CAN209 Advanced Electrical Circuits and Electromagnetics

Lecture 10 Frequency Response of LTI Systems

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OUTLINE

- > Review: Reactance and Impedance
- > Frequency Response
 - ✓ Definition
 - ✓ Filter Types
 - ✓ Decibel Scale
- First-order Circuits
 - ✓ Series Circuits (*RL* & *RC*)
 - ✓ Parallel Circuits (*RL* & *RC*)
- > Second-order RLC Circuits (series & parallel)

1.1 SINE (COSINE) WAVES

For a sinusoidal (or AC) voltage:

$$v(t) = V_p \sin(\omega t + \varphi)$$

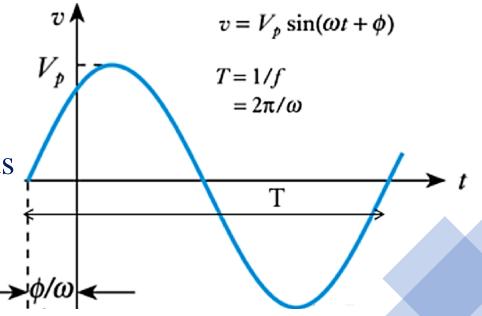
where V_p is the peak voltage, ω is the angular frequency, φ is the phase angle.

If φ is in radians, then the time delay is given by:

$$t_d = \frac{\varphi}{\omega}$$

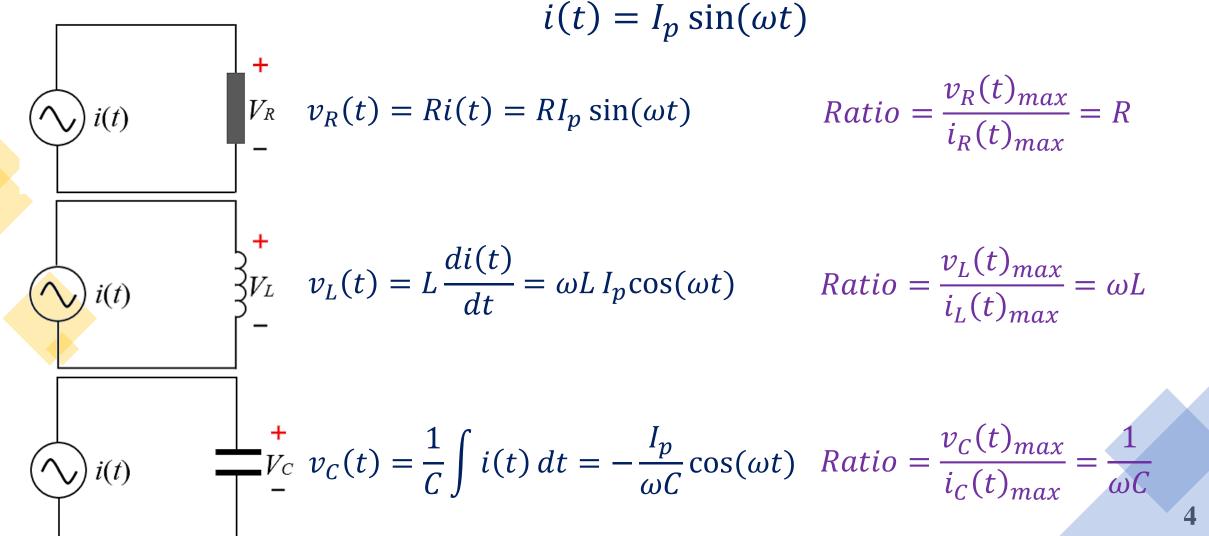
Remember that one cycle of a sinewave corresponds to a phase angle of 2π radians or 360° .

Cosine waves are the same as sine waves, expect that cosine wave is 90° advance in phase.



1.2 V-I RELATIONSHIPS

Consider that a sinusoidal current is applied for the following circuits:



1.3 REACTANCE & IMPEDANCE

The ratio of peak voltage to peak current is a measure of how a component opposes the electricity flow.

In a resistor, this ratio is the resistance *R*.

In an inductor or a capacitor, it is termed the reactance *X*.

Reactance can be viewed as 'resistance' for capacitors or inductors. It only considers peak voltage and current in a component.

Impedance of a two-terminal circuit element is the ratio of the complex representation of the sinusoidal voltage between its terminals, to the complex representation of the current flowing through it.

For example (for a capacitor):

$$Z_C = \frac{v_C(t)}{i_C(t)} = \frac{-\frac{I_p}{\omega C}\cos(\omega t)}{I_p\sin(\omega t)} = \frac{1}{j\omega C}$$

1.3 REACTANCE & IMPEDANCE

The use of complex number allows us to treat capacitors and inductors in a similar way to resistor.

Component	Reactance	Impedance
Resistor	$X_R = 0$	$Z_R = R$
Inductor	$X_L = \omega L$	$Z_L = j\omega L$
Capacitor	$X_C = \frac{1}{\omega C}$	$Z_C = \frac{1}{j\omega C}$

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BACKGROUND

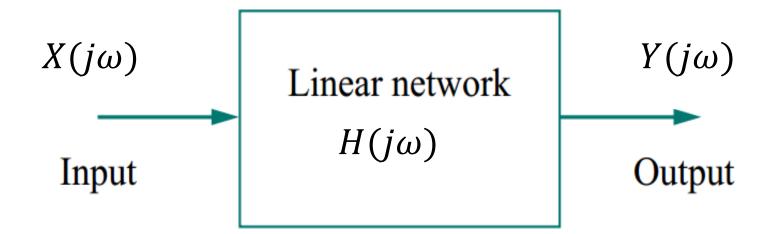
When we process a signal, usually, we're trying to enhance the meaningful part, and reduce the noise:

- Spectrum helps us to understand which part is meaningful, and which part is noise.
- \therefore Convolution (a.k.a. filtering) is the tool we use to perform the enhancement.
- : Frequency Response of a filter tells us exactly which frequencies it will enhance, and which it will reduce.

2.1 DEFINITION

The **steady state response** of a system for an input <u>sinusoidal</u> signal is known as the **frequency response**. Frequency response $H(j\omega)$ is the frequency-dependent ratio of a phasor output $Y(j\omega)$ (an element voltage or current) to a phasor input $X(j\omega)$ (source voltage or current).

In LTI systems, the analysis in frequency domain is easier than in time domain.

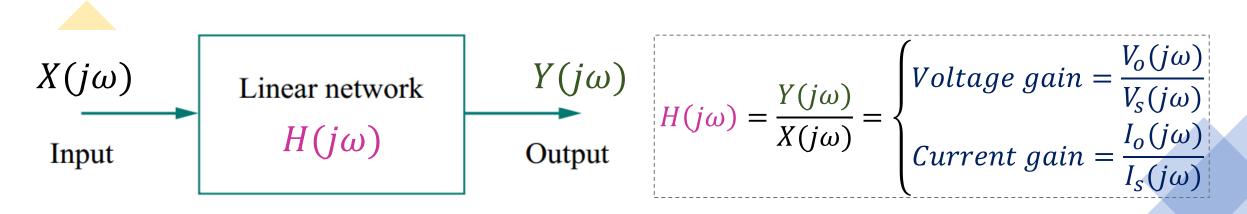


2.1 DEFINITION

The frequency response is evaluated from the sinusoidal transfer function by replacing s by $j\omega$ in the system transfer function T(s). The transfer function $H(j\omega)$ has both magnitude and phase angle.

Consider a linear system with a sinusoidal input: $x(t) = Asin(\omega t)$

Under steady-state, the system output as well as the signals at all other points in the system are sinusoidal. The steady-state output is: $y(t) = Bsin(\omega t + \theta)$



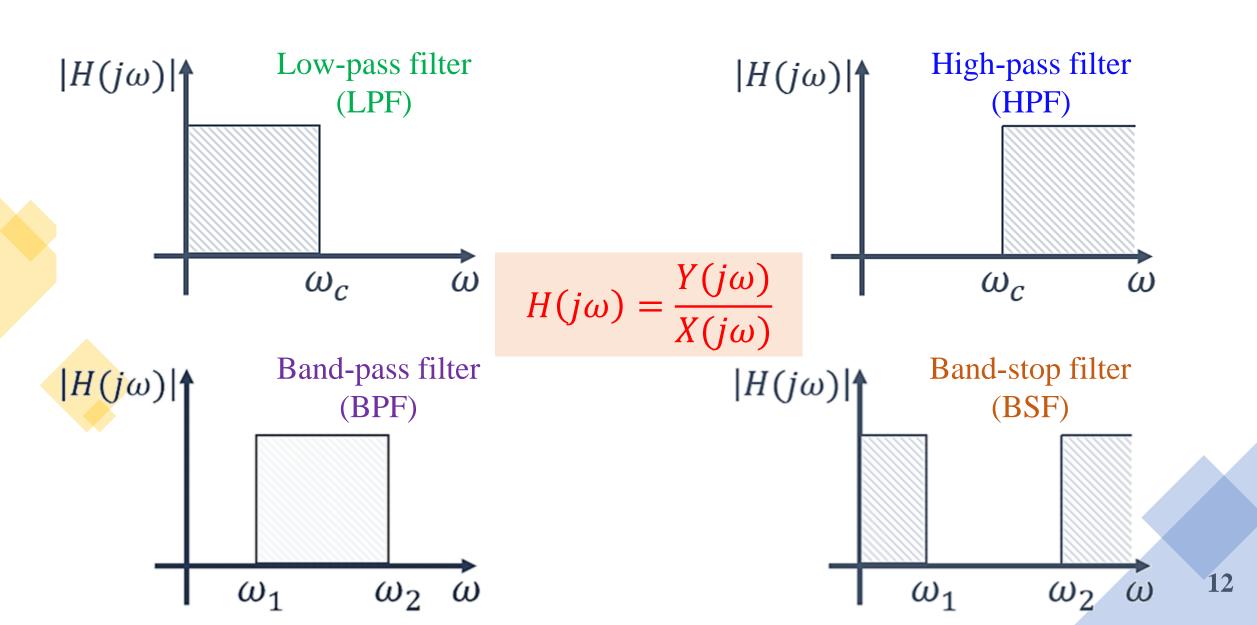
2.1 DEFINITION

The *frequency response* of a circuit is the <u>variation</u> in its behaviour with change in signal's <u>frequency</u>.

- It may also be considered as the variation of the gain and phase with frequency.
- A circuit's response depends on types of elements in the circuit, the way all elements connected, and impedances of the elements.
- Carefully selecting circuit elements (values and connections) enables us to construct a circuit that only passes those input signals which reside in the desired frequency range *frequency selective circuits*.

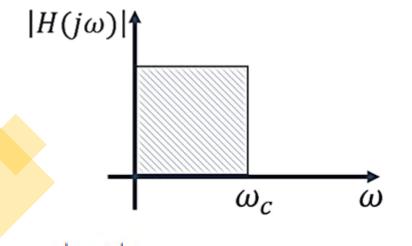
These frequency-selective circuits are also called *filters*.

2.2 TYPES

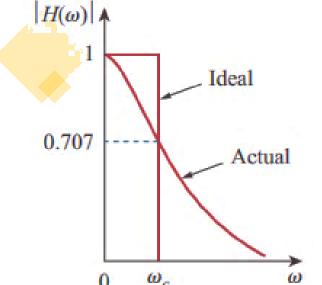


LOW PASS FILTER

A low-pass filter can pass low frequencies and stop high frequencies.



- Fideally, frequency components which are smaller than ω_c can pass, and frequency components which are greater than ω_c will be stopped.
- $\triangleright \omega_c$ is called the **cut-off frequency**.
- Practically, a filter always changes gradually from the pass band to the stop band or vice versa.

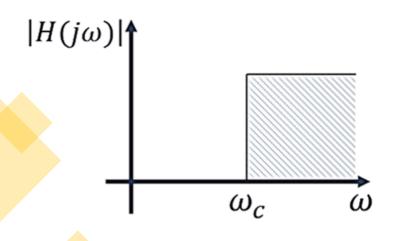


 ω_c is defined as the frequency whose magnitude of the frequency response is decreased by $1/\sqrt{2}$ from its maximum value:

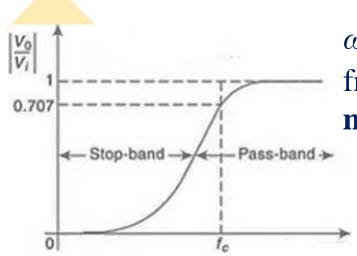
$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}H_{max}$$

HIGH PASS FILTER

A high-pass filter can pass high frequencies and stop low frequencies.



- Fideally, frequency components which are greater than ω_c can pass, and frequency components which are smaller than ω_c will be stopped.
- $\triangleright \omega_c$ is called the **cut-off frequency**.
- Practically, a filter always changes gradually from the pass band to the stop band or vice versa.



 ω_c is defined as the frequency whose magnitude of the frequency response is decreased by $1/\sqrt{2}$ from its maximum value:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}H_{max}$$

2.3 DECIBEL SCALE

In communication systems, *gain* is measured in *bels*. Historically, the *bel* is used to measure the ratio of two level of **power**:

$$Gain = Number\ of\ bels = \log_{10} \frac{P_2}{P_1}$$

The unit *decibel* (dB) provides us with a unit of less magnitude.

It is given by:

$$G(dB) = 10 \log_{10} \frac{P_2}{P_1} = 20 \log_{10} \frac{V_2}{V_1}$$

Properties of Logarithms:

$$\log P_1 P_2 = \log P_1 + \log P_2$$

$$\log P_1 / P_2 = \log P_1 - \log P_2$$

$$\log P^n = n \log P$$

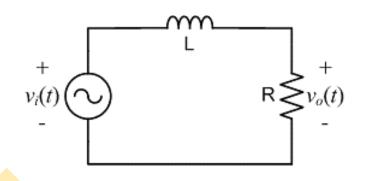
$$\log 1 = 0$$

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L&C behaviour in an ac circuit

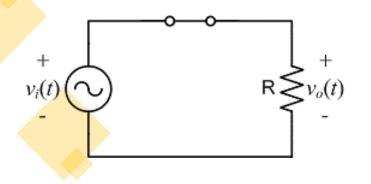
	Low	High
	frequency	frequency
Inductor	Short-circuit	Open-circuit
Capacitor	Open-circuit	Short-circuit



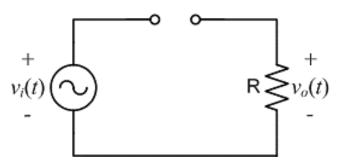
Input: sinusoidal voltage source $v_i(t)$

Output: voltage on resistor $v_o(t)$

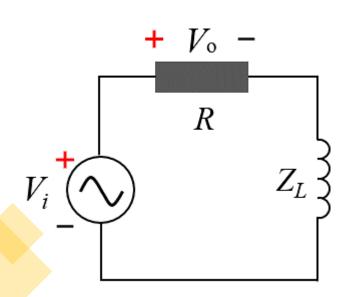
The frequency of the source increases gradually (from a very low frequency).



At low frequency $(X_L = \omega L << R)$, the inductor's impedance is very small compared with the resistor's impedance, and the inductor effectively functions as a short circuit.

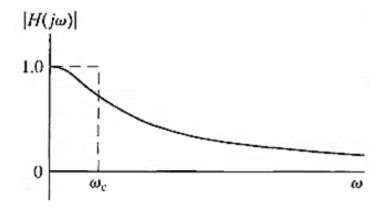


At high frequencies $(X_L = \omega L >> R)$, X_L is very large compared with resistor's impedance. So, the inductor functions as an open circuit, effectively blocking the flow of current in the circuit.



Applying KVL to the circuit:

$$v_i(j\omega) = v_o(j\omega) + (j\omega L) \frac{v_o(j\omega)}{R}$$



$$Z_{L}$$

$$z_{l}$$

$$z_{l}$$

$$z_{l}$$

$$v_{i}(j\omega) = v_{o}(j\omega) + (j\omega L) \frac{v_{o}(j\omega)}{R}$$

$$v_{o}(j\omega) = \frac{1}{1 + j\omega\left(\frac{L}{R}\right)} = \frac{R/L}{j\omega + R/L} = H(j\omega)$$

$$\begin{cases} |H(j\omega)| = \frac{R/L}{\sqrt{\omega^2 + (R/L)^2}} \\ \theta(j\omega) = -\arctan(\frac{\omega L}{R}) \end{cases}$$

The maximum value occurs at $\omega = 0$: $|H_{max}| = |H(j0)| = 1$

So, the cut-off frequency can be solved by:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H_{max}| = \frac{1}{\sqrt{2}} = \frac{R/L}{\sqrt{{\omega_c}^2 + (R/L)^2}}$$

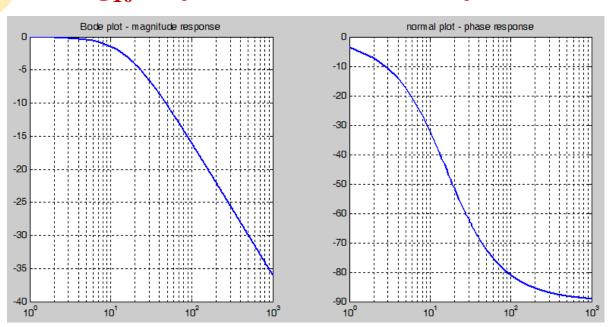
$$\omega_c = \frac{R}{L}$$
 or $f_c = \frac{\omega_c}{2\pi} = \frac{R}{2\pi L}$

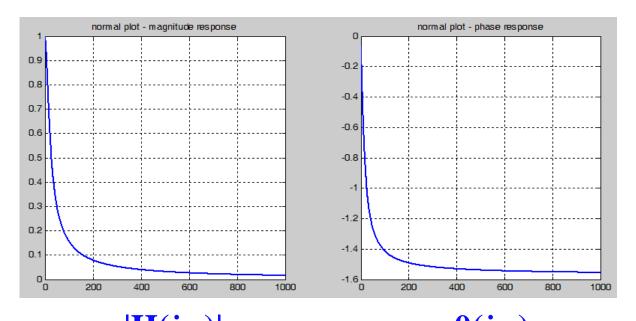
The frequency range required in a frequency response is often so wide; it is inconvenient to use a linear scale for the frequency axis.

To solve this, we introduce the log axis.

 $20\log_{10}(|\mathbf{H}(j\omega)|)$

 $\theta(j\omega)$



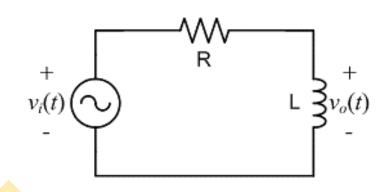


$$|\mathbf{H}(j\omega)| \qquad \qquad \mathbf{\theta}(j\omega)$$

$$\begin{cases} |H(j\omega_c)| = \frac{1}{\sqrt{2}} \\ 20log_{10}(|H(j\omega_c)|) = 20log_{10}\left(\frac{1}{\sqrt{2}}\right) = -3dB \end{cases}$$
Rodo plot:

Bode plot:

 $20\log_{10}(|\mathbf{H}(j\omega)|)$ vs f in \log ; $\theta(j\omega)$ in degree vs f in \log .



Input: sinusoidal voltage source $v_i(t)$

Output: voltage on inductor $v_o(t)$

The frequency of the source increases gradually (from a very low frequency).

Applying KVL:

$$v_{i}(j\omega) = v_{o}(j\omega) + R \frac{v_{o}(j\omega)}{j\omega L}$$

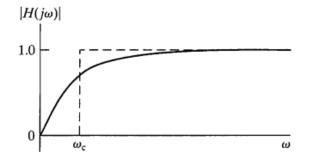
$$\frac{v_{o}(j\omega)}{v_{i}(j\omega)} = \frac{1}{1 + R/j\omega L} = \frac{j\omega}{j\omega + R/L} = H(j\omega)$$

$$\begin{cases} |H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}} \\ \theta(j\omega) = \arctan(\frac{R}{\omega L}) \end{cases}$$

The cut-off frequency:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H_{max}| = \frac{1}{\sqrt{2}} = \frac{\omega_c}{\sqrt{{\omega_c}^2 + (R/L)^2}}$$

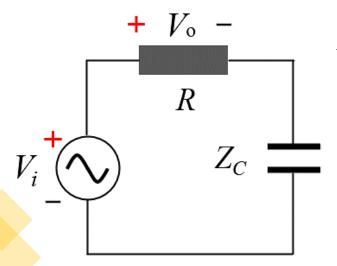
$$\omega_c = \frac{R}{L}$$
 or $f_c = \frac{\omega_c}{2\pi} = \frac{R}{2\pi L}$



SUMMARY

RL circuits	Frequency response	Cut-off frequency	Bode plot (Magnitude)
V_i	$H(j\omega) = \frac{v_o}{v_i} = \frac{R/L}{j\omega + R/L}$	$\omega_c = \frac{R}{L}$ $f_c = \frac{R}{2\pi L}$	$ H(j\omega) $ 1.0 0 ω_{c}
R Z_L V_o	$H(j\omega) = \frac{v_o}{v_i} = \frac{j\omega}{j\omega + R/L}$	$\omega_c = \frac{R}{L}$ $f_c = \frac{R}{2\pi L}$	$ H(j\omega) $ 0 ω_{c}

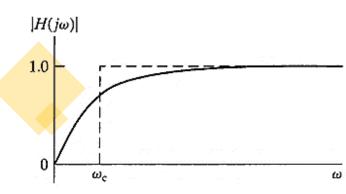
3.2 SERIES RC CIRCUIT



Applying KVL:

$$v_i(j\omega) = v_o(j\omega) + \frac{1}{j\omega C} \frac{v_o(j\omega)}{R}$$

$$\therefore H(j\omega) = \frac{v_o(j\omega)}{v_i(j\omega)} = \frac{j\omega RC}{1 + j\omega RC}$$



$$|H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (1/RC)^2}}$$

The cut-off frequency:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H_{max}| = \frac{1}{\sqrt{2}} = \frac{\omega_c}{\sqrt{\omega_c^2 + (1/RC)^2}}$$

$$\omega_c = \frac{1}{RC}$$
 or $f_c = \frac{1}{2\pi RC}$

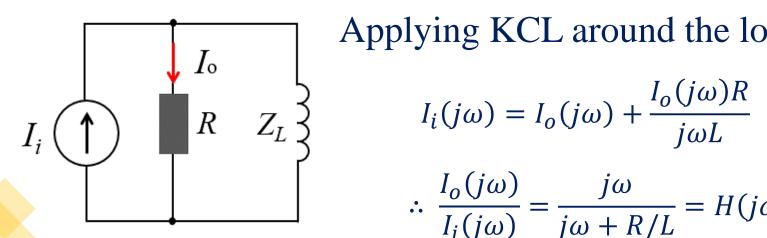
SUMMARY

RC circuits	Frequency response	Cut-off frequency	Bode plot (Magnitude)
R Z_C V_o			
V_i Z_C	$H(j\omega) = \frac{v_o}{v_i} = \frac{j\omega RC}{1 + j\omega RC}$	$\omega_c = \frac{1}{RC}$ $f_c = \frac{1}{2\pi RC}$	$ H(j\omega) $ 0 ω_{c} ω

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3.3 PARALLEL RL CIRCUI

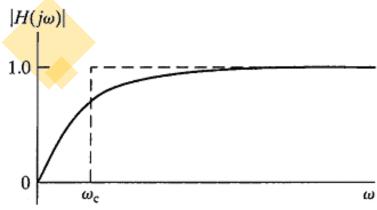


Applying KCL around the loop:

$$I_i(j\omega) = I_o(j\omega) + \frac{I_o(j\omega)R}{j\omega L}$$

$$\therefore \frac{I_o(j\omega)}{I_i(j\omega)} = \frac{j\omega}{j\omega + R/L} = H(j\omega)$$

The cut-off frequency:



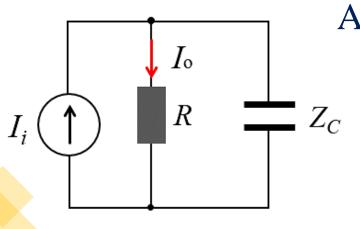
$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H_{max}| = \frac{1}{\sqrt{2}} = \frac{\omega_c}{\sqrt{{\omega_c}^2 + {(R/L)}^2}}$$

$$\omega_c = \frac{R}{L}$$
 or $f_c = \frac{R}{2\pi L}$

3.3 SUMMARY

RL circuits	Frequency response	Cut-off frequency	Frequency response plot (magnitude)
I_i R Z_L			
I_i I_o R Z_L	$H(j\omega) = \frac{I_o}{I_i} = \frac{j\omega}{j\omega + R/L}$	$\omega_c = \frac{R}{L}$ $f_c = \frac{R}{2\pi L}$	$ H(j\omega) $ 0 $\omega_{\rm c}$

3.4 PARALLEL RC CIRCUIT



Applying KCL around the loop:

$$I_{i}(j\omega) = I_{o}(j\omega) + \frac{I_{o}(j\omega)R}{\frac{1}{j\omega C}}$$

$$\therefore \frac{I_{o}(j\omega)}{I_{i}(j\omega)} = \frac{1}{1 + j\omega RC} = H(j\omega)$$

$$\therefore \frac{I_o(j\omega)}{I_i(j\omega)} = \frac{1}{1 + j\omega RC} = H(j\omega)$$

The cut-off frequency:

$$|H(j\omega)|$$

$$0$$
 ω_{c}
 ω

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H_{max}| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1 + (\omega_c RC)^2}}$$

$$\omega_c = \frac{1}{RC}$$
 o

$$\omega_c = \frac{1}{RC}$$
 or $f_c = \frac{1}{2\pi RC}$

3.4 SUMMARY

RC circuits	Frequency response	Cut-off frequency	Frequency response plot (magnitude)
I_i I_o R Z_C	$H(j\omega) = \frac{I_o}{I_i} = \frac{1}{1 + j\omega CR}$	$\omega_c = \frac{1}{RC}$ $f_c = \frac{1}{2\pi RC}$	$ H(j\omega) $ 0 ω_{c} ω
I_{i} R I_{c} Z_{C}			

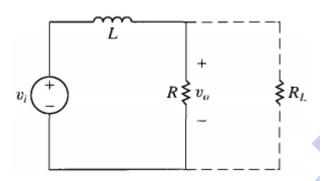
QUIZ 1

- 1. A voltage source supplies a signal of constant amplitude, from 0 to 40 kHz, to an RC lowpass filter. A load resistor, connected in parallel across the capacitor, experiences the maximum voltage at:
 - (a) d.c.

(b) 10 kHz

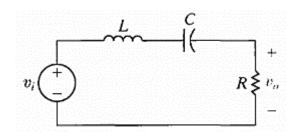
(c) 20 kHz

- (d) 50 kHz
- 2. Add a load resistor R_L to the RL filter as shown in the figure. Are the cut-off frequencies ω_c different? Are the peak pass-band gains different?
 - (a) Different; Same
- (b) Same; Same
- (c) Different; Different (d) Same; Different



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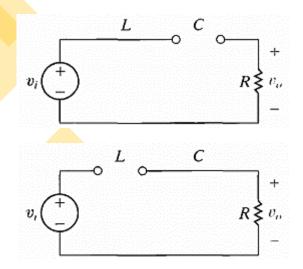


Changes to the source frequency result in changes to the impedance of the capacitor and the inductor:

$$Z = R + Z_L + Z_C = R + j(X_L - X_C)$$

$$Z_C = -jX_C = \frac{-j}{\omega C}$$

$$Z_C = -jX_C = \frac{-j}{\omega C}$$
At $\omega = 0$, the capacitor behaves like an open circuit, and the inductor behaves like a short circuit.



At $\omega = \infty$, the capacitor behaves like a short circuit, and the inductor behaves like an open circuit.

What will happen in the frequency region between $\omega = 0$ and $\omega = \infty$?

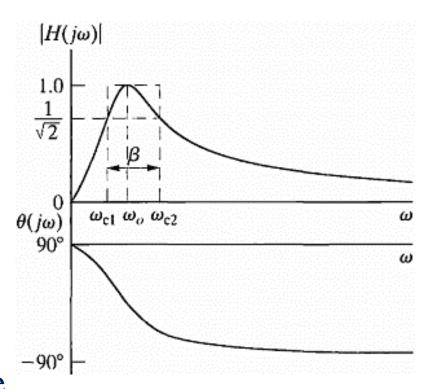
Between $\omega = 0$ and $\omega = \infty$, both the capacitor and the inductor have finite impedances.

In this region, voltage supplied by the source will drop across both inductor and capacitor, but some voltages will be delivered to the resistor.

Because

$$X_L - X_C = \omega L - \frac{1}{\omega C}$$

At some points, the impedance of the capacitor and the impedance of the inductor have equal magnitudes but opposite signs => they cancel out, causing the output voltage to equal the source voltage.



* RESONANCE

Resonance occurs in any system that has a complex conjugate pair of poles; it is the cause of oscillations of stored energy from one form to another.

• It is the phenomenon that allows frequency discrimination in communications networks.

Resonance is a condition in an *RLC* circuit where the capacitive and inductive reactances are <u>equal</u> in magnitude, thereby resulting in a <u>purely</u> <u>resistive impedance</u>.

Resonant circuits (series or parallel) are useful for constructing filters, as their frequency response can be highly frequency selective.

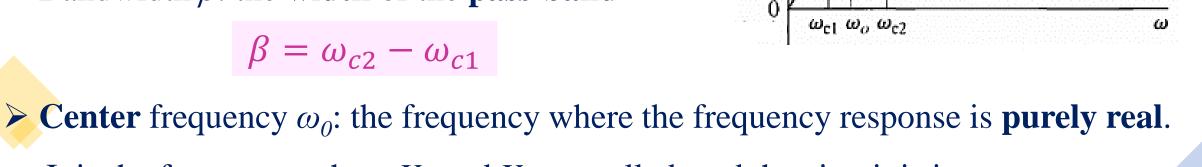
• They are used in various applications such as selecting the desired stations in radio and TV receivers.

* RESONANCE

Some definitions useful in the frequency response analysis:

- ➤ Pass-band & stop-band;
- \triangleright Cut-off frequencies ω_{c1} and ω_{c2} (half-power (3dB)_{1.0} frequencies)
- \triangleright Bandwidth β : the **width** of the **pass-band**

$$\beta = \omega_{c2} - \omega_{c1}$$



 $|H(j\omega)|$

It is the frequency where X_L and X_C cancelled, and the circuit is in **resonance**.

$$\omega_0 = \sqrt{\omega_{c1}\omega_{c2}}$$

> Quality factor: measure the "sharpness" of the resonance

* QUALITY FACTOR

At resonance, the reactive energy in the circuit oscillates Amplitude between the inductor and the capacitor. The *Q*-factor relates the **maximum or peak energy** stored to the energy dissipated in the circuit *per cycle* of oscillation:

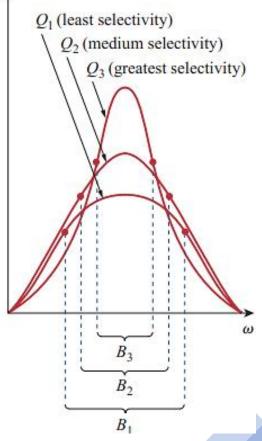
$$Q = 2\pi \frac{Peak \ energy \ stored \ in \ the \ circuit}{Energy \ dissipated \ by \ the \ circuit \ in \ 1T \ at \ resonance}$$

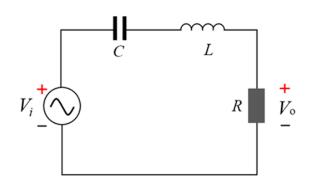
It is also regarded as a measure of the energy storage property of a circuit in relation to its energy dissipation property.

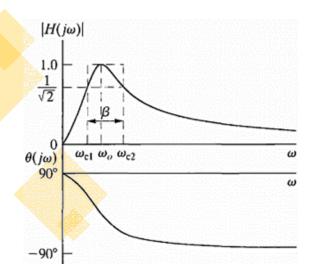
$$Q = 2\pi \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2RT_0} = \frac{2\pi Lf_0}{R} = \frac{L\omega_0}{R} = \frac{1}{\omega_0 RC}$$

The relationship between bandwidth β and Q is:

$$Q = \frac{\omega_0}{\beta}$$







$$\theta(j\omega) = \arctan\left[\frac{\frac{\omega R}{L}}{\left(\frac{1}{LC}\right) - \omega^2}\right]$$

Applying KVL:
$$V_{i} = V_{o} + \frac{V_{o}}{R} \left(j\omega L + \frac{1}{j\omega C} \right) \qquad \therefore H(j\omega) = \frac{V_{o}}{V_{i}} = \frac{j\omega\left(\frac{R}{L}\right)}{(j\omega)^{2} + j\omega\left(\frac{R}{L}\right) + (\frac{1}{LC})}$$

Center frequency ω_0 ($H(j\omega)$ is pure real):

$$j(\omega_0 L - \frac{1}{\omega_0 C}) = 0 \qquad \therefore \omega_0 = \frac{1}{\sqrt{LC}}$$

The cut-off frequency:

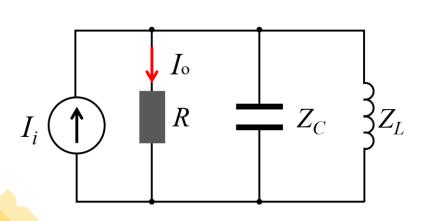
$$|H(j\omega_c)| = \frac{\left(\frac{\omega_c R}{L}\right)}{\sqrt{\left[\left(\frac{1}{LC}\right) - \omega_c^2\right]^2 + \left(\frac{\omega_c R}{L}\right)^2}} = \frac{1}{\sqrt{2}}|H_{max}| = \frac{1}{\sqrt{2}}$$

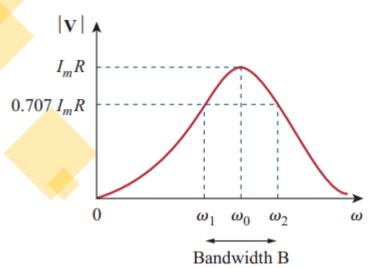
$$Q$$
-racto

$$\theta(j\omega) = \arctan\left[\frac{\frac{\omega R}{L}}{\left(\frac{1}{LC}\right) - \omega^2}\right] \qquad \text{Bandwidth } \beta: \qquad Q\text{-factor:}$$

$$\beta = \omega_{c2} - \omega_{c1} = \frac{R}{L} \qquad Q = \frac{\omega_0}{\beta} = \frac{(1/\sqrt{LC})}{(R/L)} = \sqrt{\frac{L}{R^2C}}$$

4.2 PARALLEL RLC CIRC





$$I_{o} = \frac{V}{R} = V \cdot G \qquad I_{in} = V \cdot \tilde{Y} = V \left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)$$

$$R \qquad = \frac{I_{o}}{I_{in}} = \frac{1}{R\left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)} = \frac{1}{1 + j\left(\omega RC - \frac{R}{\omega L}\right)}$$

$$\therefore H(j\omega) = \frac{I_o}{I_{in}} = \frac{1}{R\left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)} = \frac{1}{1 + j\left(\omega RC - \frac{R}{\omega L}\right)}$$

Resonance occurs at:

$$\omega_0 C - \frac{1}{\omega_0 L} = 0 \qquad \therefore \omega_0 = \sqrt{\frac{1}{LC}}$$

Cut-off frequencies:
$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H_{max}| = \frac{1}{\sqrt{2}}$$

$$\therefore \omega_{c1} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \left(\frac{1}{LC}\right)}, \quad \omega_{c2} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \left(\frac{1}{LC}\right)}$$

Bandwidth β :

Q-factor:

$$\beta = \omega_{c2} - \omega_{c1} = \frac{1}{RC} \qquad Q = \frac{\omega_0}{\beta} = \omega_0 RC$$

$$Q = \frac{\omega_0}{\beta} = \omega_0 RC$$

QUIZ 2

1. In a series *RLC* circuit, which of these quality factors has the sharpest magnitude response curve near resonance?

(a)
$$Q = 20$$

(b)
$$Q = 12$$

(c)
$$Q = 8$$

(d)
$$Q = 4$$

2. How much inductance is needed to resonate at 5 kHz with a capacitance of 12nF?

(a) 84.43 mH

(b) 3.33 mH

(c) 84.43 H

(d) 11.84 H

There is no substitute for hard work.

Thomas A. Edison

NEXT

First-Order Transient Response (with dc Sources)