MTH102 Solution to Tutorial 05
Continuous random variables

Question 1

The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10, \\ 0 & x \leq 10. \end{cases}$$

(a) Find $P(X > 20)$.

(b) Find the cumulative distribution function of X.

(c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours?

Answer:

(a)

$$P(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} dx = \frac{1}{2}.$$

(b)

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & x < 10, \\ \int_{10}^{x} \frac{10}{t^2} dt = 1 - \frac{10}{x} & x \geq 10. \end{cases}$$

(c)

$$P(X \geq 15) = 1 - F(15) = \frac{2}{3}.$$

Let Y be the number of devices which will function for at least 15 hours, then Y has a binomial distribution $b(6, 2/3)$. Therefore,

$$P(X \geq 3) = \sum_{k=3}^{6} \binom{6}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{6-k} = \frac{656}{729}.$$

Question 2

A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} c(1-x)^2 & 0 < x < 1, \\ 0 & \text{otherwise}. \end{cases}$$

(a) Determine the constant c.

(b) Find the mean and variance of X.

(c) What must the capacity of the tank be so that the probability of the supply's being exhausted in a given week is 0.008?

Answer:

(a)
$$1 = \int_0^1 c(1-x)^2 dx = \frac{c}{3},$$
which implies that $c = 3$.

(b)
$$E(X) = \int_0^1 3x(1-x)^2 dx = \frac{1}{4}.$$
$$Var(X) = E(X^2) - [E(X)]^2 = \int_0^1 3x^2(1-x)^2 dx - \frac{1}{16} = \frac{3}{80}.$$

(c) Let the capacity be a thousands of gallons. The setting implies that
$$P(X \geq a) = 0.008,$$
i.e.
$$\int_a^1 3(1-x)^2 dx = (1-a)^3 = 0.008.$$
Hence $a = 0.8$.

Question 3

A point is chosen at random on a line segment of length L. Find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

Answer:

The point X is uniformly distributed over $(0, L)$. Then the desired probability is
$$P\left(\left\{0 < X < \frac{L}{5}\right\} \cup \left\{\frac{4}{5}L < X < L\right\}\right) = \frac{2}{5}.$$

Question 4

A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it is between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10.

Answer:

Let X be the distance from the shot to the target, and $g(X)$ be the number of points scored. Then X is uniformly distributed over $(0, 10)$ with the pdf

$$f(x) = \begin{cases} \frac{1}{10} & 0 < x < 10, \\ 0 & \text{otherwise.} \end{cases}$$

The setting of problem gives that

$$g(X) = \begin{cases} 10 & 0 \le X \le 1, \\ 5 & 1 < X \le 3, \\ 3 & 3 < X \le 5, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} E[g(X)] &= \int_{-\infty}^{\infty} g(x) f(x) dx \\ &= \int_0^1 10 \cdot \frac{1}{10} dx + \int_1^3 5 \cdot \frac{1}{10} dx + \int_3^5 3 \cdot \frac{1}{10} dx \\ &= \frac{13}{5}. \end{aligned}$$

Question 5

The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is

(a) the probability that a repair time exceeds 2 hours?

(b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

Answer:

Let X be the time required to repair a machine.

(a)

$$P(X > 2) = \int_2^\infty \lambda e^{-\lambda x} dx = e^{-1}.$$

(b)

$$P(X > 10|X > 9) = \frac{P(X > 10)}{P(X > 9)} = \frac{e^{-5}}{e^{-4.5}} = e^{-0.5}.$$

Question 6

The width of a slot of a duralumin forging is (in inches) normally distributed with $\mu = 0.9$ and $\sigma = 0.004$. The specification limits were given as 0.9 ± 0.005. Use the table of standard normal distribution in Appendix to solve the following questions, rounding your answers to the 4th decimal place if necessary.

(a) What percentage of forgings will be defective?

(b) What is the maximum allowable value of σ that will permit no more than 2 in 100 defectives when the widths are normally distributed with $\mu = 0.9$ and σ?

Answer:

Let X be the width.

(a)

$$P(|X-0.9| > 0.005) = 2P\left(\frac{X-0.9}{0.004} > \frac{0.005}{0.004}\right) = 2(1-\Phi(1.25) \simeq 2(1-0.8944) = 0.2112.$$

(b) It is required that

$$P(|X - 0.9| > 0.005) \leq 0.02,$$

where

$$P(|X - 0.9| > 0.005) = 2\left(1 - \Phi\left(\frac{0.005}{\sigma}\right)\right).$$

Therefore,

$$\Phi(\frac{0.005}{\sigma}) \geq 0.99.$$

From the table of standard normal distribution on the last page, $\Phi(2.33) \simeq 0.99$, we have thus

$$\sigma \leq \frac{0.005}{2.33} \simeq 0.0021.$$

Question 7

A candy maker produces mints and the distribution of the weights of these mints is $N(21.37, 0.16)$. Use the table of standard normal distribution in Appendix to solve the following questions, rounding your answers to the 4th decimal place if necessary.

(a) Let X denote the weight of a single mint selected at random from the production line. Find $P(X > 22.07)$.

(b) Suppose that 15 mints are selected independently and weighed. Let Y equal the number of these mints that weigh less than 20.857 grams. Find $P(Y \leq 2)$.

Answer:

(a)

$$P(X > 22.07) = P\left(\frac{X - 21.37}{\sqrt{0.16}} > 1.75\right) = 1 - \Phi(1.75) \simeq 1 - 0.9599 = 0.0401.$$

(b)

$$\begin{aligned} P(X < 20.857) =& P\left(\frac{X - 21.37}{\sqrt{0.16}} < -1.2875\right) \\ =& \Phi(-1.2875) = 1 - \Phi(1.2875) \simeq 1 - \Phi(1.29) \simeq 1 - 0.9015 = 0.0985. \end{aligned}$$

The distribution of Y is then $b(15, 0.1)$. Therefore,

$$P(Y \leq 2) = \sum_{k=0}^{2} \binom{15}{k} 0.1^k 0.9^{15-k} = 0.8159.$$

Question 8

Miss Kim shoots at a target with a round bull's-eye of radius 1 centered at $(0, 0)$. The horizontal and vertical distances from the center to where she shoots are independently normally distributed with mean 0 and variance 3. She keeps shooting until she hits the bull's-eye.

(a) What is the probability that Miss Kim will hit the bull's-eye at the second shot?

(b) To make the game more difficult, the bull's eye is considered hit only if Miss Kim hit between 0.5 and 1 from the center. What is the expected number of attempts until she hits the bull's-eye?

Answer:

Let R be the distance from the shot to the target, then R has a Rayleigh distribution with parameter $\sigma^2 = 3$. Let Y be the number of attempts until she hit the bull's-eye, then Y has a geometric distribution with parameter p.

(a)

$$p = P(R \leq 1) = 1 - e^{-\frac{1}{6}}.$$

Hence

$$P(Y = 2) = (1 - p)p = e^{-\frac{1}{6}} - e^{-\frac{1}{3}}.$$

(b) In this case,

$$p = P(0.5 \leq R \leq 1) = P(R \leq 1) - P(R \leq 0.5) = e^{-\frac{1}{24}} - e^{-\frac{1}{6}}.$$

Hence,

$$E(Y) = \frac{1}{p} = \frac{1}{e^{-\frac{1}{24}} - e^{-\frac{1}{6}}}.$$

Question 9

A bakery sells rolls in units of a dozen. The demand X (in 1000 units) for rolls has a uniform distribution on $[2, 4]$. It costs \$2 to make a unit that sells for \$5 on the first day when the rolls are fresh. Any leftover units are sold on the second day for \$1. How many units should be made to maximize the expected value of the profit?

Answer:

Assume that the bakery makes a (in 1000 units) per day and the profit is $g(X)$. Then to maximize the profit, $a \in [2, 4]$. Hence,

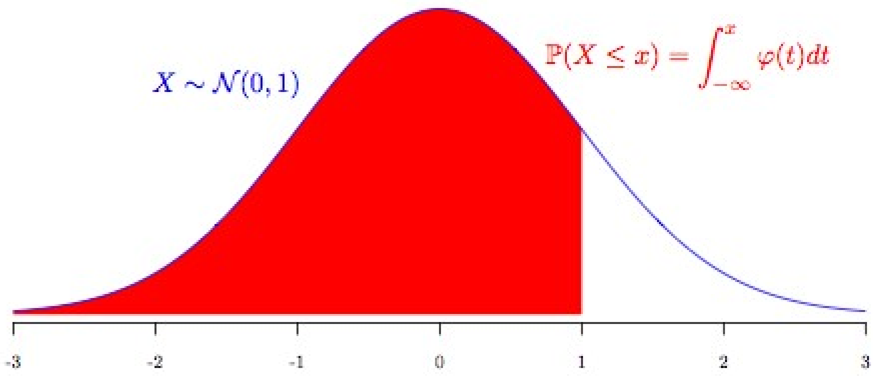
$$g(X) = \begin{cases} 3X - (a - X) = 4X - a, & \text{if } 2 \leq X \leq a, \\ 3a, & \text{if } a \leq X \leq 4. \end{cases}$$

The pdf of X is $f(x) = 1/2$ for $x \in [2, 4]$. Therefore,

$$\begin{aligned}
E[g(x)] &= \int_2^4 g(x)f(x)dx \\
&= \int_2^a \frac{1}{2}(4x - a)dx + \int_a^4 \frac{1}{2}3adx \\
&= -a^2 + 7a - 4 \\
&= -\left(a - \frac{7}{2}\right)^2 + \frac{33}{4}.
\end{aligned}$$

We then deduce that 3500 units should be made to maximize the profit.

Appendix: Table of Standard Normal Distribution



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990