CAN207 Continuous and Discrete Time Signals and Systems

Lecture-12

Solving Differential Equations and Block Diagram

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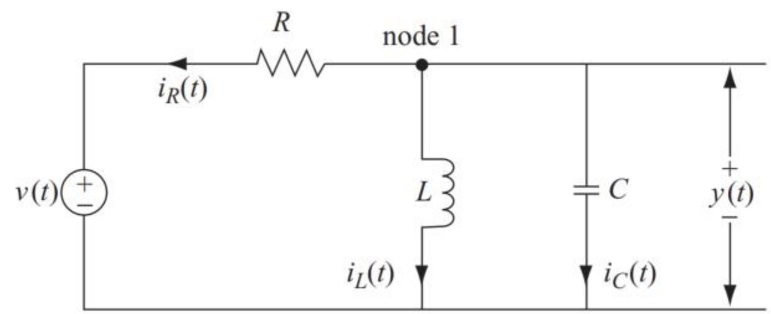
1.1 Differential equations for CT systems



- A simple electrical circuit
  - Passive components: R, L and C
  - Input: voltage signal v(t)
  - Output: voltage on capacitor y(t)

- Mathematical model
  - Currents on 3 branches are:

$$i_R = \frac{y(t) - v(t)}{R}$$

$$i_L = \frac{1}{L}\int_{-\infty}^{t} y(\tau)\mathrm{d}\tau$$

$$i_C = C\frac{\mathrm{d}y}{\mathrm{d}t}.$$

  - Apply the Kurchhoff's current law:

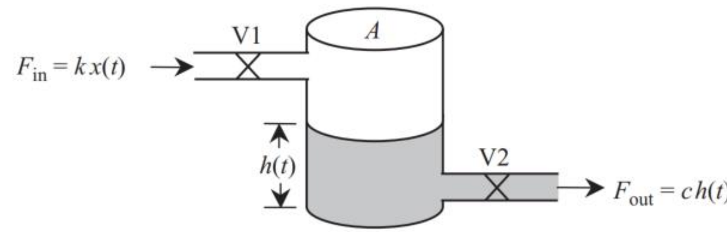
$$\frac{y(t) - v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} y(\tau)\mathrm{d}\tau + C\frac{\mathrm{d}y}{\mathrm{d}t} = 0,$$

  - Take derivative on both sides, get:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{LC}y(t) = \frac{1}{RC}\frac{\mathrm{d}v}{\mathrm{d}t}.$$

1.1 Differential equations for CT systems

- A mechanical water pump
  - Input: rate of flow $F_{in} = k\, x(t)$
  - Output: outlet flow rate $F_{out} = c\, h(t)$
  - Total volume of the water inside the tank is $V(t) = A\, h(t)$



k is the linearity constant;
x(t) is the controlling voltage;
c is the outlet flow constant;
h(t) is the height of water level.

- Mathematical model
  - the rate of change in the volume of the stored water:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = F_{\text{in}} - F_{\text{out}} = kx(t) - ch(t).$$

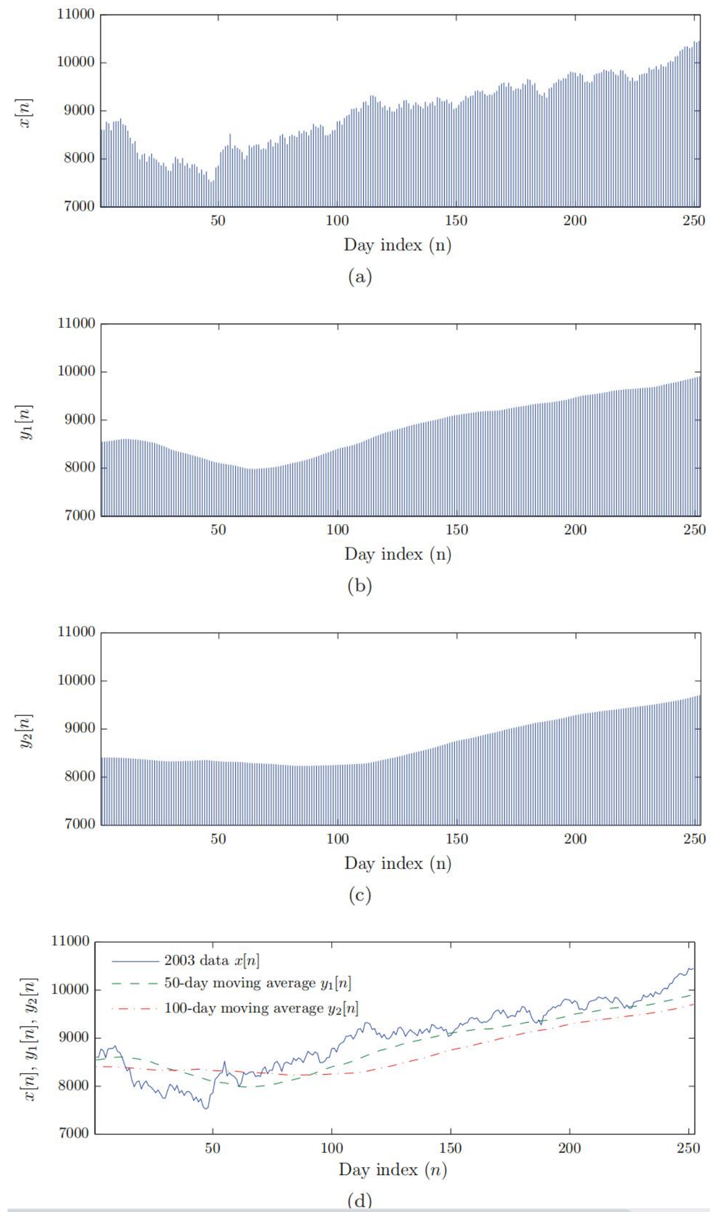
  - Expressing V(t) as the product of the cross-sectional area A of the water tank and the height h(t) of the water yields:

$$A\frac{\mathrm{d}h}{\mathrm{d}t} + ch(t) = kx(t),$$

1.2 Difference equations for DT systems

- A moving average filter
  - Input: a sequence of numbers (such as data from stock market)
  - Output: a sequence of average values of M adjacent numbers.

- Mathematical model
  - Taking average: $y[n]=\dfrac{1}{M}\sum_{k=0}^{M-1}x(n-k)$



1.2 Difference equations for DT systems

- A spatial averaging filter
  - Input: a 2D array x[m, n] is the intensity of a monochrome image
  - Output: a 2D array y[m, n] is the averaged intensity of every pixel

Original Image

- Mathematical model
  - a weighted average of the intensities of the pixels in the neighborhood

$$y[m,n] = \frac{1}{4}(x[m,n] + x[m,n-1] + x[m-1,n] + x[m-1,n-1])$$

Filtered Image

1.3 Summary

- Linear Constant-Coefficient Differential Equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- Linear Constant-Coefficient Difference Equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

2.1 Differential equations

- N^{th} order differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \ldots\ldots \quad (1)$$

- A more compact representation: denoting $\frac{d}{dt}$ by D:

$$a_N D^N y(t) + a_{N-1} D^{N-1} y(t) + \ldots + a_1 D y(t) + a_0 y(t)$$
$$= b_M D^M x(t) + b_{M-1} D^{M-1} x(t) + \ldots + b_1 D x(t) + b_0 x(t)$$

- D is the differential operator, so it can be written as:

$$\underbrace{(a_N D^N + a_{N-1} D^{N-1} + \ldots + a_0)}_{Q(D)} y(t) = \underbrace{(b_N D^M + b_{N-1} D^{M-1} + \ldots + b_0)}_{P(D)} x(t)$$

- or simply $$Q(D)y(t) = P(D)x(t) \quad \ldots\ldots \quad (2)$$
  - where Q(D) is the N^{th}-order differential operator, P(D) is the M^{th}-order differential operator, and the a_k and b_k are constants.

2.1 Solving differential equations

- Output y(t) has two components:

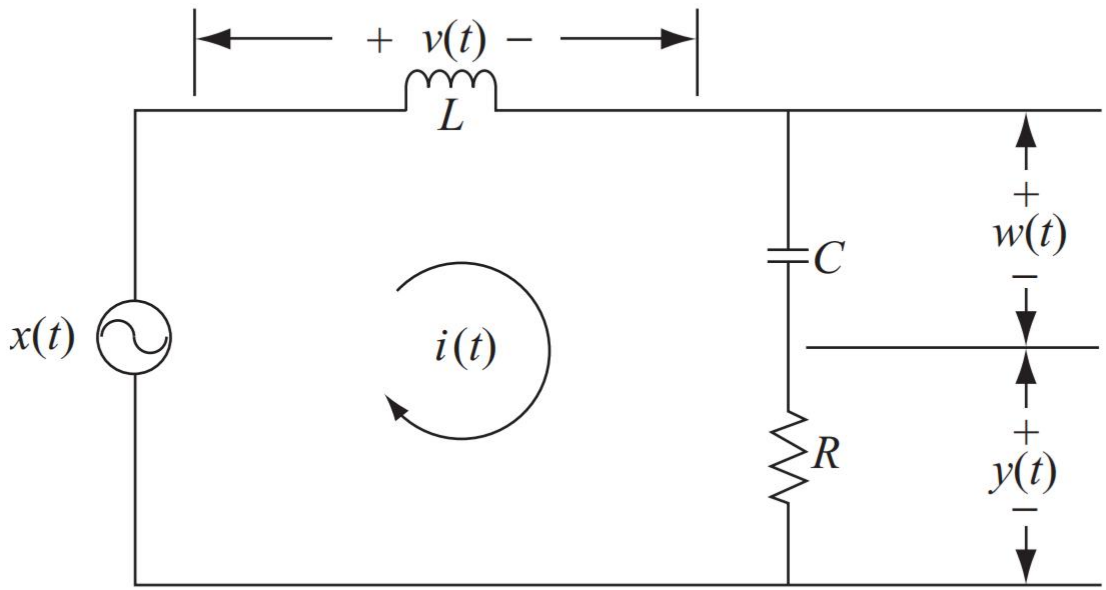
$$y(t) = \quad y_{zi}(t) \quad + \quad y_{zs}(t)$$

  - $y_{zi}(t)$: ***zero-input response*** of the system
    - the response produced by the system because of the initial conditions (and not due to any external input);
    - evaluated by solving a homogeneous equation $Q(D)y(t) = 0$;
    - decays to zero as t → ∞.

  - $y_{zs}(t)$: ***zero-state response*** of the system
    - arises due to the input signal (and doesn't depend on the initial states of the system);
    - evaluated by assuming the initial states of the system to be zero;
    - defines the steady-state value of the output.

2.1 Solving differential equations - example 1

- Consider the RLC series circuit shown below. Assume that the inductance L = 0 H (i.e. the inductor does not exist in the circuit), resistance R = 5 Ω, and capacitance C = 1/20 F.
- Determine the output signal y(t) when the input voltage is given by x(t) = sin(2t) and the initial voltage $y(0^-)$ = 2 V across the resistor.



2.1 Solving differential equations - eg.1 - solution

- Differential equation of this circuit:

$$\frac{dy}{dt} + 4y(t) = \frac{dx}{dt} = 2\cos(2t) \qquad \ldots\ldots\ldots \qquad (3)$$

- Zero-input response is obtained by solving

$$\frac{dy}{dt} + 4y(t) = 0$$

  – Characteristic equation:
    - Assume $y(t) = Ae^{st}$, then $sAe^{st} + 4Ae^{st} = 0$
    - $s + 4 = 0$ has a root at $s =- 4$.
  – The zero-input response is given by $y_{zi}(t) = Ae^{-4t}$, where A is unknown.
  – Using initial condition $y(0^-) = Ae^{-4\cdot 0} = 2$, find $A = 2$.
  – So the zero-input response is $y_{zi}(t) = 2e^{-4t}$.

2.1 Solving differential equations - eg.1 - solution

- Zero-state response is obtained by solving equation (3) with a zero initial condition $y(0^-) = 0$.
  - The particular component for sinusoidal input is
$$y_{zs}^{(p)}(t) = K_1\cos(2t) + K_2\sin(2t)$$
  - The homogeneous component is similar
$$y_{zs}^{(h)}(t) = Ce^{-4t}, \text{ where C is unknow.}$$
  - Substitute $y_{zs}^{(p)}(t)$ into equation (3), get $K_1 = 0.4$ and $K_2 = 0.2$.
  - The overall zero-state response:
$$y_{zs}(t) = Ce^{-4t} + 0.4\cos(2t) + 0.2\sin(2t)$$
  - Using $y(0^-) = 0$, find $C =- 0.4$, so
$$y_{zs}(t) =- 0.4e^{-4t} + 0.4\cos(2t) + 0.2\sin(2t)$$
- Total response
$$y(t) = y_{zi}(t) + y_{zs}(t) = \mathbf{1.6}e^{-4t} + \mathbf{0.4}\cos(2t) + \mathbf{0.2}\sin(2t)$$

2.1 Differential equations - homogeneous solution

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

- Homogeneous solution
  - "guess" solution of the form $y_h(t) = Ae^{st}$
  - Substitute back to the homogeneous equation

$$\sum_{k=0}^{N} a_k A s^k e^{st} = 0 \quad \rightarrow \quad N\ roots\ s_i,\ i = 1, \ldots, N$$

  - Characteristic equations:

$\sum_{k=0}^{N} a_k s^k = 0$, solved to get *N* values s_i

  - Homogeneous solution has the form:

$$y_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \ldots + A_N e^{s_N t}$$

  - Need *N* initial states to decide the *N* coefficients A_i

2.1 Differential equations - particular solution

- The zero-state response $y_{zs}(t)$ depends upon the input signal x(t) subject to zero initial conditions.
- The zero-state response consists of two components:

$$\boldsymbol{y_{zs}(t) = y_{zs}^{(p)}(t) + y_{zs}^{(h)}(t)}$$

  – the ***homogeneous component*** $\boldsymbol{y_{zs}^{(h)}(t)}$ is obtained by following the procedure used to solve for the zero-input response but with zero initial conditions.

  – the ***particular component*** $\boldsymbol{y_{zs}^{(p)}(t)}$ is obtained from a look-up table as shown below.

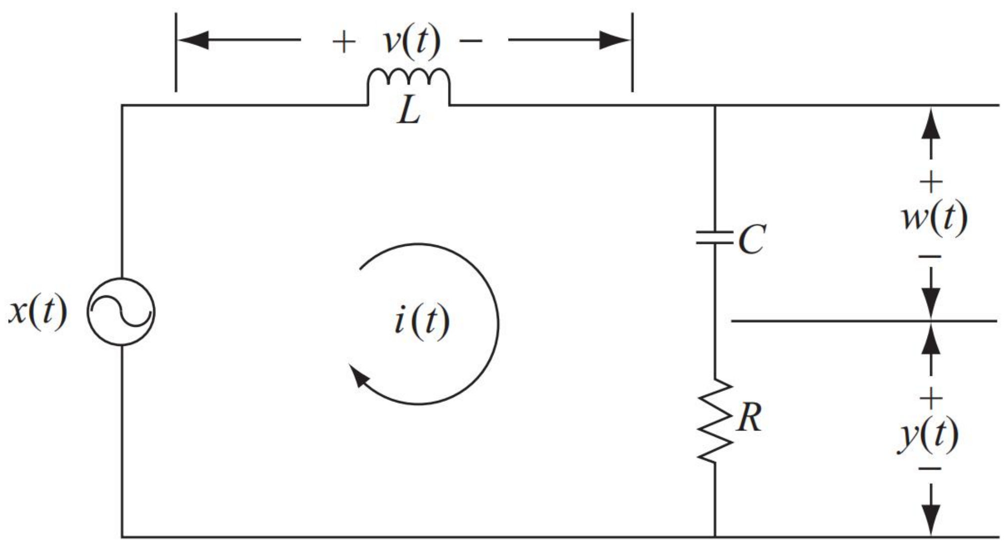
    - The constant C is determined such that $y_{zs}(t)$ satisfies the system's differential equation.

Table C.1. Zero-state response corresponding to common input signals

Input	Particular component of the zero-state response
Impulse function, $K\delta(t)$	$C\delta(t)$
Unit step function, $Ku(t)$	$Cu(t)$
Exponential, Ke^{-at}	Ce^{-at}
Sinusoidal, $A\cos(\omega_0 t + \phi)$	$C_0\cos(\omega_0 t) + C_1\sin(\omega_0 t)$

2.1 Solving differential equations - example 2

- Consider the same electrical circuit as example 1, as shown below. The values of inductance, resistance, and capacitance are set to L = 1/12H, R = 7/12Ω, and C = 1F. The circuit is assumed to be open before t = 0, i.e. no current is initially flowing through the circuit. However, the capacitor has an initial charge of 5 V. Determine
  a) the zero-input response $w_{zi}(t)$ of the system;
  b) the zero-state response $w_{zs}(t)$ of the system;
  c) the overall output $w(t)$;
- Notice: the input signal is given by $x(t) = 2e^{-t}u(t)$ and the output $w(t)$ is measured across capacitor C.



2.1 Solving differential equations - eg.2 - solution

- Differential equation of this circuit:

$$\frac{d^2w(t)}{dt^2} + 7\frac{dw(t)}{dt} + 12w(t) = 12x(t)$$

- Zero-input response:
  - using characteristic equation and initial conditions find:

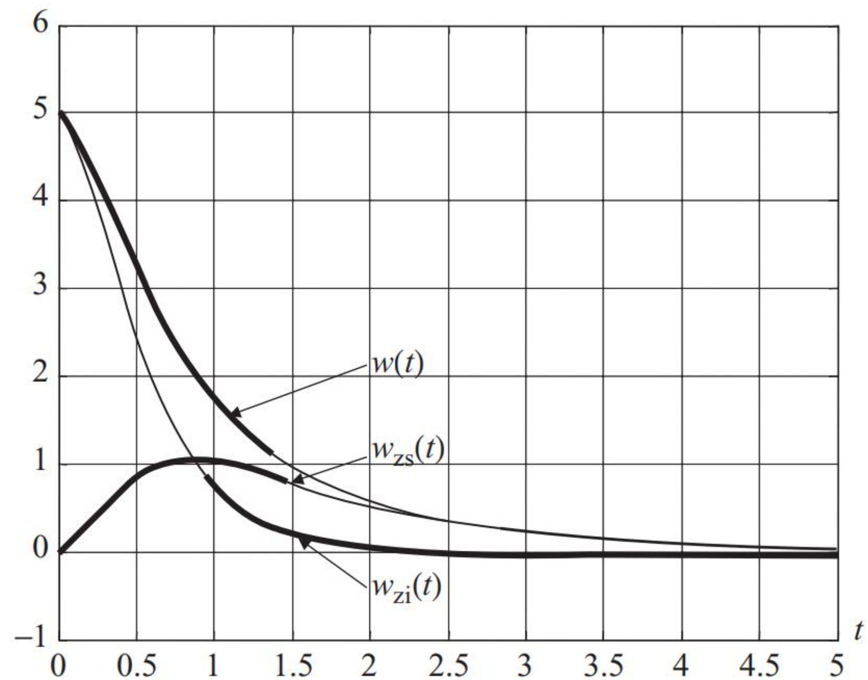
$$w_{zi}(t) = (20e^{-3t} - 15e^{-4t})u(t)$$

- Zero-state response:
  - the homogeneous component is the same form:

$$w_{zs}^{(h)}(t) = C_1e^{-4t} + C_2e^{-3t}$$

  - the particular component for $x(t) = 2e^{-t}u(t)$ is of the form $w_{zs}^{(p)}(t) = Ke^{-t}u(t)$
  - substituting back to the equation and using initial conditions to solve for C_1, C_2 and K

$$w_{zs}(t) = w_{zs}^{(h)}(t) + w_{zs}^{(p)}(t) = (8e^{-4t} - 12e^{-3t} + 4e^{-t})\,u(t)$$



- The overall response is

$$w(t) = w_{zi}(t) + w_{zs}(t) = (-7e^{-4t} + 8e^{-3t} + 4e^{-t})\,u(t)$$

2.1 Solving differential equations - example 3

- Notice: change the input to $x(t) = 2(1 - e^{-t})u(t)$.
- Find the overall output $w(t)$.
- Differential equation of this circuit: $\frac{d^2w(t)}{dt^2} + 7\frac{dw(t)}{dt} + 12w(t) = 12x(t)$
- Zero-input response: $w_{zi}(t) = (20e^{-3t} - 15e^{-4t})u(t)$
- Zero-state response:
  - the homogeneous component: $w_{zs}^{(h)}(t) = C_1e^{-4t} + C_2e^{-3t}$
  - the particular component for $x(t) = 2u(t) - 2e^{-t}\,u(t)$ is of the form

$$w_{zs}^{(p)}(t) = (K_1 + K_2e^{-t})\,u(t)$$

  - substituting back to the equation and using initial conditions to solve:

$$w_{zs}(t) = w_{zs}^{(h)}(t) + w_{zs}^{(p)}(t) = (2 - 4e^{-t} + 4e^{-3t} - 2e^{-4t})\,u(t)$$

  - The overall response is

$$w(t) = w_{zi}(t) + w_{zs}(t) = (2 - 4e^{-t} + 24e^{-3t} - 17e^{-4t})\,u(t)$$

2.2 Decomposing the whole response

- 1. Decomposition 1:
  - Zero-input response $(20e^{-3t} - 15e^{-4t})u(t)$
  - Zero-state response $(2 - 4e^{-t} + 4e^{-3t} - 2e^{-4t})\, u(t)$

- 2. Decomposition 2:
  - Natural response $(20e^{-3t} - 15e^{-4t} + 4e^{-3t} - 2e^{-4t})\, u(t)$
  - Forced response $(2 - 4e^{-t})\, u(t)$

- 3. Decomposition 3:
  - Transient response $(20e^{-3t} - 15e^{-4t} - 4e^{-t} + 4e^{-3t} - 2e^{-4t})\, u(t)$
  - Steady-state response $2\, u(t)$

Quiz 1

- A first-order differential equation is given as:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

  - where y(t) denotes the output of the system and x(t) is the input.
- Find y(t) when the input signal is $x(t) = Ke^{3t}u(t)$ and initial rest condition with $y(0) = 0$.

Hint:

Input	Particular component of the zero-state response
Impulse function, $K\delta(t)$	$C\delta(t)$
Unit step function, $Ku(t)$	$Cu(t)$
Exponential, Ke^{-at}	Ce^{-at}
Sinusoidal, $A\cos(\omega_0 t + \phi)$	$C_0\cos(\omega_0 t) + C_1\sin(\omega_0 t)$

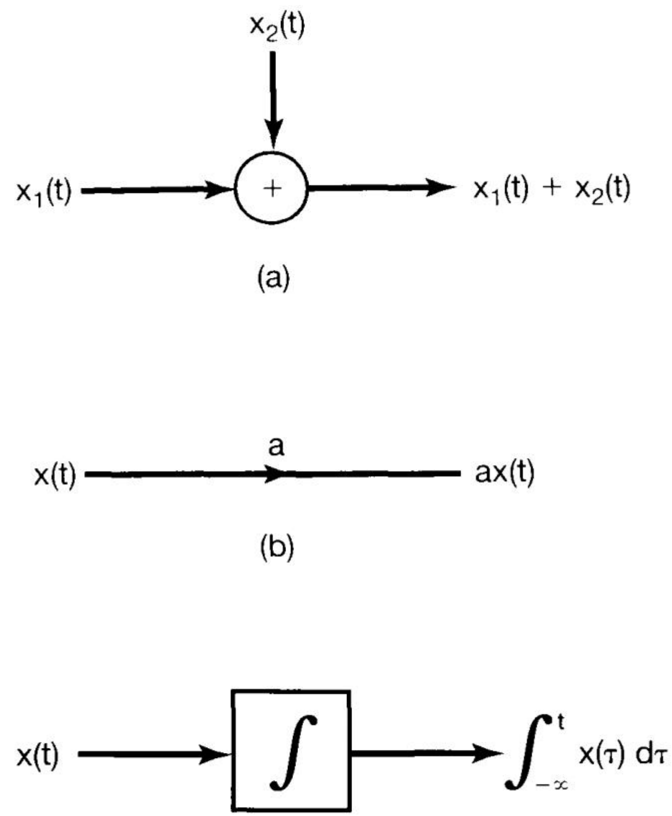
3.1 Block diagrams for differential equations

- Systems described by LCCDE can be represented in terms of block diagram interconnections of elementary operations.

- For CT systems, the commonly used elementary building blocks are ***adder***, ***multiplier*** and ***integrator***.

- Example 1: draw the block diagram of the 1st order LCCDE:

$$\frac{dy(t)}{dt} + ay(t) = \frac{dx(t)}{dt} + bx(t)$$



3.2 Direct form I

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0\,y = b_2\frac{d^2x}{dt^2} + b_1\frac{dx}{dt} + b_0\,x$$

- 1. Take 3 integrals on both sides

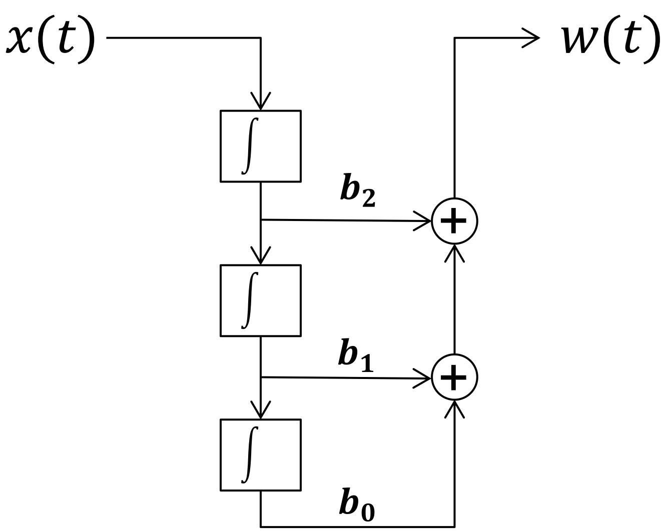
$$y(t) + a_2\int y(t)dt + a_1\iint y(t)dt + a_0\iiint y(t)dt = b_2\int x(t)dt + b_1\iint x(t)dt + b_0\iiint x(t)dt$$

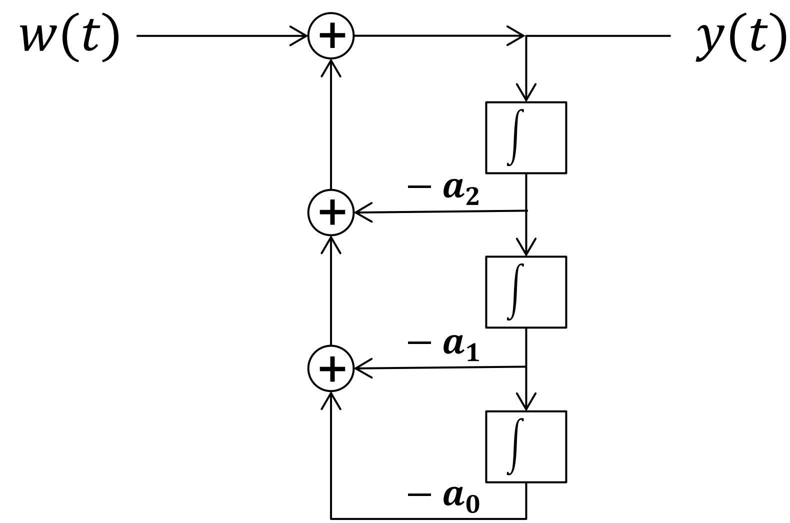
- 2. Draw the input side:

$$w(t) = b_2\int x + b_1\iint x + b_0\iiint x$$

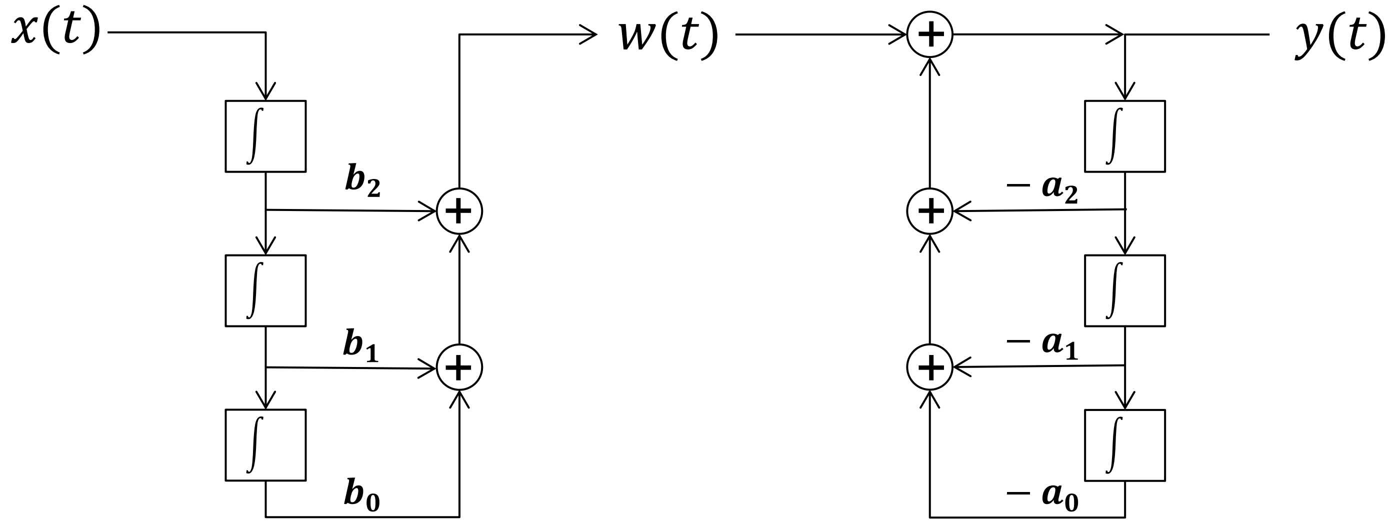
3. Draw the output side:

$$w(t) - a_2\int y - a_1\iint y - a_0\iiint y = y(t)$$

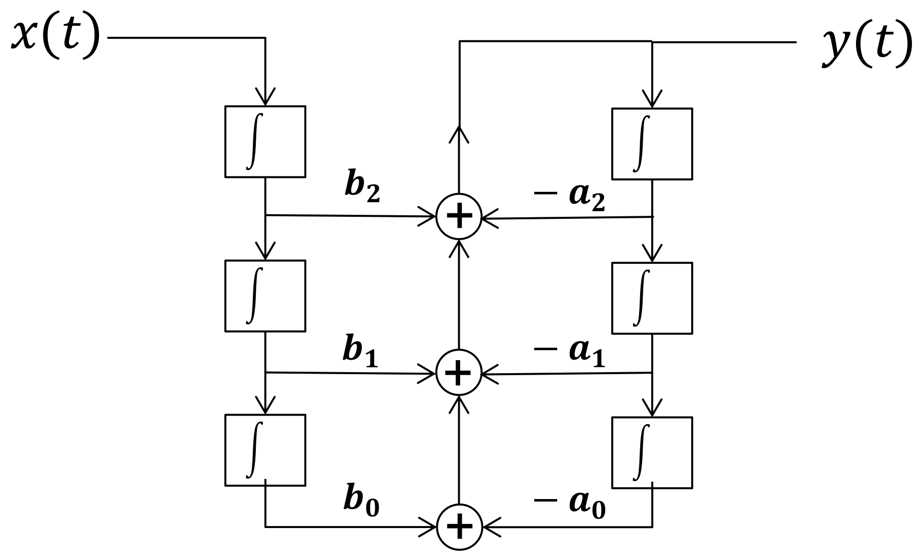




- Connect the centre:



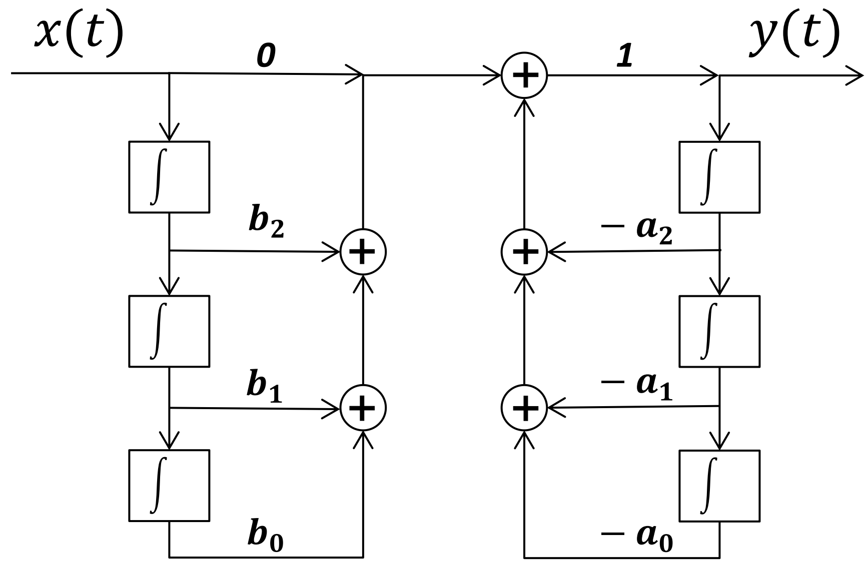
- Combine the common adders:



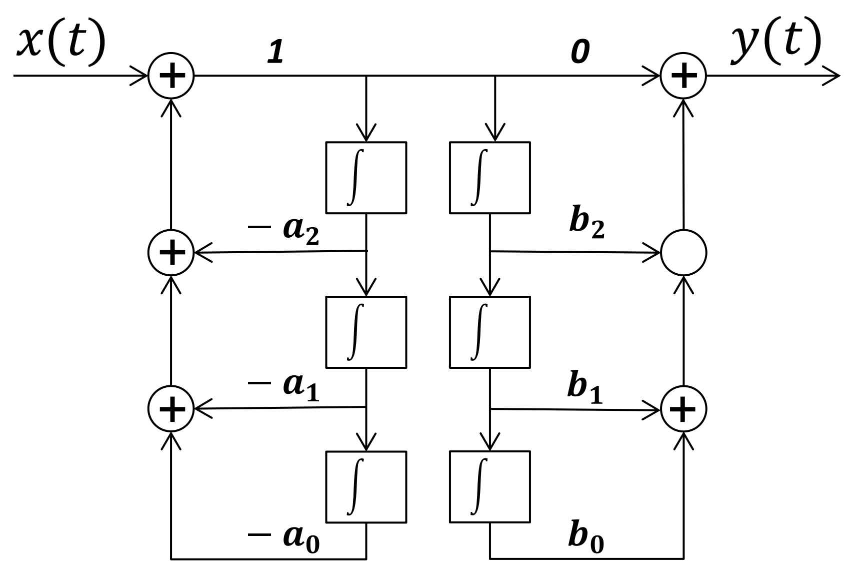
- This form is often referred to as the ***direct form I*** implementation => ***2N*** integrators are used.

3.2 Direct form II

- To save the usage of integrator:
  - Direct form I:

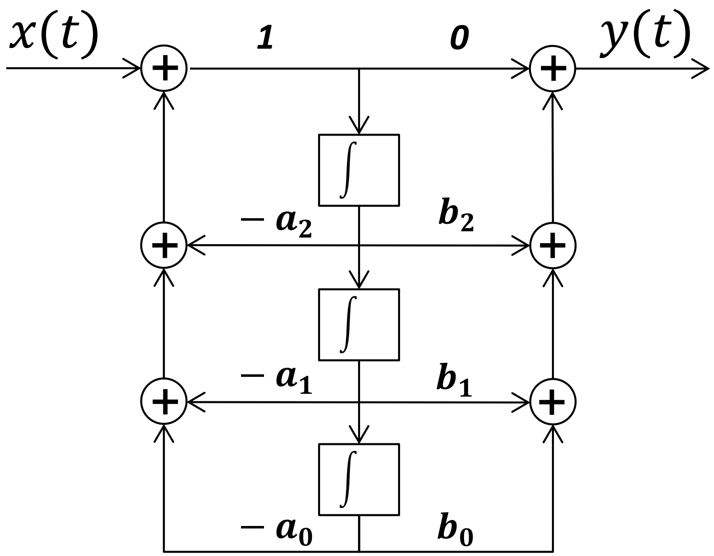


  - Interchanging the order of the two segments of Direct Form I:



3.2 Direct form II

- Combining the two chains of integrators.



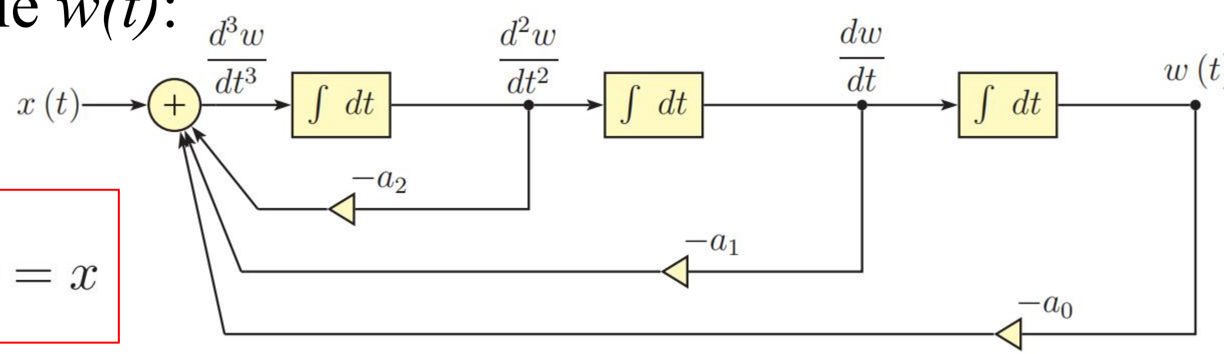
- This form is often referred to as the ***direct form II*** implementation => ***N*** integrators are used.
- Direct form II is also called the ***Canonic form***.

3.3 Horizontal Block diagrams (optional)

- Consider a third-order differential equation in the form:

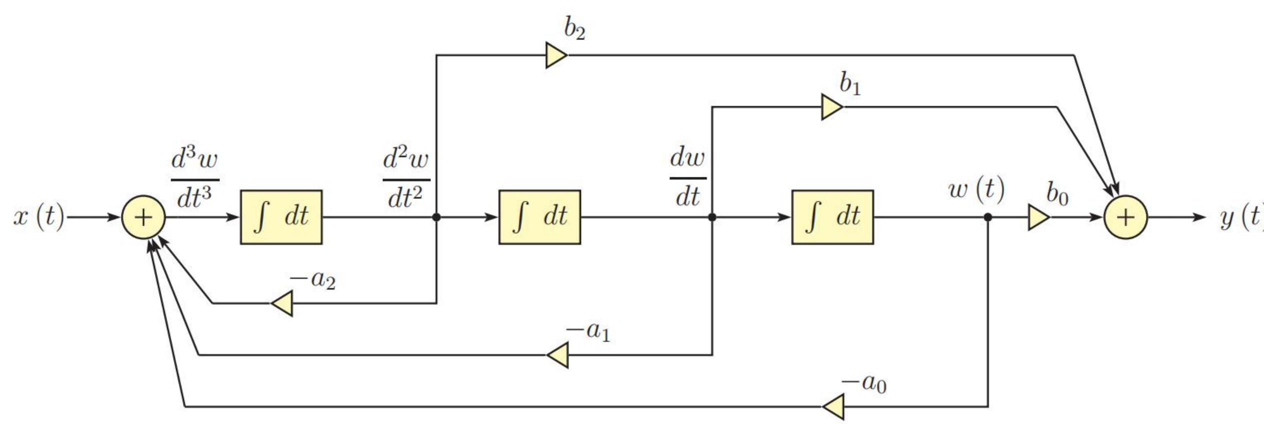
$$\boxed{\frac{d^3y}{dt^3}+a_2\frac{d^2y}{dt^2}+a_1\frac{dy}{dt}+a_0\,y}=\boxed{b_2\frac{d^2x}{dt^2}+b_1\frac{dx}{dt}+b_0\,x}$$

  - Intermediate variable *w(t)*:



$$\boxed{\frac{d^3w}{dt^3}+a_2\frac{d^2w}{dt^2}+a_1\frac{dw}{dt}+a_0\,w=x}$$

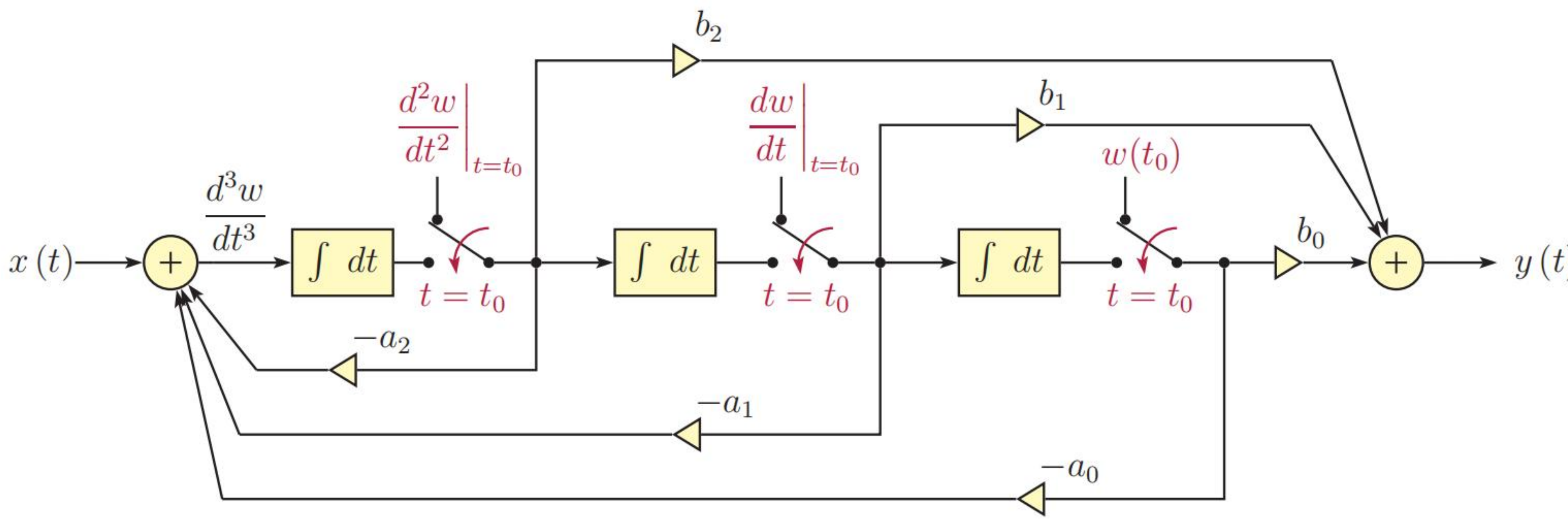
  - On the other side:



$$\boxed{y=b_2\frac{d^2w}{dt^2}+b_1\frac{dw}{dt}+b_0\,w}$$

3.3 Imposing initial conditions (optional)

- Imposing initial conditions
  - initial values of *y(t)* and its first N-1derivatives need to be converted to corresponding initial values of *w(t)* and its first N−1 derivatives;
  - Afterwards, appropriate initial value can be imposed on the output of each integrator



Quiz 2

- Draw block diagram for causal LTI systems described by the following difference and differential equations:

  a) $y(t) =- \left(\frac{1}{2}\right)\frac{dy(t)}{dt} + 4x(t)$

  b) $\frac{dy(t)}{dt} + 3y(t) = x(t)$

  c) $\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 17\frac{dy}{dt} + 13y = x + 2\frac{dx}{dt}$

Next ...

- Unilateral Laplace Transform
- Solving LCCDE using ULT