# CAN102 Electromagnetism and Electromechanics

# Lecture-9 Static Magnetic Fields I


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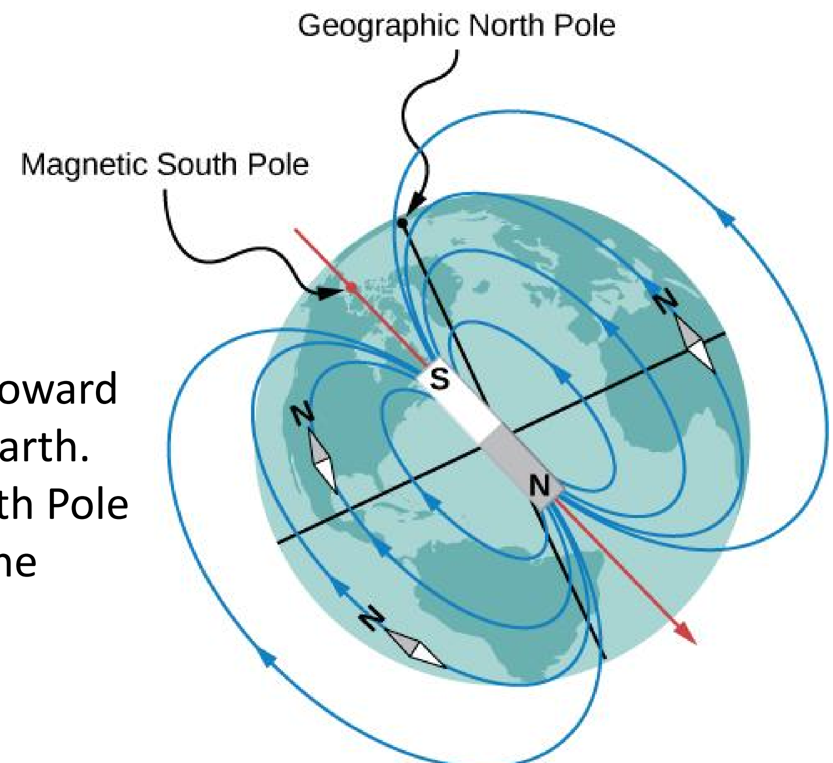
# *Outline*

- Fundamentals of Magnetic Fields
  - What is a magnetic field
  - Sources of the magnetic fields
- Biot-Savart Law
- Gauss's Law for Magnetic Field
- Magnetic field Loop Theorem – Ampere's Law
  - Integral and Differential forms
  - Application: find magnetic field for given current sources

# 1.1 What is a magnetic field?

- What is a magnetic field?
  - A field of force;
  - More precisely – a space in which the magnetic force is experienced by a moving charged particle.

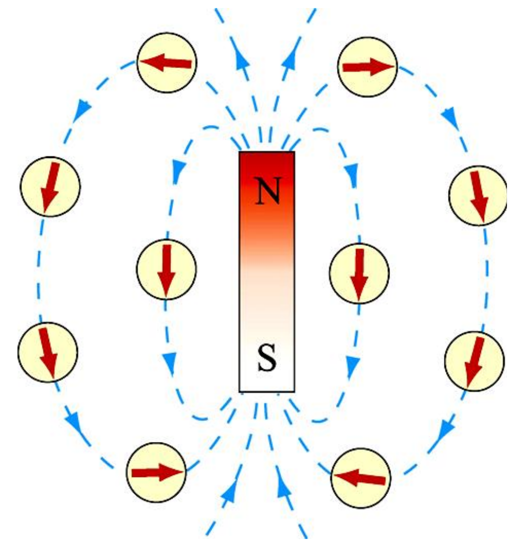
- To describe a magnetic field, we use
  - **B** – "Magnetic Flux Density"
  - **H** – "Magneitc Field Intensity"



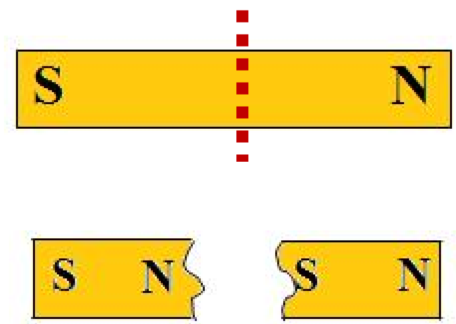

- The north pole of a compass needle points toward the south pole of the magnetic field inside Earth.
- It also points toward Earth's geographic North Pole because the geographic North Pole is near the magnetic south pole.

# 1.2 Sources of Magnetic Fields

- Source of a magnetic field:
  - 1. Permanent magnets
    - A bar magnet is a source of a magnetic field.
    - The bar magnet consists of two poles: the north (N) & the south (S).
    - Magnetic fields are strongest at the poles.
    - The magnetic field lines leave from N & enter S.
    - The like poles repel each other while the opposite poles attract.

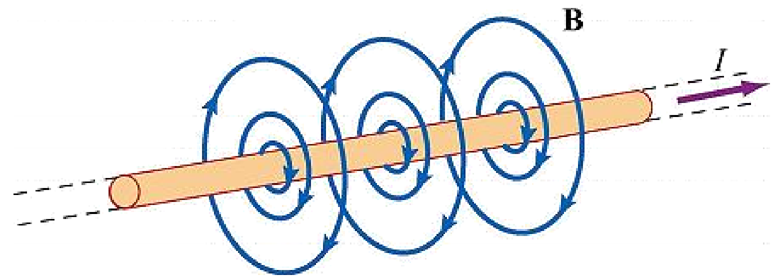


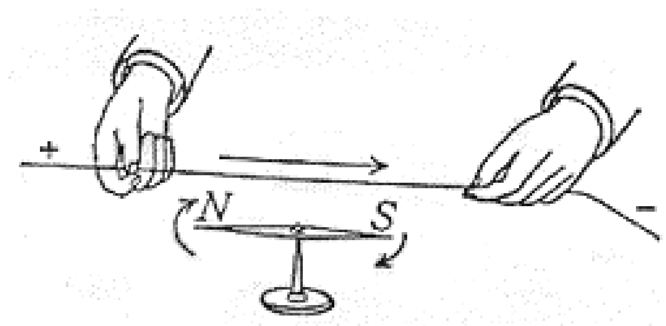

Bar magnets are dipoles!



# 1.2 Sources of Magnetic Fields

- Source of a magnetic field:
  - 2. Electric current
    - An electric current produces a magnetic field as it flows through a wire.




In 1820, a Danish physicist Oersted, realised that current flowing in a wire made the needle of a compass swing. The direction depends on the direction of the current:

Right-hand rule for the magnetic field due to the current *I*:

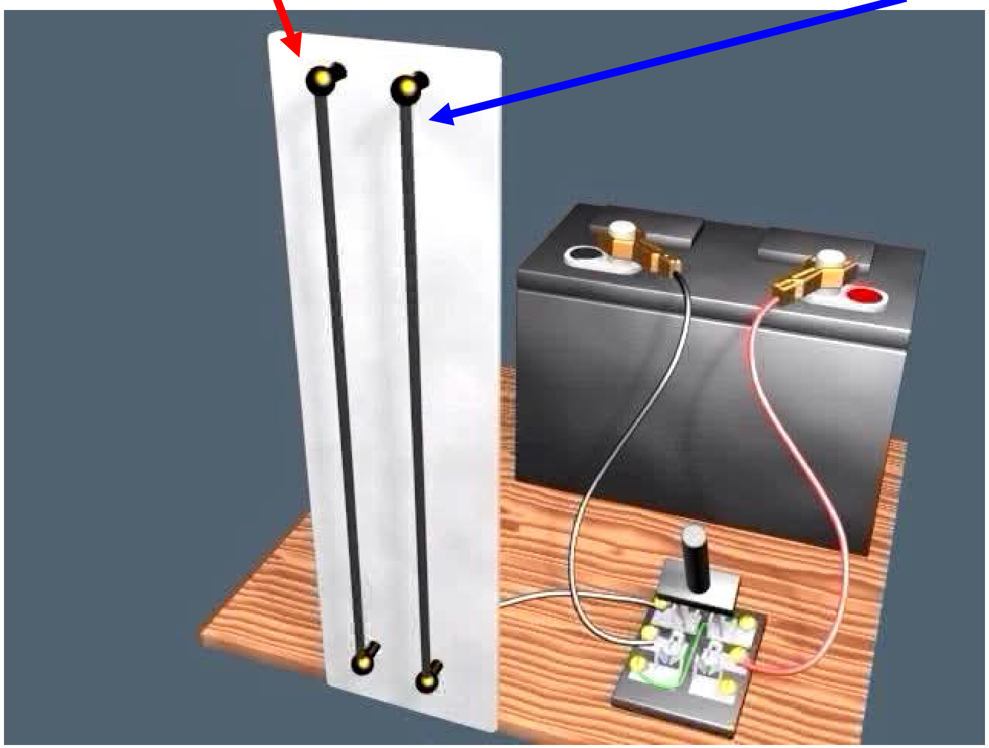
- ✓ Point the <u>thumb</u> of the right hand in the direction of the *I*.
- ✓ The <u>four fingers</u> curl around the current element in the direction of the magnetic field lines.

# 1.3 Early Studies - Ampere's exp.

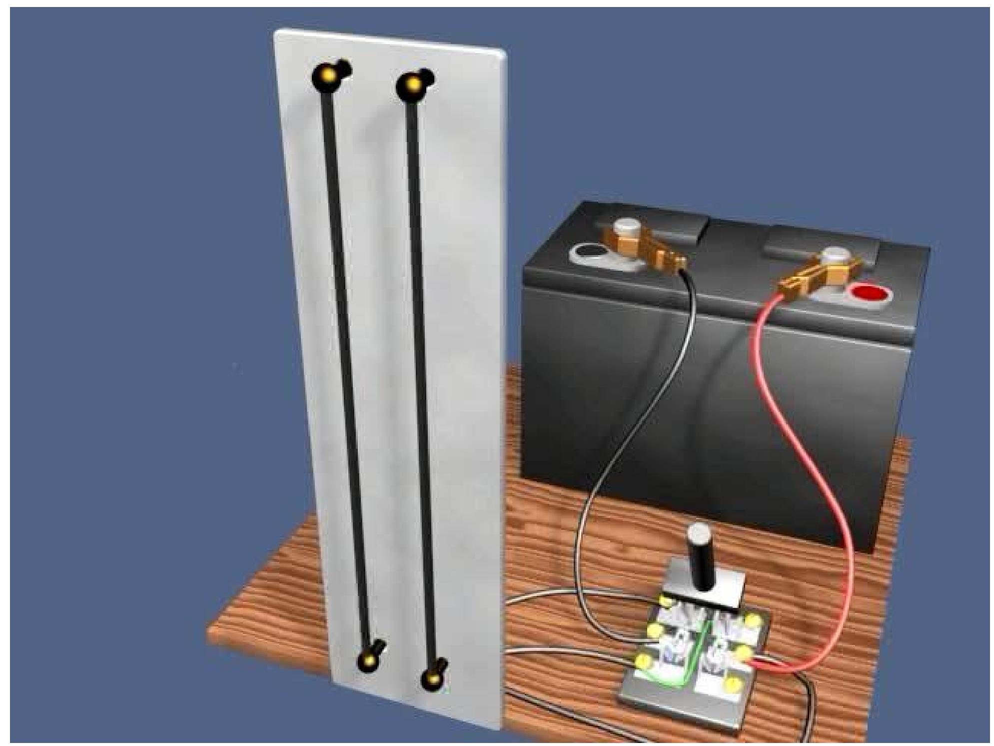
**Two current-carrying wires to exert force on each other.**

A current-carrying wire produces a magnetic field.

In a magnetic field, a wire carrying a current experiences a net force.



**Parallel** – current in same direction

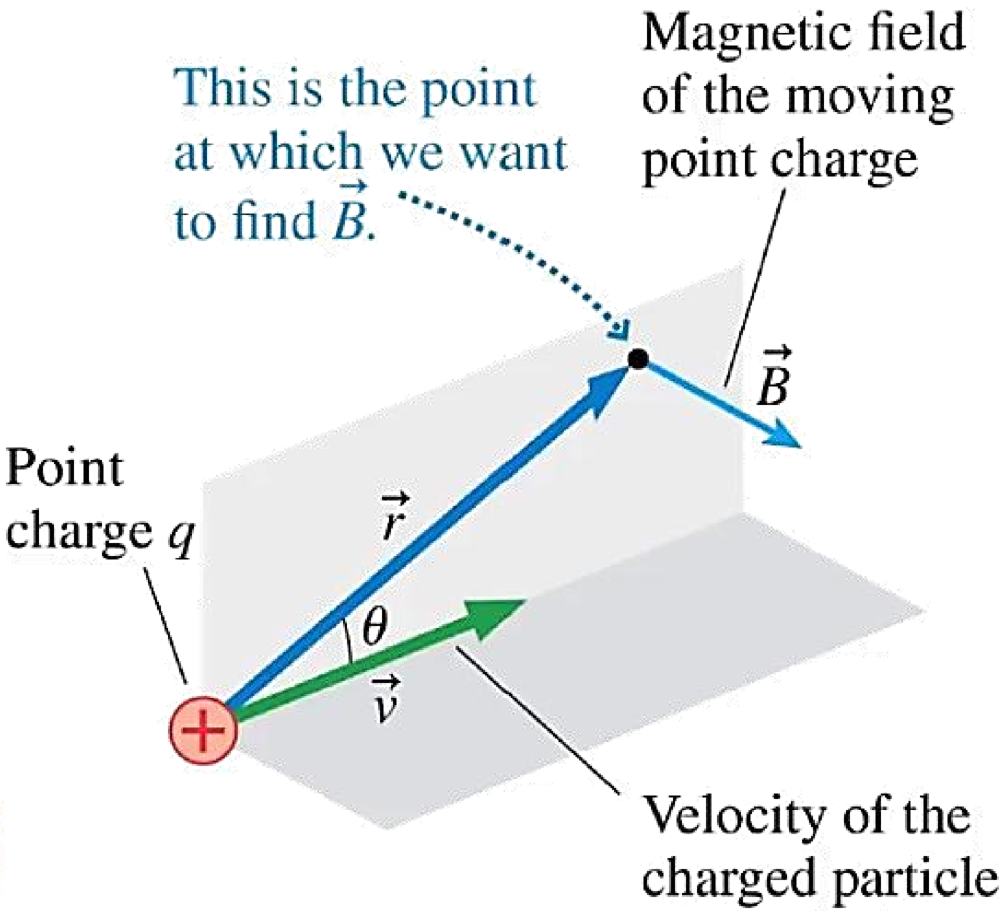


**Series** – current in different direction

# 2.1 H-field from Moving Charge

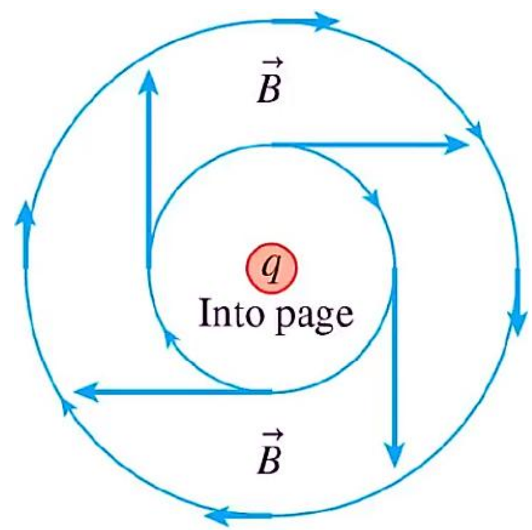
Not required

- The magnetic field can be created by a single point charge $q$ **moving** with a constant velocity.




Experiment shows that a moving point charge gives rise to a magnetic field and this field is determined by the following equation:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2}$$

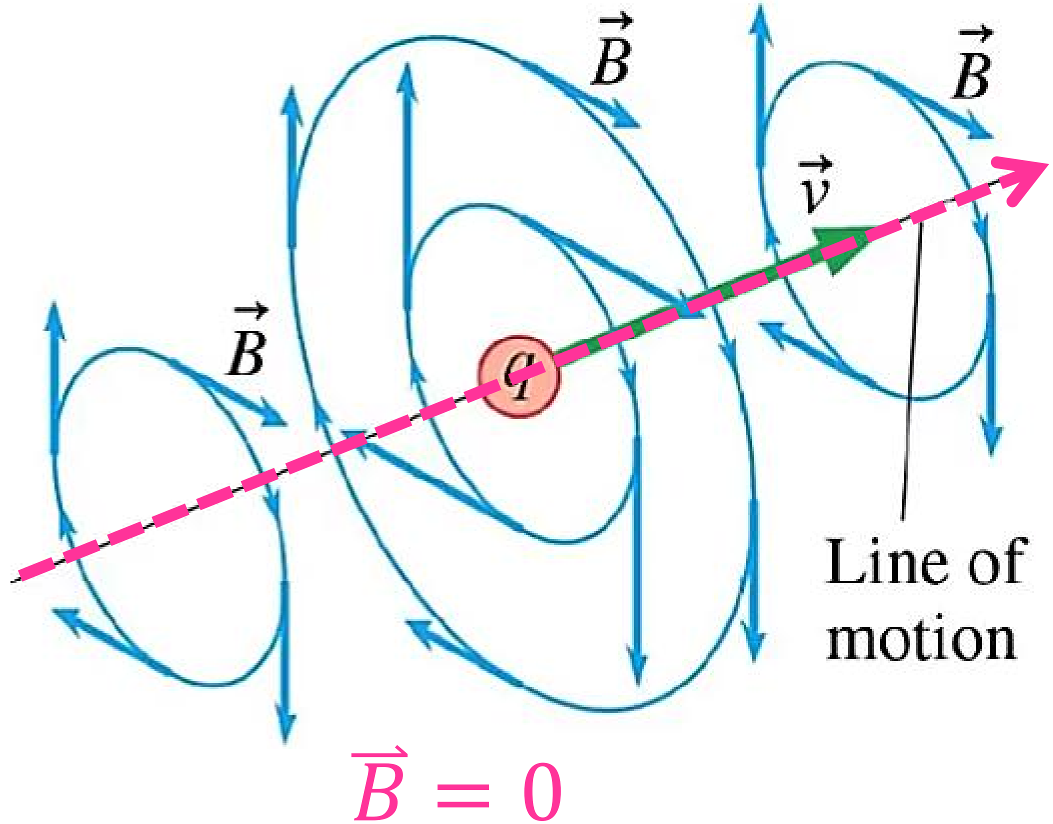



$\hat{r}$: the direction from the source point (the charge $q$ that causes the field) to the observation point of the field.

# 2.1 H-field from Moving Charge

Not required

- Field strength at point P (location of charge is O):
  - Field stronger, if P is closer to charge;
  - Equally strong for P on circles around path;
  - Field stronger, if OP is perpendicular to velocity vector $\vec{v}$;
  - Zero for points on the line of velocity vector $\vec{v}$.




$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2}$$

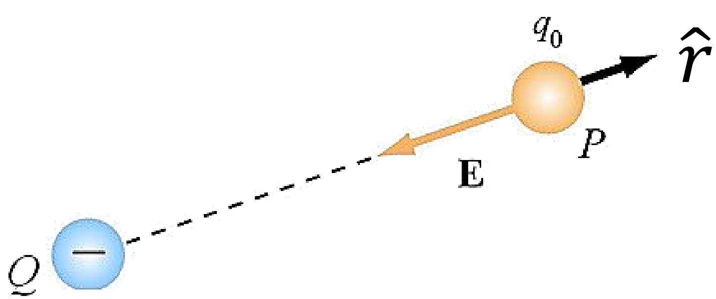
# 2.1 Comparison

Not required

- **Electric Field**
- A distribution of electric charge at rest/moving creates an E-field in the surrounding space.
- An electric charge produces an electric field:

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{r}$$

- The E-field exerts a force on any other charges that are presented in the field:

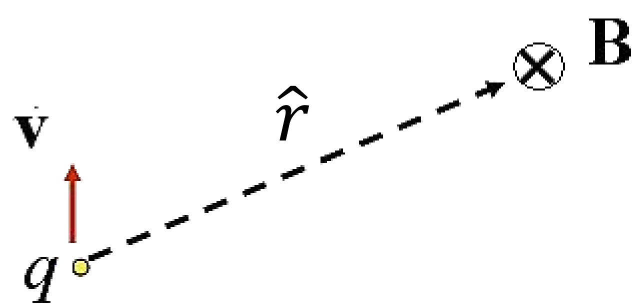



$$\vec{F}_E = q\vec{E}$$

- **Magnetic Field**
- A moving charge or a current creates a magnetic field in the surrounding space in addition to its electric field.
- Moving charge with a **constant** velocity produces magnetic field:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

- The magnetic field exerts a force on any other moving charge or current that are presented in the field.

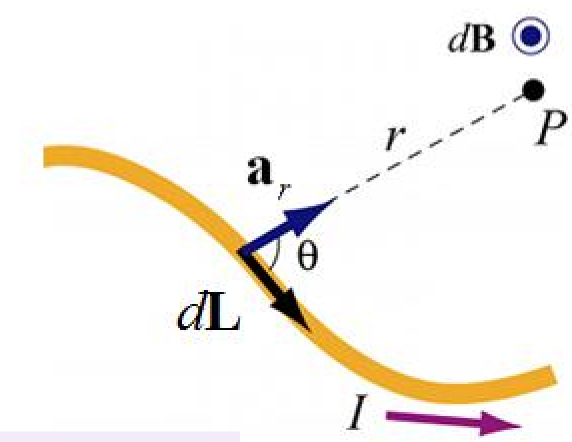


# 2.2 Biot-Savart Law

- French scientists Jean Biot and Felix Savart arrived at an expression that results the magnetic flux density **B** at a point in space to the current $I$ that generates **B**, known as the Biot-Savart Law.
- It states that the differential magnetic flux density $d\vec{B}$ generated by a steady current $I$ flowing through a different length $d\vec{l}$ is given by:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\boxed{I d\vec{l} \times \hat{r}}}{\boxed{r^2}} = \frac{\mu_0}{4\pi}\frac{I d\vec{l} \times \vec{r}}{r^3}$$

$$= \frac{\vec{r}}{r}$$




- Adding up these contributions to find the magnetic field:

$$\vec{B} = \int_L d\vec{B} = \frac{\mu_0}{4\pi}\int_L \frac{I d\vec{l} \times \vec{r}}{r^3}$$

SI Unit: N/(Cm/s) = N/(Am) = Vb/m² = T (Tesla)

# 2.2 Comparison between E and H fields

$$\vec{B} = \frac{\mu}{4\pi}\frac{q\vec{v}\times\vec{r}}{r^3} = \frac{\mu_0\mu_r}{4\pi}\frac{Id\vec{l}\times\vec{r}}{r^3}$$

- Magnetic field intensity **H**
  - unit: A/m
- Magnetic flux density **B**
  - unit: T (Wb/m$^2$)

$$\boldsymbol{B} = \mu\boldsymbol{H} = \mu_0\mu_r\boldsymbol{H}$$

  - where $\mu$ is the ***permeability*** of the material;
  - $\mu_0$ for free sapce is

$$\mu_0 = 4\pi \times 10^{-7}\ H/m$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{r} = \frac{Q\vec{r}}{4\pi\varepsilon_0\varepsilon_r r^3}$$

- Electric field intensity **E**
  - unit: V/m
- Electric flux density **D**
  - unit: C/m$^2$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} = \varepsilon_0\varepsilon_r\boldsymbol{E}$$

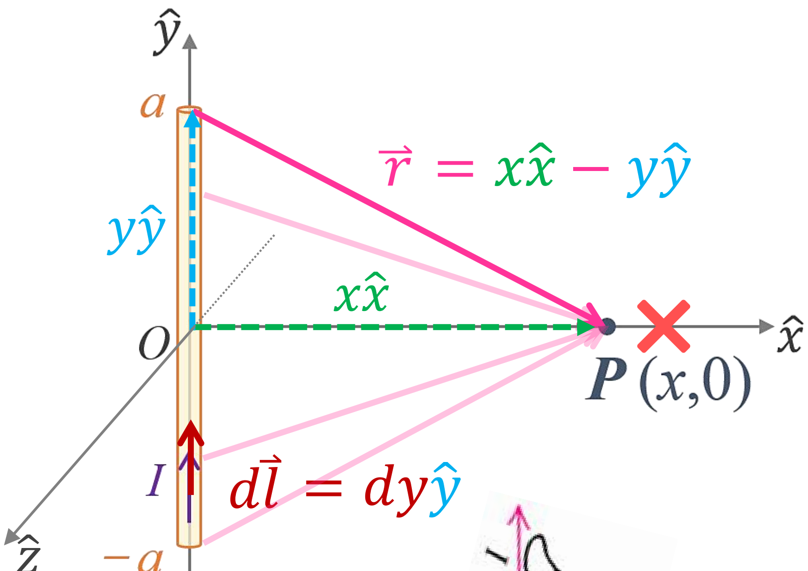
  - where $\varepsilon$ is the ***permittivity*** of the material;
  - $\varepsilon_0$ for free sapce is

$$\varepsilon_0 = 8.854 \times 10^{-12}\ F/m$$

# Example 1: A Straight Wire

$$\int \frac{1}{\left(\sqrt{r^2+z^2}\right)^3}dz = \frac{z}{r^2\sqrt{r^2+z^2}}$$

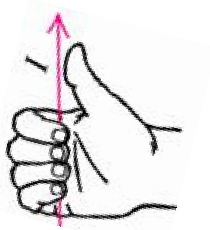
- A current element is placed at the origin of the rectangular coordinate system in the free space. The current element is oriented along the positive y-axis, its current is *I*, and its length is *2a*. Determine the magnitude of magnetic flux density due to the current element at the point *P*.

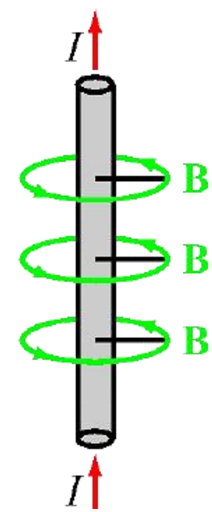



$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{d\vec{l}\times\vec{r}}{r^3} = \frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{dy\hat{y}\times(x\hat{x}-y\hat{y})}{r^3}$$

$$= \frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{x}{\left(\sqrt{x^2+y^2}\right)^3}dy(-\hat{z})$$

$$= \frac{\mu_0 I}{2\pi}\frac{a}{x\sqrt{x^2+a^2}}(-\hat{z})$$



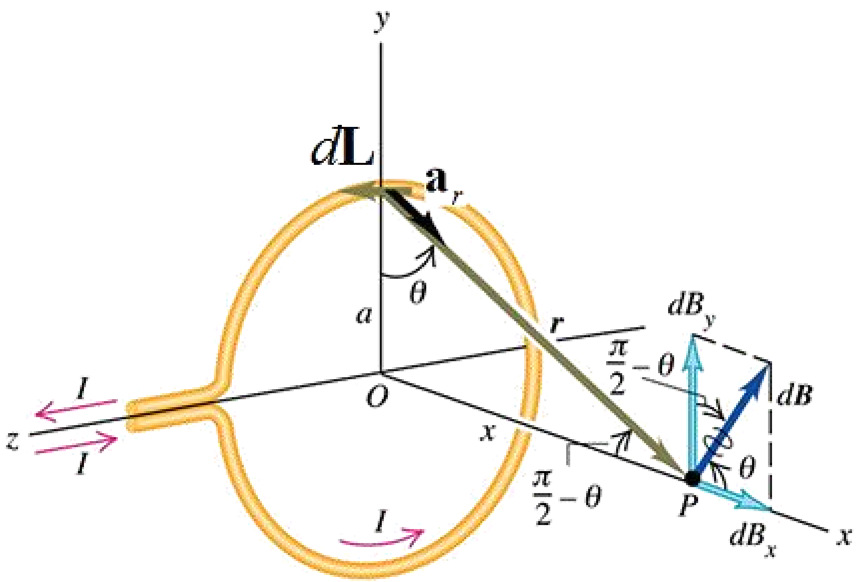




When a⟶∞, then $\vec{B} = \dfrac{\mu_0 I}{2\pi x}(-\hat{z})$

# Quiz 1: A Circular Loop

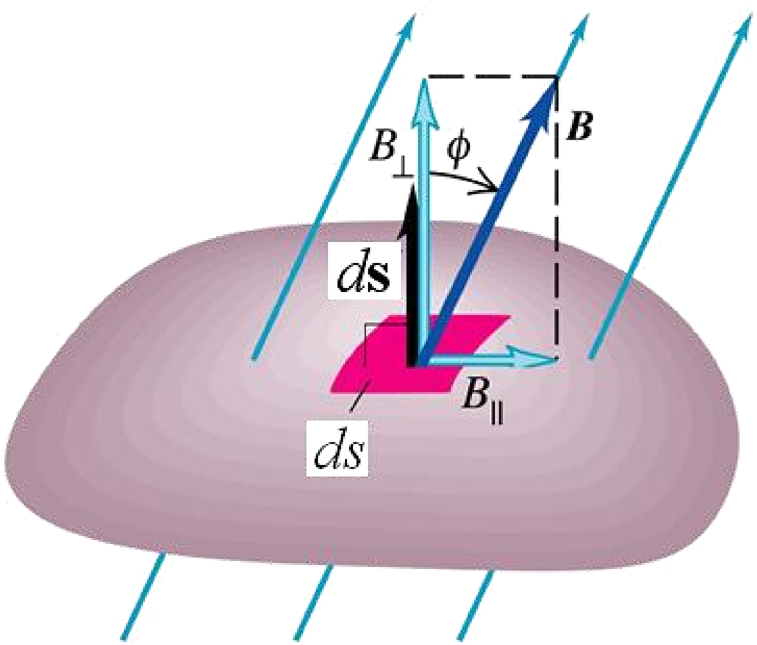
- A circular loop of radius $a$ in the $y$-$z$ plane carries a steady current $I$. What is the magnetic flux density at a point $P$ on the axis, at a distance $x$ from the center?

Rotational symmetry about *x*-axis
→ Only *x*-component remains



# 3.1 Magnetic Flux

- Magnetic flux $d\Phi_B$ through an element of area $d\vec{s}$ is defined as:




$$d\Phi_B = \vec{B} \cdot d\vec{s} \qquad \Phi_E = \int \vec{E} \cdot d\vec{s}$$

The total magnetic flux through the surface is the sum of the contributions from the individual area elements:

$$\Phi_B = \iint \vec{B} \cdot d\vec{s}$$

The SI unit of magnetic flux is:

$$1Wb = 1T \cdot m^2 = 1N \cdot m \cdot A^{-1}$$

# 3.2 Gauss's Law for Magnetic Field

- The **integral form** of Gauss's Law for magnetic field:

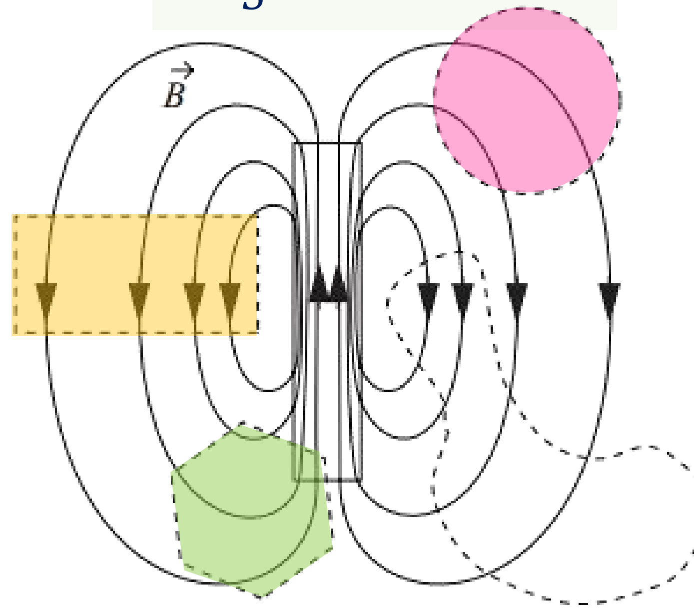
$$\oiint_S \vec{B} \cdot d\vec{s} = 0$$

- The total magnetic flux passing through any **closed** surface is zero.
- Gauss's law for magnetic fields arises directly from the lack of isolated magnetic poles ('magnetic monopoles') in nature.
  - To date, all efforts to detect magnetic monopoles have failed, and every magnetic north pole is accompanied by a magnetic south pole, no matter how small they are.
  - In other words, if you have a real or imaginary closed surface of any size or shape, the total magnetic flux through that surface must be 0.

# 3.2 Gauss's Law

- Like the electric flux $\Phi_E$, the magnetic flux $\Phi_B$ through a surface may be thought of as the "amount" of magnetic field "flowing" through the surface.
  - When you think about the number of magnetic field lines through a surface, don't forget that magnetic fields are continuous in space, and that "number of field lines" only has meaning once you've established a relationship between the number of lines you draw and the strength of the field.
  - No matter what shape of surface you choose, and no matter where in the magnetic field you place that surface, you'll find that the number of field lines entering the volume enclosed by the surface is exactly equal to the number of field lines leaving that volume.

$$\oiint_S \vec{B} \cdot d\vec{s} = 0$$




The **net** magnetic flux passing through any **closed surface** must be **zero** because magnetic field lines always form complete loops.

# 3.3 Differential Form

Vector fields with zero divergence are called "solenoidal" fields.
All static magnetic fields are solenoidal.

- **The Differential Form**
- Recall the 'Divergence Theorem', we have

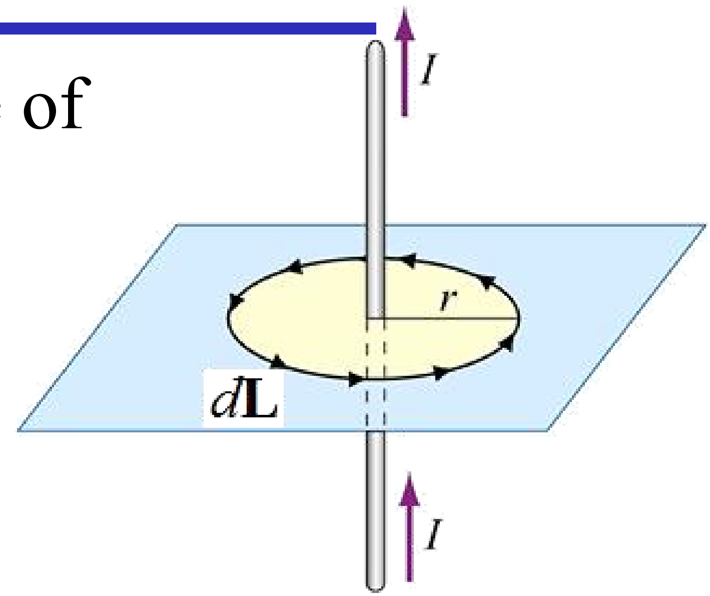
$$\iiint_V \nabla \cdot \vec{B} dv = \oiint_S \vec{B} \cdot d\vec{s} = 0$$

- This must be true for any volume V bounded by a surface S, so the two integrands must be equal.
- Therefore, at any point in space, we have

$$\nabla \cdot \vec{B} = 0$$

The divergence of the magnetic field – the tendency of the magnetic field to either "flow" away or towards a point, is **zero**.

# 4.1 Ampere's Law




- Moving charges or currents are the source of magnetism.
  - Consider the magnetic field caused by a long straight wire carrying a current *I* in free space.

- The field at a distance *r* from the wire:
  - Take the line integral of $\vec{H}$ around one such circle with radius *r*, then:

$$\oint_C \vec{H}\cdot d\vec{l} = Hr\oint_C d\varphi\,(\hat{\varphi}\cdot\hat{\varphi}) = H2\pi r = I_{enc}$$

$$\vec{H} = \frac{I}{2\pi r}\hat{\varphi}$$

# 4.1 Ampere's Law

- Ampere's circuital law (Ampere's Law):
- The line integral of the magnetic field intensity $\vec{H}$ around a closed path equals the current enclosed.

$$\oint_C \vec{H} \cdot d\vec{l} = I_{enc}$$

**Integral Form**

- "Enclosing" is done by the path $C$ around which the magnetic field is integrated.

An electric current through a surface produces a circulating magnetic field around any path that bounds that surface.

- Recall the "Curl (Stoke's) Theorem":

$$\iint_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

$$\oint_C \vec{H} \cdot d\vec{l} = \iint_S \nabla \times \vec{H} \cdot d\vec{s} = \iint_S \vec{J} \cdot d\vec{s}$$

$$\nabla \times \vec{H} = \vec{J} \quad \text{or} \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

**Differential Form**

A circulating magnetic field is produced by an electric current.

# 4.2 Comparison

- Ampere's law in magnetism is analogous to Gauss's law in electrostatics.
- In order to apply them, the system must possess certain symmetry.

| | | |
|---|---|---|
| Biot-Savart Law | $d\vec{H} = \frac{1}{4\pi}\frac{Id\vec{l} \times \hat{r}}{r^2}$ | General current source<br>*e.g.*: finite wire |
| Ampere's Law | $\oint_C \vec{H} \cdot d\vec{l} = I_{enc}$ | Current source has certain symmetry<br>*e.g.*: infinite wire |

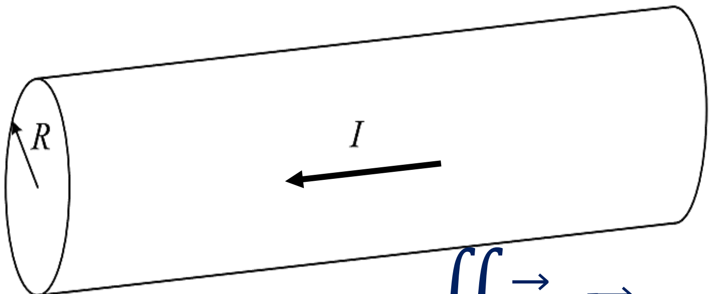
- Ampere's law is applicable to the following current configurations:

1. Infinitely long straight wires carrying a steady current $I$
2. Infinitely large sheet of thickness $b$ with a current density $J$
3. Infinite solenoid (螺线管)
4. Toroid (螺线圈)

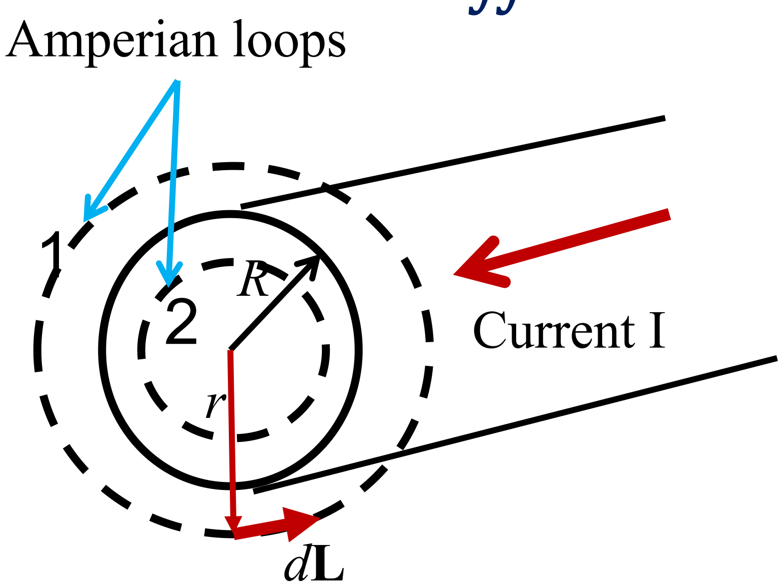
# Example 2: Long Straight Wire

$$\oint_C \vec{H}\cdot d\vec{l} = I_{enc}$$

- Consider a long straight wire of radius *R* carrying a current *I* of **uniform** current density. Find the magnetic field intensity everywhere.




$$I = \iint \vec{J}\, d\vec{s}$$




Outside the wire $r \geq R$:

$$LHS = H_\varphi \oint dl = 2\pi r H_\varphi$$

$$RHS = I$$

$$\vec{H} = \frac{I}{2\pi r}\hat{\varphi}$$

Inside the wire $r < R$:

$$LHS = H_\varphi \oint dl = 2\pi r H_\varphi$$

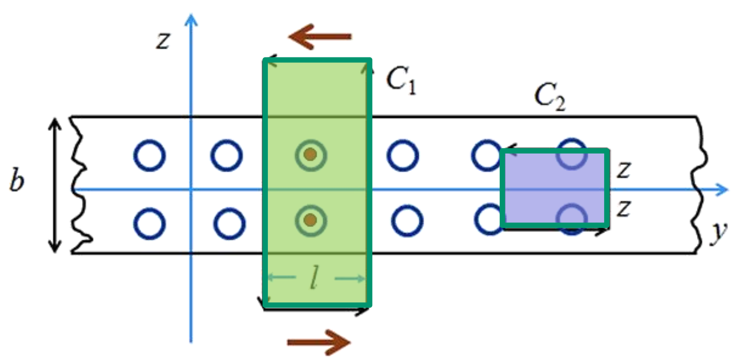
$$RHS = \left(\frac{\pi r^2}{\pi R^2}\right) I$$

$$\vec{H} = \frac{Ir}{2\pi R^2}\hat{\varphi}$$

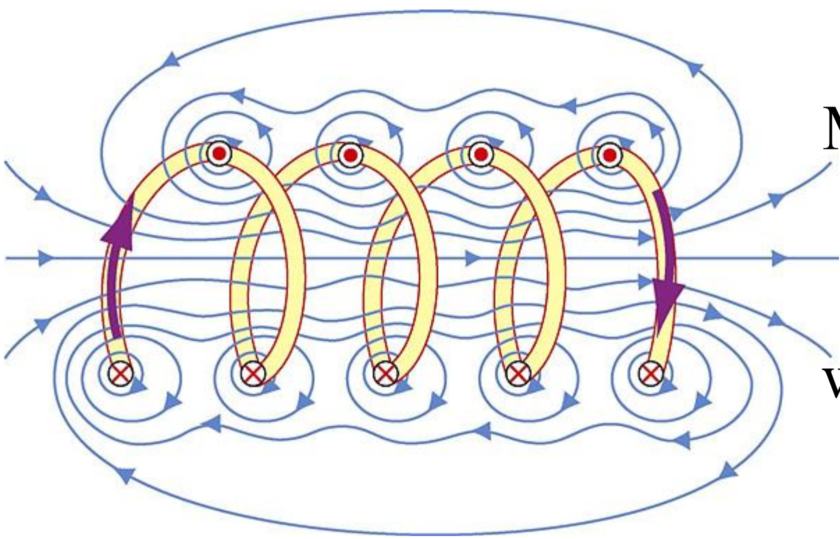
# Quiz 2: Infinite Large Sheet

- Consider an infinitely large plane of thickness $b$ lying on the $xy$ plane with a **uniform** current $I$ (with current density $J_0$ in $\hat{x}$ direction). Find the magnetic field intensity everywhere.

Hint:
Take it as a set of parallel wires carrying currents in the +*x*-direction.
The *z*-component vanishes after adding up the contributions from all wires.

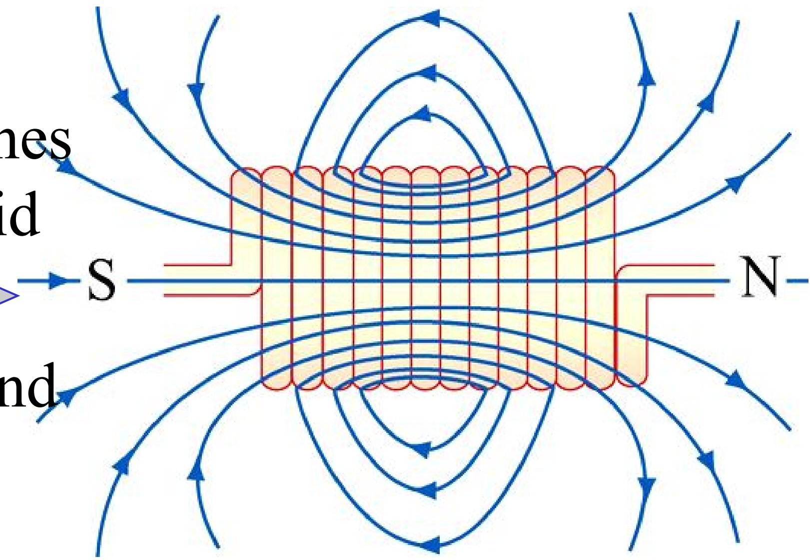


# Example 3: Solenoid

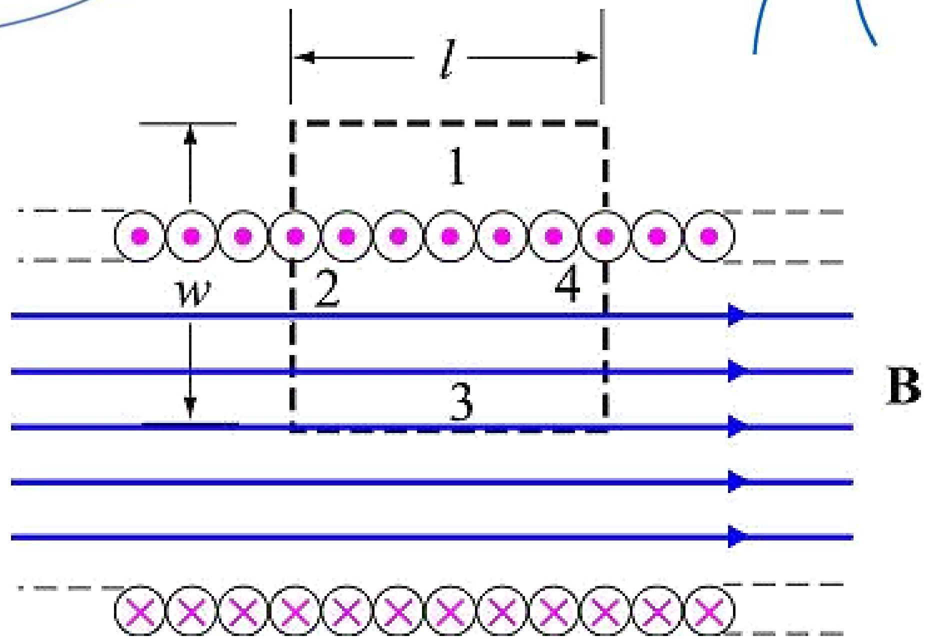


Magnetic field lines of a thin solenoid

wire **tightly** wound




Ideal solenoid (infinite)

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# Example 3: Solenoid (cont.)

$$\oint_C \vec{H} \cdot d\vec{l} = I_{enc}$$

The Amperian loop: 1→2→3→4

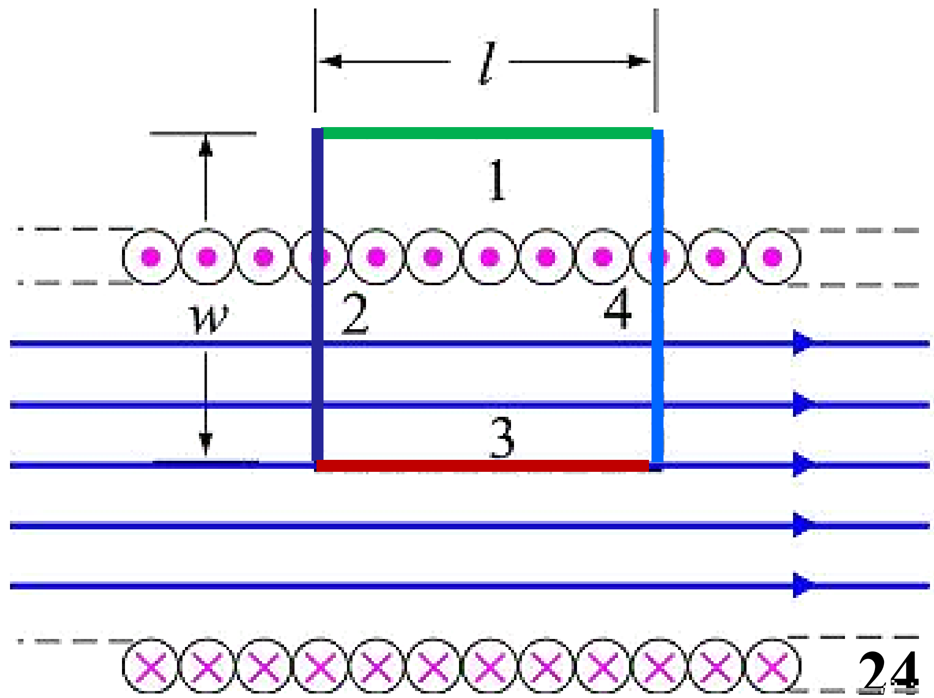
$$LHS = \int_1 \vec{H} \cdot d\vec{l} + \int_2 \vec{H} \cdot d\vec{l} + \int_3 \vec{H} \cdot d\vec{l} + \int_4 \vec{H} \cdot d\vec{l}$$

$$= 0 + 0 + H\int_3 dl + 0 = Hl$$

$$RHS = NI$$

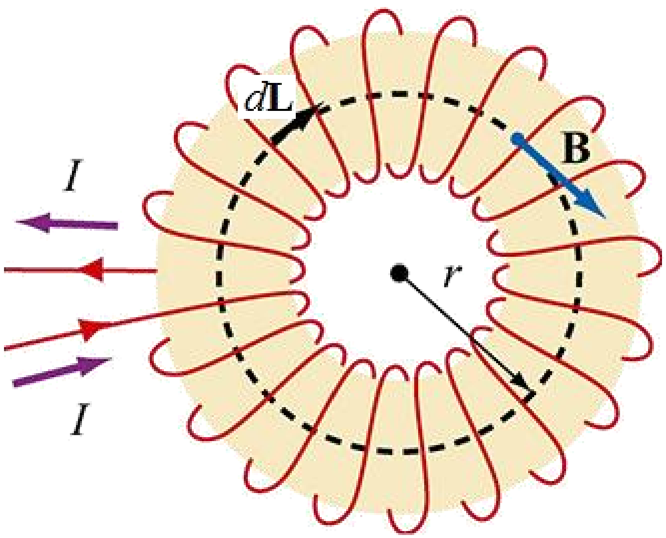
$$\Rightarrow H = \frac{NI}{l}$$

Direction: towards to the right



# Quiz 3: Toroid

- Consider a toroid which consists of $N$ **tightly wound** turns of wire carrying a current $I$. Find the magnetic field intensity everywhere.



# 4.3 Procedure

$$\oint_C \vec{H} \cdot d\vec{l} = I_{enc}$$

- **Steps:**

1. Identify regions in which to calculate the magnetic field intensity **H**
2. Direction: **Right-hand rule**
3. Choose Amperian loop: **High Symmetry**
4. Calculate the integral $LHS = \oint_C \vec{H} \cdot d\vec{l}$
5. Calculate current enclosed by the loop $RHS = I_{enc}$
6. Apply Ampere's law $\oint_C \vec{H} \cdot d\vec{l} = I_{enc}$

# Summary

## MAXWELL'S EQUATIONS – STATIC FIELDS

| Law | Integral | Differential | Physical meaning |
|---|---|---|---|
| Gauss's law for E-field | $\oiint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon}$ | $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$ | Electric flux through a closed surface is proportional to the charges enclosed |
| E-field Loop Theorem | $\oint_C \vec{E} \cdot d\vec{l} = 0$ | $\nabla \times \vec{E} = 0$ | Work done by moving a charge in the E-field along a closed loop is 0 |
| Gauss's law for H-field | $\oiint_S \vec{B} \cdot d\vec{s} = 0$ | $\nabla \cdot \vec{B} = 0$ | The total magnetic flux through a closed surface is 0 |
| H-field Loop Theorem | $\oint_C \vec{H} \cdot d\vec{l} = \mathrm{I}$ | $\nabla \times \vec{H} = \vec{J}$ | The H-field produced by an electric current is proportional to the current |

# Next ...

- Visualisation of Magneitc Fields
  - Magnetic field lines
  - Comparison with electric field lines
- Magnetic Forces
  - on a moving charge
  - on a current-carrying wire
- Magnetic materials
  - Permeability
  - Classification and ferromagnetic materials
- Boundary Conditions