CAN207 Continuous and Discrete Time Signals and Systems

Lecture 15

DTFT Properties and Applications

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Content

- 1. Definitions
  - Frequency mapping
  - Existence of DTFT (convergence)
- 2. DTFT properties
  - Periodicity, Linearity, ...
  - Convolution property
  - Modulation property
  - Duality

1.0 DTFT Definition

- The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is defined by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

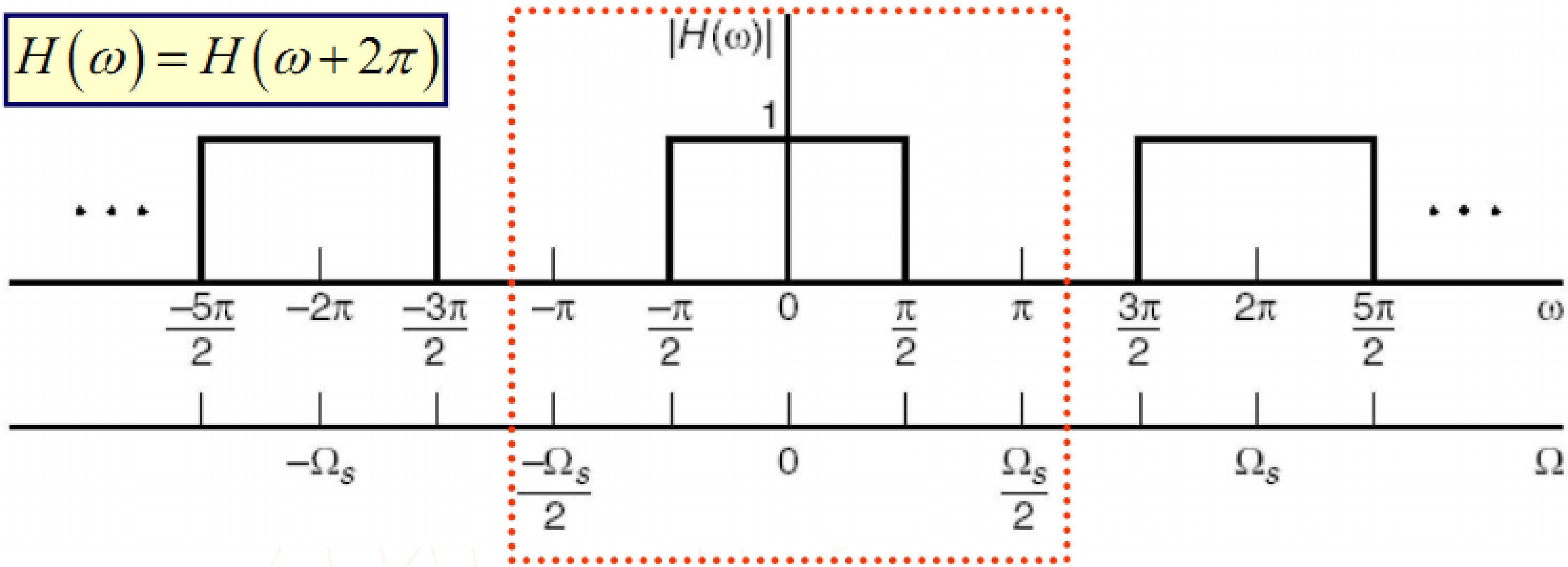
  – DTFT $X(e^{j\omega})$ of a sequence x[n] is a continuous function of ω

- Inverse Discrete-Time Fourier Transform - the Fourier coefficients {x[n]} can be computed from $X(e^{j\omega})$ using

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$$

1.1 Frequency Mapping

$$\boxed{H(\omega) = H(\omega + 2\pi)}$$



– The discrete frequency 2π corresponds to the sampling frequency Ω_s used to sample the original continuous signal x(t) to obtain x[n].

No worries, this will be elaborated in details in the "sampling" lecture later…

1.2 Existence of DTFT

- In the case of finite-length sequences, the sum defining the DTFT has a finite number of terms, thus the DTFT always exists.
- In the general case, where one or both of the limits on the sum in the definition are infinite, the DTFT sum may diverge (become infinite).
- A sufficient condition for the existence of the DTFT of a sequence x[n] is

$$\left|X(e^{j\omega})\right| \leqslant \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Proof:

$$\begin{aligned}
\left|X(e^{j\omega})\right| &= \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|, \\
&\leqslant \sum_{n=-\infty}^{\infty} \left|x[n]e^{-j\omega n}\right|, \\
&= \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}|, \\
&= \sum_{n=-\infty}^{\infty} |x[n]|.
\end{aligned}$$

1.2 Existence of DTFT

- Example: Find the DTFT of $x[n] = r^n e^{j\omega_0 n} u[n]$, and determine the condition to guarantee its existence.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} r^n e^{j\omega_0 n} u[n] e^{-j\omega n},$$

$$= \sum_{n=0}^{\infty} r^n e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=0}^{\infty} \left[r e^{-j(\omega-\omega_0)} \right]^n$$

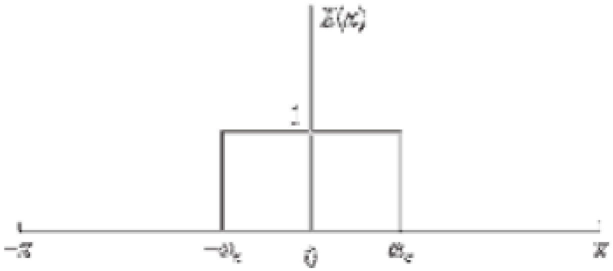
$$= \frac{1}{1 - r e^{-j(\omega-\omega_0)}}, \quad |r e^{-j(\omega-\omega_0)}| < 1$$

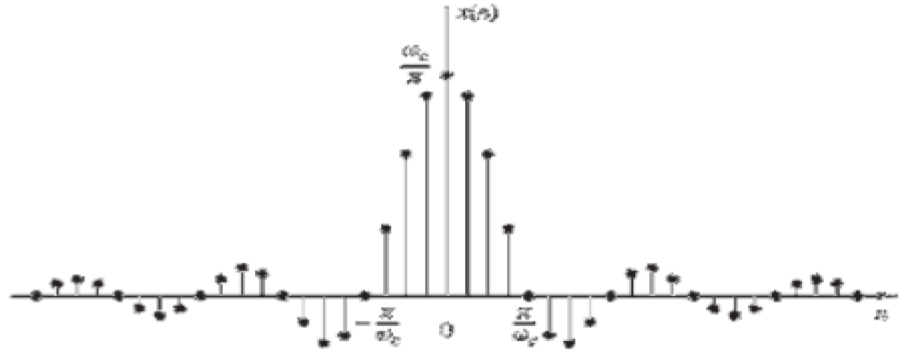
$$= \frac{1}{1 - r e^{-j(\omega-\omega_0)}}, \quad |r| < 1.$$

1.2 Special case

$$X(\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & \omega_c\le|\omega|\le\pi\end{cases} \quad \overset{\mathcal{F}^{-1}}{\Longleftrightarrow} \quad x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\omega)e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}e^{j\omega n}d\omega=\frac{\sin\omega_c n}{\pi n}$$

(with $n\neq 0$)





- The sequence {x(n)} is not absolutely summable $\sum_{n=-\infty}^{\infty}|x(n)|\nless\infty$
- But it is mean-square convergent $\sum_{n=-\infty}^{\infty}|x(n)|^2<\infty$

2.1 DTFT Property - Periodicity

- The DTFT of a discrete sequence is periodic with the period 2π, that is

$$X(\omega) = X(\omega + 2\pi k) \quad \text{for any integer } k$$

- The periodicity of DTFT can be easily verified from the definition:

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega + 2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-j(2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega)n} = X(\omega)$$

2.2 DTFT Property - Linearity

- Linearity
  - Given $x_1[n]$ and $X_1(\omega)$ form a DTFT pair, and $x_2[n]$ and $X_2(\omega)$ form another DTFT pair i.e.

$$x_1[n] \xleftrightarrow{\text{DTFT}} X_1(\omega)$$

$$x_2[n] \xleftrightarrow{\text{DTFT}} X_2(\omega)$$

  - We can show that

$$ax_1[n] + bx_2[n] \xleftrightarrow{\text{DTFT}} aX_1(\omega) + bX_2(\omega)$$

  - Example:

$$x[n] = 0.8^n u[n] + 2(-0.5)^n u[n]$$

2.3 DTFT Property - Time-reversal

- Time-reversal: A reversal of the time domain variable causes a reversal of the frequency variable

$$x[-n] \xleftrightarrow{\text{DTFT}} X(-\omega)$$

2.4 DTFT Properties - Conjugate symmetry

- Conjugation

$$x^*[n] \xleftrightarrow{\text{DTFT}} X^*(-\omega) \qquad x^*[-n] \xleftrightarrow{\text{DTFT}} X^*(\omega)$$

- Conjugate Symmetry
  - 1. If x[n] is real: $X(\omega) = X^*(-\omega)$

$$|X(\omega)| = |X\,(-\omega)| \qquad X_R(\omega) = X_R(-\omega)$$

$$\varphi(\omega) = -\varphi\,(-\omega) \qquad X_I(\omega) = -X_I(-\omega)$$

  - 2. If $x[n] = x_{even}[n] + x_{odd}[n]$

$$x_{even}[n] \xleftrightarrow{\text{DTFT}} X_{real}(\omega)$$

$$x_{odd}[n] \xleftrightarrow{\text{DTFT}} jX_{imag}(\omega)$$

2.5 DTFT Properties - Shifting

- Time Domain Shifting (TD Delay) => FD Phase Shift

$$x[n-M] \xleftrightarrow{\text{DTFT}} e^{-j\omega M} X(\omega)$$

  – Note that the magnitude spectrum is unchanged by time shift.

- Frequency Domain Shifting => TD Phase Shift

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\text{DTFT}} X(\omega - \omega_0)$$

- Example: Find DTFT of $x[n] = A\cos(\omega_0 n + \varphi)\, \alpha^n u[n]$, with $|\alpha| < 1$.

2.6 DTFT Properties - Differencing

- Differencing in Time

$$x[n] - x[n-1] \xleftrightarrow{\text{DTFT}} \left(1 - e^{-j\omega}\right)X(\omega)$$

- Differentiation in Frequency

$$nx[n] \xleftrightarrow{\text{DTFT}} j\frac{dX(\omega)}{d\omega}$$

2.7 DTFT Properties - Parseval Theorem

- Parseval Theorem: The energy of the signal , whether computed in TD or FD, is the same!

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Energy density spectrum of the signal

$$\Updownarrow$$

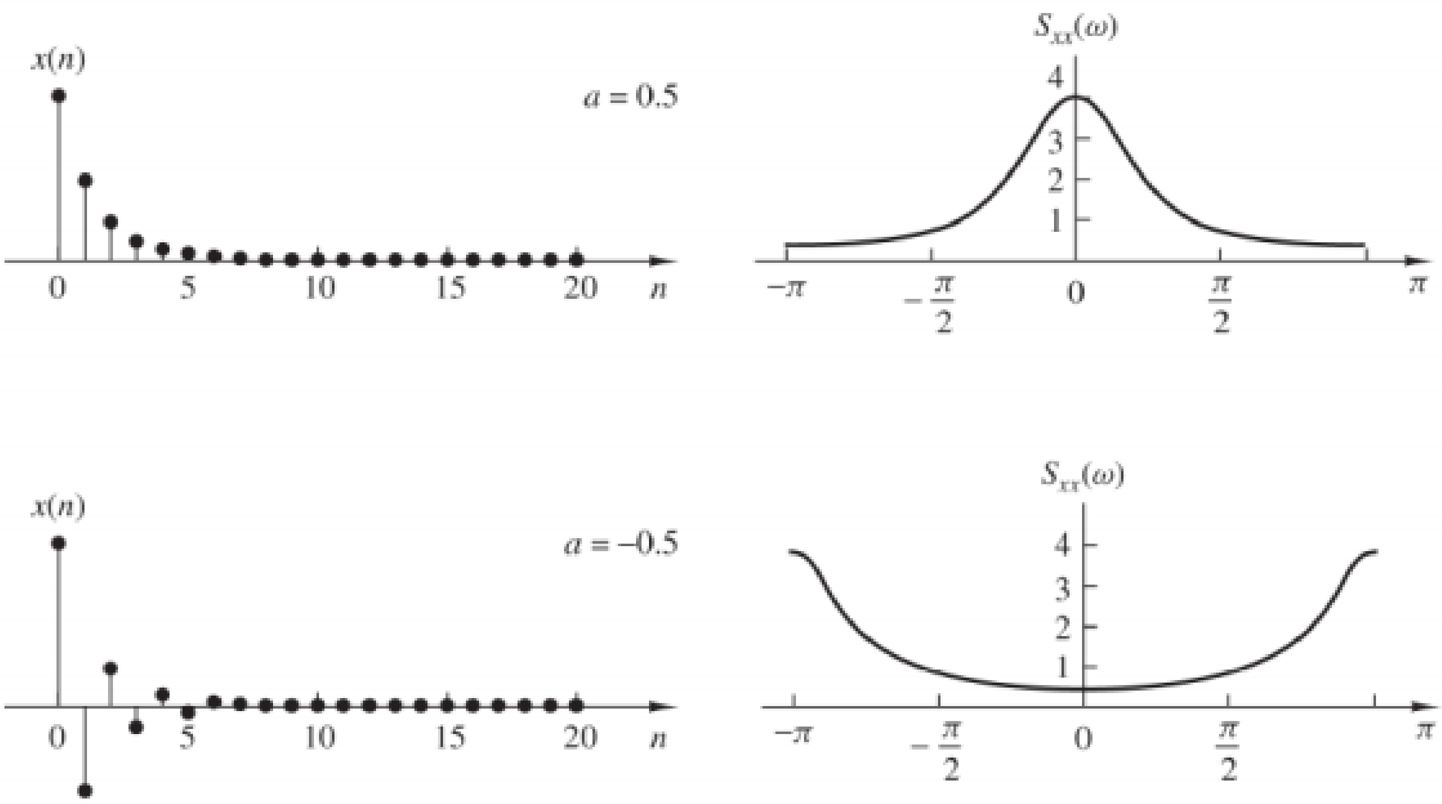
$$\sum_{n=-\infty}^{\infty} x[n]\cdot x^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cdot X^*(\omega) d\omega$$

2.7 DTFT Properties - Parseval Theorem

- Example - Determine and sketch the energy density spectrum of the signal

$$x(n) = a^n u(n), \qquad -1 < a < 1$$

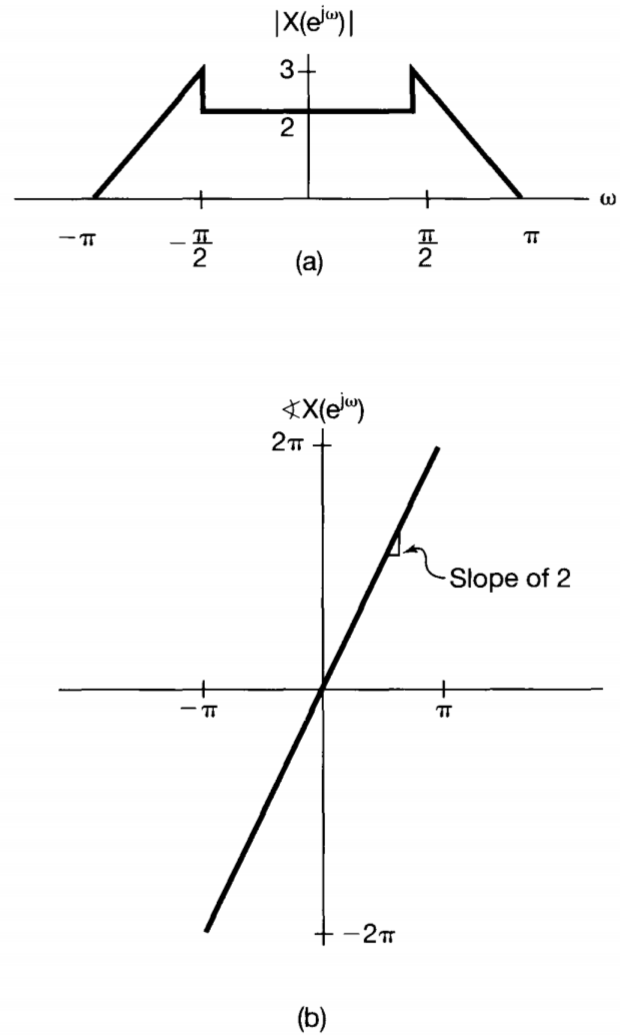
- Result:



(a) (b)

Quiz 1

- Example: Consider the sequence x[n] whose Fourier transform $X(e^{j\omega}) = X(\omega)$ is depicted for $-\pi \leq \omega \leq \pi$.
- Determine whether or not, in the time domain, x[n] is periodic, real, even, and/or of finite energy.



2.8 DTFT Properties - Convolution Property

- Convolution in TD = multiplication in FD

$$y[n] = x[n] * h[n] \xleftrightarrow{\text{DTFT}} X(\omega) \cdot H(\omega) \ = Y(\omega)$$

  - The convolution property illustrates the system's response to input, in TD, the output is the result of convolution, and in FD, the output is the result of multiplication;
  - By designing the $H(\omega)$ carefully, we can pass certain frequency components, then make $|H(\omega)| \cong 1$; stop certain frequency components, i.e. $|H(\omega)| \cong 0$. This is the concept of ***FILTERING***

2.8 DTFT Properties - Convolution Property

- Example: Consider an LTI system with impulse response

$$h[n] = a^n u[n], \qquad |a| < 1$$

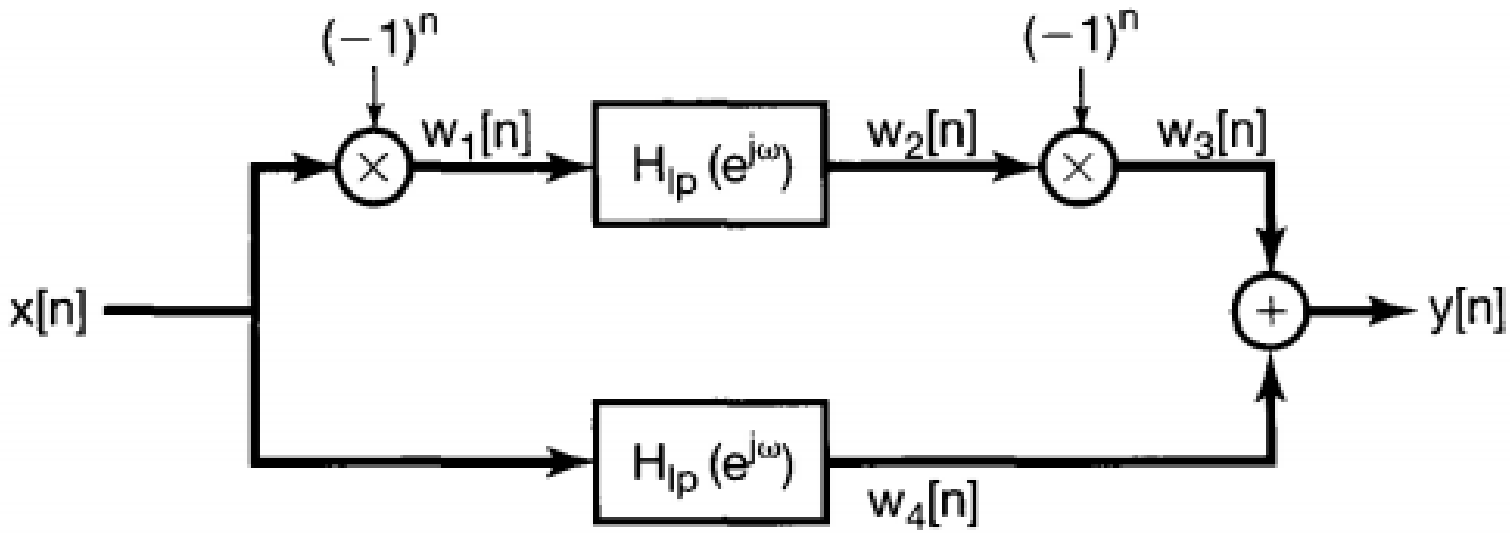
- Suppose that the input to this system is

$$x[n] = \beta^n u[n], \qquad |\beta| < 1$$

- Find the output $y[n]$.

2.8 DTFT Properties - Convolution Property

- Example: Consider the system shown in below figure with input *x[n]* and output *y[n]*. The LTI systems with frequency response $H_{1p}(\omega)$ are ideal lowpass filters with cutoff frequency $\pi/4$ and unity gain in the passband.
- Find the overall frequency response of this system.



(a)

2.8 DTFT Properties - Multiplication Property

- Multiplication in TD = convolution integral in FD

$$x[n] \cdot h[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\gamma) \cdot H(\omega - \gamma) d\gamma$$

  - h[n] can be considered as either system impulse response or another signal;
  - This property is also called the modulation property, since it involves the modulation of one signal x[n] with the other h[n];

2.8 DTFT Properties - Multiplication Property

- Example: Find the Fourier transform $X(e^{j\omega})$ of a signal $x[n] = x_1[n]\, x_2[n]$ where:

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$$

2.9 Duality - DTFS (Optional)

- Since the Fourier series coefficients ak of a periodic signal x[n] are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series.
- The duality property for DTFS implies that the Fourier series coefficients for the periodic sequence a_k are the values of (1/N)x[-n] (i.e., are proportional to the values of the original signal, reversed in time).
- Example:

$$x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ \frac{5}{9}, & n = \text{multiple of } 9 \end{cases} \quad \xleftrightarrow{\text{DTFS}} \quad a_k = \begin{cases} 1/9, & |k| \le 2 \\ 0, & 2 < |k| \le 4 \end{cases}$$

with period N=9

$$g[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & 2 < |n| \le 4. \end{cases} \quad \xleftrightarrow{\text{DTFS}} \quad b_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ \frac{5}{9}, & k = \text{multiple of } 9 \end{cases}$$

with period N=9

2.9 Duality - DTFT (Optional)

- Recall: Duality for CTFT
  - For a transform pair

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

    - implies that

$$X(t) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega)$$

  - Time domain and frequency domain are symmetric.
    - This property suggests if signal A's frequency spectrum is signal B, then signal B's frequency spectrum takes a form similar to signal A.
  - Using linear frequency f instead of angular frequency ω, there is:

$$X(t) \xleftrightarrow{\mathcal{F}} x(-f)$$

- for DTFT
  - the duality is between the DTFT and CTFS

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega,$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n},$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_0 t}dt.$$

2.9 Duality - DTFT (Optional)

- Example:

$$g(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & T_1 < |t| \le \pi \end{cases} \xleftrightarrow{\text{CTFS}} a_k = \frac{\sin(kT_1)}{k\pi}$$

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}. \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \pi/2 \\ 0 & \pi/2 < |\omega| \le \pi \end{cases}$$

2.9 Duality - Summary (Optional)

- Summary

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

duality

Quiz 2

- If $X(e^{j\omega})$ is the DTFT of the sequence $x[n] = \{3, 1, -4, 0, -5, 2, 1;\ -4 \le n \le 2\}$. Calculate the values of following expressions without calculating the DTFT:
  - (a) $X(e^{j0})$;
  - (b) $X(e^{j\pi})$;
  - (c) $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$;
  - (d) $\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$;
  - (e) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega$.

Next ...

- DTFT pairs
- Inverse DTFT
- DTFT of LTID systems
- Concept of filtering