CAN207 Continuous and Discrete Time Signals and Systems

Lecture 19

Z-Transform_Part 1

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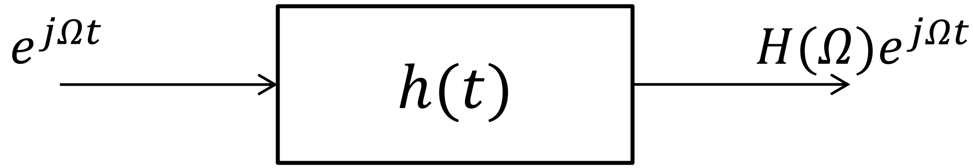
Content

- 1. Definition of z-transform
  - Eigenfunctions
  - Relationship between DTFT and z-transform
  - Visualisation of DTFT and z-transform
  - s-plane to z-plane
- 2. Region of Convergence (ROC)
  - Definition and graphical depiction
  - Zeros and Poles (Zero-pole plot)
  - ROC properties
- 3. Commonly use z-transform pairs

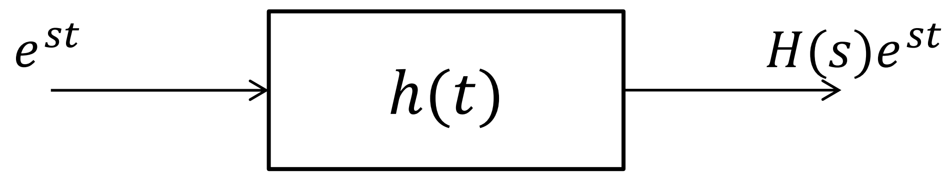
1.1 Eigen functions in CT systems

- A Continuous-Time (LTIC) system:



  - where $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\Omega)$ is a CTFT pair.

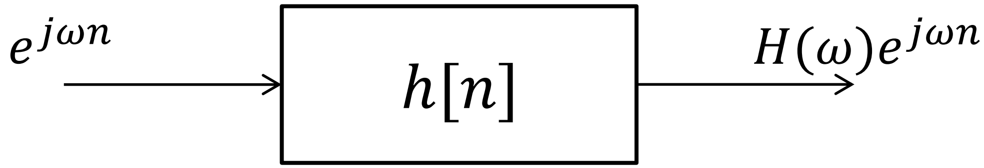
- It can be generalised to:



  - where $h(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s)$ is a Laplace transform pair.
  - s is the complex frequency, relating to the analogue angular frequency by: $s = \sigma + j\Omega$

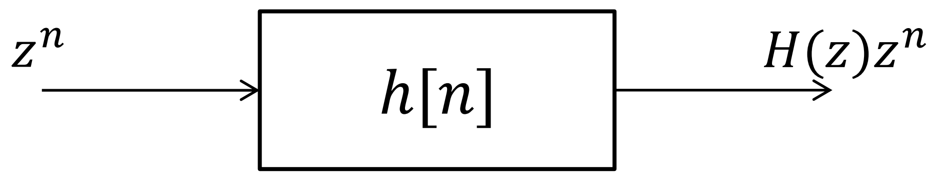
1.1 Eigen functions in DT systems

- In a Discrete-Time (LTID) system:



  - where $h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$ is a DTFT pair.

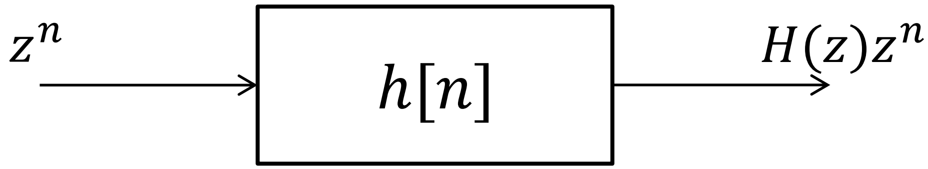
- It can be generalised to:



  - where $h[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z)$ is a z-transform pair.
  - z is another complex frequency, relating to the digital angular frequency by: $z = re^{j\omega}$

1.1 Eigen function for z-transform

- Consider z^n as the input to the DT system.



- The output can be calculated by "convolution sum":

$$z^n * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} = z^n \boxed{\underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}_{H(z)}}$$

definition of z-trans

  - $z^n \to z^n H(z)$
    - $H(z)$ is not a function of n, so it could be considered as the eigen value, while z^n is the eigen function of the LTID system.

1.2 Relationship between DTFT and z-transform

- Definition equation of DTFT:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

- Definition equation of z-transform ($z = re^{j\omega}$):

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=-\infty}^{\infty} h[n] r^{-n} e^{-j\omega n}$$

  – This is denoted as $h[n] \overset{z}{\longleftrightarrow} H(z)$, or $h[n] = \mathcal{Z}\{H(z)\}$
  – Relationship between DTFT and z-transform?

*This is often referred to as the **bilateral** z-transform.

1.2 Relationship between DTFT and z-transform

- 1. DTFT is the z-transform of h[n] evaluated on the unit circle

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \Bigg|_{z=e^{j\omega}}$$

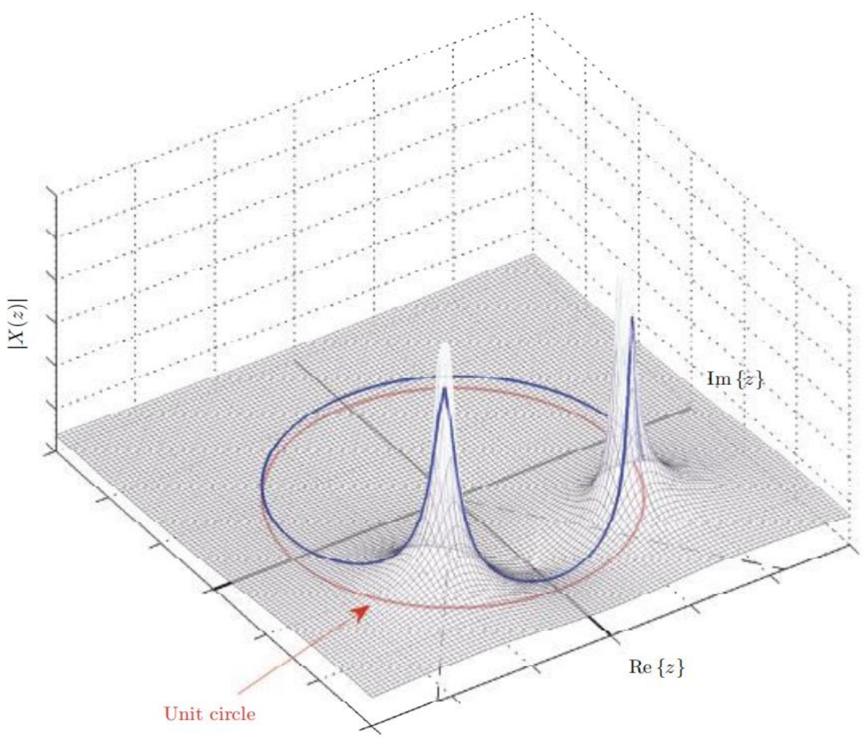
$$= H(z)\big|_{z=e^{j\omega}}$$

- 2. z-transform is the DTFT of r^{-n}-scaled $h[n]$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=-\infty}^{\infty} h[n] r^{-n} e^{-j\omega n}$$

$$= DTFT\{h[n] r^{-n}\}$$

1.3 Visualisation - two examples



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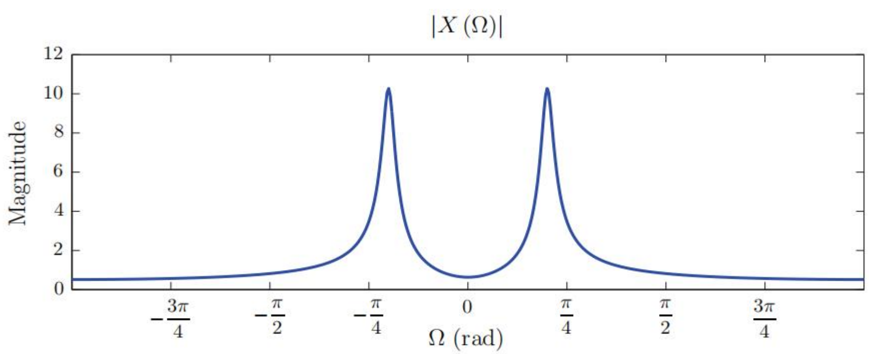

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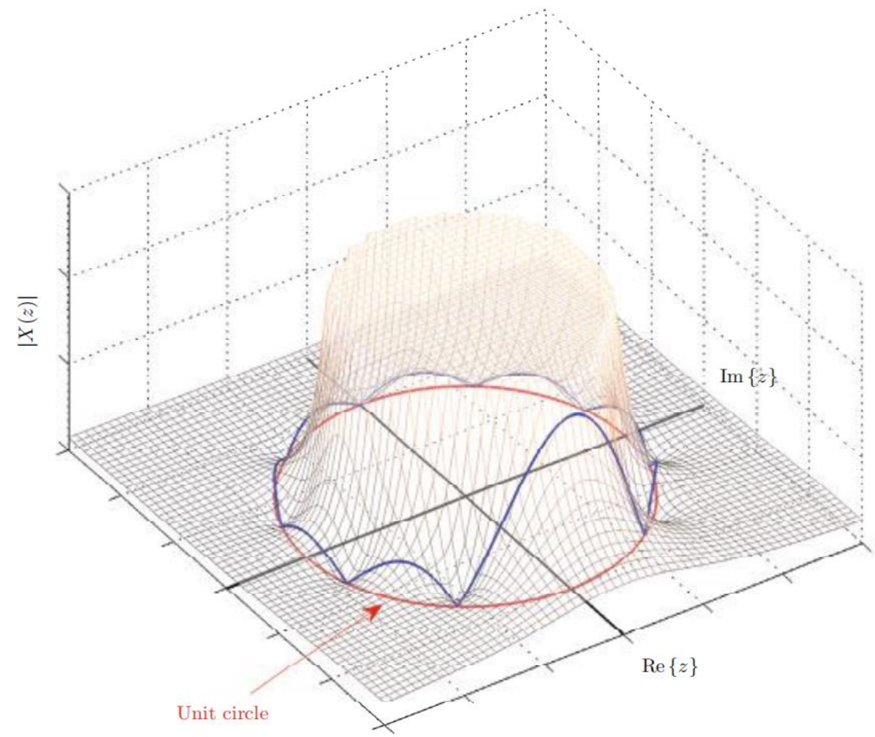
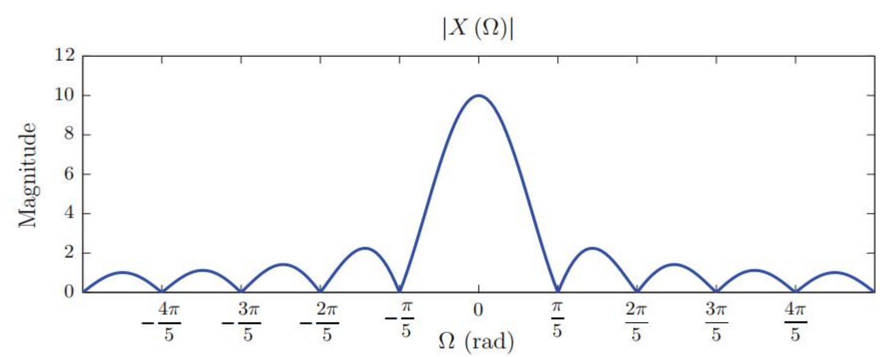


(a)

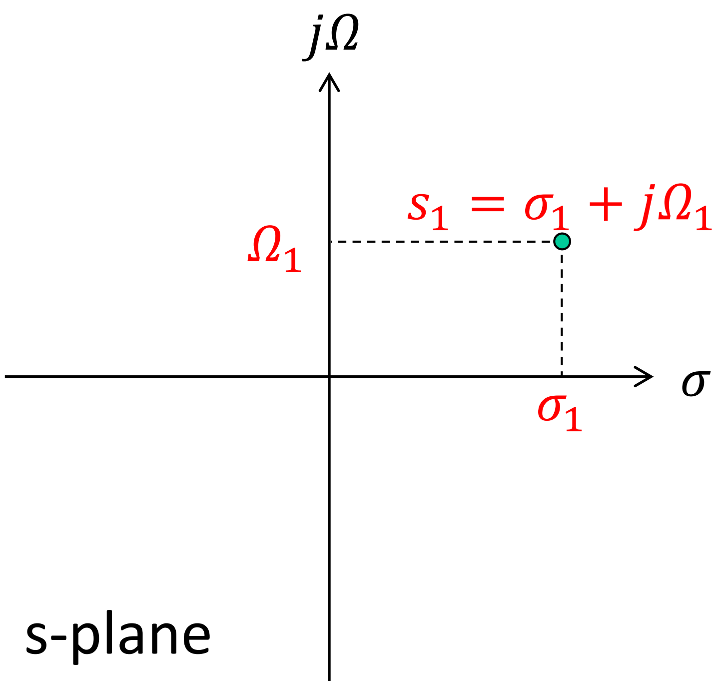


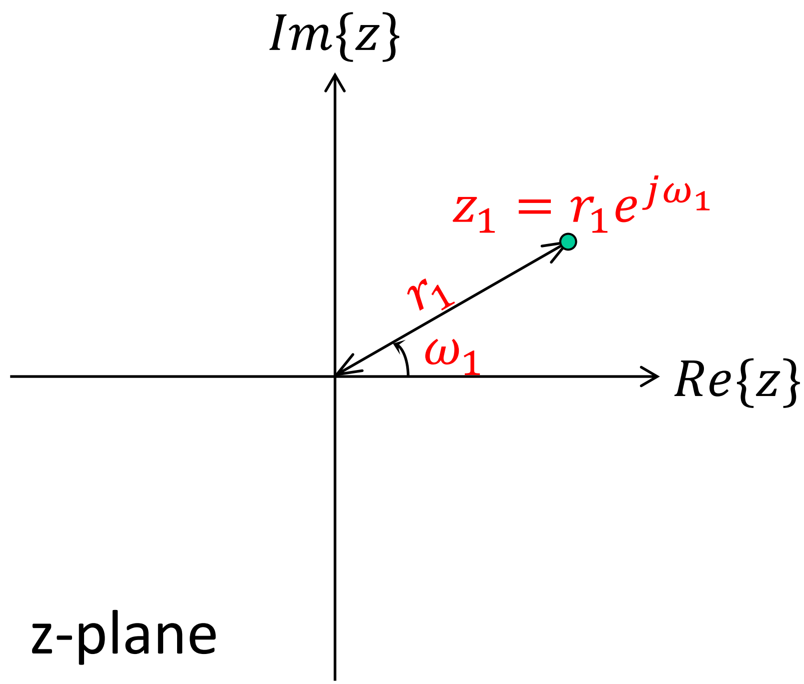
(b)

1.4 Complex frequencies 's' and 'z'

Analogue complex frequency $s = \sigma + j\Omega$

Digital complex frequency $z = re^{j\omega}$





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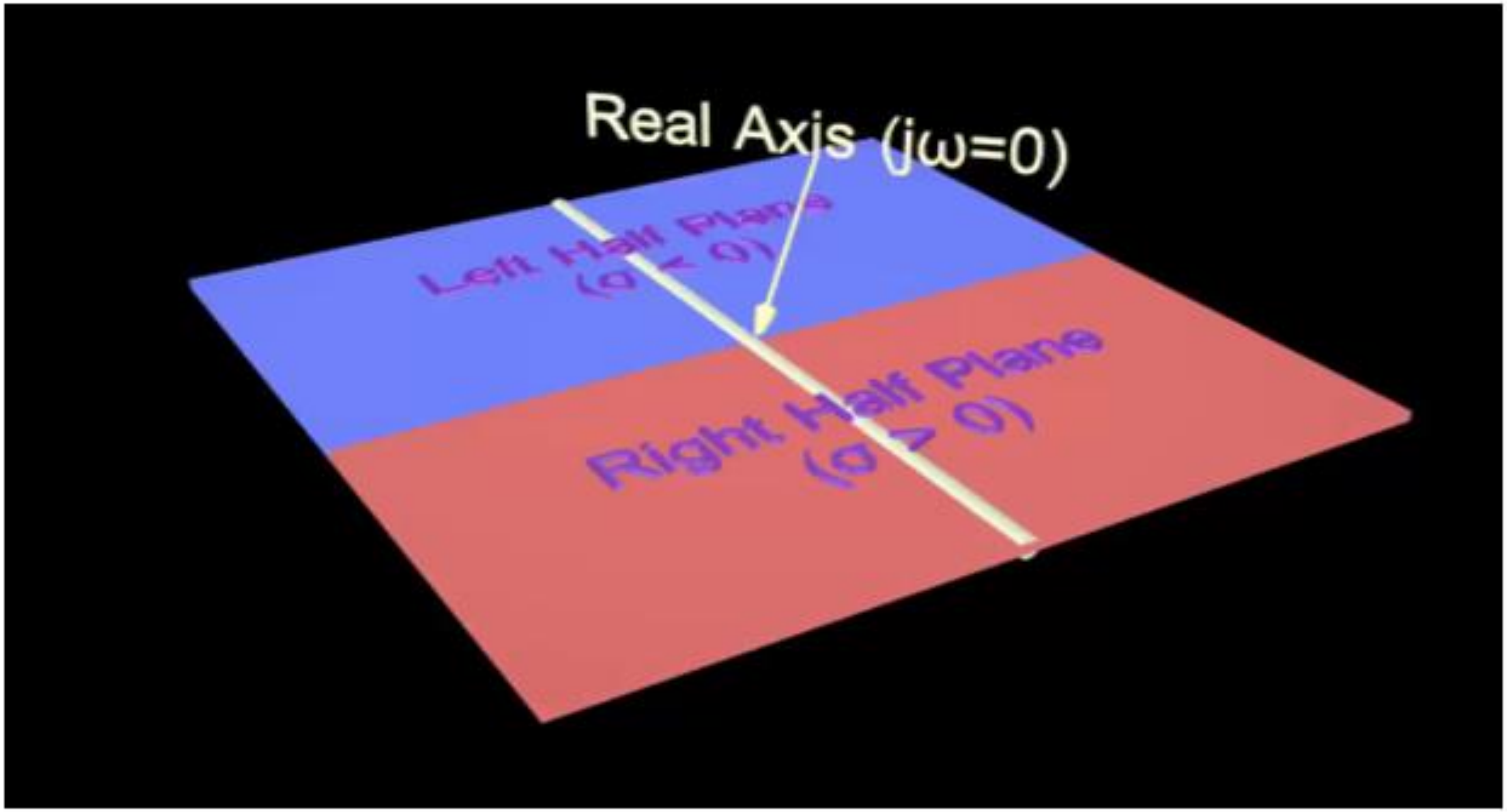
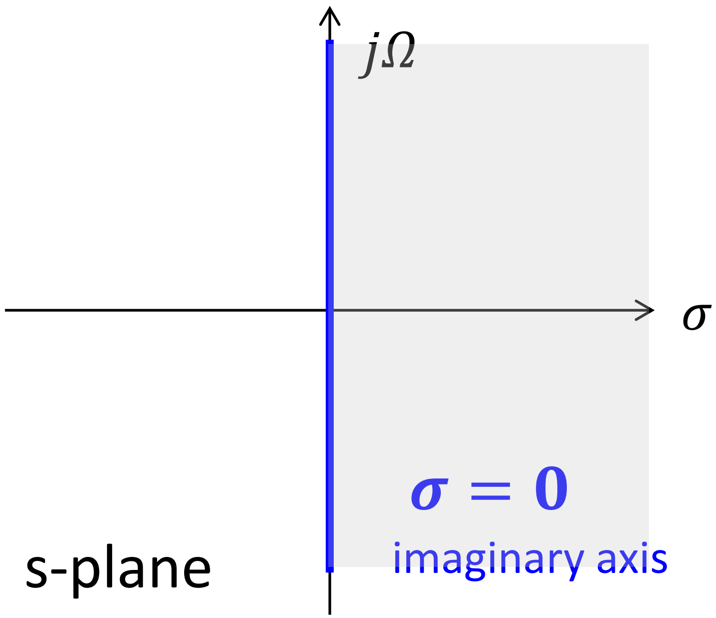
1.4 s-plane to z-plane

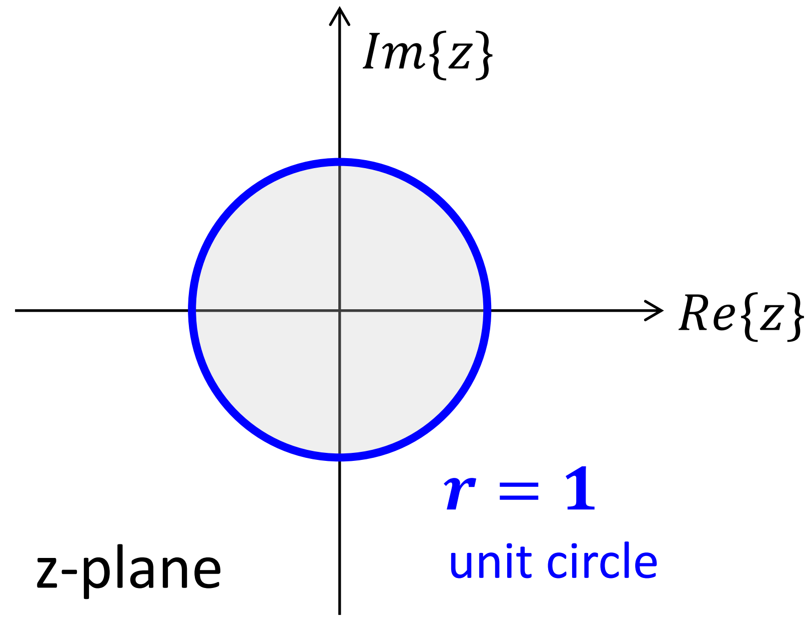


https://www.youtube.com/watch?v=4PV6ikgBShw

1.4 s-plane to z-plane

$$z = e^s = e^{\sigma + j\Omega} = e^{\sigma} e^{j\Omega}$$





- Other important mapping points:

$\sigma = 0, \Omega = 2k\pi$	$z = 1$
$\sigma = -\infty$	$z = 0$
$\sigma = 0, \Omega = (2k+1)\pi$	$z = -1$
$\sigma = +\infty$	$r = \infty$

2.1 Why do we need another transform?

- Think about all the transforms you have seen so far
  - Laplace transform, Fourier series, CTFT, DTFT and DFT
- Why do we need another one?
  - Convergence issues with the Fourier transforms:

    The DTFT of a sequence exists if and only if the sequence x[n] is absolutely summable, that is, if

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty}$$

  - DTFT may not exist for certain signals of practical interest or some analytical signals, whose frequency analysis can therefore not be obtained through DTFT

2.1 Z-Transform

- A generalization of the DTFT leads to the z-transform that may exist for many signals for which the DTFT does not.
  - DTFT is in fact a special case of the z-transform
  - …just like the CTFT is a special case of Laplace’s transform.
- Importance of z-transform
  - The use of z-transform techniques permits simple algebraic manipulations
  - The z-transform has become an important tool in the analysis and design of digital filters
  - The representation of an LTI discrete-time system in the z-domain is given by its transfer function which is the z-transform of the impulse response of the system

2.2 Region of Convergence (ROC)

- Just like the DTFT, z-transform also has its own convergence requirements: $x[n]r^{-n}$ must be absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

- For a given sequence, the set R of values of z for which its z-transform converges is called the region of convergence (ROC).
  - The area where the above condition is satisfied defines the ROC, which in general is an annular region of the z-plane

    $$R^- < |z| < R^+, \qquad \text{where } 0 \leq R^- < R^+ \leq \infty$$
  - The z-transform must always be specified with its ROC !

DTFT exists only when ROC include |z|=1, the unit circle!

Example 1

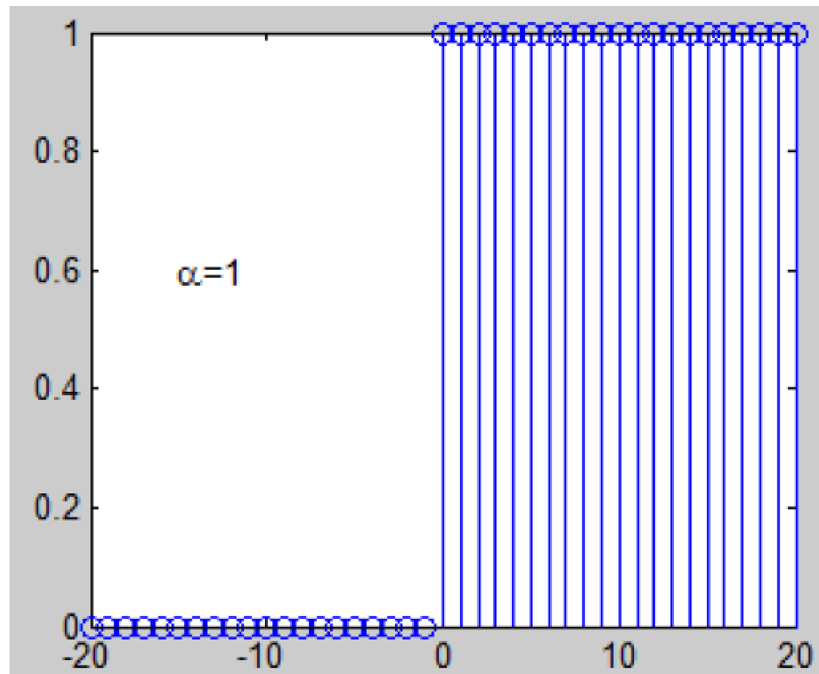
- Determine the z-transform and the corresponding ROC of the unit step sequence $u[n]$

$$U(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \ldots + z^{-n} + \ldots$$

which converges to

$$U(z) = \frac{1}{1 - z^{-1}}, \qquad \text{for } |z^{-1}| < 1$$

$$= \frac{z}{z-1}, \qquad \text{for } |z| > 1$$



- The region of convergence is the annular region in the z-plane

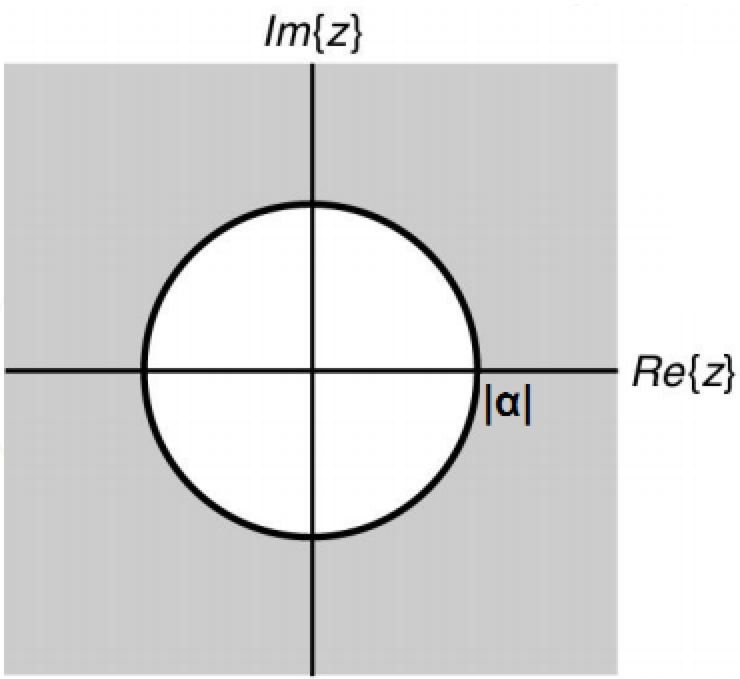
$$1 < |z| < \infty$$

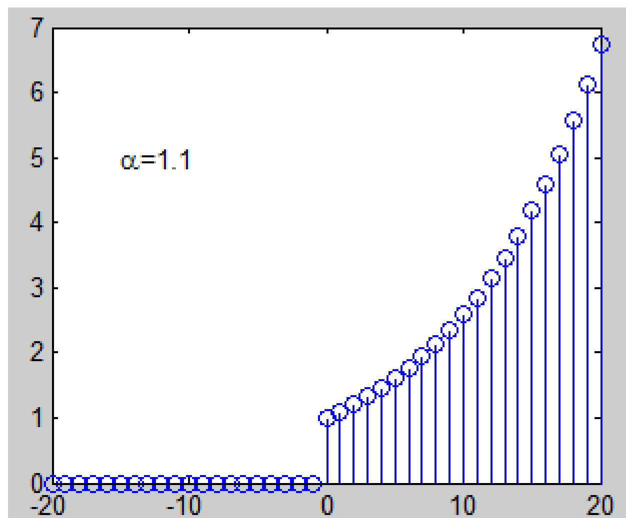
Example 2

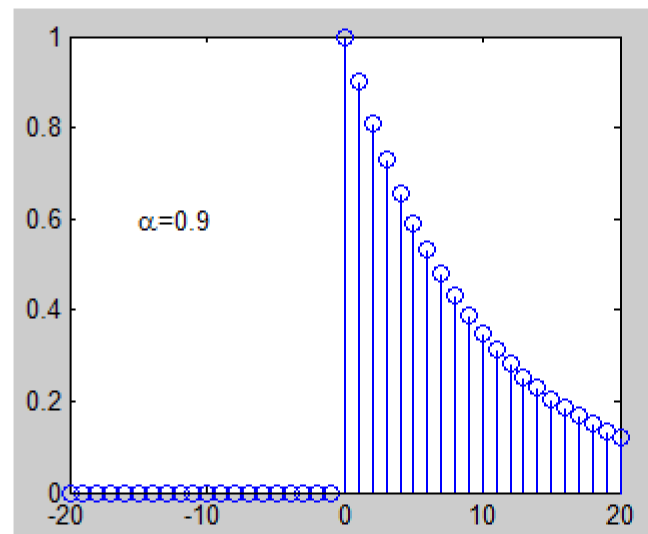
- Determine the z-transform and the corresponding ROC of the causal sequence $x[n] = \alpha^n u[n]$ (<u>right-sided</u>)

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \Longrightarrow X(z) = \frac{1}{1 - az^{-1}}, \qquad \text{for } |az^{-1}| < 1$$

$$= \frac{z}{z - a}, \qquad \text{for } |z| > |a|$$







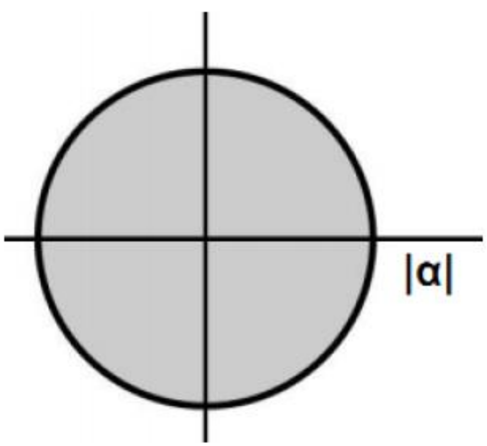
Example 3

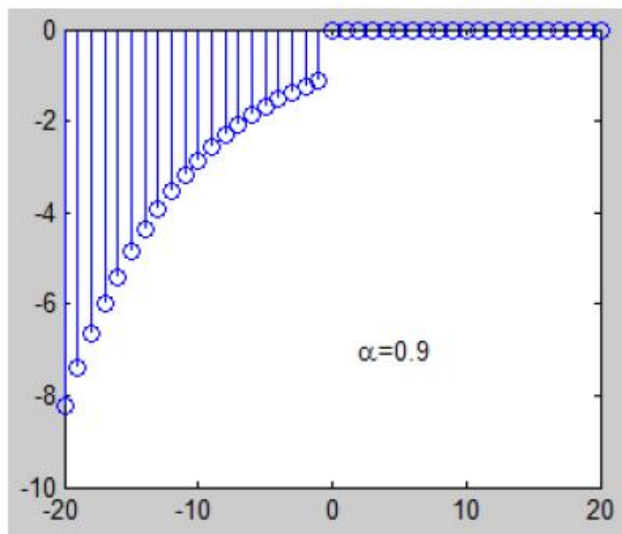
Is the same with that in previous slide, but with different ROC

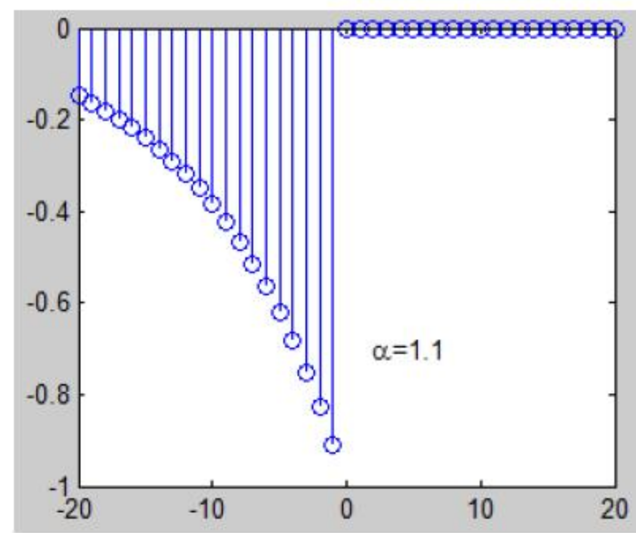
- Now consider the anti-causal $y[n] = -\alpha^n u[-n-1]$ (<u>left-sided</u>)

$$Y(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m$$

$$= -\left(\sum_{m=0}^{\infty} a^{-m} z^m - 1\right) = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad \text{for } |z| < |a|$$







Comparing Eg. 2 and Eg. 3

- Impulse responses: $x[n] = \alpha^n u[n]$ and $y[n] = -\alpha^n u[-n-1]$
- Transfer functions:

$$X(z) = \frac{z}{z-a}, \qquad \text{for } |z| > |a|$$

$$Y(z) = \frac{z}{z-a}, \qquad \text{for } |z| < |a|$$

  - The z-transforms of the two sequences x[n] and y[n] are identical even though the two parent sequences are different
  - Only way a unique sequence can be associated with a z-transform is by specifying its ROC
  - Both transfer functions have a pole at $z = \alpha$, which make the transfer function asymptotically approach to infinity at this value. Therefore, $z = \alpha$ is not included in either of the ROCs.

2.2 Zeros, Poles and ROC

- Rational system:

$$X(z) = \frac{N(z)}{D(z)} = \frac{z}{z-\alpha}, \qquad for\ |z| > |\alpha|$$

- In the X(z) given above, z = 0 is its ***zero***, and z = α is its ***pole***.
- The circle with the radius of ***α*** is called the ***pole circle***. A system may have many poles, and hence many pole circles.
- For right sided sequences, the ROCs extend outside of the outermost pole circle, whereas for left sided sequences, the ROCs are the inside of the innermost pole circle.
- For two-sided sequences, the ROC will be the intersection of the two ROC areas corresponding to the left and right sides of the sequence.

2.2 Overlapping ROCs

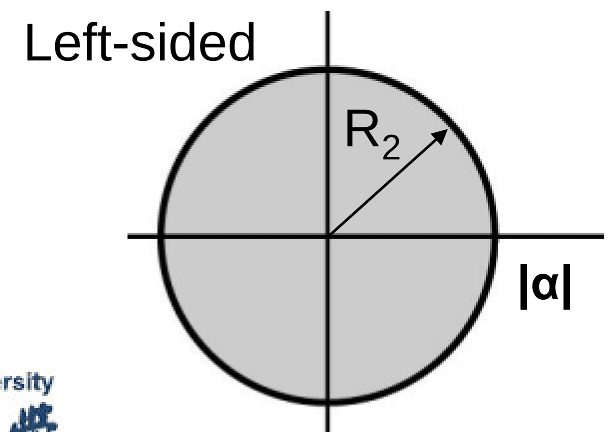
- For double sided sequence:

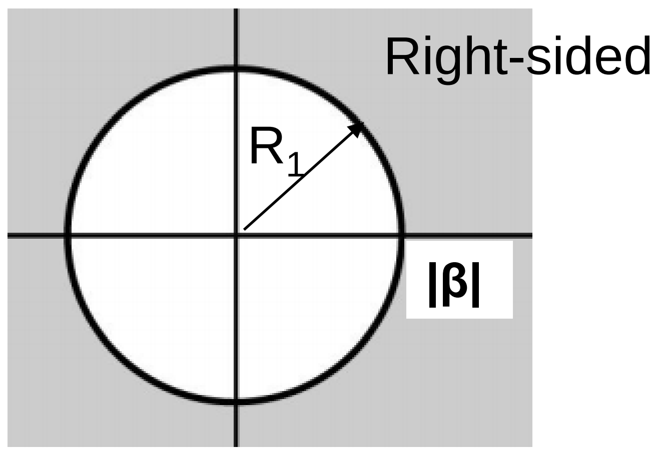
$$x[n] = \beta^n u[n] - \alpha^n u[-n-1]$$

- Its z-transform is:

$$X(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

  - Two poles of the transfer function: $|z| = |\alpha|$and $|z| = |\beta|$
  - ROC: $|z| > |\beta|$ and $|z| < |\alpha|$

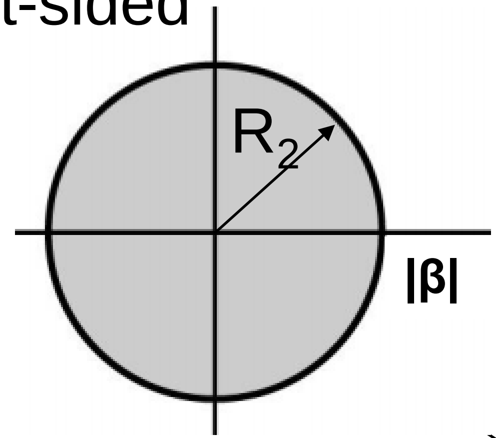




2.2 Overlapping ROCs

- When $R_1 < R_2$

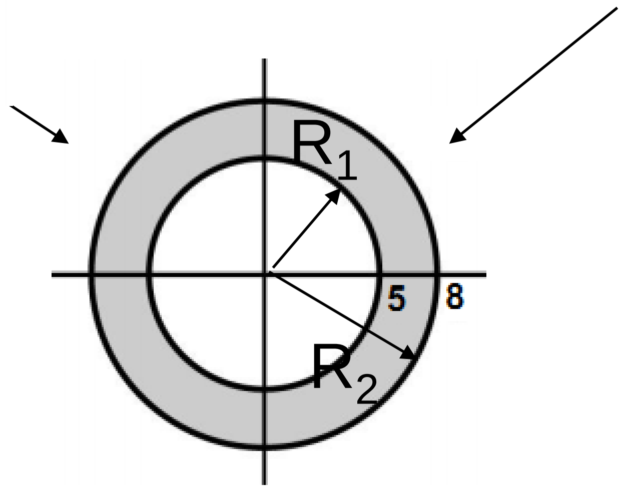
Left-sided



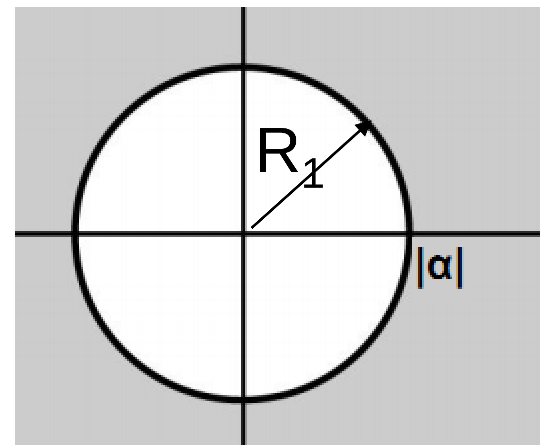
ROC of a left-sided sequence is inside of a circular area

$$R_1 < |z| < R_2$$

$$if\ 0 \leq R_1 < R_2 \leq \infty$$



Right-sided



ROC of a right-sided sequence is outside of a circular area

- When $R_1 > R_2$
  - No valid ROC => z-transform doesn't exist.

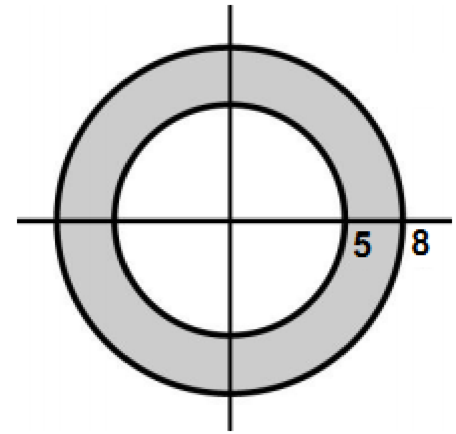
Example 4

- Consider $x[n] = 5^n u[n] - 8^n u[-n-1]$

$$X(z) = \frac{z}{z-5} + \frac{z}{z-8}$$

  – Corresponding ROCs are |z|>5 and |z|<8

  – Therefore the ROC for this signal is the annular region 5<|z|<8

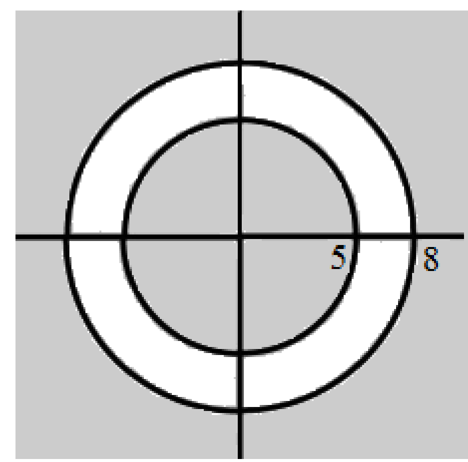


- Consider $x[n] = 8^n u[n] - 5^n u[-n-1]$

$$X(z) = \frac{z}{z-5} + \frac{z}{z-8}$$

  – Corresponding ROCs are |z|<5 and |z|>8

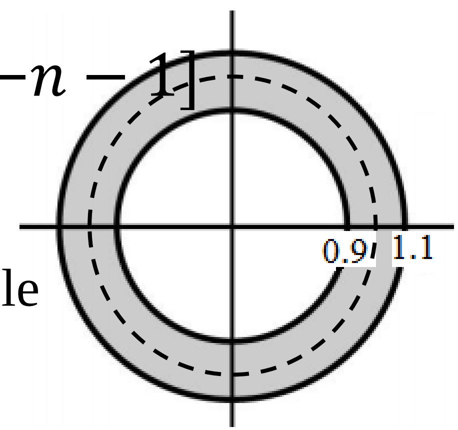
  – Therefore, the z-transform of this sequence does not exist!



2.2 Existence of DTFT and z-transform

- Since DTFT is the z-transform evaluated on the unit circle, that is for $z = e^{j\omega}$, DTFT of a sequence exists if and only if the ROC includes the unit circle!
  - The DTFT for $x[n] = 5^n u[n] - 8^n u[-n-1]$ clearly does not exist, since the ROC does not include the unit circle!

  - Consider the sequence $x[n] = 0.9^n u[n] - 1.1^n u[-n-1]$
    - Its transfer function is: $$X(z) = \frac{z}{z-0.9} + \frac{z}{z-1.1}$$
    - with the ROC as $0.9<|z|<1.1$, which includes the unit circle
    - Therefore, the DTFT of x[n] exists

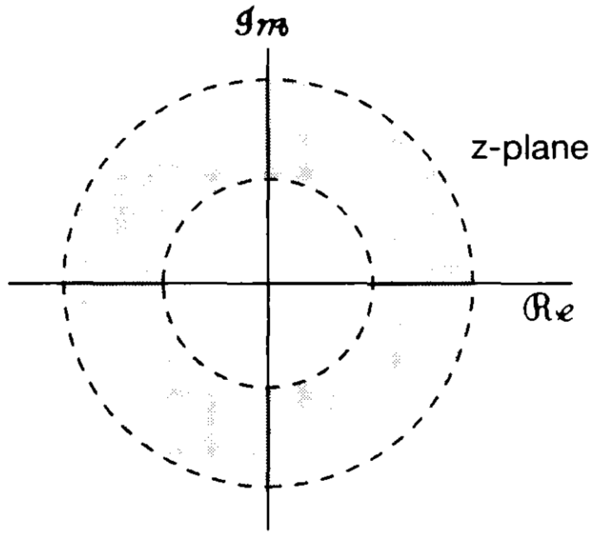


The existence of DTFT is not a guarantee for the existence of the z-transform either!

2.3 ROC Properties

- **Property 1**: The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

$$\sum_{n=-\infty}^{+\infty} |x[n]| r^{-n} < \infty.$$



- **Property 2**: The ROC does not contain any poles.
  - By definition of poles, $X(p) = \infty$

2.3 ROC Properties

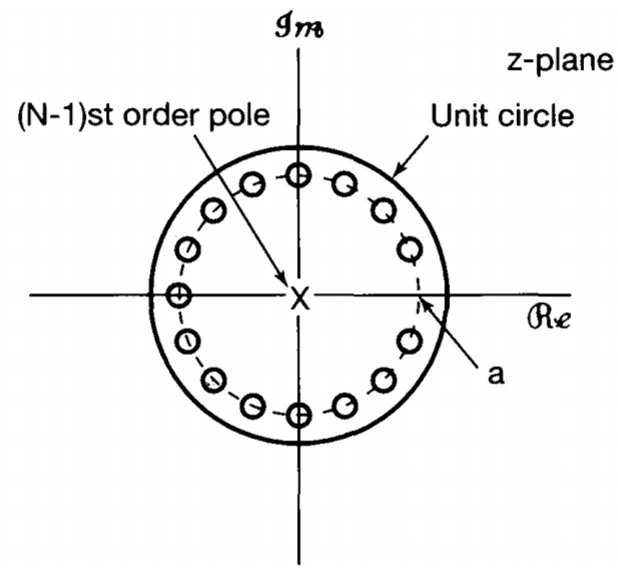
- **Property 3**: If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$.
  - Example: $\delta[n]$ and $\delta[n-1]$
  - Example:

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, a > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$$

$$= \sum_{n=0}^{N-1} (az^{-1})^n$$

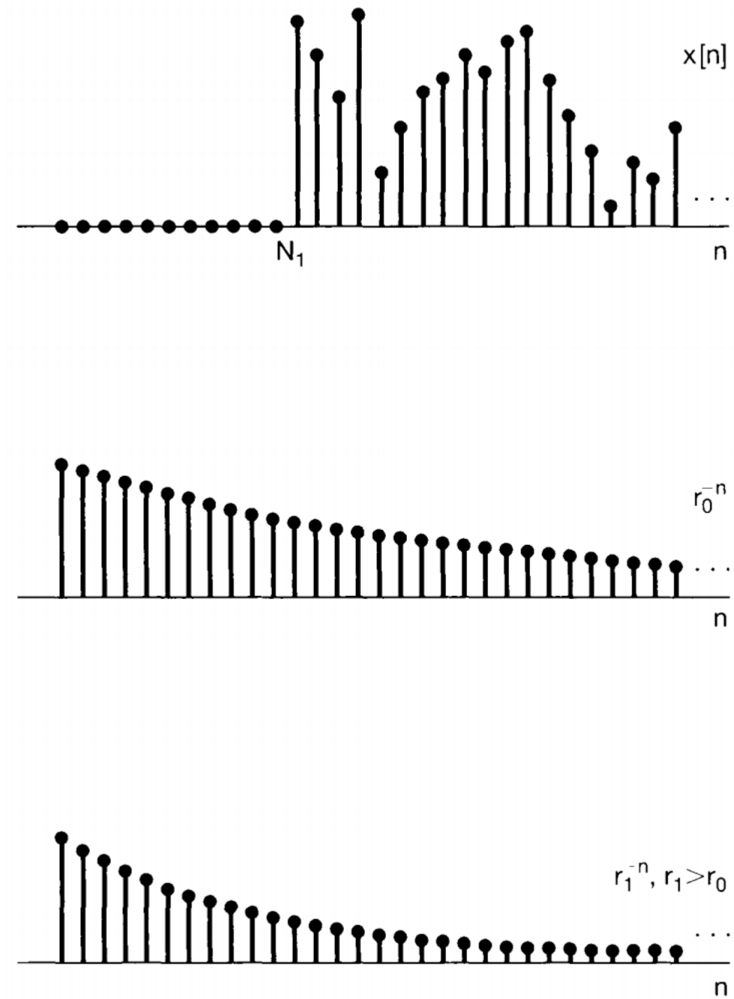
$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}.$$



2.3 ROC Properties

- **Property 4**: If $x[n]$ is a right -sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.

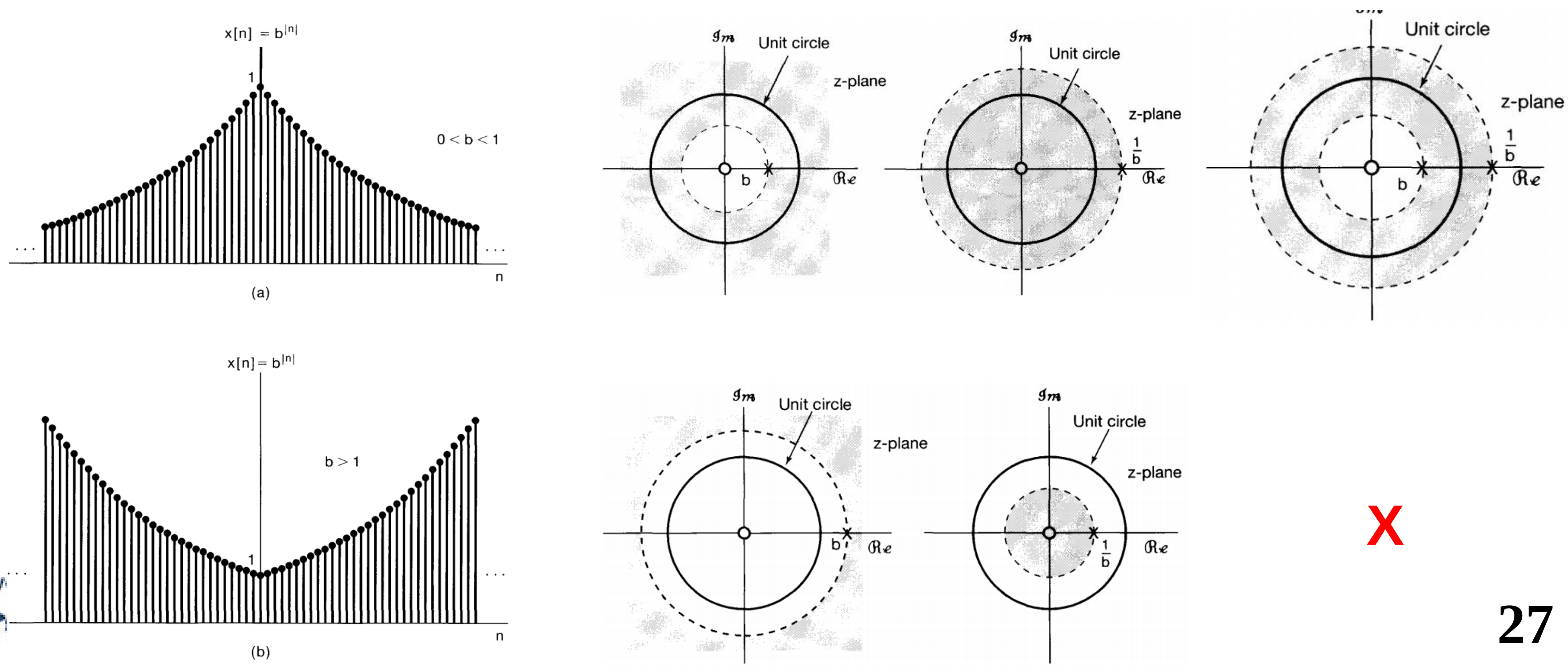
- **Property 5**: If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| < r_0$ will also be in the ROC.



2.3 ROC Properties

- **Property 6**: If $x[n]$ is a double-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.
  - Example:

$$x[n] = b^{|n|}, \qquad b > 0$$



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2.3 ROC Properties

- **Property 7**: If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.
- **Property 8**: If the z-transform $X(z)$ of $x[n]$ is rational, and if x[n] is right sided, then the ROC is the region in the z-plane outside the outermost pole.
  - Furthermore, if $x[n]$ is causal (i.e., if it is right sided and equal to 0 for $n < 0$), then the ROC also includes $z = \infty$.
- **Property 9**: If the z-transform $X(z)$ of $x[n]$ is rational, and if x[n] is left sided, then the ROC is the region in the z-plane inside the innermost nonzero pole.
  - i.e., inside the circle of radius equal to the smallest magnitude of the poles, other than any at $z = 0$ and extending inward to and possibly including $z = 0$.
  - Furthermore, if $x[n]$ is anticausal (i.e., left sided and equal to 0 for $n > 0$), then the ROC also includes $z = 0$.

Quiz 1

- Determine the z-transform and the corresponding ROC of the signal

$$x[n] = \{3.7,\ 1.3,\ \underset{\substack{\uparrow \\ n=0}}{-1.5},\ 3.4,\ 5.2\}$$

- Solutions:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= x[-2]\, z^{2} + x[-1]\, z^{1} + x[0] + x[1]\, z^{-1} + x[2]\, z^{-2}$$

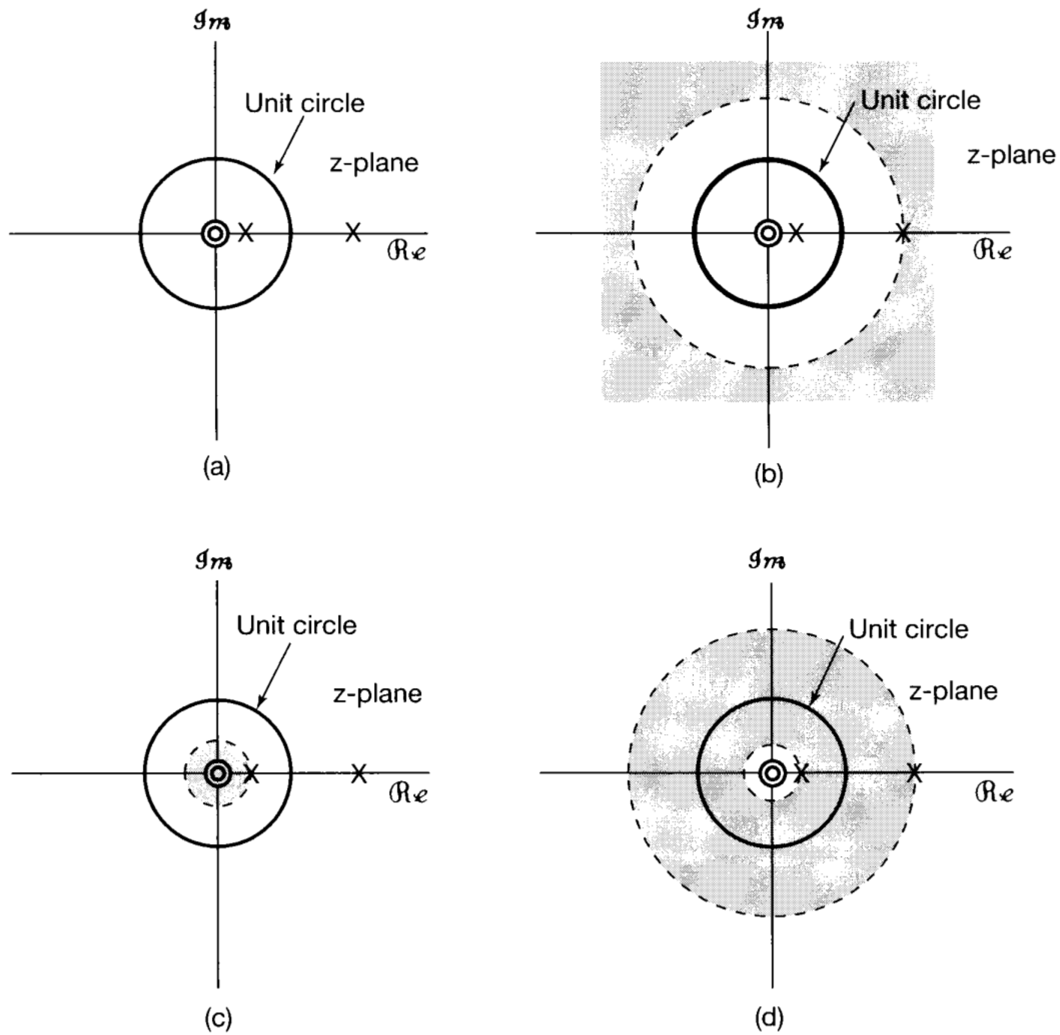
$$= 3.7\, z^{2} + 1.3\, z^{1} - 1.5 + 3.4\, z^{-1} + 5.2\, z^{-2}$$

ROC: the transform converges at every point in the *z*-plane with the two exceptions, namely the origin and infinity, therefore the ROC is the whole z-plane expect the origin and infinity.

Quiz 2

- Find all of the possible ROCs that can be connected with the function

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right)}$$



3. Commonly used z-transform pairs

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Next ...

- Following of Z-transform
  - 4. Properties of z-transform
  - 5. Inverse Z-transform
  - 6. Geometric Evaluation of DTFT based on z-transform
  - 7. Unilateral z-transform
  - 8. Analysis of LTID systems using z-transform