# CAN207 Continuous and Discrete Time Signals and Systems

## Lecture 22

## DFT (Discrete Fourier Transform)

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- 1. Definition of DFT
  - 1.1 DFT definition
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# 1.1 From DTFT to DFT

- DTFT is an important tool in digital signal processing, as it provides the spectral content of a discrete time signal.
  - However, the computed spectrum, $X(\omega)$ is a continuous function of $\omega$, and therefore cannot be computed using a computer
  - We need a method to compute the spectral content of a discrete time signal and have a spectrum – actually a discrete function
- A straightforward solution: Simply sample the frequency variable $\omega$ of the DTFT in frequency domain in the $[0, 2\pi)$ interval.
  - If we want $N$ points in the frequency domain, then we divide $\omega$ in the $[0, 2\pi)$ interval into $N$ equal intervals.
  - Then the discrete values of $\omega$ are $0, \frac{2\pi}{N}, \frac{2\times 2\pi}{N}, \frac{3\times 2\pi}{N}, \ldots, \frac{(N-1)\times 2\pi}{N}$

# 1.1 Discrete Fourier Transform (DFT)

- The simplest relation between a length-N sequence $x[n]$, defined for $0 \leq n \leq N-1$, and its DTFT $X(\omega)$ is obtained by uniformly sampling $X(\omega)$ on the $\omega$-axis $0 \leq \omega < 2\pi$ with

$$\omega_k = \frac{2\pi k}{N}, \qquad k = 0,1,\ldots,N-1$$

- From the definition of DTFT, we thus have

$$X(z) = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = X\left(e^{j\frac{2\pi k}{N}}\right)$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}, \qquad k = 0,1,\ldots,N-1$$

Twiddle factor

- Using the notation $W_N = e^{-j\frac{2\pi}{N}}$, the DFT is usually expressed as

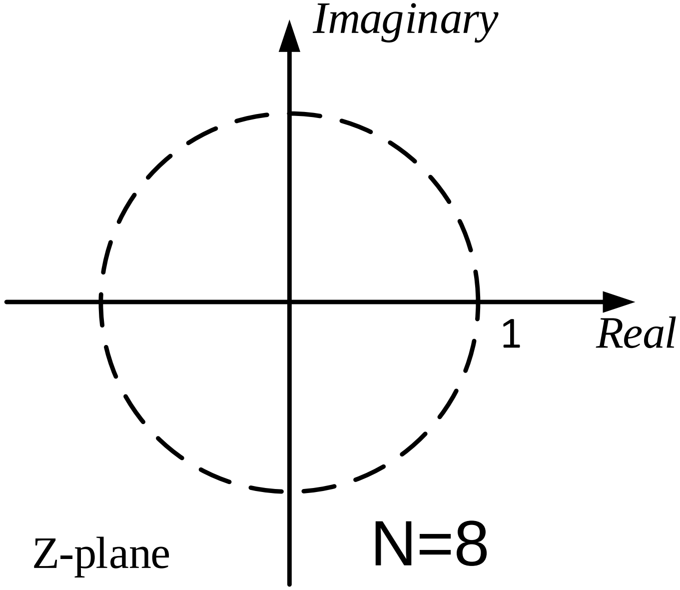
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0,1,\ldots,N-1$$

# 1.1 Twiddle Factor $W_N$

- The twiddle factor $W_N$ is important in DFT

$$W_N = e^{-j\frac{2\pi}{N}} \qquad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, 0 \le k \le N-1$$

  – Complex exponential wheel

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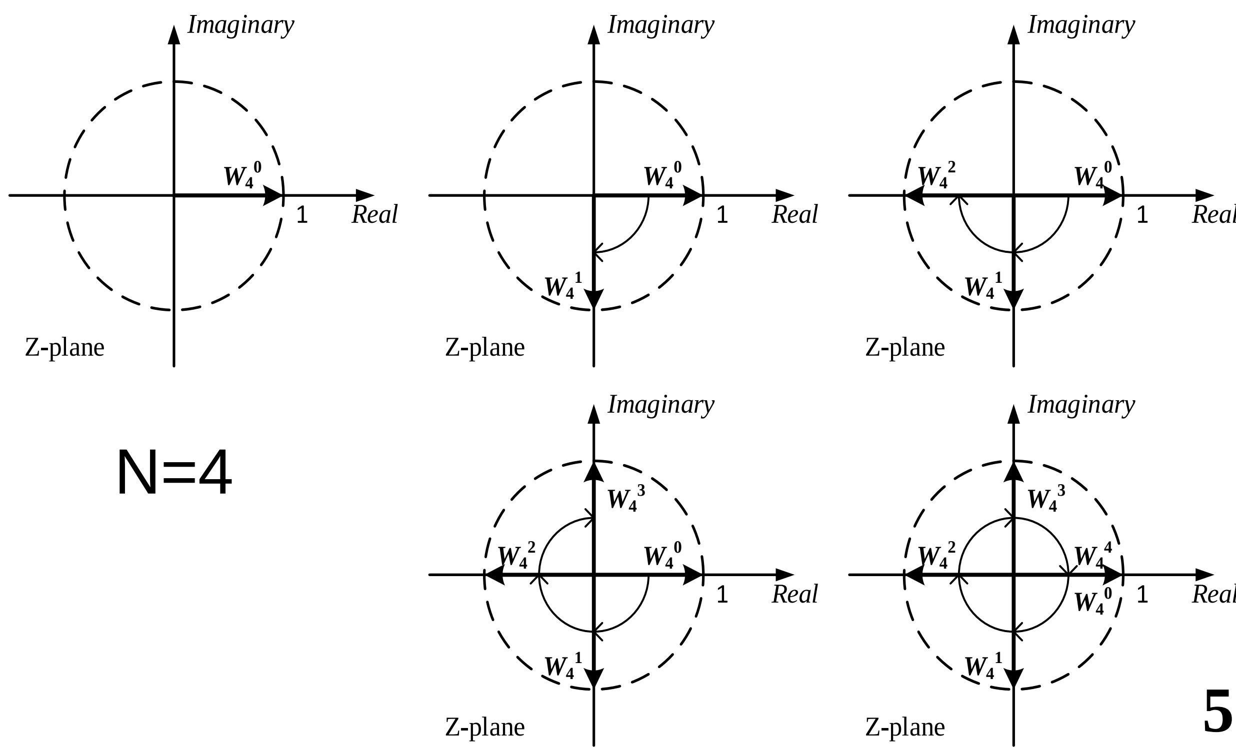



N=4

# 1.2 Inverse Discrete Fourier Transform (IDFT)

- The IDFT is defined as

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi n}{N}k}, \qquad n = 0,1,\ldots,N-1$$

- Using the notation $W_N = e^{-j\frac{2\pi}{N}}$, the IDFT is usually expressed as

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad n = 0,1,\ldots,N-1$$

- To verify the above expression, we multiply both sides of IDFT equation by $e^{-j\frac{2\pi kn}{N}}$ and sum the result from $n = 0$ to N-1.

# 1.2 DFT Pair

- The analysis equation

$W_N = e^{-j\frac{2\pi}{N}}$

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0,1,\ldots,N-1}$$

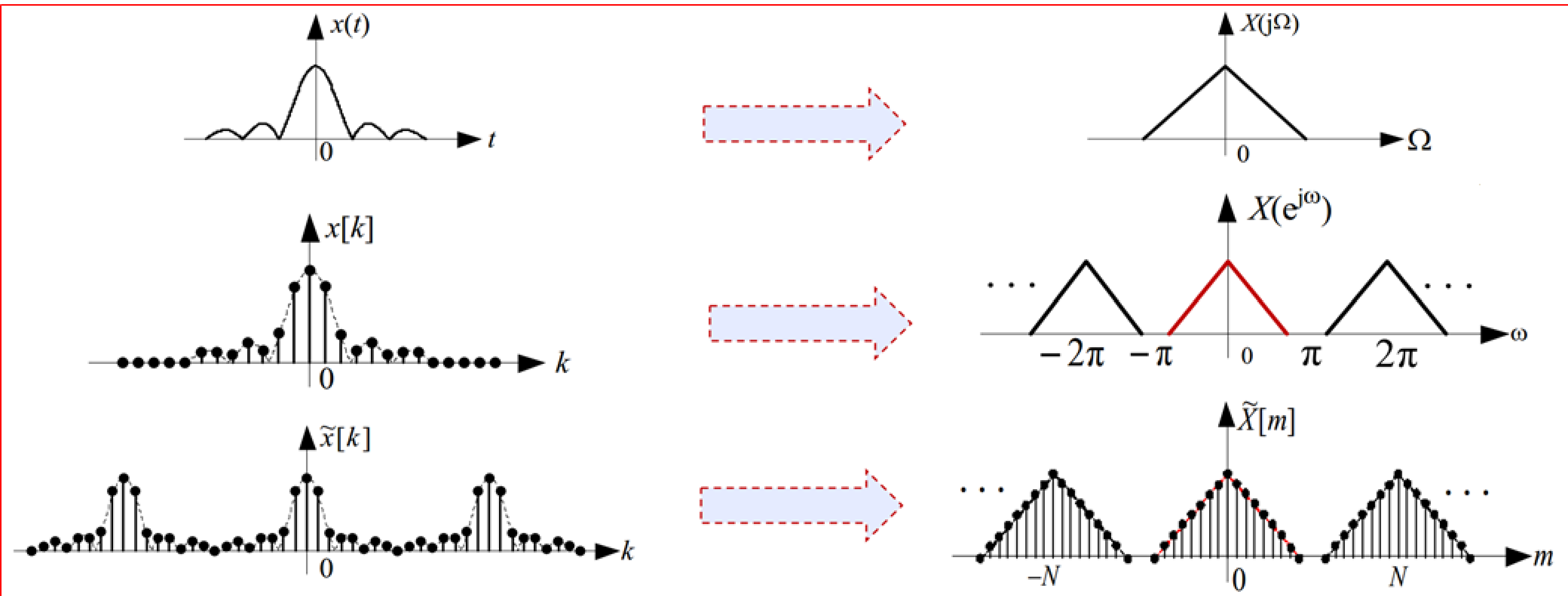
- The synthesis equation

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad n = 0,1,\ldots,N-1}$$

- The DFT pair is denoted as $x[n] \overset{DFT}{\longleftrightarrow} X[k]$

# 1.3 CTFT - DTFT - DFT

| | | | |
|---|---|---|---|
| Time Domain | $\boldsymbol{x(t)}$ | $\boldsymbol{x[k]}$ | $\boldsymbol{\tilde{x}[k]}$ |
| | $\updownarrow$ CTFT | $\updownarrow$ DTFT | $\updownarrow$ DFT |
| Frequency Domain | $\boldsymbol{X(j\Omega)}$ | $\boldsymbol{X(e^{j\omega})}$ | $\boldsymbol{X[m]}$ |

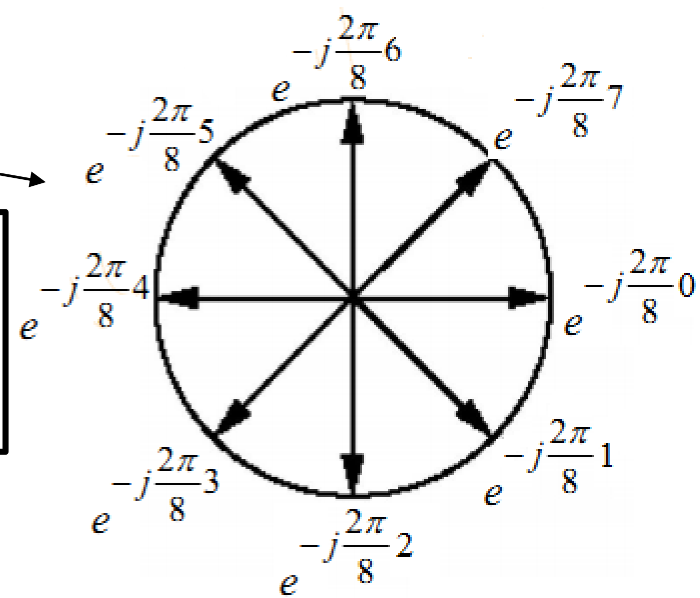


# 2.1 Computing DFT

- For any given k, the DFT is computed by multiplying each x[n] with each of the complex exponentials $W_N^{nk} = e^{-j2\pi nk/N}$ and then adding up all these components
- If, for example, we wish to compute an 8-point DFT, the complex exponentials are 8 unit vectors placed at equal distances from each other on the unit circle

Complex exponential wheel

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0,1,\ldots,N-1}$$



# *Example 1*

- Find the DFT of a 4-point sequence $x_4[n]=\{1,1,1,1;\ n=0,1,2,3\}$

# Example 2

- Find the 8-point DFT of the zero-padded sequence $x_8[n]=\{1,1,1,1,0,0,0,0;\ n=0,1,2,3,4,5,6,7\}$

# Compare Example 1 and 2

- Find the DTFT of $x_4[k]$ and $x_8[k]$

$$X(\omega) = e^{-j\frac{3}{2}\omega}\left[2cos\left(\frac{\omega}{2}\right) + 2\cos\left(\frac{3\omega}{2}\right)\right]$$

- $x_4[k] = \{\mathbf{1}, 1,1,1\}$

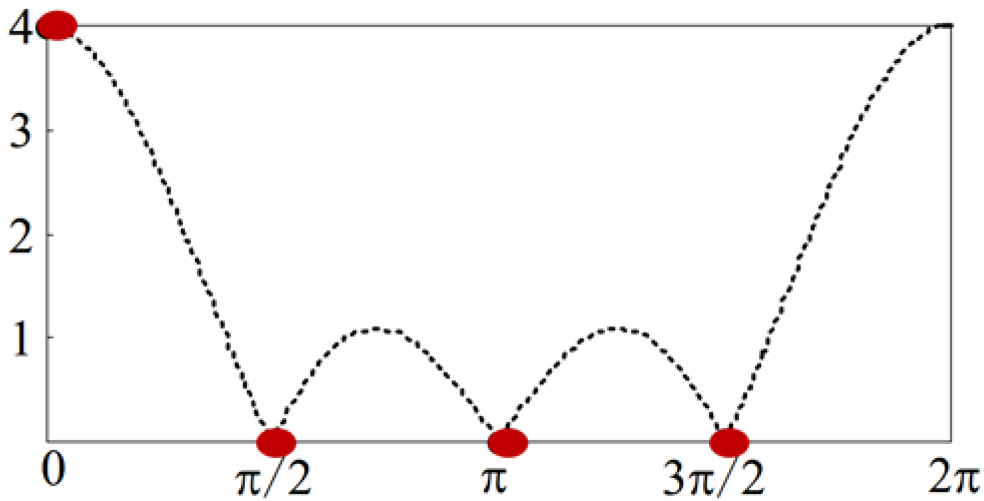
$$\Longrightarrow X_4[m] = X(\omega)\Big|_{\omega=\frac{2\pi m}{4}}, \quad \omega = 0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}$$

- $x_8[k] = \{\mathbf{1}, 1,1,1,0,0,0,0\}$

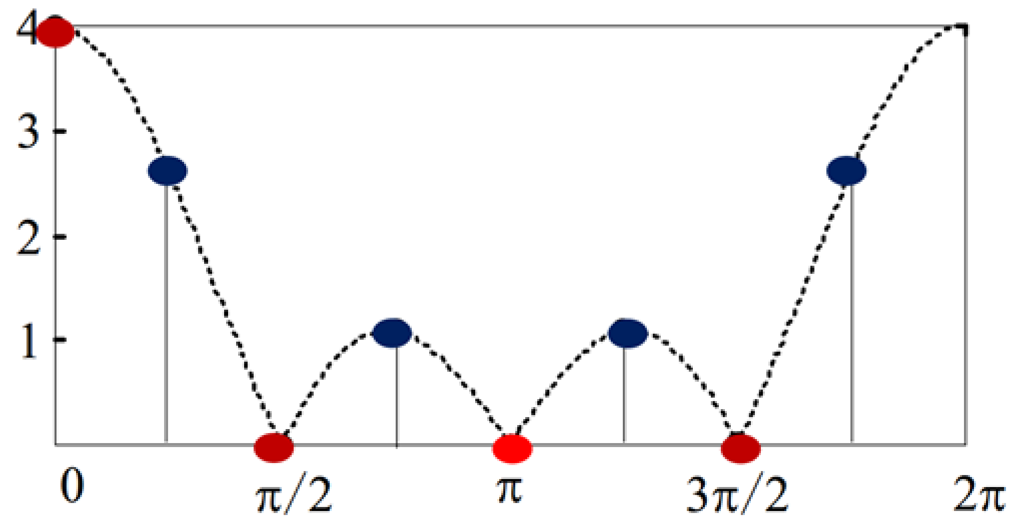
$$\Longrightarrow X_8[m] = X(\omega)\Big|_{\omega=\frac{2\pi m}{8}}, \quad \omega = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$$

# Compare Example 1 and 2

$$X_4[m] = X(\omega)\Big|_{\omega=\frac{2\pi m}{4}}$$




$$X_8[m] = X(\omega)\Big|_{\omega=\frac{2\pi m}{8}}$$




- Zero-padded sequence provides more sampling points in frequency domain, i.e. more details in the spectrum.

# Other DFT Pairs

- Example 3:

$$x[n] = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{other } n \end{cases}$$

- Example 4:

$$x[n] = \delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & \text{other } n \end{cases}$$

- Example 5:

$$g[n] = \cos\left(\frac{2\pi rn}{N}\right), \qquad 0 \le r \le N - 1$$

# 3. DFT Properties

- Like the DTFT, the DFT also satisfies a number of properties which are useful in signal processing
- Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different

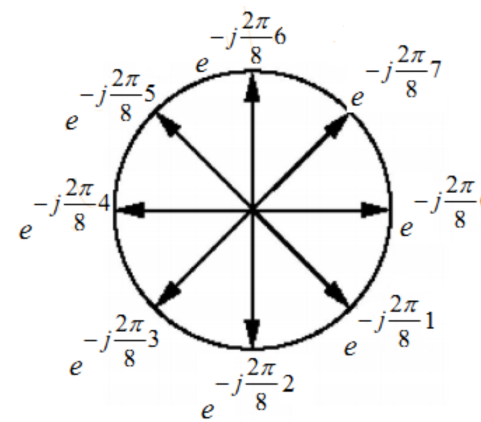
| Type of Property | Length-$N$ Sequence | $N$-point DFT |
|---|---|---|
| | $g[n]$<br>$h[n]$ | $G[k]$<br>$H[k]$ |
| Linearity | $\alpha g[n] + \beta h[n]$ | $\alpha G[k] + \beta H[k]$ |
| Circular time-shifting | $g[\langle n - n_o \rangle_N]$ | $W_N^{kn_o} G[k]$ |
| Circular frequency-shifting | $W_N^{-k_o n} g[n]$ | $G[\langle k - k_o \rangle_N]$ |
| Duality | $G[n]$ | $N g[\langle -k \rangle_N]$ |
| $N$-point circular convolution | $\sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N]$ | $G[k]H[k]$ |
| Modulation | $g[n]h[n]$ | $\frac{1}{N} \sum_{m=0}^{N-1} G[m] H[\langle k - m \rangle_N]$ |

# 3.1 Periodicity in DFT

- DFT is periodic in both *time* and *frequency* domains
  - Even though the original time domain sequence to be transformed is not periodic!
- Periodicity can be explained by
  - Mathematically: both the analysis and synthesis equations are periodic by N.

$$x[n+N] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j\frac{2\pi k(n+N)}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}e^{j\frac{2\pi kN}{N}} = x[n]$$

$$X[k+N] = \sum_{k=0}^{N-1} x[n]e^{-j\frac{2\pi(k+N)n}{N}} = \sum_{k=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}e^{-j\frac{2\pi Nn}{N}} = X[k]$$




  - From the complex exponential wheel: there are only N vectors around a unit circle, the transform will repeat itself every N points.

# 3.2 Parseval's relation

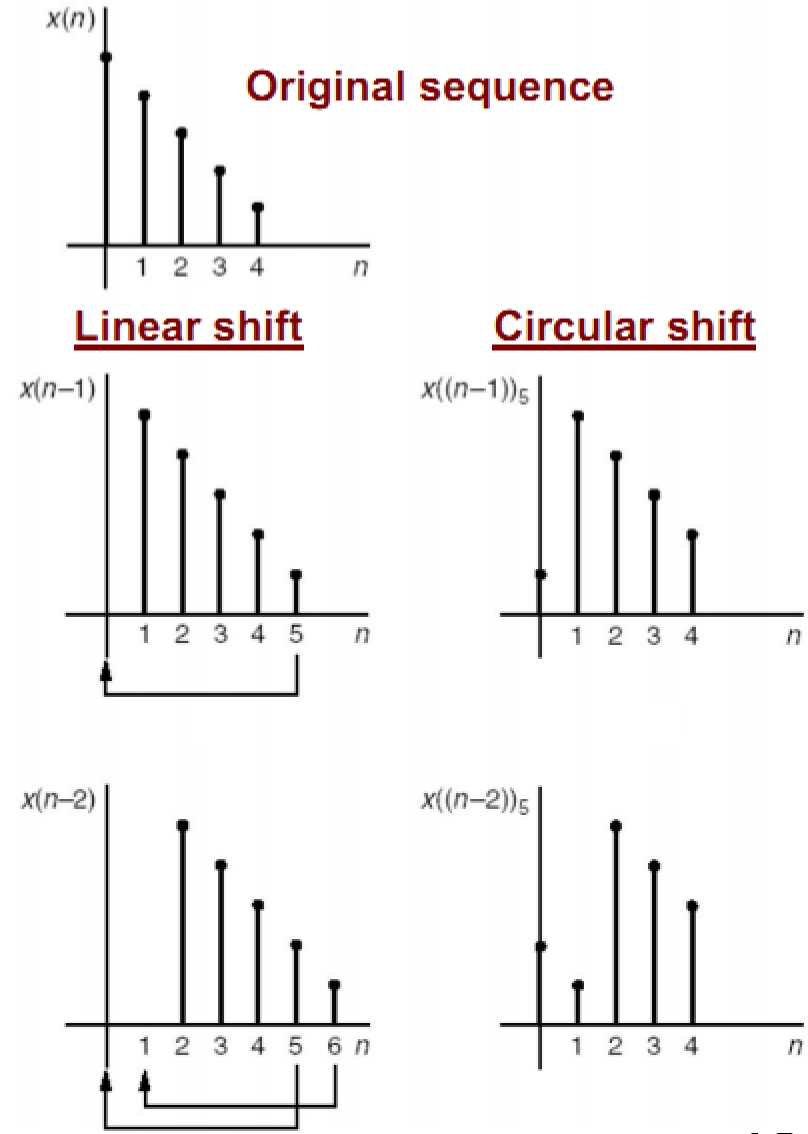
- Similar to DTFT, DFT also holds the Parseval's relation:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

  - The energy conservation in time and frequency domain is valid for DFT too.

# 4.1 Circular shift

- A circularly shifted sequence is denoted by $x[n-m]_{\mathrm{mod}\ N}$
  - $[\quad]_{\mathrm{mod}\ N}$ denotes modulo operation;
  - m is the amount of shift;
  - N is the length of the previously determined base interval
- To obtain a circularly shifted sequence:
  - first linearly shift the sequence by m
  - then rotate the sequence in such a manner that the shifted sequence remain in the same interval originally defined by N.



# 4.1 Circular shift

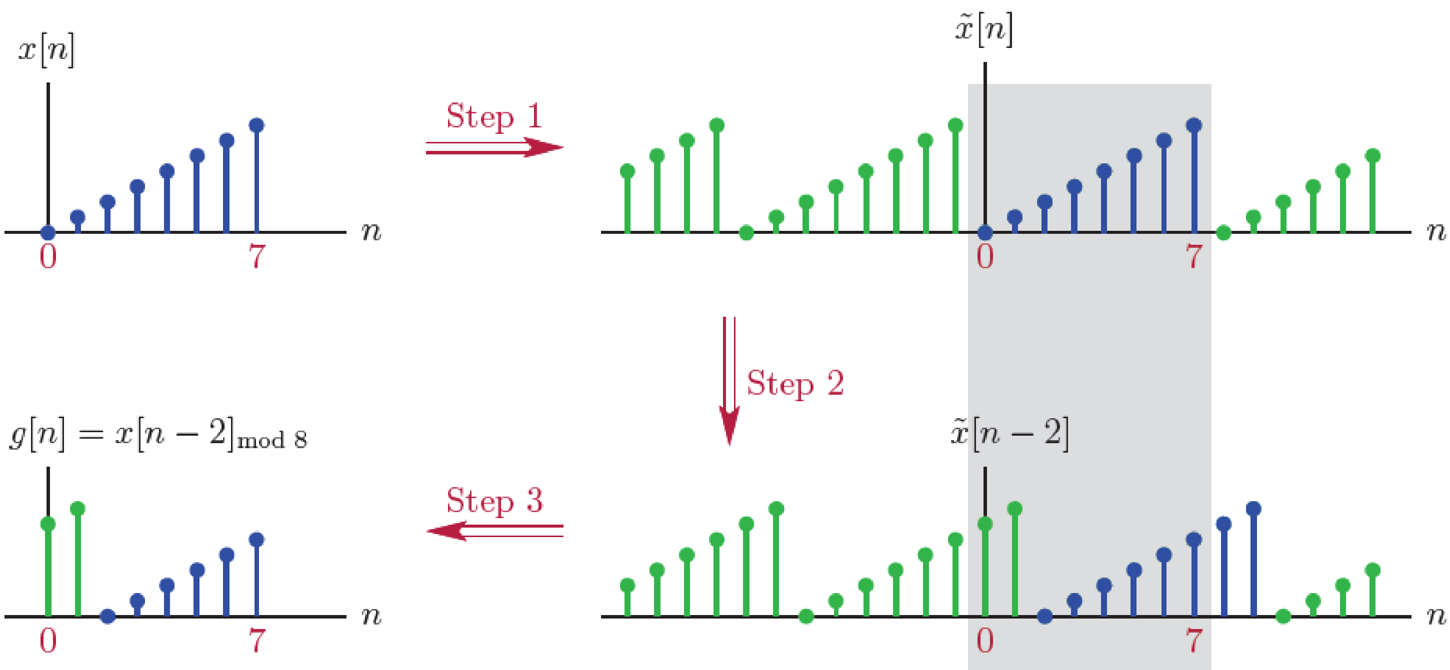
**Step 1.** Obtain periodic extension $\tilde{x}[n]$ from $x[n]$;

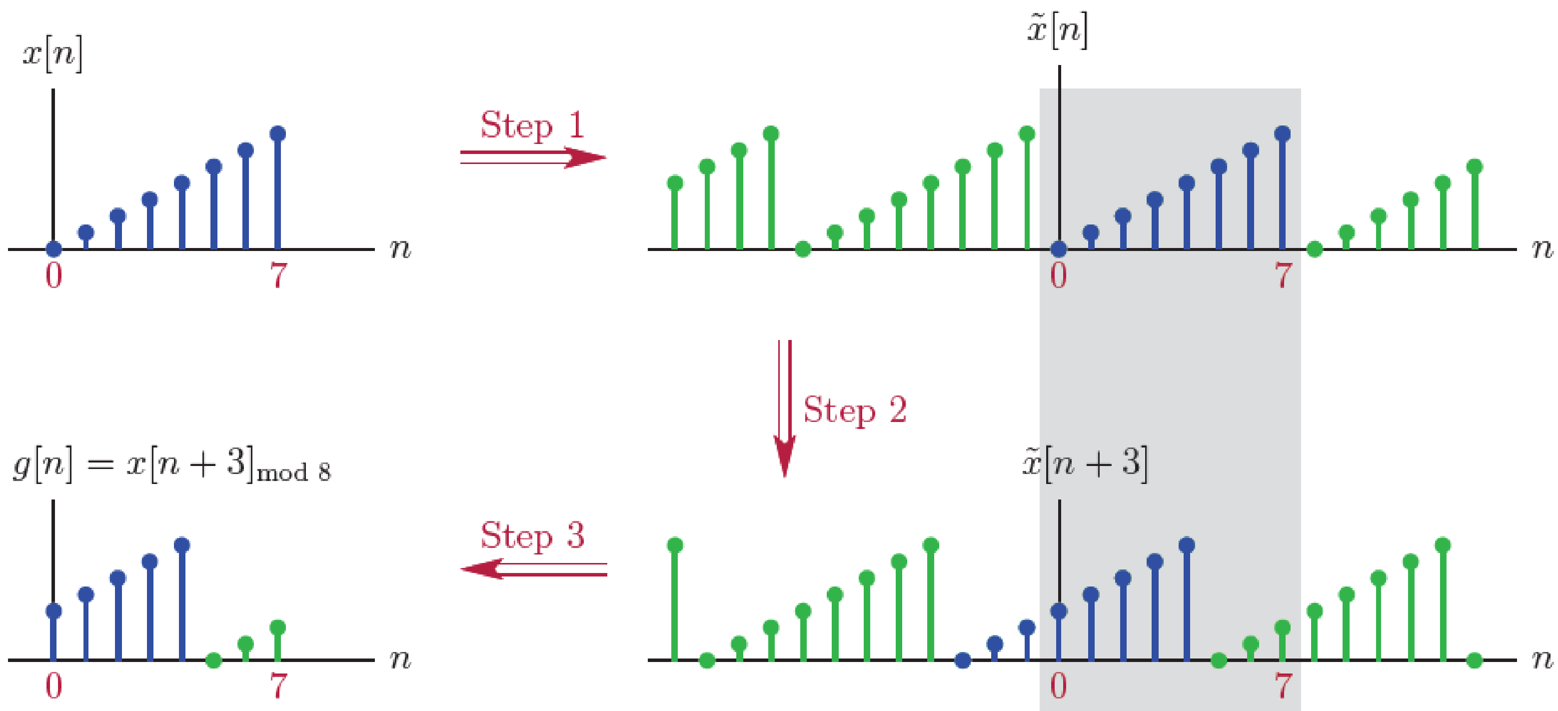
**Step 2.** Apply a time shift to $\tilde{x}[n]$ to obtain $\tilde{x}[n-m]$. $m$ may be positive or negative;

**Step 3.** obtain an length-N signal $g[n]$ by extracting the main period of $\tilde{x}[n-m]$ from 0 to N-1.

The resulting signal $g[n]$ is a circularly shifted version of $x[n]$, i.e.

$$g[n] = x[n-m]_{\text{mod } N}$$





$x[n]$
0
7
$n$
Step 1
$\tilde{x}[n]$
0
7
$n$
Step 2
$g[n] = x[n+3]_{\text{mod } 8}$
0
7
$n$
Step 3
$\tilde{x}[n+3]$
0
7
$n$

# 4.2 Circular reversal

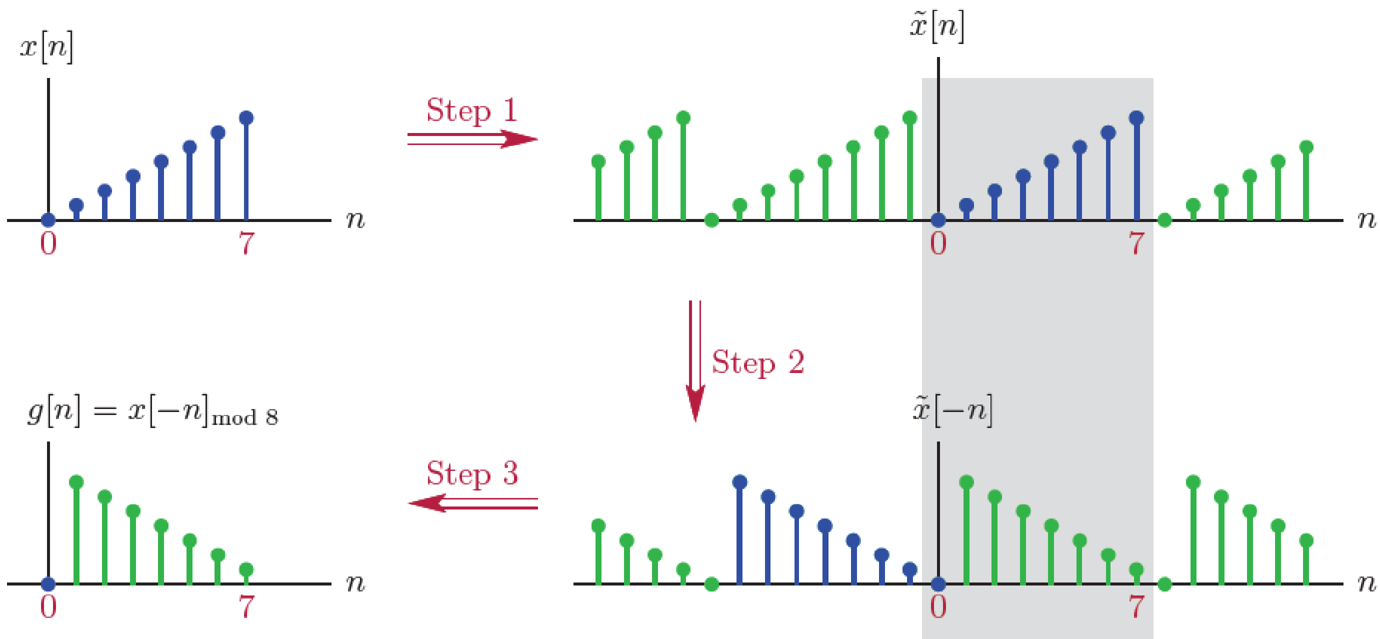
**Step 1.** Obtain periodic extension $\tilde{x}[n]$ from $x[n]$;

**Step 2.** Apply a time reversal operation to $\tilde{x}[n]$ to obtain $\tilde{x}[-n]$;

**Step 3.** obtain an length-N signal $g[n]$ by extracting the main period of $\tilde{x}[-n]$ from 0 to N-1.

The resulting signal $g[n]$ is a circularly reversed version of $x[n]$, i.e.

$$g[n] = x[-n]_{\text{mod } N}$$




Circular time reversal of a length-8 signal.

# Time Reversal and Conjugation Property

## Time reversal

For a transform pair

$$x[n] \overset{\text{DFT}}{\longleftrightarrow} X[k]$$

it can be shown that

$$x[-n]_{\text{mod } N} \overset{\text{DFT}}{\longleftrightarrow} X[-k]_{\text{mod } N}$$

## Conjugation property

For a transform pair

$$x[n] \overset{\text{DFT}}{\longleftrightarrow} X[k]$$

it can be shown that

$$x^*[n] \overset{\text{DFT}}{\longleftrightarrow} X^*[-k]_{\text{mod } N}$$

# *Circularly Conjugate Symmetry*

A length-N signal is circularly conjugate symmetric if it satisfies

$$x^*[n] = x[-n]_{\text{mod } N}$$

or circularly conjugate antisymmetric if it satisfies

$$x^*[n] = -x[-n]_{\text{mod } N}$$

For any signal $x[n]$, it can be decomposed into two components:

$$x_E[n] = \frac{x[n]+x^*[-n]_{\text{mod } N}}{2} \text{ (conjugate symmetric component)}$$

$$x_O[n] = \frac{x[n]-x^*[-n]_{\text{mod } N}}{2} \text{ (conjugate antisymmetric component)}$$

So that

$$x[n] = x_E[n]+x_O[n]$$

In similar manner, its DFT can also be decomposed into

$$X[k] = X_E[k]+X_O[k]$$

# Symmetry of DFT

Symmetry properties of the DFT:

$$x[n]:\ \text{Real},\quad \text{Im}\{x[n]\} = 0 \implies X^*[k] = X[-k]_{\text{mod } N}$$

$$x[n]:\ \text{Imag},\quad \text{Re}\{x[n]\} = 0 \implies X^*[k] = -X[-k]_{\text{mod } N}$$

$$x^*[n] = x[-n]_{\text{mod } N} \implies X[k] : \text{Real}$$

$$x^*[n] = -x[-n]_{\text{mod } N} \implies X[k] : \text{Imag}$$

Consider a length-N signal $x[n]$ that is complex-valued. It can be written as sum of real part and imaginary part, i.e., $x[n] = x_r[n] + jx_i[n]$, then

$$x_r[n] + jx_i[n] \overset{\text{DFT}}{\longleftrightarrow} X_E[k] + X_o[k]$$

Or, it can be written as the sum of conjugate symmetric part and conjugate antisymmetric part, i.e. $x[n] = x_E[n] + x_O[n]$, then

$$x_E[n] + x_O[n] \overset{\text{DFT}}{\longleftrightarrow} X_r[k] + jX_i[k]$$

# Example using Symmetry Property of DFT

The DFT of a length-4 signal $x[n]$ is given by

$$X[k] = \{ \underset{\substack{\uparrow \\ k=0}}{(2+j3)}, (1+j5), (-2+j4), (-1-j3) \}$$

Without computung $x[n]$ first, determine the DFT of $x_r[n]$, the real part of $x[n]$.

Solution: We know from the symmetry properties of the DFT that the transform of the real part of $x[n]$ is the conjugate symmetric part of $X[k]$:

$$\text{DFT}\{x_r[n]\} = X_E[k] = \frac{X[k] + X^*[-k]_{\text{mod } N}}{2}$$

The complex conjugate of the time reversed transform is

$$X^*[-k]_{\text{mod } 4} = \{ \underset{\substack{\uparrow \\ k=0}}{(2-j3)}, (-1+j3), (-2-j4), (1-j5) \}$$

$$X_E[k] = \frac{X[k] + X^*[-k]_{\text{mod } N}}{2}$$

$$X_O[k] = \frac{X[k] - X^*[-k]_{\text{mod } N}}{2}$$

| $k$ | $X[k]$ | $X[-k]_{\text{mod } 4}$ | $X^*[-k]_{\text{mod } 4}$ | $X_E[k]$ | $X_O[k]$ |
|---|---|---|---|---|---|
| 0 | 2+j3 | 2+j3 | 2-j3 | 2 | j3 |
| 1 | 1+j5 | -1-j3 | -1+j3 | j4 | 1+j |
| 2 | -2+j4 | -2+j4 | -2-j4 | -2 | j4 |
| 3 | -1-j3 | 1+j5 | 1-j5 | -j4 | -1+j |

Recall:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad n = 0,1,\ldots,N-1$$

In this case: $N = 4$, $W_N = e^{-j\frac{2\pi}{N}} = e^{-j\frac{\pi}{2}} = -j.$

$$x_r\,[n] = \frac{1}{N}\sum_{k=0}^{3} X_E[k]W_N^{-kn}$$

The conjugate symmetric component of $X[k]$ is

$$X_E[k] = \{\underset{\substack{\uparrow \\ k=0}}{2},\ j4,\ -2,\ -j4\}$$

The real part of $x[n]$ can be found as the inverse transform of $X_E[k]$:

$$x_r[n] = \{\underset{\substack{\uparrow \\ n=0}}{0},\ -1,\ 0,\ 3\}$$

# 4.3 Circular convolution

- The convolution operation involves folding and shifting, we need to redefine the convolution for circularly shifted signals:

$$y[k] = x_1[k] \circledast_N x_2[k] = \sum_{n=0}^{N-1} x_1[\langle n \rangle_N] x_2[\langle k-n \rangle_N]$$

- Expressed in matrix form, take N=4 as an example:

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} x_2[0] & x_2[3] & x_2[2] & x_2[1] \\ x_2[1] & x_2[0] & x_2[3] & x_2[2] \\ x_2[2] & x_2[1] & x_2[0] & x_2[3] \\ x_2[3] & x_2[2] & x_2[1] & x_2[0] \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ x_1[3] \end{bmatrix}$$

# *4.3 Circular convolution*

- Example: Compute the linear and circular convolution of the following sequences x[n]=[**1** 2 3 4], h[n]=[**5** 6 7]

# *Linear VS circular convolution*

- All LTI systems are based on the principle of **linear convolution**, as the output of an LTI system is the linear convolution of the system impulse response and the input to the system, which is equivalent to the product of the respective DTFTs in the frequency domain.
- However, if we use DFT instead of DTFT (so that we can compute it using a computer), then the result appear to be invalid:
  - DTFT is based on linear convolution, and DFT is based on circular convolution, and they are not the same!
  - For starters, they are not even of equal length: For two sequences of length N and M, the linear convolution is of length N+M-1, whereas circular convolution of the same two sequences is of length max(N,M), where the shorter sequence is zero padded to make it the same length as the longer one.

# Quiz

- Calculate the linear convolution and circular convolution of the following sequence:
  - a) $\{x[k]\}=\{\underline{1}, 2, 1\}$, $\{h[k]\}=\{\underline{1}, 0, 2, 0, 1\}$;
  - b) $\{x[n]\}=\{1, \underline{2}, 1\}$, $\{h[n]\}=\{1, \underline{0}, 2, 0, 1\}$;

# *Next ...*

- Revision