CAN207 Continuous and Discrete Time Signals and Systems

Lecture-10

Laplace Transform

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Content

- 1. Introduction to Laplace Transform
  - Eigenfunctions: $e^{j\omega t}$ and e^{st}
  - Definition of Laplace transform (also called s-transform)
- 2. Region of Convergence (ROC)
  - Zeros and Poles
  - ROC propreties
  - Causality
  - Stability

1.1 Review of CTFS and CTFT

- Review CTFS/CTFT

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Eigenfunction
  – The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude

$$e^{j\omega_k t} \rightarrow H(\omega_k)e^{j\omega_k t}$$

1.2 Eigenfunction - more general

- Eigen function
  - Proof for the previous complex exponential:

$$e^{j\omega_k t} \to \int_{-\infty}^{\infty} h(\tau)e^{j\omega_k(t-\tau)}d\tau = e^{j\omega_k t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega_k\tau}d\tau = e^{j\omega_k t}H(\omega_k)$$

  - where $e^{j\omega_k t}$ is a special complex exponential, with magnitude fixed at 1, and the exponent is pure imaginary.
  - It can be generalized by extending the pure imaginary exponent to a common complex number $s = \sigma + j\omega$, where σ is the real part, and $j\omega$ is the imaginary part (same as the $j\Omega_k$ before), now:

$$e^{st} \to \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}_{} = e^{st}H(s)$$

Transfer function = Laplace transform $\boldsymbol{H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt}$

1.3 Definition of Laplace transform

- The Laplace transform* is

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

  - where $s = \sigma + j\omega$.

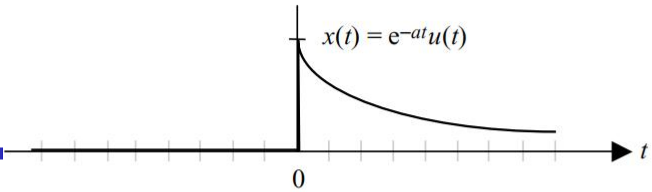
- The Laplace transform is closely related to CTFT by:

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

  - which suggests that the Laplace transform is the Fourier transform of $x(t)e^{-\sigma t}$;
  - The value of σ would affect the convergence of the CTFT of $x(t)e^{-\sigma t}$.

*This is often referred to as the bilateral Laplace transform.

Example 1



- Let's look at $x(t) = e^{-at}u(t)$
- Recall its CTFT

$$X(\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = \frac{1}{j\omega + a}$$

  – where the integration converges when $a > 0$.

- The Laplace transfrom of $x(t)$ is

$$X(s) = X(\sigma + j\omega) = \int_0^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt = -\frac{e^{-(\sigma+a)t}e^{-j\omega t}}{j\omega + (\sigma + a)}\Bigg|_0^{\infty} = \frac{1}{j\omega + \sigma + a}$$

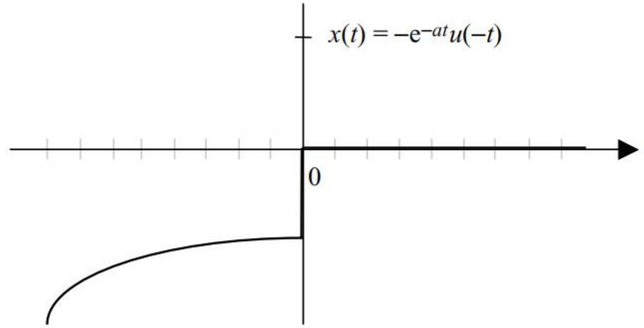
  – where the integration converges when $\sigma + a > 0$.

- In other words, the Laplace transform of $x(t)$ is

$$X(s) = \frac{1}{s+a}, \quad \boldsymbol{\mathcal{R}e\{s\} > -a}$$

  – In this example, the Laplace transform of $x(t)$ exist even if the CTFT of $x(t)$ does not exist (when $a < 0$).

Example 2



- Let's look at $x(t) =- e^{-at}u(-t)$
- The Laplace transfrom of $x(t)$ is

$$X(s) =- \int_{-\infty}^{0} e^{-(s+a)t}dt = \frac{e^{-(s+a)t}}{s+a}\Big|_{-\infty}^{0}$$

  – For convergence, we need $\mathcal{Re}\{s\} + a < 0$, such that $e^{-(\sigma+a)t}$ goes to zero when t goes to $-\infty$.

  – Therefore, $X(s) = \frac{1}{s+a}$, $\boldsymbol{\mathcal{Re}\{s\} <- a}$

- We see that the Laplace transform expressions for $e^{-at}u(t)$ and $- e^{-at}u(-t)$ are both $\frac{1}{s+a}$, but the ranges of $\mathcal{Re}\{s\}$ that ensures convergence are different.
- In other words, to obtain the Laplace transform, we need to specify both the expression of $X(s)$ and the region of convergence (ROC) that tells us the range of values of s when $X(s)$ is valid.

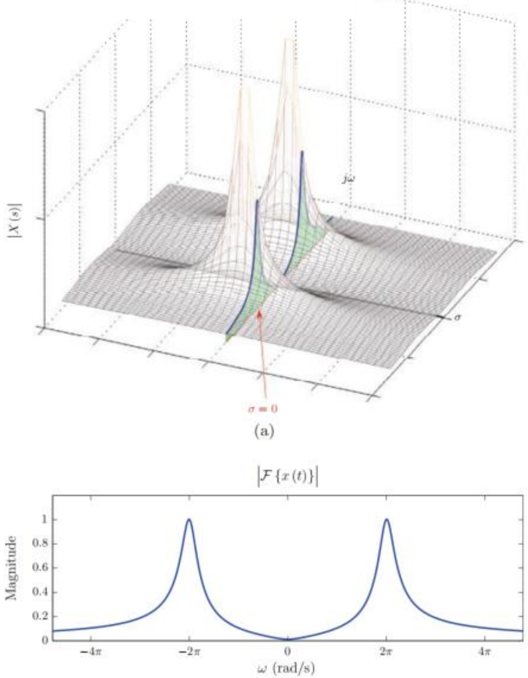
1.4 Fourier Transform VS Laplace Transform

- Fourier Transform VS Laplace Transform

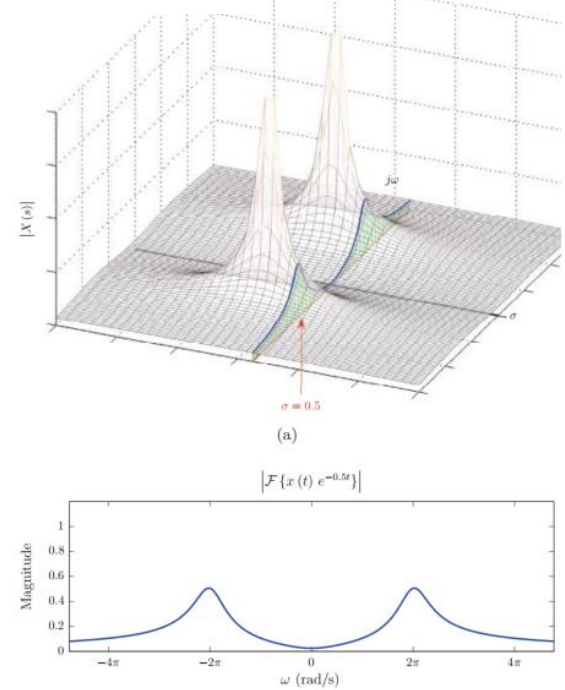
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad\qquad X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- Two cases:

$$X(s)\Big|_{s=0+j\omega} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \mathcal{F}\{x(t)\}$$



$$X(s)\Big|_{s=\sigma_1+j\omega} = \mathcal{F}\left\{x(t)\, e^{-\sigma_1 t}\right\}$$



Importance of Laplace Transform

- The Laplace Transform is a tool used to convert an operation of a real time domain variable t into an operation of a complex domain variable s;
- By operating on the transformed complex signal rather than the original real signal it is often possible to substantially simplify a problem involving:
  - Linear Differential Equations
  - Convolutions
  - Systems with Memory
- Operations on signals involving linear differential equations may be difficult to perform strictly in the time domain
- These operations may be Simplified by:
  - Converting the signal to the Complex Domain
  - Performing Simpler Equivalent Operations
  - Transforming back to the Time Domain

2.1 Convergence

- Finding the Laplace Transform requires integration of the function from minus infinity to infinity

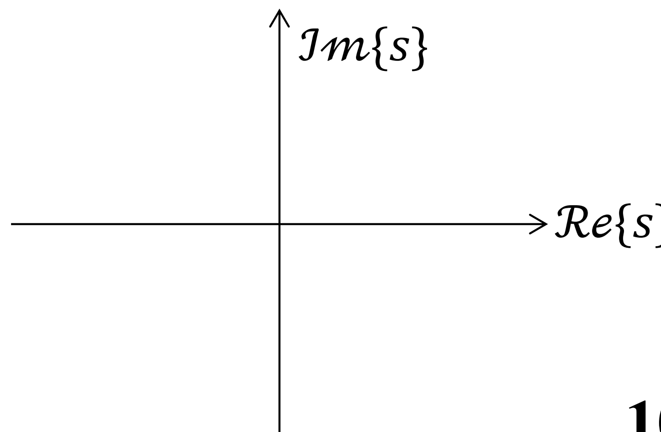
$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

  – For X(s) to exist, the integral must converge;

  – Convergence means that the area under the integral is finite;

  – Laplace Transform, X(s), exists only for a set of points in the ***s*-domain** called the Region of Convergence (ROC)

    - *s* is the complex frequency;
    - *s*-domain is the complex plane.



– In general, $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, $\mathcal{Re}\{s\}, s \in ROC$.

2.1 Convergence - Magnitude of X(s)

- For a complex X(s) to exist, it's magnitude must converge

$$|X(s)| < \infty$$

- By replacing s with $\sigma + j\omega$, it can be rewritten as:

$$|X(s)| = \left|\int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt\right| \leq \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}e^{-j\omega t}|dt$$

$$\leq \int_{-\infty}^{\infty} |x(t)||e^{-\sigma t}||e^{-j\omega t}|dt$$

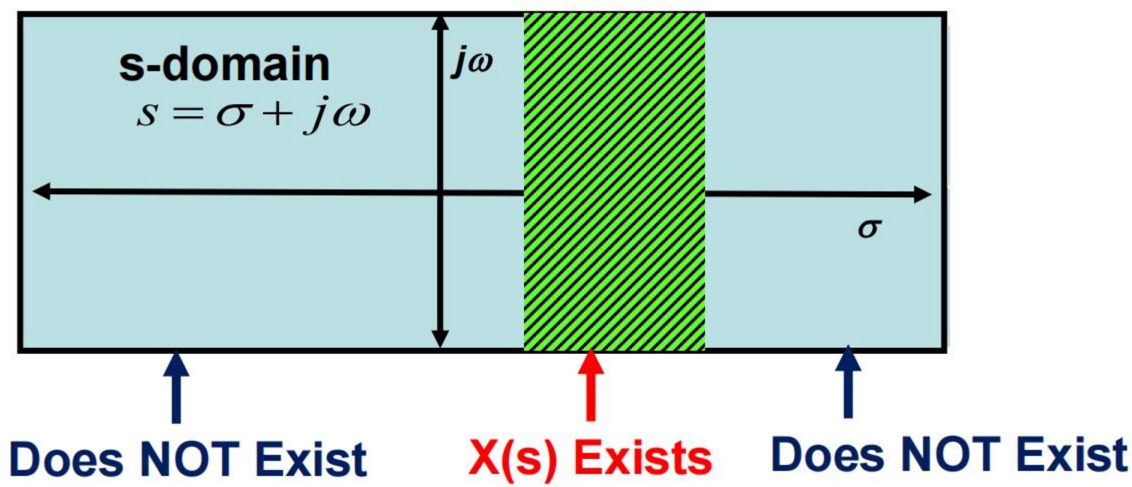
  - $|e^{-\sigma t}|$ is a real number, therefore $|e^{-\sigma t}| = e^{-\sigma t}$;
  - $|e^{-j\omega t}|$ is a complex number with a magnitude of 1;
  - Therefore, the Magnitude Bound of X(s) is dependent only upon the magnitude of x(t) and the real part of s:

$$|X(s)| \leq \int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt$$

The Region of Convergence (ROC, range of complex frequency) is defined as the region where the real part of *s* meets this criteria

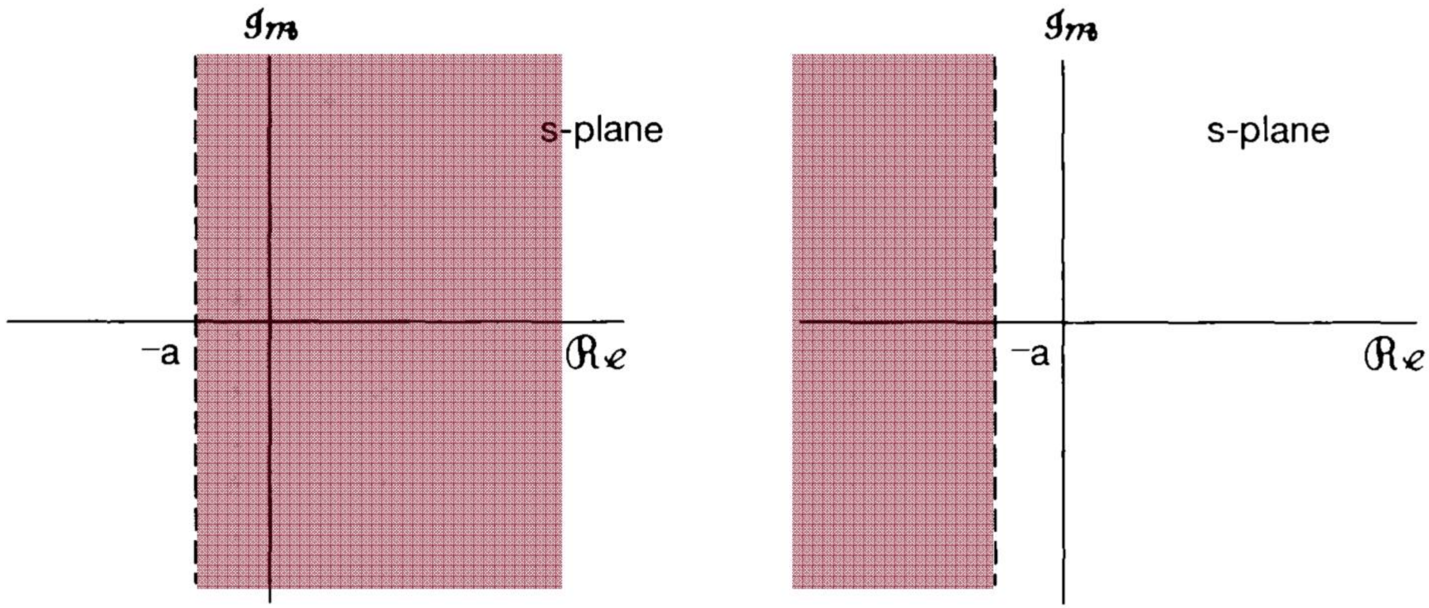
2.1 ROC Graphical Depiction

- The *s*-domain can be graphically depicted as a 2D plot of the real and imaginary portions of *s*;
- In general the ROC is a stripe in the complex *s*-domain;



2.1 ROC plots of Example 1 & 2

- The ROC of the Laplace transforms of $e^{-at}u(t)$ and $-e^{-at}u(-t)$ can be plotted in the complex plane (also called s-plane) as:



For $e^{-at}u(t)$, ROC is $\mathcal{R}e\{s\} > -a$. For $-e^{-at}u(-t)$, ROC is $\mathcal{R}e\{s\} < -a$.

right-sided ROC left-sided ROC

Quiz 1

- 1. Find the Laplace transform of

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

- 2. Find the Laplace transform of

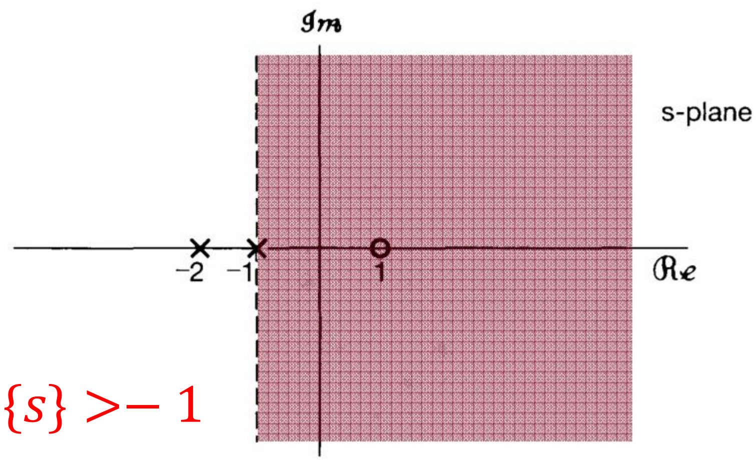
$$x(t) = e^{-|t|}$$

2.2 Zeros and Poles

- In the previous example

$$X(s) = \frac{s-1}{(s+2)(s+1)}, \mathcal{R}e\{s\} > -1$$

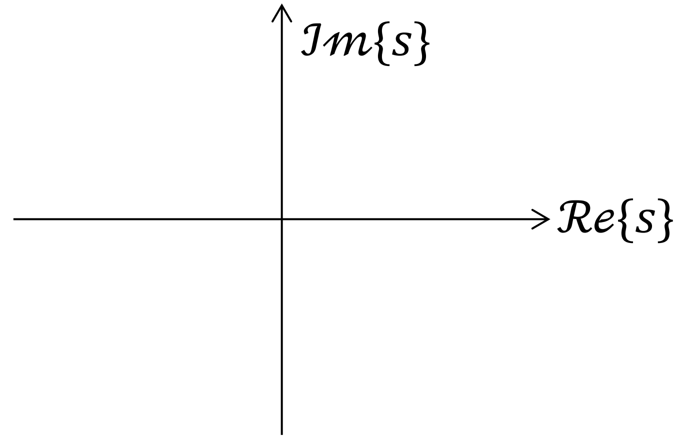
  - we see that $X(1) = 0$ and $X(s) \to \infty$ when $s = -1, -2$.
  - also, $X(s)$ goes to zero if s goes to infinity.
- The values of s that makes $X(s) = 0$ are called the zeros;
- The values of s that makes $X(s) = \infty$ are called the poles;
- The zeros and the poles (apart from those at infinity) can be plotted in the s-plane alongside with the ROC:
  - "o" for zeros.
  - "x" for poles.



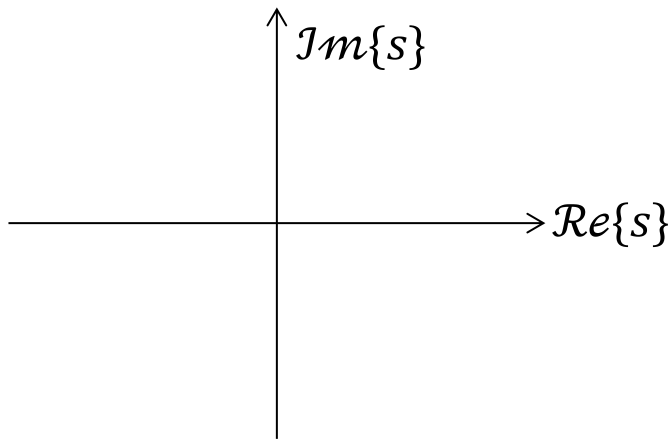
$$X(s) = \frac{s-1}{(s+2)(s+1)}, \mathcal{R}e\{s\} > -1$$

2.2 Zeros and Poles

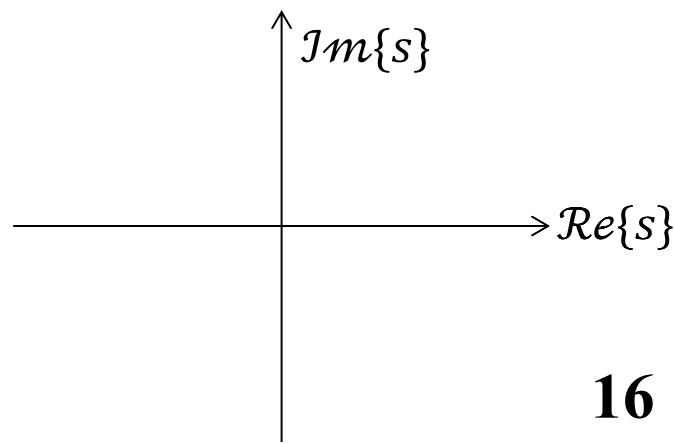
- Determine the poles and zeros of the following systems:
  - a) $H(s) = \frac{(s+4)(s+5)}{s^2(s+2)(s-2)}$



  - b) $H(s) = \frac{s^2+1}{s^2+2s+1}$



  - c) $H(s) = \frac{2s+5}{s^2+s-6}$



Quiz 2

- 1. Find the Laplace transform of

$$x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$$

- 2. Find the Laplace transform of

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

2.3 ROC Properties

- The ROC can provide some information about the time-domain function.

- For example, we saw that the Laplace transforms of $e^{-at}u(t)$ and $-e^{-at}u(-t)$ have the same algebraic expression, but the ROCs $\mathcal{R}e\{s\} > -a$ and $\mathcal{R}e\{s\} < -a$ are completely different.

- In fact, time-domain characteristics of a function would impose constraints to the ROC of the Laplace transform.

- Let's see the properties of ROC.

2.3 ROC Properties

- Property 1: The ROC of $X(s)$ consists of stripes parallel to the jω-axis in the s-plane.
  - This can be seen because the Laplace transform of $x(t)$ is equivalent to the Fourier transform of $x(t)e^{-\sigma t}$, where only $\mathcal{Re}\{s\} = \sigma$ from s would affect the convergence.

- Property 2: For rational Laplace transforms, the ROC does not contain any poles.
  - A rational Laplace transform means $X(s) = \frac{N(s)}{D(s)}$, where N(s) and D(s) are polynomials of s. If the ROC contains a pole, then D(s) = 0 at that pole and X(s) goes to infinity.

2.3 ROC Properties

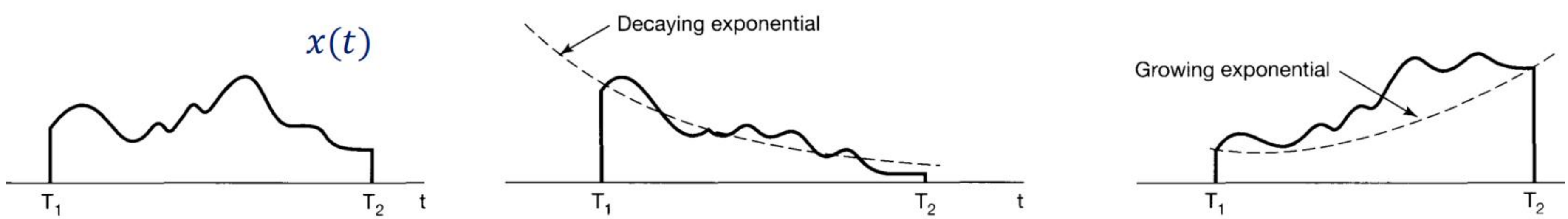
- Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

- Again, the Laplace transform of $x(t)$ is equivalent to the Fourier transform of $x(t)e^{-\sigma t}$.

- When $x(t)$ is absolutely integrable (i.e. $\int_{-\infty}^{\infty} |x(t)|dt < \infty$) and of finite duration, either

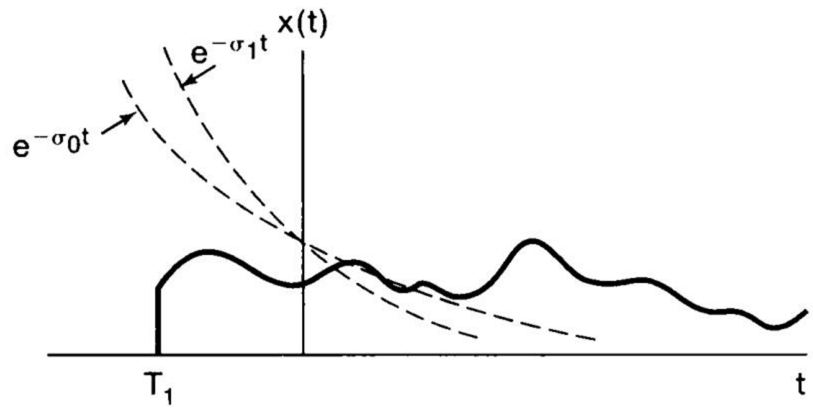
$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)|dt \text{ for } \sigma > 0$$

$$\text{or } \int_{T_1}^{T_2} |x(t)|e^{-\sigma t}dt \ < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)|dt \ \text{ for } \sigma < 0$$



2.3 ROC Properties

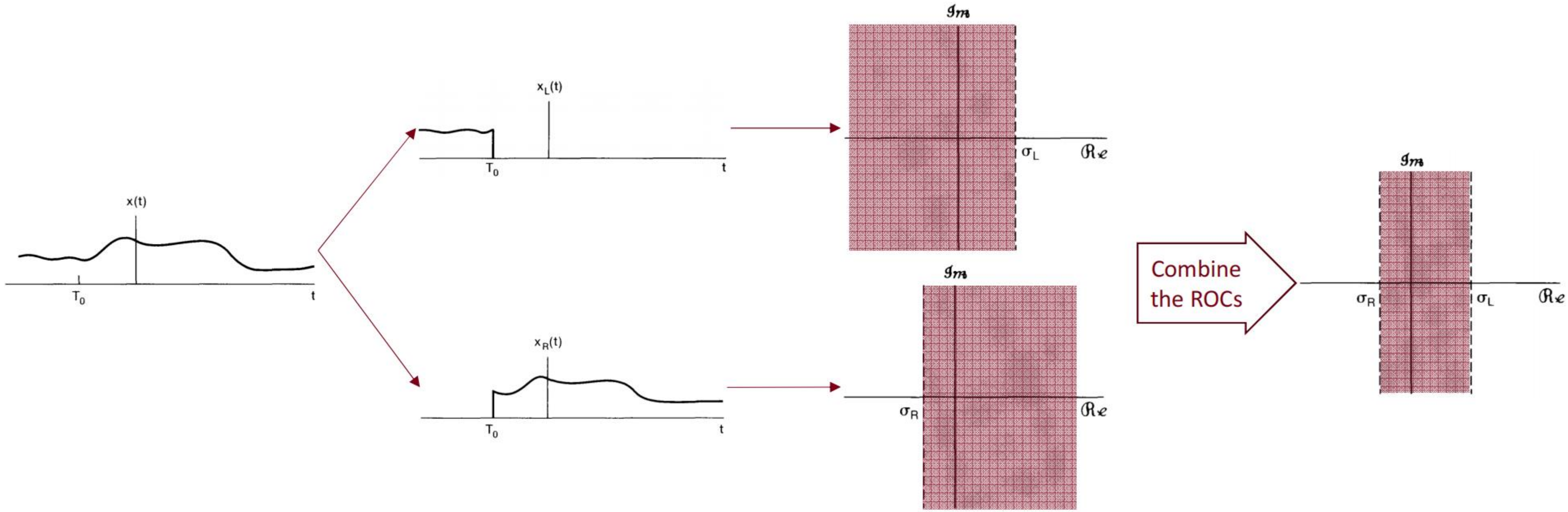
- Property 4: If $x(t)$ is right sided and the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e\{s\} > \sigma_0$ will also be in the ROC (i.e. right half of the s-plane).
  - as mentioned before, a right sided signal means $x(t) = 0$ for $t < T_1$, where T_1 is some finite time.
  - similarly, a left sided signal means $x(t) = 0$ for $t > T_1$.



- Property 5: If $x(t)$ is left sided and the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e\{s\} < \sigma_0$ will also be in the ROC (i.e. left half of the s-plane).

2.3 ROC Properties

- Property 6: If $x(t)$ is two sided (i.e. $x(t)$ is of infinite extent for both $t > 0$ and $t < 0$), and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a stripe in the s-plane that includes the line $\mathcal{R}e\{s\} = \sigma_0$.
  - This can be seen by decomposing a two-sided signal into a right-sided signal and a left-sided signal and then apply Properties 4 and 5.

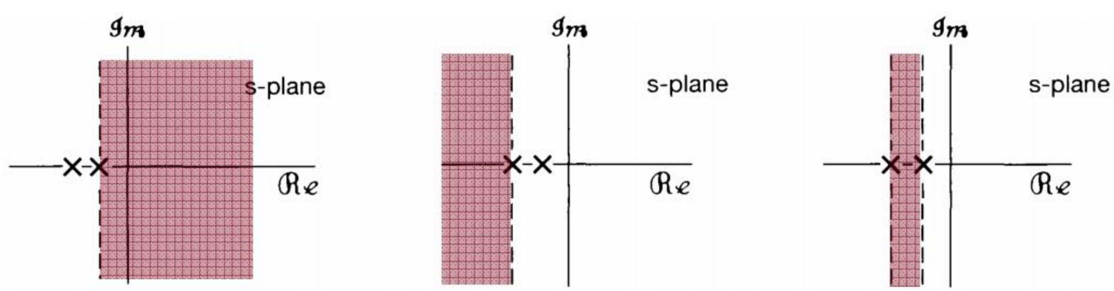


2.3 ROC Properties

- <u>Property 7</u>: If the Laplace transform $X(s)$ of $x(t)$ is rational, then:
  - its ROC is bounded by poles or extends to infinity.
  - in addition, no poles of $X(s)$ are contained in the ROC.

- <u>Property 8</u>: If the Laplace transform $X(s)$ of $x(t)$ is rational, then:
  - if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole;
  - if $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole;

- Example: For $X(s) = \frac{1}{(s+1)(s+2)}$, there can be three possible pole-zero plots:



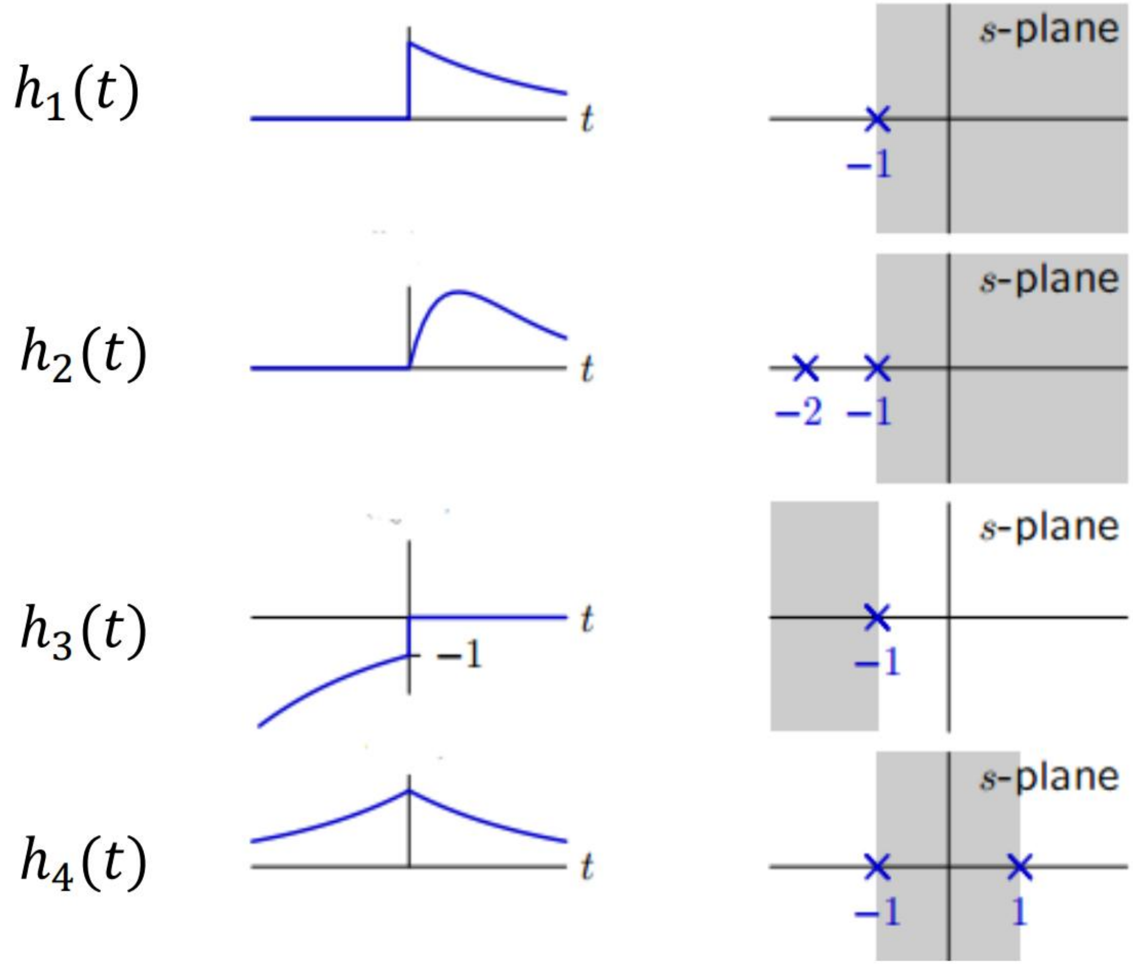
2.4 Causality

- The ROC associated with the system function for a causal system is a right-half plane.
- For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

- Example: consider the following systems:
  - Case 1: $h(t) = e^{-t}u(t)$
  - Case 2: $h(t) = e^{-|t|}$
  - Case 3: $H(s) = \frac{e^{s}}{s+1}, \mathcal{R}e\{s\} > -1$

2.4 Causality

Causality implies that the ROC is to the right of the rightmost pole, but the converse is not in general true, unless the system function is rational.

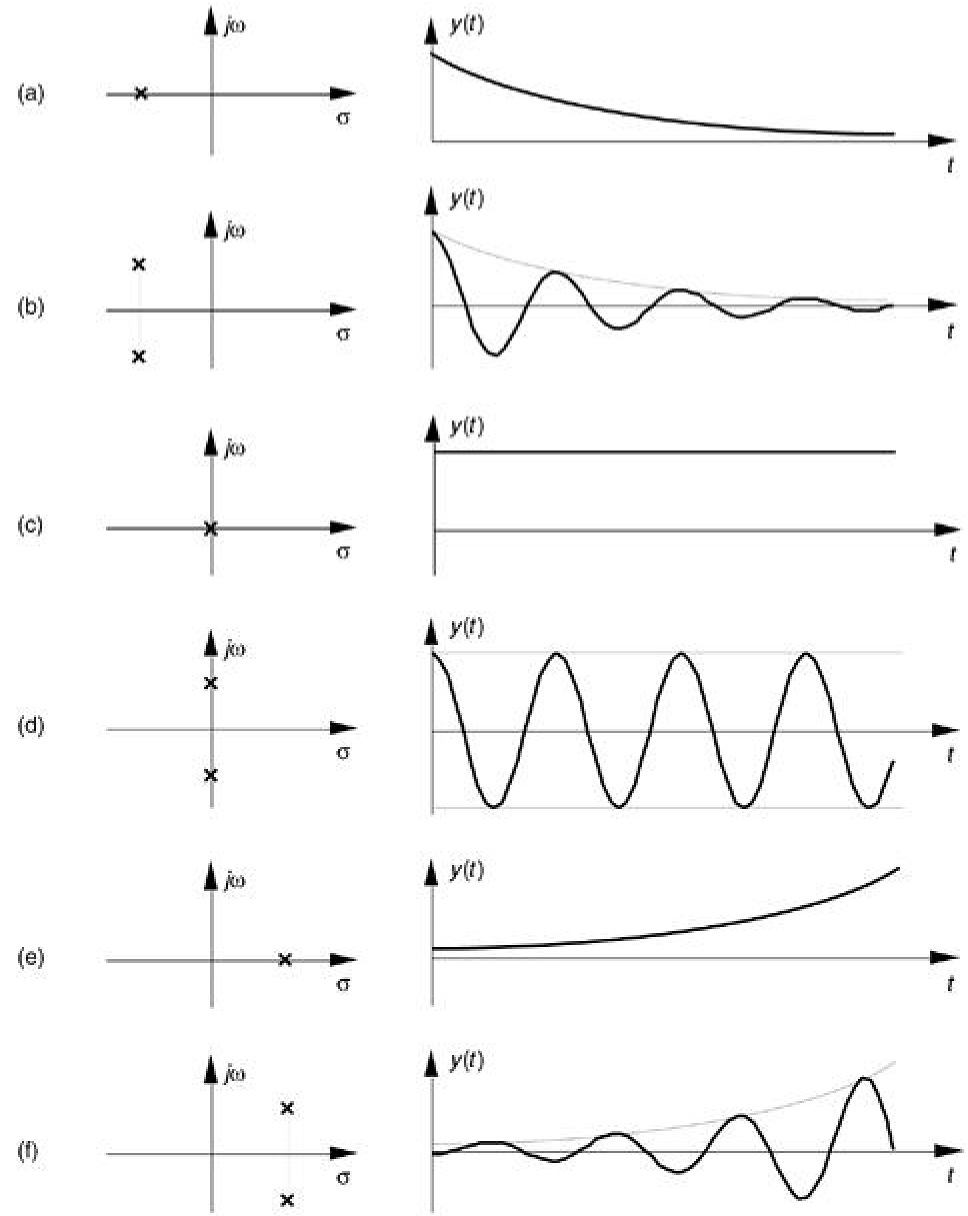


2.5 Stability

- Recall the definition of BIBO stable: A system is referred to as BIBO stable if an arbitrary bounded-input signal always produces a bounded-output signal.
- A LTI system is stable if and only if the ROC of its system function $H(s)$ includes the $j\omega$-axis (i.e. $\mathcal{R}e\{s\} = 0$)
  - Because: stability of the system = $h(t)$ is absolutely integrable $\rightarrow H(\omega)$ converge $\rightarrow$ ROC of $H(s)$ includes $j\omega$-axis
- Example: consider the following systems:
  - Case 1: $h(t) = e^{-t}u(t)$
  - Case 2: $h(t) = e^{-|t|}$
  - Case 3: $H(s) = \frac{e^s}{s-1}, \mathcal{R}e\{s\} > 1$

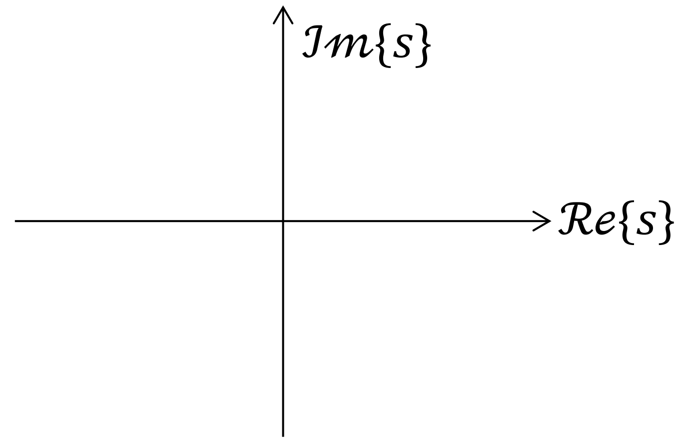
2.5 Stability

A causal system with rational system function H(s) is stable if and only if all the poles of H(s) lie in the left-half of the s-plan –i.e., all of the poles have negative real parts.

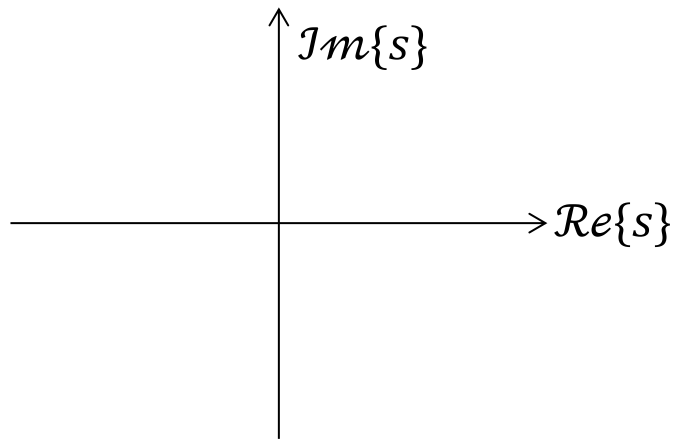


Quiz 3

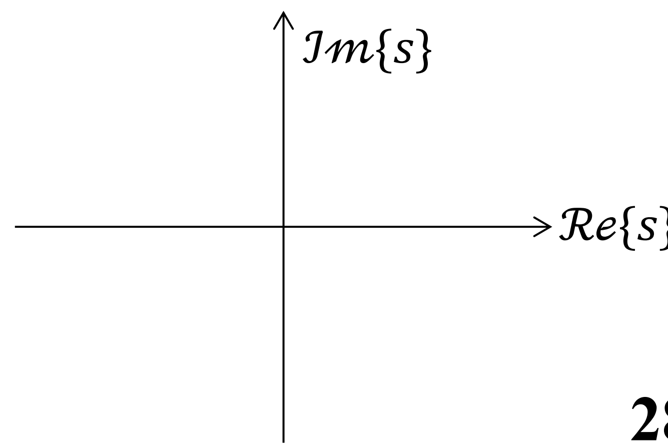
- Assuming that the systems are causal, determine if the systems are BIBO stable:
  - a) $H(s) = \dfrac{(s+4)(s+5)}{s^2(s+2)(s-2)}$



  - b) $H(s) = \dfrac{s^2+1}{s^2+2s+1}$



  - c) $H(s) = \dfrac{2s+5}{s^2+s-6}$



Next ...

- Inverse Laplace Transform
- Geometric Evaluation of CTFT based on LT
- Unilateral Laplace Transform
- Analysis of LTIC systems using LT

List of Abbreviations

- FS (CTFS) - Fourier Series
- FT (CTFT) - Fourier Transform
- LT - Laplace Transforms
- ROC - Region of Convergence
- Re - Real part
- Im - Imaginary part
- TD - Time Domain
- FD - Frequency Domain
- sD - s-Domain (complex frequency domain)