Basic *RC* and *RL* Circuits

EEE103 ELECTRICAL CIRCUITS (Part 2)
Week 8
S1, 2023/24

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Content

- The source-free *RC* circuit
- The source-free *RL* circuit
- The general *RC/RL* circuits
- The unit-step function
- The driven *RC* circuit
- The driven *RL* circuit

The circuit response

- The current-voltage relations of capacitors and inductors are described by differential equations.
- Circuit analysis with capacitors and inductors will need to solve differential equations.
- There are two ways to describe circuit response:

    1. The steady-state response (does not change with time) and the transient response (change with time):

    $$v_{complete} = v_{steady-state} + v_{transient} \qquad [1]$$

    2. The natural response (describe the behavior without external sources) and the forced response (the additional response from external sources):

    $$v_{complete} = v_{natural} + v_{forced} \qquad [2]$$

- We will focus on the second way to describe circuit response in this session.

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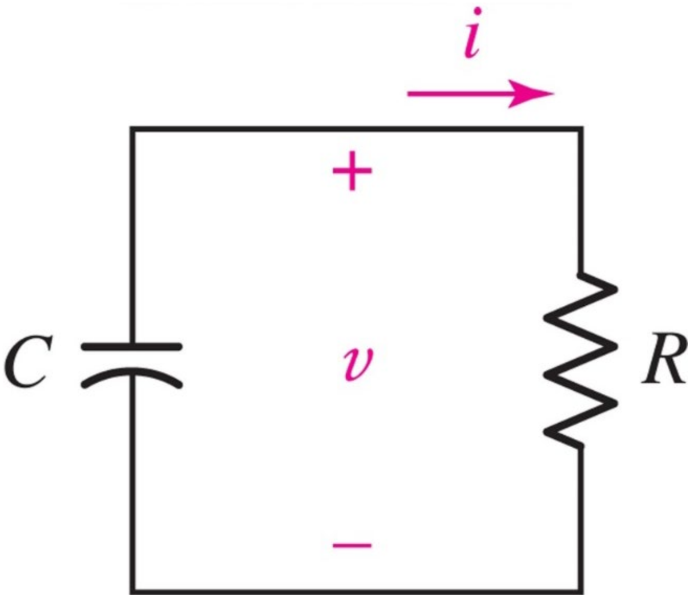
The Source-free RC circuit

Find the natural response for the *RC* circuit with the initial condition $v(0) = \mathrm{V}_0$.



- Step 1: apply KCL:

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad [1]$$

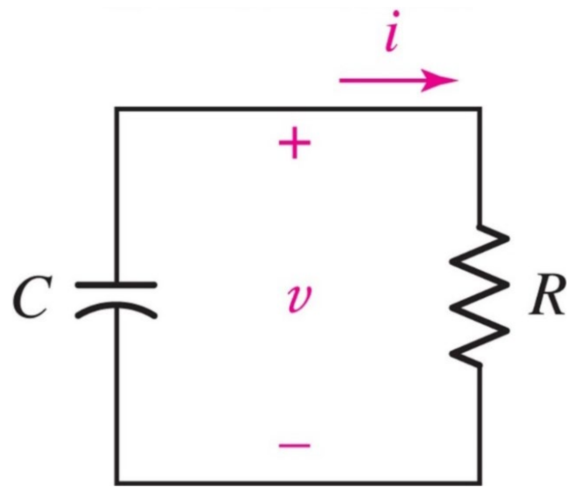
- Step 2: two solutions :

    1. Solution by direct integration
    2. The general approach to the solution of first-order differential equations

Note: direct integration works well for source-free RC circuit, but it is limited to cases where variable cannot be separated.

The Source-free RC circuit

1. Solution by direct integration



$$C\frac{dv}{dt} + \frac{v}{R} = 0 \quad [1]$$

Separating voltage and time variables:

$$\frac{dv}{v} = -\frac{1}{RC}dt \qquad [2]$$

Integrating both sides:

$$\int_{v(0)}^{v(t)} \frac{dv'}{v} = \int_0^t -\frac{1}{RC}\, dt' \qquad [3]$$

Following integration and initial condition $v(0) = \mathrm{V}_0$:

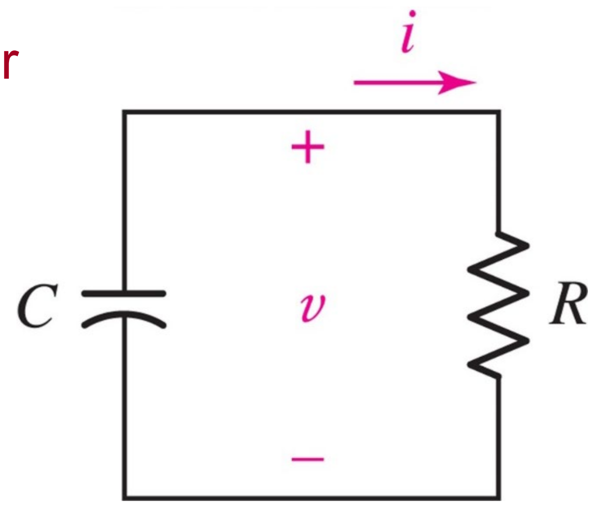
$$\ln v(t) - \ln v(0) = -\frac{1}{RC}(t-0) \qquad [4]$$

Taking the exponential of both sides and solving for $v(t)$:

$$v(t) = \mathrm{V}_0 e^{-t/RC} \qquad [5]$$

The Source-free *RC* circuit

2. The general approach to the solution of first-order differential equations



$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad [1]$$

For the linear first order ordinary differential equations we are concerned with in *RC/RL* circuits, we can assume $v(t)$ in Eq.[1] is an exponential function:

$$v(t) = Ae^{st} \qquad [2]$$

where A and s are unknown constants. Substituting Eq.[2] into Eq.[1], we have:

$$C\frac{d(Ae^{st})}{dt} + \frac{Ae^{st}}{R} = 0 \quad [3] \quad \Rightarrow \quad \left(sC + \frac{1}{R}\right)Ae^{st} = 0 \quad [4]$$

In order to satisfy this equation to all values of time, we choose:

$$\left(sC + \frac{1}{R}\right) = 0 \quad [5] \quad \Rightarrow \quad s = -\frac{1}{RC} \quad [6]$$

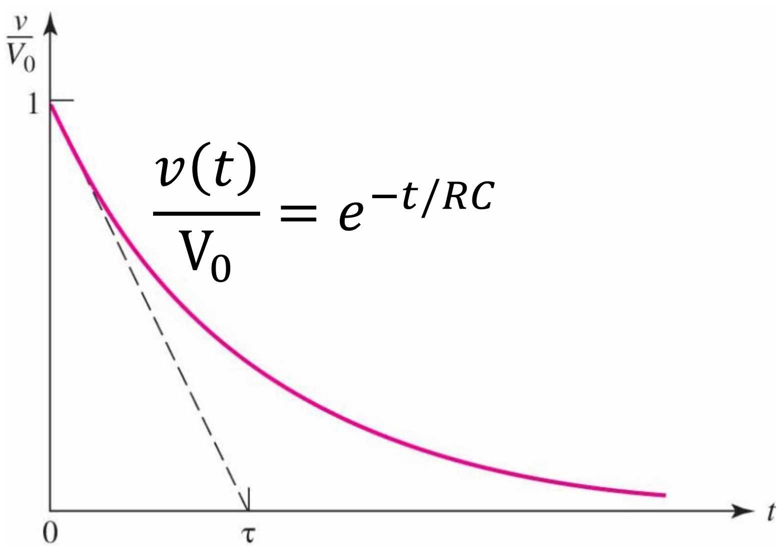
Substitute Eq.[6] back into Eq.[2] with initial condition $v(0) = \mathrm{V}_0$, we obtain $v(t)$:

$$v(t) = \mathrm{V}_0 e^{-t/RC} \qquad [7]$$

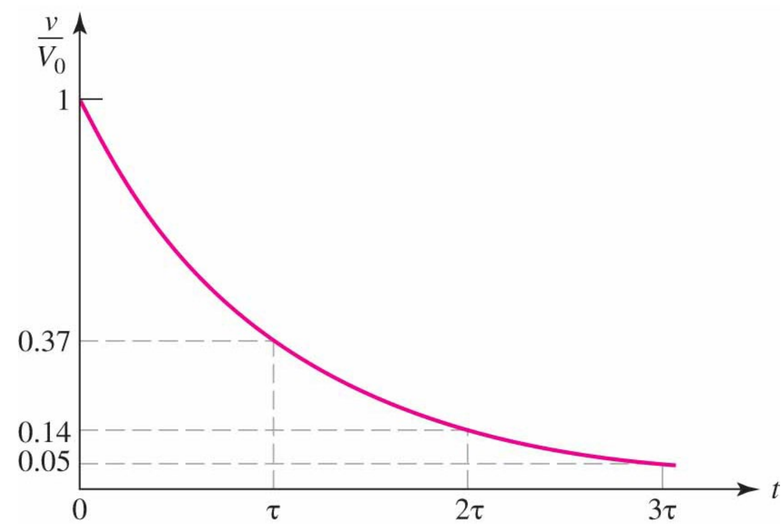
The Natural Response of the *RC* circuit

- Define the time that would be required for the voltage to drop to zero if it continued to drop at its initial rate as the time constant.
- The time constant $\tau = RC$ reflects the rate of decay of the voltage response of the RC circuit
- After five time constants, the voltage is less than 1 percent of its original value, which is negligible.

$$v(t) = \mathrm{V}_0 e^{-t/RC} = \mathrm{V}_0 e^{-t/\tau}$$

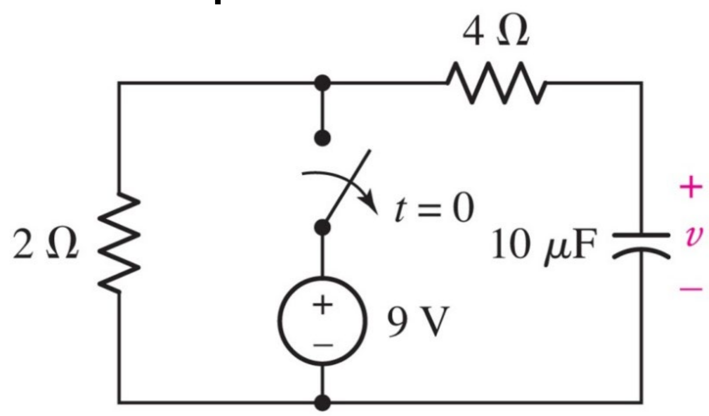


$$\frac{v(t)}{\mathrm{V}_0} = e^{-t/RC}$$

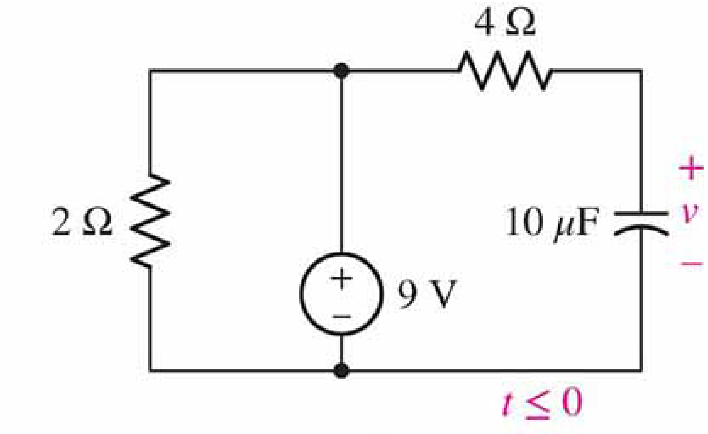


Example 1:

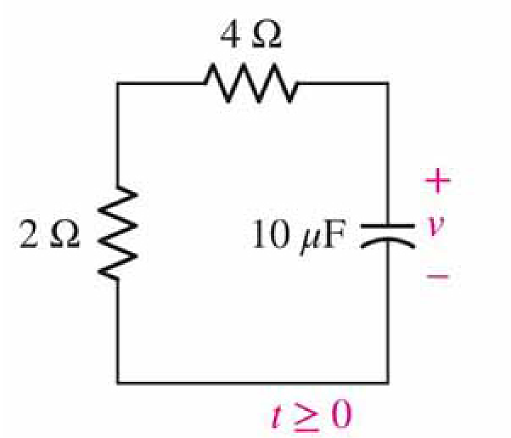
Find $v(t)$ at t = 200 μs.



$t \leq 0$

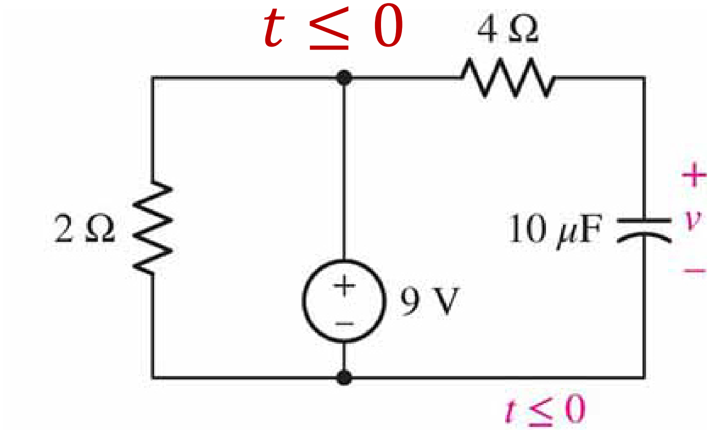


$t \geq 0$



Solution to Example 1

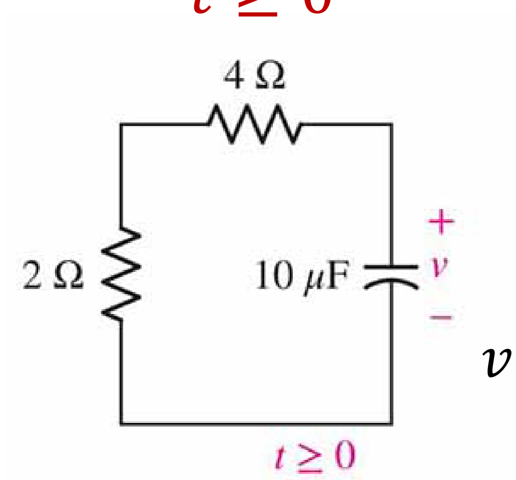
$t \leq 0$



The purpose of this step is to obtain an initial capacitor voltage $v(t = 0)$.

$$v(0) = \mathrm{V}_0 = 9\ \mathrm{V}$$

$t \geq 0$



$$v(t) = \mathrm{V}_0 e^{-t/\tau}$$

$$\tau = RC = (2+4)(10 \times 10^{-6}) = 60 \times 10^{-6}\mathrm{s}$$

$$v(t) = 9e^{-t/60\times10^{-6}}\mathrm{V}$$

$$v(t = 200\mu s) = 9e^{-200\times10^{-6}/60\times10^{-6}}\mathrm{V} = 321.1\ \mathrm{mV}$$

Less than 4% of its maximum value.

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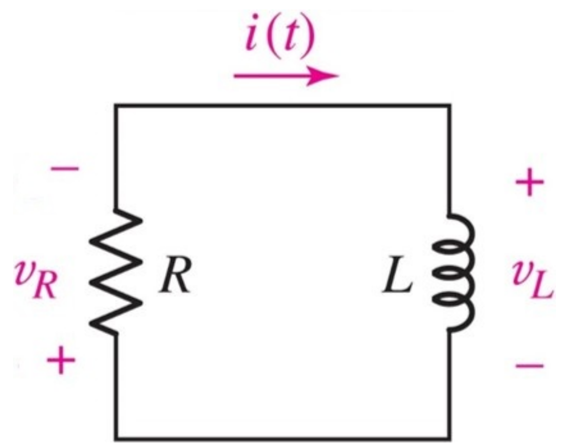
The Source-free *RL* Circuit

Find the natural response for the *RL* circuit with the initial condition $i(0) = \mathrm{I}_0$.



Applying KVL:

$$L\frac{di}{dt} + Ri = 0 \qquad [1]$$

Solving 1st order differential equation :

Step 1: Assuming $i(t) = Ae^{st}$

Step 2: Substituting $i(t) = Ae^{st}$ into $L\frac{di}{dt} + Ri = 0$, solve $s = -\frac{R}{L}$.

Step 3: Applying initial condition $i(0) = \mathrm{I}_0$ into $i(t) = Ae^{-tR/L}$,

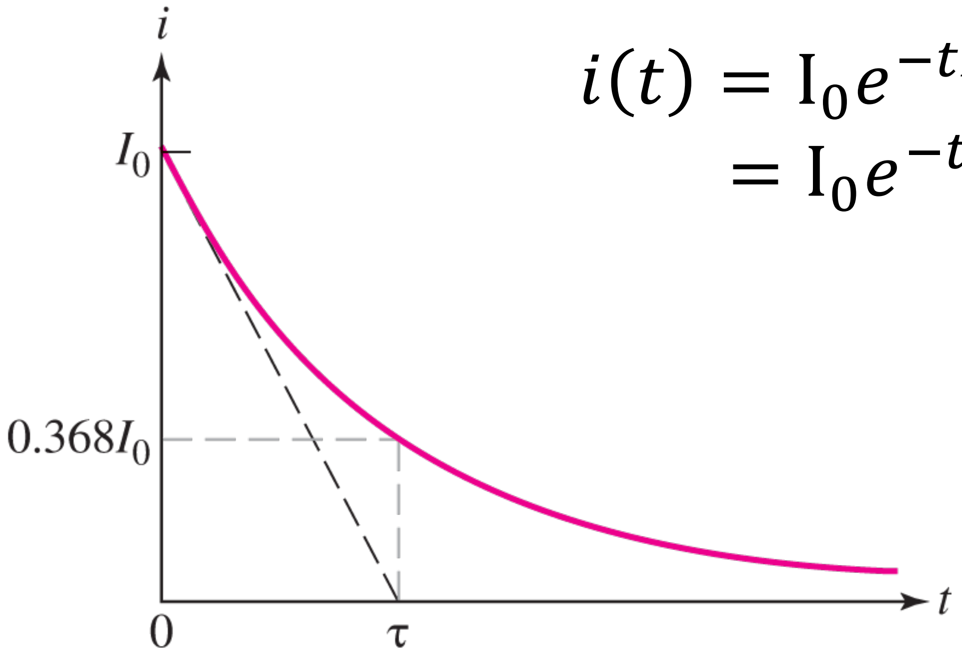
Solve $A = \mathrm{I}_0$.

Thus:

$$i(t) = \mathrm{I}_0 e^{-tR/L} \qquad [2]$$

The Natural Response of the *RL* circuit

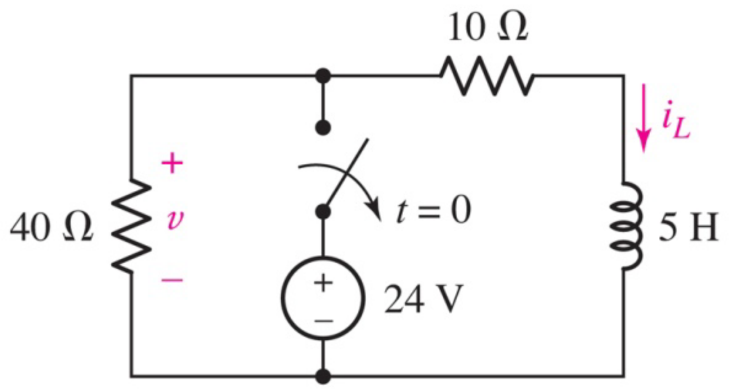
- Define the time that would be required for the current to drop to zero if it continued to drop at its initial rate as the time constant.
- The time constant $\tau = L/R$ reflects the rate of decay of the current response of the *RL* circuit.



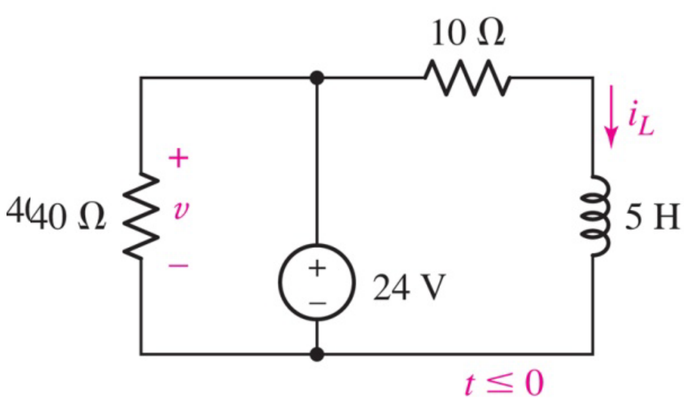
$$i(t) = \mathrm{I}_0 e^{-tR/L}$$
$$= \mathrm{I}_0 e^{-t/\tau}$$

Example 2

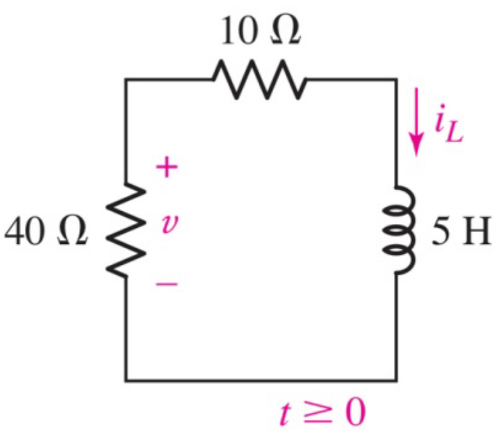
Find $v(t)$ at t = 200 ms.



$t \leq 0$

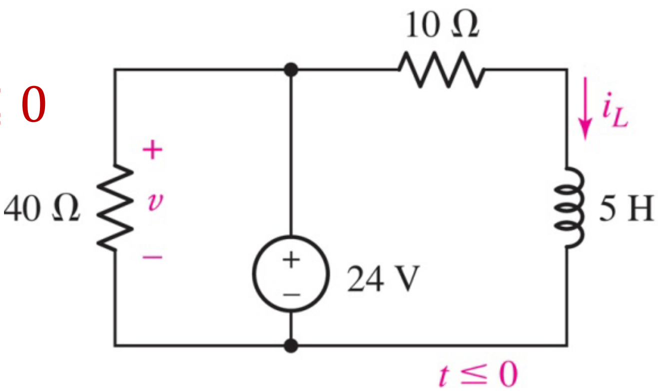


$t \geq 0$



Solution to Example 2

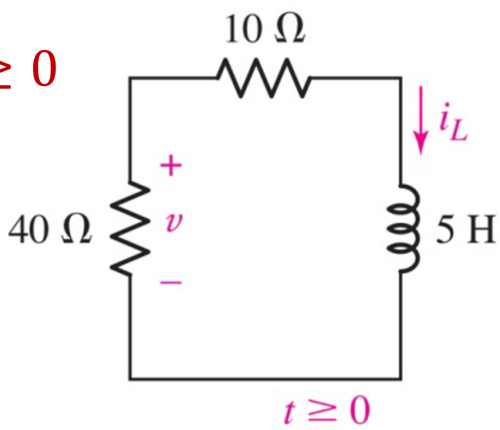
$t \leq 0$



The purpose of this step is to obtain an initial inductor current $i(t=0)$.

$$i(0) = \mathrm{I}_0 = \frac{24}{10} = 2.4\ \mathrm{A}$$

$t \geq 0$



$$i(t) = \mathrm{I}_0 e^{-t/\tau}$$

$$\tau = \frac{L}{R} = 5/(10+40) = 0.1\mathrm{s}$$

$$i(t) = 2.4e^{-10t}\mathrm{A}$$

$$v = 10i(t) + L\frac{di(t)}{dt}$$

$$= 24e^{-10t} - 120e^{-10t}$$

$$= -96e^{-10t}$$

$$v(t = 200ms) = -96e^{-10\times200m} = -12.99\mathrm{V}$$

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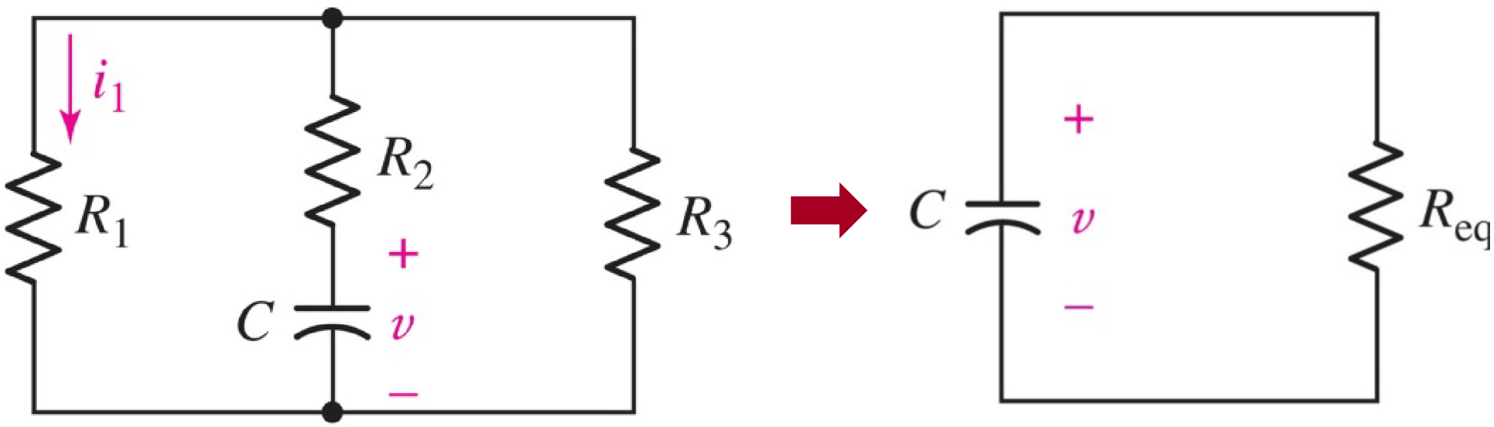
General *RC/RL* Circuits

- Regardless of how many resistors we have in the circuit, we obtain a single time constant (either $\tau = RC$ or $\tau = L/R$) when only one energy storage element is present.
- Many of the *RC* or *RL* circuits for which we would to find the natural response contain more than a single resistor and capacitor/inductor.
- It is possible to replace the two-terminal resistive network which is across the capacitor or inductor terminals with an equivalent resistor.
- The effective time constant is given by:

$$\tau = R_{eq}C \qquad \text{or} \qquad \tau = L/R_{eq}$$

General *RC* circuits

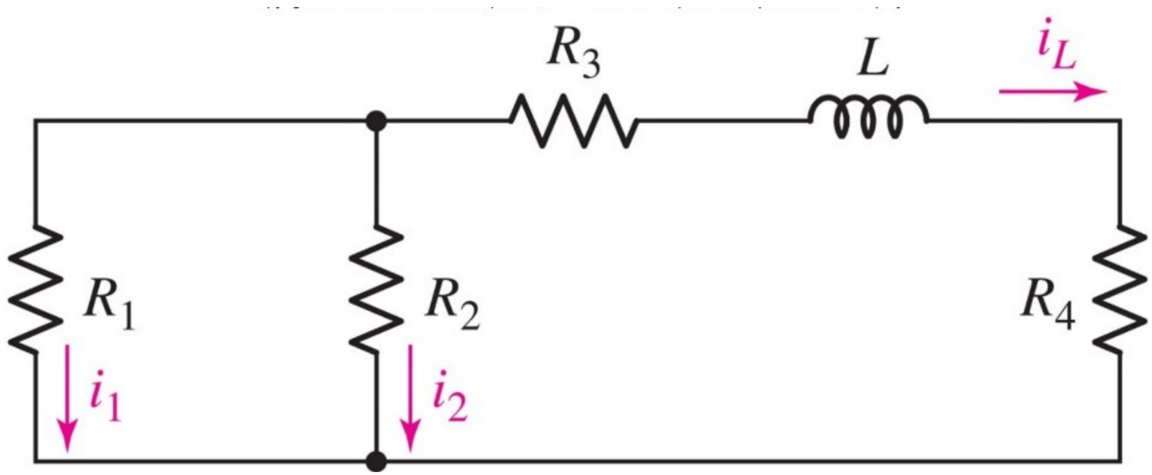
The time constant of a single-capacitor circuit will be $\tau = R_{eq}C$. Please find the equivalent resistance R_{eq} which is seen by the capacitor.



$$R_{eq} = \frac{R_1 R_3}{R_1 + R_3} + R_2$$

General *RL* circuits

The time constant of a single-inductor circuit will be $\tau = L/R_{eq}$. Please find the equivalent resistance R_{eq} which is seen by the inductor.



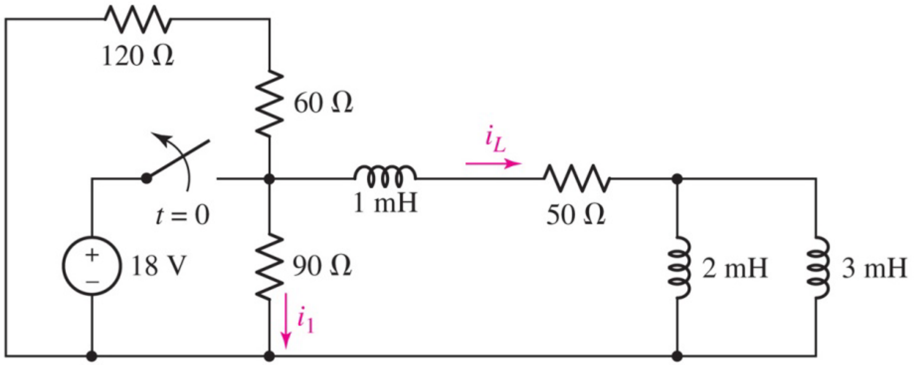
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} + R_2 + R_3$$

1st Order Response Observations

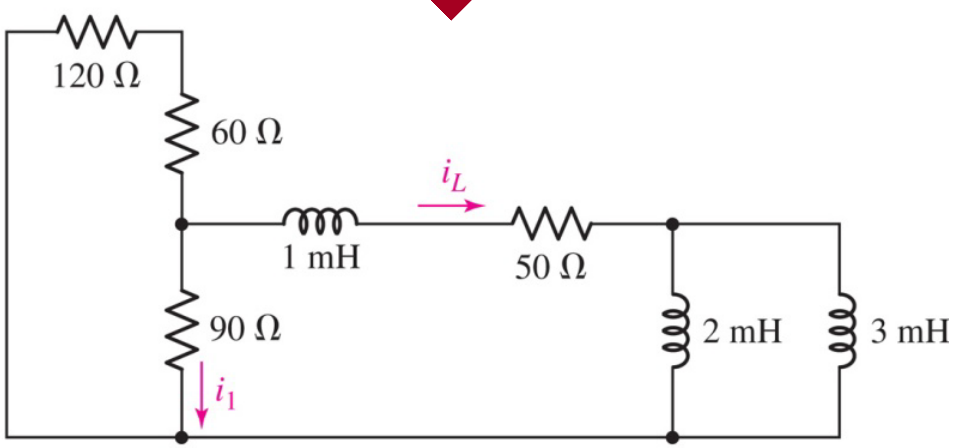
- The voltage across a capacitor cannot change instantaneously, but the current through the capacitor can change instantaneously. In other words, the capacitor voltage just before and after the switching is the same, which is $v(0^-) = v(0^+)$.
- The current through a inductor cannot change instantaneously, but the voltage across the inductor can change instantaneously. In other words, the inductor current just before and after the switching is the same, which is $i(0^-) = i(0^+)$.
- Resistor voltage (or current) just before the switching $v(0^-)$ can be different from the voltage after the switching $v(0^+)$.
- All the voltages and current in an *RC* or *RL* circuit follow the same natural response $e^{-t/\tau}$.

Example 3

Determine both $i_1(t)$ and $i_L(t)$ in the circuit below for $t > 0$.



$t \geq 0$



Solution to Example 3

- For $t \leq 0$, the current though the $1\ \mathrm{mH}$ inductor is $i_L(t)$:

$$i_L(0^-) = \frac{18}{50} = 0.36\mathrm{A}$$

- For $t \geq 0$, the voltage source is disconnected, the R_{eq} and L_{eq} can be represented by:

$$L_{eq} = \frac{2 \times 3}{2+3} + 1 = 2.2\ \mathrm{mH}$$

$$R_{eq} = \frac{90 \times (60+120)}{90 + (60+120)} + 50 = 110\ \Omega$$

- The time constant:

$$\tau = \frac{L_{eq}}{R_{eq}} = \frac{2.2\mathrm{m}}{110} = 20\ \mathrm{\mu s}$$

Solution to Example 3

- Since $i_L(0^-) = i_L(0^+) = 0.36\text{A}$,

For $t > 0$

$$
\begin{aligned}
i_L(t) &= i_L(0^+)e^{-t/\tau} \\
&= 0.36e^{-t/20\,\mu}\text{A} \\
&= 0.36e^{-50000t}
\end{aligned}
$$

For $t > 0$

$$
\begin{aligned}
i_1(t) &= \frac{180}{180+90}(-i_L(t)) \\
&= -\frac{2}{3} \times 0.36e^{-50000t}\text{A} \\
&= -0.24e^{-50000t}
\end{aligned}
$$

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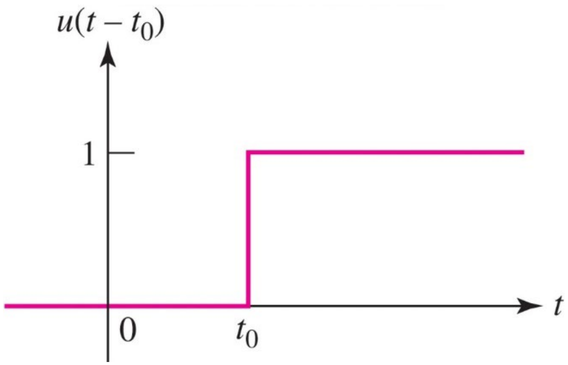
The Unit Step Function

- We have analyzed natural response on the circuits without sources (From all the previous examples, the voltage sources are disconnected when $t > 0$) .
- The circuits will be further analyzed on the response that occurs when the dc sources are suddenly applied
- The Unit Step function $u(t)$ is a convenient notation to represent a sudden change.

The Unit Step Function

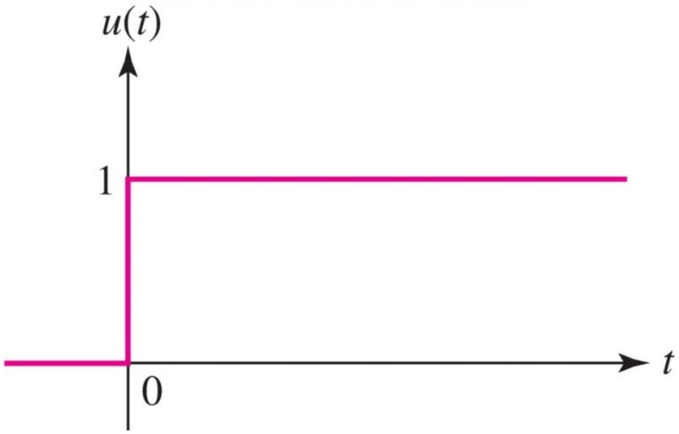
- The unit-step function is defined as a function of time which is zero for all values of its argument less than zero and which is unity for all positive values of its argument. The mathematical definition of the unit-step function is:

$$u(t-t_0)=\begin{cases}0 & t<t_0\\ 1 & t>t_0\end{cases}$$



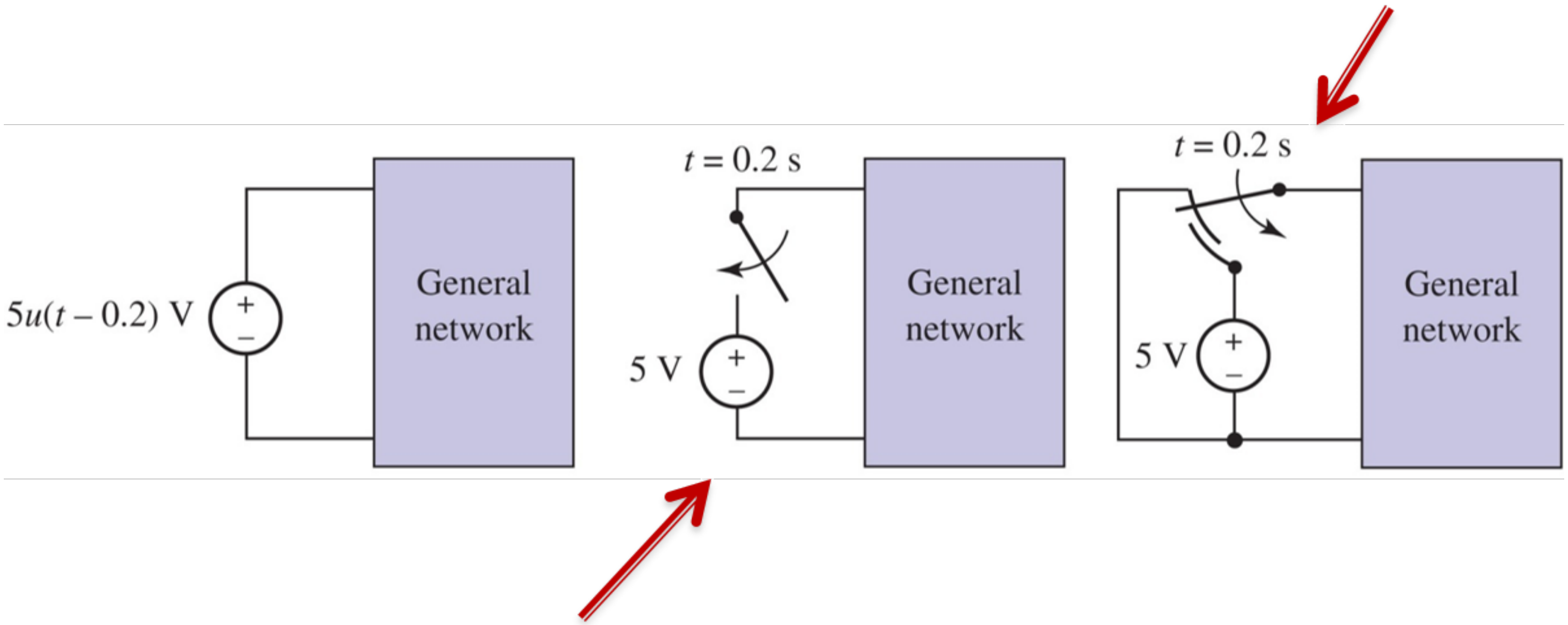
- In the case $t_0=0$, the corresponding unit-step function is represented by $u(t)$, which is represented as:

$$u(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$$



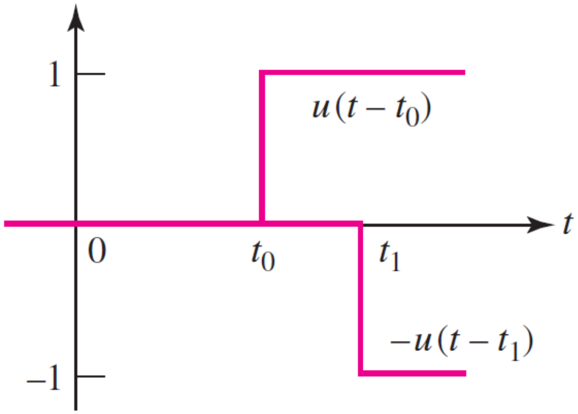
Switches and Steps

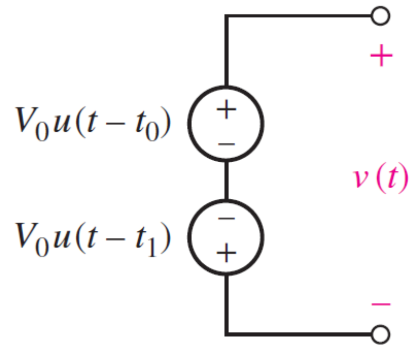
The unit step models a double-throw switch.



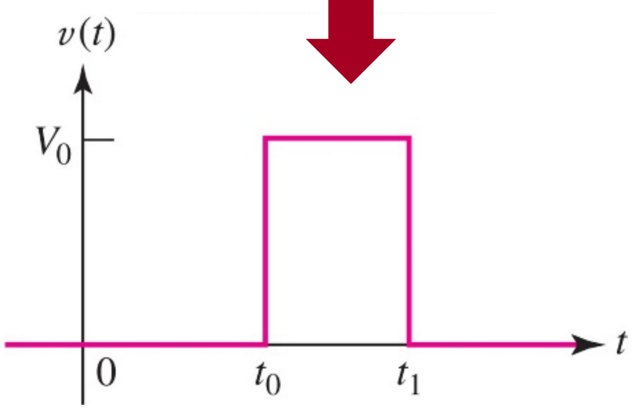
A single-throw switch is open circuit for $t < 0$, not short circuit.

Modelling Pulses Using $u(t)$

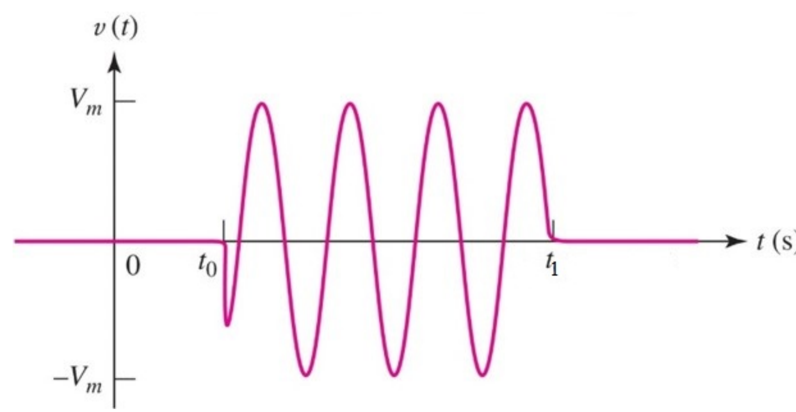




$$V_0 u(t-t_0) - V_0 u(t-t_1)$$



Rectangular pulse



$$v(t) = V_m[u(t-t_0) - u(t-t_1)]sin(295 \times 10^6 t)$$

Sinusoidal pulse

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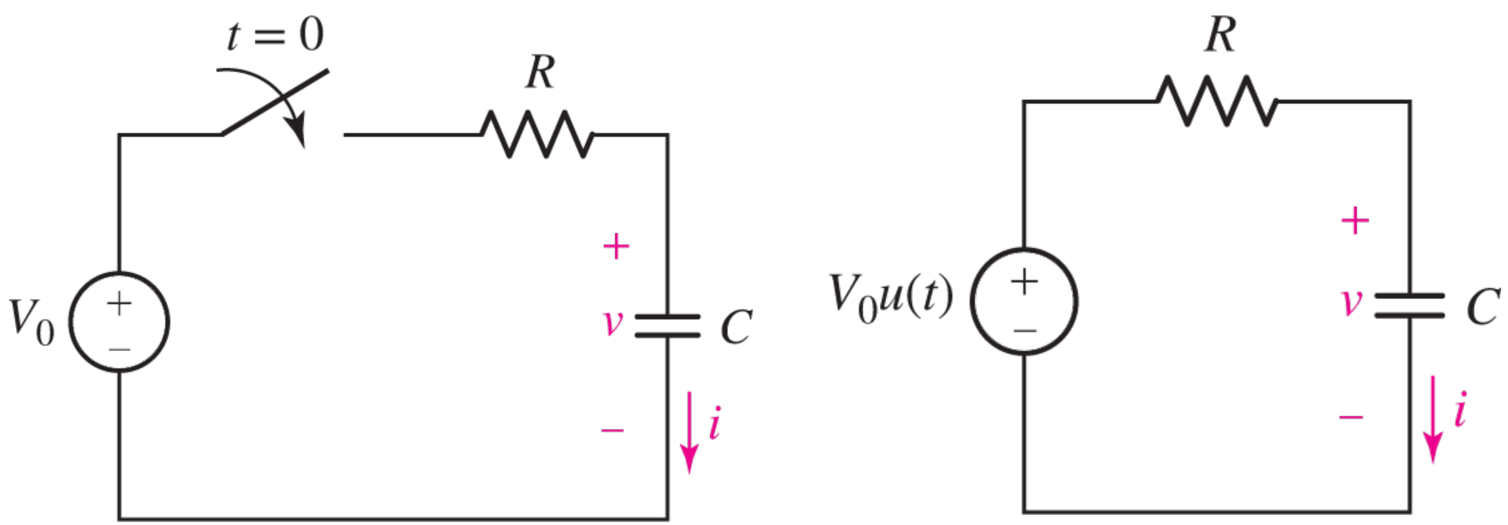
The Driven *RC* circuit

- The two circuits shown both have $v(t) = 0$ for $t < 0$ and are also the same for $t > 0$.
- We now have to find both the natural response and the forced response due to the source V_0 to form the complete response.

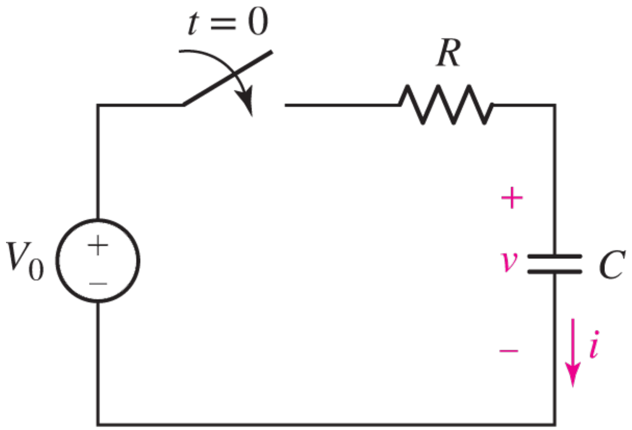


The Driven *RC* circuit

Find the complete response for the *RC* circuit with the switch closed at $t = 0$.

- Applying KCL for the resistor and capacitor current, we get:

$$C\frac{dv}{dt} + \frac{v - V_0}{R} = 0 \text{ (for } t > 0) \qquad [1]$$



- Rearranging:

$$RC\frac{dv}{dt} + v = V_0 \qquad [2]$$

$$v_{complete} = v_{natural} + v_{forced}$$

The Driven *RC* circuit

We are dealing with a mathematical question:

Solve a first order ordinary linear inhomogeneous differential equation.

Summary:

Step 1: Get the natural response (general solution) by setting the forcing term of the differential equation to zero, and try the natural response in form of Ae^{st}.

Step 2: Get the forced response (particular solution) by “guessing” from steady state ($t = \infty$).

Step 3: Find the complete solution by combining the natural response and the forced response.

$$v = v_n + v_f$$

Step 4: Apply initial/boundary conditions to find A.

The Driven *RC* circuit

Step 1: Get the natural response (general solution) by setting the forcing term of the differential equation to zero, and try the natural response in form of Ae^{st}.

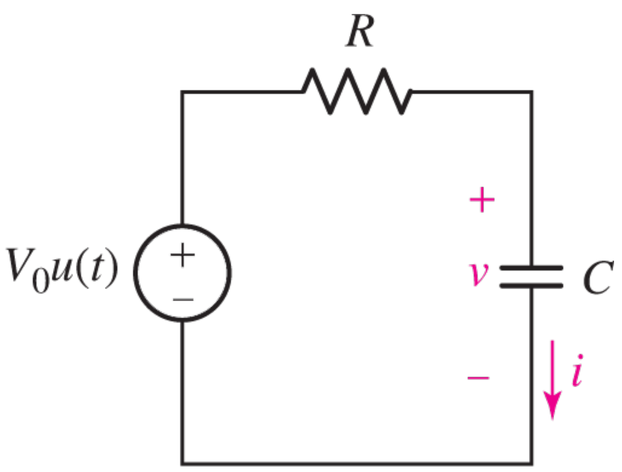
The natural response (The general solution):

Removing the source term from Eq.[1]:

$$RC\frac{dv}{dt} + v = 0 \qquad [2]$$

Try the natural response in form of Ae^{st}(repeating the same steps of previous source-free circuits):

$$v_n = Ae^{-t/RC} \qquad [3]$$



$$RC\frac{dv}{dt} + v = V_0 \qquad [1]$$

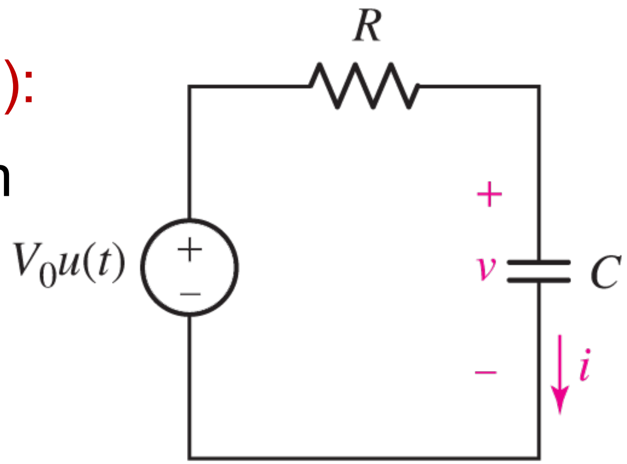
The Driven *RC* circuit

Step 2: Get the forced response (particular solution) by "guessing" from steady state ($t = \infty$).

The forced response (the particular solution):

The forced response can be "guessed" from steady state, which is

$$v_f = v(t = \infty) = \mathrm{V}_0$$



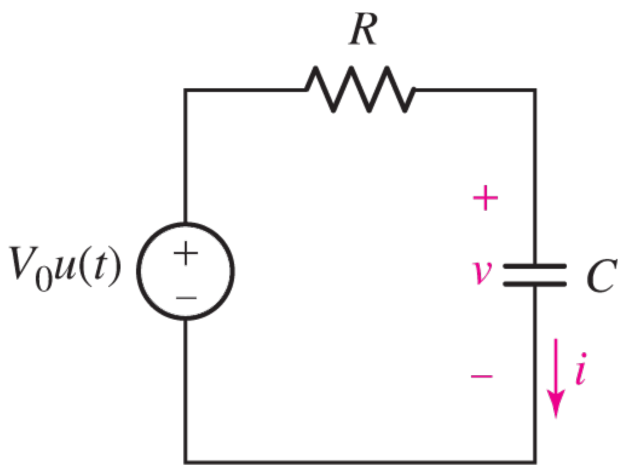
The Driven *RC* circuit

Step 3: Find the complete solution by combining the natural response and the forced response.

The complete response:

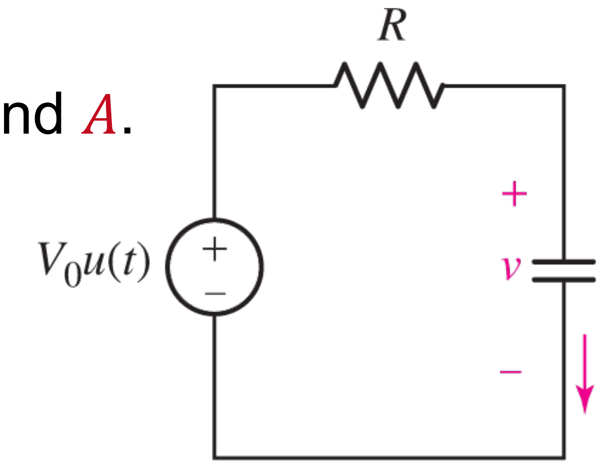
The complete solution is given by:

$$v = v_n + v_f = Ae^{-t/RC} + \mathrm{V}_0$$



The Driven *RC* circuit

Step 4: Apply initial/boundary conditions to find A.



The voltage of the capacitor cannot jump,

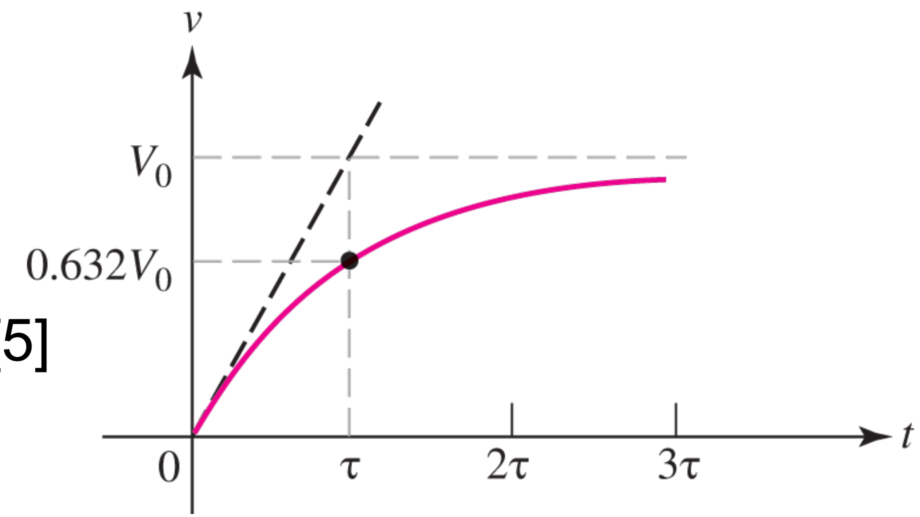
$$v(0^-) = v(0^+) = 0 \qquad [2]$$

Substituting Eq.[2] into Eq.[1],

$$v(t) = Ae^{-t/RC} + V_0 \qquad [1]$$

$$v(0^+) = Ae^{-(0)/RC} + V_0 \qquad [3]$$

$$A = -V_0 \qquad [4]$$



Hence,

$$v(t) = V_0 - V_0e^{-t/RC} \text{ for } t > 0 \qquad [5]$$

The Driven *RC* circuit: a general way

Step 1: Get the natural response:

$$v_n = Ae^{-t/RC} = Ae^{-t/\tau} \qquad [1]$$

Step 2: Get the forced response:

$$v_f = v(t=\infty) = v(\infty) \qquad [2]$$

Step 3: The complete solution:

$$v = v_n + v_f = Ae^{-t/\tau} + v(\infty) \qquad [3]$$

Step 4: Apply initial/boundary conditions to find A,

$$v(0^+) = Ae^{-(0)/RC} + v(\infty) \qquad [4]$$

$$A = v(0^+) - v(\infty) \qquad [5]$$

Finally,

$$v = [v(0^+) - v(\infty)]e^{-t/\tau} + v(\infty) \qquad [6]$$

General procedure for *RC* circuit

New Four Steps for general cases:

Step 1: Simply the circuit to determine R_{eq} and C_{eq}, and calculate the time constant $\tau_{eq} = R_{eq}C_{eq}$.
(with the consideration of all the independent voltage sources as short circuit and all the independent current sources as open circuit).

Step 2: Determine the initial condition $v(0^+)$ or $i(0^+)$.
(recall the requirement that any capacitor voltage satisfies $v(0^-) = v(0^+)$).

Step 3: Determine the final condition $v(\infty)$ or $i(\infty)$.

Step 4: The final response is given by:

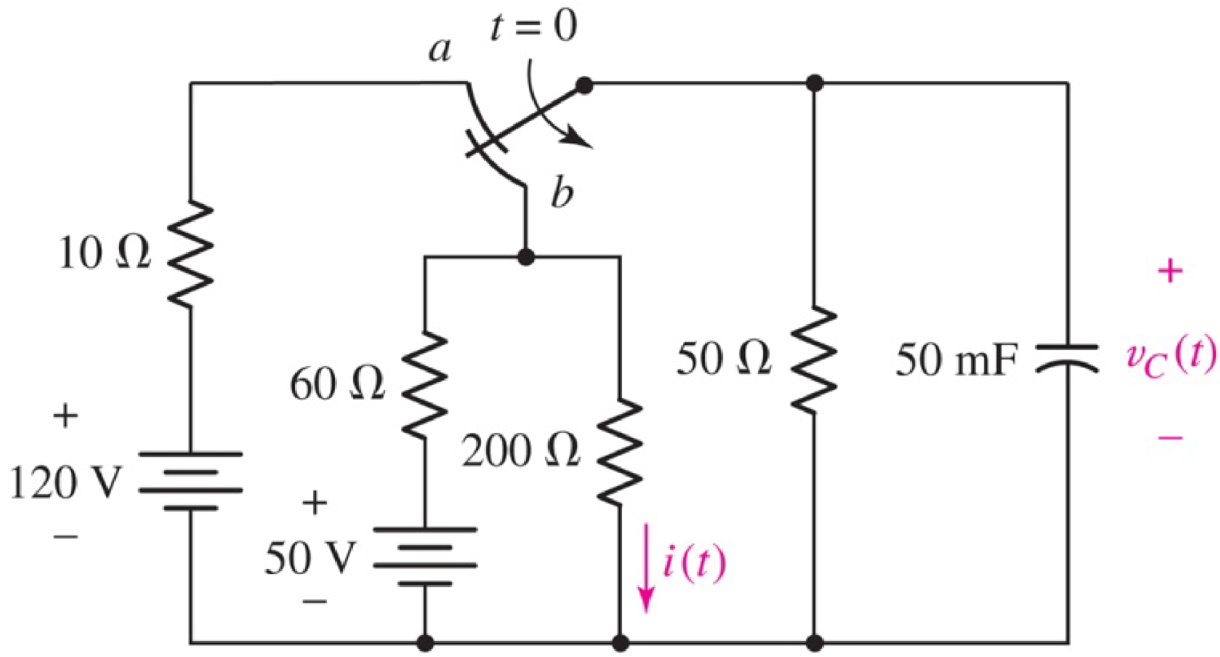
$$v = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

OR

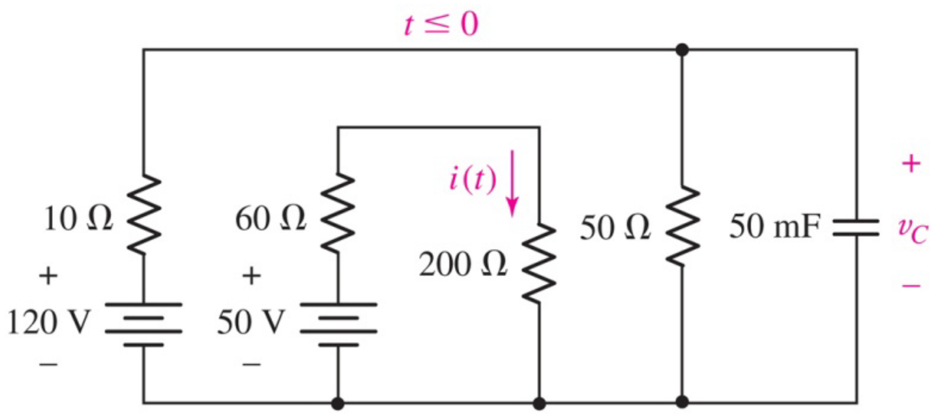
$$i = i(\infty) + [i(0^+) - i(\infty)]e^{-t/\tau}$$

Example 4

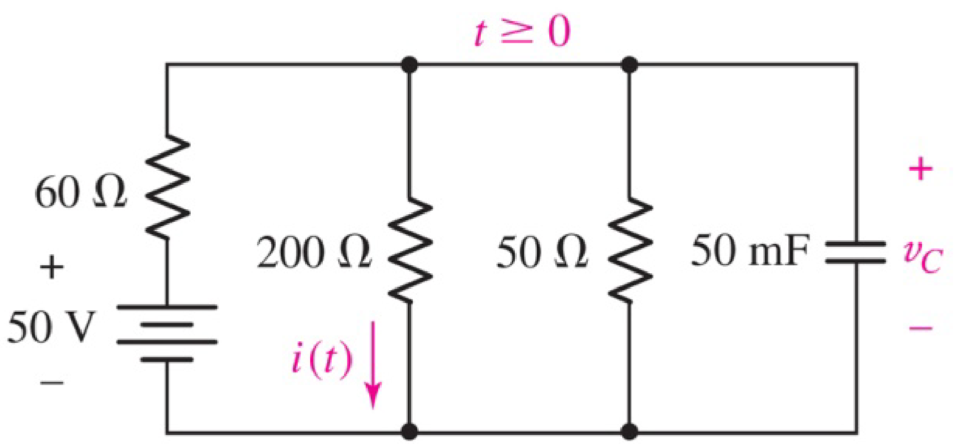
Find the capacitor voltage $v_c(t)$ for all the time.



Solution to Example 4



For Step 2



For Step 1, 3

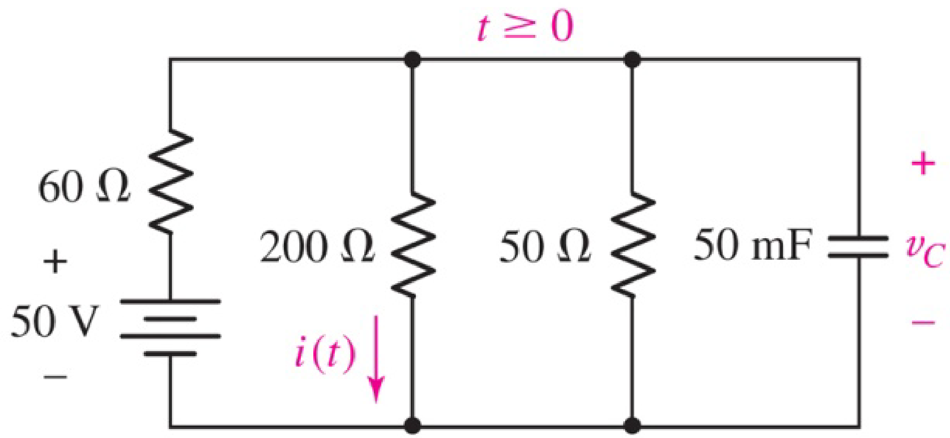
Solution to Example 4

Step 1: Simply the circuit to determine R_{eq} and C_{eq}, and calculate the time constant $\tau_{eq} = R_{eq}C_{eq}$.

- Replace the 50V source with a short circuit and evaluate the equivalent resistance to find the time constant.

$$
\begin{aligned}
R_{eq} &= 60\Omega||200\Omega||50\Omega \\
&= \frac{1}{\frac{1}{50}+\frac{1}{200}+\frac{1}{60}} \\
&= 24\ \Omega
\end{aligned}
$$

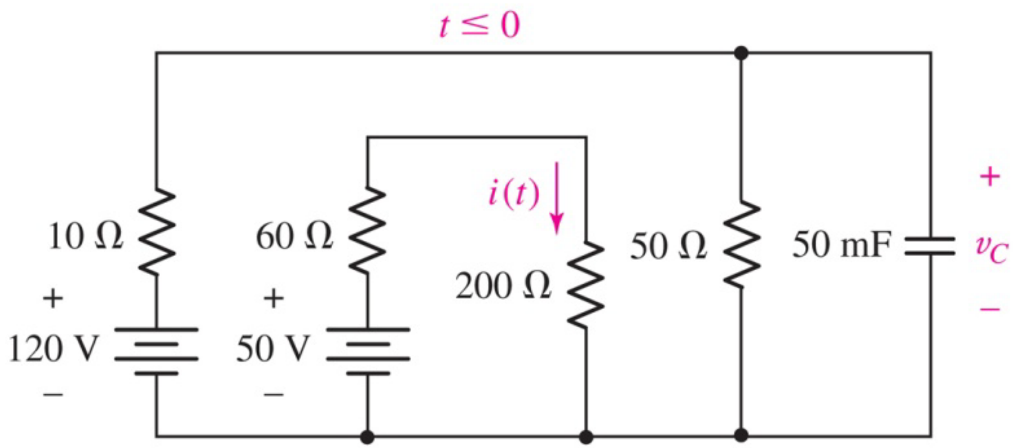
$$
\begin{aligned}
\tau_{eq} &= R_{eq}C \\
&= 24 \times 0.05 \\
&= 1.2\ \text{s}
\end{aligned}
$$



Solution to Example 4

Step 2: Determine the initial condition $v(0^+)$.

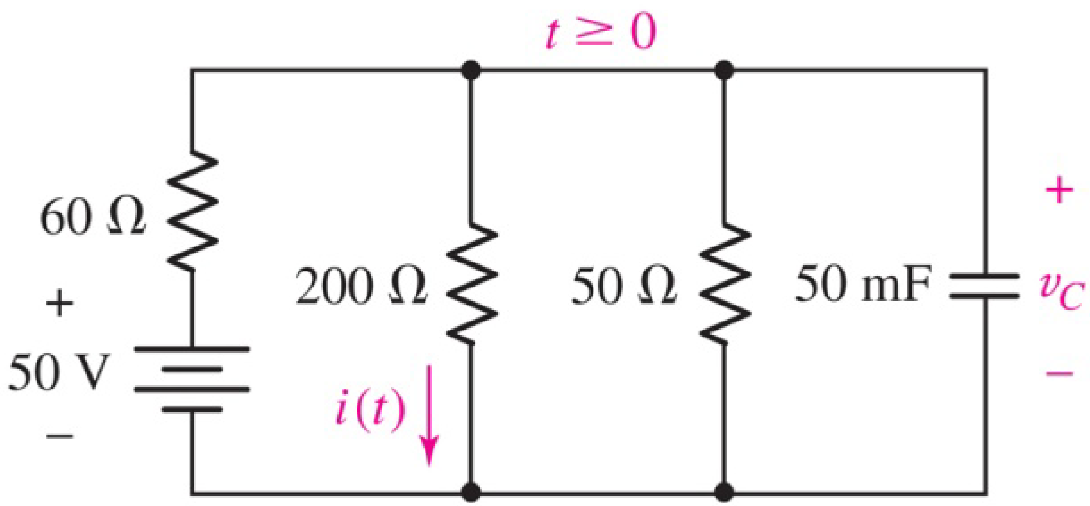
$$v_c(0^+) = v_c(0^-) = \frac{50}{50+10}(120) = 100\text{V}$$



Solution to Example 4

Step 3: Determine the final condition $v(\infty)$.

$$v_c(\infty) = \frac{50||200}{50||200 + 60}(50) = 20\text{V}$$



Solution to Example 4

Step 4: The final response is given by:

$$v = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

For $t \geq 0$

$$v_c = v_c(\infty) + [v_c(0^+) - v_c(\infty)]e^{-t/\tau}$$

$$= 20 + (100 - 20)e^{-t/1.2}$$

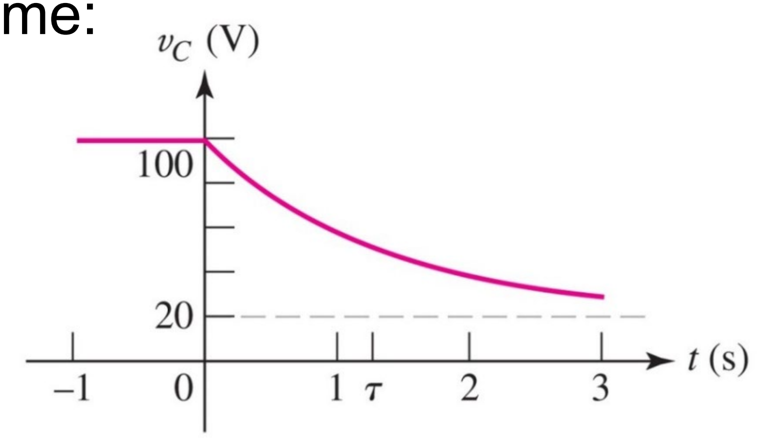
$$= 20 + 80e^{-t/1.2}\text{V}$$

For $t < 0$

$$v_c = 100\text{ V}$$

Thus, the capacitor voltage for all the time:

$$v_c(t) = \begin{cases} 20 + 80e^{-t/1.2}\text{V}, & t \geq 0 \\ 100\text{V}, & t < 0 \end{cases}$$



OR

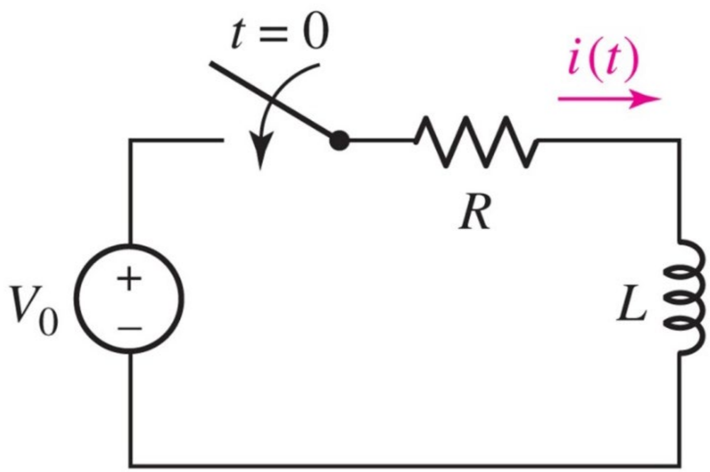
$$v_c(t) = 100u(-t) + (20 + 80e^{-\frac{t}{1.2}})u(t)\text{V}$$

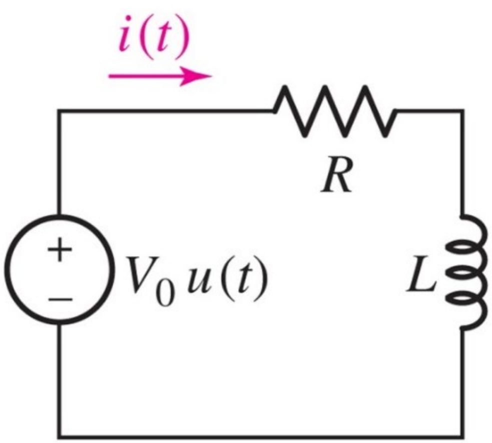
Content

- The source-free *RC* circuit
- The source-free *RL* circuit
- The general *RC/RL* circuits
- The unit-step function
- The driven *RC* circuit
- The driven *RL* circuit

The Driven *RL* circuit

- The two circuits shown both have $i(t) = 0$ for $t < 0$ and are also the same for $t > 0$.
- We now have to find both the natural response and the forced response due to the source V_0 to form the complete response.

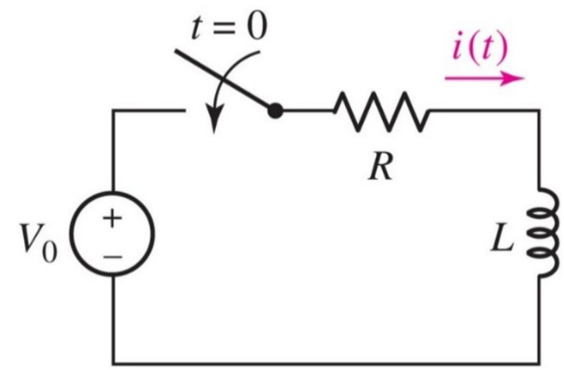




The Driven *RL* circuit

Find the complete response for the *RL* circuit with the switch closed at $t = 0$.

Repeating the same steps with the driven *RC* circuit, the complete response for the driven *RL* circuit is:

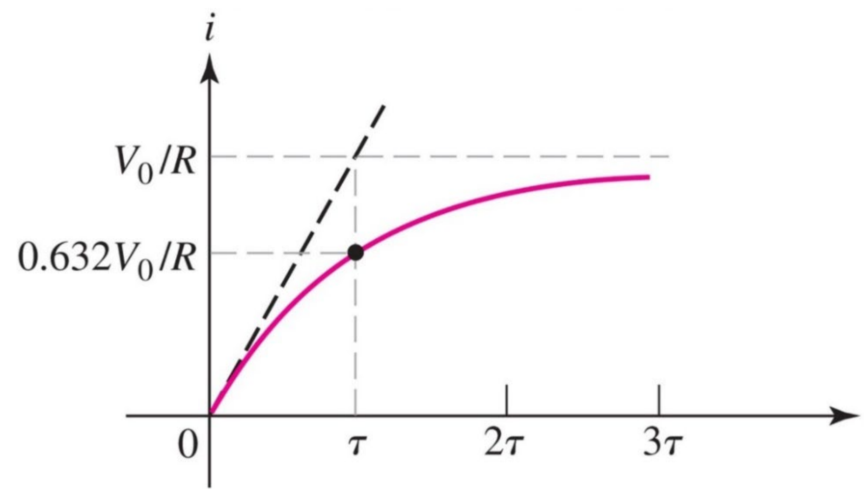


$$i(t) = \frac{V_0}{R} - \frac{V_0}{R} e^{-tR/L} \text{ for } t > 0 \quad \text{OR} \quad i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{tR}{L}}\right)u(t)$$

Same format

The complete response for driven *RC* circuit:

$$v(t) = V_0 - V_0 e^{-t/RC} \text{ for } t > 0$$



General procedure for *RL* circuit

Step 1: Simply the circuit to determine R_{eq} and L_{eq}, and calculate the time constant $\tau_{eq} = L_{eq}/R_{eq}$.

(with the consideration of all the independent voltage sources as short circuit and all the independent current sources as open circuit).

Step 2: Determine the initial condition $v(0^+)$ or $i(0^+)$. (recall the requirement that any inductor current satisfies $i(0^-) = i(0^+)$).

Step 3: Determine the final condition $v(\infty)$ or $i(\infty)$.

Step 4: The final response is given by:

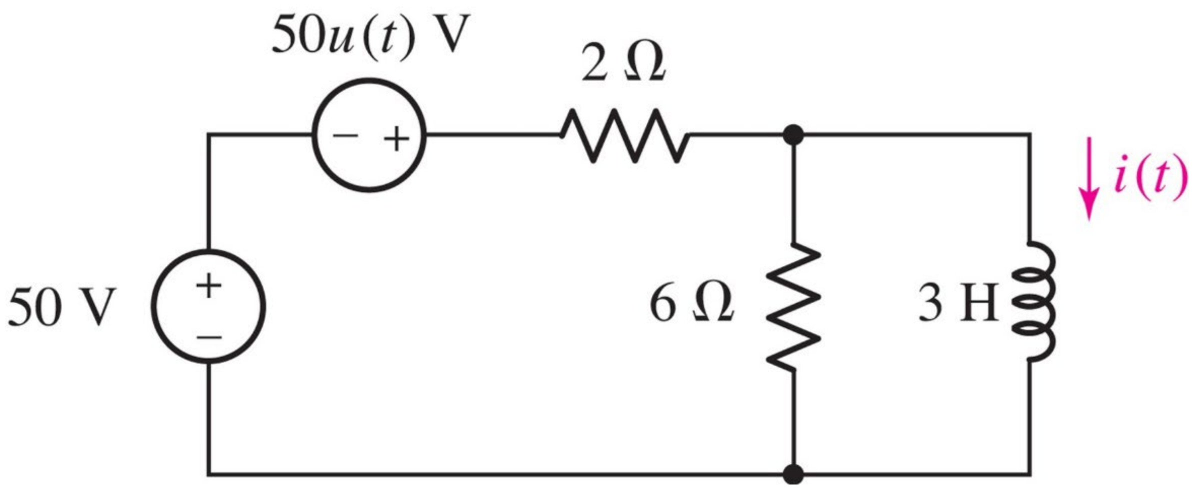
$$v = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

OR

$$i = i(\infty) + [i(0^+) - i(\infty)]e^{-t/\tau}$$

Example 5

Determine $i(t)$ for all values of time in the circuit below:



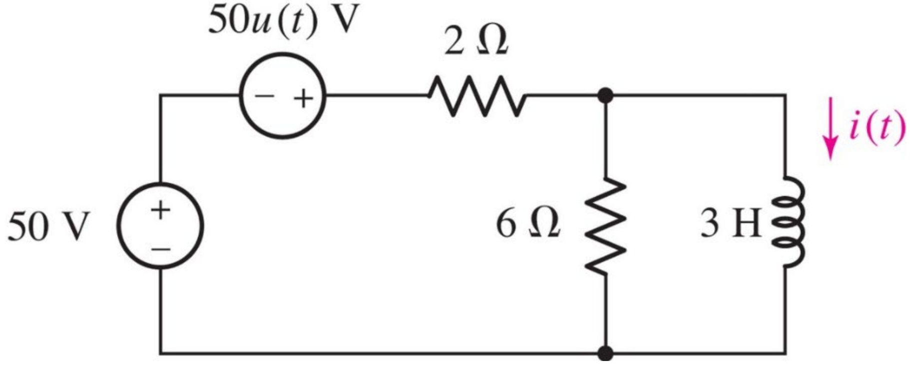
Solution to Example 5

Step 1: Simply the circuit to determine R_{eq} and L_{eq},
and calculate the time constant $\tau_{eq} = L_{eq}/R_{eq}$.

- Replace all the independent voltage sources with short circuit, and evaluate the equivalent resistance to find the time constant.

$$R_{eq} = 2\Omega||6\Omega = \frac{2\times 6}{2+6} = 1.5\Omega$$

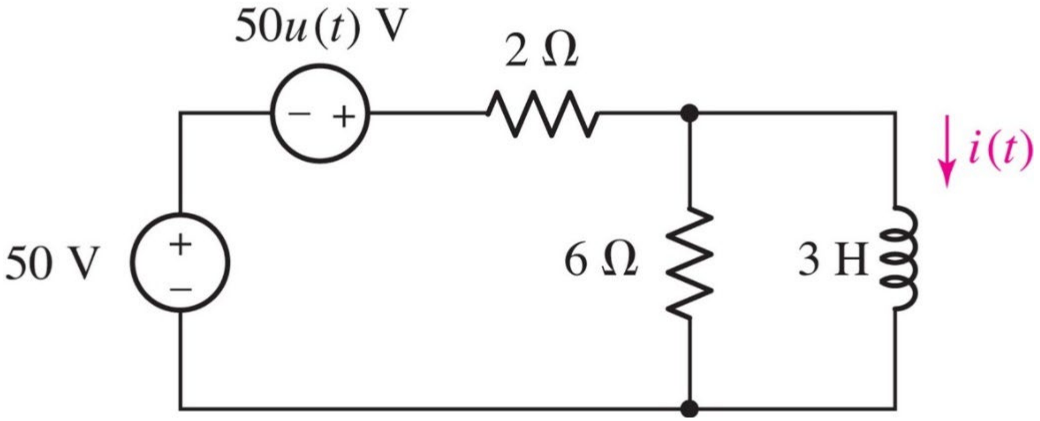
$$\tau = L_{eq}/R_{eq} = 3/1.5 = 2s$$



Solution to Example 5

- Step 2: Determine the initial condition $i(0^+)$.

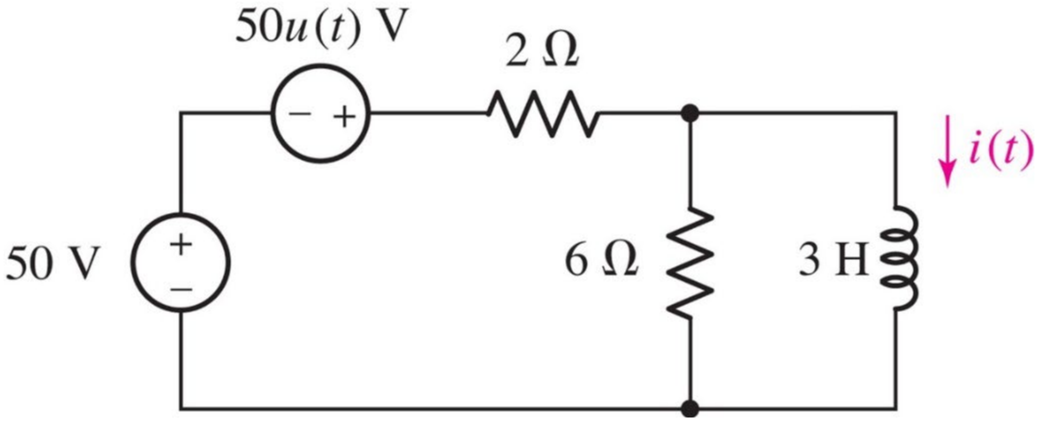
$$i(0^+) = i(0^-) = \frac{50}{2} = 25\text{A}$$



Solution to Example 5

- Step 3: Determine the final condition $i(\infty)$.

$$i(\infty) = \frac{50 + 50}{2} = 50\text{A}$$



Solution to Example 5

- Step 4: The final response is given by:

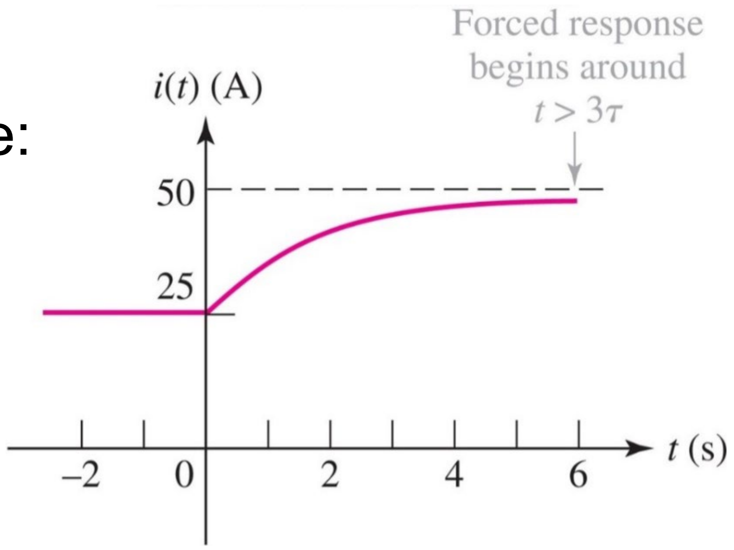
$$i = i(\infty) + [i(0^+) - i(\infty)]e^{-t/\tau}$$

For $t \geq 0$

$$i(t) = i(\infty) + [i(0^+) - i(\infty)]e^{-t/\tau}$$
$$= 50 + (25 - 50)e^{-t/2}$$
$$= 50 - 25e^{-t/2}\,\mathrm{A}$$

For $t < 0$

$$i(t) = 25\ \mathrm{A}$$



Thus the inductor current for all the time:

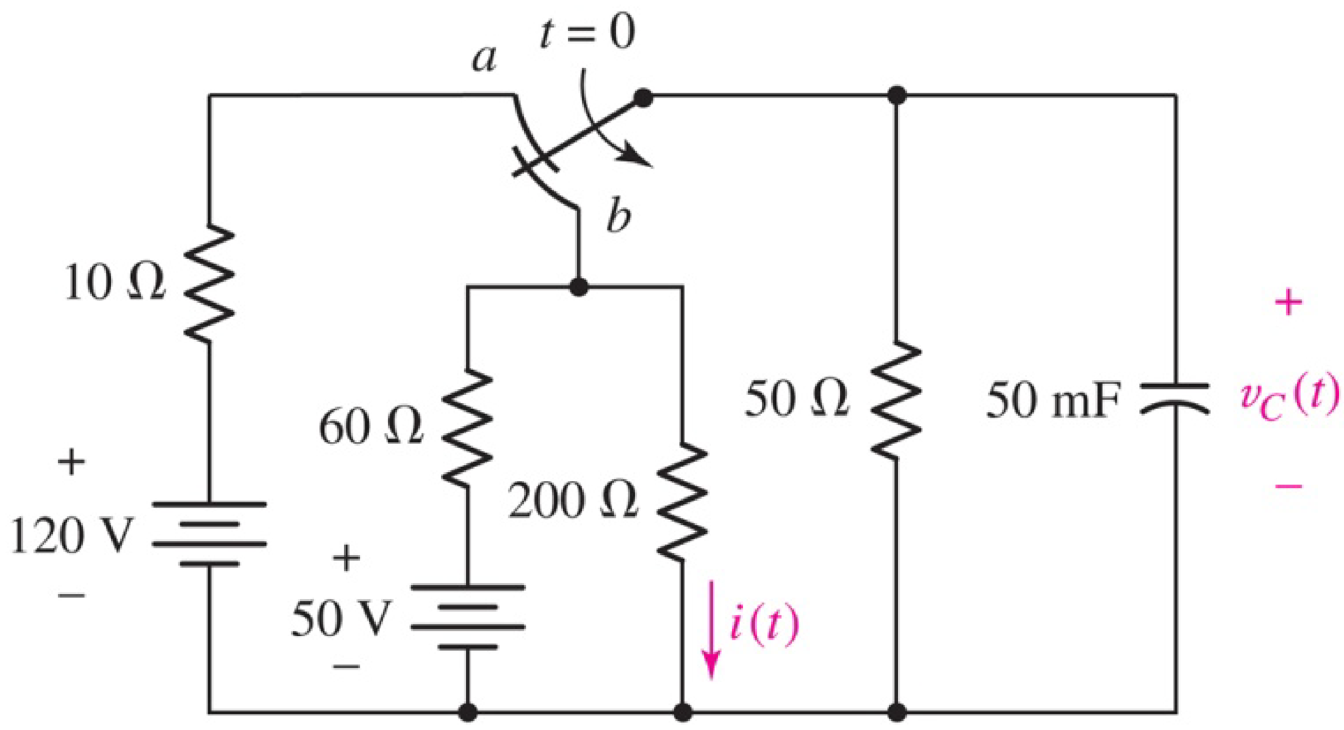
$$i(t) = \begin{cases} 50 - 25e^{-t/2}\,\mathrm{A}, & t \geq 0 \\ 25\ \mathrm{A}, & t < 0 \end{cases}$$

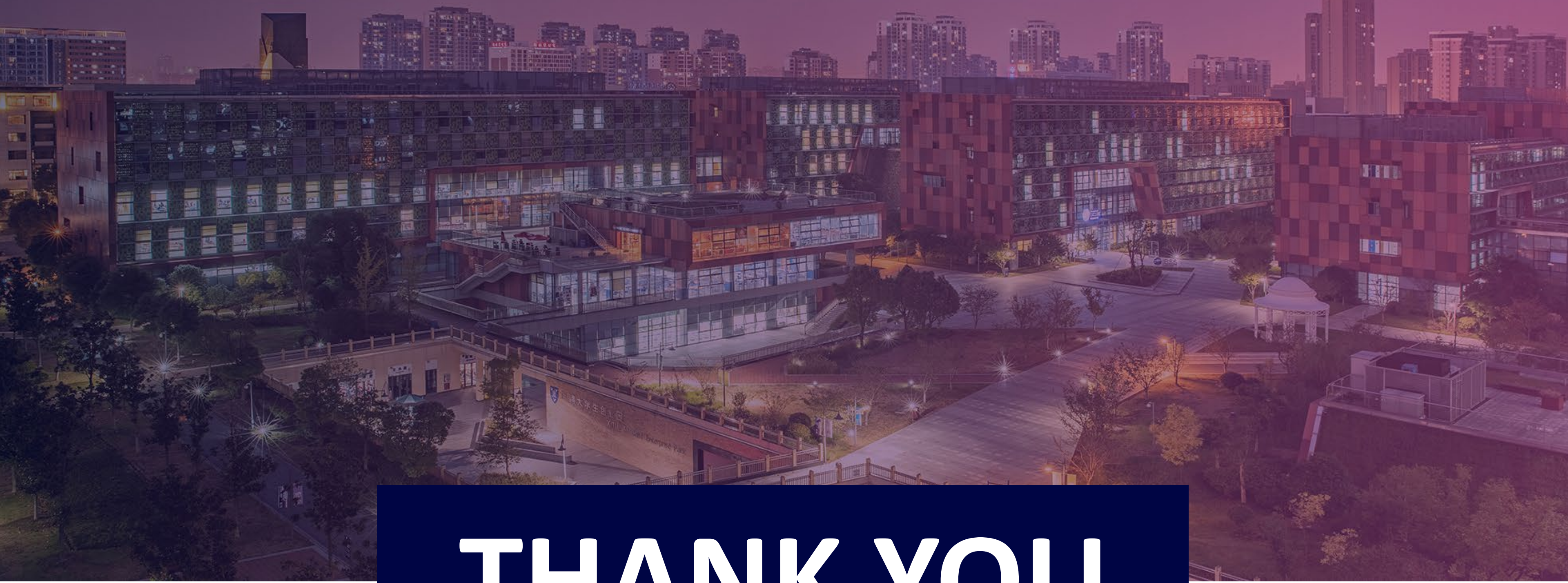
OR

$$i(t) = 25u(-t) + \left(50 - 25e^{-\frac{t}{2}}\right)u(t)\ \mathrm{A}$$

Homework (Example 4 continued)

Find the current $i(t)$ through the 200Ω resistor all the time.





THANK YOU

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