CAN209 Advanced Electrical Circuits and Electromagnetics

Lecture 11 First Order Transient Response

Dr. Zhenzhen Jiang

Assistant Professor, FHEA

Faculty: School of Advanced Technology

Email: zhenzhen.jiang02@xjtlu.edu.cn

Xi'an Jiaotong-Liverpool University
西交利物浦大學

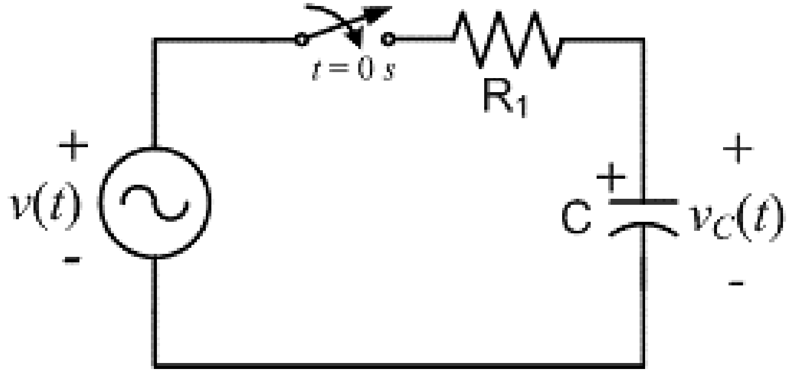
OUTLINE

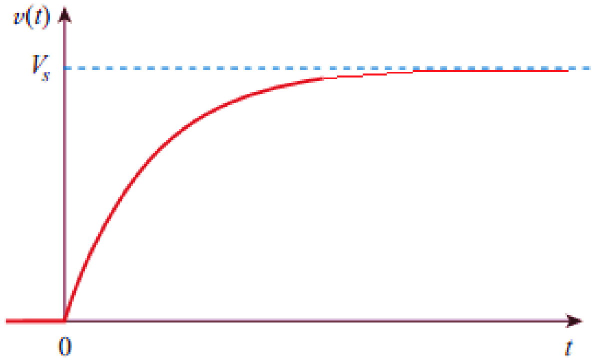
- ➢ Transient & Steady State
- ➢ First-order Transient Analyses
  - Source-free Circuit – Natural Response
    - ✓ RL Circuits
    - ✓ RC Circuits
  - Driven Circuit – Forced/Step Response
    - ✓ RL Circuits
    - ✓ RC Circuits

1.1 TRANSIENT VS STEADY-STATE

Transient

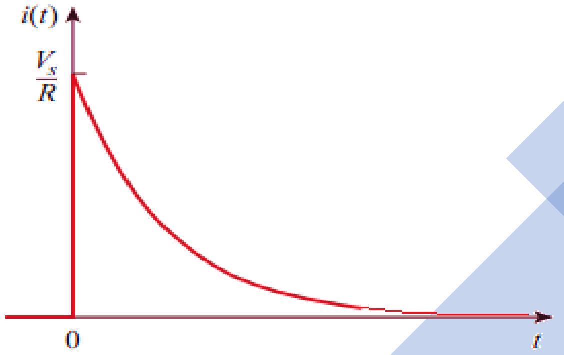
When applying a DC voltage to a series connected capacitor C and resistor R circuit, there is a short period immediately after connected the voltage, during which the current flowing in the circuit and voltages across C and R are changing. These changing values are called ***transients***.





Steady-state

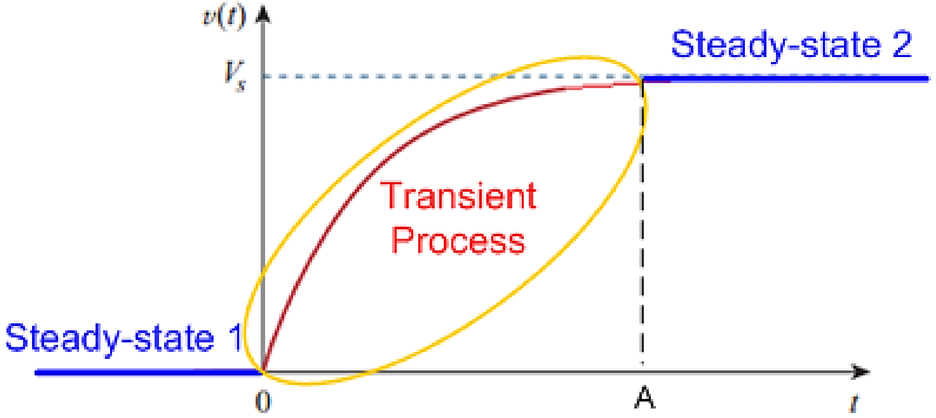
After a long period, when C is fully charged, it will act as open-circuit, and there is no current flowing in the circuit.



1.2 TRANSIENT

After the switching, there is a transient process in the circuit.

The transient characteristics describes the behaviour of the circuit during the transition from one steady state condition to another.



the instant of switching

$t = 0$

$t = A$

$t < 0$: original steady state

$0 < t < A$: transient process

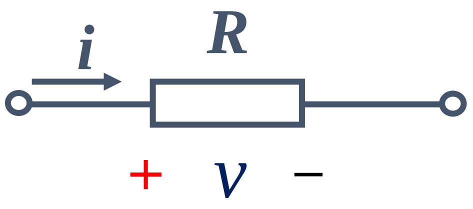
$t > A$: new steady state

time

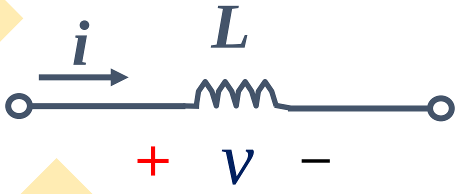
$t = 0^-$: final values of the original steady state

$t = 0^+$: initial values after switching, used in the DEs as the initial values

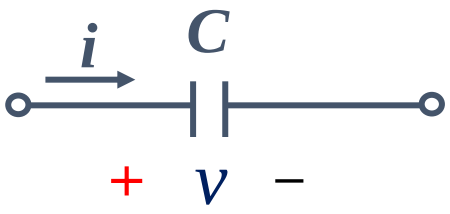
1.3 COMPONENTS



$$i = \frac{v}{R}$$



$$v = L\frac{di}{dt}$$



$$i = C\frac{dv}{dt}$$

- ✓ L & C are energy storage elements
- ✓ Circuits contained L & C can be represented by DEs
- ✓ The order of the circuit is equal to the number of L AND C in the circuit
- ✓ A **first** order circuit has **only ONE** energy storage element, i.e., one L or one C
- ✓ There are two types of 1st order circuits:

RL circuit & *RC* circuit

- ✓ Applying Kirchhoff's laws to these circuits results in DEs involving voltage or current, which are first-order.

1.3 TRANSIENT RESPONSE

Excitation

There are **two** ways to excite a circuit:

- ✓ Initial conditions of the storage elements – Source-Free Circuits
  (Energy stored in the capacitor, Energy stored in the inductor)

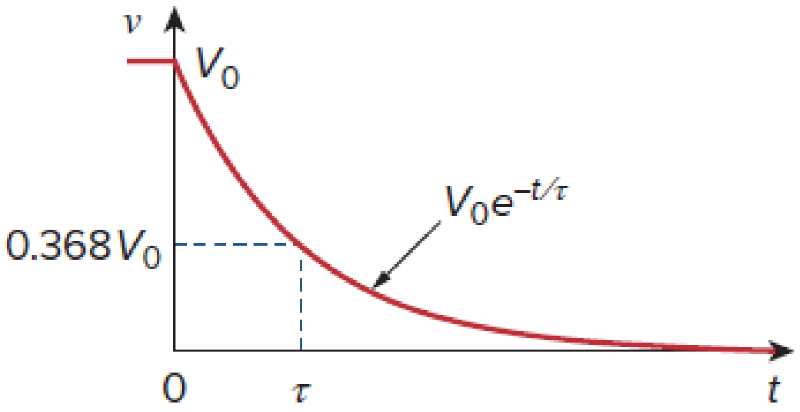
- ✓ Independent sources – Forced Excitation (Driven) circuits
  (DC sources, Sinusoidal sources, Exponential Sources)

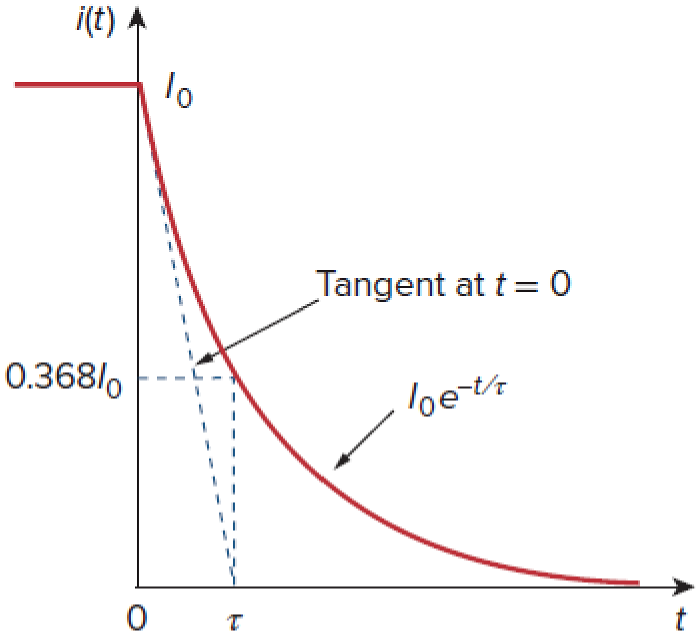
1.3.1 SOURCE-FREE CIRCUIT

Natural Response

The natural response of a circuit refers to the behaviour (in terms of voltage or current) with **no external sources** of excitation.

The circuit has a response **only** because of the energy initially stored in the energy storage elements (*i.e.*, capacitor or inductor).





1.3.2 DRIVEN CIRCUIT

Forced/Step Response

When we consider independent sources acting on a circuit, part of the response will resemble the nature of the particular source (or forcing function) used; this part of the response, called the particular solution, the *step* response, or the *forced* response.

The response of a circuit to the sudden application of a constant voltage or current source, describing the charging behaviour of the circuit.

Step (charging) response and natural (discharging) response show how the signal in a digital circuit switches between Low and High with time.

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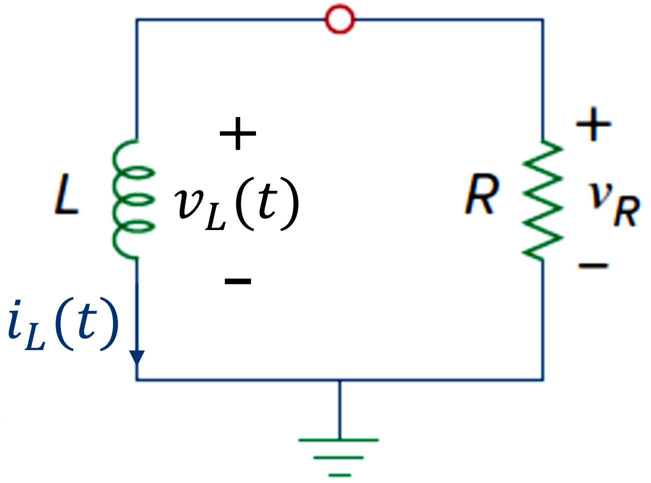
2.1 SOURCE-FREE RL



A circuit with series connection of a resistor R and inductor L.

Current $i(t)$ through the inductor is considered as response of this system.

At $t = 0$, assume the inductor has an initial current I_0.

- $t < 0$: the circuit has a current source, a resistor R_1, a switch and an inductor.
  Assuming the circuit is in steady-state, so the inductor is working as an ideal wire, and the current $i_L(t) = i_s(t)$.
- The switch is moved from ***a*** to ***b*** at $t = 0$.
- $t > 0$: there is no source in the circuit. The response is due to the initial energy stored in the inductor and the physical characteristics of the circuit, not due to any external voltage or current sources.

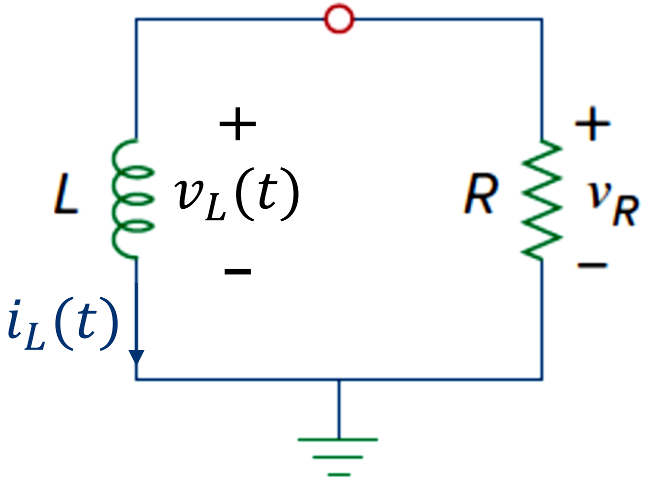
2.1 SOURCE-FREE RL

Natural Response of RL

Assume the value of $i_L(t)$ at $t = 0$ as I_0.

Applying KVL around the loop:

$$-v_L + v_R = -L\frac{di_L(t)}{dt} - Ri_L(t) = 0 \Rightarrow \frac{di_L}{dt} + \frac{R}{L}i_L = 0$$



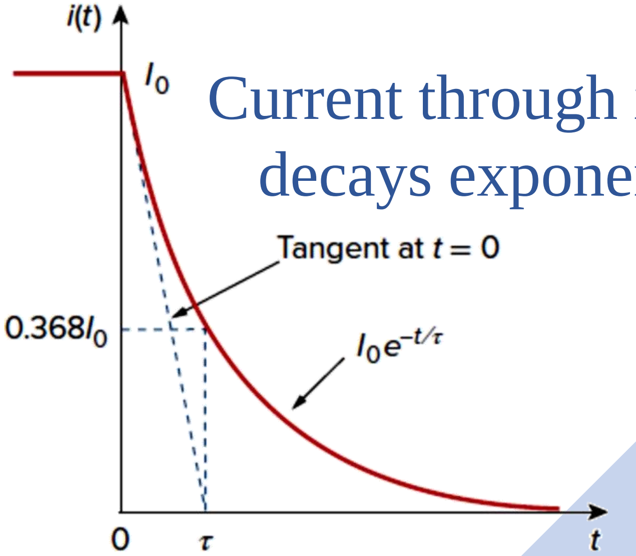
Find $i_L(t)$ which satisfies this equation and has the value I_0 at $t = 0$:

Solving the equation, we get:

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\frac{R}{L}\int_0^t dt$$

$$\therefore i_L(t) = I_0 e^{-Rt/L} = I_0 e^{-t/\tau}$$

where $\tau = L/R$, is the time constant for the RL circuit.



Current through inductor decays exponentially

EQUIVALENT RESISTANCE

- Regardless of how many resistors we have in the circuit, we obtain a single time constant (either $\tau = L/R$ or $\tau = RC$) when only one energy storage element is present.
- The value needed for *R* is in fact the Thévenin equivalent resistance **seen by our energy storage element** *L* or *C*.

2.1 SOURCE-FREE RL

Power Dissipation

At $t = 0^-$, the energy stored in the inductor is

$$w_L(t=0) = \frac{1}{2} L {I_0}^2$$

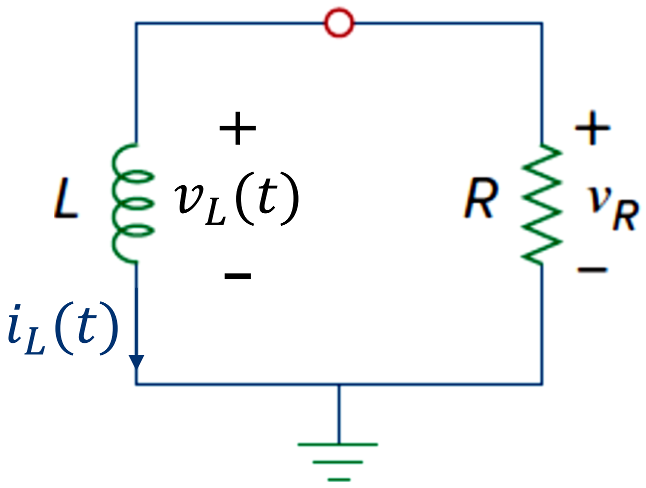
After switching, the power being dissipated in R is:

$$p_R = v_R i = {I_0}^2 R e^{-2Rt/L}$$

So the total energy turned into heat in the resistor is:

$$w_R = \int_0^\infty p_R dt = {I_0}^2 R \int_0^\infty e^{-2Rt/L} dt$$

$$= {I_0}^2 R \left(\frac{-L}{2R}\right) e^{-2Rt/L} \Big|_0^\infty = \frac{1}{2} L {I_0}^2$$



$$i_L(t) = I_0 e^{-Rt/L}$$

ALL the energy initially stored in the inductor is dissipated by the resistor.

APPLICATIONS

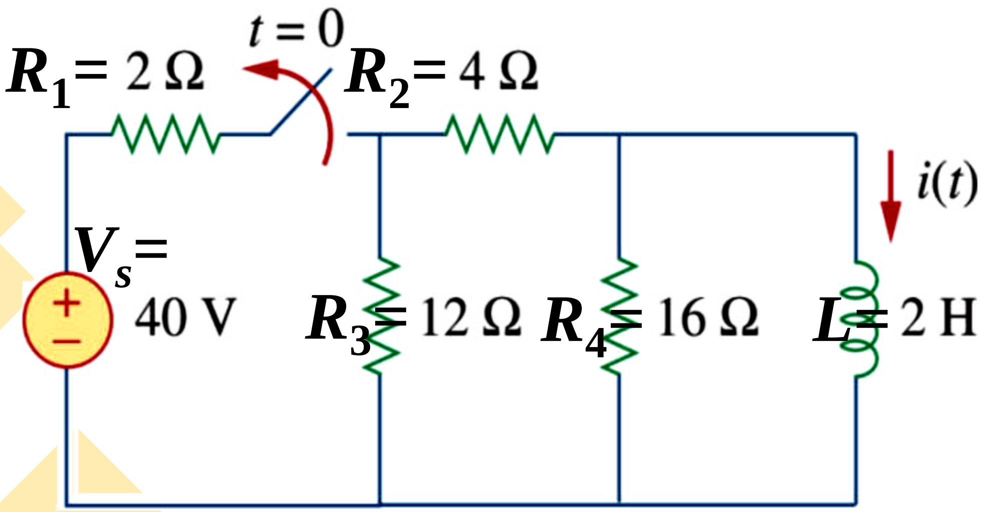
Tubelight choke

Pulse Generators

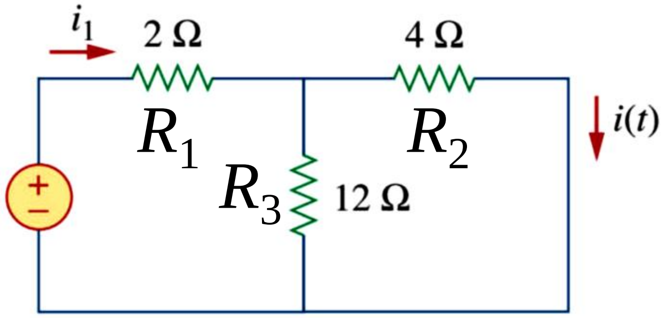
Electronic filter

QUIZ 1

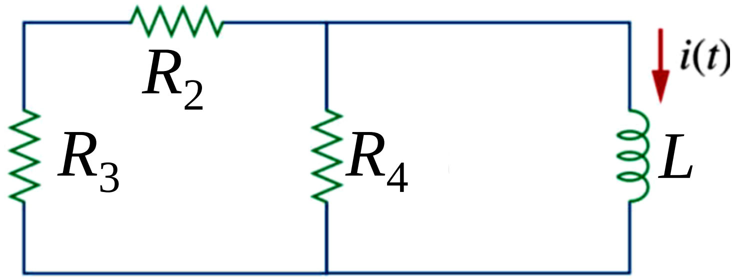
The switch in the circuit of the shown figure has been closed for a long time. At $t = 0$ the switch is opened. Calculate $i(t)$ for $t > 0$.



$t < 0$:

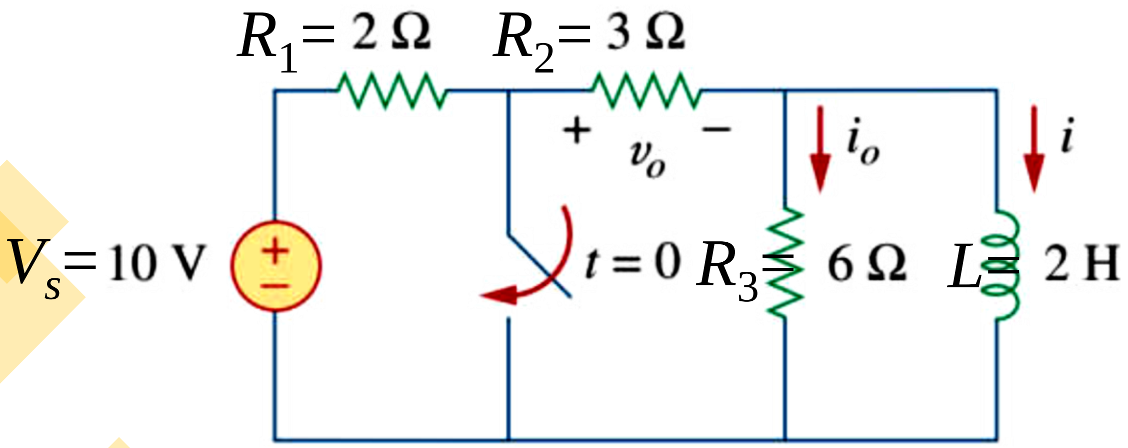


$t > 0$:

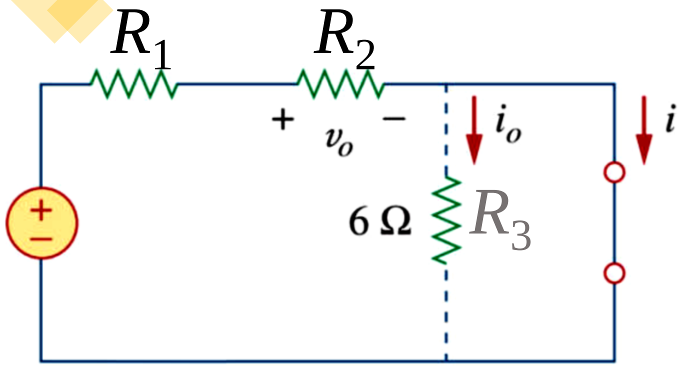


QUIZ 2

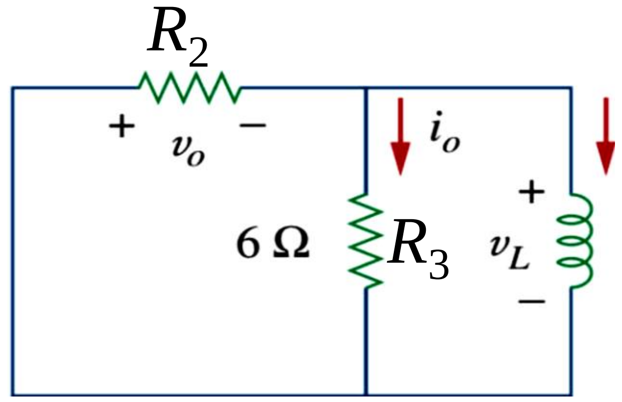
The switch in the circuit of the shown figure has been closed for a long time. At $t = 0$ the switch is opened. Calculate $i(t)$ and $v_0(t)$ for $t > 0$.



$t < 0$:



$t > 0$:



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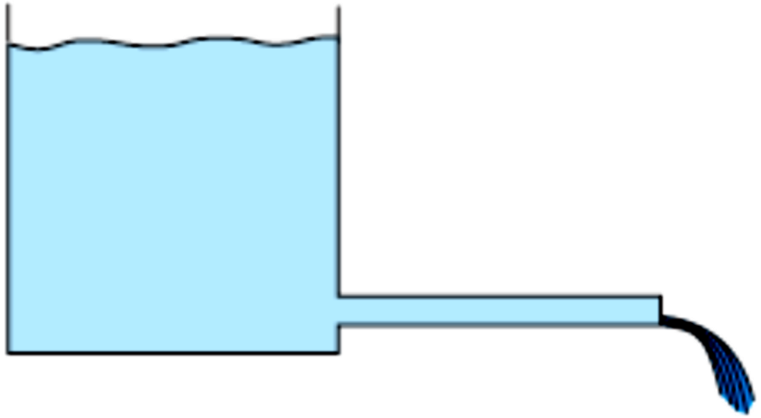
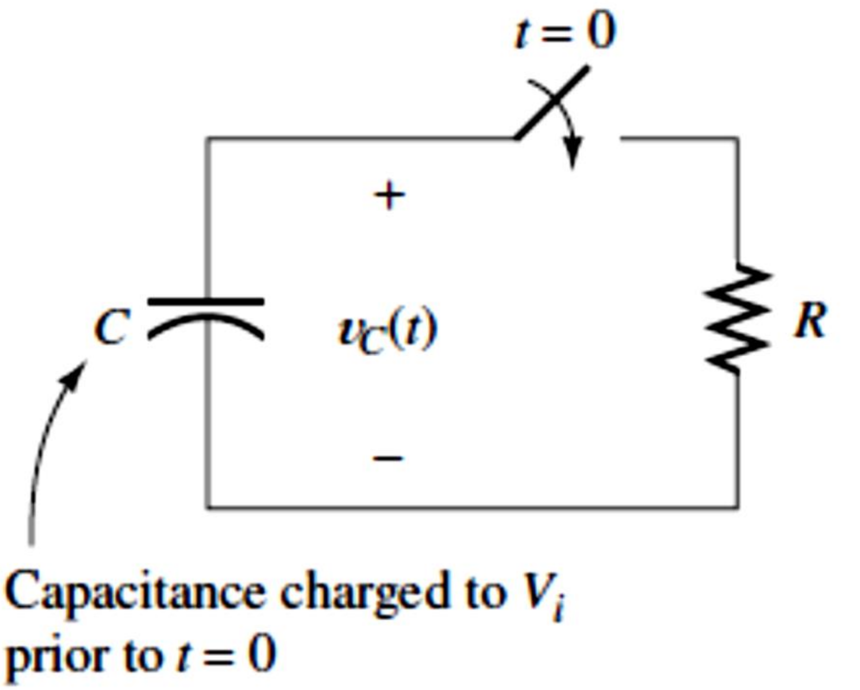
EXAMPLE



Analogous

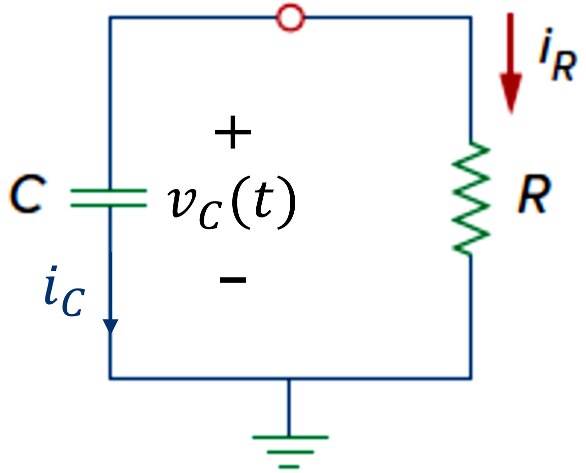


Fluid-flow analogy: water tank emptying through a small pipe

Electrical circuit: Capacitor discharging through resistance

2.2 SOURCE-FREE RC

Natural Response of RC



- A source-free *RC* circuit occurs when its d.c. source is suddenly disconnected.
- The energy already stored in the capacitor(s) is released to the resistor(s) & dissipated.

- RC source-free circuit is analysed from its initial voltage $v(0) = V_0$ and time constant τ.

Applying KCL, we get:

$$i_C + i_R = C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0$$

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = 0$$

Solving the equation:

$$v_C(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$$

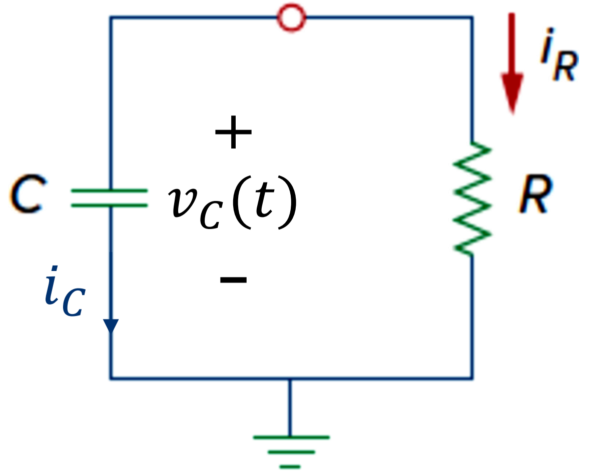
where τ = RC is the time constant for the RC circuit.

2.2 SOURCE-FREE RC

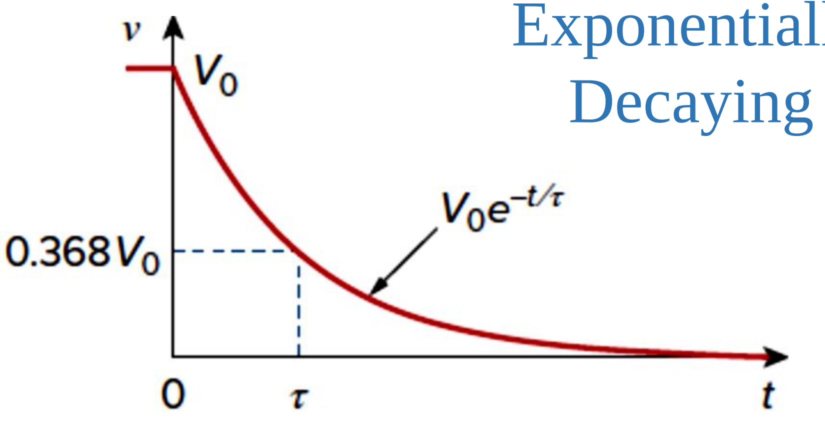
Time constant τ = RC

$$v_C(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$$

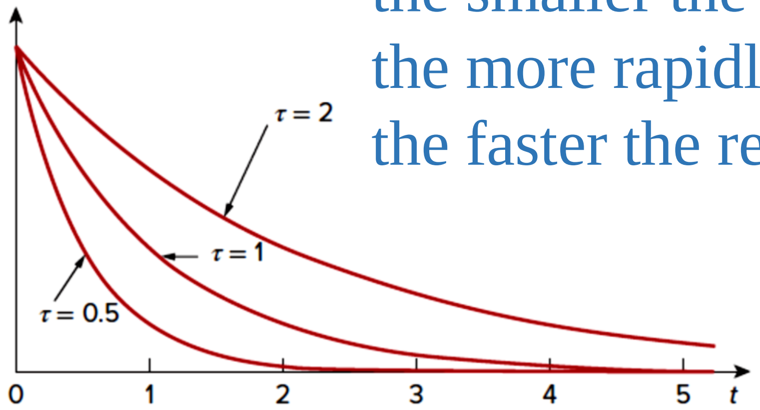
The time constant of a circuit is the **time** required for the response to decay to a factor of **1/e** or **36.8%** of its initial value.



The voltage $v_C(t)$ is less than 1% of V_0 after 5τ, thus, it is customary to assume the capacitor is fully discharged after 5τ. In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time.



Exponentially Decaying



the smaller the τ,
the more rapidly the voltage drops,
the faster the response.

2.2 SOURCE-FREE RC

Power Dissipation

At $t = 0^-$, the energy stored in the capacitor is

$$w_c(t=0) = \frac{1}{2} C{V_0}^2$$

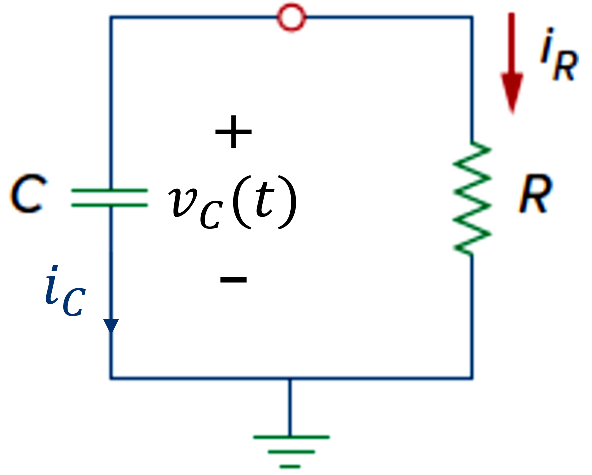
After switching, the power being dissipated in R is:

$$p_R = v_R i = \frac{{V_0}^2 e^{-2t/RC}}{R}$$

So the total energy turned into heat in the resistor is:

$$w_R = \int_0^\infty p_R dt = \frac{{V_0}^2}{R} \int_0^\infty e^{-2t/RC} dt$$

$$= -\frac{{V_0}^2}{2R} RCe^{-2t/RC} \Big|_0^\infty = \frac{1}{2} C{V_0}^2$$



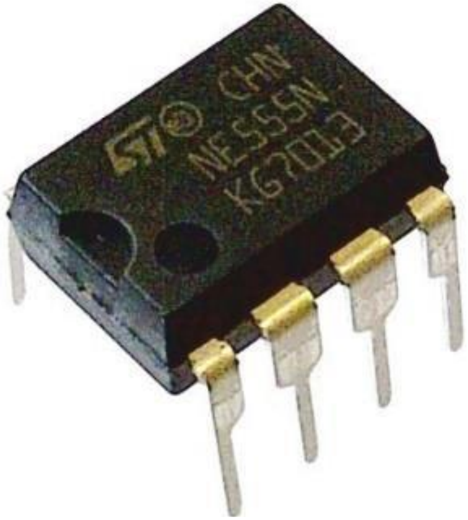
$$v_C(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$$

ALL the energy initially stored in the capacitor is dissipated by the resistor.

APPLICATIONS

Timers

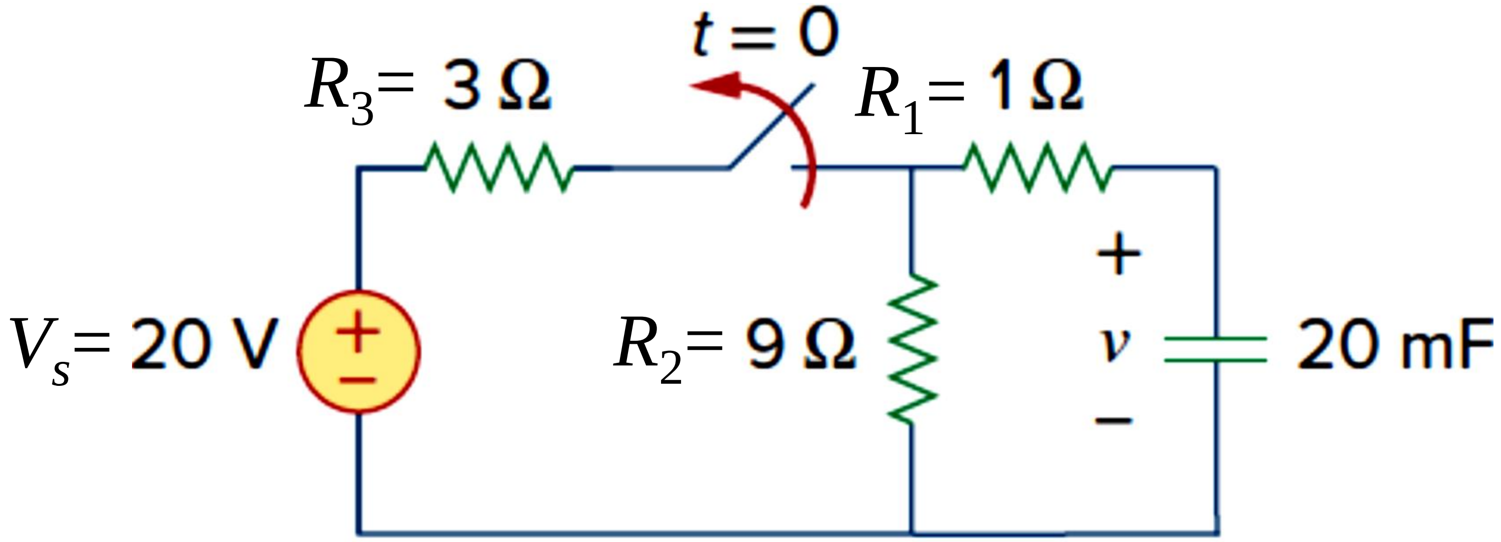
Oscillators



555 Timer circuits

QUIZ 3

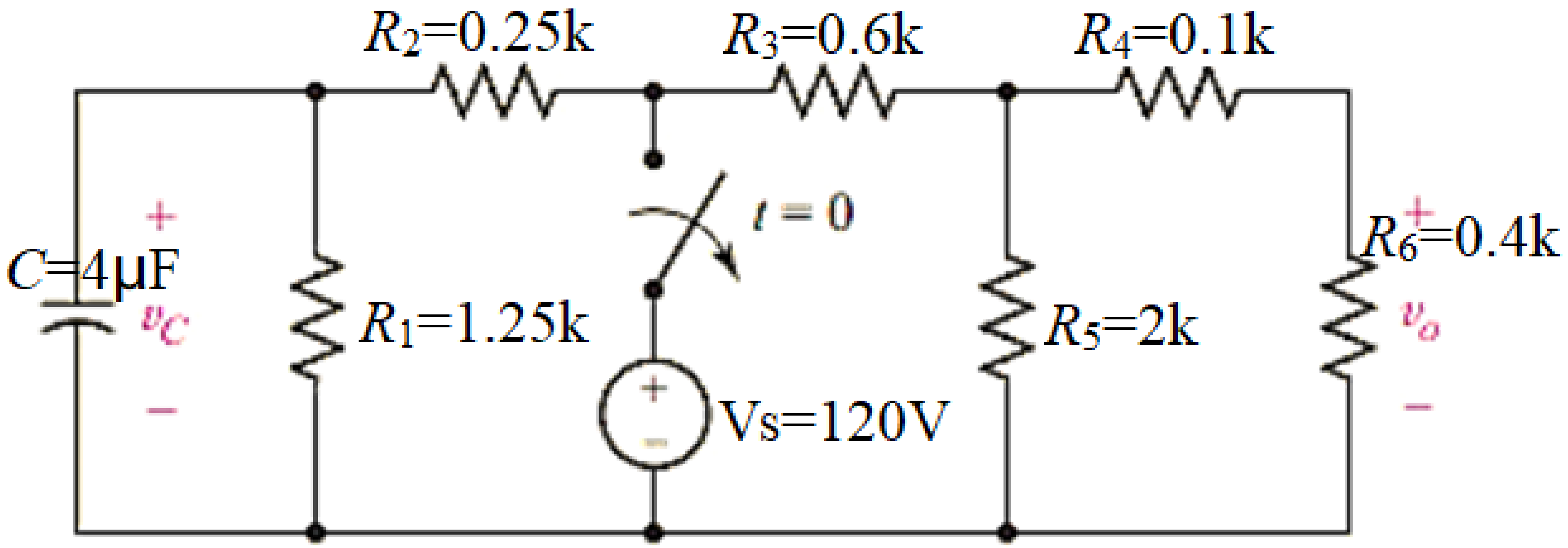
The switch in the circuit is closed for a long time. At $t = 0$ s, the switch is opened. Calculate $v(t)$ for $t > 0$ and the energy stored in the capacitor before opening of the switch.



QUIZ 4

Find the values of v_C and v_o in the circuit at :

(a) $t = 0^-$; (b) $t = 0^+$; (c) $t = 1.3$ ms.



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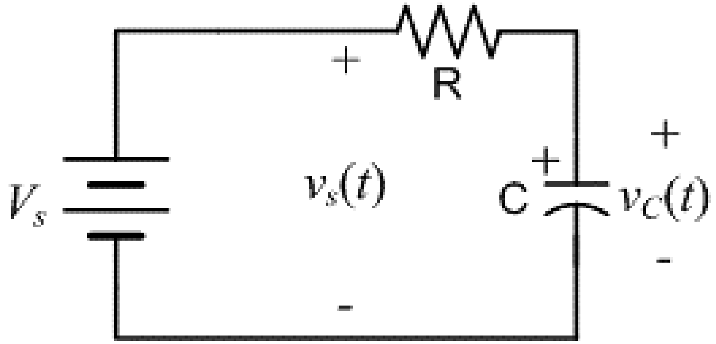
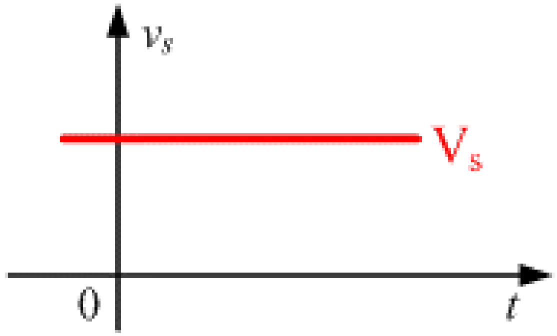
3 DRIVEN CIRCUITS

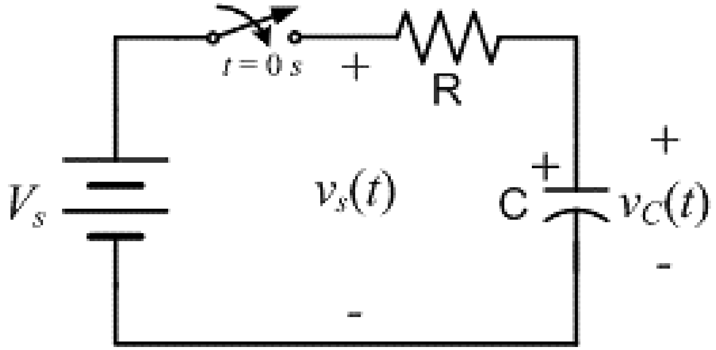
d.c. sources include d.c. voltage source and d.c. current source. In this module, they are constant values all the time.

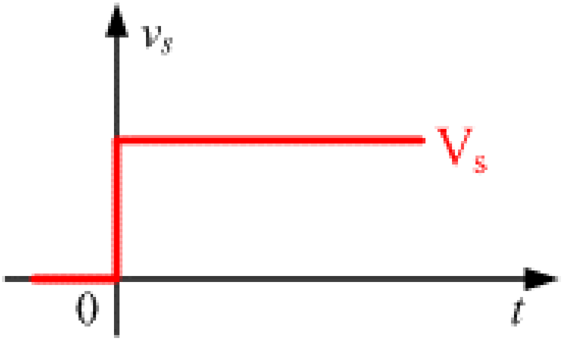




$v_s(t) = V_s$

Step function: the DC sources are applied to the circuit suddenly (usually at $t = 0$ s).





$v_s(t) = V_s u(t)$

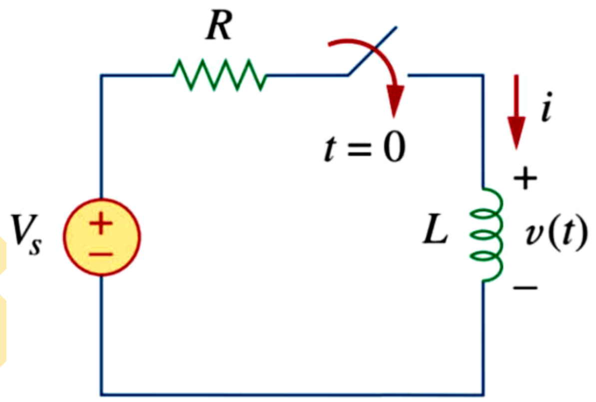
The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t.

3.1 DRIVEN RL

The goal is to find the inductor current I as the circuit response.

Let the response be the **sum** of the transient (natural) response and the forced (steady-state) response.

$$i = i_n + i_f$$

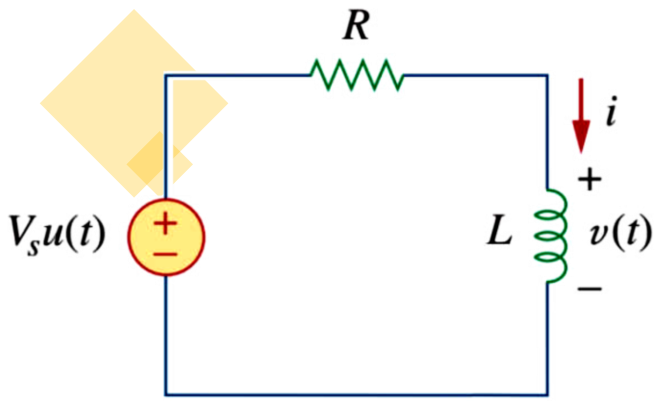


The transient response is always a decaying exponential, that is:

$$i_n = Ae^{-t/\tau} \qquad \tau = L/R$$

The steady-state response is the value of the current a long time after the switch is closed. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage appears across R.

Thus, the steady-state response is:



$$i_f = \frac{V_s}{R} \qquad \therefore i = i_n + i_f = Ae^{-t/\tau} + \frac{V_s}{R}$$

Let I_0 be the initial current through the inductor. Since the current through the inductor cannot change immediately at $t = 0$:

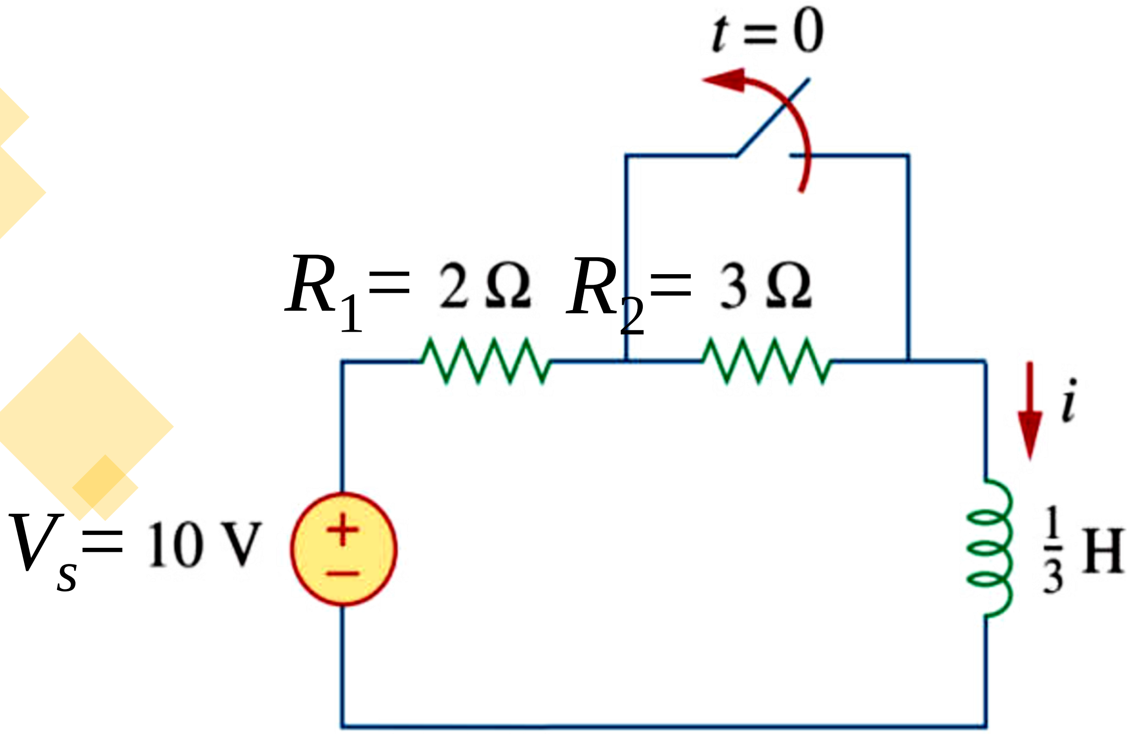
$$i(0^+) = i(0^-) = I_0 = A + \frac{V_s}{R} \qquad \therefore A = I_0 - \frac{V_s}{R}$$

Complete Response

$$\therefore i = \frac{V_s}{R} + (I_0 - \frac{V_s}{R})e^{-\frac{t}{\tau}} = i(\infty) + (i(0) - i(\infty))e^{-t/\tau}$$

QUIZ 5

The switch in the circuit of the shown figure has been closed for a long time. At $t = 0$ the switch is opened. Calculate $i(t)$ for $t > 0$.



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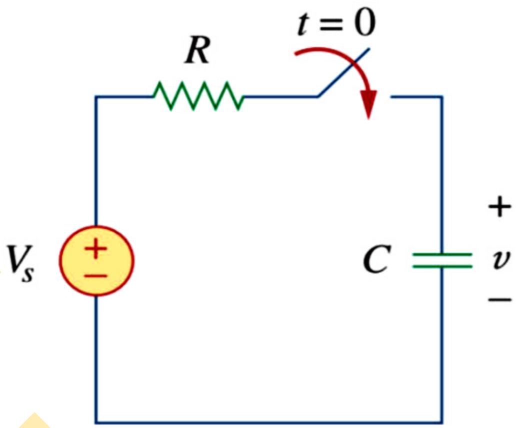
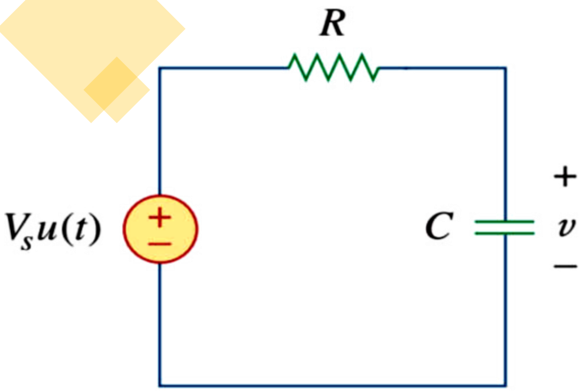
3.2 DRIVEN RC

The goal is to find the capacitor voltage $v_c(t)$ as the circuit response.

When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modelled as a step function, and the response is known as a step response.





Initial condition: $v_C(0^-) = v_C(0^+) = V_0$

Applying KCL ($t > 0$), we get:

$$i_C - i_R = C\frac{dv_C}{dt} - \frac{V_s u(t) - v_C}{R} = 0$$

$$\Rightarrow \quad \frac{dv_C}{dt} + \frac{1}{RC}v_C = \frac{V_s}{RC}u(t) = \frac{V_s}{RC} \quad \Rightarrow \quad \frac{1}{v_C - V_s}dv_C = -\frac{1}{RC}dt$$

Integrating both sides and then taking the exponential of both sides:

$$\frac{v_C - V_s}{V_0 - V_s} = e^{-t/RC}$$

$$\therefore v_C = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}} = v(\infty) + (v(0) - v(\infty))e^{-t/\tau}$$

Complete Response $\quad \tau = RC$

4.1 SOURCE-FREE CIRCUITS

RL: $$\frac{di_L}{dt} + \frac{R}{L} i_L = 0$$

RC: $$\frac{dv_C}{dt} + \frac{1}{RC} v_C = 0$$

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = 0$$

Source free 1st order differential equation

General solution for the source free 1st ODE is:

$$x(t) = x_0 e^{-t/\tau}$$

where x_0 is the initial value, τ is time constant (sec)

RL: τ = L/R

RC: τ = RC

Time constant represents the time it would take the voltage or current to get to it's final value if the initial rate stayed constant.

4.1 SOURCE-FREE CIRCUITS

1. Be careful with the **polarity** of **voltage** across the capacitor and the **direction** of the **current** through the inductor.

   ✓ v & i are defined strictly according to the **passive sign convention**.

2. The capacitor voltage is always continuous so that $v_C(0^+)=v_C(0^-)$, and the inductor current is always continuous so that $i_L(0^+)=i_L(0^-)$.

   ✓ where $t=0^-$ denotes the time just before a switching event and $t = 0^+$ is the time just after the switching event (assume the switching event takes place at $t = 0$)

Thus, in finding initial conditions, we first focus on those variables that cannot change abruptly:

capacitor voltage and inductor current

4.2 DRIVEN CIRCUITS

Complete Response = Transient Response + Steady-state Response

$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}$$

$x(0)$: initial voltage at $t = 0^+$

$x(\infty)$: final or steady-state value

For Driven RC circuit:

1. find the initial capacitor voltage $v_C(0)$
2. find the final capacitor voltage $v_C(\infty)$
3. find the time constant τ = RC

For Driven RL circuit:

1. find the initial inductor current $i_L(0)$
2. find the final inductor current $i_L(\infty)$
3. find the time constant τ = L/R

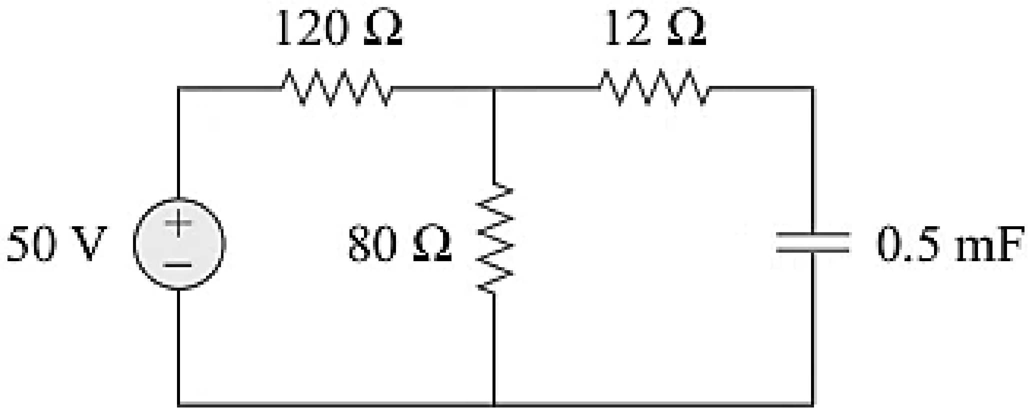
QUIZ 6

(Q1) A source free RL circuit has $R = 2\ \Omega$ and $L = 4$ H. The time needed for the inductor current to decrease to 36.8% of its steady-state value is:

(a) 0.5 s (b) 1 s (c) 2 s (d) 4 s

(Q2) The time constant for the RC circuit is:

(a) 30 ms
(b) 6 ms
(c) 5.5 ms
(d) 66 ms



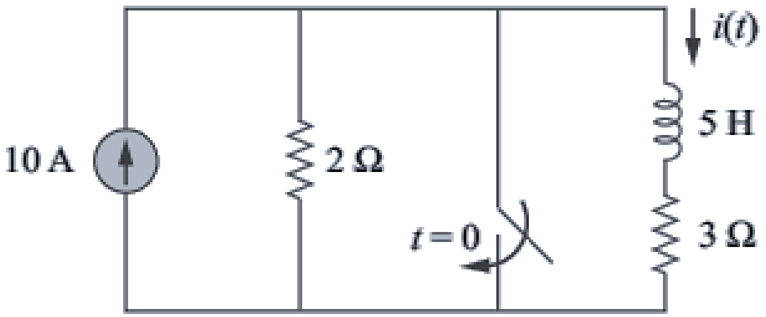
QUIZ 7

1 A driven RL circuit has $R = 2\ \Omega$ and $L = 4$ H. The time needed for the inductor current to charge to 36.8% of its steady-state value is

(a) 0.9 s (b) 1 s (c) 2 s (d) 4 s

2.1 For the circuit as shown, the inductor currents before $t = 0$ is:

(a) 10 A (b) 6 A (c) 4A (d) 0A



2.2 For the same circuit, the inductor currents at $t = \infty$ is:

(a) 10 A (b) 6 A (c) 4A (d) 0A

NEXT…

Second-Order Circuits:

Natural Response