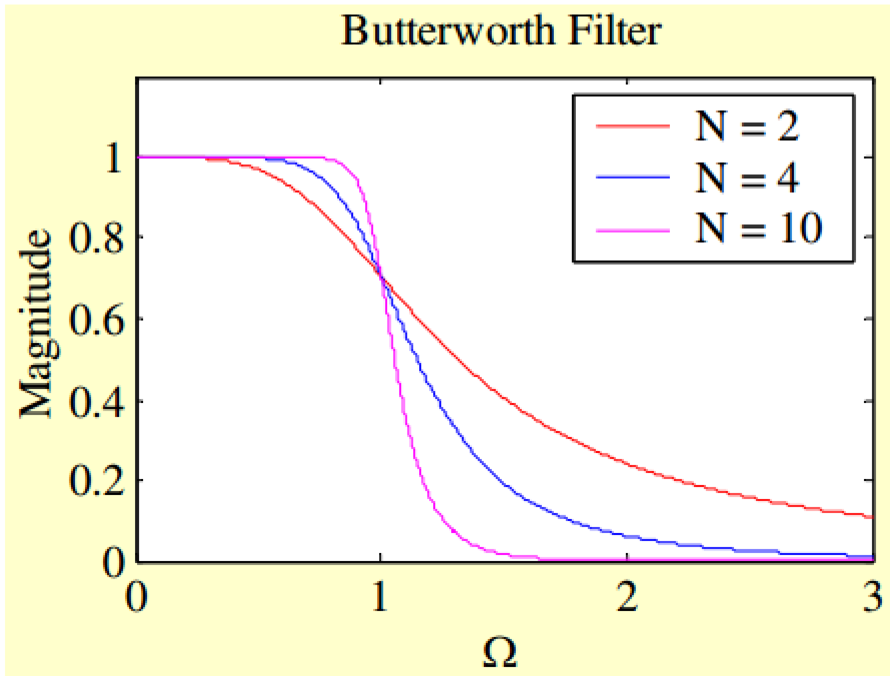
  – Ω_c is the 3-dB cutoff frequency ($20log|H(\Omega_c)| = -3\ dB$), N is the filter order;

  – The most interesting property of this function is that the first 2N-1 derivatives of this function are zero at **Ω**=0.

$$\left.\frac{d^k}{d\Omega^k}\left(\left|H_a(j\Omega)\right|^2\right)\right|_{\Omega=0} = 0, \quad 1 \le k \le 2N-1$$

  – Order N increasing:
    - Reducing transition band;
    - Increasing smoothness near Ω=0.

The Butterworth approximation is also called a *maximally flat* approximation.



3.2 The Butterworth Filters (Optional)

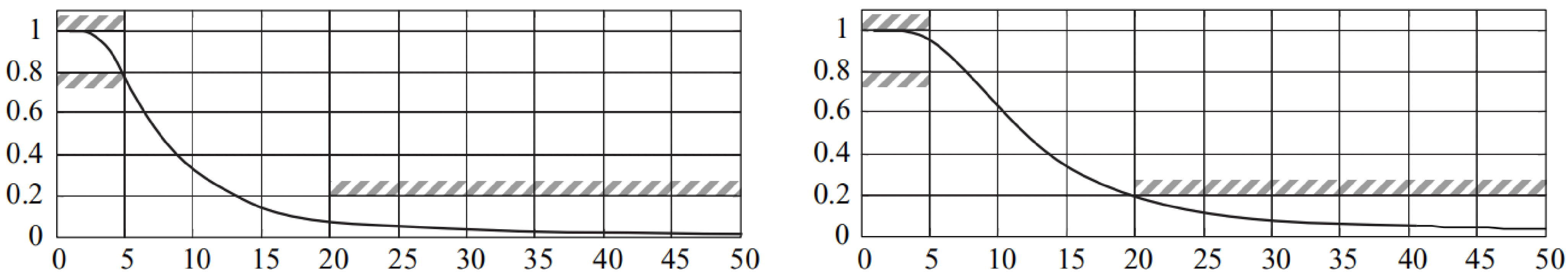
- Example: Design a Butterworth lowpass filter with the following specifications:
  - pass band ($0 \le |\omega| \le 5$ radians/s) $0.8 \le |H(\omega)| \le 1$;
  - stop band ($|\omega| > 20$ radians/s) $|H(\omega)| \le 0.20$.

- Design steps:
  - Step 1: Determine the order N of the Butterworth filter.
  - Step 2: Determine the transfer function for the normalized Butterworth filter of order N.
  - Step 3 Determine the cut-off frequency ω_c.
  - Step 4 Determine the transfer function H(s) from the normalized Butterworth filter H(S).

3.2 The Butterworth Filters (Optional)

- Example: Design a Butterworth lowpass filter with the following specifications:
  - pass band ($0 \le |\omega| \le 5$ radians/s) $0.8 \le |H(\omega)| \le 1$;
  - stop band ($|\omega| > 20$ radians/s) $|H(\omega)| \le 0.20$.

- Results:



(a) satisfies the constraint at the pass-band edge frequency

(b) satisfies the constraint at the stop-band edge frequency

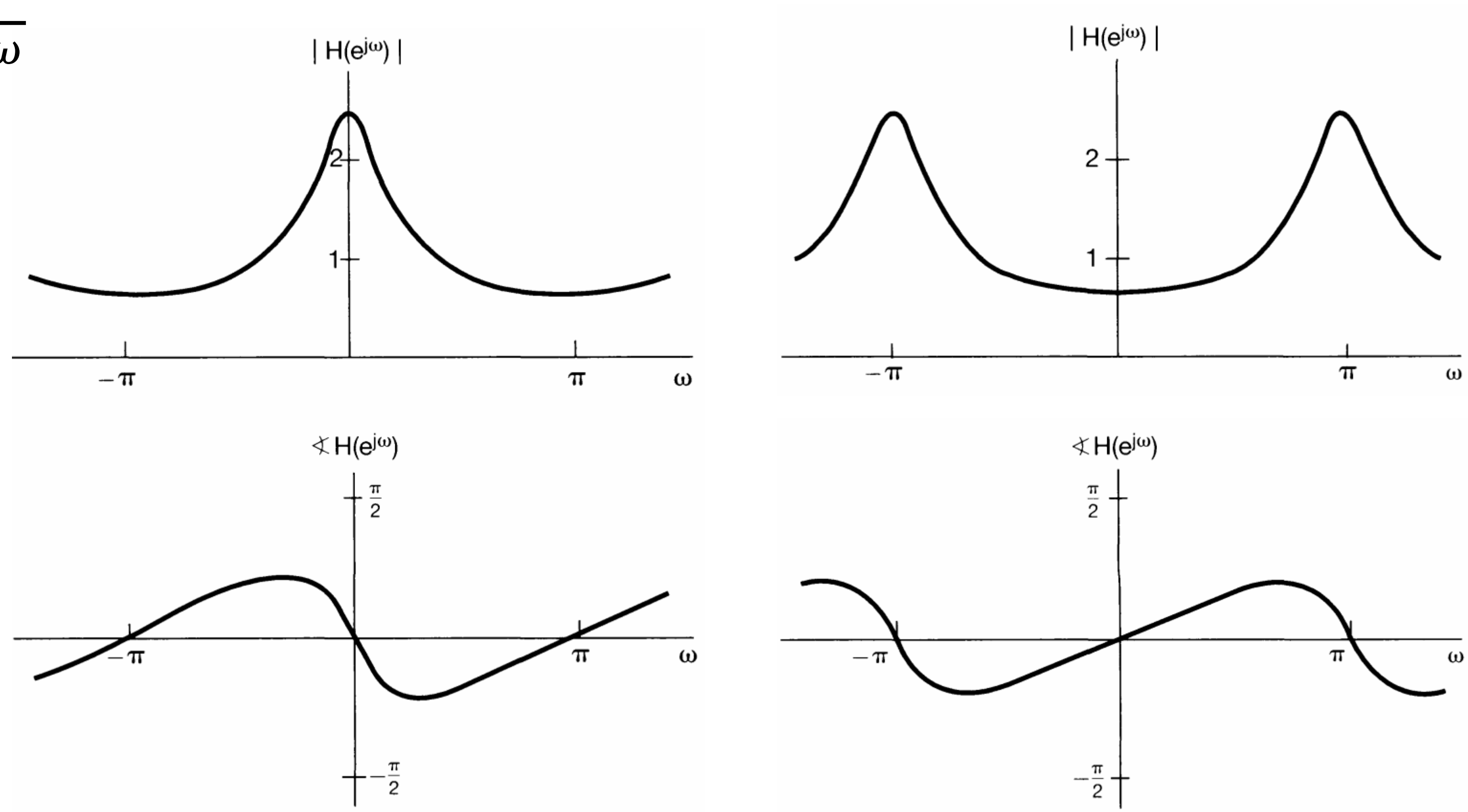
4.1 Recursive filter

- The DT counterpart of sec.3.1 is the LTID systems described by the first-order difference equation:

$$y[n] - ay[n-1] = x[n]$$

$$H(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

$a = 0.6$ $\qquad$ $a = -0.6$



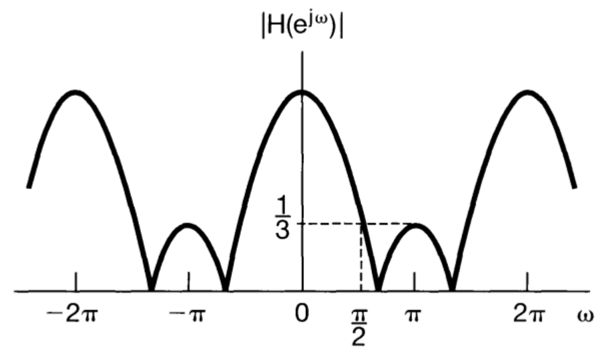
- It depends on its own value at earlier time, so it's called the **recursive** filter.

4.1 Non-recursive filter

- If the output doesn't depend on its own previous value, then it's called **non-recursive**:

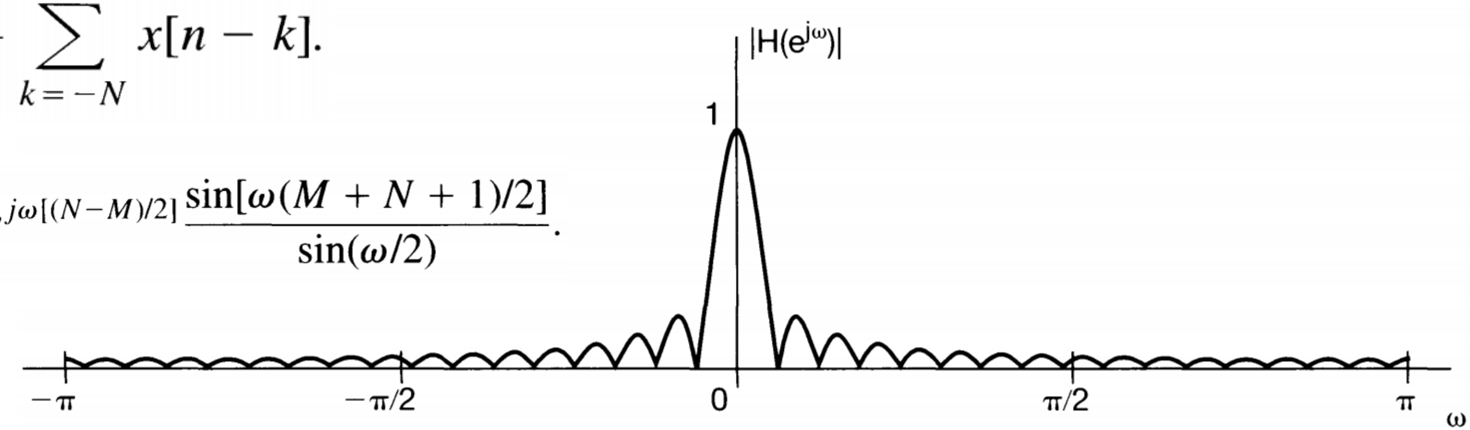
$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$$

$$H(\omega) = \frac{1}{3}(1 + 2\cos\omega)$$



$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k].$$

$$H(e^{j\omega}) = \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}.$$



4.2 FIR systems

- If the impulse response h[n] of a system is of finite length, that system is referred to as a finite impulse response (FIR) system

$$h[n] = 0 \ \text{ for } n < N_1 \text{ and } n > N_2, \quad N_1 < N_2$$

- The output of such a system can then be computed as a finite convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

4.2 FIR systems

- For a causal LTI system h[n]:

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k] = \sum_{k=0}^{M} h[k]x[n-k] = \sum_{j=0}^{M} b_j x[n-j]$$

- Compare with the CCLDE:

$$y[n] = \cancel{\sum_{i=0}^{N} a_i y[n-i]} + \sum_{j=0}^{M} b_j x[n-j]$$

- FIR systems are also called non-recursive systems, where the output can be computed from the current and past input values only – without requiring the values of previous outputs

4.2 IIR systems

- If the impulse response is of infinite length, then the system is referred to as an infinite impulse response (IIR) system.
  - These systems cannot be characterized by the convolution sum due to infinite sum.
  - Instead, they are typically characterized by linear constant coefficient difference equations (LCCDEs)

$$y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j]$$

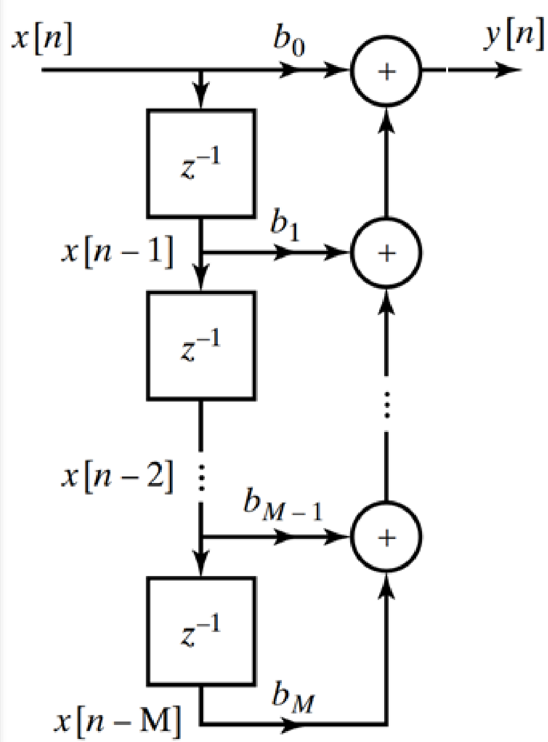
  - There output y[n] relies on both current and previous input x[n-j], and also previous output y[n-i]. It is called a ***recursive system***

4.2 FIR vs. IIR – Implementation

- Direct Forms of FIR and IIR are drawn below, according to their LCCDE:

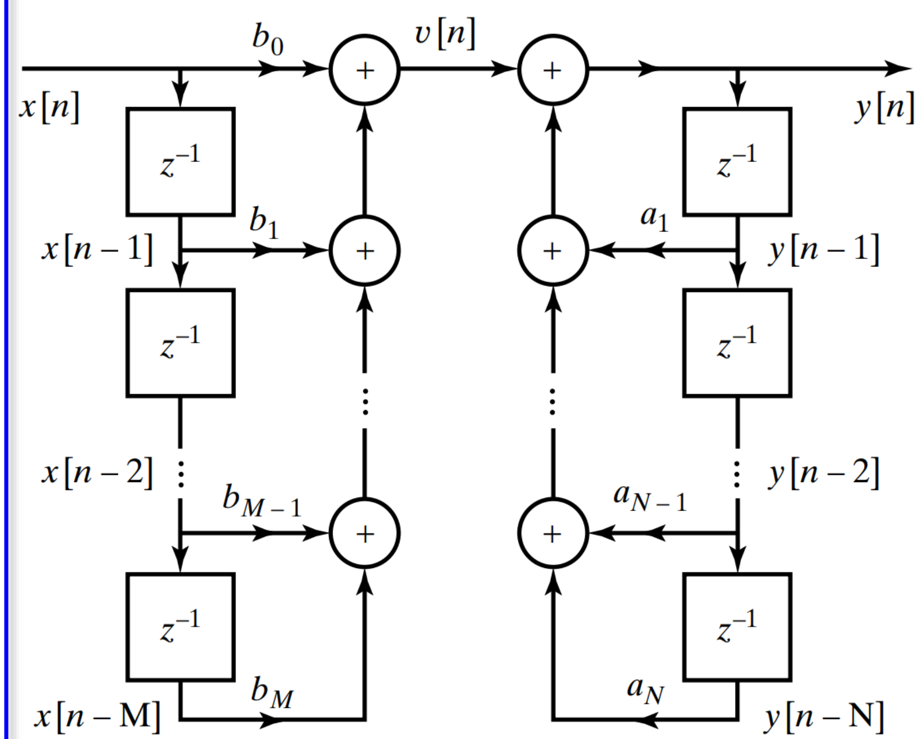
FIR

$$y[n] = \sum_{j=0}^{M} b_j x[n-j]$$



IIR

$$y[n] = \sum_{i=1}^{N} a_i y[n-i] + \sum_{j=0}^{M} b_j x[n-j]$$



Next ...

- Sampling
- z-transform
  - Generalization from DTFT
  - ROC
  - Compare with s-transform (Laplace transform)
  - Zeros and poles
  - Graphical evaluation