EEE103 ELECTRICAL CIRCUITS

WEEK4-HANDY CIRCUIT ANALYSIS TECHNIQUES

Ye Wu
ye.wu@xjtlu.edu.cn
Xi'an Jiaotong-Liverpool University
西交利物浦大学

CONTENT

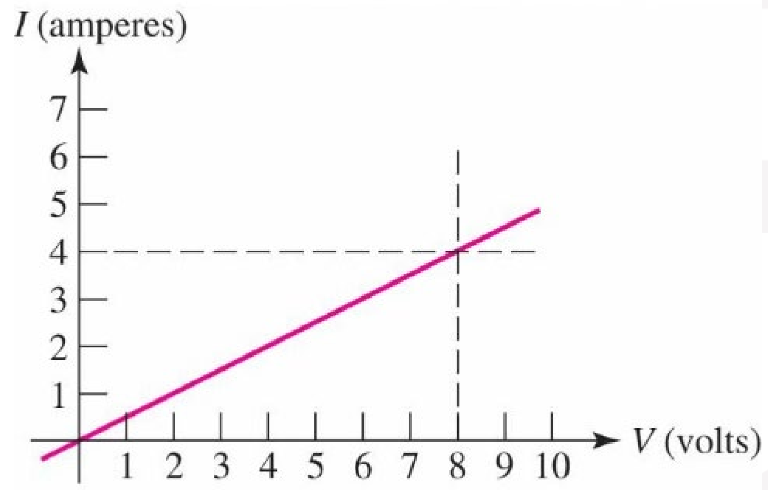
- Linearity and Superposition
- Source Transformations
- Thévenin and Norton Equivalent Circuits
- Maximum Power Transfer
- Delta-Wye Conversion

Linear Elements and Circuits

Linear element: Passive element has a linear voltage-current relationship:

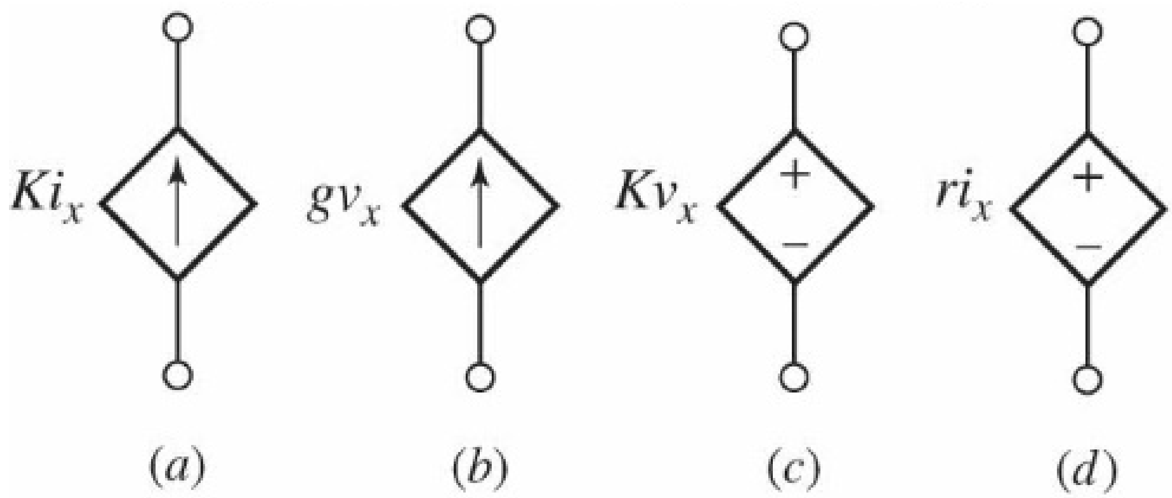
- if $i(t)$ produces $v(t)$, then $Ki(t)$ produces $Kv(t)$
- if $i_1(t)$ produces $v_1(t)$ and $i_2(t)$ produces $v_2(t)$, then $i_1(t) + i_2(t)$ produces $v_1(t) + v_2(t)$,
- *R*esistors are linear elements

$$v(t) = Ri(t)$$



Linear Elements and Circuits

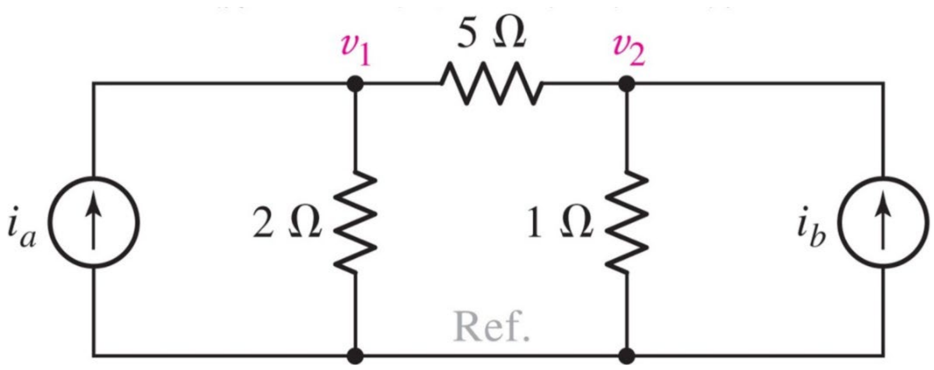
Linear dependent source: dependent current or voltage source whose output current or voltage is proportional only to the first power of a specified current or voltage variable in the circuit (or to the sum of such quantities).



Linear circuit: A circuit has only independent sources, linear dependent sources, and linear elements

The Superposition Concept[1]

For the circuit shown, we have 2 independent source i_a, i_b



Question: How much of v_1 is due to source *a*, and how much is because of source *b*?

If the two sources are i_a, i_b, we get v_1, v_2

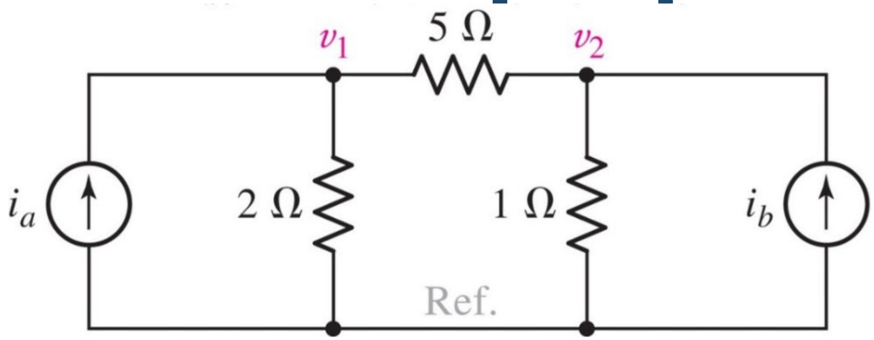
Apply KCL to node 1: $\frac{v_1 - v_2}{5} + \frac{v_1}{2} - i_a = 0$

Apply KCL to node 2: $\frac{v_2 - v_1}{5} + \frac{v_2}{1} - i_b = 0$

$$\rightarrow \begin{cases} 0.7v_1 - 0.2v_2 = i_a \\ -0.2v_1 + 1.2v_2 = i_b \end{cases}$$

Sources i_a, i_b: forcing functions;
Nodal voltages v_1, v_2: response functions(or simply responses).

The Superposition Concept[1]



$$\rightarrow \begin{cases} 0.7v_1 - 0.2v_2 = i_a \\ -0.2v_1 + 1.2v_2 = i_b \end{cases}$$

Experiment X: i_{ax}, i_{bx}, new v_{1x}, v_{2x} $\rightarrow \begin{cases} 0.7v_{1x} - 0.2v_{2x} = i_{ax} \\ -0.2v_{1x} + 1.2v_{2x} = i_{bx} \end{cases}$

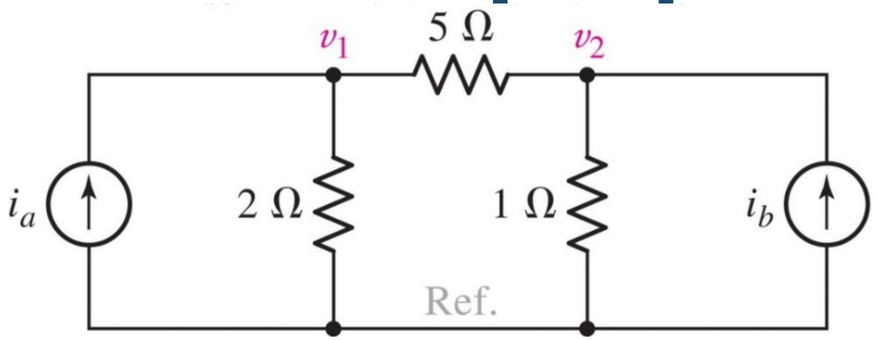
Experiment Y: i_{ay}, i_{by}, new v_{1y}, v_{2y} $\rightarrow \begin{cases} 0.7v_{1y} - 0.2v_{2y} = i_{ay} \\ -0.2v_{1y} + 1.2v_{2y} = i_{by} \end{cases}$

Add X and Y: $\rightarrow \begin{cases} 0.7(v_{1x} + v_{1y}) - 0.2(v_{2x}+v_{2y}) = i_{ax} + i_{ay} \\ -0.2(v_{1x} + v_{1y}) + 1.2(v_{2x}+v_{2y}) = i_{bx} + i_{by} \end{cases}$

If we have: $i_{ax} + i_{ay} = i_a$ & $i_{bx} + i_{by} = i_b$

$$\rightarrow v_1 = v_{1x} + v_{1y}, v_2 = v_{2x} + v_{2y}$$

The Superposition Concept[1]



$$\rightarrow \begin{cases} 0.7v_1 - 0.2v_2 = i_a \\ -0.2v_1 + 1.2v_2 = i_b \end{cases}$$

Experiment X: $i_{ax} = i_a,\ i_{bx} = 0$

$$\rightarrow \begin{cases} 0.7v_{1x} - 0.2v_{2x} = i_a \\ -0.2v_{1x} + 1.2v_{2x} = 0 \end{cases}$$

Experiment Y: $i_{ay} = 0,\ i_{by} = i_b$

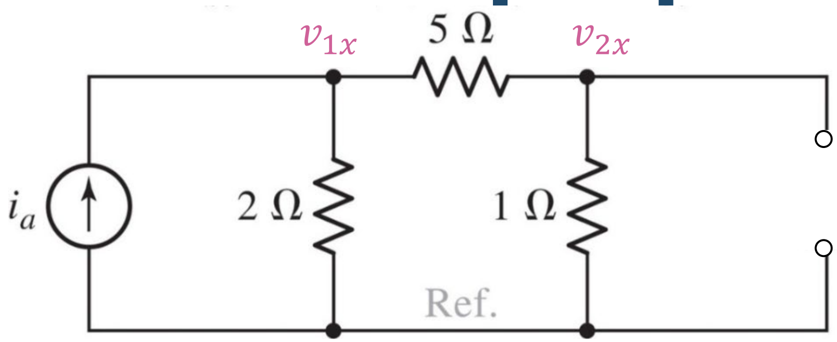
$$\rightarrow \begin{cases} 0.7v_{1y} - 0.2v_{2y} = 0 \\ -0.2v_{1y} + 1.2v_{2y} = i_b \end{cases}$$

Add X and Y:

$$\rightarrow \begin{cases} 0.7(v_{1x} + v_{1y}) - 0.2(v_{2x}+v_{2y}) = i_a \\ -0.2(v_{1x} + v_{1y}) + 1.2(v_{2x}+v_{2y}) = i_b \end{cases}$$

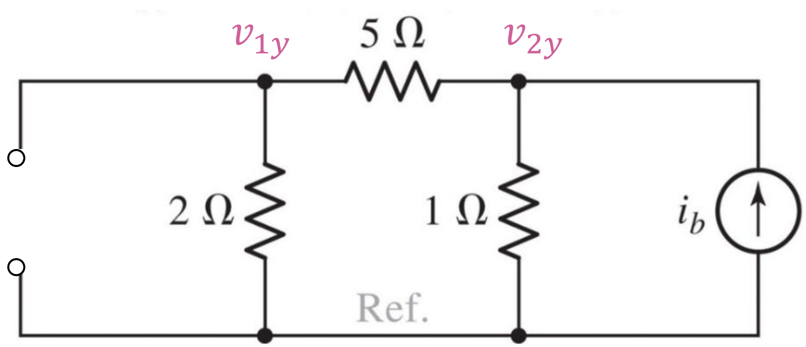
$\rightarrow v_1 = v_{1x} + v_{1y}\ , v_2 = v_{2x} + v_{2y}$

The Superposition Concept 1



If : $i_{ax} = i_a,\ i_{bx} = 0$ ➔ $\begin{cases} 0.7v_{1x} - 0.2v_{2x} = i_a \\ -0.2v_{1x} + 1.2v_{2x} = 0 \end{cases}$

Apply KCL to node 1: $\frac{v_{1x}}{6} + \frac{v_{1x}}{2} - i_a = 0$



If : $i_{ay} = 0,\ i_{by} = i_b$ ➔ v_{1y}, v_{2y}

Question: How much of v_1 is due to source *a*, and how much is because of source *b*?

Apply KCL to node 2: $\frac{v_{2y}}{1} + \frac{v_{2y}}{7} - i_b = 0$

$v_1 = v_{1x} + v_{1y}$

The Superposition Theorem

In a linear network, the **voltage across** or the **current through** any element may be calculated by *adding algebraically* all the individual voltages or currents caused by the separate independent sources acting "alone", that is, with

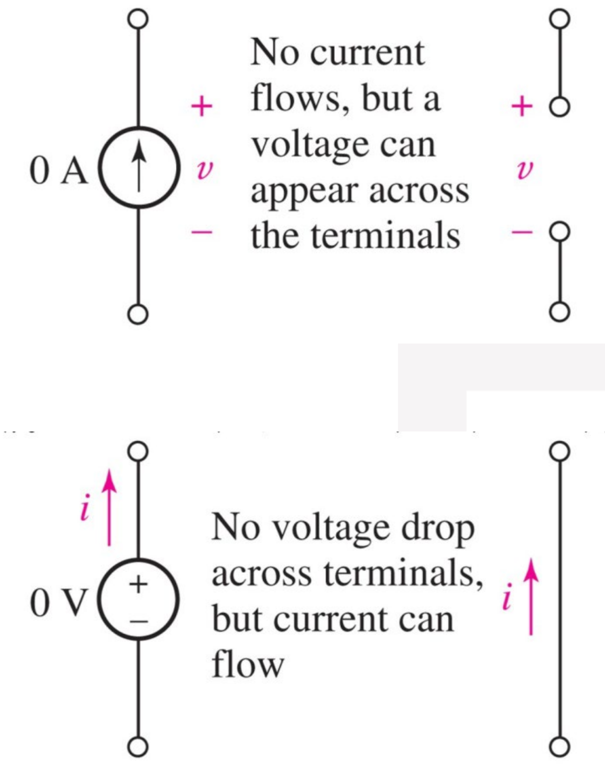
- all other independent voltage sources replaced by short circuits
- all other independent current sources replaced by open circuits

Applying Superposition

Leave one source ON and turn all other sources OFF:

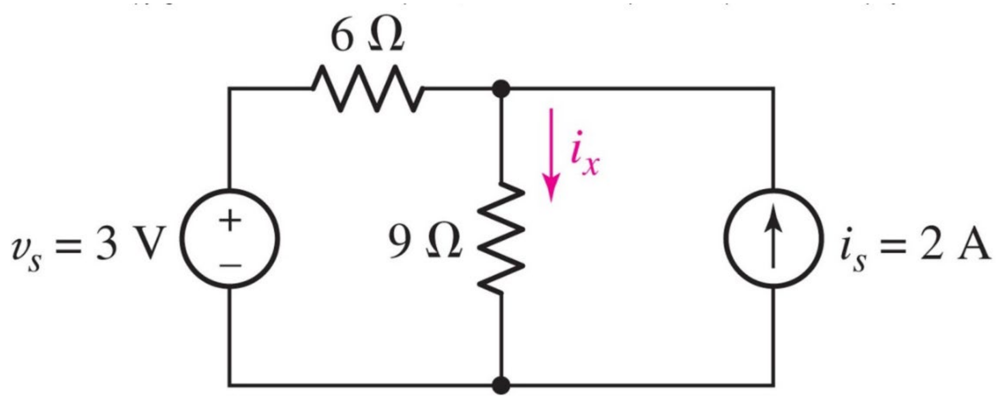
- current sources: *set i*=0*.*
- These become *open circuits.*
- voltage sources: *set v*=0*.*
- These become *short circuits.*
- *Find the response from this source.*

Add the resulting responses to find the total response.

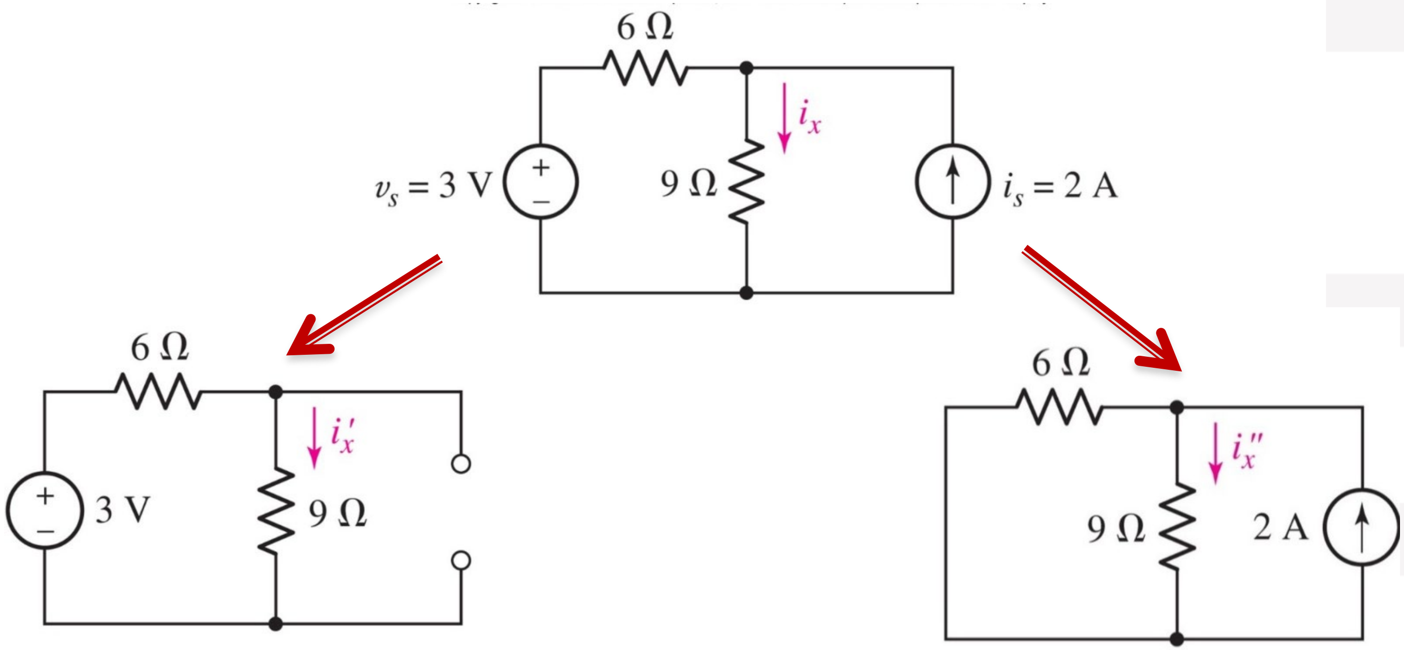


Superposition Example

Use superposition to solve for the current i_x



Superposition Example



First, turn the current source off:

$$i_x' = \frac{3}{6+9} = 0.2\,\mathrm{A}$$

Then, turn the voltage source off:

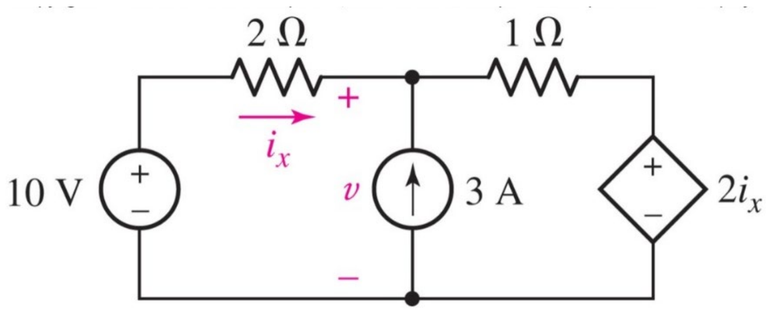
$$i_x'' = \frac{6}{6+9}(2) = 0.8\,\mathrm{A}$$

Finally, combine the results:

$$i_x = i_x' + i_x'' = 0.2 + 0.8 = 1.0\,\mathrm{A}$$

Superposition with a Dependent Source [1]

Use superposition to solve for the current i_x



Nodal analysis:

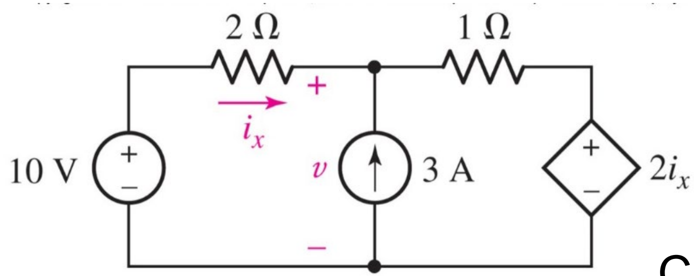
Apply KCL at node 1: $\frac{v_1-10}{2}+\frac{v_1-2i_x}{1}=3$

Relate i_x with v_1 : $i_x=\frac{10-v_1}{2}$

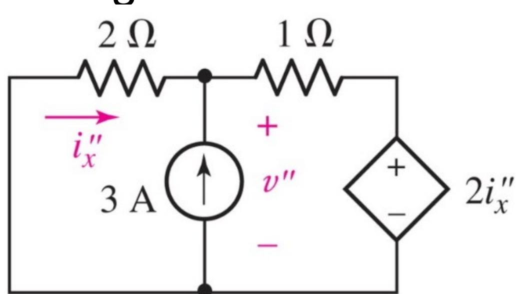
Solve: $v=v_1=7.2$ V, $i_x=1.4$ A

Superposition with a Dependent Source$_2$

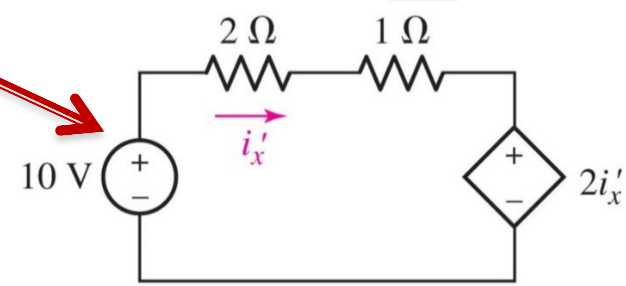
Use superposition to solve for the current i_x



Voltage source off



Current source off



Apply KCL at node 1: $\frac{v''}{2} + \frac{v'' - 2i''_x}{1} = 3$

Relate i''_x with v'' : $i''_x = \frac{-v''}{2}$

Solve: $i''_x = -0.6$ A

$$-10 + 2i'_x + i'_x + 2i'_x = 0$$

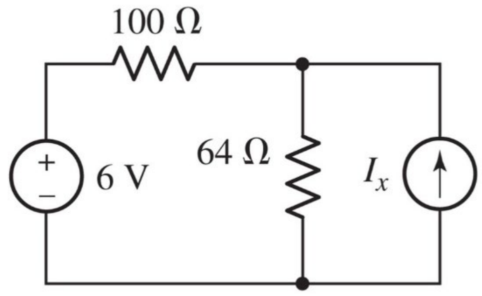
$i'_x = 2$ A

$$i_x = i_x' + i_x'' = 2 + (-0.6) = 1.4 \text{ A}$$

When applying superposition to circuits with *dependent* sources, these *dependent* sources are never "turned off."

Example: Power Ratings

Each resistor is rated to a maximum of 250 mW. Determine the maximum *positive* current to which the source I_x can be set before any resistor exceeds its power rating.

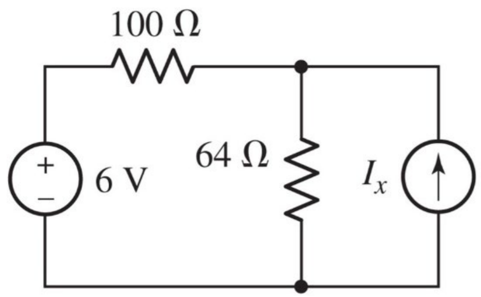


The maximum current of each resistor

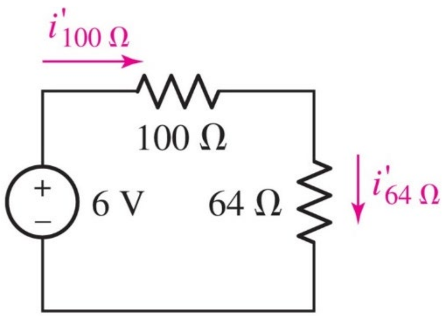
$$i_{max} = \sqrt{\frac{P_{max}}{R}}, \quad i_{100\Omega} < 50mA, \ i_{64\Omega} < 62.5mA$$

$$V_{max} = \sqrt{P_{max}R}, \quad V_{100\Omega} < 5V, \ V_{64\Omega} < 4V$$

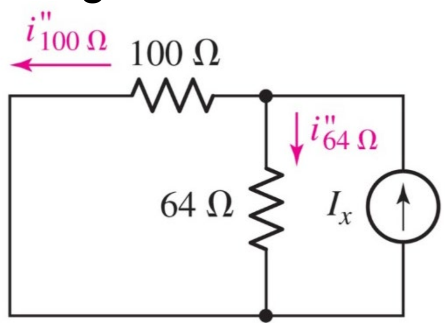
Example: Power Ratings



Current source off



Voltage source off



$i'_{100\Omega} = i'_{64\Omega} = 36.59m\text{A}$

$$i''_{100\Omega} = \frac{64}{100+64} I_x = 0.39 I_x$$

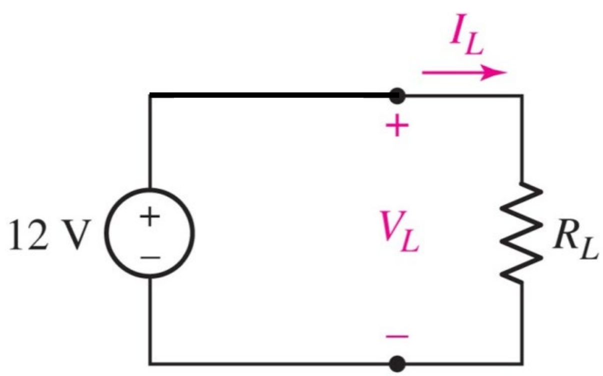
$$i''_{64\Omega} = \frac{100}{100+64} I_x = 0.61 I_x$$

$$i''_{100\Omega} - i'_{100\Omega} < 50mA$$
$$i'_{64\Omega} + i''_{64\Omega} < 62.5mA$$

$$Answer: I_x < 42.49 \text{ mA}$$

Practical Voltage Sources

Ideal voltage sources: a first approximation model for a battery.



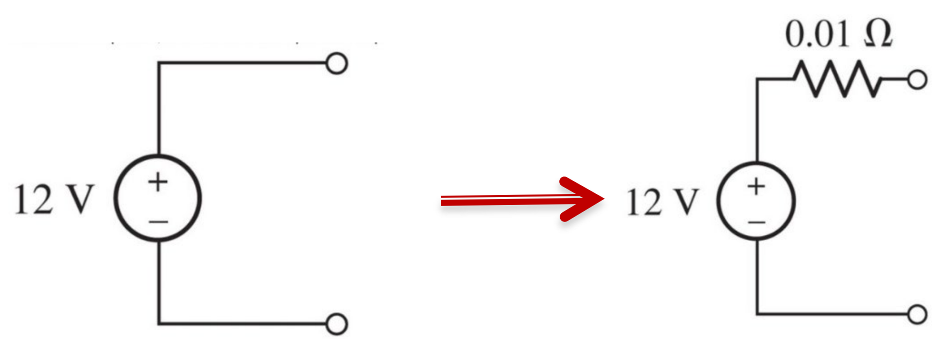
If R_L=1Ω, I_L=?

If R_L=1μΩ, I_L=?

If R_L=0, I_L=?

Why do real batteries have a current limit and experience voltage drop as current increases?

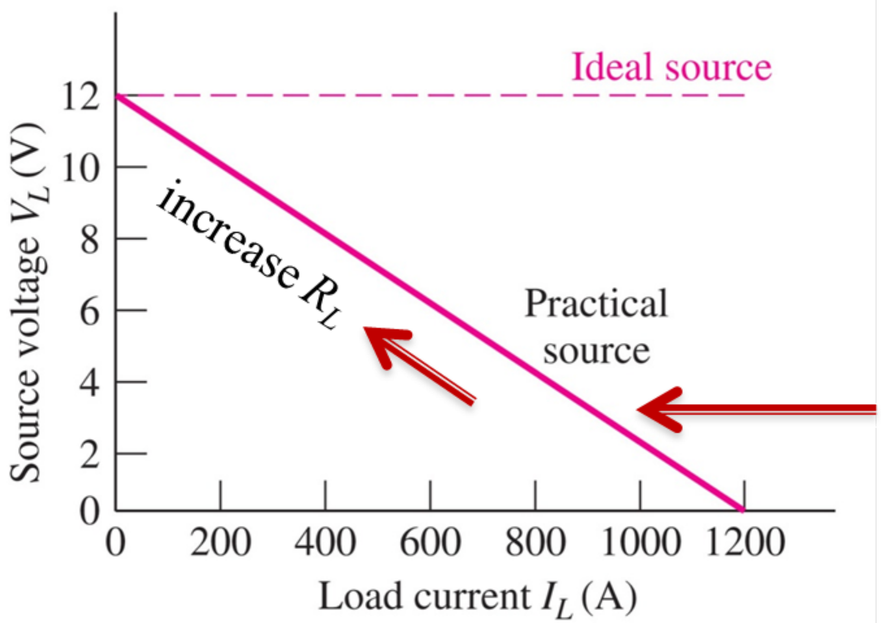
Two battery models:

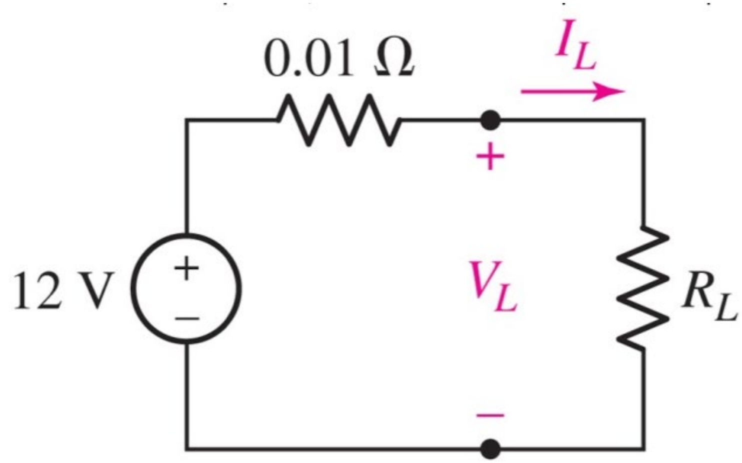


Practical Source: Effect of Connecting a Load

The practical voltage source model:

$$V_L = 12 - 0.01\, I_L$$

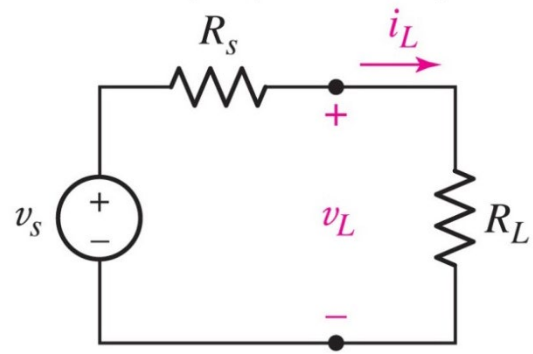




This line represents all possible R_L

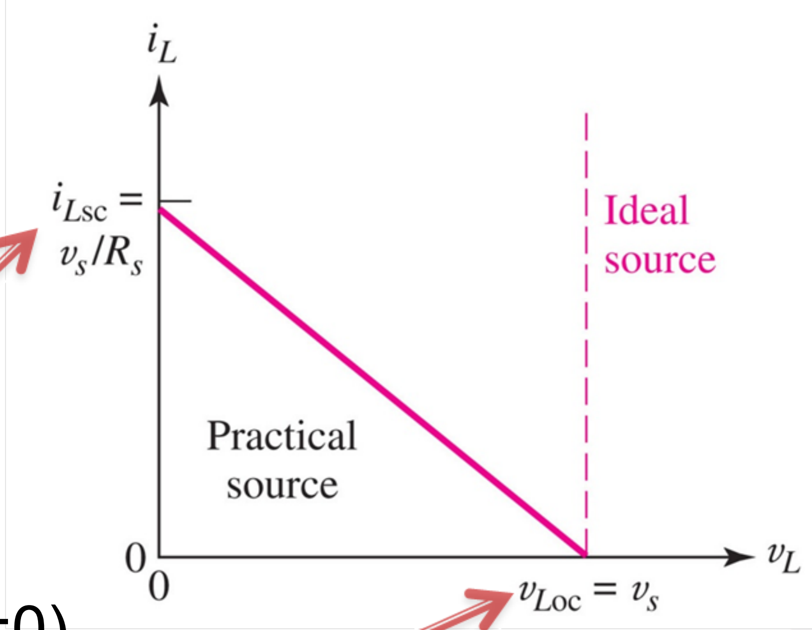
Practical Voltage Source

The source has an internal resistance or output resistance, which is modeled as R_s



The linear relationship between v_L and i_L:

$$v_L = v_s - R_s i_L$$

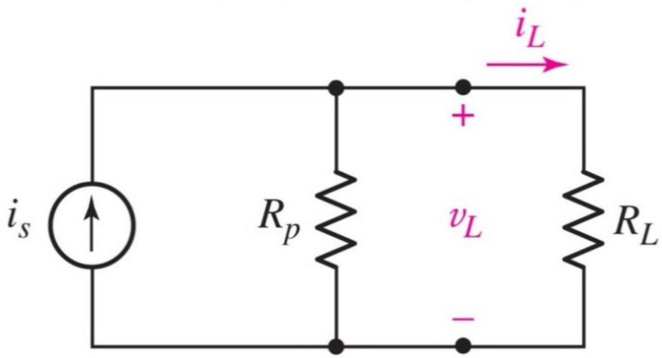


short circuit current (when R_L=0)

open circuit voltage (when R_L=∞)

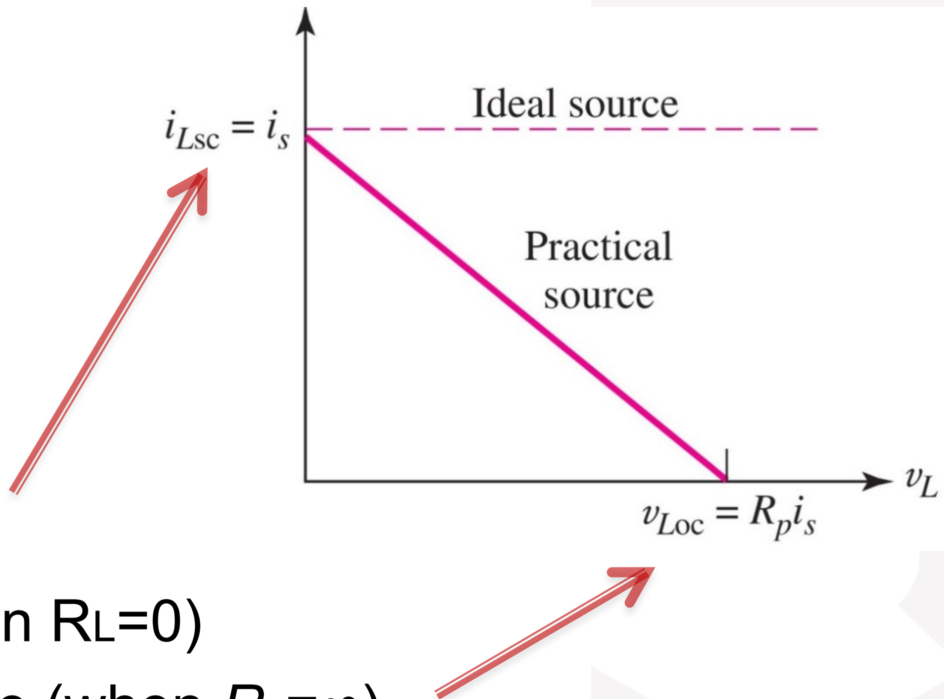
Practical Current Source

The source has an internal *parallel* resistance which is modeled as R_p



The linear relationship between v_L and i_L:

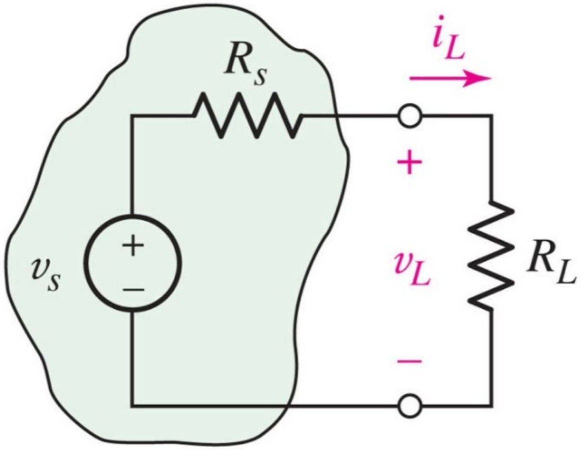
$$i_L = i_s - \frac{v_L}{R_p}$$

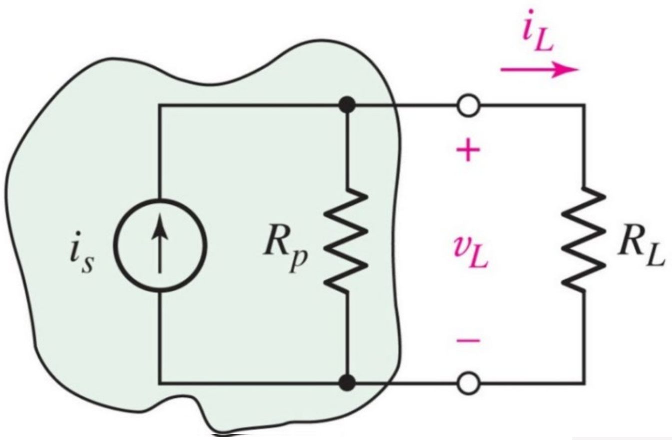


short circuit current (when RL=0)

open circuit voltage (when R_L=∞)

Source Transformation and Equivalent Sources





$$v_L = v_s \frac{R_L}{R_s + R_L}$$

$$v_L = i_s \frac{R_p}{R_p + R_L} \cdot R_L$$

The sources are equivalent if

$$\boxed{R_s = R_p \text{ and } v_s = i_s R_s}$$

Source Transformation

The circuits (a) and (b) are equivalent at the terminals.

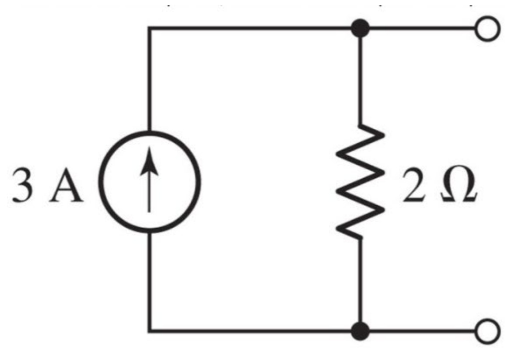
$$R_s = R_p = 2\Omega$$

$$v_s = i_s R_s = 3A * 2\Omega = 6\text{V}$$

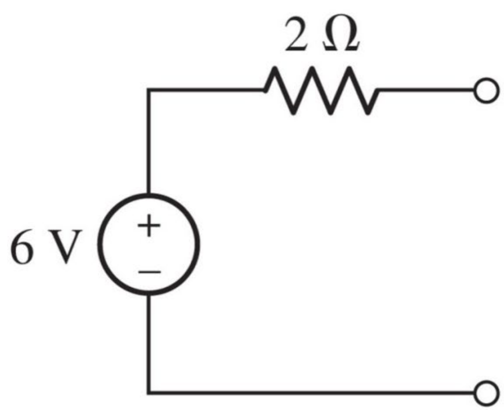
If given circuit (a), but circuit (b) is more convenient, switch them!

This process is called

source transformation.



(a)



(b)

The head of the current source arrow corresponds to the "+" terminal of the voltage source.

Source Transformation

If $R_L = 4\Omega$:

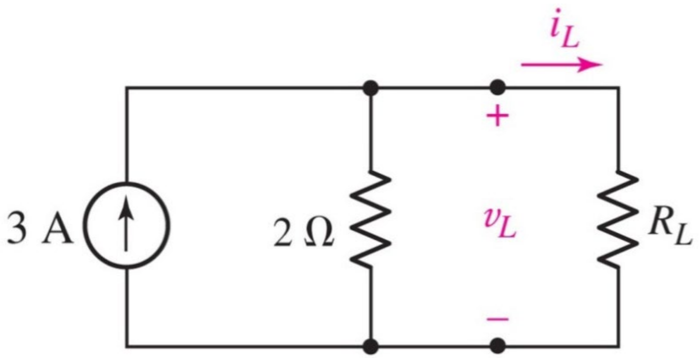
(a) Load terminal:

$i_L = \frac{R_s}{R_s+R_L} i_s = 1A$, $V_L = i_L R_L = 4V$,

$P_{R_L} = i_L v_L = 4W$

Inside the practice source:

$P_{3\text{A}} = -3\text{A} * v_L = -12W$, $P_{R_s} = {v_L^2}/{R_s} = 8W$



(a)

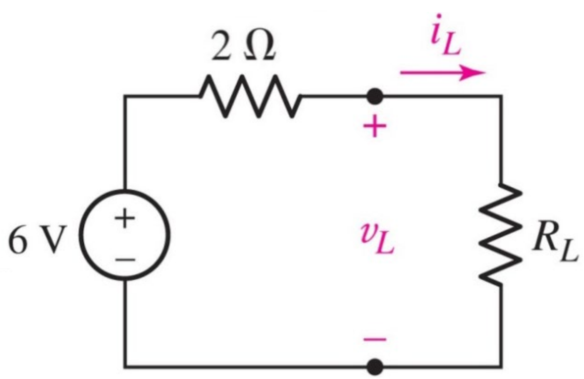
(b) Load terminal:

$i_L = \frac{v_s}{R_s+R_L} = 1A$, $V_L = i_L R_L = 4V$,

$P_{R_L} = i_L v_L = 4W$

Inside the practice source:

$P_{6V} = -6V * i_L = -6W$, $P_{R_s} = i_L^2 * R_s = 2W$

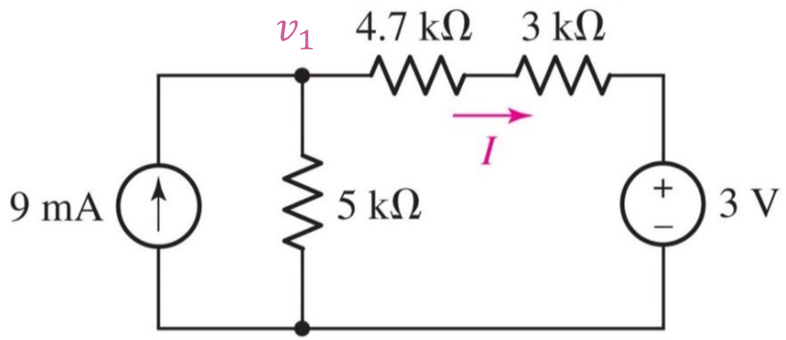


(b)

The two practical sources are equivalent only with respect what transpires at the load terminals; they are not equivalent internally!

Example: SourceTransformation

Calculate the current *I* in the circuit below:



Method 1:

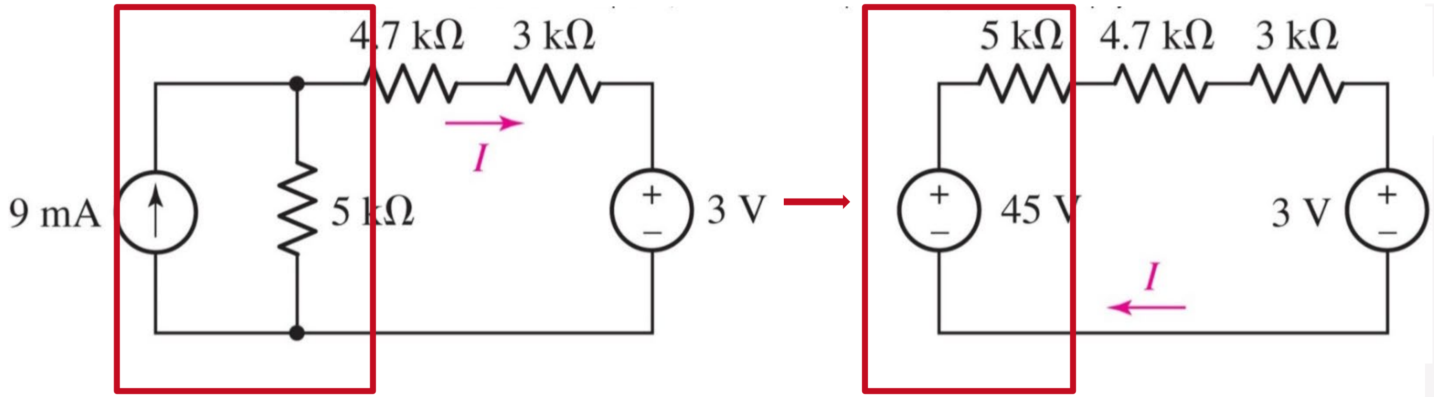
Apply KCL: $9mA - \frac{v_1}{5k\Omega} - \frac{v_1 - 3V}{4.7k\Omega + 3k\Omega} = 0$

$$v_1 = 28.465\ V$$

$$I = \frac{v_1 - 3V}{7.7k\Omega} = 3.307mA$$

Example: SourceTransformation

Calculate the current *I* in the circuit below using source transformation



Method 2 : source transformation

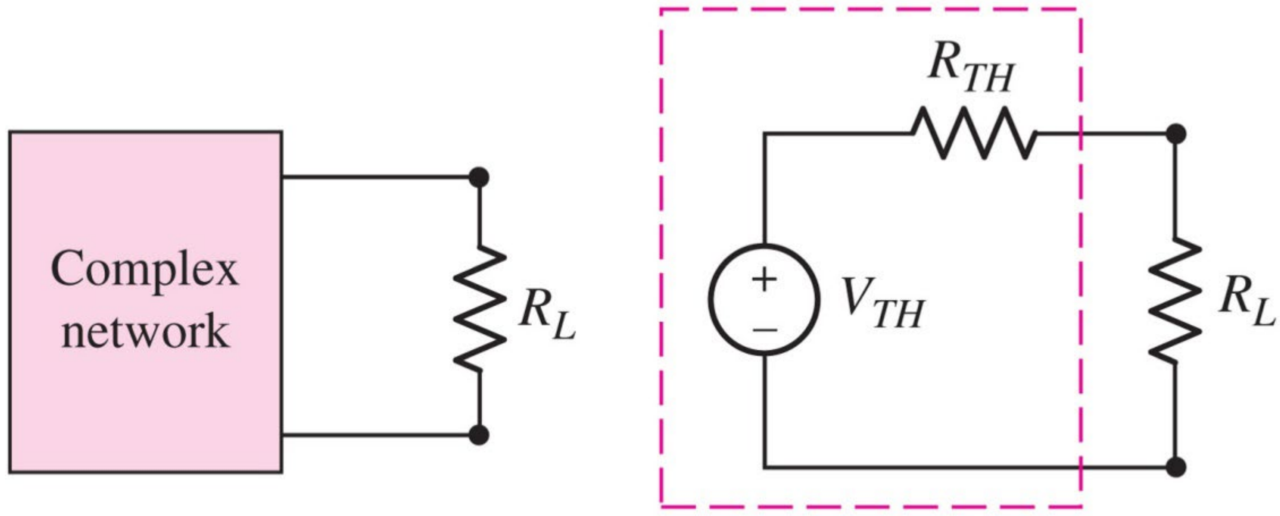
Equivalent voltage source:

$$R_s = R_p = 5k\Omega,\ V_s = i_s R_s = 45V$$

$$I = (45 - 3) / (5 + 4.7 + 3) = 3.307\ \text{mA}$$

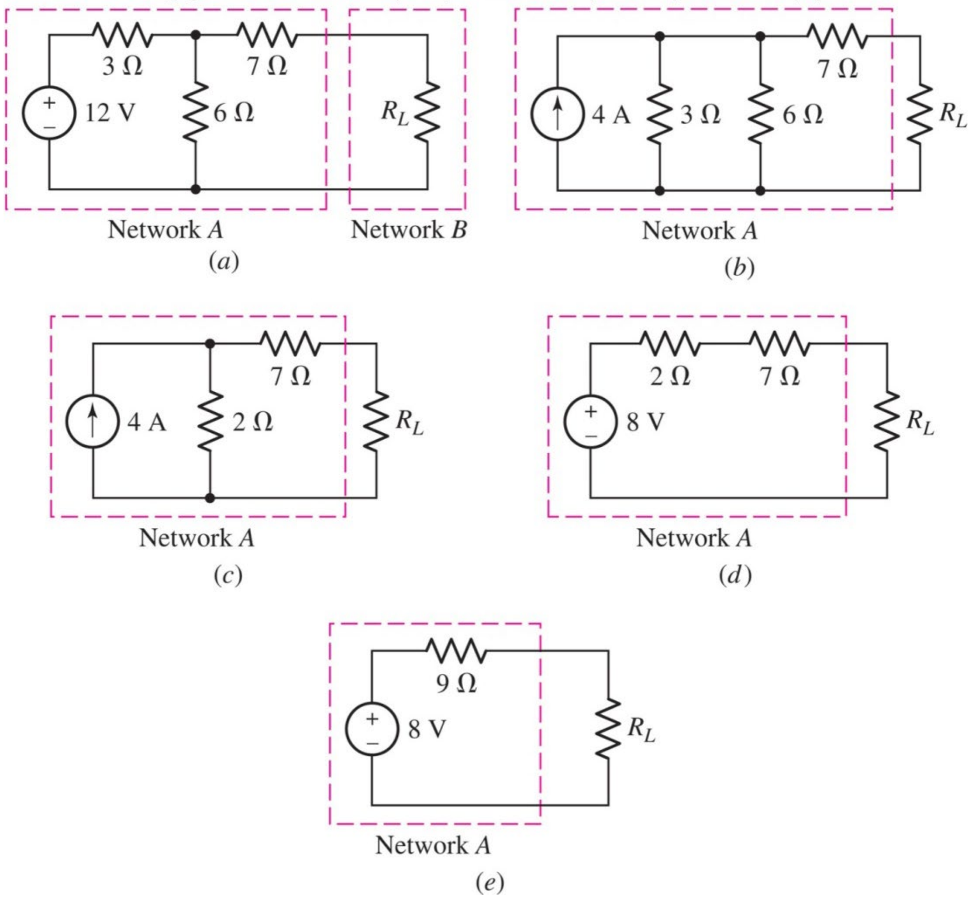
Thévenin Equivalent Circuits

Thévenin's theorem: a linear network can be replaced by its Thévenin equivalent circuit, as shown below:



Thévenin Equivalent using Source Transformation

We can repeatedly apply source transformation on network A to find its Thévenin equivalent circuit.



(a) to (b) Source transformation:

$$R_p = R_s = 3\Omega, i_s = \frac{v_s}{R_s} = 4A$$

(c) to (d) Source transformation:

$$R_s = R_p = 2\Omega, v_s = i_s R_s = 8V$$

This method has limitations- not all circuits can be source transformed.

Finding the Thévenin Equivalent

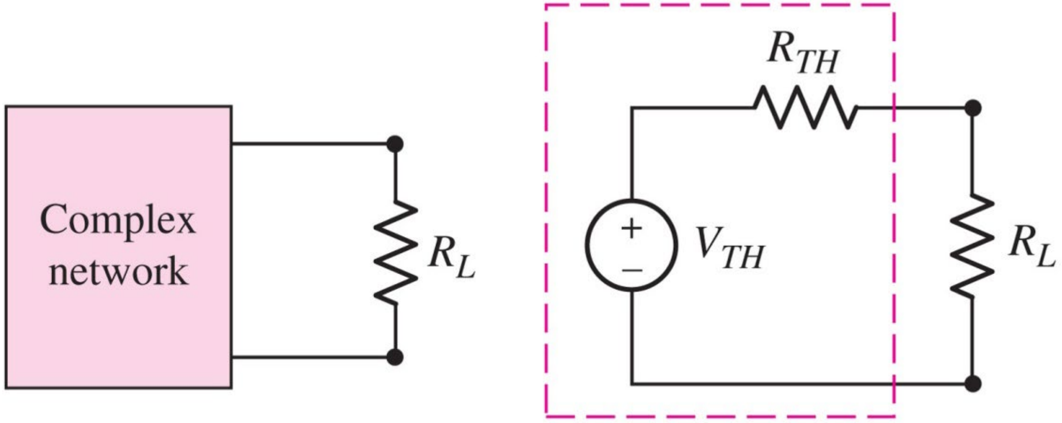
Step1: Disconnect the load ($R_L = \infty$). Find the open circuit voltage v_{oc}

Step2: Find the equivalent resistance R_{eq} of the network with all independent sources turned off.

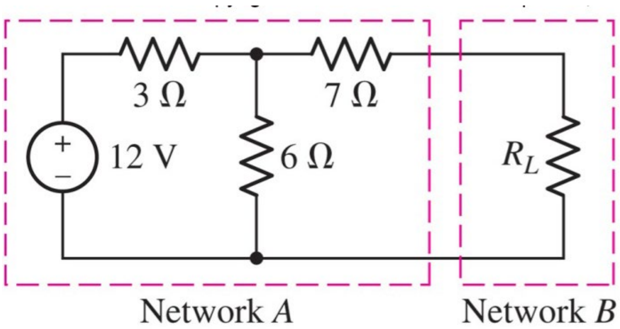
Then:

$$V_{TH} = v_{oc} \text{ and}$$

$$R_{TH} = R_{eq}$$

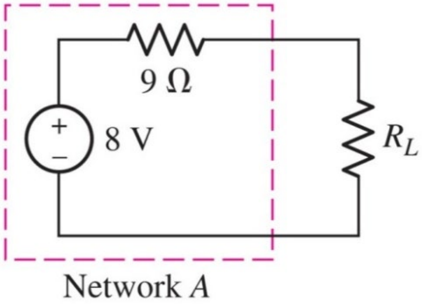


Thévenin Example

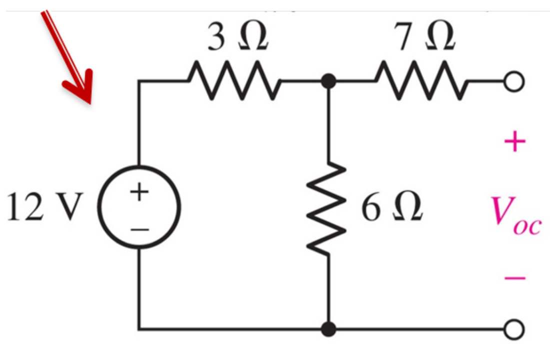


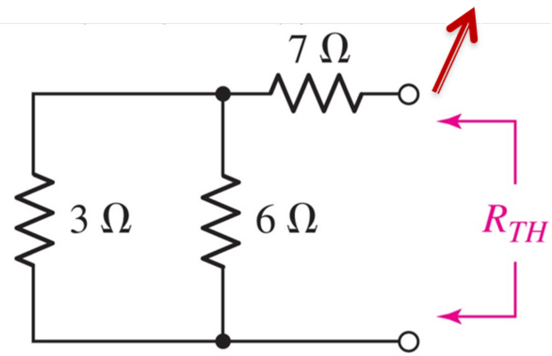
$$V_{TH} = v_{oc} \text{ and}$$

$$R_{TH} = R_{eq}$$



Disconnect the load ($R_L = \infty$). Find the open circuit voltage v_{oc}





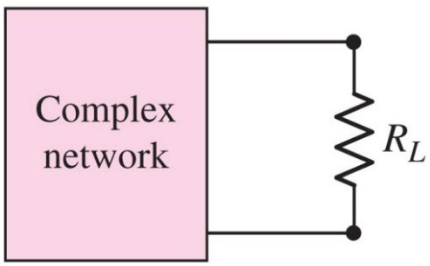
Find R_{eq} of the network with all independent sources turned off.

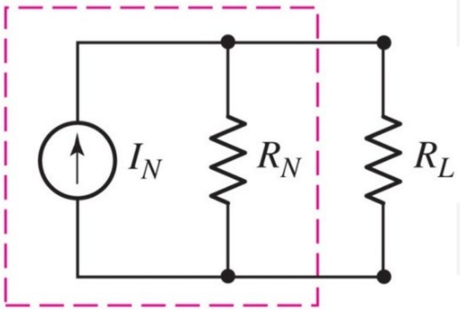
$$Voc = 12\frac{6}{3+6} = 8V$$

$$R_{TH} = 3||6 + 7 = 9\Omega$$

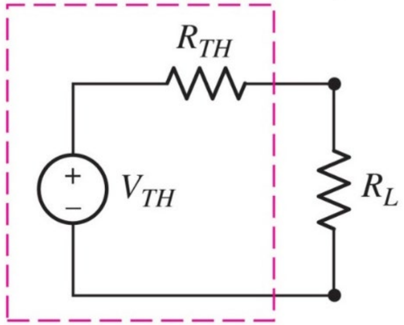
Norton Equivalent Circuits

Norton's theorem: a linear network can be replaced by its Norton equivalent circuit, as shown below:





The Thévenin and Norton equivalents are source transformations of each other!



$$R_{TH} = R_N = R_{eq} \text{ and } v_{TH} = i_N R_{eq}$$

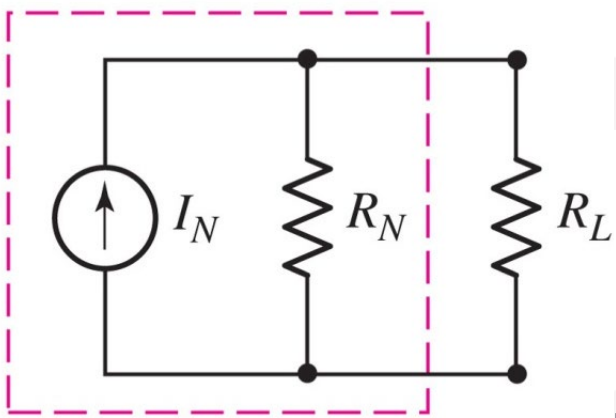
Finding the Norton Equivalent

Step1: Replace the load with a short circuit. ($R_L = 0$) Find the short circuit current i_{sc}

Step2: Disconnect the load ($R_L = \infty$). Find the equivalent resistance R_{eq} of the network with all independent sources turned off.

Then:

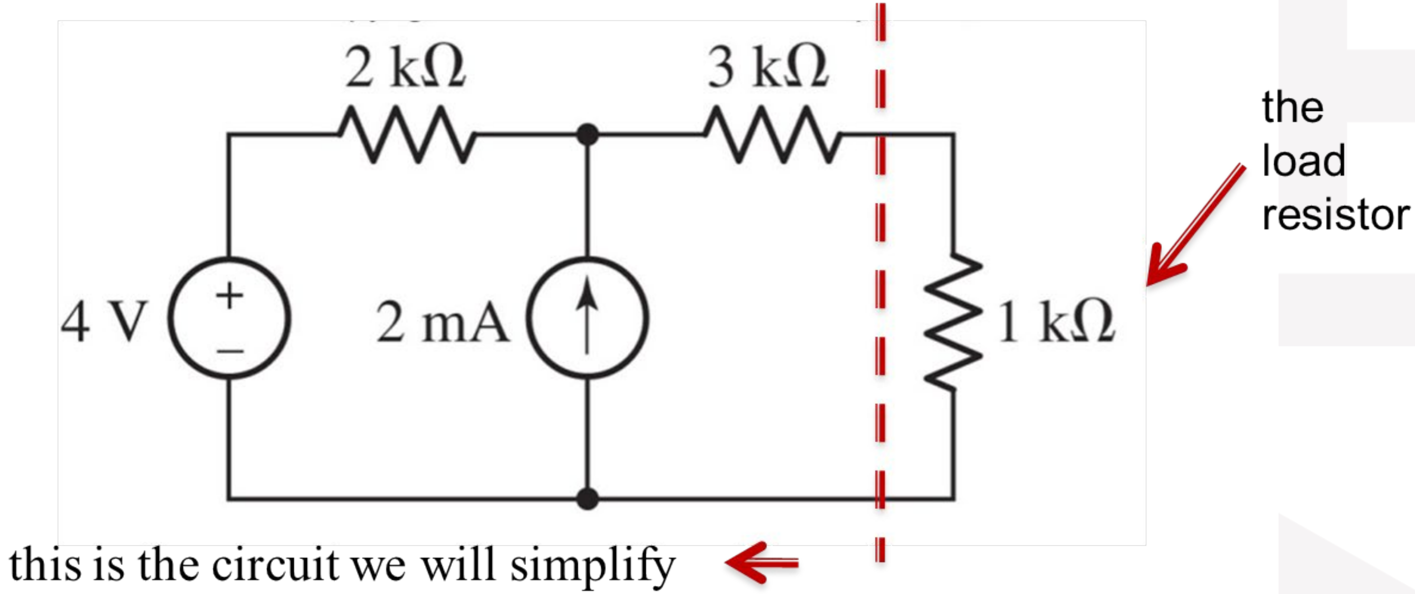
$$I_N = i_{sc} \text{ and } R_N = R_{eq}$$



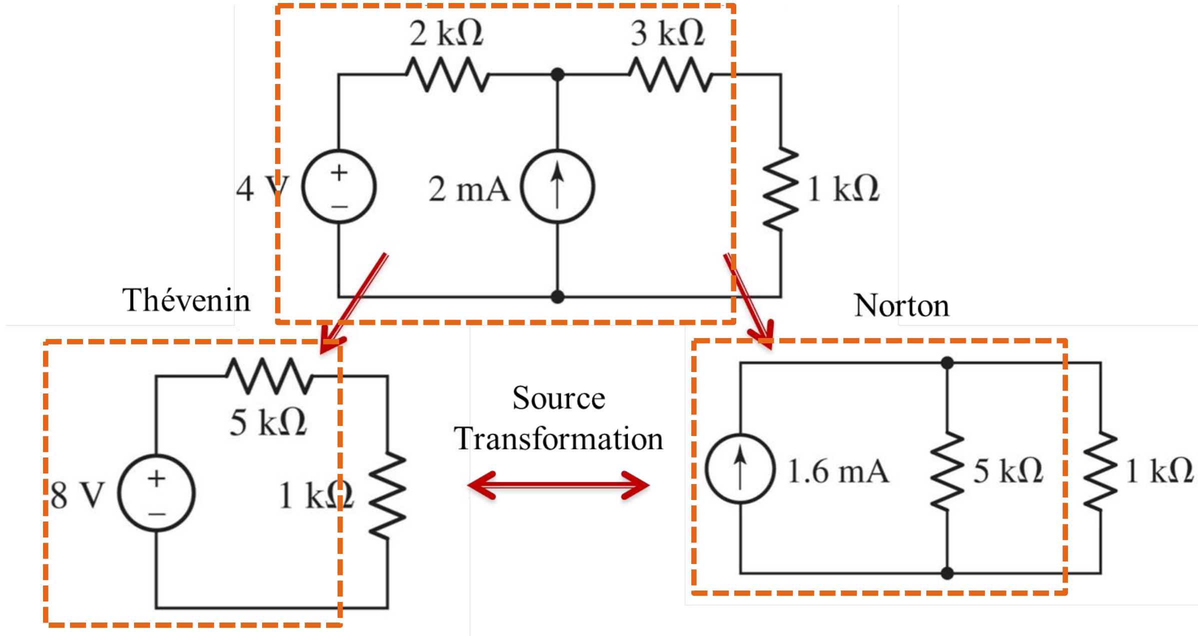
Example: Norton and Thévenin 1

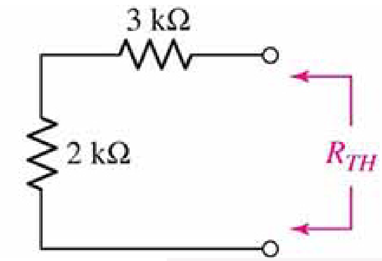
Find the Thévenin and Norton equivalents for the network faced by the 1 kΩ resistor.

Answer: next slide



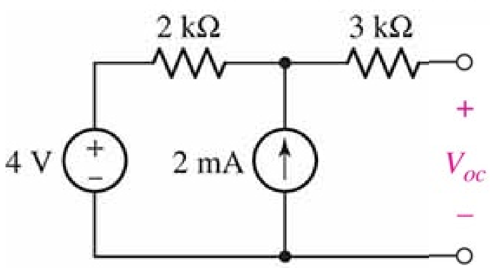
Example: Thévenin and Norton[2]





$$V_{TH} = V_{oc} = 8V$$

$$R_{TH} = 5k\Omega$$



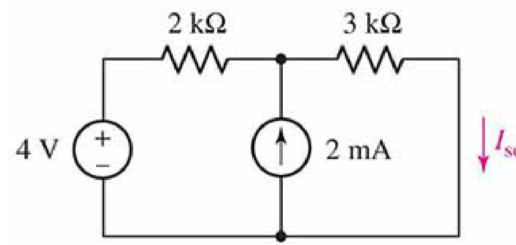
Source Transformation:

$$R_p = R_s = 5k\Omega$$

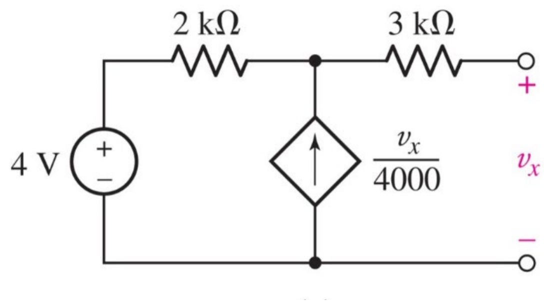
$$i_s = \frac{v_s}{R_s} = 1.6mA$$

Norton Equivalent:

$$I_N = I_{sc} = 1.6mA$$



Thévenin Example: Handling Dependent Sources [1]

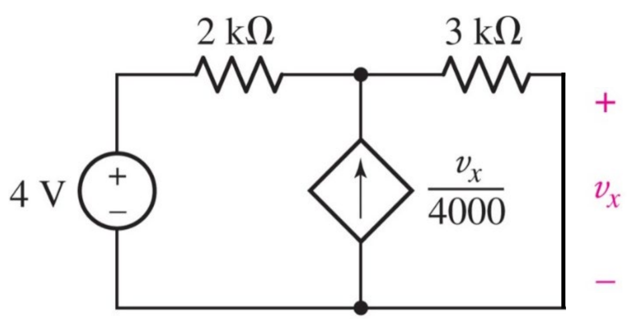


Thévenin:

Apply KVL: $-4 + \left(-\frac{v_x}{4000}\right) * 2000 + v_x = 0$

$V_{oc} = v_x = 8V$

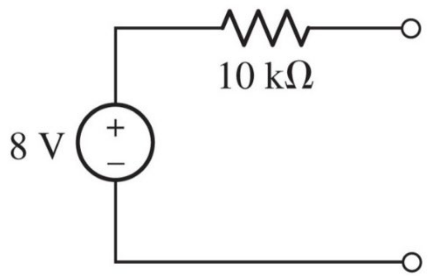
$R_{TH} = ?$



Norton:

$$v_x = 0, I_{sc} = \frac{4}{2000 + 3000} = 0.8mA$$

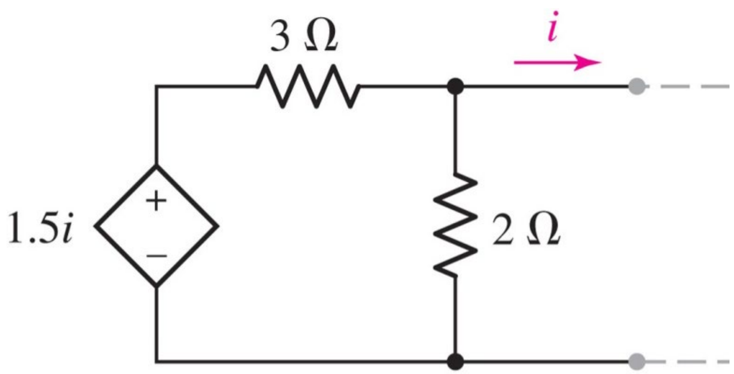
$$R_{TH} = \frac{V_{oc}}{I_{sc}} = 10k\Omega$$

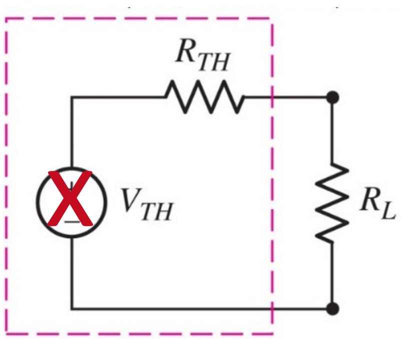


One method to find the Thévenin equivalent of a circuit with a dependent source: find V_{TH} and I_N and solve for $\boxed{R_{TH} = V_{TH} / I_N}$

Thévenin Example: Handling Dependent Sources [2]

Finding the ratio V_{TH} / I_N fails when both quantities are zero

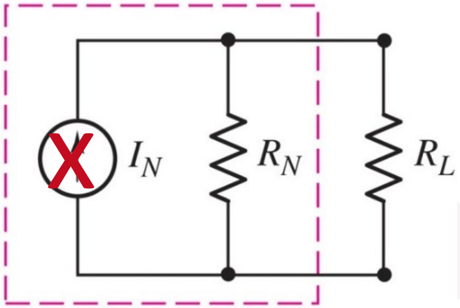




$R_{TH} = ?$

Open circuit:

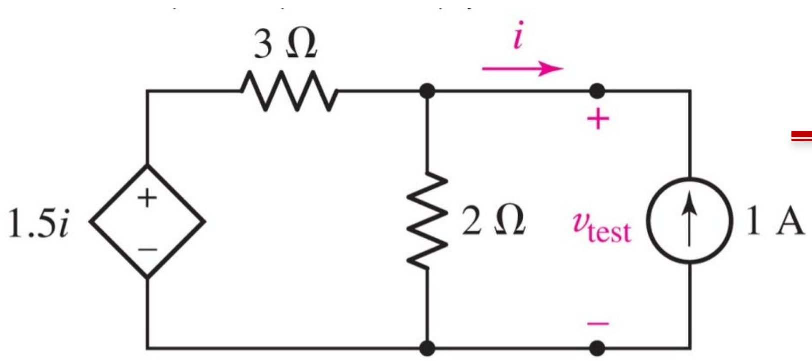
$i = 0 \longrightarrow V_{oc} = 0$



Short circuit:

$-1.5i + 3i = 0 \rightarrow \ I_{sc} = i = 0$

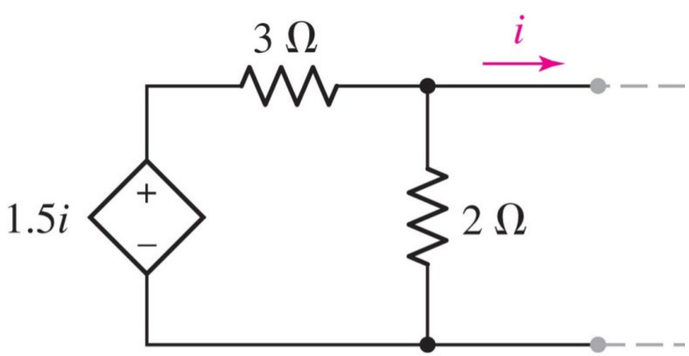
Thévenin Example: Handling Dependent Sources [3]

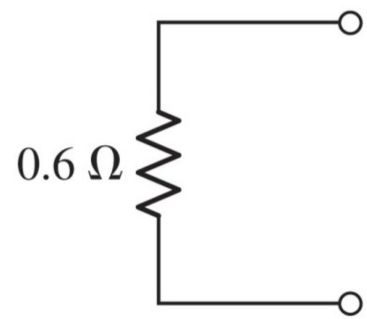


$$\frac{v_{test}}{2}+\frac{v_{test}-(1.5i)}{3}=1$$

$$i=-1$$

Solve: $v_{test}=0.6\ \text{V}$, and so $R_{TH}=0.6\ \Omega$

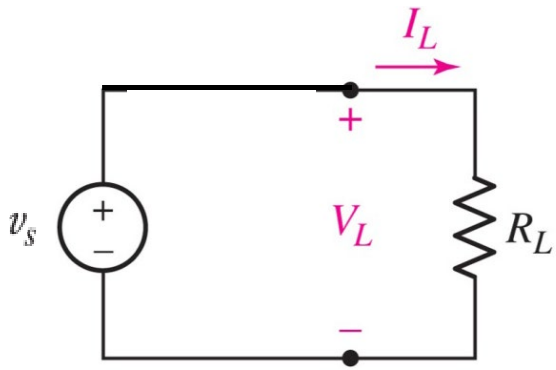


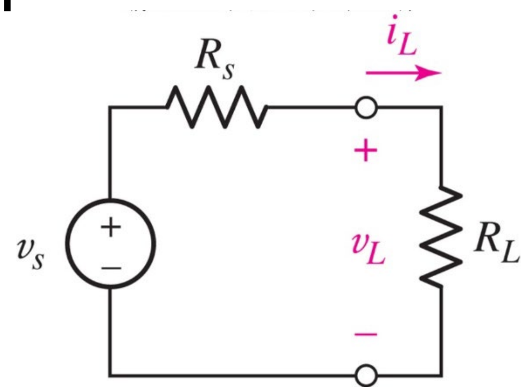


One method to find the Thévenin equivalent of a circuit with a dependent source: apply a test source

Maximum Power Transfer

What load resistor will allow the **practical source** to deliver the maximum power to the load?





Idear source:

$$p_L = \frac{v_s^2}{R_L}$$

Pratical source:

$$p_L = i_L^2 R_L = \frac{v_s^2 R_L}{(R_s + R_L)^2}$$

[solve $dp_L/dR_L = 0$, check $R_L = 0$ & $R_L = \infty$]

Answer: $R_L = R_s$

$$p_{\text{max|delivered to load}} = \frac{v_s^2}{4R_s}$$

Maximum Power Transfer

Maximum power transfer theorem:

An independent voltage source in series with a resistance Rs (or an independent current source in parallel with a resistance Rs) delivers maximum power to a load resistance R_L such that R_L = Rs.

An alternative expression:(In terms of the Thévenin equivalent resistance of a network):

A network delivers maximum power to a load resistance R_L when R_L is equal to the Thévenin equivalent resistance of the network (R_{TH}).

$$p_{\max}\big|_{\text{delivered to load}} = \frac{v_s^2}{4R_s} = \frac{v_{TH}^2}{4R_{TH}}$$

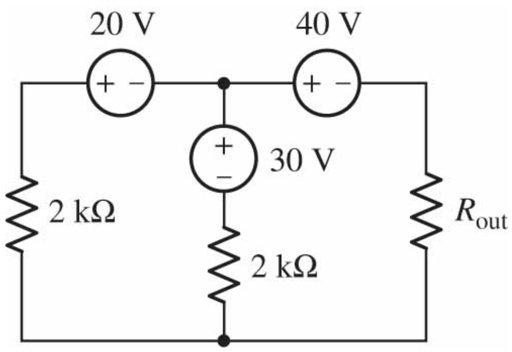
Example: Maximum Power Transfer

Consider the circuit below:

(a) What is the maximum power that can be delivered to Rout?

(b) If Rout = 3 kΩ, find the power delivered to it.

(c) What two different values of R_{out} will have exactly 20 mW delivered to them?

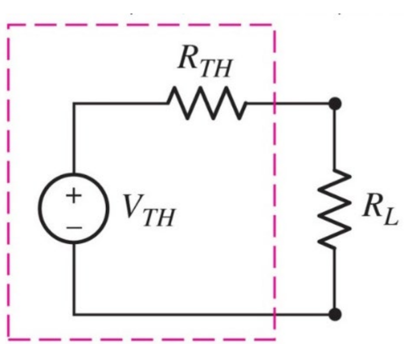


Open circuit:

In single loop: $\frac{v+20}{2} + \frac{v-30}{2} = 0$

$V_{TH} = V_{oc} = v - 40 = -35V$

$R_{TH} = 1k\Omega$



(a)When $R_{out} = R_{TH}$ the load get the maximum power

$$p_{max} = (\frac{V_{TH}}{R_{TH} + R_{out}})^2 R_{out} = 306mW$$

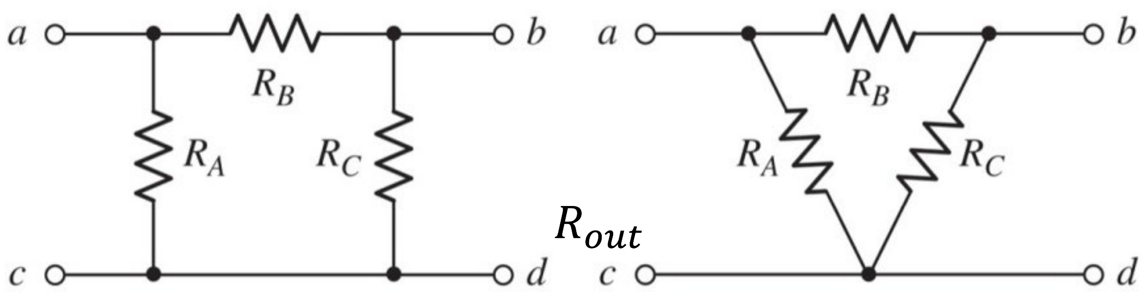
(b) $p = (\frac{V_{TH}}{R_{TH}+R_{out}})^2 R_{out} = 230mW$

(c)$(\frac{V_{TH}}{R_{TH}+R_{out}})^2 R_{out} = 20mW$

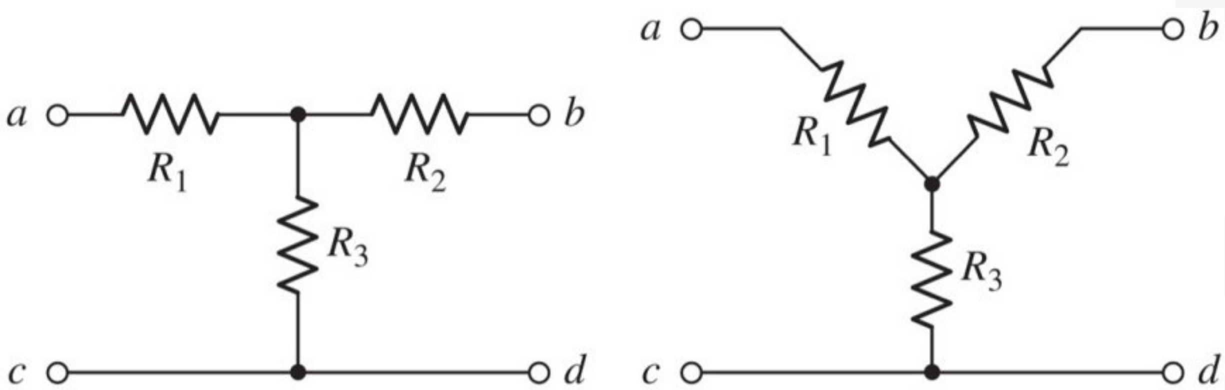
$R_{out1} = 59.2k\Omega, R_{out2} = 16.88\Omega$

Δ-Y (delta-wye) Conversion[1]

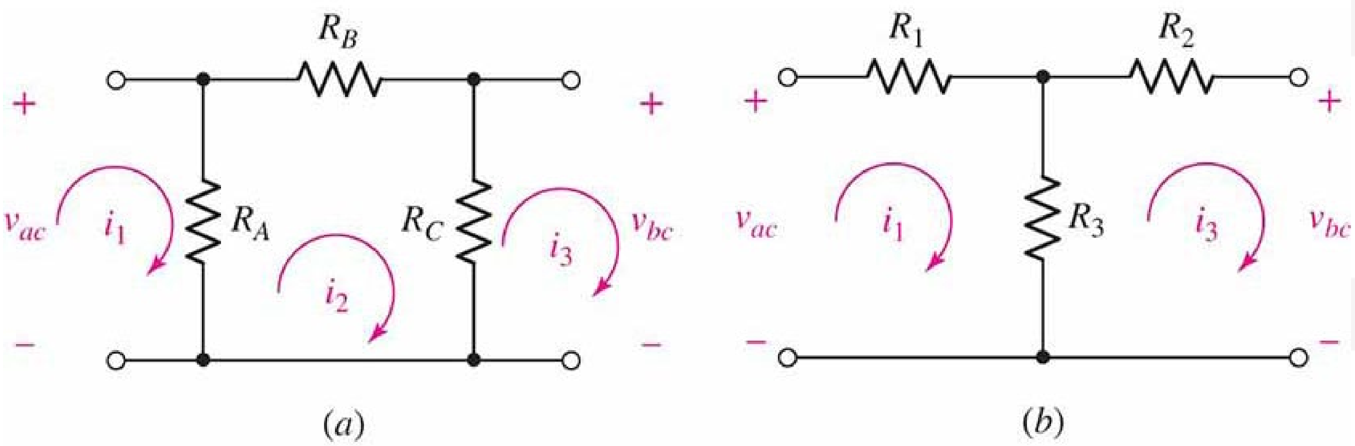
The following resistors form a Δ:



The following resistors form a Y:



Δ-Y (delta-wye) Conversion [2]



If the two networks are equivalent, then the terminal voltages and currents must be equal.

$$
\begin{aligned}
R_A i_1 - R_A i_2 &= v_{ac} \\
-R_A i_1 + (R_A + R_B + R_C) i_2 - R_C i_3 &= 0 \\
-R_C i_2 + R_C i_3 &= -v_{bc}
\end{aligned}
$$

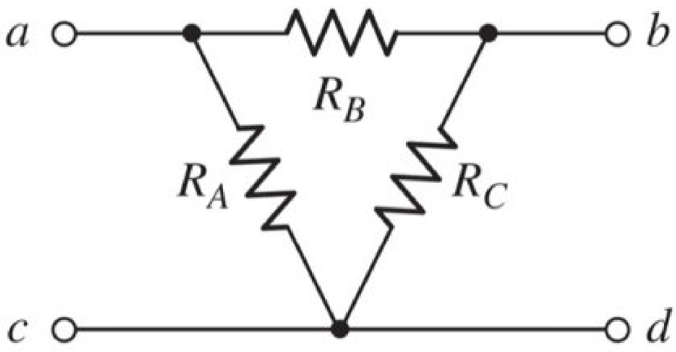
$$
\begin{aligned}
(R_1 + R_3) i_1 - R_3 i_3 &= v_{ac} \\
-R_3 i_1 + (R_2 + R_3) i_3 &= -v_{bc}
\end{aligned}
$$

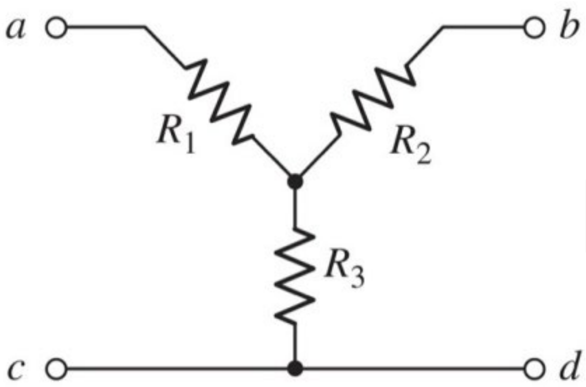
Remove i_2

$$\left(R_A - \frac{R_A^2}{R_A + R_B + R_C}\right) i_1 - \frac{R_A R_C}{R_A + R_B + R_C} i_3 = v_{ac}$$

$$-\frac{R_A R_C}{R_A + R_B + R_C} i_1 + \left(R_C - \frac{R_C^2}{R_A + R_B + R_C}\right) i_3 = -v_{bc}$$

Δ-Y (delta-wye) Conversion 3





This Δ is equivalent to the Y if

$$R_A = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

$$R_B = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

$$R_C = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$

This Y is equivalent to the Δ if

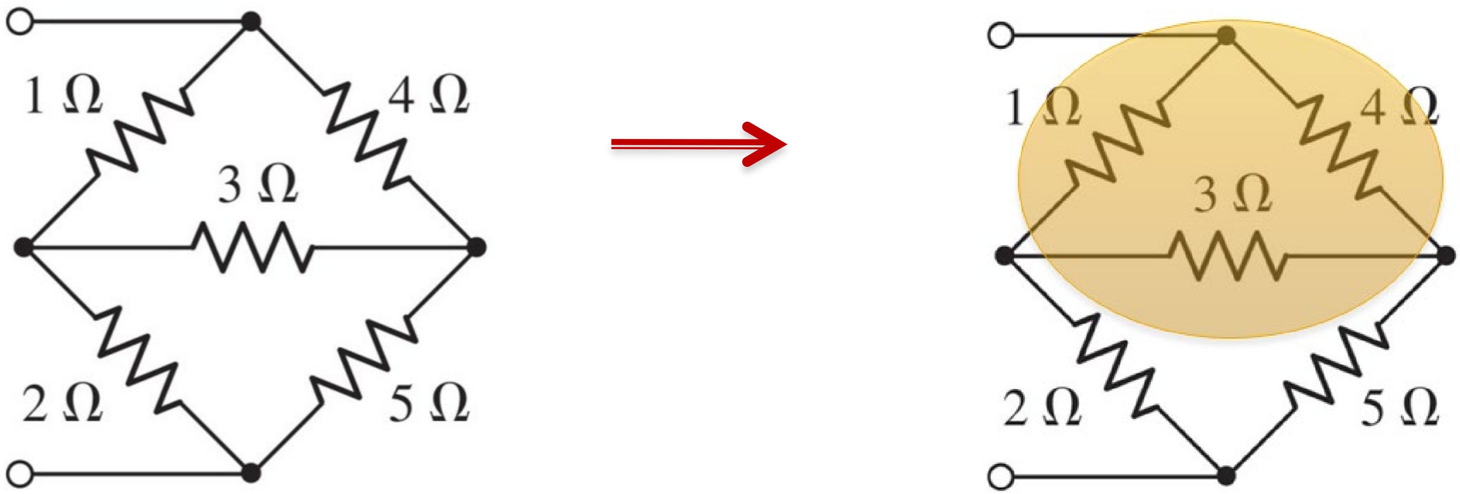
$$R_1 = \frac{R_AR_B}{R_A + R_B + R_C}$$

$$R_2 = \frac{R_BR_C}{R_A + R_B + R_C}$$

$$R_3 = \frac{R_CR_A}{R_A + R_B + R_C}$$

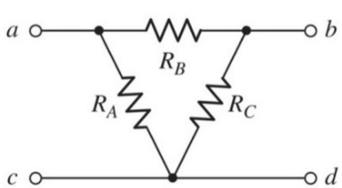
Example: Δ-Y Conversion 1

How do we find the equivalent resistance of the following network? Convert a Δ to a Y

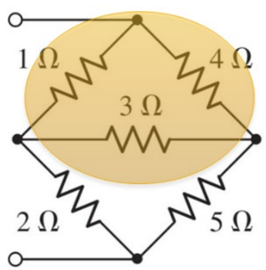


Example: Δ-Y Conversion 2

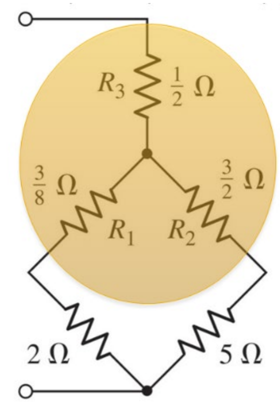
How do we find the equivalent resistance of the following network? Convert a Δ to a Y

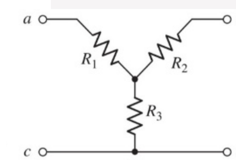


$R_A = 1\Omega$
$R_B = 3\Omega$
$R_C = 4\Omega$

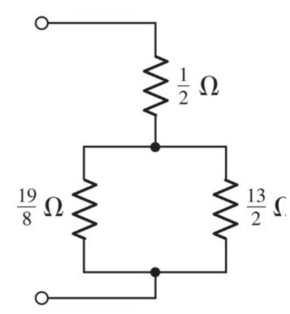


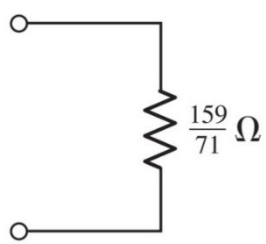
Use the Δ to Y equations





$$R_1 = \frac{R_A R_B}{R_A + R_B + R_C}$$
$$R_2 = \frac{R_B R_C}{R_A + R_B + R_C}$$
$$R_3 = \frac{R_C R_A}{R_A + R_B + R_C}$$

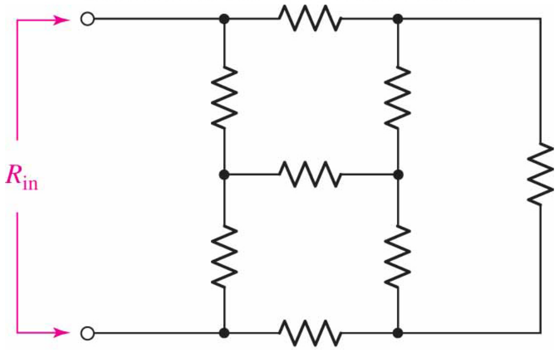




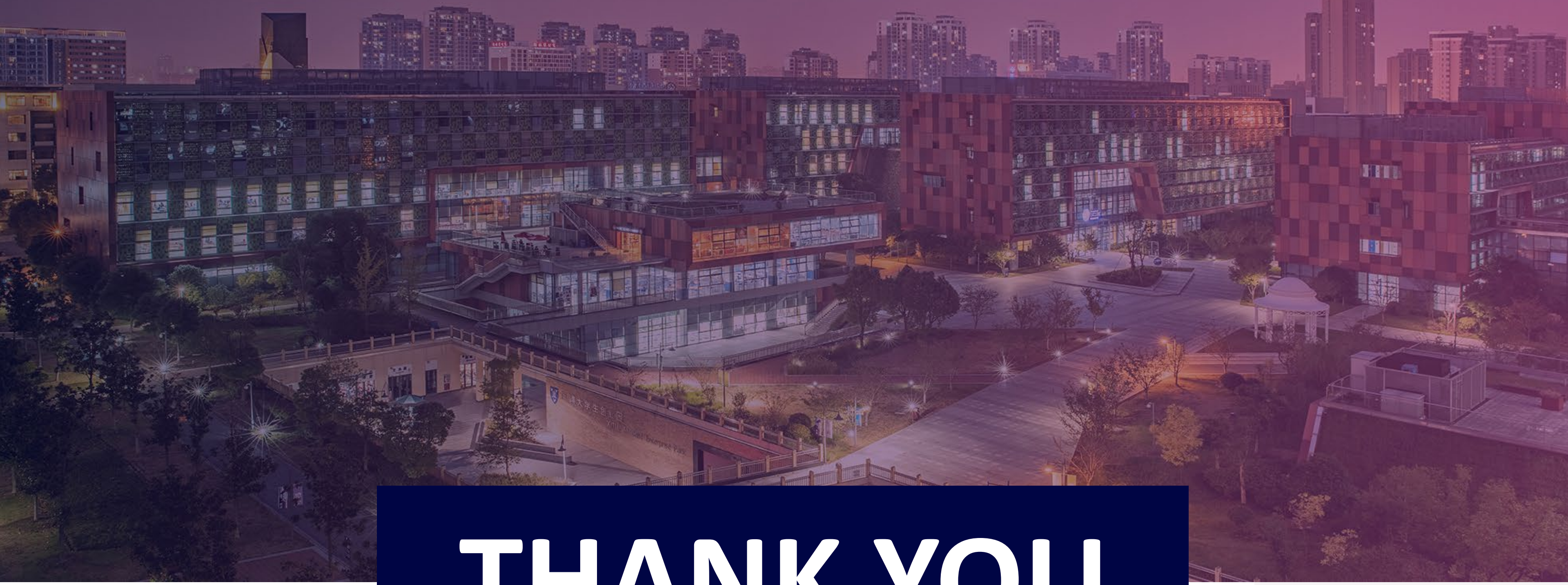
Use standard serial and parallel combinations

Example: Δ-Y Conversion 3

Use the technique of Y–Δ conversion to find the Thévenin equivalent resistance of the circuit



Each R is 10 Ω



THANK YOU

Xi'an Jiaotong-Liverpool University
西交利物浦大學