CAN207 Continuous and Discrete Time Signals and Systems

Lecture 14

Discrete-Time Fourier Series and Transform

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Review

	Time Domain			Frequency Domain		
	Periodic	Continuous	Finite	Periodic	Continuous	Finite
CTFS						
CTFT						
Laplace						
DTFS						
DTFT						
z-trans.						
DFT						

Content

- 1. Discrete-Time Fourier Series (for periodic sequences)
  - Review the concepts of eigenfunction for LTI systems
  - Definition of DTFS
  - Examples

- 2. Discrete-Time Fourier Transform (for aperiodic sequences)
  - From DTFS to DTFT
  - Definition of DTFT
  - Examples
  - DTFT of periodic signals (optional)

Recall Lect.5_p.4

- LTI systems possess the ***superposition property***.
  - Input (linearly combined) → Output (linearly combined)
- Strategy:
  - Decompose input signal into a linear combination of *basic signals*;
  - Choose basic signals so that responses are easy to compute.
- Basic signals?

delayed impulses	⟷	convolution	in Time Domain
complex exponentials	⟷	Fourier analyses	in Frequency Domain

For CT signals: recall Lect.7_p.10

$$\overset{\phi_k(t)}{\underset{e^{j\Omega_k t}}{\longrightarrow}} \boxed{h(t)\overset{\mathcal{F}}{\leftrightarrow}H(\Omega)} \overset{\lambda\phi_k(t)}{\underset{H(\Omega_k)e^{j\Omega_k t}}{\longrightarrow}}$$

- The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude

$$\phi_k(t) = e^{j\Omega_k t} \rightarrow H(\Omega_k)e^{j\Omega_k t}$$

  – where the complex amplitude factor $H(\Omega_k)$ is a function of the frequency Ω_k.

  – Proof:

$$e^{j\Omega_k t} \rightarrow \int_{-\infty}^{\infty} h(\tau)e^{j\Omega_k (t-\tau)}d\tau = \underbrace{\boxed{e^{j\Omega_k t}}}_{\text{eigen-function}}\underbrace{\boxed{\int_{-\infty}^{\infty} h(\tau)e^{-j\Omega_k \tau}d\tau}}_{H(\Omega_k)\ \text{eigen-value}}$$

  - A signal for which the system output is a (possibly complex) constant times the input is referred to as an ***eigen-function*** of the system, and the amplitude factor is referred to as the system's ***eigen-value***.

1.1 Eigenfunction for DT signals

- Consider a set of basic signals (complex exponentials)
- Then the LTID (Linear Time-Invariant Discrete) system has the eigenfunction property:

$$\underset{e^{j\omega_k n}}{\overset{\phi_k[n]}{\longrightarrow}} \boxed{h[n] \overset{\mathcal{F}}{\leftrightarrow} H(\omega)} \underset{H(\omega_k)e^{j\omega_k n}}{\overset{\lambda\phi_k[n]}{\longrightarrow}}$$

If we put a complex exponential into the system

The response is a complex exponential at the same complex frequency, and multiplied by an appropriate factor / constant (depending on the frequency)

- Proof:

$$e^{j\omega_k n} \xrightarrow{convolution} \sum_{r=-\infty}^{\infty} h[r]e^{j\omega_k(n-r)} = e^{j\omega_k n} \underbrace{\sum_{r=-\infty}^{\infty} h[r]e^{-j\omega_k r}}_{}$$

frequency response of the system $\leftarrow H(\omega_k)$

1.2 Definition of DTFS

$$CTFS: x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

- For a discrete-time periodic signal:
  - $x[n]$: periodic
  - period: N
  - fundamental frequency: $\omega_0 = \frac{2\pi}{N}$
  - fundamental harmonics (with fundamental frequency ω_0): $e^{jk\omega_0 n}$
    - $e^{jk\omega_0 n}$ are all periodic with the common period N
    - although the fundamental period of each is different
  - $x[n]$ could be built as a linear combination of these $e^{jk\omega_0 n}$ as:

$$x[n] = \sum_k a_k e^{jk\omega_0 n}$$

    - For CTFS: this decomposition involves infinite terms, i.e. $k \in (-\infty, \infty)$
    - For DTFS: the range for k here is different, due to a significant difference between CT and DT exponentials.

1.2 Definition of DTFS - Synthesis equation

- Complex exponentials
  - CT version $e^{jk\Omega_0 t}$ is periodic for t, but not periodic for k;
  - DT version $e^{jk\omega_0 n}$ is both periodic for n and for k

$$e^{jk\omega_0 n} = e^{jk\omega_0 (n+N)} = e^{j(k+N)\omega_0 n}$$

    - So the range of k is from 0 up to N-1, N values in total;
    - There are only N distinct complex exponentials.

- Therefore, the linear combination of complex exponentials:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

synthesis equation of DTFS

N equations of N unknowns

  - only relates to N distinct complex exponentials;
  - a_k only has N distinct values;
  - k is ranging over N continuous integers, typically 0 to N-1.

1.2 Definition of DTFS - Analysis equation

- The weighting a_k of each frequency component:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}$$

analysis equation of DTFS

could be solved from the N equations of N unknowns

  - a_k only has N distinct values;
  - $a_k = a_{k+N}$
  - Comparing with CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \quad \text{and} \quad a_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}dt$$

    - To build up x(t), infinite complex exponentials are needed
      - for DTFS, only N complex exponentials are needed
    - As the inverse to the summation, the analysis eq. is an integration
      - for DTFT, the inverse to the summation is still a summation

1.2 Definition of DTFS - Summary

- Synthesis equation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

- Analysis equation

$$a_k = \sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}$$

- Main difference to CT: periodicity
- Reason for the difference:
- Convergence: no issue, always exists

x[n]	periodic in n
$e^{jk\omega_0 n}$	periodic in n
$e^{jk\omega_0 n}$	periodic in k
a_k	periodic in k

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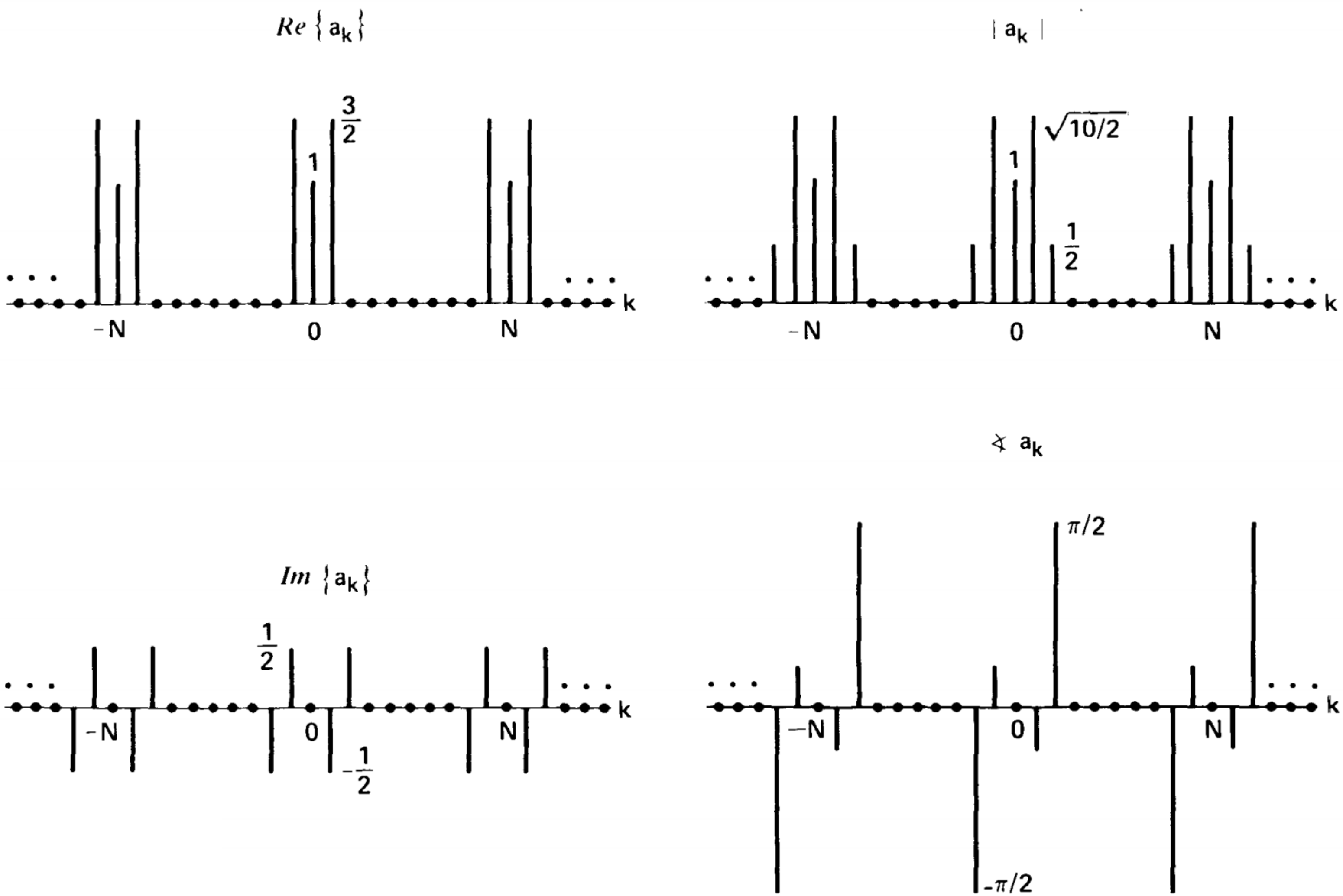
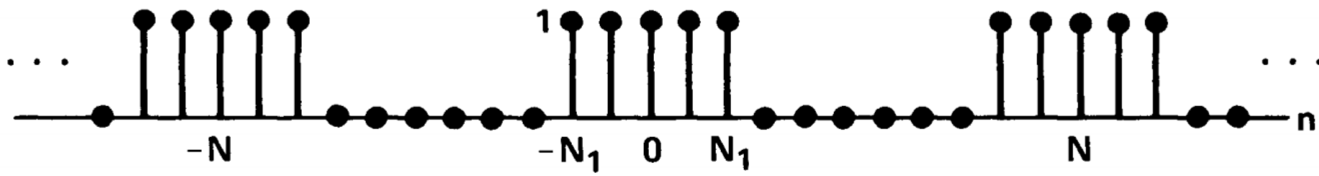
1.3 DTFS - Example 1

$$x[n] = 1 + \sin\omega_0 n + 3\cos\omega_0 n + \cos(2\omega_0 n + \pi/2)$$



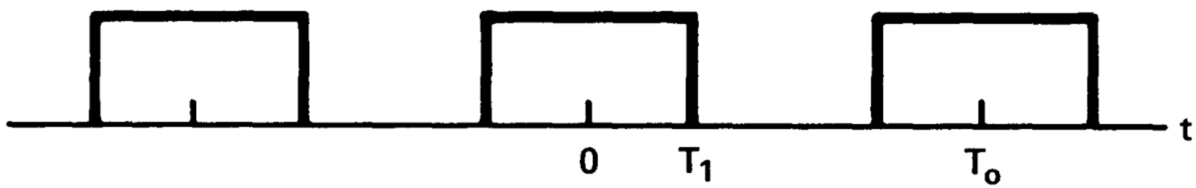
1.3 DTFS - Example 2

- The DT square sequence



$$Na_k = \left.\frac{\sin[(2N_1+1)\cdot\frac{\omega}{2}]}{\sin(\frac{\omega}{2})}\right|_{\omega=\frac{2\pi k}{N}}$$

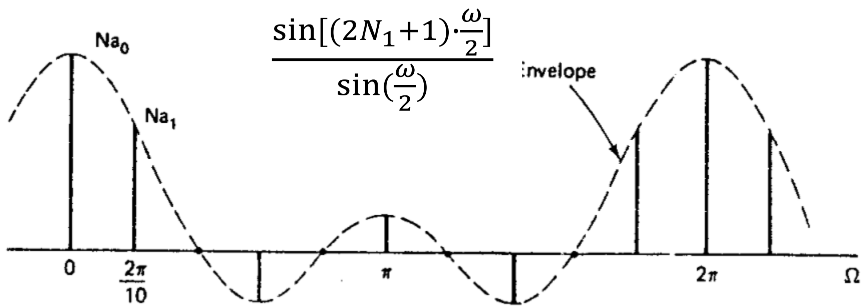
- Comparing with the CT-square wave:



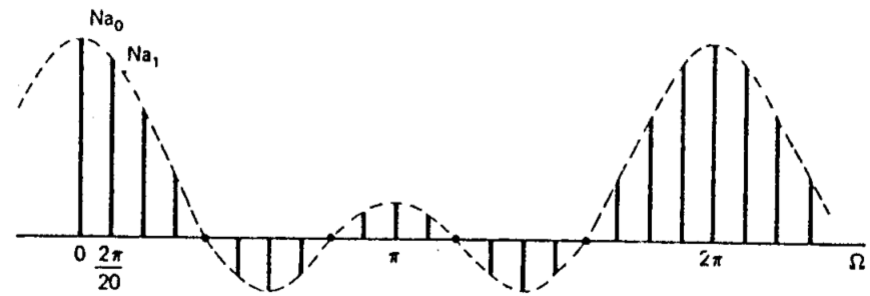
$$T_0 a_k = \left.\frac{2\sin\Omega T_1}{\Omega}\right|_{\Omega=\frac{2\pi k}{T_0}} = \left.\frac{\sin[2T_1\cdot\frac{\Omega}{2}]\Omega}{\frac{\Omega}{2}}\right|_{\Omega=\frac{2\pi k}{T_0}}$$

1.3 DTFS - Example 2

- DTFS a_k as samples of an envelope:

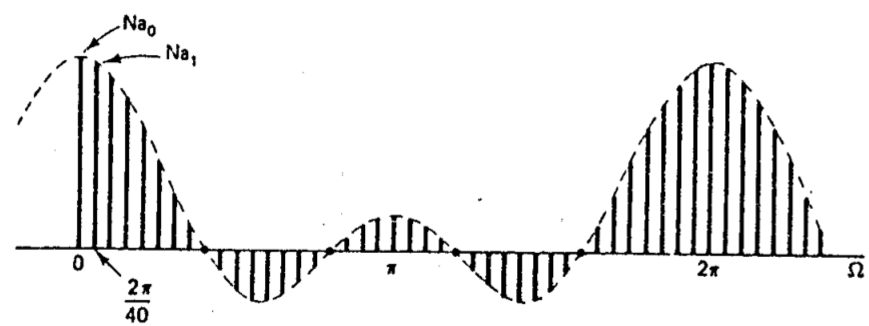


N=10



N=20

As the period increases, the envelope remains the same and the samples representing the Fourier series coefficients become more closely spaced.



N=40

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Quiz 1

- Considering a DT impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

- Find its DTFS and draw the spectrum.

2.1 Develop DTFT from DTFS

- Recall the Continuous-Time approach:

1. $x(t)$ APERIODIC

   - construct periodic signal $\tilde{x}(t)$ for which one period is $x(t)$

   - $\tilde{x}(t)$ has a Fourier series

   - as period of $\tilde{x}(t)$ increases, $\tilde{x}(t) \rightarrow x(t)$ and Fourier series of $\tilde{x}(t) \rightarrow$ Fourier Transform of $x(t)$

2.1 Develop DTFT from DTFS

- For the Discrete-time, exactly the same:

1. $x[n]$ APERIODIC

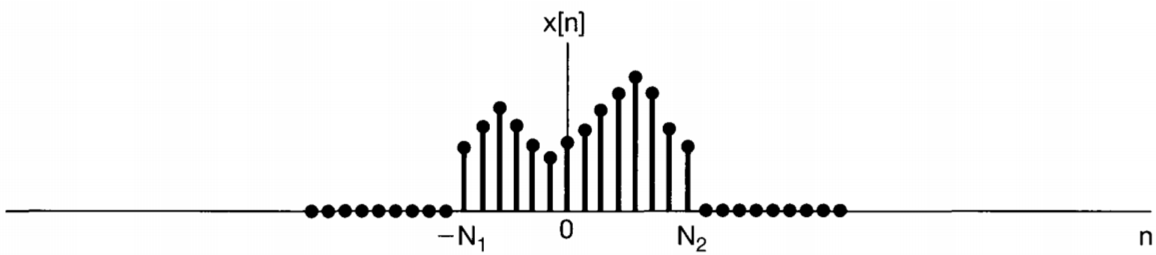
   - construct periodic signal $\tilde{x}[n]$ for which one period is $x[n]$

   - $\tilde{x}[n]$ has a Fourier series

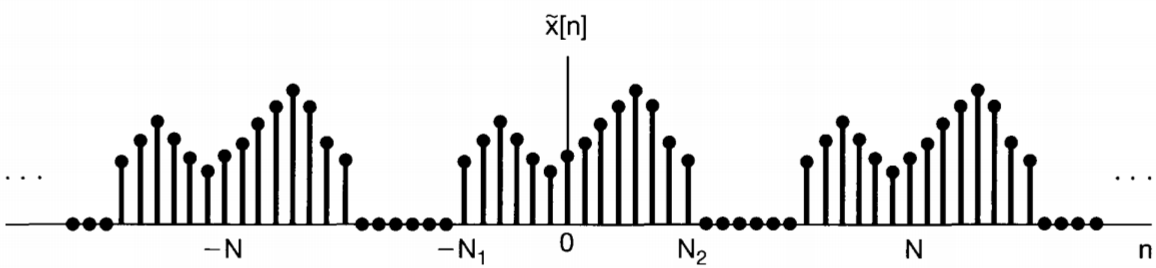
   - as period of $\tilde{x}[n]$ increases,
     $\tilde{x}[n] \rightarrow x[n]$ and Fourier series of
     $\tilde{x}[n] \rightarrow$ Fourier Transform of $x[n]$

2.1 Develop DTFT from DTFS

- Consider a general aperiodic sequence x[n] that is of finite duration (Figure a).
- Construct a periodic sequence $\tilde{x}[n]$ for which x[n] is one period (Figure b)



(a)



(b)

$\tilde{x}[n] = x[n] \quad |n| < \frac{N}{2}$

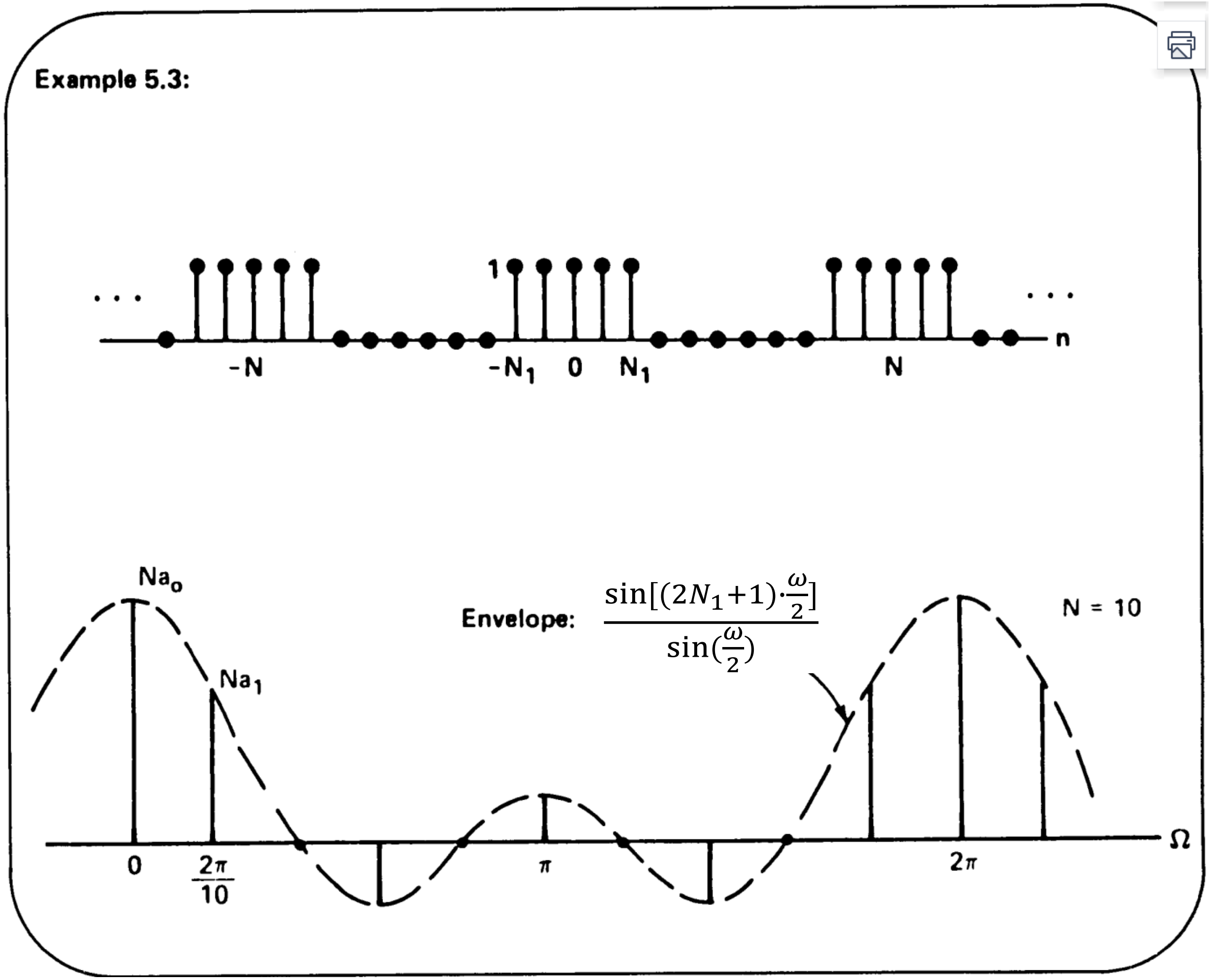
As $N \rightarrow \infty \quad \tilde{x}[n] \rightarrow x[n]$

– let $N \rightarrow \infty$ to represent x[n]

– use Fourier series to represent $\tilde{x}[n]$

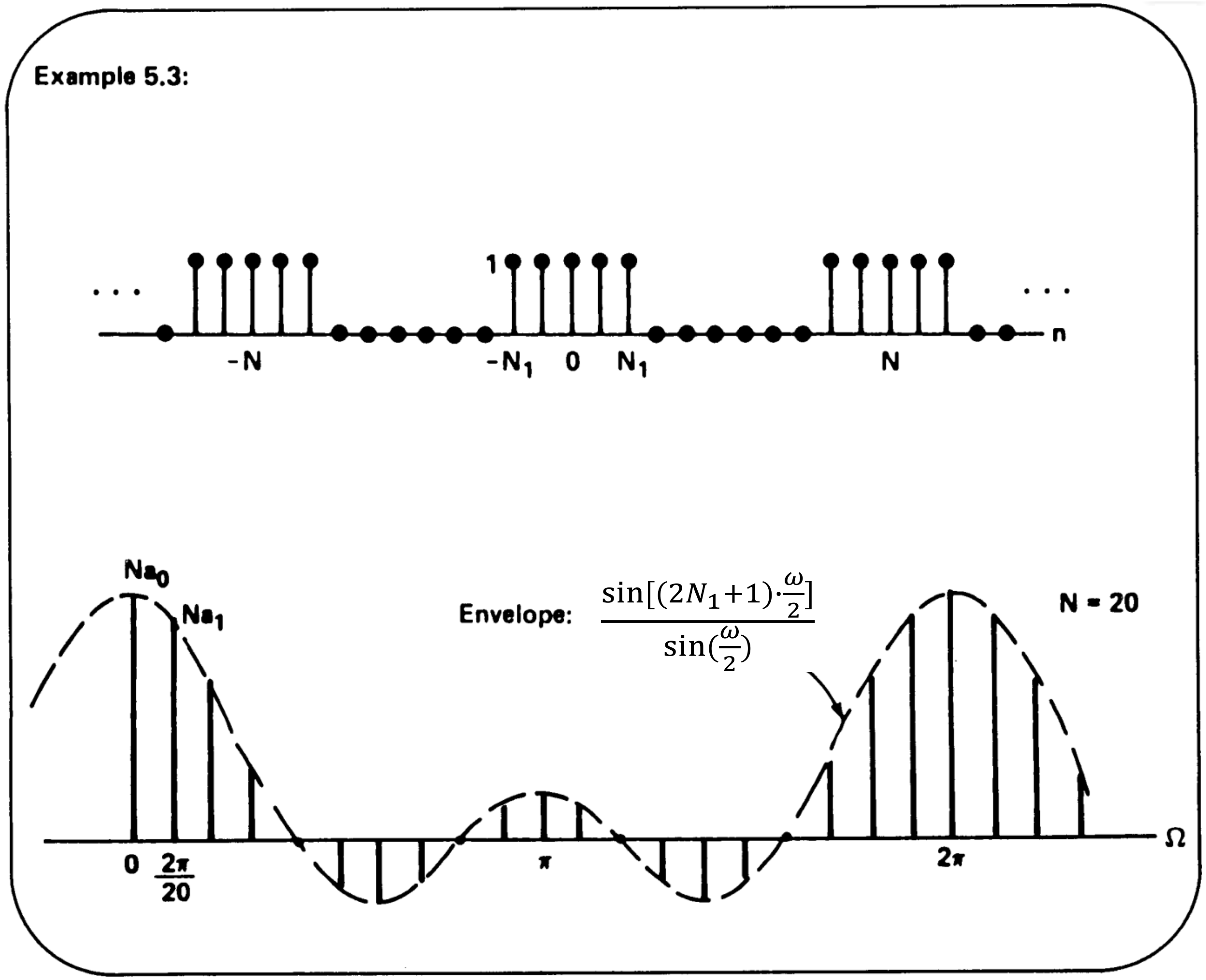
2.1 Develop DTFT from DTFS

- Consider an aperiodic square sequence x[n]



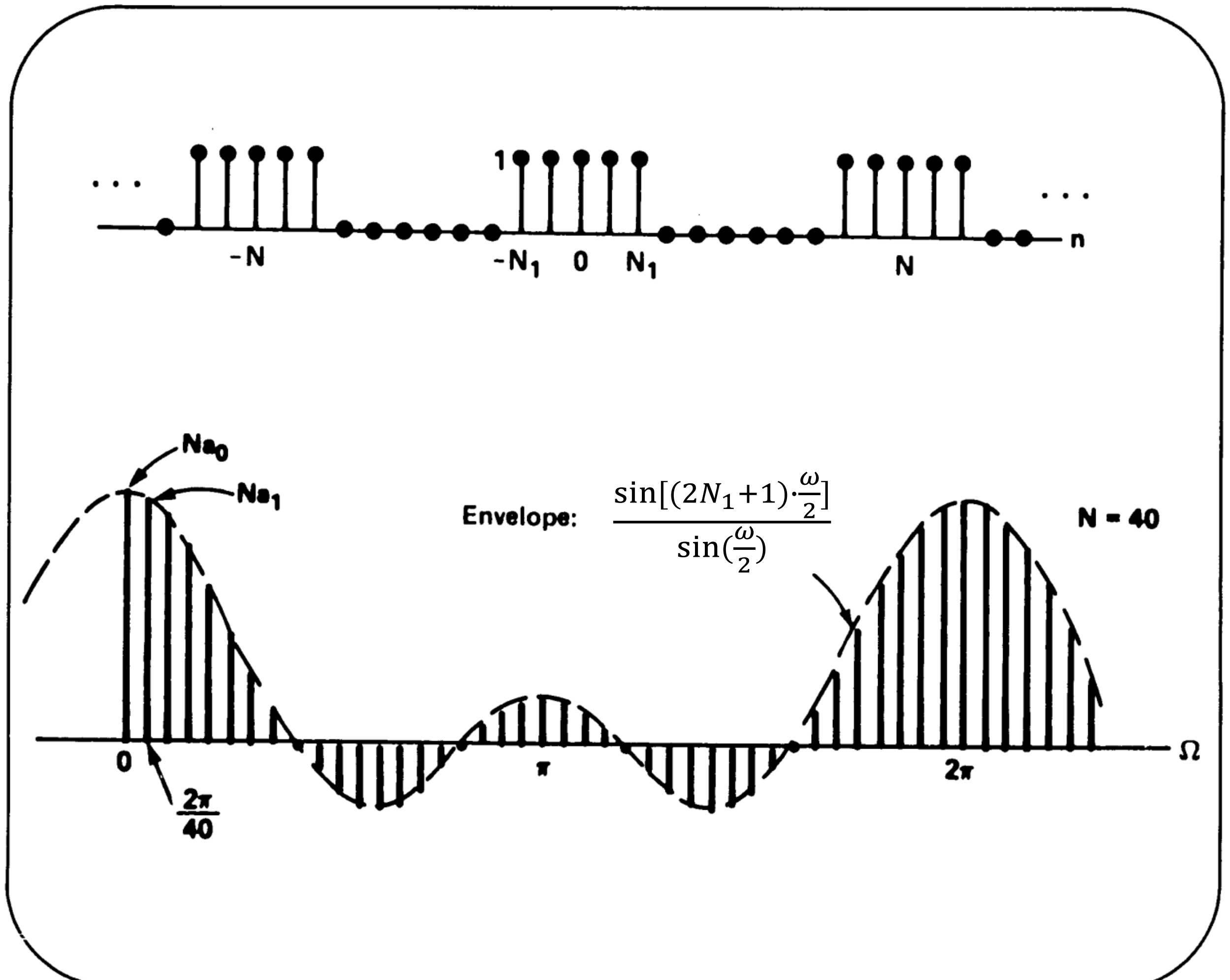
2.1 Develop DTFT from DTFS

- Consider an aperiodic square sequence x[n]



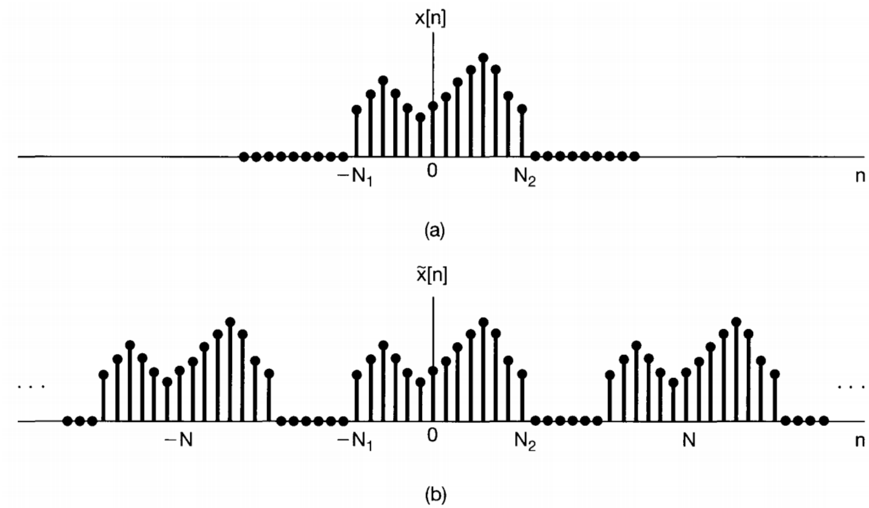
2.1 Develop DTFT from DTFS

- Consider an aperiodic square sequence x[n]



2.2 Definition of DTFT

- For the periodic sequence $\tilde{x}[n]$



$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n}$$

- Since $x[n] = \tilde{x}[n]$ over a period, so $\tilde{x}[n]$ can be replaced by $x[n]$ in the summation:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n}$$

  – Notice x[n] is zero outside the interval $-N_1 \le n \le N_2$.

2.2 Definition of DTFT (cont.)

- Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ Ananlysis equation of DTFT

- The coefficients a_k are proportional to samples of $X(e^{j\omega})$, i.e.

$$a_k = \frac{1}{N}X(e^{jk\omega_0})$$

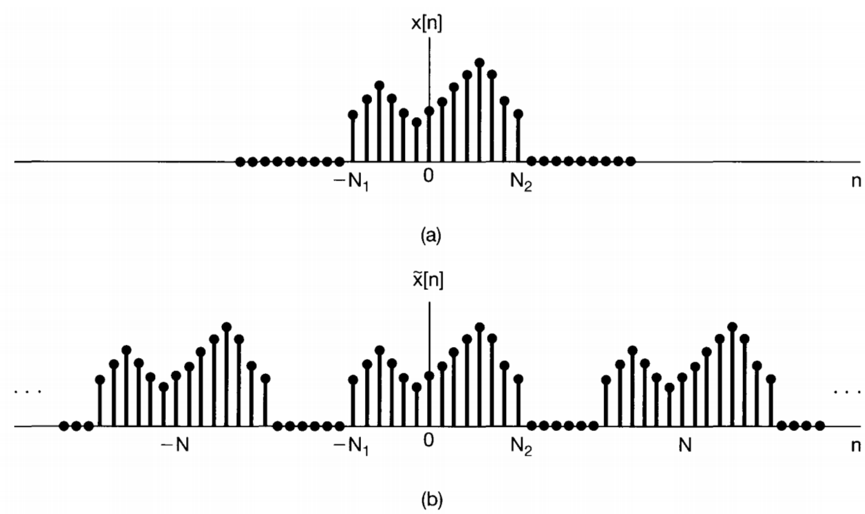
  – where $\omega_0 = \frac{2\pi}{N}$ is the spacing of the samples in the frequency domain.

- Then the periodic sequence $\tilde{x}[n]$ has

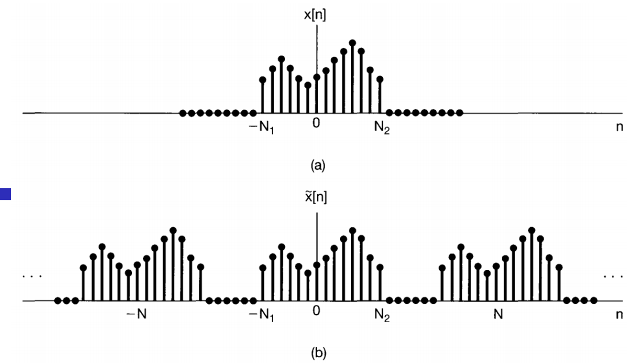
$$\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}$$

  – since $\omega_0 = \frac{2\pi}{N}$, it can be rewritten as

$$\tilde{x}[n] = \frac{1}{2\pi}\sum_{k=\langle N\rangle} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0$$

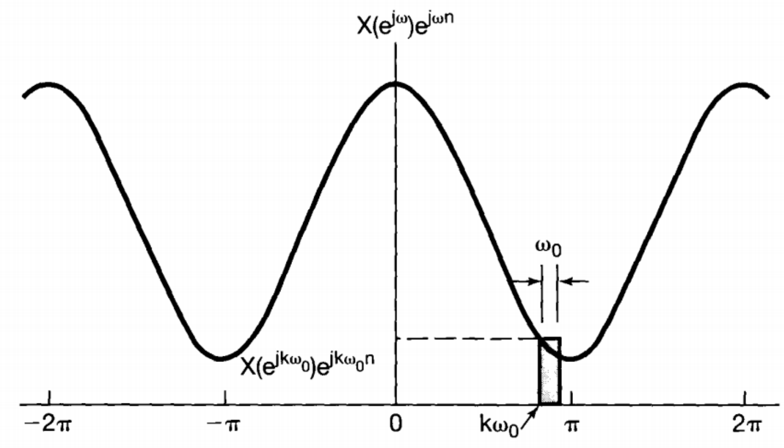


2.2 Definition of DTFT (cont.)



$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

- Consider $X(e^{j\omega})e^{j\omega n}$ as sketched:
  - $X(e^{j\omega})$ is periodic in ω with period 2π.



- As $N \to \infty$, summation changes to integral, $k\omega_0 \to \omega$, $\omega_0 \to d\omega$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

Synthesis equation of DTFT

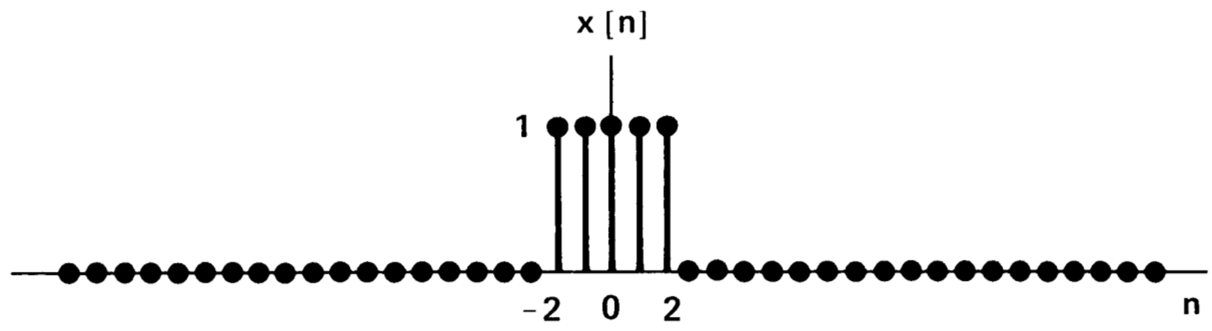
- Combining the pre-defined:

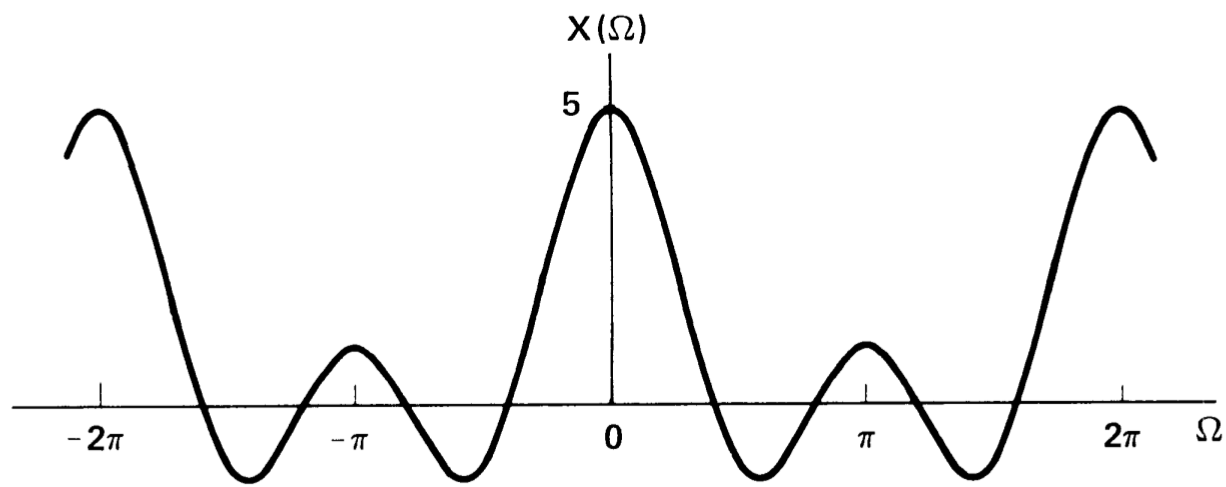
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Analysis equation of DTFT

2.3 DTFT - Example 1

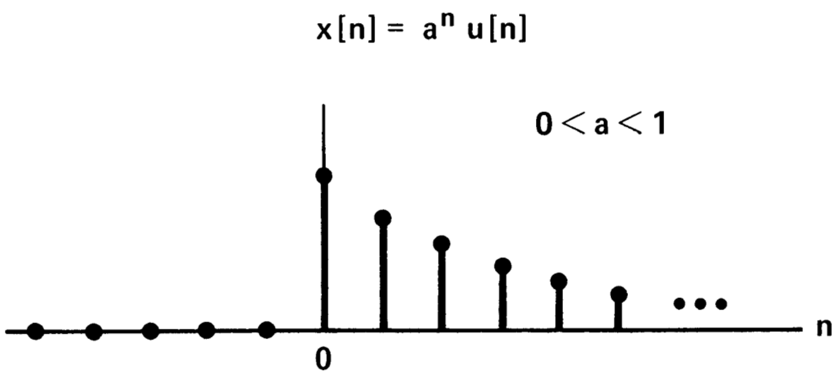
- The square sequence

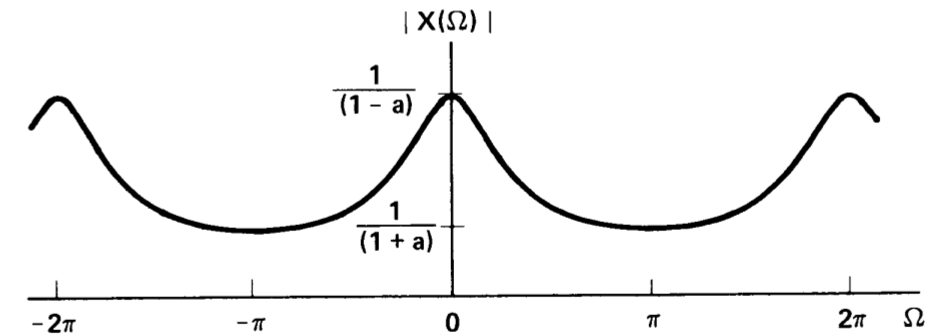


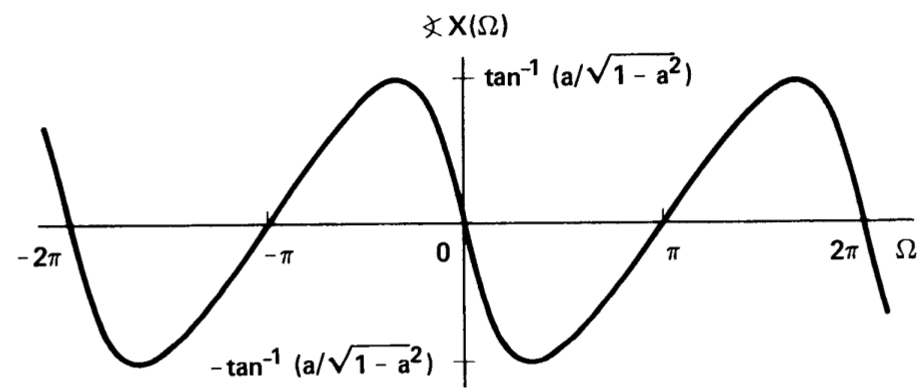


2.3 DTFT - Example 2

- The real exponential sequence







2.4 DTFT of periodic signals (optional)

- Recall the CTFS and CTFT relationship in Lect. 8, p. 18

2. $\tilde{x}(t)$ PERIODIC, $x(t)$ REPRESENTS ONE PERIOD

   - Fourier series coefficients of $\tilde{x}(t)$

     = $(1/T_o)$ times samples of Fourier transform of $x(t)$

3. $\tilde{x}(t)$ PERIODIC

   - Fourier transform of $\tilde{x}(t)$ defined as impulse train:

$$\tilde{X}(\Omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \, \delta(\Omega - k\Omega_0)$$

2.4 DTFT of periodic signals (optional)

- DTFT of a periodic signal can also be obtained from DTFS

2. $\tilde{x}[n]$ PERIODIC, $x[n]$ REPRESENTS ONE PERIOD

   - Fourier series coefficients of $\tilde{x}[n]$

     = (1/N) times samples of Fourier

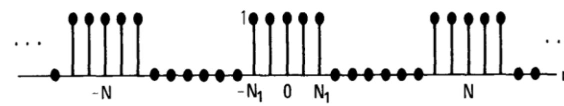
     transform of $x[n]$

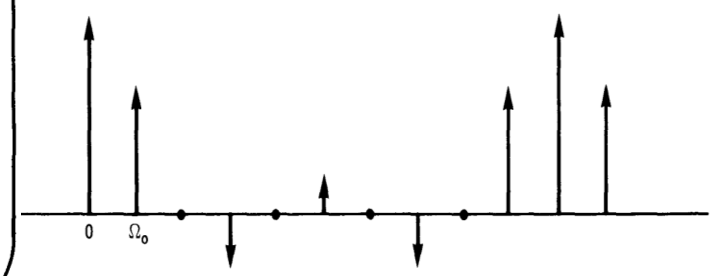
3. $\tilde{x}[n]$ PERIODIC

   - Fourier transform of $\tilde{x}[n]$ defined as

     impulse train:

$$\tilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \, \delta(\omega - k\omega_0)$$



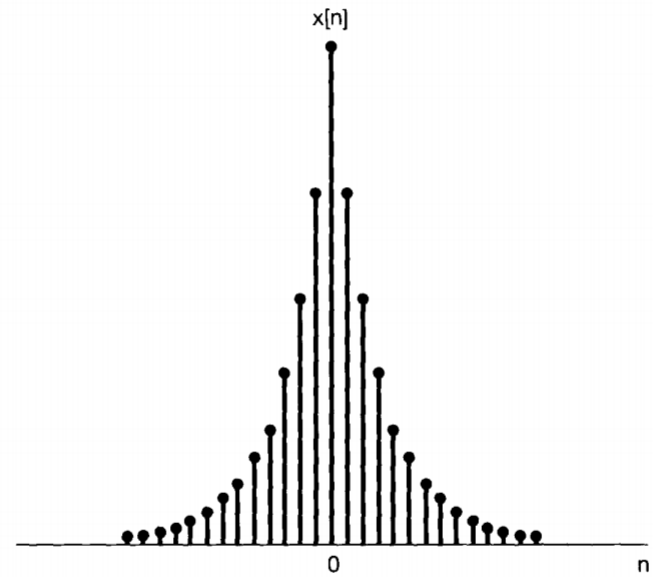


Quiz 2

- A signal x[n] is given as

  $$x[n] = a^{|n|}, \qquad |a| < 1$$

- Find its Fourier transform and sketch the magnitude.



Next ...

- More about Discrete-Time Fourier Transfrom
  - Properties
  - Commonly used pairs
  - Frequency spectrum of discrete-time systems