CAN207 Continuous and Discrete Time Signals and Systems

Lecture 16

DTFT in LTID Systems and Filtering

Qing Liu

Qing.Liu@xjtlu.edu.cn

Room SC340

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- 1. Commonly used DTFT pairs
  - Use DTFT properties when needed
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  - Definition and calculation
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  - Impulse response vs. Frequency response
  - Magnitude and phase spectrum
- 4. Concept of Filtering
  - What is filtering?
  - Frequency-shaping vs Frequency-selective filters

1.1 Important DTFT Pairs - Impulse Signals

- 1. Impulse Function

$$\mathcal{DTFT}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \delta[0] \cdot e^{-j\omega 0} = 1$$

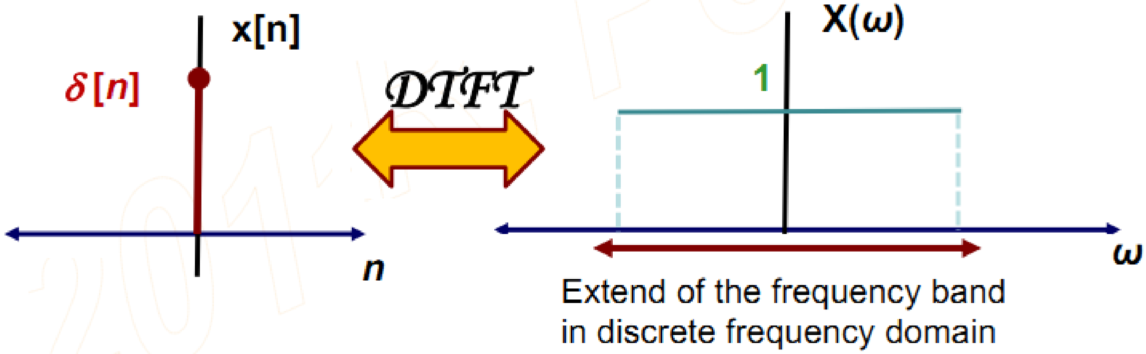
- 2. Delayed Impulse Function

$$\mathcal{DTFT}\{\delta[n-n_0]\} = \sum_{n=-\infty}^{\infty} \delta[n-n_0]e^{-j\omega n}$$

$$= \delta[n_0 - n_0] \cdot e^{-j\omega n_0} = e^{-j\omega n_0}$$

1.2 Important DTFT Pairs - Impulse Train

- 1. The DTFT of the impulse function is "1" over the entire frequency band.

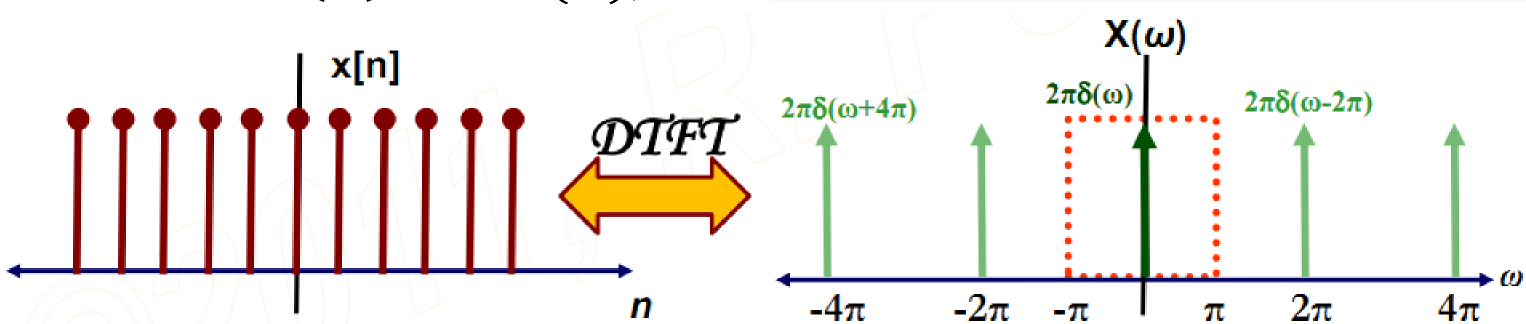


- 2. Constant Function

$$X(\omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\mathcal{F}^{-1}\left\{2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)\right\}$$
$$= 2\pi \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)\right] e^{j\omega n} d\omega$$
$$= e^{j0n} = 1$$

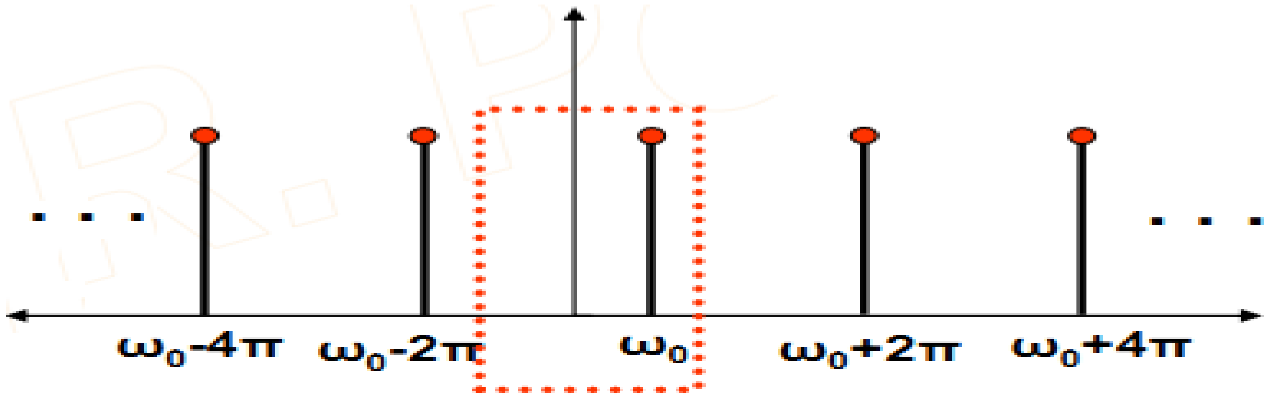
  - Note that x[n]=1 is not absolutely summable;
  - But its DTFT still exists: $X(\omega) = 2\pi\delta(\omega)$;



1.3 Important DTFT Pairs - Complex Exponential

- The complex exponential

$$x[n]=e^{j\omega_0 n} \Leftrightarrow X(\omega)=2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0+2\pi k)$$

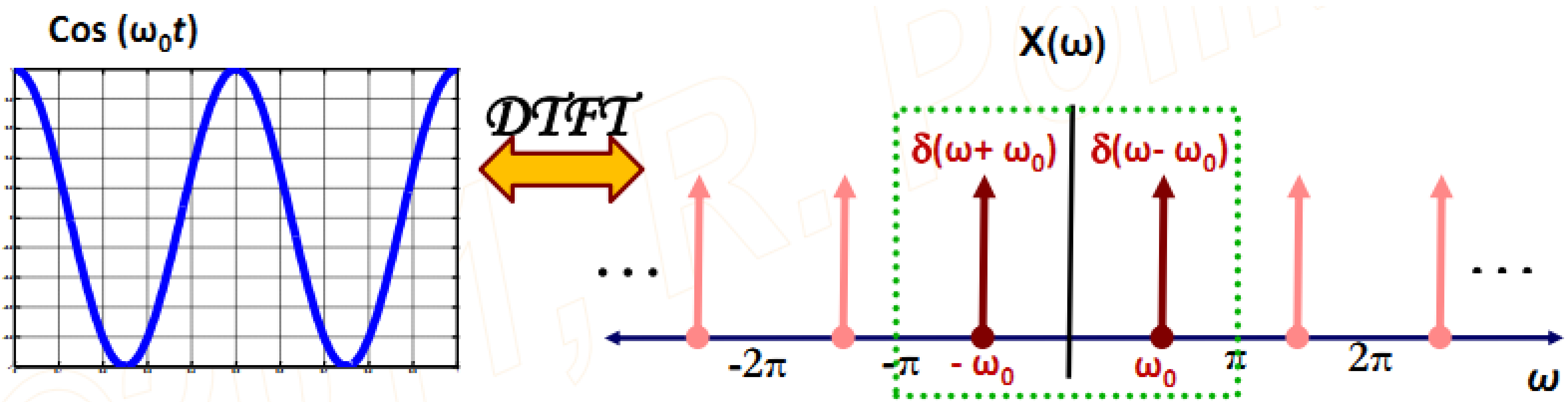


  - We are only interested in [-π, π] range, where there is only one spectral component.
  - Hence, the spectrum of a single complex exponential at a specific frequency is an impulse at that frequency.

1.4 Important DTFT Pairs - Sinusoidal Signals

- The sinusoid

$$x[n]=\cos(\omega_0 n)\overset{\Im}{\Leftrightarrow}\pi\sum_{m=-\infty}^{\infty}\delta(\omega-2\pi m-\omega_0)+\pi\sum_{m=-\infty}^{\infty}\delta(\omega-2\pi m+\omega_0)$$



  – The above expression can also be obtained from the DTFT of the complex exponential through the Euler's formula.

$$e^{j\omega_0 n}\overset{\Im}{\Leftrightarrow}2\pi\sum_{m=-\infty}^{\infty}\delta(\omega-\omega_0\pm 2\pi m)$$

Xi'an Jiaotong-Liverpool University
西交利物浦大学

1.5 Important DTFT Pairs - Real Exponential

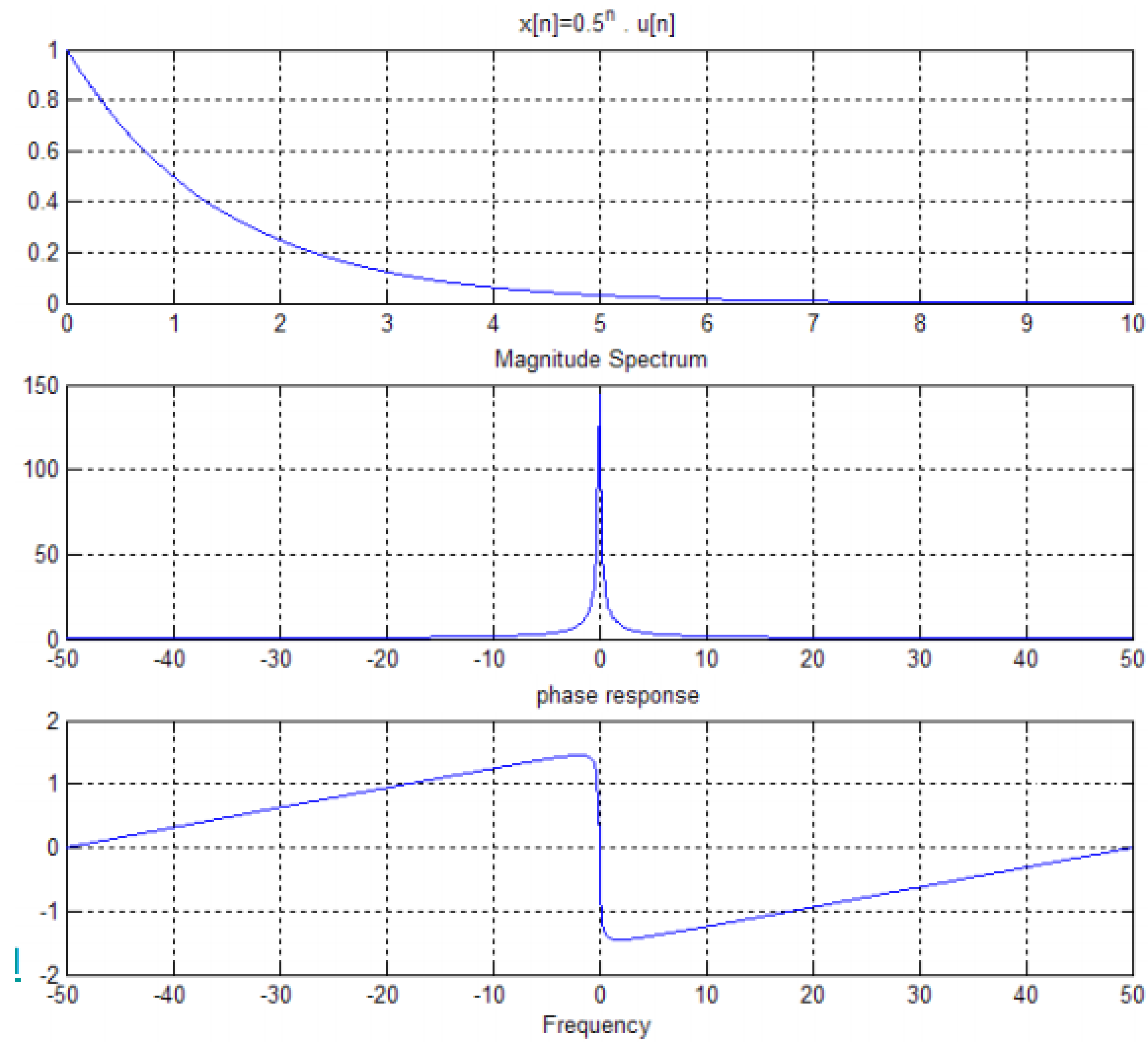
- The real exponential

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

$$\overset{\Im}{\Leftrightarrow}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1 - \alpha e^{-j\omega}}$$



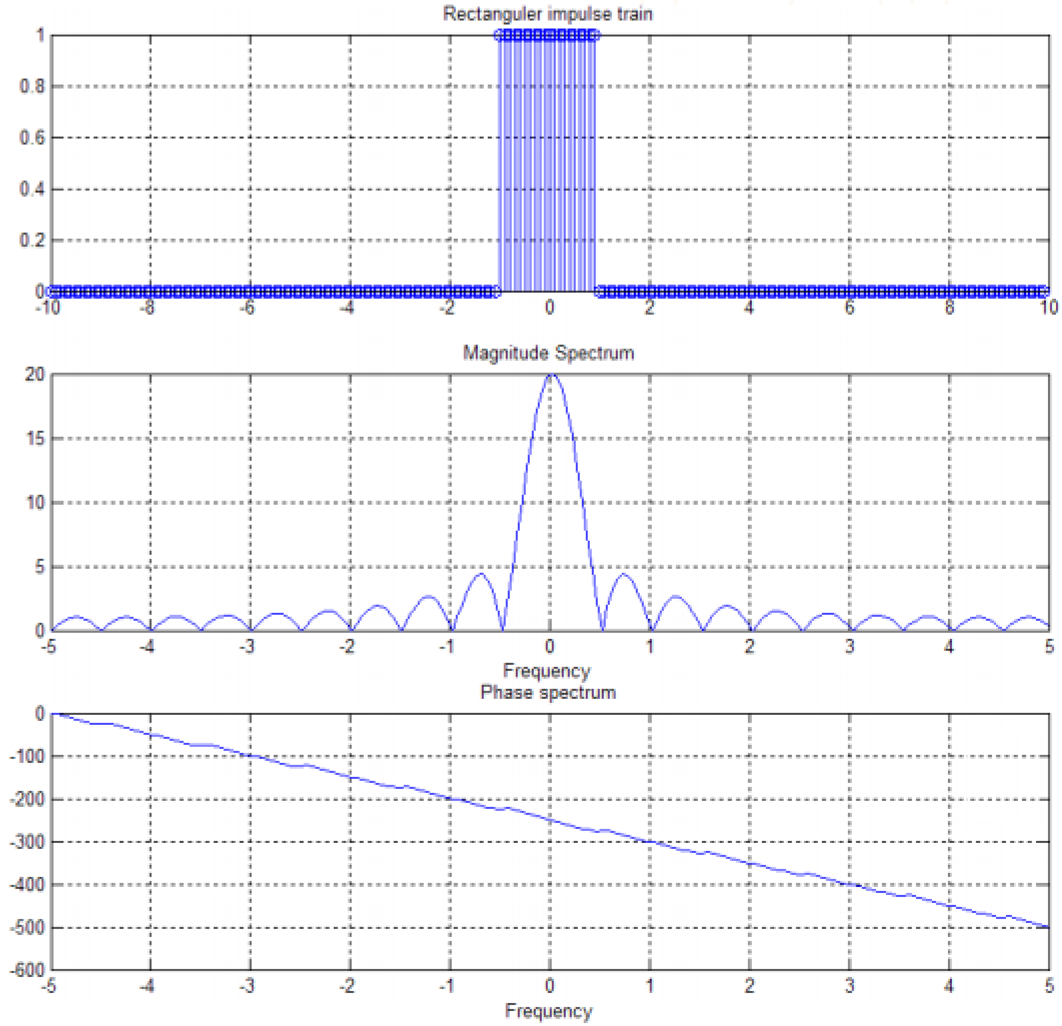
1.6 Important DTFT Pairs - Rectangular

- 6. Rectangular pulse train

$$x[n] = \text{rect}_M[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$\overset{\mathfrak{F}}{\Leftrightarrow}$$

$$\sum_{n=-M}^{M} e^{-j\omega n} = \frac{\sin\left(M + 1/2\right)\omega}{\sin\left(\omega/2\right)}, \quad \omega \neq 0$$



Quiz 1

- Find the DTFT of the following signals:

$$a)\ x[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1]$$

$$b)\ x[n] = \left(\frac{\sin\frac{\pi n}{5}}{\pi n}\right)\cos\frac{7\pi n}{2}$$

2.1 Inverse DTFT - Uniqueness

- The DTFT is a unique relationship between $x[n]$ and $X(\omega)$.
  - Two different signals cannot have the same DTFT.
  - If we know a DTFT representation, we can start in either the time or frequency domain and easily write down the corresponding representation in the other domain.
  - The uniqueness property implies we can always go back and forth between the time-domain and frequency domain representations.
- To find the TD sequence from a known Fourier transform, the most straight forward way is to compare the FD expression $X(\omega)$ to its definition equation.
- Example: Find the inverse DTFT of

$$X(\omega) = 2\cos(2\omega)$$

2.1 Inverse DTFT - Uniqueness

- The uniqueness property implies we can always go back and forth between the time-domain and frequency domain representations.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

  – But we seldom use this definition equation in practical calculation.

- Most frequently used pair:

$$a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1 - ae^{-j\omega}}, \qquad |a| < 1$$

  – Partial Fraction Expansion

2.2 Inverse DTFT - Partial Fraction Expansion

- Recall Lect. 11, P.18
- Consider a rational transform in the form

$$X(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{(1-\alpha_1 e^{-j\omega})(1-\alpha_2 e^{-j\omega})\ldots(1-\alpha_N e^{-j\omega})}$$

  – where the poles $\alpha_1, \alpha_2, \ldots, \alpha_N$ are distinct.
  – the order of the numerator polynomial of $e^{-j\omega}$ is less than the order of the denominator polynomial.

- The transform $X(e^{j\omega})$ can be expanded into partial fractions in the form

$$X(e^{j\omega}) = \frac{k_1}{1-\alpha_1 e^{-j\omega}} + \frac{k_2}{1-\alpha_2 e^{-j\omega}} + \ldots + \frac{k_N}{1-\alpha_N e^{-j\omega}}$$

  – the coefficients $k_1, k_2, \ldots, k_N$ are called the residues of the partial fraction expansion. They can be computed by

$$k_i = (1-\alpha_i e^{-j\omega})\, X(e^{j\omega})\Big|_{e^{j\omega}=\alpha_i} \qquad i = 1, 2, \ldots, N$$

2.2 PFE - Example

- Find the inverse DTFT of the following transforms:

  a) $X(e^{j\omega}) = \dfrac{1}{(1-\alpha_1 e^{-j\omega})(1-\alpha_2 e^{-j\omega})}$

  b) $X(e^{j\omega}) = \dfrac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}$

Quiz 2

- Find the Inverse DTFT of the following transforms

  a) $X(\omega) = 1 + 3e^{-j\omega} + 2e^{-j2\omega} - 4e^{-j3\omega} + e^{-j10\omega}$

  b) $X(\omega) = \frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-j2\omega}}$

3.1 DTFT in LTID systems

- LTID system in Time domain
  - Impulse response $h[n]$:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

  - LCCDE (Linear Constant Coefficient Difference Equation)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- LTID system in Frequency domain
  - Frequency response $H(e^{j\omega})$ or simply $H(\omega)$, also called "***transfer function***" or "***system function***".
  - What's the relationship between $H(\omega)$, $h[n]$ and the LCCDE?

3.1 DTFT in LTID systems

- 1. Based on the “eigenfunction” concept

$$\sum_{r=-\infty}^{\infty} h[r]e^{j\omega_k(n-r)} = e^{j\omega_k n} \underbrace{\sum_{r=-\infty}^{\infty} h[r]e^{-j\omega_k r}}_{H(\omega_k)}$$

  – Therefore, the frequency response $H(\omega)$ is the DTFT of the impulse response $h[n]$: $h[n] \xleftrightarrow{DTFT} H(\omega)$

- Example: An LTI system with impulse response $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$ is connected in parallel with another causal LTI system with impulse response $h_2[n]$. The resulting parallel interconnection has the frequency response $H(\omega) = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}}$. Find $H_2(\omega)$.

3.1 DTFT in LTID systems

- 2. Considering the "convolution property"

$$y[n] = x[n] * h[n] \overset{DTFT}{\longleftrightarrow} X(\omega)H(\omega) = Y(\omega)$$

  - So the frequency response can be obtained by converting every term in LCCDE to its frequency counterparts, get:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

    - since $y[n-k] \overset{DTFT}{\longleftrightarrow} e^{-j\omega k} Y(\omega)$

$$\Rightarrow Y(\omega) \sum_{k=0}^{N} a_k e^{-j\omega k} = X(\omega) \sum_{k=0}^{M} b_k e^{-j\omega k}$$

$$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

3.1 DTFT in LTID systems

- Example: Consider a causal LTI system that is characterized by the difference equation

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

- Find the impulse response of it.

3.2 Magnitude and phase spectra

- The Fourier transfer function $H(\omega)$ provides a complete description of an LTID system.
  - In most cases, $H(\omega)$ is a complex function of the angular frequency ω.
  - Usually expressed as

$$H(\omega) = |H(\omega)|e^{j\measuredangle H(\omega)}$$

    - $|H(\omega)|$ is referred to as the "magnitude spectrum"
    - $\measuredangle H(\omega)$ is referred to as the "phase spectrum"

Quiz 3

1. Let h[n] and g[n] be the impulse responses of two stable discrete-time LTI systems that are inverses of each other. What is the relationship between the frequency responses of these two systems?
2. Consider a causal LTI system described by the following difference equations. Determine the ***impulse response*** of the inverse system and the ***difference equation*** that characterizes the inverse.

$$y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{1}{4}x[n-1] - \frac{1}{8}x[n-2]$$

4.1 Concept of Filtering

- In many signal processing applications the need arises to change the strength or the relative significance, of various frequency components in a given signal.
  - Linear time-invariant systems that change the shape of the spectrum are often referred to as *frequency-shaping* filters.
  - Systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others are referred to as *frequency-selective* filters.

- This act of changing the relative amplitudes of frequency components in a signal is referred to as ***filtering***, and the system that facilitates this is referred to as a ***filter***.

4.1 Concept of Filtering

- Recall the concept of eigenfunction
- For LTIC system:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \longrightarrow \boxed{H(j\Omega)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\Omega_0) a_k e^{jk\Omega_0 t}$$

- For LTID system:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y[n] = \sum_{k=0}^{N-1} H(e^{jk\omega_0}) a_k e^{jk\omega_0 n}$$

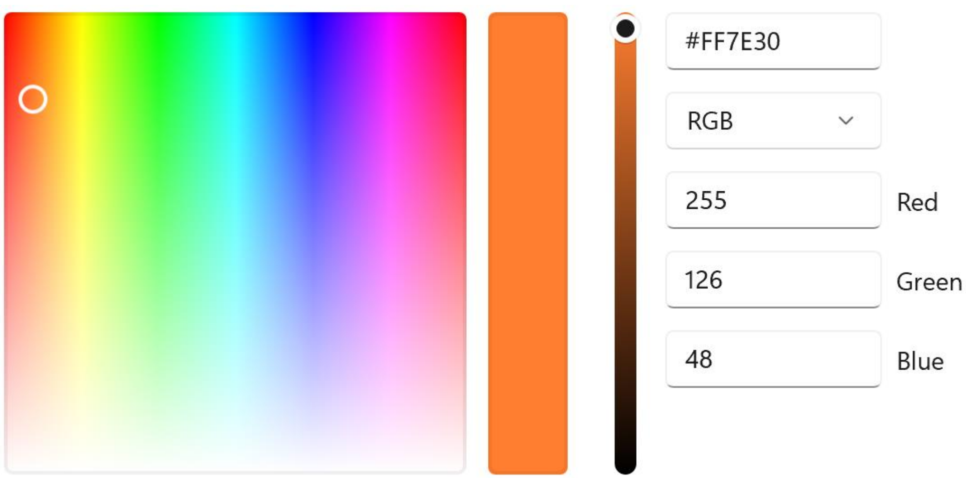
- For a specific frequency $k\Omega_0$ or $k\omega_0$, the system modify this frequency component by multiplying $H(jk\Omega_0)$ or $H(e^{jk\omega_0})$ to it.
  – Therefore, by adjusting the complex values of $H(\cdot)$, we can shape the signals frequency spectrum, in terms of magnitude and phase.

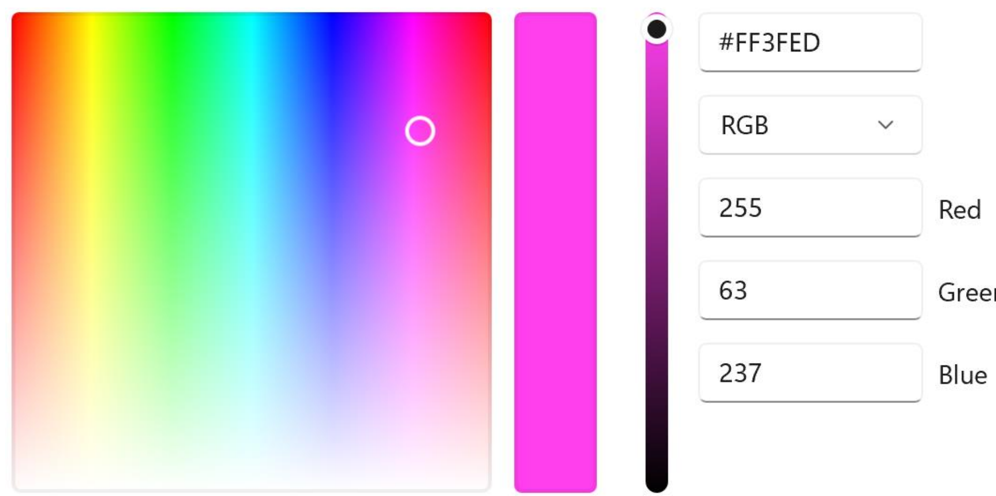
4.1 Example - Color analysis and synthesis

- The three primary colors (Red, Green and Blue) can be considered as a set of basis.
- Therefore, any color could be formed from a linear combination of these three:

$$C = rR + gG + bB$$

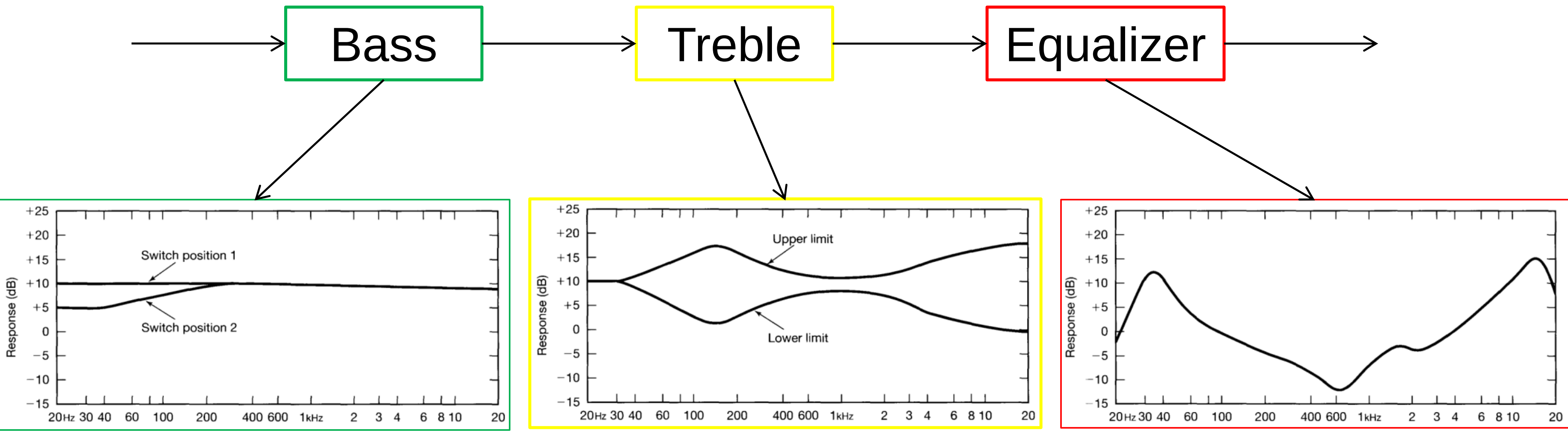
- Example:





4.2 Frequency-shaping Filters

- In audio systems, the equalizing circuit includes those cascaded filtering stages as shown below:



(a)
Low frequency filter controlled by a two-position switch

(b)
Upper and lower frequency limits on an adjustable shaping filter

(c)
Fixed frequency response of the equalizer stage

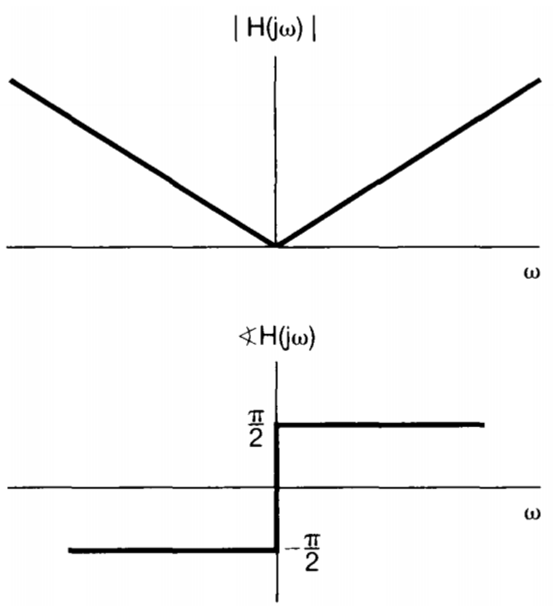
4.2 Frequency-shaping Filters

- Consider a system, which performs derivative of the input:

$$x(t) \longrightarrow \boxed{H(j\Omega)} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

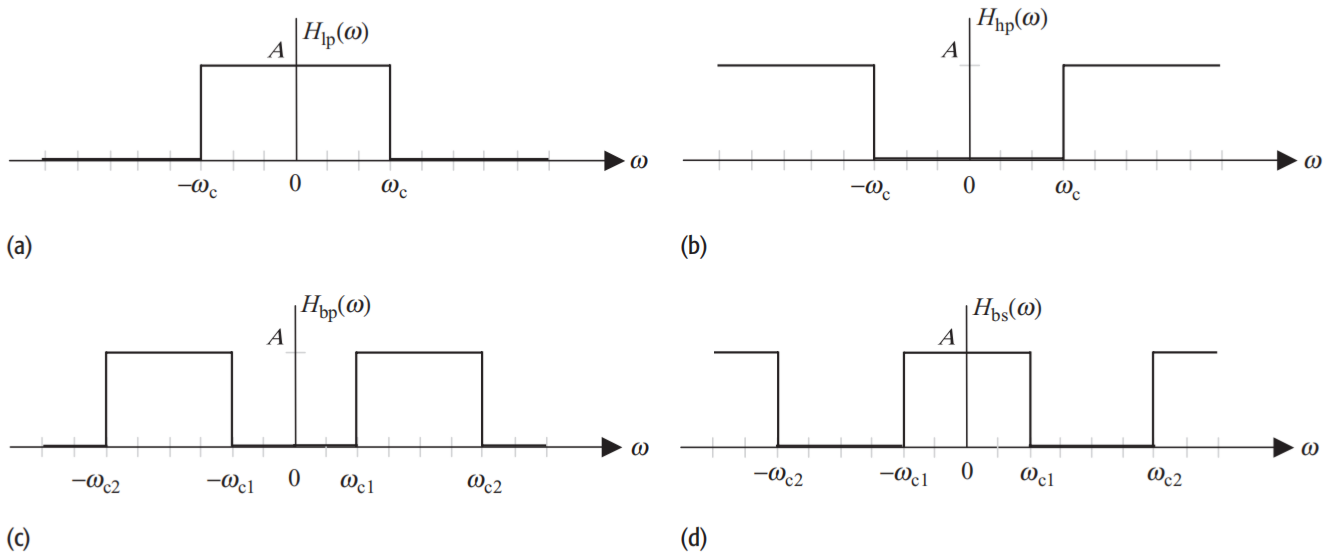
- The frequency response of the system is:

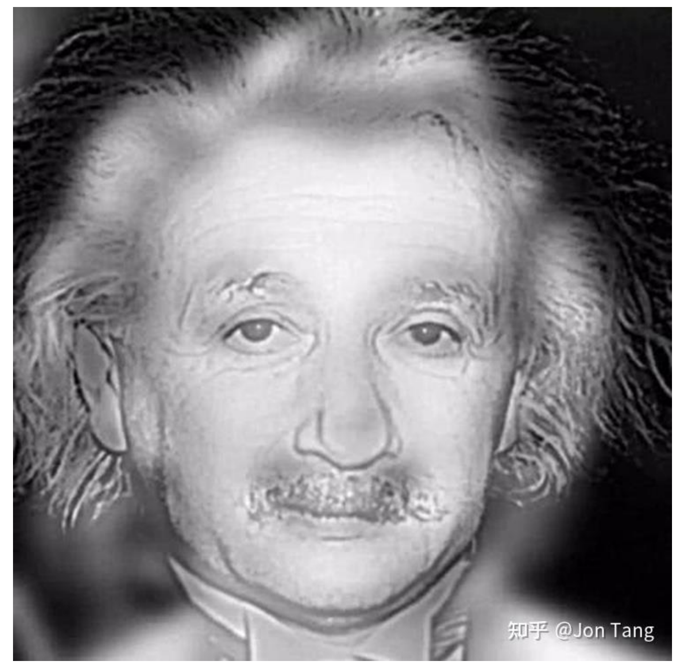
$$H(j\Omega) = j\Omega$$



4.3 Frequency-selective Filters

- Frequency-selective filters are a class of filters specifically intended to accurately or approximately select some bands of frequencies and reject others.
  - Will be explained in detail in next lecture.





Next ...

- Filtering
  - Continuous-Time vs. Discrete-Time filtering
  - CT and DT Filters examples
  - Example of CT filters: Butterworth filters
  - Important DT filters: FIR vs. IIR filters.