Xi'an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

2024-25 Semester 2

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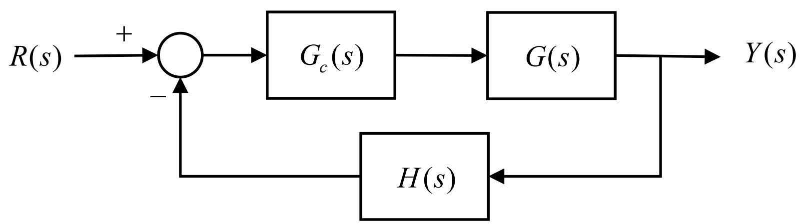
Lecture 18

Outline

Root Locus Method

- ❑ The Root Locus Concept
- ❑ Root Locus Plotting Procedure
- ❑ **Root Locus Using Matlab**
- ❑ **Parameter Design using the Root Locus Method**
- ❑ PID Controllers
  - Concept
  - PID Tuning
- ❑ Design Examples

Fundamentals of Root Locus Method/Analysis



Closed-loop TF: $T_{CL}(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)H(s)}$

(Open) Loop TF: $T_L(s) = G_c(s)G(s)H(s)$

Characteristic function $\Delta(s) = 1 + G_c(s)G(s)H(s)$

Assume $G_c(s)G(s)H(s) = \frac{kb(s)}{a(s)} = kL(s)$, where $a(s) = \prod_{j=1}^{n}(s+p_j)\,, b(s) = \prod_{i=1}^{m}(s+z_i)\,, and\ 0 < k < \infty$:

$$T_{CL}(s) = \frac{G_c(s)G(s)}{1+\frac{kb(s)}{a(s)}}$$

$$T_L(s) = \frac{kb(s)}{a(s)}$$

$a(s) = 0$ → OL poles

$b(s) = 0$ → OL zeros

CL system poles: $a(s) + kb(s) = 0$

- By comparing the poles-zeroes of the two systems, we can establish that:
  - when k is zero, the CL poles coincide with the LTF poles (or simply, OL poles).
  - When k is ∞, the CL poles coincide with the LTF zeros (or simply, OL zeros) .
  - So, it is logical to expect that, for $0 < k < \infty$, as k increases from 0 to *inf*, the CL poles are moving from the OL poles towards the OL zeroes. The traces of these CL poles form the loci of the CL characteristic equation's roots, a.k.a. "root locus".

Example 18.1 – Full example

Obtain the root locus for the closed-loop system with the following loop transfer function, as k varies for $0 \leq k < \infty$:

$$G_c(s)G(s)H(s) = \frac{k}{s(s+4)(s+4+j4)(s+4-j4)}$$

Poles: 0, -4, -4+j4, -4-j4
Zeroes: None

7+1 rules: S N R O A B A

- Symmetrical
- N = 4 → Number of OL poles

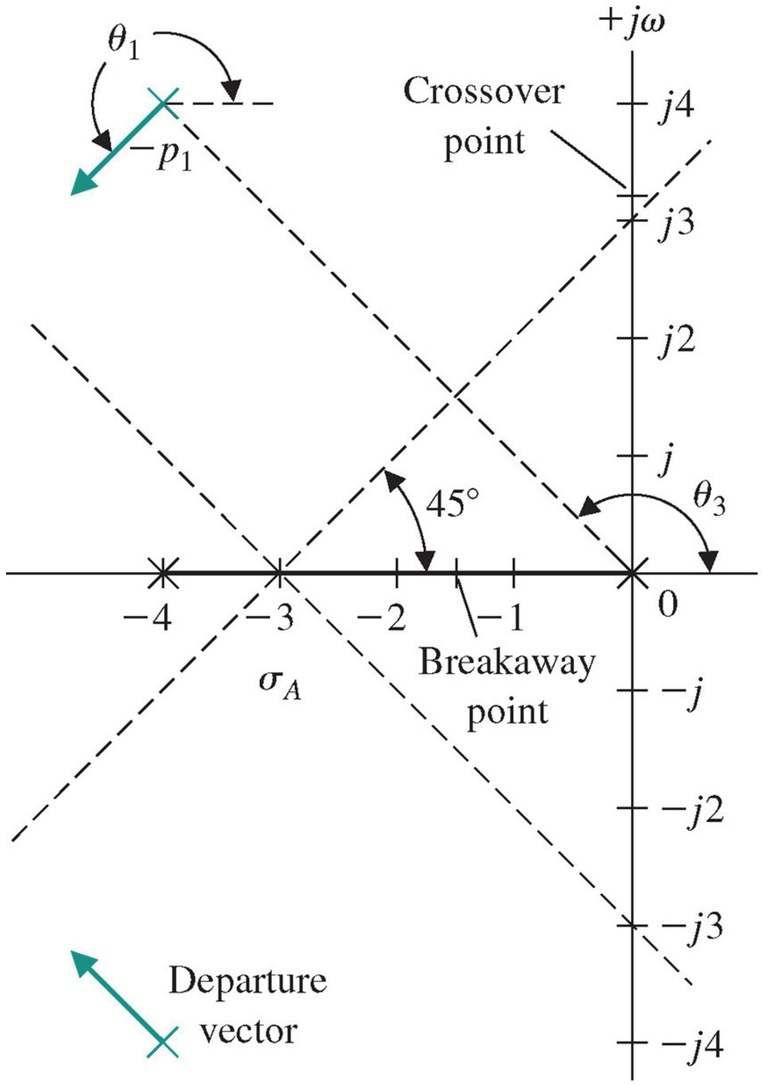
Number of OL poles *n* = 4
Number of OL zeroes *m* = 0

- R = *n* – *m* = 4 → Number of zeroes at *inf*
- *O* → *recognize this rule on the first sketch*

$$1+\frac{k}{s(s+4)(s+4+j4)(s+4-j4)}=0$$

A - Asymptotes angles and POI

$$Asymptote\ angles$$
$$=\frac{(2q+1)}{4}180^o, q=0,1,2,3$$
$$=45^o, 135^o, 225^o, 315^o$$

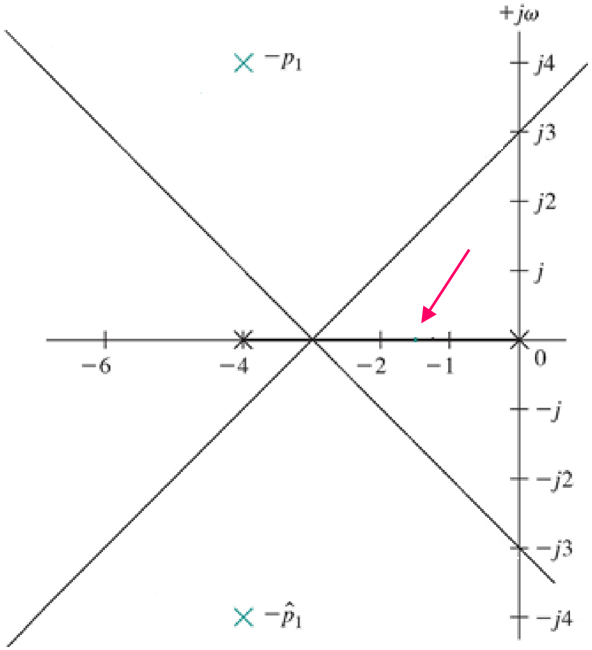


Centroid (or point of intersection @real axis):

$$\sigma_A=\frac{-4-4-4j-4+4j}{4}=-3$$

$$1 + \frac{k}{s(s+4)(s+4+j4)(s+4-j4)} = 0$$

B – Break-in break-out points (if any)



The breakaway point is estimated by evaluating

$$\frac{d}{ds}[s(s+4)(s+4+j4)(s+4-j4)] = 0$$

between $s = -4$ and $s = 0$.

Answer: $s = -1.577$.

A – Angle of departure (from complex poles, if any)

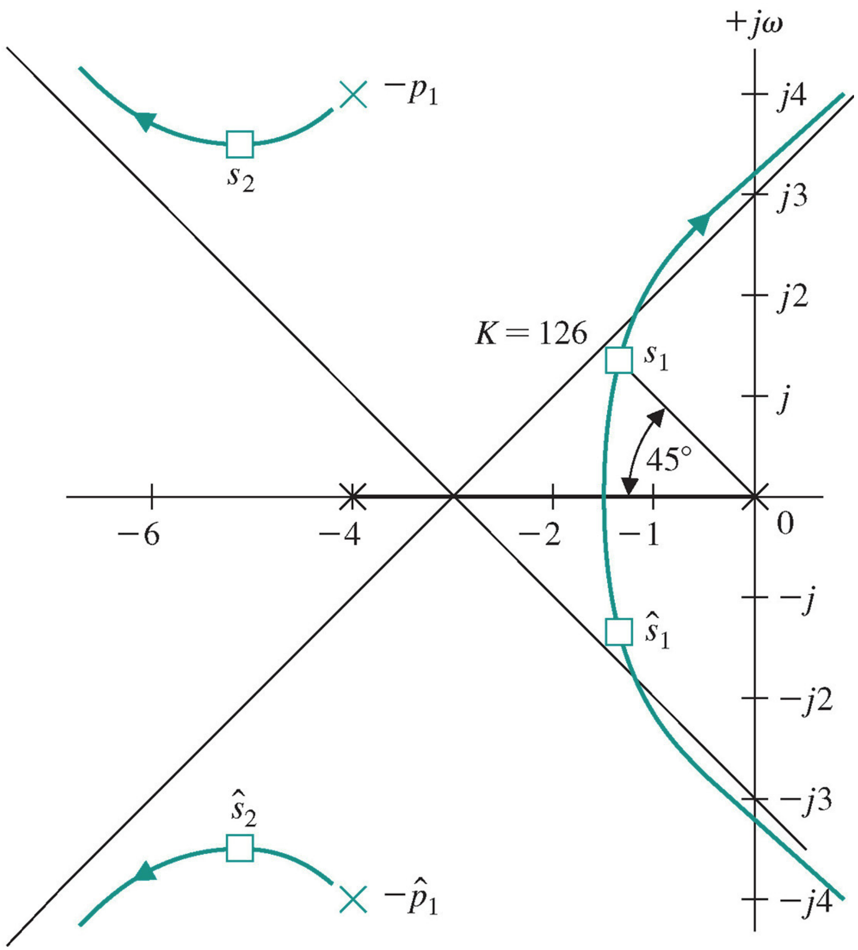
For angle of departure at complex pole $-p_1$, utilize angle criterion as follows

$$\theta_1 + 90^o + 90^o + 135^o = 180^o$$

$$\theta_1 = -135^o \equiv 225^o$$

$$1+\frac{k}{s(s+4)(s+4+j4)(s+4-j4)}=0$$

Full sketch, and crossing with the imaginary axis



8th rule: Crossing at the imaginary axis –Two ways

(a) Subs. $s = j\omega$ into $1 + kL(s) = 0$

(b) Routh Hurwitz Criterion

Subs. $s = j\omega$ into:

$$(s^2+4s)(s^2+8s+32)+k=0$$

$$(-\omega^2+4j\omega)(32-\omega^2+8j\omega)+k=0$$

Imaginary part:

$$4j\omega(32-\omega^2)-8j\omega^3=0$$
$$\omega = \pm 3.266\, rad/s$$

Real part:

$$-\omega^2(32-\omega^2)-32\omega^2+k=0$$
$$k=569$$

$$1+\frac{k}{s(s+4)(s+4+j4)(s+4-j4)}=0$$

Alternative 8th rule: Crossing at the imaginary axis – through RHC

The characteristic equation is rewritten as

$$s(s+4)(s^2+8s+32)+K=s^4+12s^3+64s^2+128s+K=0$$

Therefore, the Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 64 & K \\ s^3 & 12 & 128 & \\ s^2 & b_1 & K & \\ s^1 & c_1 & & \\ s^0 & K & & \end{array}$$

where

$$b_1=\frac{12(64)-128}{12}=53.33$$

$$c_1=\frac{53.33(128)-12K}{53.33}$$

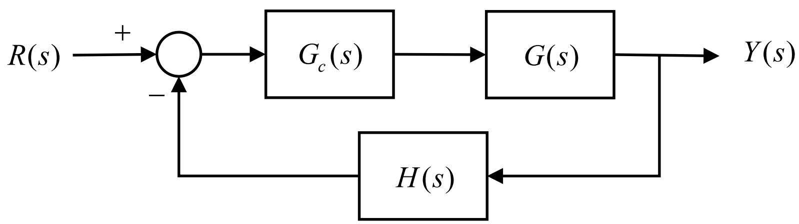
Conditions of stability: $b_1>0$, $c_1>0$, and $K>0$ $\quad\rightarrow\quad$ $0<K<568.9$

The gain K for marginally stability is $K=568.9$, and the roots for the auxiliary equation are

$$53.33s^2+568.9=53.33(s^2+10.67)=53.33(s+j3.266)(s-j3.266)$$

Therefore, the root locus crosses the $j\omega$-axis at $s=\pm j3.266$ when $K=568.9$.

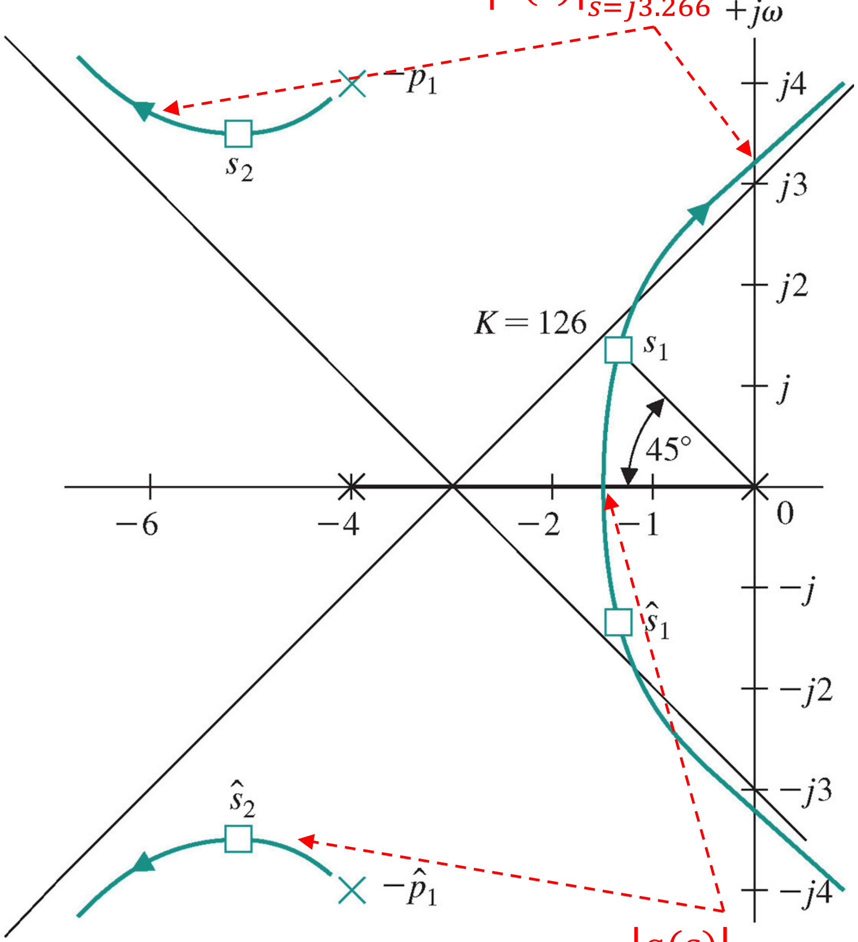
The final root locus plot:



$$T_{CL} = \frac{G_c(s)G(s)}{1+G_c(s)G(s)H(s)}$$

$$G_c(s)G(s)H(s) = \frac{k}{s(s+4)(s+4+j4)(s+4-j4)}$$

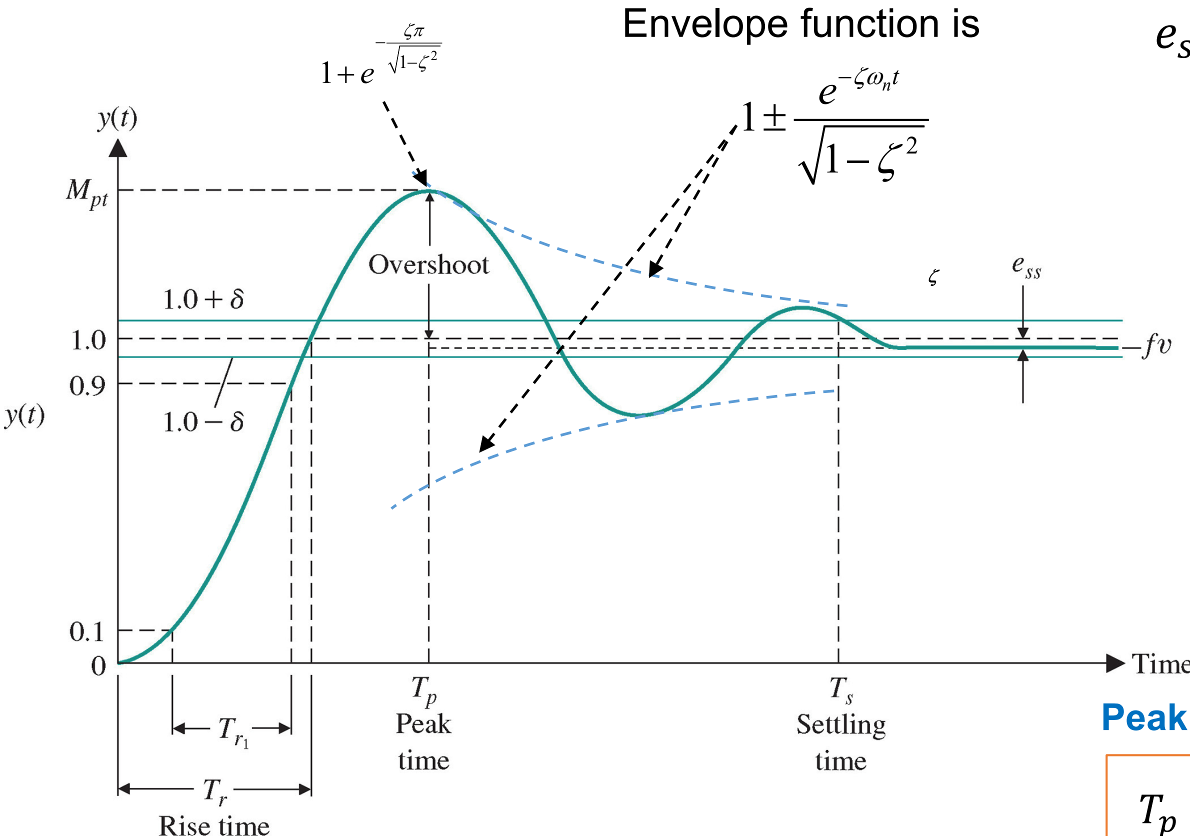
$$K = \left|\frac{a(s)}{b(s)}\right|_{s=j3.266} = 568.9$$



$$K = \left|\frac{a(s)}{b(s)}\right|_{s=-1.577}$$

Recall: 2nd order system's SS and transient characteristic performance (from Lectures 12-14)

- For a generalized second-order transfer function, we have the following:



Steady-state error/output towards unit impulse/step/ramp:

$$e_{ss} = \lim_{s\to 0} sE(s) \qquad y_{ss} = \lim_{s\to 0} sY(s)$$

2% Settling Time (for $0 \le \zeta \le 0.9$):

$$\boxed{T_s \cong 4\tau = \frac{4}{\zeta\omega_n}}$$

Percent/Maximum Overshoot (%):

$$\boxed{P.O.\ or\ M.O. = 100\% \times e^{-\zeta\pi/\sqrt{1-\zeta^2}}}$$

Peak time and rise time:

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \qquad \boxed{T_r = \frac{1}{\beta\omega_n}\tan^{-1}\left(-\frac{\beta}{\zeta}\right)}$$

Parameter Design by the Root Locus Method

- Originally, the root locus method was developed to determine the locus of roots of the characteristic equation as the system gain K is varied from zero to infinity.
  - The effect of other system parameters can be readily investigated by rearranging the characteristic equation (as demonstrated in Lecture 17).

- Then, it seems that the root locus method is a single-parameter design tool. The interesting question is: Can we use it to investigate the effect of two or more parameters?
  - The answer is yes. This method can be extended to account for two or more parameters. The process, however, may require some iteration to complete.
  - This makes the **multi-parameter design** for a CL system possible in actual system design.

Root Contours (Demo)

A family of root loci can be generated for two parameters in order to determine the total effect of varying two parameters. For example, let us determine the effect of varying α and β of the following characteristic equation:

$$s^3 + 3s^2 + 2s + \beta s + \alpha = 0$$

The root locus equation as a function of α is (set $\beta = 0$)

$$1 + \frac{\alpha}{s(s+1)(s+2)} = 0 \qquad (1)$$

The root locus as a function of β with non-zero α is

$$1 + \frac{\beta s}{s^3 + 3s^2 + 2s + \alpha} = 0 \qquad (2)$$

Note: the roots of eq.(1) become poles of eq.(2).

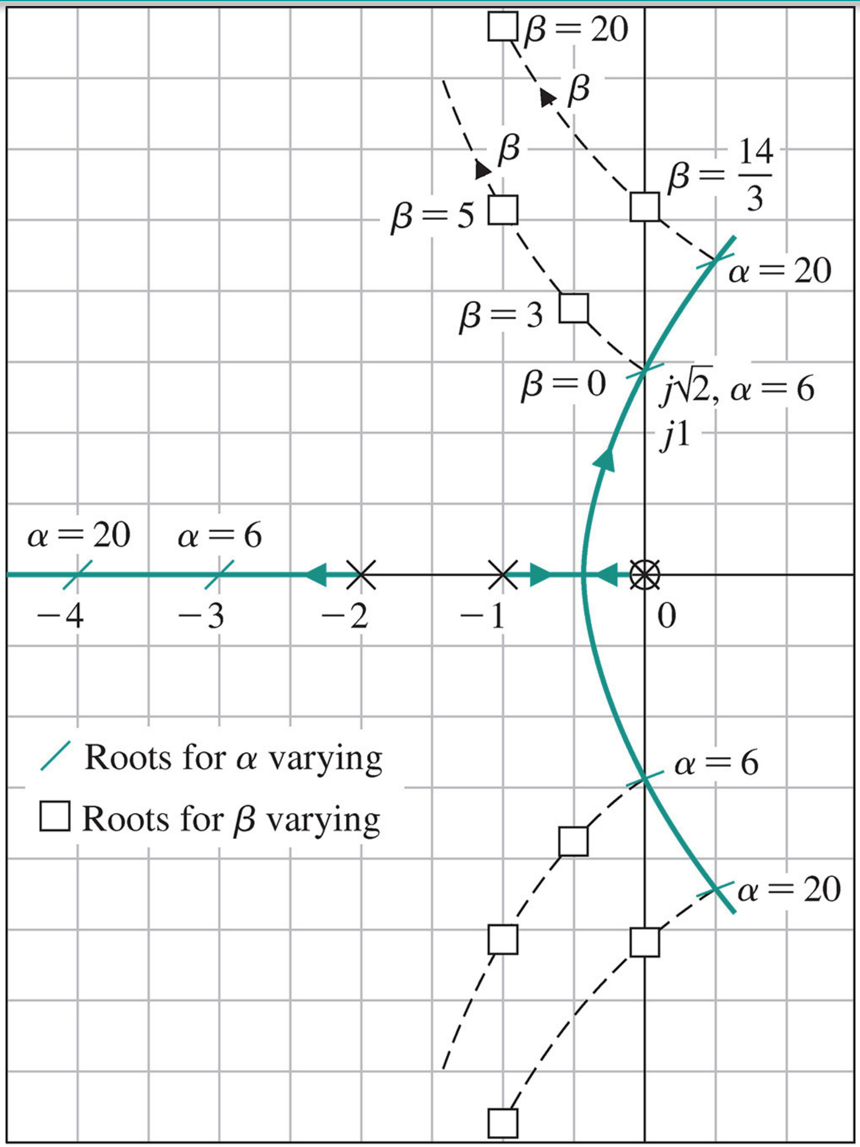
A family of loci, often called root contours can be sketched, which illustrates the effect of varying both α and β on the roots of the system's characteristic equation.

Two-parameter root locus. The loci for α varying are solid; the loci for β varying are dashed.

- Apply RLM on eq. (1), followed by applying RLM on eq. (2)

$$1 + \frac{\alpha}{s(s+1)(s+2)} = 0 \qquad (1)$$

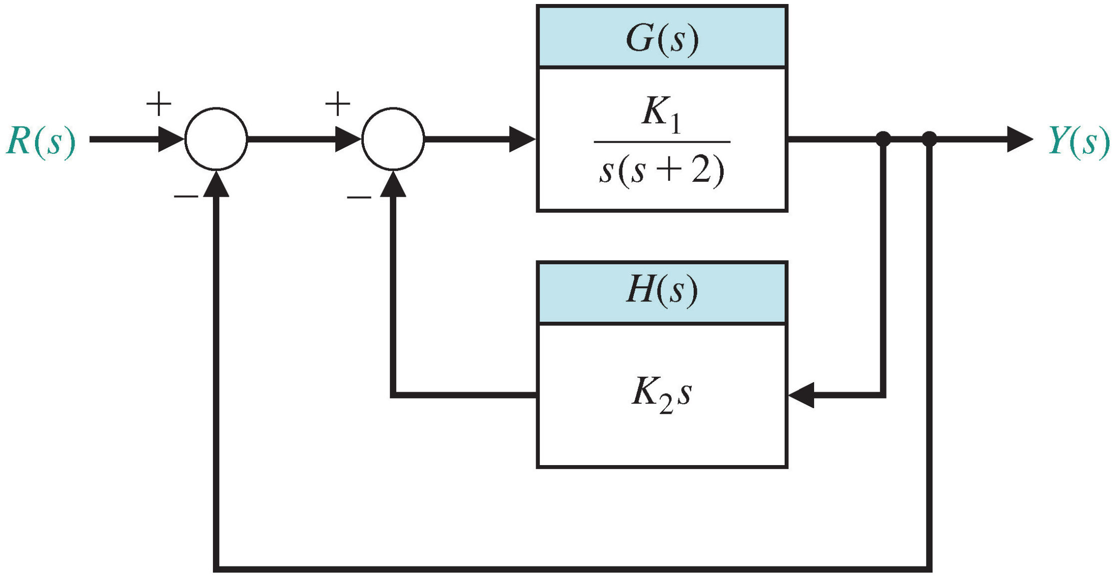
$$1 + \frac{\beta s}{s^3 + 3s^2 + 2s + \alpha} = 0 \qquad (2)$$



Example 18.2: Welding Head Control

A welding head for an auto body requires an accurate control system for positioning the welding head. The feedback control system is to be designed (i.e., values of K_1 and K_2 are to be determined) to satisfy the following specifications:

1. Steady-state error for a ramp input is $e_{ss} \leq 35\%$ of the input slope
2. Damping ratio of dominant roots is $\zeta \geq 0.707$
3. Settling time to within 2% of the final value is $T_s \leq 3s$



Solutions:

Start by defining the relevant TFs:

$$T_L = \frac{K_1}{s^2 + (K_1K_2 + 2)s} \qquad T_{CL} = \frac{K_1}{s^2 + (K_1K_2 + 2)s + K_1}$$

Step 1. determine conditions and root locations that satisfy the design specifications.

- For steady-state error requirement:

$$E(s) = R(s) - Y(s) = \frac{s^2 + (K_1K_2 + 2)s}{s^2 + (K_1K_2 + 2)s + K_1} R(s)$$

Ramp input -> $R(s) = \frac{A}{s^2}$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\,\frac{s^2 + (K_1K_2 + 2)s}{s^2 + (K_1K_2 + 2)s + K_1}\,\frac{A}{s^2} = \frac{K_1K_2 + 2}{K_1} A \le 0.35A$$

$K_2 + \frac{2}{K_1} \le 0.35$ --> we need small value of K_2.

- For damping ratio requirement:

$$\zeta \geq 0.707$$

as $\theta = \cos^{-1}\zeta$

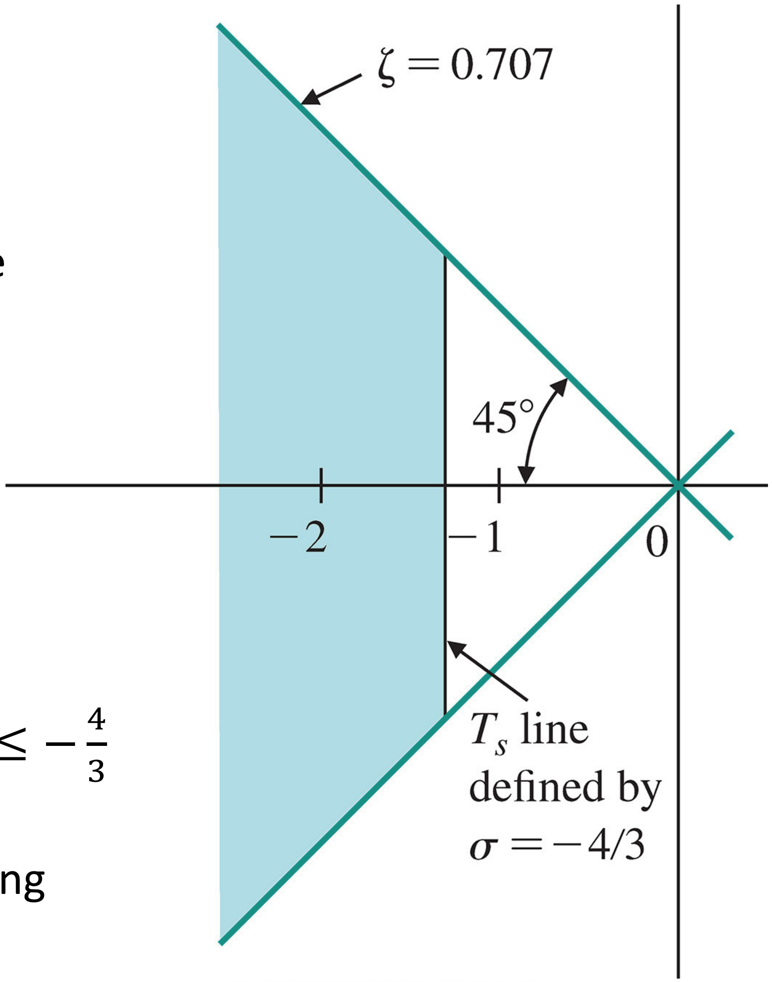
→ The roots of the closed-loop system must be below the line at 45° in the left-hand s-plane.

- For settling time requirement:

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\zeta\omega_n} \leq 3$$

$$\zeta\omega_n \geq \frac{4}{3} \quad \text{means, } -\zeta\omega_n \leq -\frac{4}{3}$$

→ We want the dominant roots (with real part being , $-\zeta\omega_n$) to lie to the left of the line $\sigma = -\frac{4}{3}$.



Step 2. Look into the root locus with one varying parameter, while setting the other parameter to be zero.

Characteristic equation for the closed-loop system:

$$\Delta(s) = s^2 + (K_1K_2 + 2)s + K_1$$

Assume $K_1 = \alpha$, $K_1K_2 = \beta$, then

$$\Delta(s) = s^2 + \beta s + 2s + \alpha$$

Set $\beta = 0$, sketch the root locus with varying α from zero to infinity

$$\Delta_1(s) = 1 + \alpha\frac{1}{s(s+2)} = 0$$

$$T_{L1} = \alpha\frac{1}{s(s+2)}$$

OL zeroes: none; OL poles: 0, -2.

S – symmetrical, N = 2, R = 2

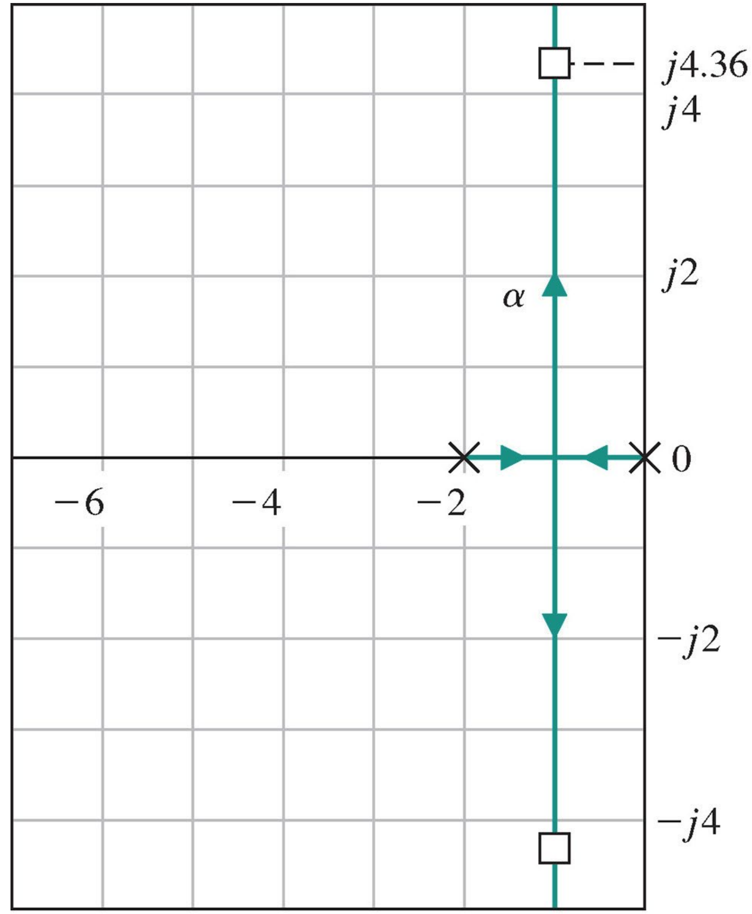
O – sketch and recognize this feature in the plot

A – P.O.I. @real axis = -1; Angles of asymptotes: 90°, 270°

B – Apply the rule, B. I. B. O. points obtained are -1.

A – not relevant here

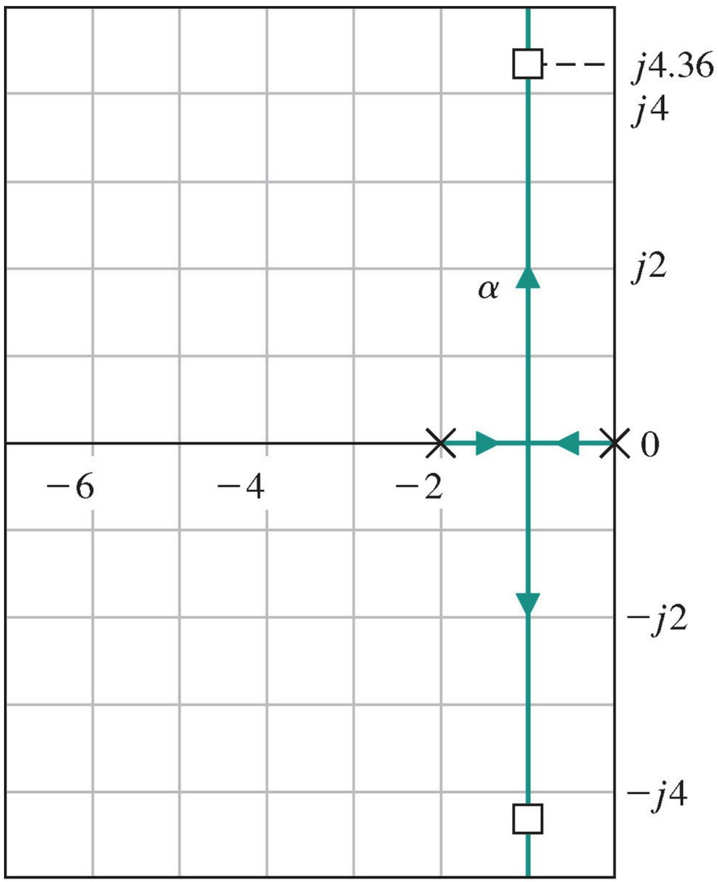
+1 rule – not relevant here (first 7 rules are sufficient)



(a)

$$\Delta(s) = s^2 + (\beta + 2)s + \alpha$$

Step 3. Select a fixed value of α, investigate the effect of another parameter by sketching the corresponding root locus.



(a)

For example, choose a gain of $K_1 = \alpha = 20$, the roots are

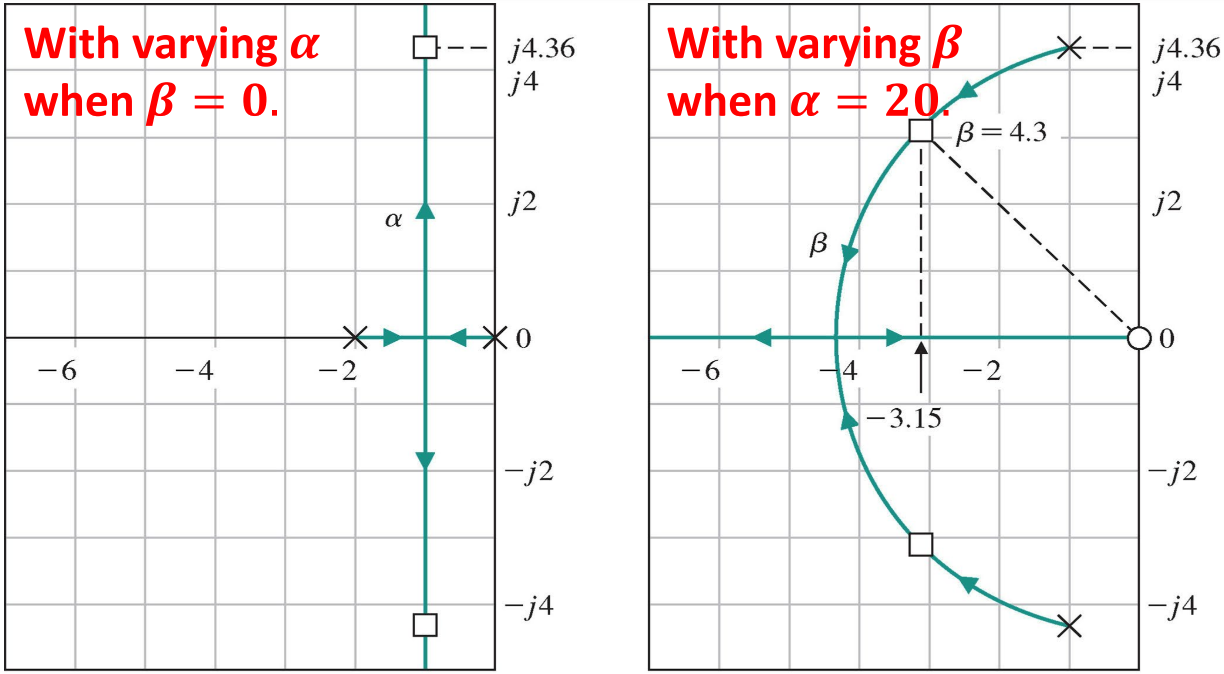
$$s = -1 \pm j4.36$$

Then the effect of varying $\beta = 20K_2$ (recall that $\beta = K_1K_2$) will be determined through the CL characteristic equation of:

$$1 + \beta \frac{s}{s^2 + 2s + 20} = 0$$

The root locus at $\alpha = 20$ for varying β can be then obtained through

$$T_{L2,\alpha=20} = \beta \frac{s}{s^2 + 2s + 20}$$



(a) (b)

$$T_{L2,\alpha=20} = \beta \frac{s}{s^2 + 2s + 20}$$

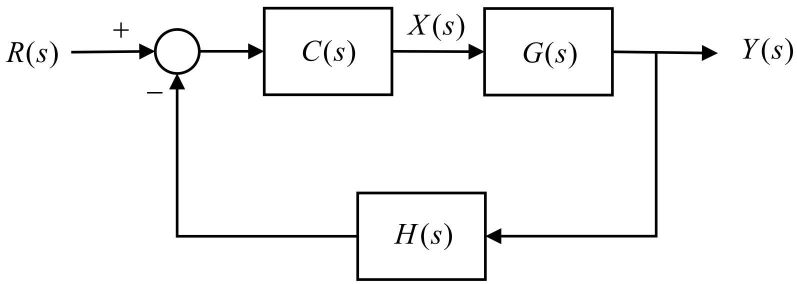
OL zeroes: 0; OL poles: -1±*j*4.36
S – symmetrical, N = 2, R = 1
O – sketch and recognize this feature
A – P.O.I. @real axis = -2; Angles of asymptotes: 180°
B – Apply the rule, B. I. B. O. points obtained are ±4.47 (ignore +ve, no locus there).
A – Angles are 192.9° (upper), –192.9° (lower)
+1 rule – not relevant here

Step 4. determine the parameter values.

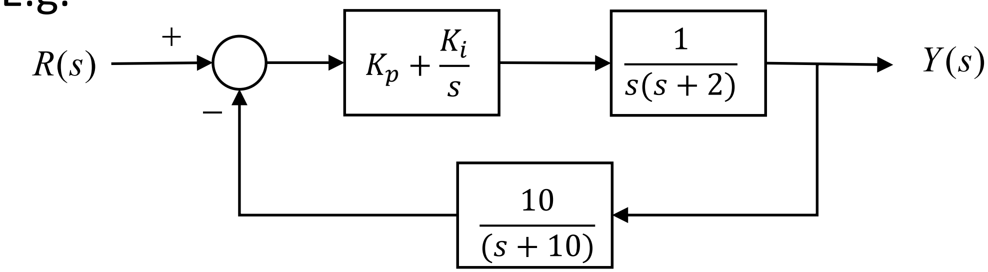
The root with $\zeta = 0.707$ are obtained when $\beta = 4.3 = K_1K_2$, the real part of these roots is $\sigma = -3.15$, then $T_s = 1.27s$. Therefore, when $K_1 = 20$, $K_2 = 0.215$, the design specifications can be met.

The root locus method can be extended to more than two parameters by extending the number of steps in the method.

Forms of Controllers (mainly FYI)



E.g.



- The ability of RLM to deal with multiple gain parameters becomes useful for more complex classical SISO controllers. This includes:
  - **PI** $K_p + K_i/s$
  - **PD** $K_p + K_d s$
  - **PID** $K_p + K_i/s + K_d s$
  - **Lead compensator** $(s\text{-}z_1)/(s\text{-}p_1)$, $|z_1| < |p_1|$
  - **Lag compensator** $(s\text{-}z_2)/(s\text{-}p_2)$, $|z_2| > |p_2|$
  - **Lead-lag compensator** $(s\text{-}z_1)(s\text{-}z_2)/(s\text{-}p_1)(s\text{-}p_2)$
  - **Cascaded controller, and others**

Effect of Time delay

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

- Time delay may have critical impacts on closed-loop dynamical control.
- Time delay (T) can be represented in s-domain as e^{-sT}. One of the commonly used approximation is

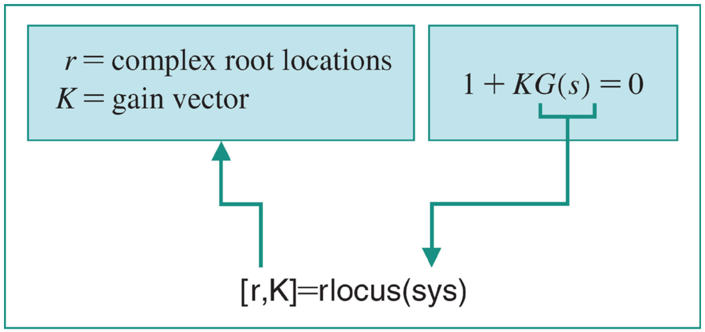
$$e^{-sT} = \frac{e^{-sT/2}}{e^{sT/2}} \approx \frac{1 - \frac{Ts}{2}}{1 + \frac{Ts}{2}}$$

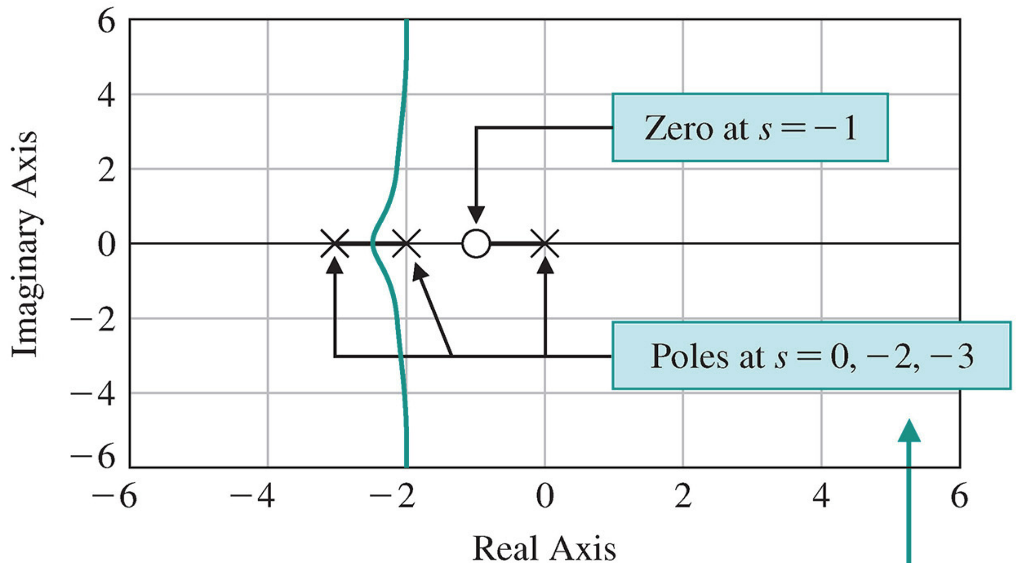
- E.g. $G_c(s)G(s)H(s) = \frac{ke^{-sT}}{s+1}$, and approximate $e^{-sT} \approx \frac{1-\frac{Ts}{2}}{1+\frac{Ts}{2}}$. The CL system performance can be analysed using RLM through

$$G_c(s)G(s)H(s) = \frac{k(2 - Ts)}{(s+1)(2 + Ts)}$$

Root Locus Using Matlab

The **rlocus** function.





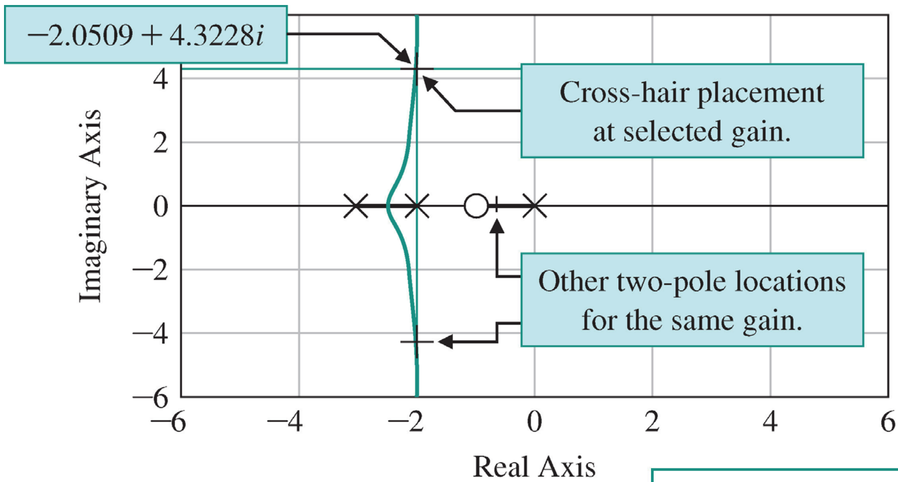

```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); rlocus(sys)
```

Generating a root locus plot.

```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); [r,K]=rlocus(sys);
```

Obtaining root locations r associated with various values of the gain K.

Using the **rlocfind** function.




```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); rlocus(sys)
>>rlocfind(sys)

   Select a point in the graphics window

    selected_point =
       -2.0509 + 4.3228i
    ans =
        20.5775
```

rlocfind follows the rlocus function.

Value of K at selected point

Example 18.3 (in-class)

$$\frac{\Delta y(s)}{\Delta \delta(s)} = \frac{7(s + 0.05)}{s^3 + 0.1s^2 - 2.4s + 0.05}$$

- Figure above shows the ballistic missile used in an military test launch. The linearized transfer function (at a particular operation point) that relates the output altitude $y(t)$ to the input thrust chamber deflection angle $\delta(t)$ is show above.
- A junior engineer (who never attend MEC208!) simply proposed a proportional controller for the CL system just to get the job done! The missile underwent a test launch but the mission failed terribly!
- Can you provide technical explanation on the failure, and possibly propose an alternative solution? (open question)

Next Lecture

- **In our next lecture:** we will see some PID tuning and design examples, and their relationship with Root Locus.

- **What you can do from now till the next lecture:** revise the material, further reading, and group study.

- **How to get in touch**: through LMO Module's "*General question and answer forum*" section or during my weekly consultation hour(s).