CAN207 Continuous and Discrete Time Signals and Systems

Lecture 21

Z-Transform_Part 3

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Content

- 6. Analysis of LTID systems using z-transform
  - Impulse response $h[n]$, LCCDE $y[n] \ldots x[n]$ and system transfer function $H(z)$
  - Zeros and poles of $H(z)$
  - Causality and stability
  - Geometric Evaluation of DTFT based on zero-pole locations
  - System behavior
- 7. Block diagram representation
  - Review of direct forms (I and II)
  - Cascade and parallel form
- 8. Unilateral z-transform (optional)
  - Definition and properties (initial value theorem)
  - Applications (difference equations with initial values)

6. z-Transform and the LTID systems

- The input-output relationship and its impulse response could be related through the convolution sum in time-domain:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- Based on the convolution property, they can also be related by the multiplication of their z-transforms in complex domain:

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

$$Y(z) = X(z)\,H(z)$$

- The function $H(z)$ is the ***system function*** (***transfer function***) of the system. It also represents a complete description of the LTID system.

6.1 Relating the system function to LCCDE

- LCCDE (Linear Constant Coefficient Difference Equation):

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Take z-transform of both sides:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

- So the system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

6.1 Relating the system function to LCCDE

- Finding the system function from the difference equation:
    1. Separate the terms of the difference equation so that y[n] and its time-shifted versions are on the left of the equal sign, and x[n] and its time-shifted versions are on the right of the equal sign as in equation (1).
    2. Take the z-transforms of both sides of the difference equation, and use the time-shifting property of the z-transform as in equation (2).
    3. Determine the system function as the ratio of Y(z) to X(z) as in equation (3).
    4. If the impulse response is needed, it can now be determined as the inverse z-transform of H(z).

- LCCDE corresponds to a LTI system only if all initial conditions are zero.
    - i.e. in determining the system function from LCCDE, all initial conditions must be assumed to be zero.

Quiz 1

- 1. Finding the system function and impulse response from the difference equation

$$y[n] = 0.4y[n-1] + 0.12y[n-2] + x[n] - x[n-1]$$

- 2. Finding the difference equation from the system function

$$H(z) = \frac{z^2 - 5z + 6}{z^3 + 2z^2 - z - 2}$$

6.2 Zeros and Poles

- The system function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{N(z)}{D(z)}$$

  – **Zeros ($\boldsymbol{z_i}$)**: The zeros of the transfer function $H(z)$ of an LTID system are finite locations in the complex z-plane, where $|H(z)| = 0$.
    - location of zeros can be obtained by solving $N(z) = \sum_{k=0}^{M} b_k z^{-k} = 0$;
    - Since $N(z)$ is an Mth-order polynomial, it has M roots leading to M zeros.

  – **Poles ($\boldsymbol{p_i}$)**: The poles of the transfer function $H(z)$ of an LTID system are at locations in the complex z-plane, where $|H(z)| \to \infty$.
    - location of poles can be obtained by solving $D(z) = \sum_{k=0}^{N} a_k z^{-k} = 0$;
    - Since $D(z)$ is an Nth-order polynomial, it has N roots leading to N poles.

6.2 Zeros and Poles

- For the given system function:

$$H(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- It normally has M zeros and N poles.
  - In some cases, the location of a pole may coincide with the location of a zero. => the pole and zero will cancel each other, and the actual number of poles and zeros will be reduced.
  - To find zeros and poles, factorise the function of z:

$$H(z) = \frac{N(z)}{D(z)} = \frac{(z - z_1)(z - z_2)\ldots(z - z_M)}{(z - p_1)(z - p_2)\ldots(z - p_N)}$$

  - or alternatively as:

$$H(z) = \frac{N(z)}{D(z)} = z^{M-N} \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})\ldots(1 - z_M z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})\ldots(1 - p_N z^{-1})}$$

Quiz 2

- Determine the poles and zeros of the following systems and plot them on the complex z-plane.

1. $H(z) = \frac{z}{z^2-3z+2}$
2. $H(z) = \frac{1}{(1-0.1z^{-1})(2-0.8z^{-1})(2z+1.8)}$
3. $H(z) = \frac{z^2-3z+2}{z^4-1}$

6.3.1 Causality

- A causal LTI system: $h[n] = 0 \quad \text{for} \quad n < 0$

- **Principle 1**: A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.
  - ROC of a right-sided system is the exterior of a circle in the z-plane, but may or may not include infinity;
  - But for causal system, $H(z) = \sum_{n=0}^{+\infty} h[n] z^{-n}$ doesn't include any positive powers of z, so its ROC includes infinity.

- **Principle 2**: A discrete-time LTI system with rational system function $H(z)$ is causal if and only if:
  - (a) the ROC is the exterior of a circle outside the outermost pole;
  - (b) with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

Example

- 1. Consider a system with system function whose algebraic expression is

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$$

Is it a causal system?

- 2. Consider a system with system function:

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \qquad |z| > 2$$

Is it a causal system?

6.3.2 Stability

- The stability of a LTID system is equivalent to its impulse response being absolutely summable.
  - In this case, the DTFT of $h[n]$ converges (exists), and consequently, the ROC of $H(z)$ must include the unit circle.

- Principle: An LTI system is stable if and only if the ROC of its system function $H(z)$ includes the unit circle, $|z| = 1$.

- Example: A system transfer function is given as:

$$H(z) = \frac{(z-1)(z+2)}{(z-0.5)(z-2)}$$

List all the possible ROCs and determine which one is for a stable system.

6.3.3 Stable and Causal system

- Combining the requirement for both, that is:

A causal LTI system with rational system function $H(z)$ is stable if and only if all of the poles of $H(z)$ lie inside the unit circle, i.e., they must all have magnitude smaller than 1.

- Example: a stable system is characterized by the system function

$$H(z)\frac{z(z-1)}{(z-0.8)(z+1.2)(z-2)}$$

Determine the impulse response of the system.

Quiz 3

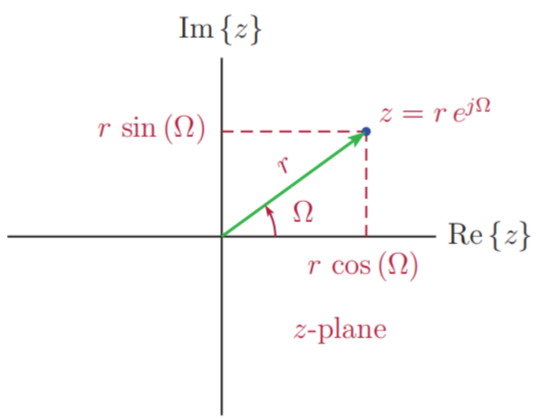
- A LTID system with a pair of poles at $p_{1,2} = 0.4 \pm 0.8j$ is characterized by the difference equation

$$y[n] = x[n-1] + 3x[n-2] + 2x[n-3] \\ +2.3y[n-1] - 2y[n-2] + 1.2y[n-3]$$

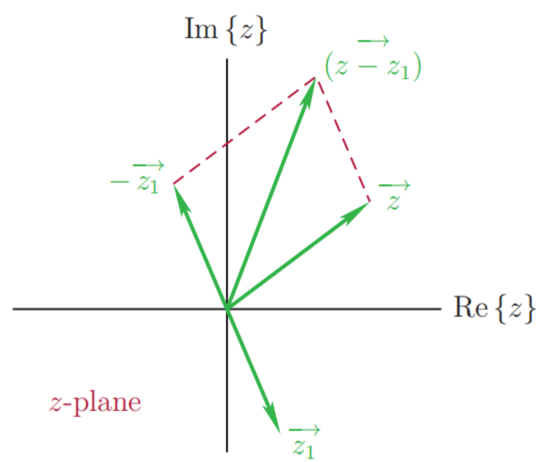
Comment on the stability of this system.

6.4.1 Graphical Interpretation of zero-pole plot

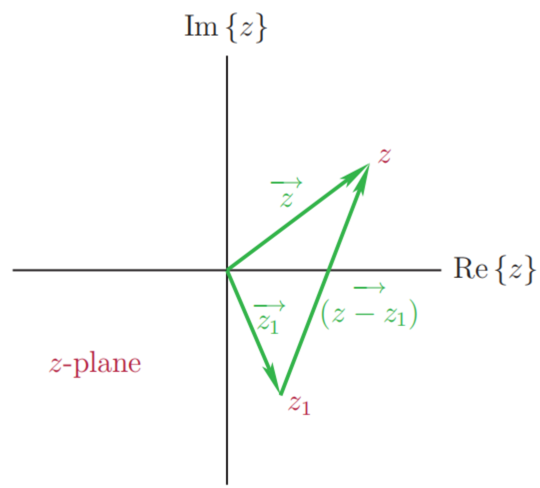
- Graphical Interpretation of the zero-pole plot
  - The complex variable z can be represented as a point in the z-plane;
  - Alternatively, a complex number can also be thought of as a vector in the complex plane;
  - The vector $\overrightarrow{z - z_1}$ is drawn with an arrow that starts at the point z_1 and ends at the point z.



$z = re^{j\omega}$



Vector $\overrightarrow{z - z_1}$



Alternative rep. of vector $\overrightarrow{z - z_1}$

6.4.1 Graphical Interpretation of zero-pole plot

- Consider a first-order system:

$$H(z) = \frac{z}{z-a}, \qquad ROC: |z| > |a|$$

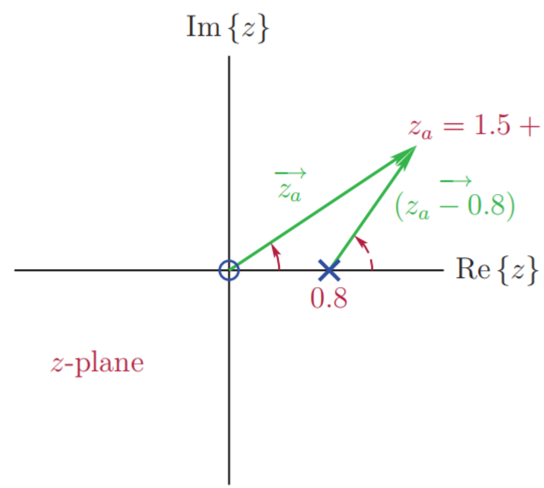
- In vector form, the system function can be written as the ratio of two vectors:

$$\overrightarrow{H(z)} = \frac{\vec{z}}{\overrightarrow{z-a}}$$

- Suppose we need to evaluate the system at a specific point $z = z_a$, the magnitude and phase at that point are computed as:

$$\left|\overrightarrow{H(z_a)}\right| = \frac{|\overrightarrow{z_a}|}{|\overrightarrow{z_a - a}|}$$

$$\measuredangle H(z_a) = \measuredangle\overrightarrow{z_a} - \measuredangle\overrightarrow{z_a - a}$$



- Example: $a = 0.8$ and $z_a = 1.5 + j1$

6.4.2 Geometric Evaluation of DTFT

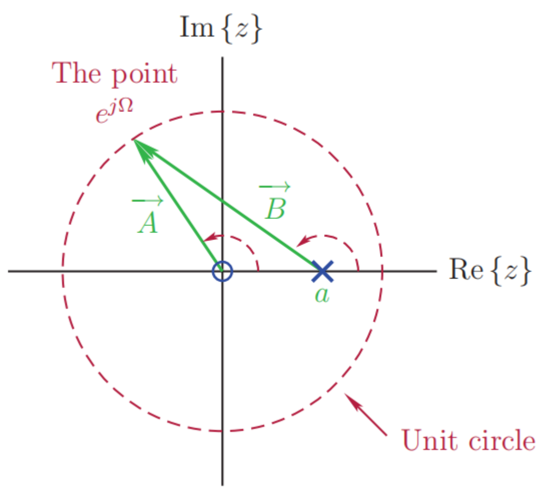
- DTFT-based frequency response $H(\omega)$ of a LTID system can be obtained from the z-domain system function by evaluating $H(z)$ at each point on the unit circle of the z-plane.
- Example: continuing with the first-order system function

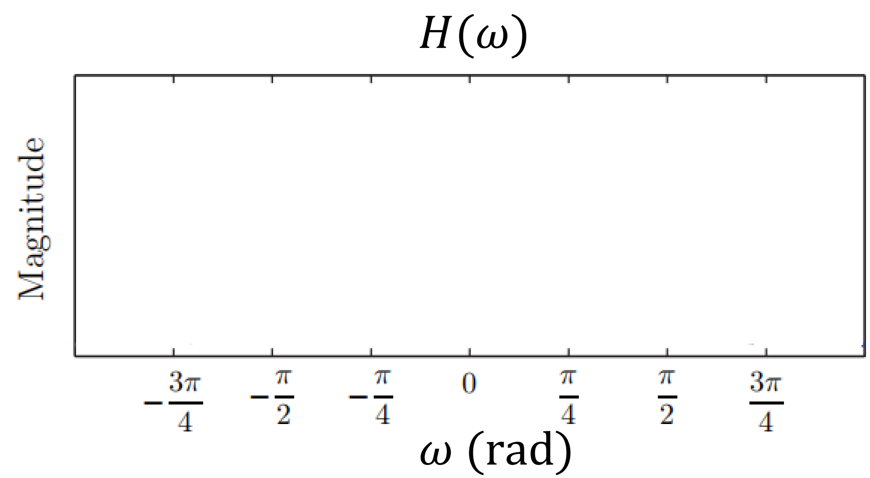
$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - a}$$

$a = 0.8$

$\vec{A} = \overrightarrow{e^{j\omega}}$

$\vec{B} = \overrightarrow{(e^{j\omega} - a)}$





6.4.2 Geometric Evaluation of DTFT

- Consider a more general system function in the form:

$$H(z) = K\frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$$

  – M zeros and N poles;

  – The magnitude of the system function is:

$$\left|\overrightarrow{H(z_a)}\right| = K\frac{|\overrightarrow{z_a-z_1}||\overrightarrow{z_a-z_2}|\ldots|\overrightarrow{z_a-z_M}|}{|\overrightarrow{z_a-p_1}||\overrightarrow{z_a-p_2}|\ldots|\overrightarrow{z_a-p_N}|}$$

  – The phase is:

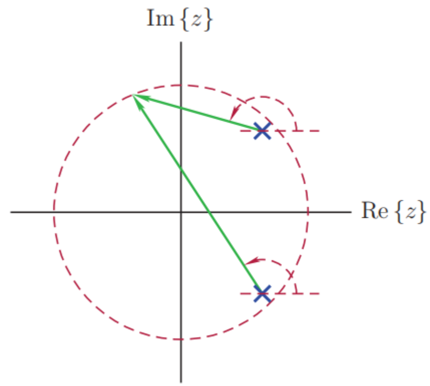
$$\measuredangle H(z_a) = \measuredangle\overrightarrow{z_a-z_1} + \measuredangle\overrightarrow{z_a-z_2} + \ldots + \measuredangle\overrightarrow{z_a-z_M}$$
$$-\measuredangle\overrightarrow{z_a-p_1} - \measuredangle\overrightarrow{z_a-p_2} - \ldots - \measuredangle\overrightarrow{z_a-p_N}$$

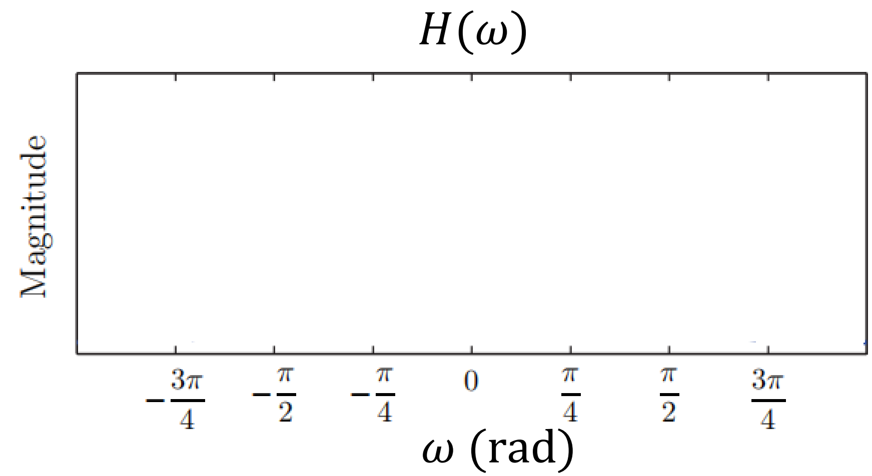
  – The vector-based graphical method described above is useful for understanding the correlation between pole-zero placement and system behavior.

6.4.2 Geometric Evaluation of DTFT

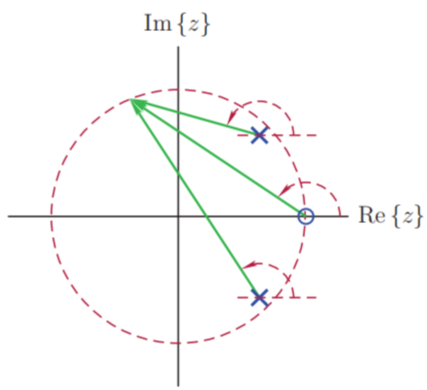
- Examples:

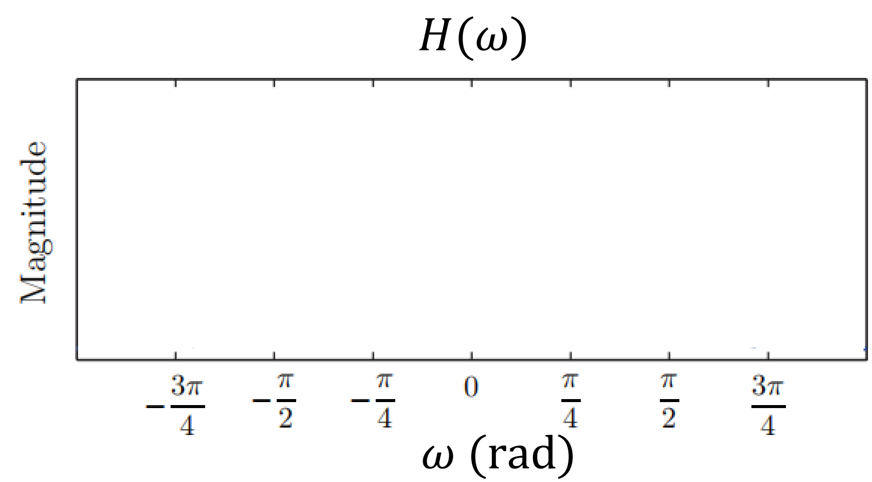
(a)





(b)

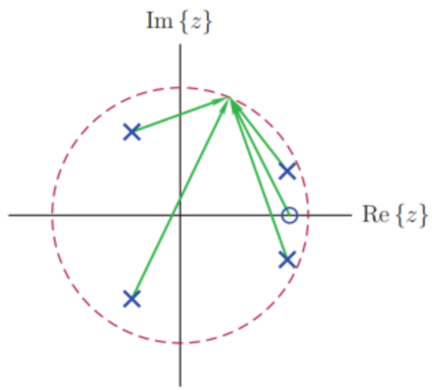


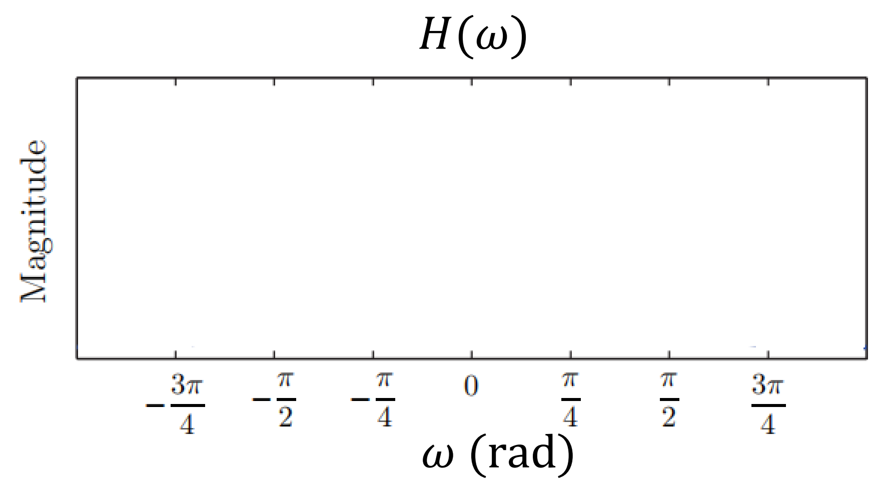


6.4.2 Geometric Evaluation of DTFT

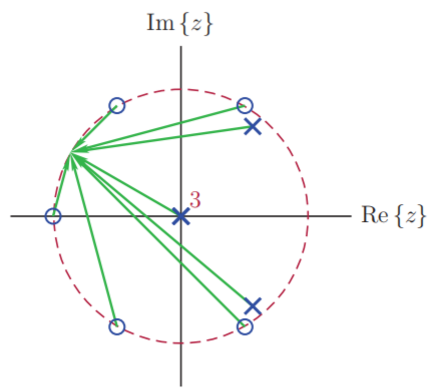
- Examples:

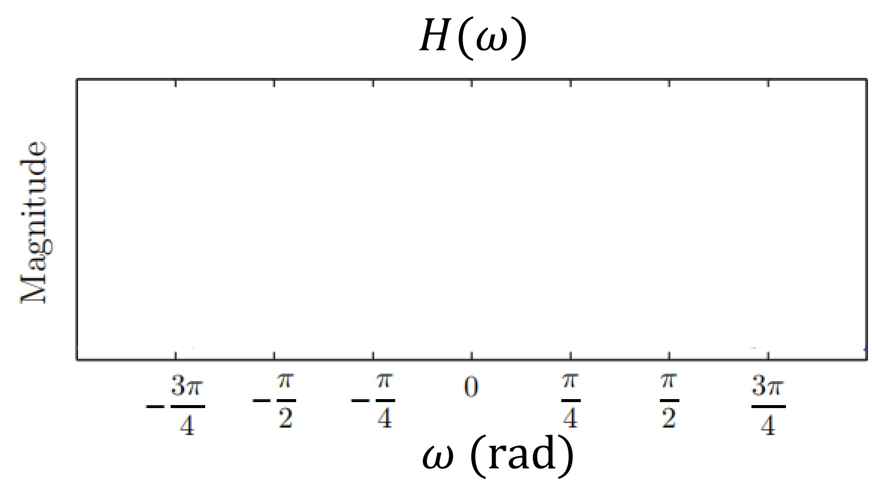
(c)





(d)





Quiz 4

- For the **causal** systems given in quiz 3, sketch the magnitude response by evaluating the zero-pole plots.

1. $H(z) = \frac{z}{z^2-3z+2}$
2. $H(z) = \frac{1}{(1-0.1z^{-1})(2-0.8z^{-1})(2z+1.8)}$
3. $H(z) = \frac{z^2-3z+2}{z^4-1}$

6.5 System Behavior

- Many properties of discrete-time LTI systems can be directly related to the system function and its characteristics.
- Here are some examples showing how z-transform properties can be used in analyzing systems.

- Example: with the information of two sets of input-output:
  1. when input is $x_1[n] = (1/6)^n u[n]$, and the corresponding output is $y_1[n] = [a(1/2)^n + 10(1/3)^n]u[n]$, where a is a real number;
  2. when input is $x_2[n] = (-1)^n$, then the output is $y_2[n] = \frac{7}{4}(-1)^n$.

  – Find the system transfer function, impulse response and LCCDE of this system.

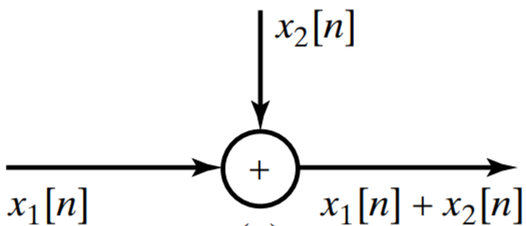
Quiz 5

- Consider a stable and causal system with impulse response $h[n]$ and rational system function $H(z)$. Suppose it is known that $H(z)$ contains a pole at $z = 1/2$ and a zero somewhere on the unit circle. The precise number and locations of all of the other poles and zeros are unknown.
- Determine whether the following statements are true or false:

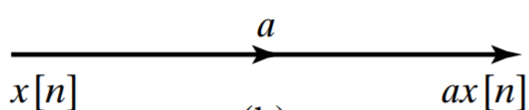
  1. $\mathcal{F}\{\left(\frac{1}{2}\right)^n h[n]\}$ converges;
  2. $H(e^{j\omega}) = 0$ for some frequencies;
  3. $h[n]$ has finite duration;
  4. $h[n]$ is real;
  5. $g[n] = n(h[n] * h[n])$ is a stable system.

7. Implementation structures for LTID system

- The implementation of an LTID system by iteratively evaluating a recurrence formula obtained from a difference equation requires that delayed values of the output, input, and intermediate sequences be available.

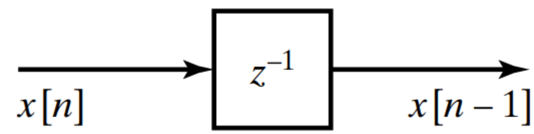


(a)



(b)

Multiplier
Multiplication of a sequence by a constant



(c)

Unit Delay

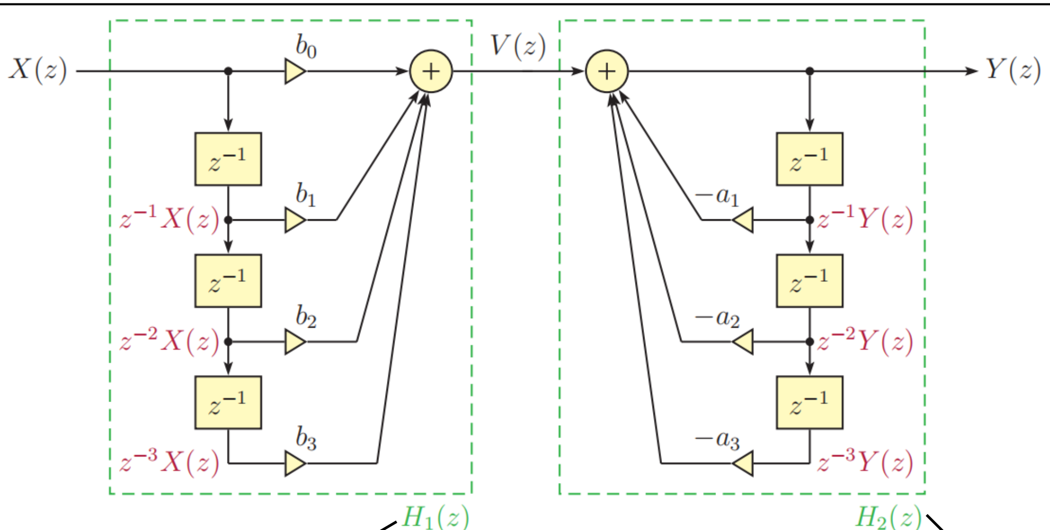
7.1.1 Direct Form I

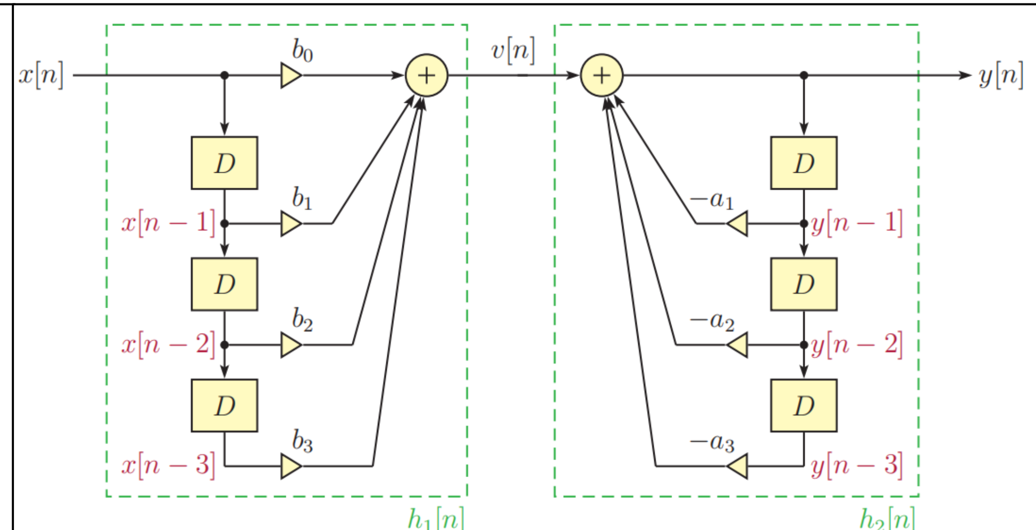
- Take a 3rd-order LTID system as example:

$$H(Z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

- It can be considered as the cascade of two parts:

$$H(Z) = \frac{Y(z)}{V(z)} \times \frac{V(z)}{X(z)} = H_1(z)\, H_2(z)$$





$$H_1(z) = \frac{V(z)}{X(s)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}$$

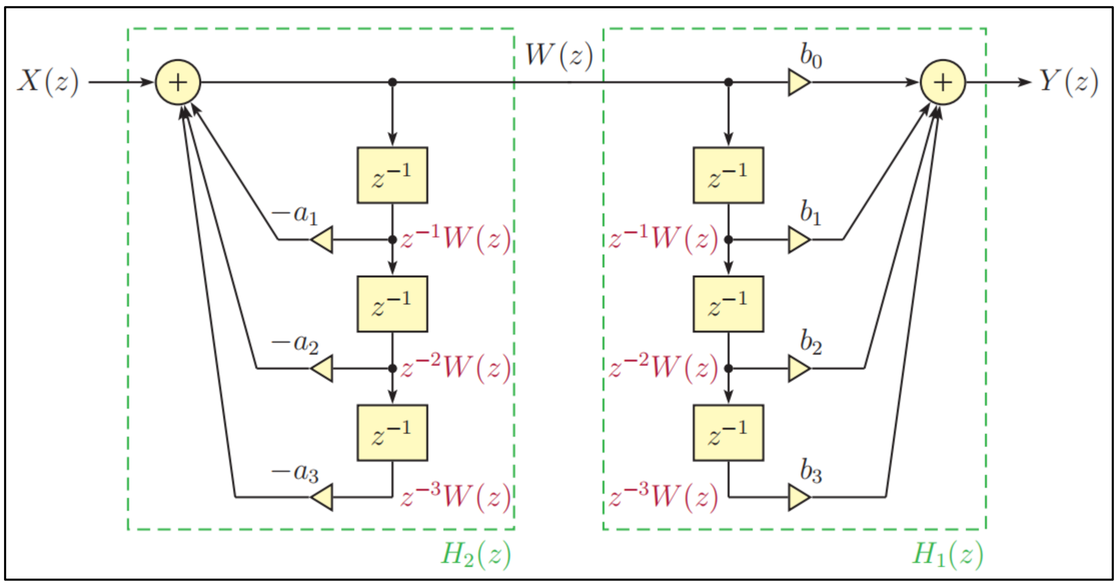
$$H_2(s) = \frac{Y(z)}{V(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

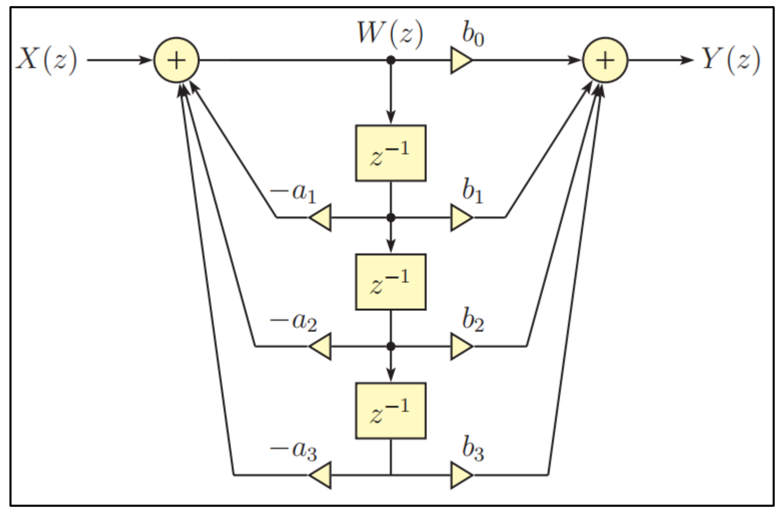
7.1.2 Direct Form II

- Exchange the sequence of the two subsystems:

$$H(Z) = H_2(z)\, H_1(z) = \frac{V(z)}{X(z)} \times \frac{Y(z)}{V(z)}$$

- Block diagram obtained by swapping the order of two subsystems:





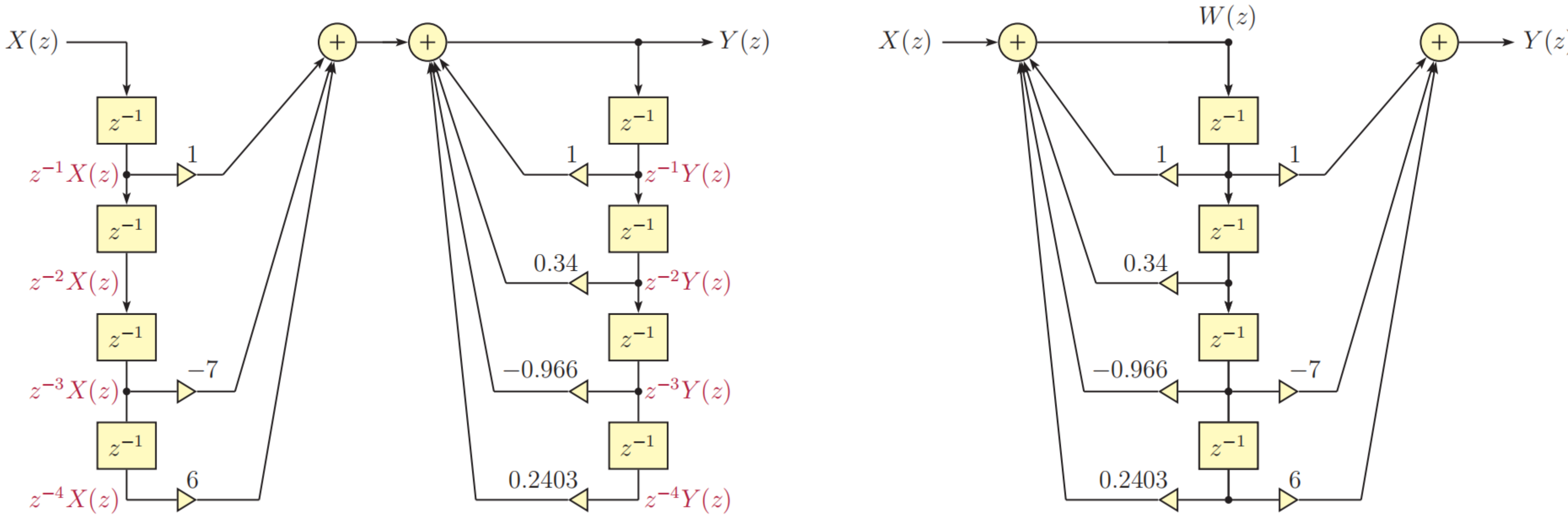
direct form II

- The middle part of the diagram has two delay lines running parallel to each other and holding identical values => they can be merged.

Example

- Draw the direct form I and II forms of a causal LTID system described by the z-domain system function

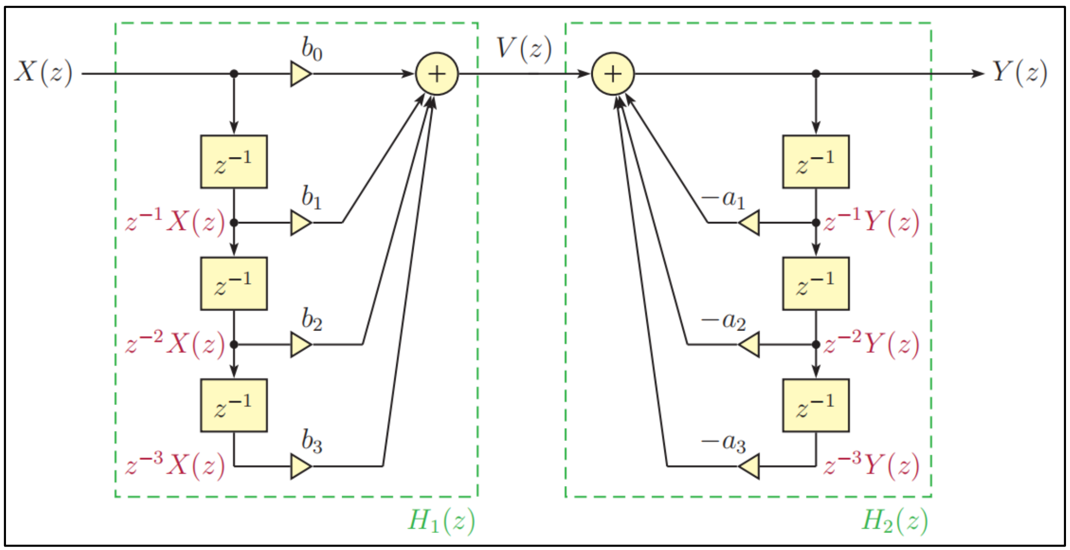
$$H(z) = \frac{z^3 - 7z + 6}{z^4 - z^3 - 0.34\,z^2 + 0.966\,z - 0.2403}$$

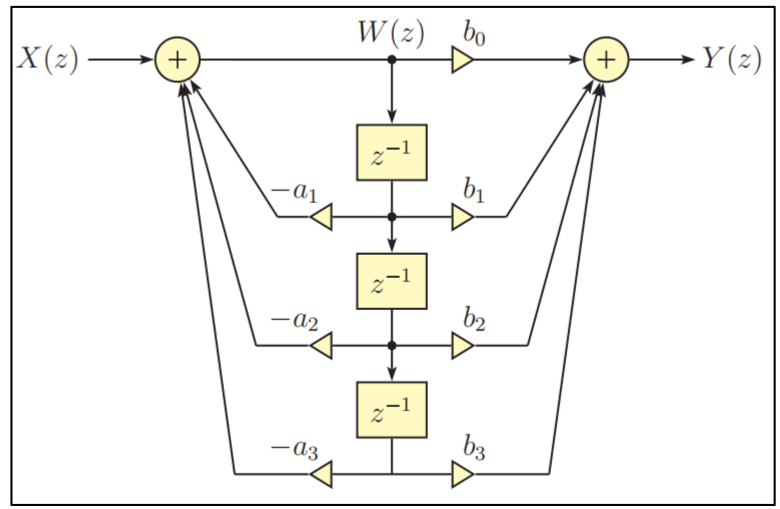


7.1 Direct Form I & II

- The direct forms are called "direct" because they direct use the coefficients in the polynomials as coefficients in the block diagrams:

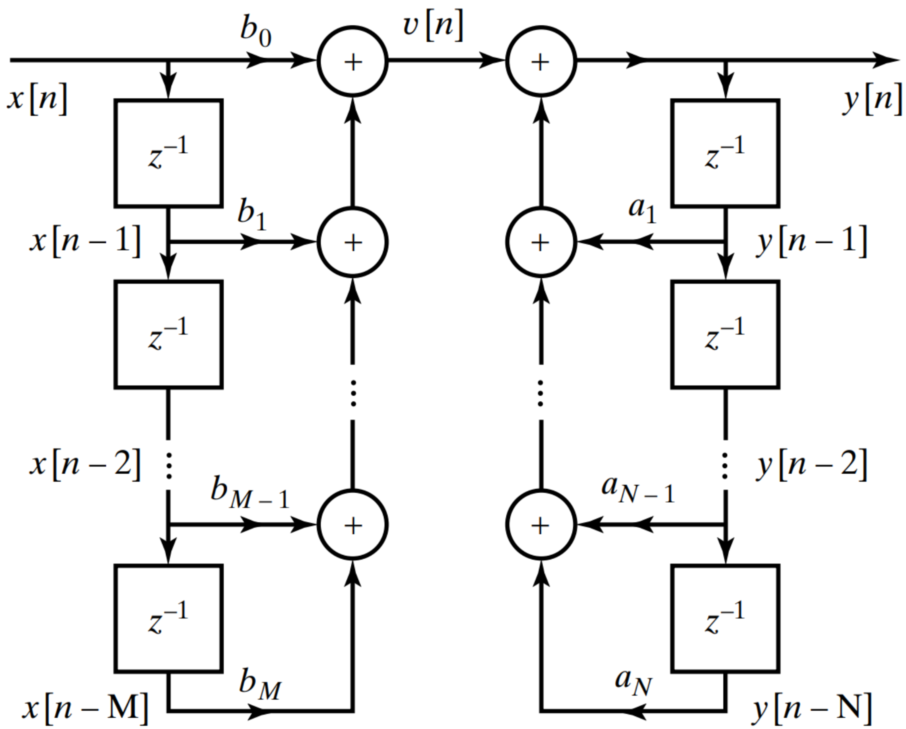
$$H(Z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}}{1+a_1z^{-1}+a_2z^{-2}+a_3z^{-3}}$$





7.1 Canonic Form

An implementation with the minimum number of delay elements is commonly referred to as a *canonic form implementation*.

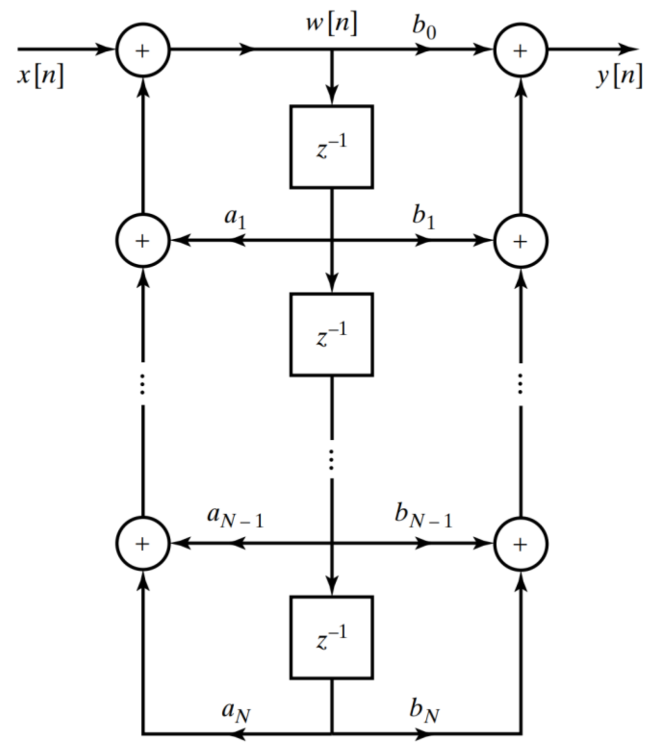


Direct Form I

No. of Adders: M+N

No. of Multipliers: M+N+1

No. of delay units: M+N



Direct Form II

No. of Adders: M+N

No. of Multipliers: M+N+1

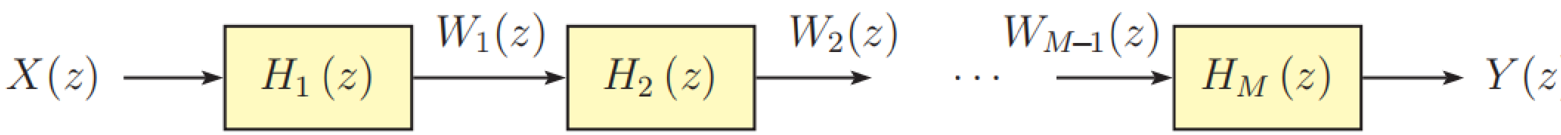
No. of delay units: max(N,M)

Canonic form

7.2.1 Cascade forms

- It is also possible to express the system function as the product of lower order sections:

$$H(Z) = H_1(z)\, H_2(z) \ldots H_M(z)$$



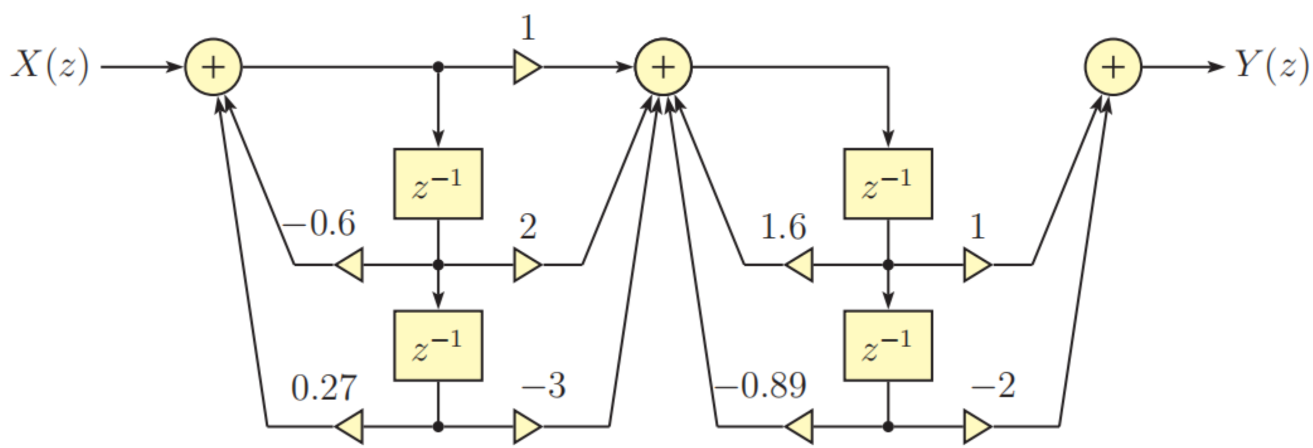
- The sub-systems could be any order, but usually are 2nd order.
  - Especially for conjugate pole and zero pairs, they are often combined to be 2nd order sub-systems;
  - Order of the numerator and denominator polynomials are usually the same.

Example

- Draw the cascade form of the following system function:

$$H(z) = \frac{(z+3)(z-1)(z-2)}{(z+0.9)(z-0.3)(z-0.8-j0.5)(z-0.8+j0.5)}$$

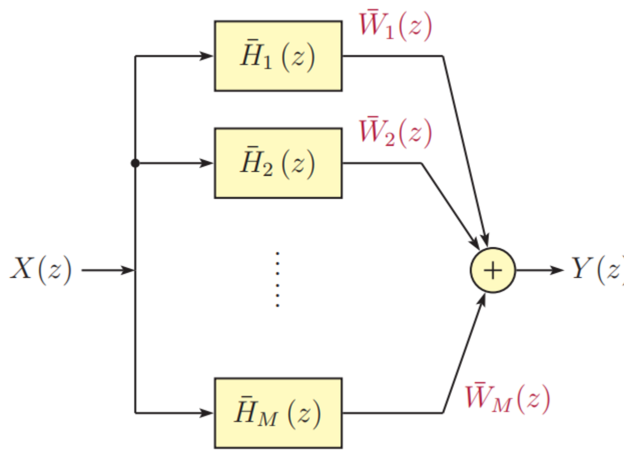
$$H_1(z) = \frac{(z+3)(z-1)}{(z+0.9)(z-0.3)} = \frac{z^2+2z-3}{z^2+0.6z-0.27} \qquad H_2(z) = \frac{z-2}{(z-0.8-j0.5)(z-0.8+j0.5)} = \frac{z-2}{z^2-1.6z+0.89}$$



7.2.2 Parallel forms

- An alternative is to express the system function as the sum of lower order sections:

$$H(Z) = \widetilde{H}_1(z) + \widetilde{H}_2(z) + \ldots + \widetilde{H}_M(z)$$



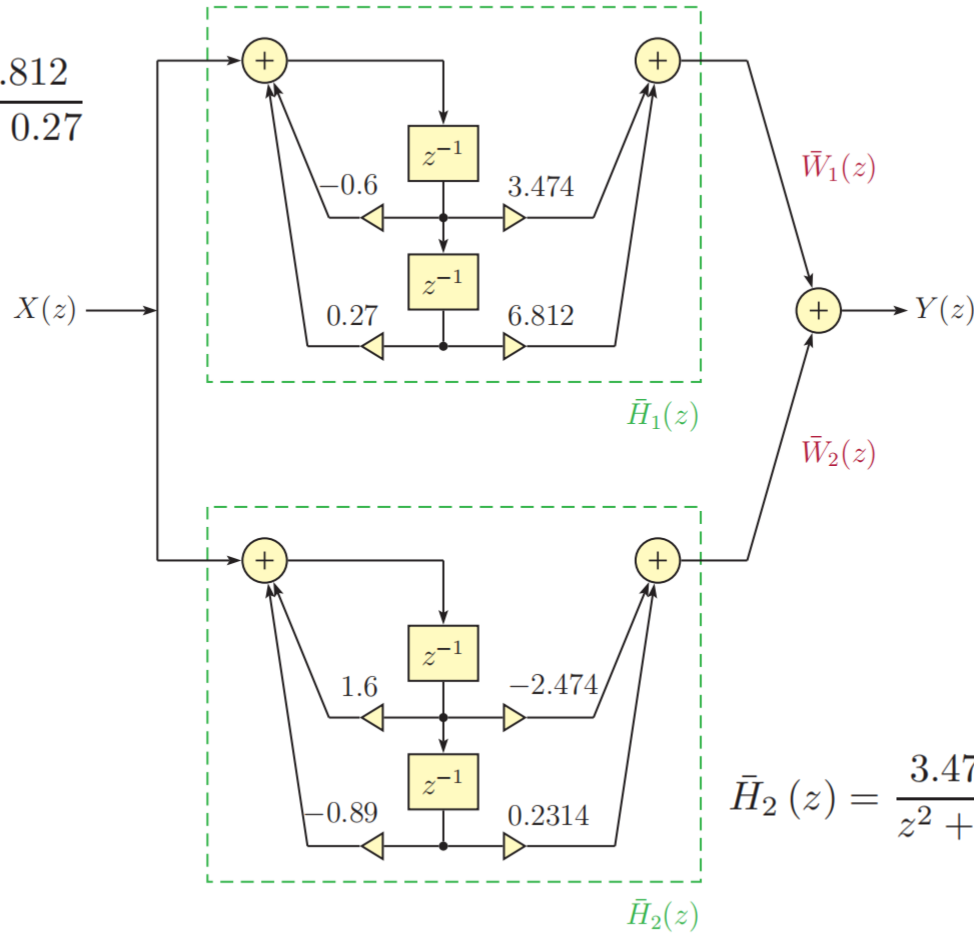
- A rational system function $H(z)$ may be sectioned using partial fraction expansion.
  - Similarly, complex poles and zeros are paired to keep the real coefficients in the expression.

Example

- Draw the parallel form of the following system function:

$$H(z) = \frac{(z+3)(z-1)(z-2)}{(z+0.9)(z-0.3)(z-0.8-j0.5)(z-0.8+j0.5)}$$

$$\bar{H}_1(z) = \frac{3.474\,z + 6.812}{z^2 + 0.6\,z - 0.27}$$



$$\bar{H}_2(z) = \frac{3.474\,z + 6.812}{z^2 + 0.6\,z - 0.27}$$

Quiz 6

- Draw the direct form I, direct form II, cascade form and parallel form of the following LCCDE.

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - [n-1]$$

8.1 Definition

- Bilateral z-transform → Unilateral z-transform

(two-sided) (one-sided)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad X^{+}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

- $X^{+}(z)$ is appropriate for problems involving causal signals and systems;
  - $X^{+}(z)$ of x[n] is identical to the two-sided transform X(z) of the sequence x[n]u[n], so the ROC of $X^{+}(z)$ are always the exterior of a circle.
  - $X^{+}(z)$ is unique for a causal signal;

8.2 Time-Shifting Property

- Almost all the properties of the bilateral transform are applicable to the unilateral transform, except the ***time-shifting*** property.
- Time-Shifting Property when the sequence is delayed by k: if $x_s[n] = x[n-k]$, when $k > 0$:

$$X_s{}^{+}(z) = x[-k] + x[-k+1]z^{-1} + \cdots + x[-1]z^{-k+1} + z^{-k}X^{+}(z)$$

$$= \sum_{m=1}^{k} x[-m]z^{-k+m} + z^{-k}X^{+}(z) = z^{-k}\left[\sum_{m=1}^{k} x[-m]z^{m} + X^{+}(z)\right]$$

- Most important one is when $k = 1$:

$$x_s[n] = x[n-1] \longrightarrow X_s{}^{+}(z) = x[-1] + z^{-1}X^{+}(z)$$

8.3 Nonzero Initial Condition Problem

- Example: Consider the causal LTI system described by the difference equation

$$y[n] - ay[n-1] = x[n]$$

with $x[n] = Au[n]$ and initial condition $y[-1] \neq 0$.

- Find the time-domain expression of $y[n]$.

8.3 Nonzero Initial Condition Problem

- Applying the unilateral transform to both sides and using the linearity and time-shifting properties, get:

$$Y^+(z) - ay[-1] - az^{-1}Y^+(z) = X^+(z) = \frac{A}{1-z^{-1}}$$

- Solving for $Y^+(z)$ yields:

$$Y^+(z) = \frac{ay[-1]}{1-az^{-1}} + \frac{A}{(1-az^{-1})(1-z^{-1})}$$

$$= \frac{ay[-1]}{1-az^{-1}} + \frac{-\frac{aA}{1-a}}{1-az^{-1}} + \frac{\frac{A}{1-a}}{1-z^{-1}}$$

- Performing the inverse unilateral transform, get

$$y[n] = \underbrace{y[-1]a^{n+1}}_{\text{ZIR}} + \underbrace{\frac{A}{1-a}(1-a^{n+1})}_{\text{ZICR}}$$

Quiz 7

- For the following difference equations and associated input and initial conditions, determine the response y[n] for $n \geq 0$ by using the unilateral z-transform:

$$y[n] + 3y[n-1] = x[n]$$

  – with $x[n] = 0.5^n u[n]$;
  – and $y[-1] = 1$.

Next ...

- The last transform: DFT (Discrete Fourier Transform)
  - DFT vs. DTFT
  - Circular convolution vs Linear convolution