CAN102 Electromagnetism and Electromechanics

Lecture-4 Static Electric Fields II

(Electrostatics)

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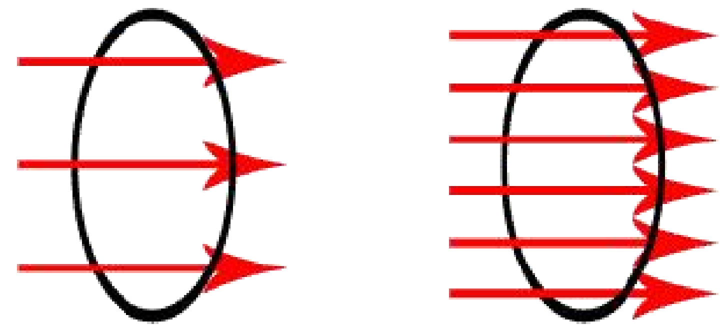
Room EE322

Outline

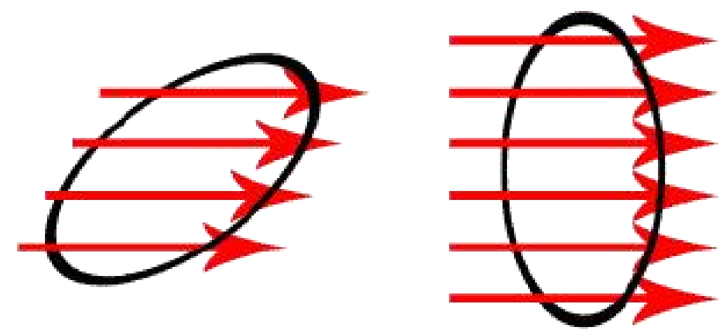
- Electric Flux
- Gauss's Law - Integral form
  - Gauss's Law
  - Flux density
  - Calculating E-field using Gauss's Law
- Gauss's Law - Differential form
  - Divergence
  - Divergence Theorem
  - Gauss's Law in differential form

1.1 What is flux?

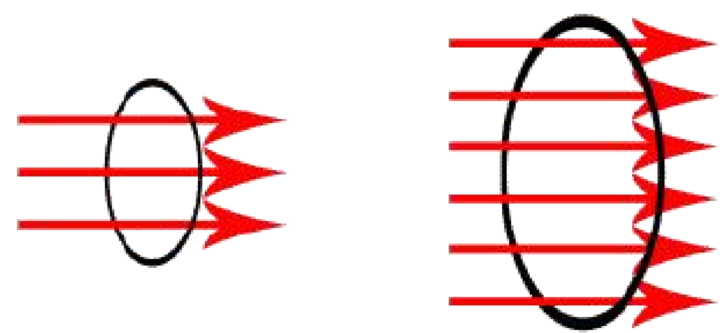
- Flux is the rate of flow of the field through a given area.
  - Flux is proportional to the density of flow;
  - Flux varies by how the boundary faces the direction of flow;
  - Flux is propotional to the area within the boundary.



Flux is proportional to the density of flow.



Flux varies by how the boundary faces the direction of flow.



Flux is proportional to the area within the boundary.

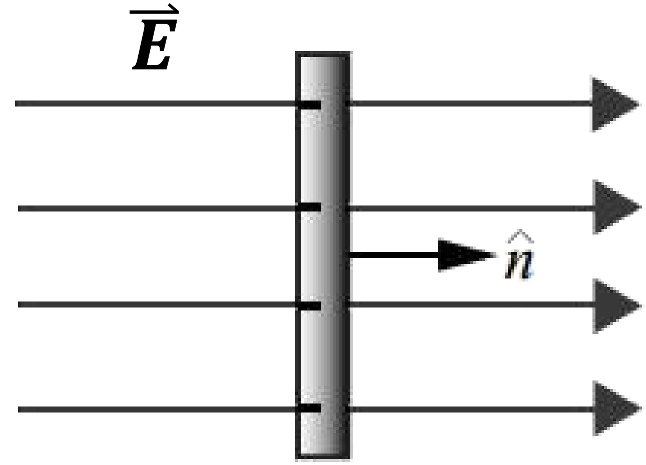
Flux visualized

The ring shows the surface boundaries. The red arrows for the field lines.

1.2 Electric Flux (电通量) - Simple Case

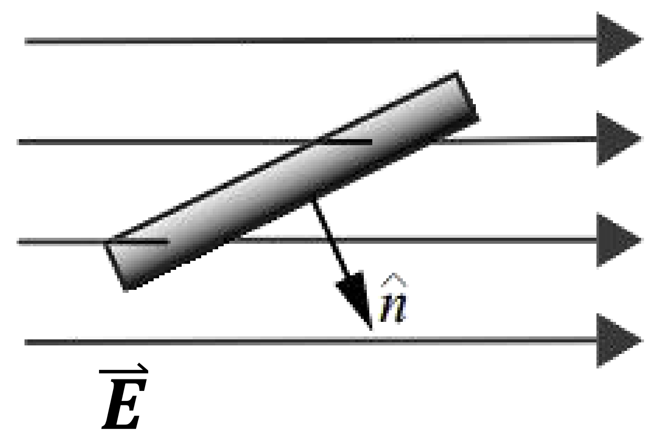
- In the simplest case of a uniform vector field $\vec{E}$ and a surface $\vec{S}$ perpendicular to the direction of the field, the electrical flux Φ is defined as the product of the field magnitude and the area of the surface:

$$\Phi = ES = \vec{E} \cdot \vec{S} = E\hat{n} \cdot S\hat{n} = ES(\hat{n} \cdot \hat{n})$$



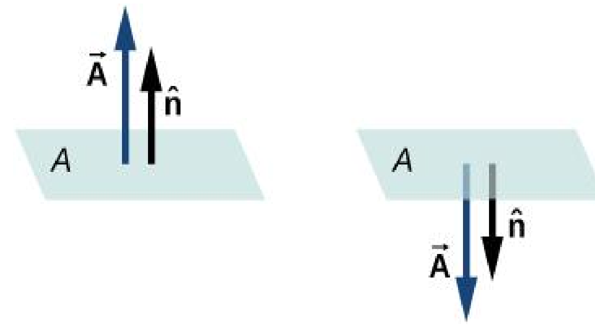
- If the vector field is uniform but **not** vertical to the surface, Φ can be determined by finding the component of $\vec{E}$ perpendicular to the surface and then multiplying that value by the surface area:

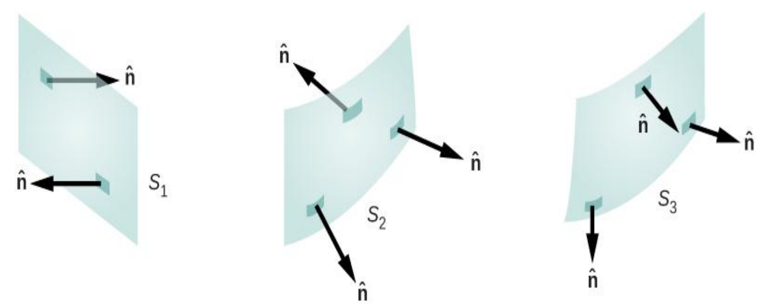
$$\Phi = \vec{E} \cdot \vec{S} = ES\cos\theta$$

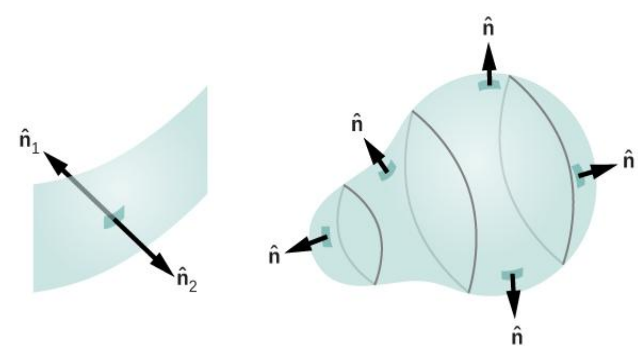


1.2 Area Vector

- The area vector of a surface has:
  - Magnitude is equal to area (A);
  - Direction is along the normal to the surface ($\hat{\boldsymbol{n}}$); that is, perpendicular to the surface;
  - The direction $\hat{\boldsymbol{n}}$ of an ***open*** surface needs to be chosen; it could be either of the two directions;
  - The direction $\hat{\boldsymbol{n}}$ of all area segments should be consistent over the entire surface.
  - The area vector of a part of a ***closed*** surface is defined to point from the inside of the closed space to the outside.



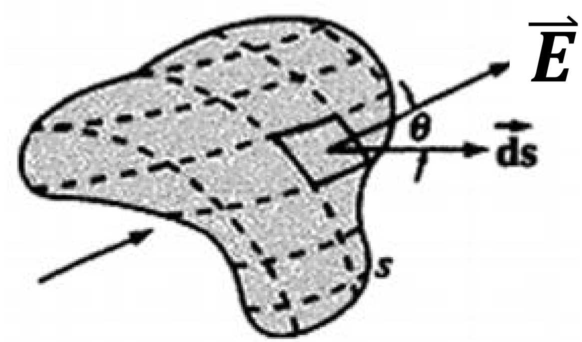




1.2 Electric Flux - General surfaces

- A large curved surface is divided into small vector areas, each one has an area of Δs_i and direction $\hat{n}_i$.
- A vector field flows through this surface having different values and different directions at any point on the surface, illustrated as $\vec{E}_i$.
- The flux flows through one area element is:

$$\Delta\Phi_i = \vec{E}_i \cdot \Delta\vec{s}_i$$



- The total flux flows through the whole surface is:

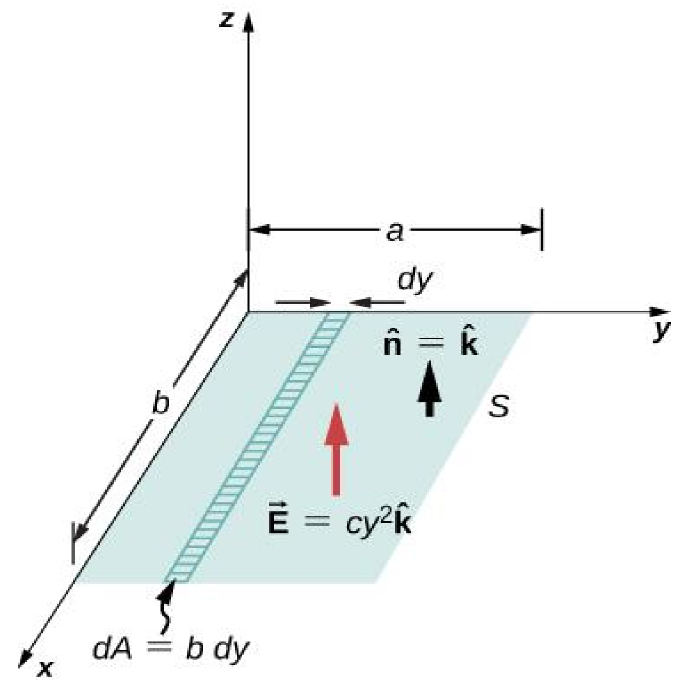
$$\Phi = \sum \vec{E}_i \cdot \Delta\vec{s}_i$$

- If the area elements are small enough, the summation becomes integration:

$$\Phi = \lim_{\Delta s_i \to 0} \sum \vec{E}_i \cdot \Delta\vec{s}_i = \iint_S \vec{E} \cdot d\vec{s}$$

Quiz 1

- What is the total flux of the electric field $\vec{\boldsymbol{E}} = cy^2\,\hat{\boldsymbol{z}}$ through the rectangular surface as shown.



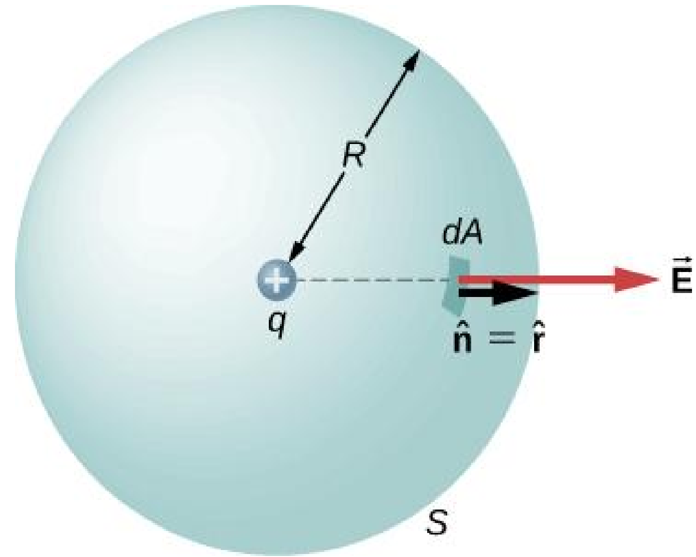
2.1 Electric flux of a positive point charge

- The electric field of a positive point charge q:

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

- The electric flux through a spherical surface around it:

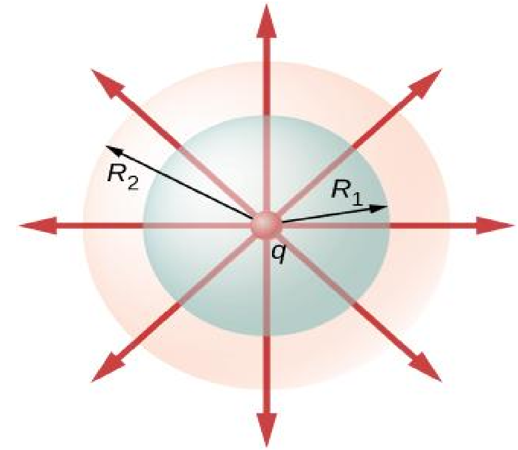
$$d\Phi = \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}} dA = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} dA$$



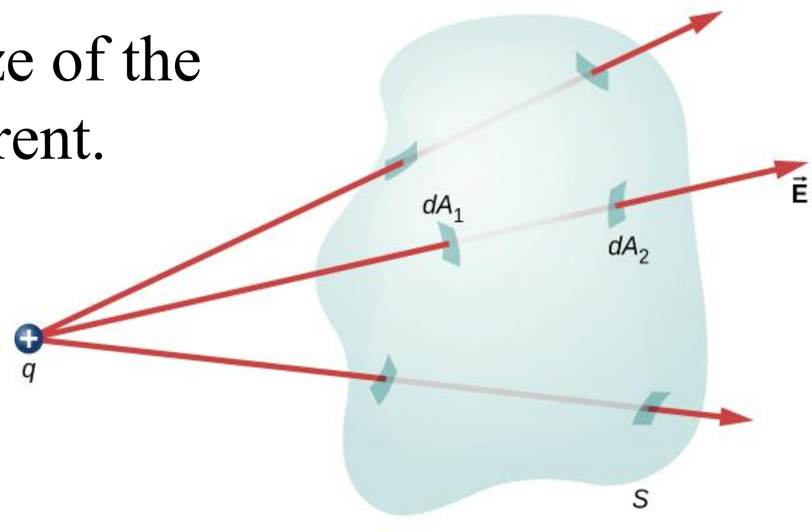
$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\oiint_S dA = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\varepsilon_0}$$

2.1 Flux and field lines

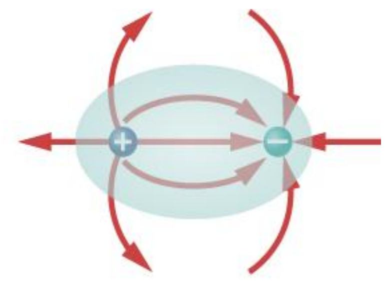
- The flux through a closed spherical surface:

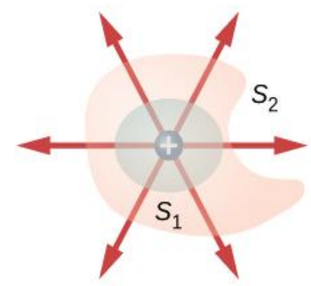


- The flux in terms of field lines:
  a) The flux due to a charge outside that surface;
  b) Charges are enclosed, but the net charge is zero;
  c) The shape and size of the surfaces are different.



(a)

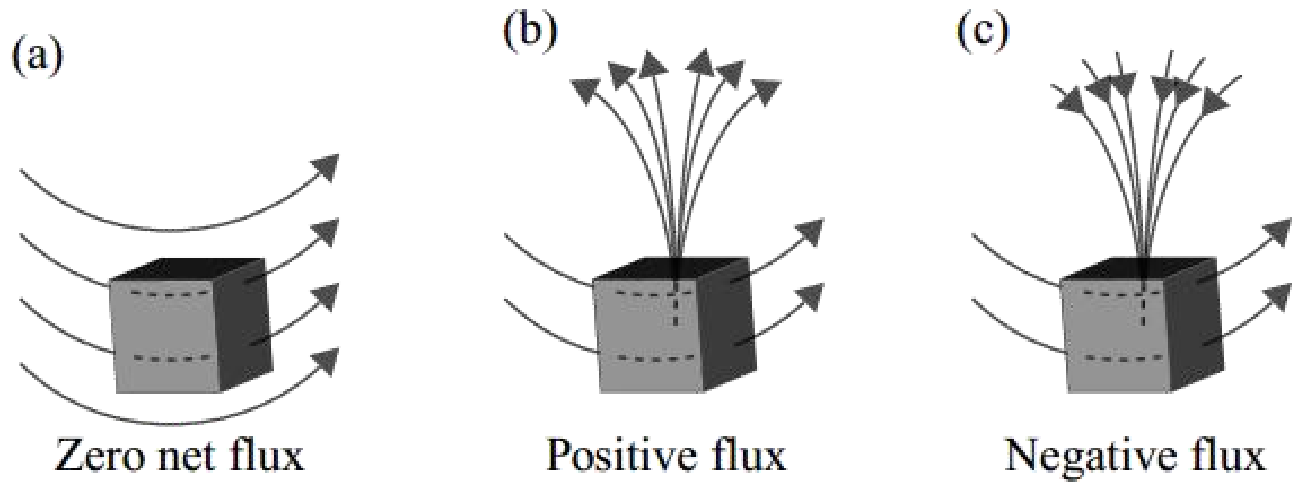

(b)



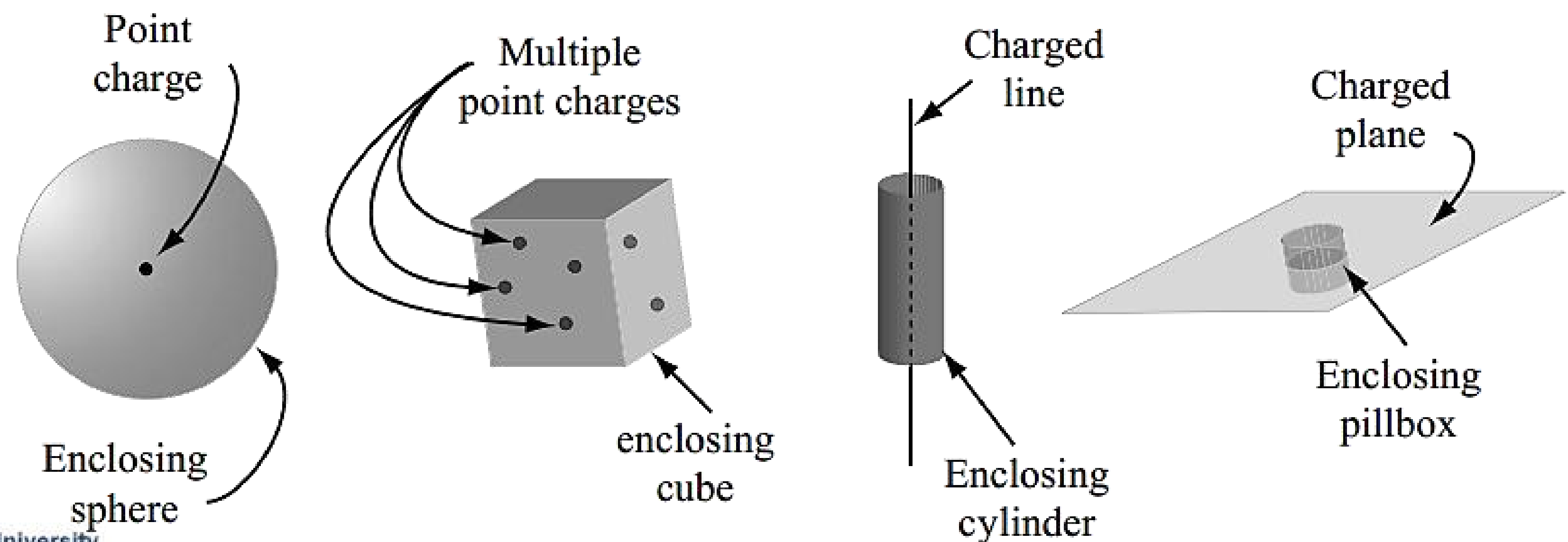
(c)

2.1 Flux & Enclosed q

- Flux lines penetrating closed surfaces:



- Surfaces enclosing known charges:



2.2 Gauss's Law

- Gauss's Law: Electric charges produce an $\vec{E}$ -field, and the flux of that field passing through any **closed** surface is proportional to the total charge **contained** within that surface.

$$\Phi = \oiint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon}$$

- Gaussian surface - the closed surface (through which the flux passing)
  - no need to be a real, physical object or surface;
  - mathematical construct of any shape, but must be closed;
  - highly symmetrical,

2.2 Flux Density and Displacement Flux

- Gauss's law can also be expressed as:

$$\varepsilon \oiint_S \vec{E} \cdot d\vec{s} = Q = \oiint_S \varepsilon\vec{E} \cdot d\vec{s} = \oiint_S \vec{D} \cdot d\vec{s} = \Psi$$

$$\boxed{Q = \Psi}$$

- ***Electric flux*** Ψ, also called ***displacement flux***
  - different from Φ ($\Psi = \varepsilon\Phi$)
  - is the number of field lines (Q) that penetrates a given surface
- ***Electric flux density*** $\vec{\boldsymbol{D}}$
  - the flux per unit area
  - relates to the E-field: $\vec{\boldsymbol{D}} = \varepsilon\vec{\boldsymbol{E}}$
  - the number of field lines per unit area (square meter)

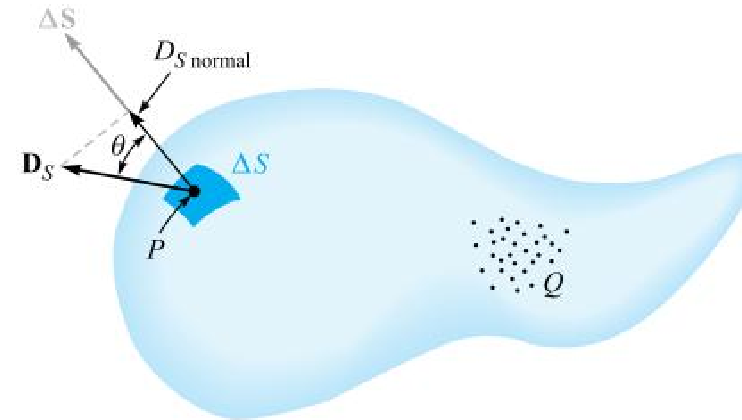
2.3 Gauss's Law - Integral Form

- Gauss's law:

$$\boxed{\Psi = Q}$$

$$\Psi = \oint\!\!\oint \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}} \qquad\qquad Q = \iiint_V \rho dv$$

$$\boxed{\oint\!\!\oint \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}} = \iiint_V \rho dv}$$

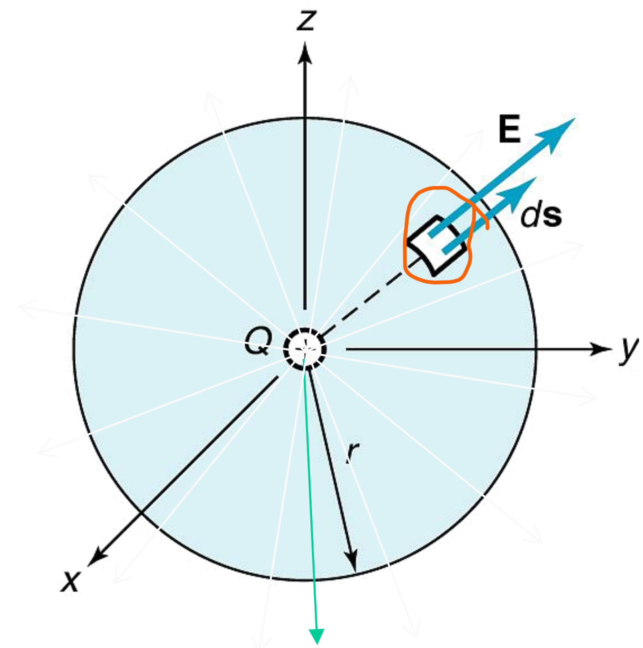


***Integral form of Gauss's Law –** The total electric flux emanating from a closed surface is numerically equal to the net positive charge inside the closed surface.*

Example: Point Charge

- Consider a positive point charge Q located at the center of a sphere of radius r in free space.
- Find the electric field on the sphere and electric flux through the sphere.

$$\boxed{\oiint \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}} = \iiint_V \rho dv}$$



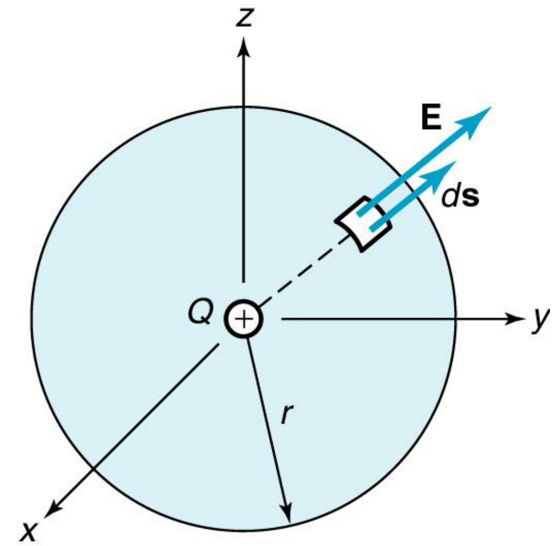
The sphere of radius r called the "**Gaussian surface**"

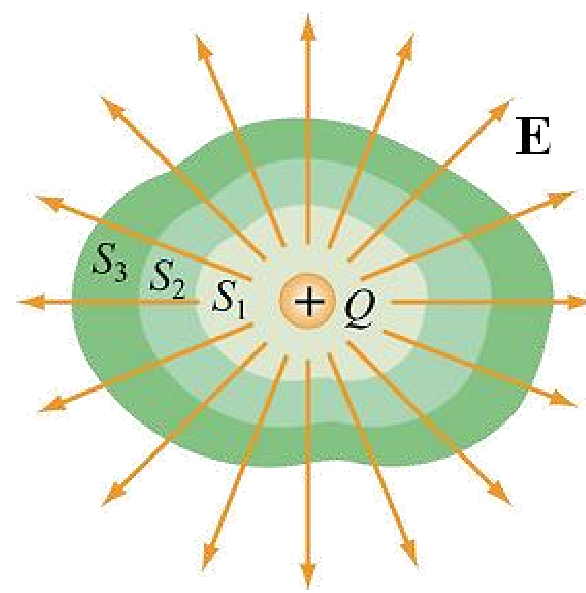
$$\left.\begin{aligned} LHS &= E_r \oiint ds = 4\pi r^2 E_r \\ RHS &= \frac{Q_{encl}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \end{aligned}\right\} \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

2.4 Gaussian Surface

- The flux is independent from the surface. The total "flux" through any of the **enclosed surfaces**, such as $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$, is the same and depends only on the amount of charge inside.
- **Gaussian Surface** is imaginary, there does not need any material object at the position of the surface.
- For a closed surface the unit vector is chosen to point in the **outward** normal direction.

Choose Gaussian Surface wisely





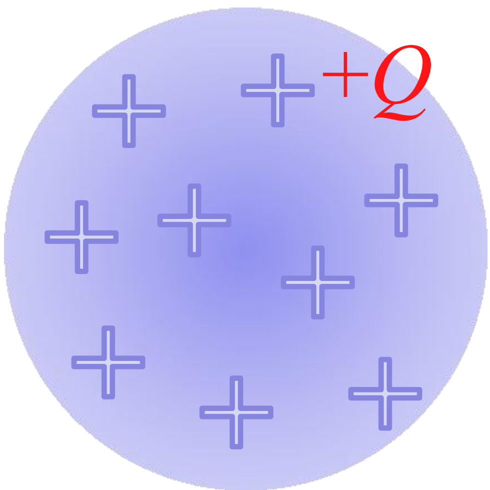
2.5 Calculating E-field using Gauss's Law

1. If you want to find the field at a particular point, then that point should lie on your Gaussian surface.
2. The Gaussian surface does not have to be a real physical surface. It is an imaginary geometric surface, such as: empty surface, embedded in a solid body, or both.
3. If the charge distribution has cylindrical or spherical symmetry, choose the Gaussian surface to be a coaxial cylinder or a concentric sphere, respectively.

Symmetry of charges	Gaussian Surface
Spherical	Concentric Sphere
Cylindrical	Coaxial Cylinder
Planar	Gaussian "Pillbox"

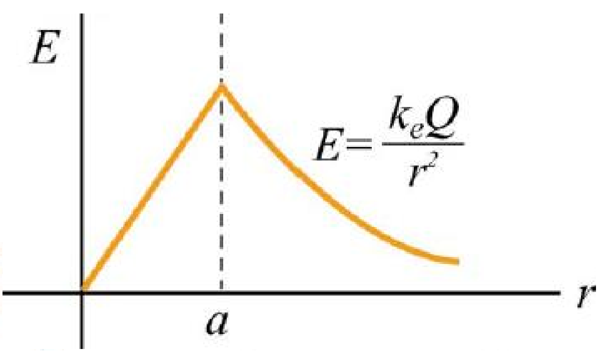
Case 1: Spherical Symmetry

- Positive charge $+Q$ **uniformly** distributed throughout **non-conducting solid sphere** of radius a.
- Find electric field every where.



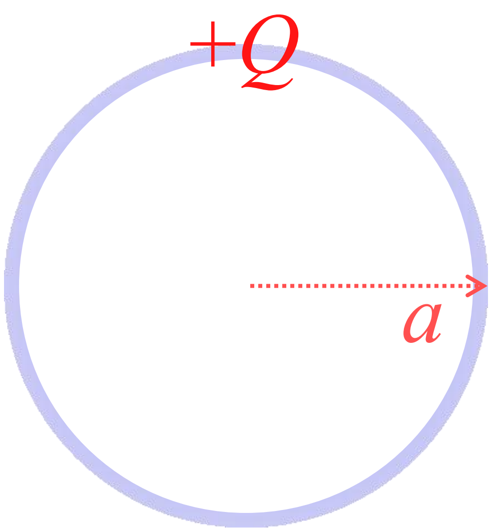
Gaussian Surface: **Sphere**

Draw a spherical Gaussian surface of radius r centred at the centre of the spherical charge distribution. r is arbitrary but is the radius for the Gaussian surface.



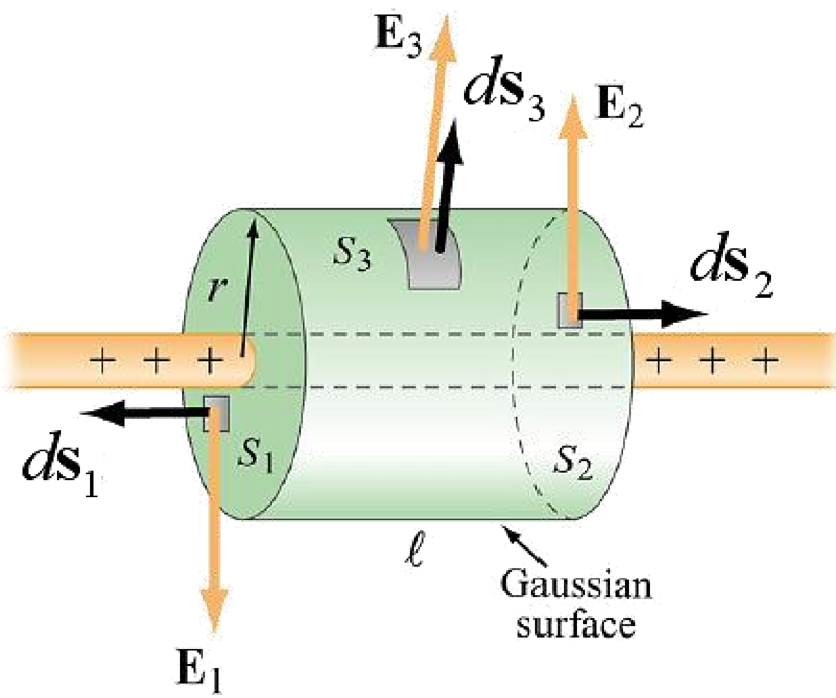
Quiz 2

- A **very thin** spherical shell of radius a has a charge $+Q$ **evenly** distributed over its surface.
- Find the electric field both inside and outside the shell in free space.



Case 2: Cylindrical Symmetry

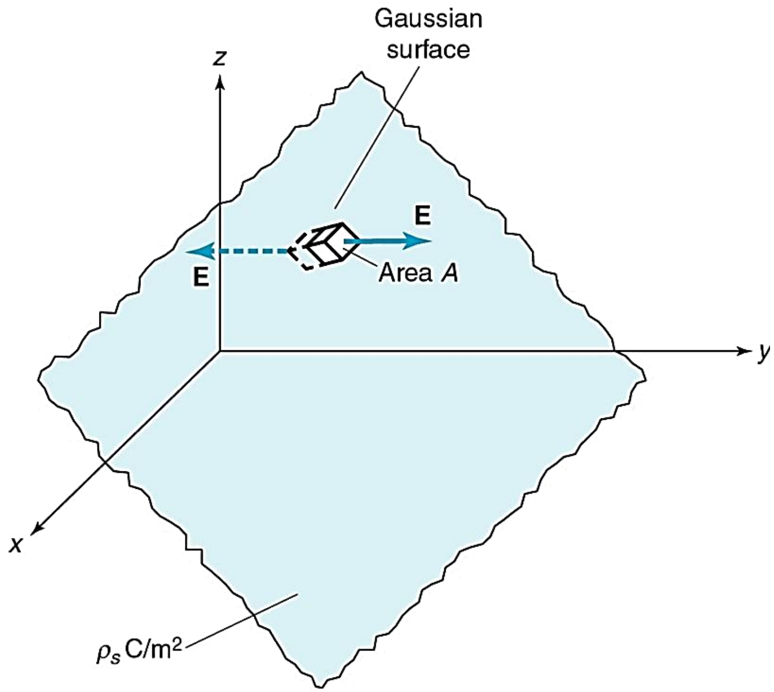
- An **infinitely long** rod of negligible radius has a **uniformly** distributed charge density λ.
- Find the electric field outside the rod in free space.

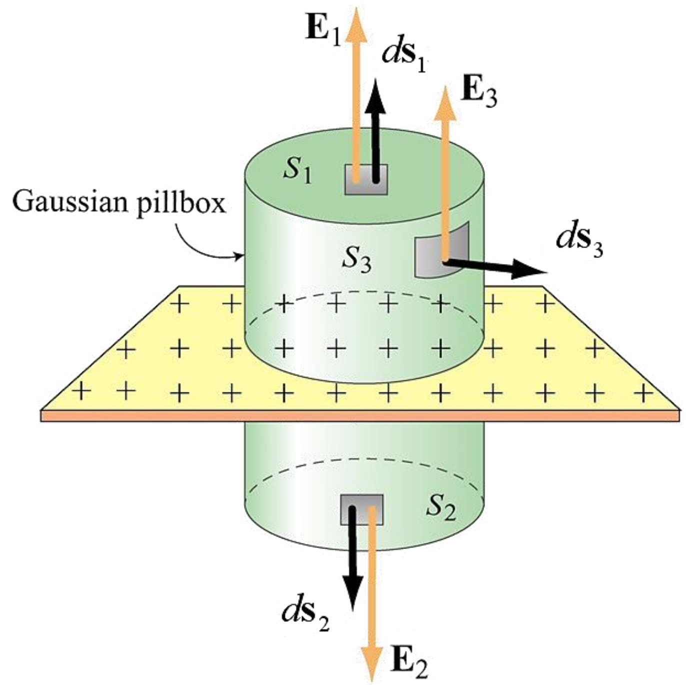


Symmetry: **Cylindrical**
Gaussian Surface: **Coaxial Cylinder**

Case 3: Planar Symmetry

- An **infinite** slab has a **uniformly** distributed charge density ρ_s. Find the electric field outside the plane in free space.





Lecture 3, p22

Symmetry: **Planar**
Gaussian Surface: **Circular Cylinder**
(faces parallel to the pane of charge)

2.5 Summary

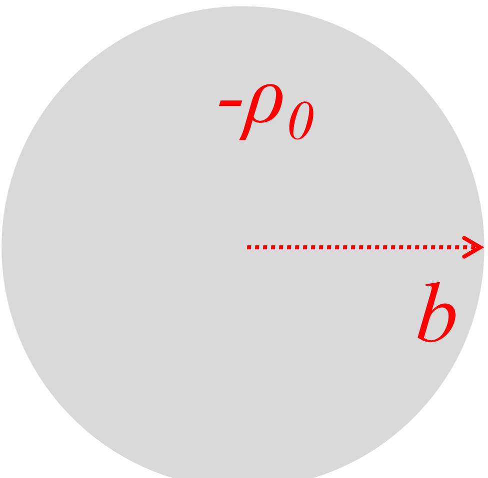
- Gauss's law provides a convenient tool for evaluating electric field. However, its application is limited only to systems that possess certain symmetry: cylindrical, planar and spherical symmetry.
- **Steps:**
  - (1) Identify the symmetry associated with the charge distribution.
  - (2) Determine the direction of **E**-field (**D**-field), and a "Gaussian surface".
  - (3) Divide the space into different regions associated with the charge distribution.
  - (4) Calculate the electric flux Φ_E through the Gaussian surface for each region.
  - (5) Equate Φ_E with Q_{enc}/ε, and deduce the magnitude of the electric field.

2.5 Typical Examples

Point charge (charge $= q$)	$\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{r}$ (at distance r from q)
Conducting sphere (charge $= Q$)	$\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{r}$ (outside, distance r from center) $\vec{E} = 0$ (inside)
Uniformly charged insulating sphere (charge $= Q$, radius $= r_0$)	$\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{r}$ (outside, distance r from center) $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Qr}{r_0^3}\hat{r}$ (inside, distance r from center)
Infinite line charge (linear charge density $= \lambda$)	$\vec{E} = \dfrac{1}{2\pi\varepsilon_0}\dfrac{\lambda}{r}\hat{r}$ (distance r from line)
Infinite flat plane (surface charge density $= \sigma$)	$\vec{E} = \dfrac{\sigma}{2\varepsilon_0}\hat{n}$

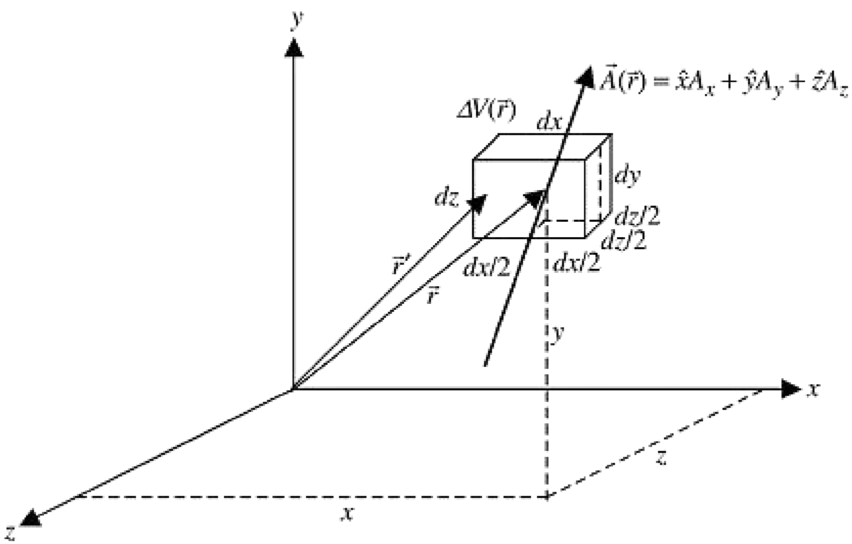
Quiz 3

- Determine the $\vec{E}$ field both inside and outside a spherical cloud of electrons with a uniform volume charge density $\rho = -\rho_0$ (where ρ_0 is a positive quantity) for $0 \le r \le b$ and $\rho = 0$ for $r > b$.



3.1 Divergence (散度)

- The ***divergence*** of a vector field at a point is the net outflux of that vector per unit volume. Thus, it gives a measure of the strength of the sources that produce the vector field.
- Denoted by $\nabla \cdot \vec{A}$ or $div(\vec{A})$, where $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ is the vector differential operator, reads as 'del'.



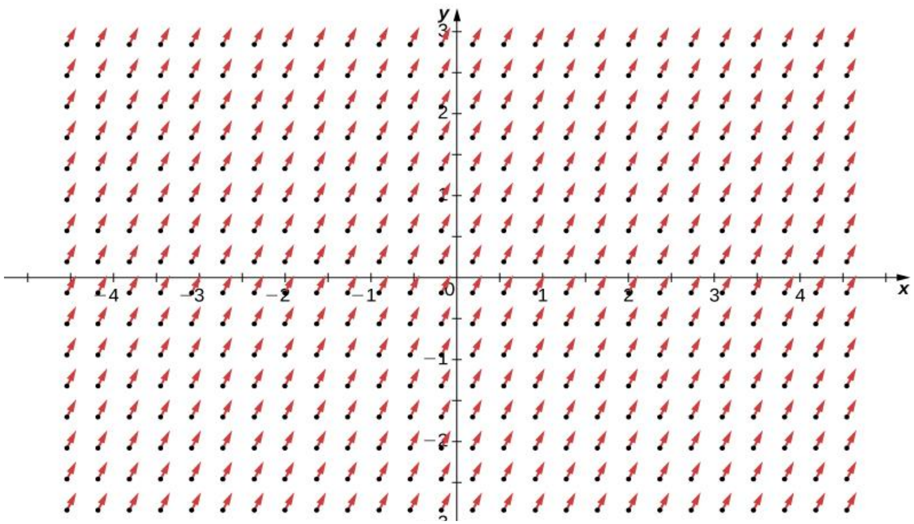
In Cartesian coordinates
$\vec{A}(\vec{r})$ is the spatial distributed vector field, then the divergence is:

$$\nabla \cdot \vec{A}(\vec{r}) = \lim_{\Delta V(r) \to 0} \frac{\oiint_S \hat{n} \cdot \vec{A}(\vec{r}) ds}{\Delta V(\vec{r})}$$

$$\nabla \cdot \vec{A}(\vec{r}) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

3.1 Divergence - Examples



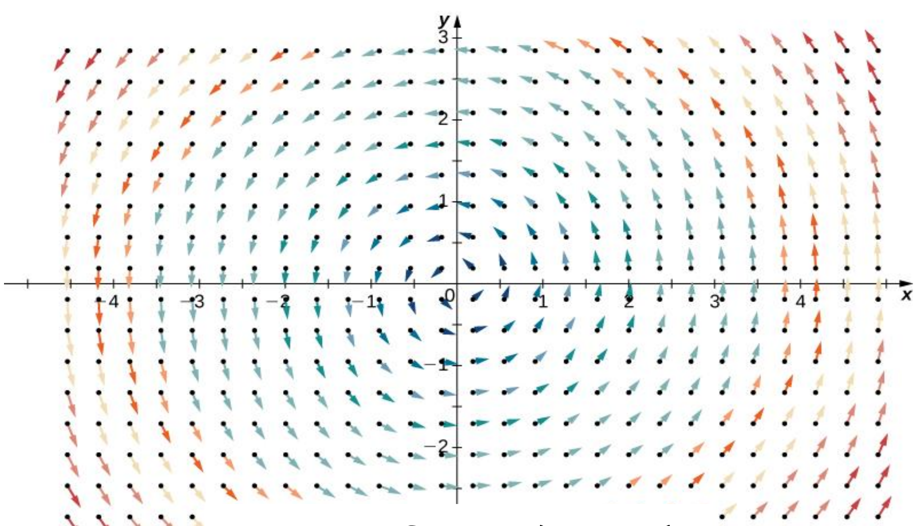
Vector field $\langle 1, 2\rangle$

- Divergence of the vector fields

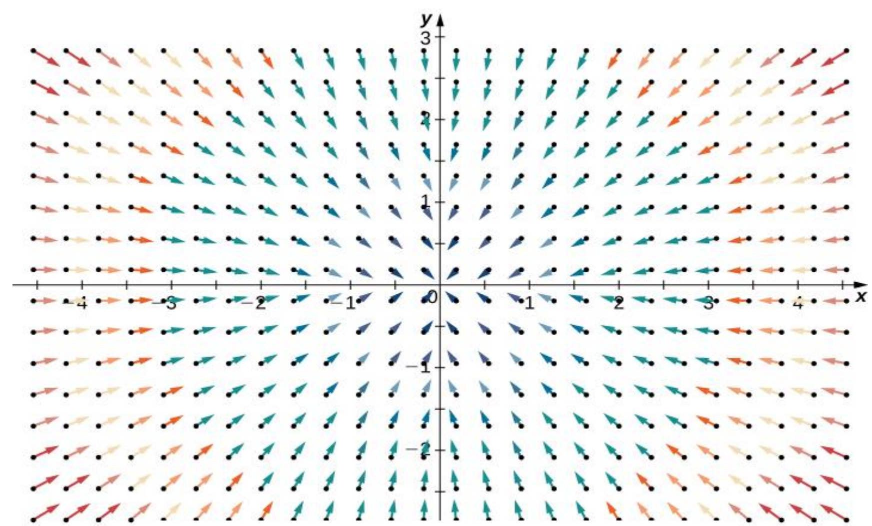
$$\mathrm{div}\,(\langle 1,2\rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0$$

$$\mathrm{div}\,(\langle -y,x\rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

$$\mathrm{div}\,(\mathbf{R}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2$$



Vector field $\langle -y, x\rangle$



Vector field $\langle -x, -y\rangle$

3.1 Divergence in different CS

- Divergence in different coordinate systems:
  - Cartesian: $\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
  - Cylindrical: $\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial r A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$
  - Spherical: $\nabla \cdot \vec{A} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{Rsin\theta}\frac{\partial (sin\theta A_\theta)}{\partial \theta} + \frac{1}{Rsin\theta}\frac{\partial A_\varphi}{\partial \varphi}$

- Some "divergence rules":
  - $\nabla \cdot a = 0$
  - $\nabla \cdot (\vec{A}_1 + \vec{A}_2) = \nabla \cdot \vec{A}_1 + \nabla \cdot \vec{A}_2$
  - $\nabla \cdot c\vec{A} = c\nabla \cdot \vec{A}$

Quiz 4

- Find the divergence of the vector field

$$\vec{F}_1 = \cos(4xy)\hat{x} + \sin(2x^2y)\hat{y}$$

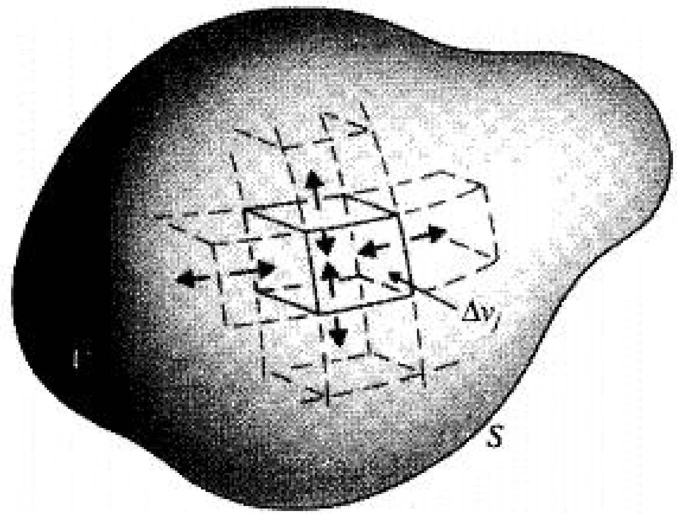
$$\vec{F}_2 = 2r^2\cos\varphi\hat{r} + \sin\varphi\hat{\varphi} + 4z^2\sin\varphi\hat{z}$$

$$\vec{F}_3 = R^3\cos\theta\hat{R} + R\theta\hat{\theta} + 2\sin\varphi\cos\theta\hat{\varphi}$$

3.1 Divergence (Gauss's) Theorem

- Divergence Theorem:
  - The net outward flux from a closed surface equals the integral of the divergence throughout the region bounded by that surface.

$$\iiint_V \nabla \cdot \vec{A}\, dv = \oiint_S \vec{A} \cdot d\vec{s}$$



  - It is also known as "Gauss's Theorem".
  - It converts a closed surface integral into an equivalent volume integral and vice versa.

3.2 Gauss's Law - Integral and Differential

- Considering the electric flux density:

$$\boxed{\oiint \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}} = \iiint_V \rho dv}$$ (From slide p.15)

- Using the Divergence Theorem:

$$\oiint_S \vec{D} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{D} dv = \iiint_V \rho dv$$

This is true for any volume *v* bounded by a surface *s*.
So, the two integrands must be equal.

- Thus, at any point in space, we have:

$$\nabla \cdot \vec{D} = \rho \quad \text{or} \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

Quiz 5

- Applying the Divergence Theorem
- Calculate the surface integral $\oiint_S \vec{F} \cdot d\vec{s}$, where S is cylinder $x^2 + y^2 = 1, 0 \leq z \leq 2$, including the circular top and bottom.

$$\vec{\boldsymbol{F}} = \langle \frac{x^3}{3} + yz, \frac{y^3}{3} - \sin(xz), z - x - y \rangle$$

Quiz 6

- The electric flux density in the region $r \leq 0.08\text{m}$ is $\boldsymbol{D} = 5r^2\hat{\boldsymbol{r}}\ \text{mC/m}^2$.
  - Find the volume charge density ρ_v for $r = 0.06\text{m}$;
  - To make $\boldsymbol{D} = 0$ for $r > 0.08\text{m}$, what surface charge density could be located at $r = 0.08\ \text{m}$?

Next ...

- Electric Potential
  - Energy and Potential
  - The potential field
  - Gradient
  - Maxwell's equation - II
  - Curl