CAN207 Continuous and Discrete Time Signals and Systems

Lecture-11

Laplace Transform _ Part 2

Zhao Wang

Zhao.wang@xjtlu.edu.cn

Room EE322

Content

- 3. Fundamental LT Pairs
  - Commonly seen LT pairs
- 4. Properties of Laplace transform
  - Useful properties (similar to CTFT)
- 5. Inverse Laplace Transform
  - Partial Fraction Expansion
  - ROC determination
- 6. Geometric Evaluation of CTFT based on LT
  - Zero-pole plot
  - Graphical interpretation

3. Fundamental LT pairs

- 1. Unit-impulse $x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

Since it does not depend on the value of s, it converges at every point in the s-plane with no exceptions. So the ROC is the entire s-plane.

- 2. Shifted unit-impulse $x(t) = \delta(t - t_0)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-st} dt = e^{-st_0}$$

  - If $\tau > 0$:
  - If $\tau < 0$:

3. Fundamental LT pairs

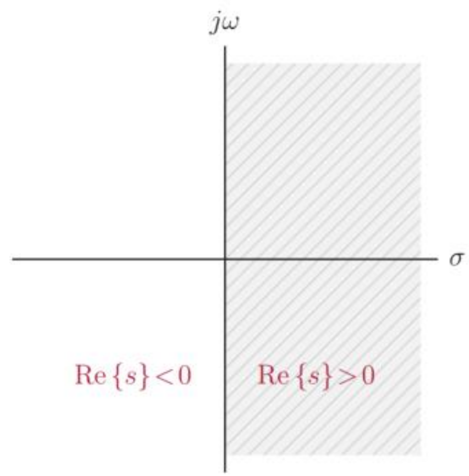
- 3. Unit-step $x(t) = u(t)$

$$X(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty}$$

$$= -\frac{1}{\sigma + j\omega}e^{-(\sigma+j\omega)t}\Big|_0^{\infty} = -\frac{1}{\sigma + j\omega}[0-1] = \frac{1}{\sigma + j\omega} = \frac{1}{s}$$

for the exponential term $e^{-\sigma t}$ to converge as t → ∞, we need σ > 0

- The transform expression is valid only for points on the right half of the s-plane. This region is shown shaded on the right.
  - Note that the transform does not converge at points on the jω axis. It converges at any point to the right of the jω axis regardless of how close to the axis it might be.



3. Fundamental LT pairs

- 4. Causal exponential signal

$$x(t) = e^{-at}u(t)$$

  - When a is real:

$$X(s) = \int_0^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt = -\left.\frac{e^{-(\sigma+a)t}e^{-j\omega t}}{j\omega + (\sigma + a)}\right|_0^{\infty} = \frac{1}{j\omega + \sigma + a} = \frac{1}{s+a}$$

  where the integration converges when $\mathcal{R}e\{s\} = \sigma > -$ a.

  - When a is complex, i.e. $a = a_r + ja_i$, then it changes:

$$X(s) = \frac{1}{a_r + ja_i + \sigma + j\omega} e^{-(\sigma+a_r)t}e^{-j(\omega+a_i)t}\Big|_0^{\infty}$$

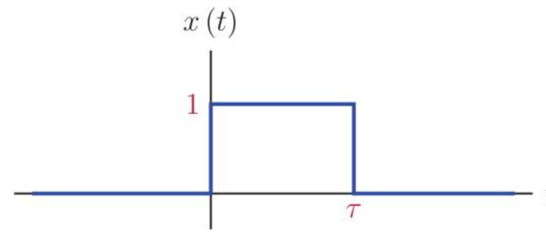
  - To converge, we need $\sigma + a_r > 0$, so the result:

$$X(s) = \frac{1}{s+a}, \mathcal{R}e\{s\} > -\mathcal{R}e\{a\}$$

3. Fundamental LT pairs

- 5. Rectangular pulse signal

$$x(t) = \Pi\left(\frac{t-\tau/2}{\tau}\right)$$



- The Laplace transform of the signal x(t) is computed as

$$X(s) = \int_0^{\tau} 1\, e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_0^{\tau} = \frac{1-e^{-s\tau}}{s}$$

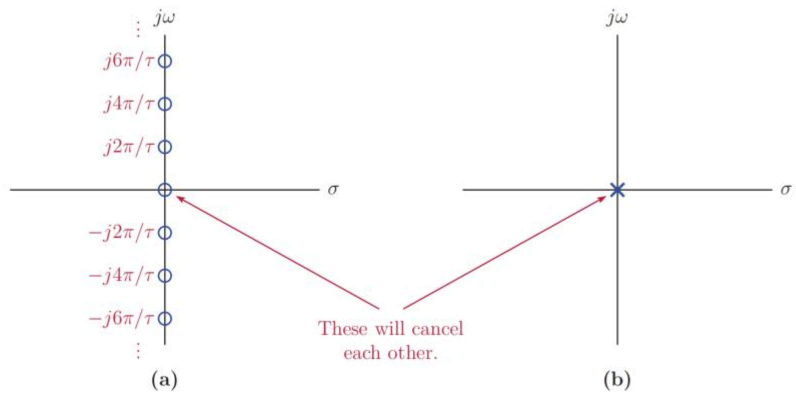
  – At a first glance we may be tempted to think that the transform X(s) might not converge at $s = 0$ since the denominator of X(s) becomes equal to zero at $s = 0$.

  – We must realize, however, that the numerator of X(s) is also equal to zero at $s = 0$.

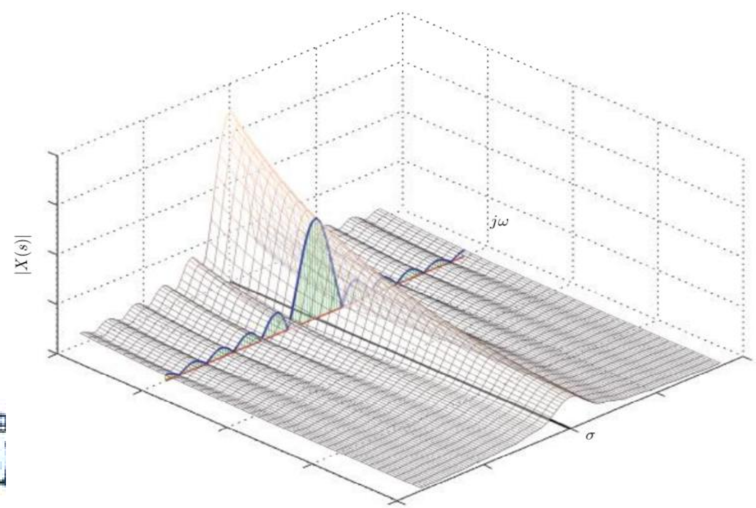
  – Using L'Hospital's rule, $X(s)|_{s=0} = \left.\frac{\tau e^{-s\tau}}{1}\right|_{s=0} = \tau$, so X(s) converges at $s = 0$.

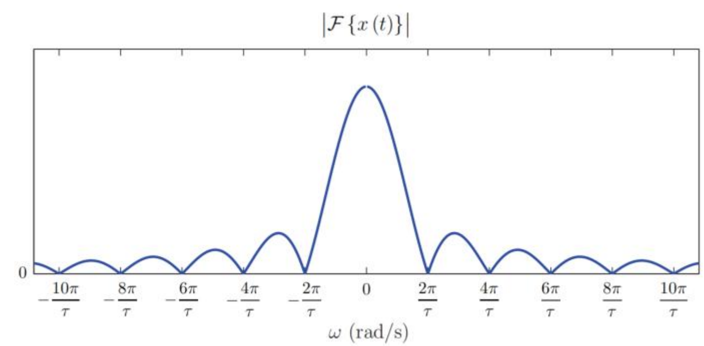
3. Fundamental LT pairs

- 5. Rectangular pulse signal $x(t) = \Pi\left(\frac{t-\tau/2}{\tau}\right)$
- Zero-pole plot:



- Magnitude of LT and FT:





Quiz 4

- Find the Laplace transform of the signal
  1. $x(t) = e^{j\omega_0 t}u(t)$
  2. $x(t) = \cos(\omega_0 t)u(t)$

4. Properties

- 1. Linearity

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} \alpha_1 X_1(s) + \alpha_2 X_2(s)$$

  - ROC: at least the overlap of the two individual ROCs, if such an overlap exists.
    - The ROC may be greater than the overlap of the two regions if the addition of the two transforms results in the cancellation of a pole.

- 2. Time-shifting

$$x(t-\tau) \overset{\mathcal{L}}{\longleftrightarrow} e^{-s\tau} X(s)$$

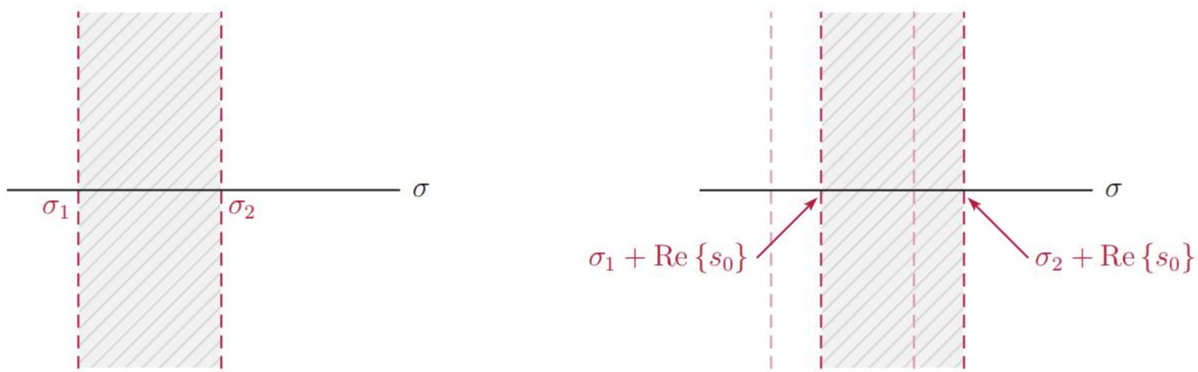
  - ROC: generally same as the ROC of X(s).
    - If the time shift makes a causal signal non-causal then the points $\sigma = \infty$ would need to be excluded from the ROC.
    - Similarly, if an anti-causal signal loses its anti-causal property as the result of a shift, then the points $\sigma = -\infty$ need to be excluded.

4. Properties

- <u>3. Shifting in the s-domain</u>

$$x(t)\, e^{s_0 t} \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0)$$

  - ROC: shifted version of the ROC for X(s), shifted horizontally by an amount equal to the real part of the parameter s_0.



- Example:

$$x(t) = e^{-2t}\cos(3t)u(t)$$

4. Properties

- 4. Scaling in time and s-domain

$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

  - ROC: scaled version of the ROC for X(s).

$$\sigma_1 < \mathrm{Re}\{s\} < \sigma_2 \longrightarrow \sigma_1 < \frac{\mathrm{Re}\{s\}}{a} < \sigma_2$$

  - Depending on the sign of the parameter a, two possibilities need to be considered for the ROC:

a. If $a > 0$: $a\,\sigma_1 < \mathrm{Re}\{s\} < a\,\sigma_2$

b. If $a < 0$: $a\,\sigma_2 < \mathrm{Re}\{s\} < a\,\sigma_1$

- Example:

$$x(t) = e^{2t}u(-t)$$

4. Properties

- <u>5. Differentiation in time domain</u>

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s)$$

  – ROC: at least equal to the ROC of the original transform.
    - If the original transform X(s) has a single pole at s = 0 that sets the boundary of its ROC, then the cancellation of that pole due to multiplication by s causes the ROC of the new transform sX(s) to be larger.

- <u>6. Differentiation in the s-domain</u>

$$t\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{dX(s)}{ds}$$

  – ROC: the same as the ROC of the original transform X(s).

- Example: unit ramp signal

$$r(t) = tu(t)$$

Xi'an Jiaotong-Liverpool University
西交利物浦大學

4. Properties

- <u>7. Convolution property</u>

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)\, X_2(s)$$

  – ROC: at least the overlap of the two individual ROCs, if such an overlap exists.
    - The ROC may be greater than the overlap of the two regions if the addition of the two transforms results in the cancellation of a pole.

- Example: a signal x(t) is fed in a system with the impulse response h(t). Determine the output y(t) using the convolution property.

$$h(t) = e^{-t}u(t)$$
$$x(t) = \delta(t) - e^{-2t}u(t)$$

4. Properties

- 8. Integration property

$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} X(s)$$

  - It can be derived that

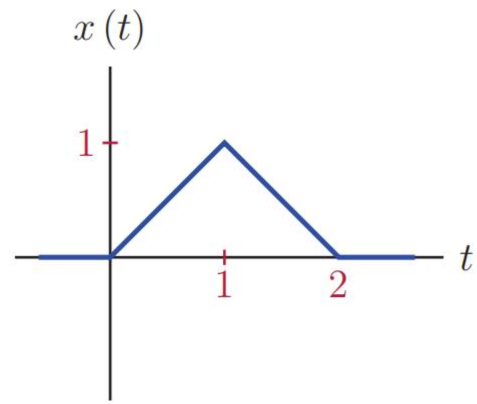
$$\mathcal{L}\left\{\int_{-\infty}^{t} x(\tau) d\tau\right\} = \mathcal{L}\{u(t)\}\mathcal{L}\{x(t)\} = \frac{1}{s} X(s)$$

  - the ROC of X(s) is $\sigma_1 < \mathcal{Re}\{s\} < \sigma_2$
  - the ROC of u(t) is $\mathcal{Re}\{s\} > 0$
  - the ROC of the integrated x(t) must be at least the overlap of the two ROCs given above.
    - It may be larger than the overlap if X(s) has a zero at s = 0 to counter the pole at s = 0 introduced by the transform of the unit-step function.

Quiz 5

- Using the properties of the Laplace transform, determine X(s) for each of the signals listed below. Also indicate the ROC in each case.

  1. $x(t) = \delta(t) + 2e^{-t}u(t)$
  2. $x(t) = e^{2(t+1)}u(-t-1)$
  3. $x(t) = u(t) - 2u(t-1)$
  4. plotted on the right



5.1 Inverse Laplace Transform

- The Inverse Laplace Transform is strictly defined as:

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

  - Strict computation is complicated and rarely used in engineering
- Practically, the Inverse Laplace Transform of a rational function is calculated using a method of table look-up:
  - based on the LT pairs $\mathcal{L}\{Ae^{-at}u(t)\} = \frac{A}{s+a}, \mathcal{Re}\{s\} >- a$ and $\mathcal{L}\{-Ae^{-at}u(-t)\} = \frac{A}{s+a}, \mathcal{Re}\{s\} <- a.$
  - a rational function of LT could be expressed as

$$X(s) = \sum_{i=1}^{N} \frac{A_i}{s+a_i}, \quad s \in ROC$$

  - then its inverse LT is a linear combination of $A_i e^{-a_i t}u(t)$ and $-A_i e^{-a_i t}u(-t)$
  - the ROC will suggest the corresponding time-domain function.

5.2 Partial Fraction Expansion

- Recall the PFE introduced in Lect. 2 (sec. 6)
  - Simpler version:

- Step 1: Factor the bottom (denominator)

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)}$$

- Step 2: Write one partial fraction for each of those factors

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A_1}{x-2} + \frac{A_2}{x+1}$$

- Step 3: Reduce the fractions to a common denominator

$$5x-4 = A_1(x+1) + A_2(x-2)$$

- Step 4: Solve for A_1 and A_2

Root for $(x+1)$ is $x = -1$

$$\begin{aligned} 5(-1) - 4 &= A_1(-1+1) + A_2(-1-2) \\ -9 &= 0 + A_2(-3) \\ A_2 &= 3 \end{aligned}$$

Root for $(x-2)$ is $x = 2$

$$\begin{aligned} 5(2) - 4 &= A_1(2+1) + A_2(2-2) \\ 6 &= A_1(3) + 0 \\ A_1 &= 2 \end{aligned}$$

$$\boxed{\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}}$$

Xi'an Jiaotong-Liverpool University 西交利物浦大学

21

1. It has to be **proper** rational expressions;

2. Single poles, i.e. no higher order of roots on the denominator.

5.2 Partial Fraction Expansion

- Complete version of PFE
- Consider a rational transform in the form

$$X(s) = \frac{B(s)}{(s-p_1)(s-p_2)\dots(s-p_N)}$$

  - where the poles $p_1, p_2, \dots, p_N$ are distinct.
  - the order of the numerator polynomial B(s) is less than the order of the denominator polynomial.
- The transform X(s) can be expanded into partial fractions in the form

$$X(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \dots + \frac{k_N}{s-p_N}$$

  - the coefficients $k_1, k_2, \dots, k_N$ are called the residues of the partial fraction expansion. They can be computed by

$$k_i = (s-p_i)\,X(s)\Big|_{s=p_i}, \qquad i = 1, 2, \dots, N$$

5.2 Partial Fraction Expansion

- Example 1: A causal signal x(t) has the Laplace transform as follows. Determine x(t) using PFE.

$$X(s) = \frac{s+1}{s(s+2)}$$

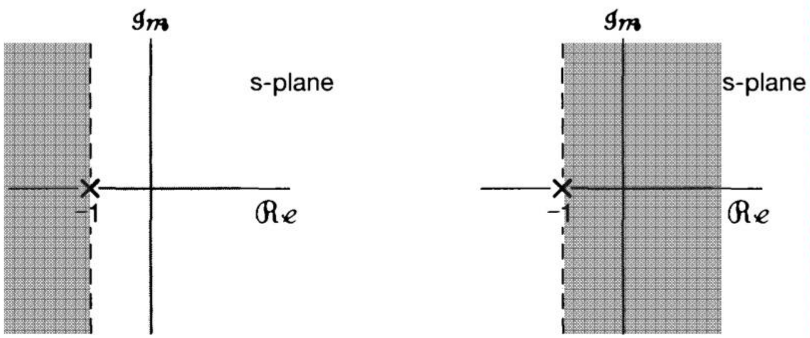
- Example 2: A signal x(t) has the Laplace transform as follows. Determine x(t) using PFE.

$$X(s) = \frac{s+1}{s(s^2+9)}, \mathcal{R}e\{s\} > 0$$

5.3 ROCs' influence

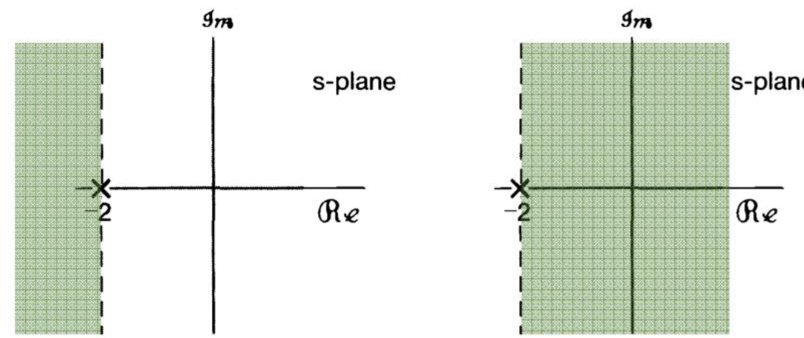
- The ROC of X(s) is the overlapping region of all partial fraction expanded sections.
- Consider an example $X(s) = X_1(s) + X_2(s) = \frac{1}{s+1} - \frac{1}{s+2}$
- the possible ROCs for them are:

For $X_1(s)$



$x_1(t) = -e^{-t}u(-t)$ $\quad$ $x_1(t) = e^{-t}u(t)$

For $X_2(s)$



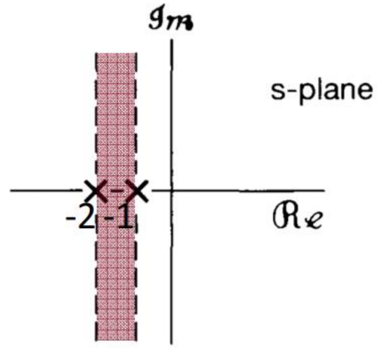
$x_2(t) = -e^{-2t}u(-t)$ $\quad$ $x_2(t) = e^{-2t}u(t)$

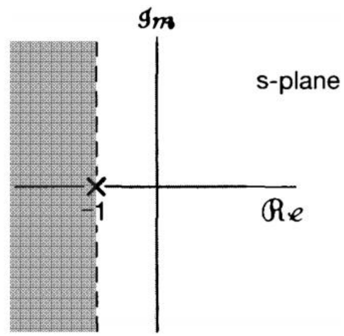
- Given the ROC of X(s), we need to select corresponding ROC_1 and ROC_2 such that $\text{ROC} = \text{ROC}_1 \cap \text{ROC}_2$

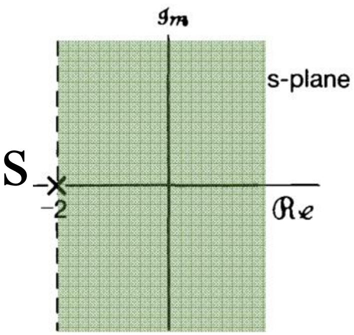
5.3 ROCs' influence

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

- For example, if the ROC of X(s) is the following



- then we know ROC1 is 

 and ROC2 is 

- such that $x(t) =- e^{-t}u(-t) - e^{-2t}u(t)$

Quiz 6

- The Laplace transform of a signal x(t) is

$$X(s) = \frac{5(s-1)}{(s+1)(s+2)(s-2)(s-3)}$$

with the ROC specified as

$$-1 < \mathcal{R}e\{s\} < 2$$

Determine x(t).

6.1 Zero-pole plot

- Recall: a rational function H(s) can be expressed in *zero-pole form* as:

$$H(s) = K\frac{(s-z_1)(s-z_2)\dots(s-z_M)}{(s-p_1)(s-p_2)\dots(s-p_N)}$$

  - The roots $z_1,\dots,z_M$ of the numerator polynomial are referred to as the ***zeros*** of the system function;
  - The roots $p_1,\dots,p_N$ of the denominator polynomial are the ***poles*** of the system function.
- A pole-zero plot for a system function is obtained by marking the poles and the zeros of the system function on the s-plane.
  - o for zeros;
  - x for poles.

Xi'an Jiaotong-Liverpool University 西交利物浦大學

6.1 Zero-pole plot

- Example: Construct a pole-zero plot for a LTIC system with system function (with a pole at -1)

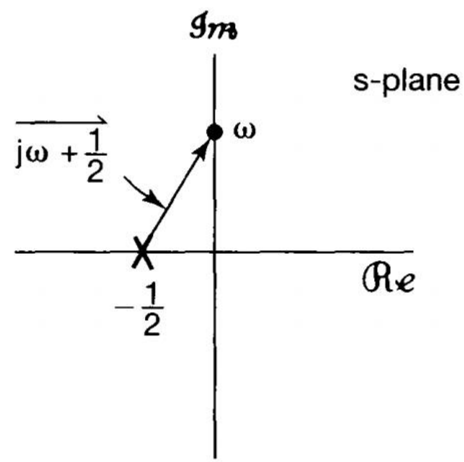
$$H(s) = \frac{s^2 + 1}{s^3 + 5s^2 + 17s + 13}$$

  - If this system is **causal**, indicate its ROC on the zero-pole plot.
  - What if this system is **stable**?

6.2 Geometric Evaluation of the Fourier Transform

- Suppose $H(s) = \frac{1}{s+0.5}, \ -0.5 < \mathcal{Re}\{s\}$;
  - The $j\omega$-axis is included in the ROC, so its FT exists as $H(\omega) = \frac{1}{j\omega+0.5}$;
- In the s-plane, $H(\omega)$ can be represented by the vector pointing from the pole at $(-0.5, 0)$ to a moving point $(0, j\omega)$ on the $j\omega$-axis as ω varies.
- For the magnitude spectrum $|H(\omega)|$:
  - the length of the vector $(0.5, j\omega)$ is $\sqrt{0.5^2 + \omega^2}$;
  - so the magnitude spectrum is the reciprocal of the length of the vector:

$$|H(\omega)| = \frac{1}{\sqrt{0.5^2 + \omega^2}}$$



- For the phase spectrum $\angle H(\omega)$:
  - the phase angle of the vector $(0.5, j\omega)$ is $\tan^{-1}(\omega/0.5)$;
  - so the phase spectrum is:

$$\angle H(\omega) = -\tan^{-1}(2\omega)$$

6.2 Geometric Evaluation of the Fourier Transform

- Assuming the system is stable, the Fourier transform-based system function H(ω) exists, and can be found by evaluating H (s) for $s = j\omega$:

$$H(\omega) = H(s)\Big|_{s=j\omega} = K\frac{(j\omega - z_1)(j\omega - z_2)\ldots(j\omega - z_M)}{(j\omega - p_1)(j\omega - p_2)\ldots(j\omega - p_N)}$$

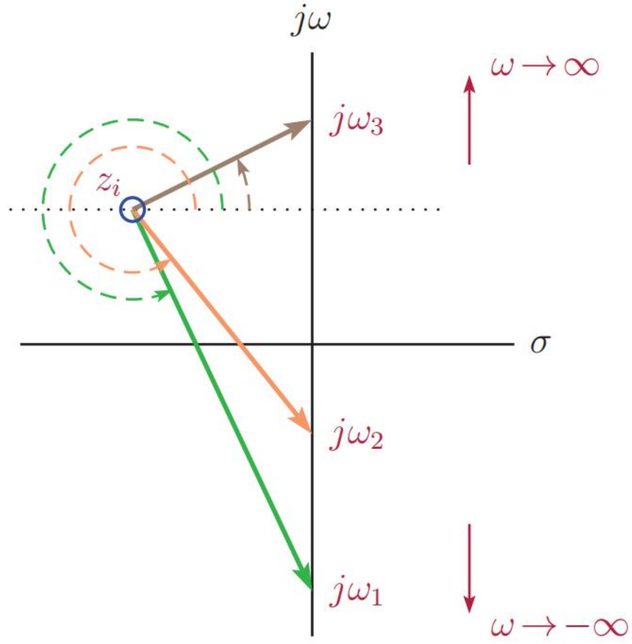
  - ω is a point on the imaginary axis in the s-plane. When ω changes, this point is moving along the $j\omega$-axis from $-\infty$ to ∞;
  - The numerator is represented as M vectors pointing from $(z_i, 0)$ to $(0, j\omega)$, so the vector is $(z_i, j\omega)$;
  - Similarly, the denominator is represented as N vectors $(p_i, j\omega)$;
- The magnitude of the frequency response at $\omega = \omega_0$ is found by

$$|H(\omega_0)| = K\frac{|B_1(\omega_0)|\,.\,|B_2(\omega_0)|\ \ldots\ |B_M(\omega_0)|}{|A_1(\omega_0)|\,.\,|A_2(\omega_0)|\ \ldots\ |A_N(\omega_0)|}$$

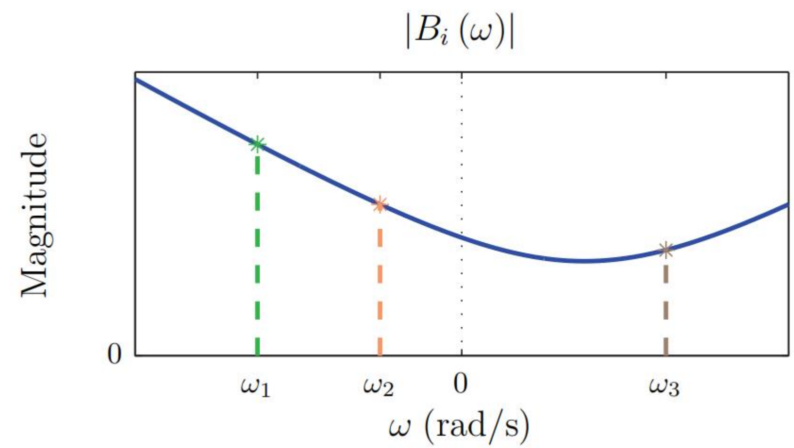
- The phase of the frequency response at $\omega = \omega_0$ is found by

$$\measuredangle H(\omega_0) = \measuredangle B_1(\omega_0) + \measuredangle B_2(\omega_0) + \ldots + \measuredangle B_M(\omega_0) - \measuredangle A_1(\omega_0) - \measuredangle A_2(\omega_0) - \ldots - \measuredangle A_N(\omega_0)$$

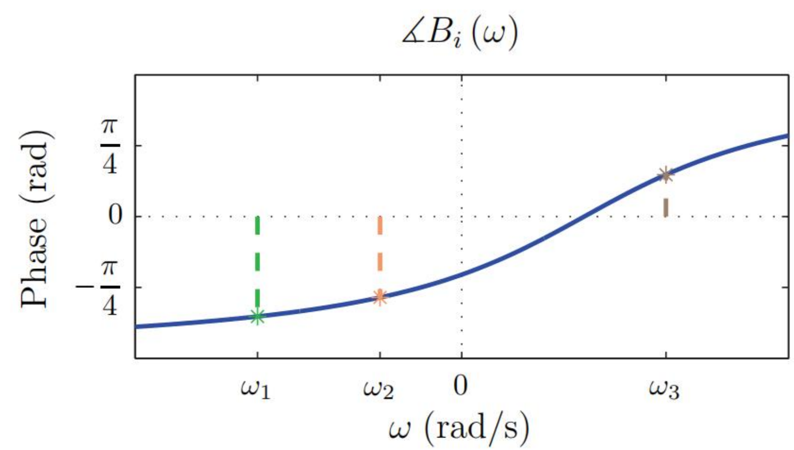
6.2 Geometric Evaluation of the Fourier Transform



(a)



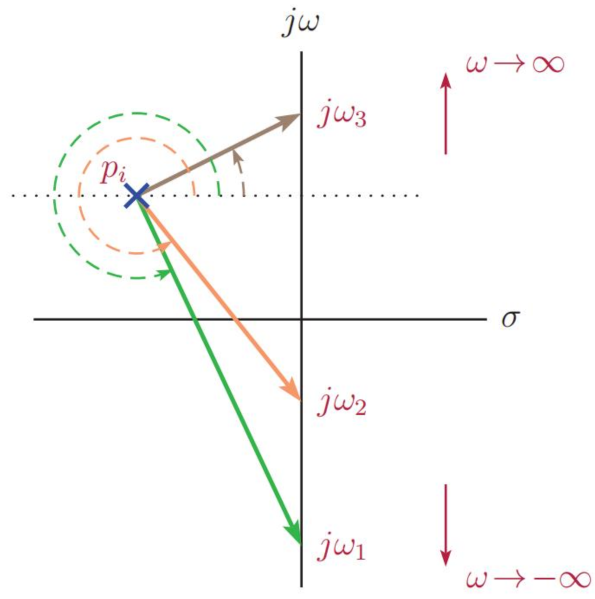
(b)



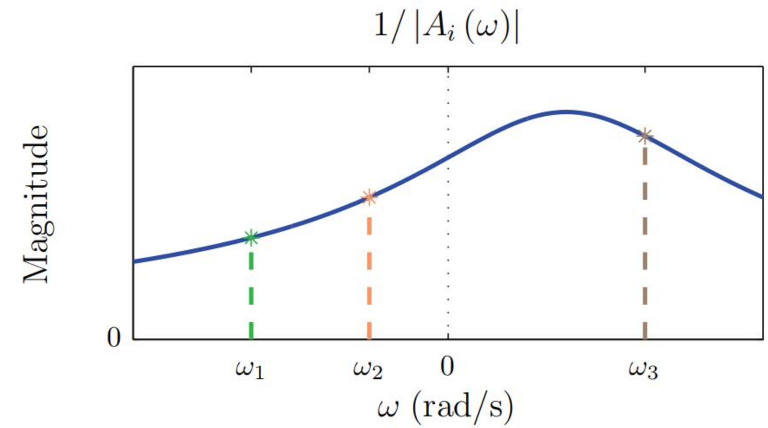
(c)

(a) Moving the tip of the vector for $B_i(\omega)$ on the $j\omega$-axis;
(b) contribution of the zero at $s = z_i$ to the magnitude of the frequency response;
(c) contribution of the zero at $s = z_i$ to the phase of the frequency response.

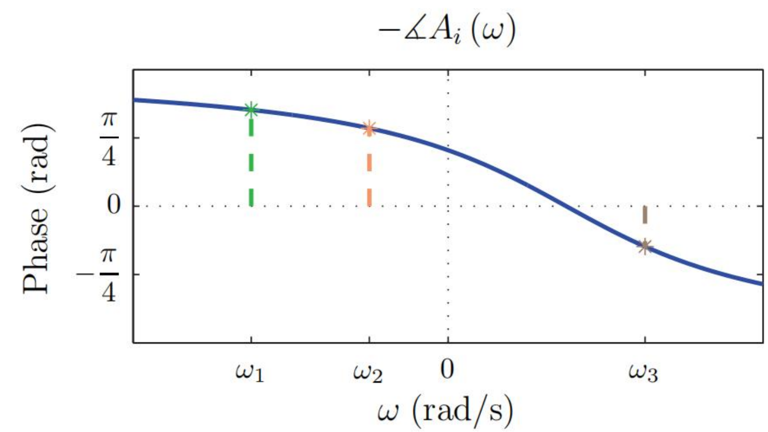
6.2 Geometric Evaluation of the Fourier Transform



(a)



(b)



(c)

(a) Moving the tip of the vector for $B_i(\omega)$ on the $j\omega$-axis;
(b) contribution of the poles at $s = p_i$ to the magnitude of the frequency response;
(c) contribution of the poles at $s = p_i$ to the phase of the frequency response.

Quiz 7

- A LTI system is described by the system function

$$H(s) = \frac{s^2 + s - 2}{s^2 + 2s + 5}$$

  – Construct a pole-zero plot;

  – Use it to determine the magnitude and the phase of the frequency response of the system at the frequency $\omega_0 = 2$ rad/s;

  – Sketch the magnitude and phase characteristics for all frequencies.

Next ...

- No NEW content in week 7
- A revision class on Wednesday