MTH102 Engineering Mathematics II

Lesson 8: Commonly used continuous random variables

Term: 2024

Outline

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1. Exponential distribution
2. Normal distribution
3. Rayleigh distribution

Definition

A continuous random variable X is said to be an *exponential* random variable (or is said to be *exponentially distributed*) if the pdf is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The cdf of X is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The mean and variance of X are

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}.$$

Applications

In practice, the exponential distribution often arises as the distribution of the length of time until some specific event occurs.

- Starting from now, the length of time until an earthquake occurs.
- The waiting time for a bus or a metro each morning.
- The waiting time if the line is occupied when calling a service center.

Poisson distribution is commonly applied to describe the number of occurrences in a given time interval. In fact, the waiting times between successive occurrences are random variables having the exponential distribution. If λ_1 and λ_2 are respectively the parameters of the Poisson distribution and the associated exponential distribution, then $\lambda_1 = \lambda_2$.

Example 1

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

(a) more than 10 minutes;

(b) between 10 and 20 minutes.

Example 2

Customers arrive in a certain shop according to an approximate Poisson distribution at a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer?

Exercise

The life time in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = \begin{cases} xe^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Compute the expected lifetime of such a tube.

Memoryless property

The exponential distribution is *memoryless*: for $t, s > 0$

$$P(X > t + s | X > s) = P(X > t).$$

Proof. We have that $P(X \leq t) = 1 - e^{-\lambda t}$ for $t \geq 0$, therefore

$$P(X > t) = e^{-\lambda t}.$$

Hence,

$$P(X > t + s | X > s) = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t).$$

Let X be waiting time for a bus (in minutes), and X has an exponential distribution with parameter λ.

Person A arrives at the bus stop earlier than Person B. When Person B arrives at the bus stop, the bus hasn't come yet and Person A has already waited s minutes. Knowing that Person A has already waited s minutes, the conditional probability that he will wait another t minutes is then $P(X > t + s | X > s)$. The probability that Person B will wait t minutes is then $P(X > t)$.

Outline

Definition

The random variable X has a **normal distribution** if its pdf is defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \; x \in \mathbb{R},$$

where $\mu, \sigma \in \mathbb{R}$ are parameters. Briefly, we say that X is $N(\mu, \sigma^2)$.
Recall that

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2}\right] dx = \sqrt{2\pi}.$$

By setting $z = (x-\mu)/\sigma$, we can check that

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz = 1.$$

The mean and variance of X are

$$E(X) = \mu, \; Var(X) = \sigma^2.$$

Standard normal distribution

If Z is $N(0, 1)$, we say that Z has a **standard normal distribution**. Moreover, the cdf of Z is

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw, \ z \in \mathbb{R}.$$

Values of $\Phi(z)$ for $z \geq 0$ are given in the Table on the last page. Because of the symmetry of its pdf, it holds that

$$\Phi(-z) = 1 - \Phi(z), \ \forall z \in \mathbb{R}.$$

In particular, $\Phi(0) = 0.5$.

Example 3

If Z is $N(0,1)$, find $P(Z \leq 1.24)$ and $P(-1.24 \leq Z \leq 2.37)$.

Using the Table on the last page,

$$\Phi(1.24) = 0.8925, \ \ \Phi(2.37) = 0.9911.$$

Hence

$$P(Z \leq 1.24) = \Phi(1.24) = 0.8925,$$

$$P(-1.24 \leq Z \leq 2.37) = \Phi(2.37) - \Phi(-1.24) = \Phi(2.37) - (1 - \Phi(1.24)) = 0.8836.$$

Normal distribution

Theorem

If X is $N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma$ is $N(0, 1)$.

Proof.

The cdf of Z is

$$\begin{aligned} P(Z \le z) &= P\left(\frac{X-\mu}{\sigma} \le z\right) = P(X \le z\sigma + \mu) \\ &= \int_{-\infty}^{z\sigma+\mu} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \\ &= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \\ &= \Phi(z), \end{aligned}$$

which is exactly the cdf of $N(0, 1)$. Hence, Z is $N(0, 1)$. □

Normal distribution

If $X \in N(\mu, \sigma^2)$, according to the above theorem, we deduce the following formula: for any $[a, b] \subseteq \mathbb{R}$,

$$
\begin{aligned}
P(a \leq X \leq b) &= P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)
\end{aligned}
$$

In particular, for any $c > 0$,

$$
\begin{aligned}
P(|X-\mu| < c\sigma) &= P(\mu - c\sigma \leq X \leq \mu + c\sigma) \\
&= \Phi(c) - \Phi(-c) \\
&= \Phi(c) - [1 - \Phi(c)] \\
&= 2\Phi(c) - 1.
\end{aligned}
$$

Example 4

Problem. The historical statistics show that the annual precipitation in Suzhou is $N(1070, 150^2)$ (unit: mm).

- Find the probability that the precipitation of the next year is between 950mm and 1250mm;
- Find the constant $C > 0$ such that the probability that the precipitation of the next year is below Cmm is 10%.

Solution. Let X be the annual precipitation. Then

$$
\begin{aligned}
P(950 \le X \le 1250) &= \Phi\left(\frac{1250-1070}{150}\right) - \Phi\left(\frac{950-1070}{150}\right) \\
&= \Phi(1.2) - \Phi(-0.8) \\
&= 0.8849 - (1 - 0.7881) \\
&= 0.673
\end{aligned}
$$

Example 4

Problem. The historical statistics show that the annual precipitation in Suzhou is $N(1070, 150^2)$ (unit: mm).

- Find the constant $C > 0$ such that the probability that the precipitation of the next year is below Cmm is 10%.

Solution. We look for C such that $P(X < C) = 0.1$, i.e.

$$P(X < C) = \Phi\left(\frac{C - 1070}{150}\right) = 0.1.$$

Therefore,

$$\Phi\left(-\frac{C - 1070}{150}\right) = 0.9.$$

Using the Table on the last page we obtain $\Phi(1.28) \approx 0.9$. Thus

$$-\frac{C - 1070}{150} \approx 1.28,$$

that is, $C \approx 878$.

Exercise

There are two routes to go to the railway station. The first route passes through the city center which is shorter but with more traffic jam. The estimated time (in minutes) follows a normal distribution $N(50, 100)$. The second route takes the ring road which is longer but with less traffic jam. The estimated time (in minutes) follows a normal distribution $N(60, 16)$.

(a) If there are 70 minutes available, which route is better?

(b) If there are 65 minutes available, which route is better?

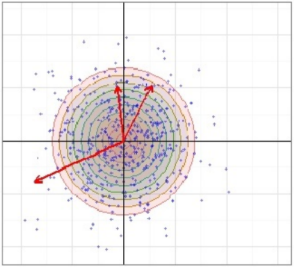
Outline

Definition

The motivation is to shoot the bull's eye and measure the shooting distance from the center:

$$R = \sqrt{X^2 + Y^2},$$

where X, Y are the coordinates of the shooting point and X, Y have normal distribution $N(0, \sigma^2)$. Then R has a *Rayleigh distribution* with parameter σ.



Rayleigh distribution

- The pdf

$$f(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, \ r > 0.$$

- The cdf

$$F(r) = P(R \leq r) = 1 - e^{-\frac{r^2}{2\sigma^2}}, \ r > 0.$$

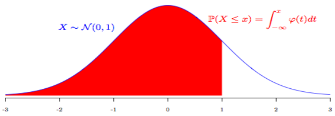
-

$$E(R) = \sigma\sqrt{\frac{\pi}{2}}, \ Var(R) = \frac{4-\pi}{2}\sigma^2.$$

Example 6

The distance, R, at which a weight thrown by Mr Jay is distributed as a Rayleigh variable with parameter $\sigma = 10m$.

1. Give $P(8 < R < 12)$.
2. If the his personal record is $20m$ and he keeps throwing until he breaks this record in one training. How many attempts in average will it take to break the record?



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Figure: cdf for standard normal r.v.