Xi'an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

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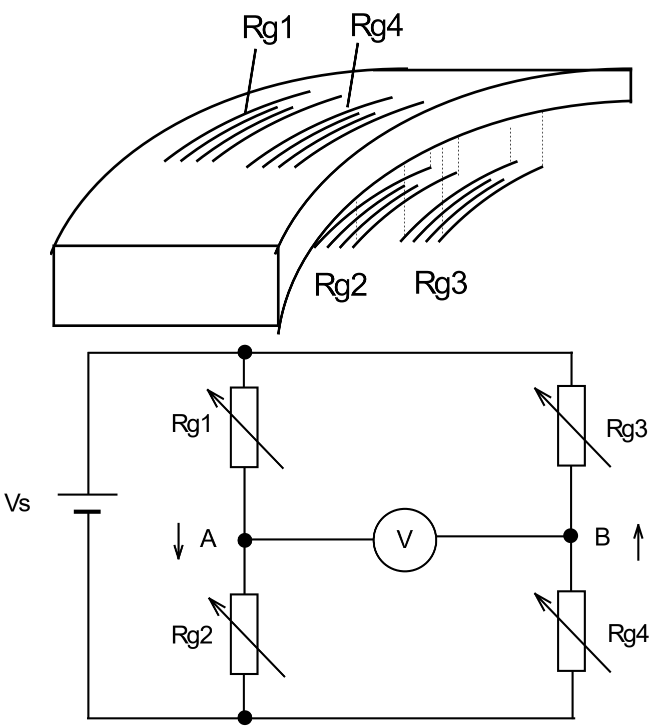
Quiz 4.1

The strain in a beam subject to tensile stress is to be measured using four strain gauges. The supply voltage to the bridge Vs is 6V, the gauge factor G=2.3, the gauges have a resistance of 200Ω each, Young's modulus of the beam is 250 x 10^9 Newtons/m^2. (hint: Young's modulus=stress/strain)

Suppose the measured output voltage from the bridge is 80μV, determine the corresponding strain and stress.



$$V_{out} = V_s G\varepsilon$$

Strain is

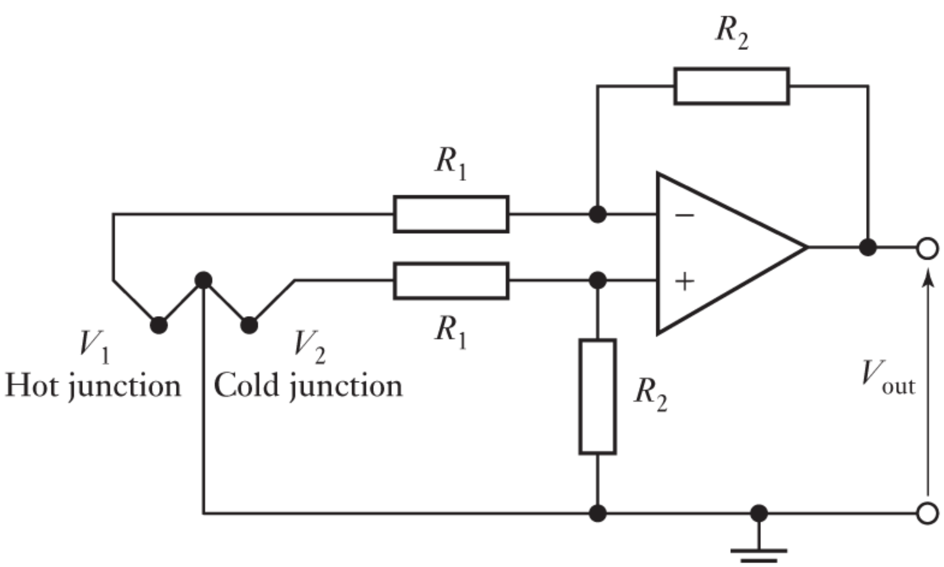
$$\varepsilon = \frac{V_{out}}{V_S G} = \frac{80 * 10^{-6}}{6 * 2.3} = 5.80 * 10^{-6}$$

Stress is

$$5.8 * 10^{-6} * 250 * 10^{9} = 1.45 * 10^{6}\ N/m^2$$

Quiz 4.2

A differential amplifier is used with a copper-constantan thermocouple sensor with sensitivity of $43\mu V/^oC$. Suppose $R_1 = 1k\Omega$ and $R_2 = 2.32k\Omega$, when the temperature difference between the two thermocouple junctions is 100°C. Find the output voltage of the circuit.



$$V_2 - V_1 = 100 * 43 * 10^{-6} = 4.3mV$$

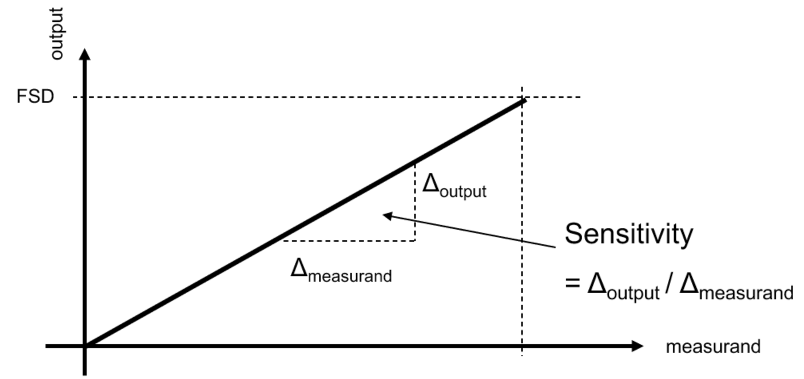
$$V_{out} = \frac{R_2}{R_1}(V_2 - V_1) = 2.32 * 4.3 = 9.976mV$$

Quiz 5.1

A thermocouple is used to measure temperature from 50 °C to 250 °C. The output voltage is then passed through an ADC for further transmission.

1) If 8 digits are used to represent the signal, what's the resolution of the system?
2) If the sensitivity of the thermocouple is 0.2 mV/ °C, to detect a signal of 0.1 mV, how many digits are required?

1) FSD=250-50=200 °C, resolution is $\frac{200}{2^8} = 0.78$ °C
2) Resolution of voltage signal is 0.1/0.2=0.5 °C, then the partition of FSD is 200/0.5=400. Because $2^9 = 512 > 400 > 256 = 2^8$, at least 9 digits are required.



Quiz 6.1

If ODE of a system is: $$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = 2r(t)$$

with the initial conditions and input: $y(0) = 1, \frac{d}{dt}y(0) = 0,\ r(t) = 1$ (step function)

How to find the time-domain response of the system?

1. Perform Laplace transform on the ODE:

$$(s^2Y(s) - sy(0) - \frac{d}{dt}y(0)) + 4(sY(s) - y(0)) + 3Y(s) = 2R(s)$$

Since $y(0) = 1, \frac{d}{dt}y(0) = 0$, R(s) = $\frac{1}{s}$:

$$Y(s) = \frac{s+4}{s^2+4s+3} + \frac{2}{s(s^2+4s+3)}$$

2. Expand the equation into partial fractions, which will yield:

$$Y(s) = \left(\frac{\frac{3}{2}}{s+1} + \frac{-\frac{1}{2}}{s+3}\right) + \left(\frac{-1}{s+1} + \frac{\frac{1}{3}}{s+3} + \frac{\frac{2}{3}}{s}\right)$$

3. Derive time-domain equation by using inverse Laplace transform:

$$y(t) = \left(\frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}\right) + \left(-e^{-t} + \frac{1}{3}e^{-3t} + \frac{2}{3}\right) = \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t} + \frac{2}{3}$$

<u>steady-state response:</u>

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{2}{3}$$

Lecture 7

Outline

Control Systems:

Mathematical Models of Systems

- ❑ Differential Equations of Physical Systems
- ❑ Linear Approximation of Physical Systems
- ❑ The Laplace Transform
- ❑ *The Transfer Function of Linear Systems*
- ❑ Block Diagram Models
- ❑ Signal-Flow Graph Models
- ❑ Simulation Tool

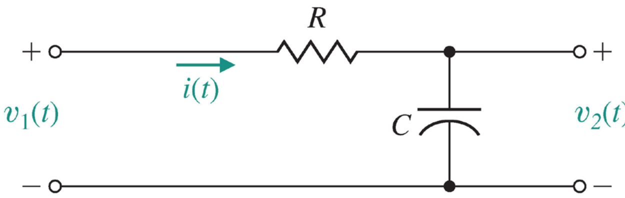
Transfer Function

The **Transfer Function (TF)** of a linear system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input variable, with all initial conditions assumed to be zero.

- TF represents the relationship describing the dynamics of the system under consideration;
- TF may be defined only for linear time-invariant **(LTI)** systems.
  - A time-varying system, whose parameters are time-dependent, and the Laplace transform may not be utilized.
- A transfer function does not describe any information concerning the internal structure of the system and its behavior.
  - A state-space model provides more information about the internal system, which will be introduced later.

Transfer function of RC Circuit

- U-I relation about capacitor:



$$i = C\frac{dU}{dt}$$

$$I(s) = sCU(s) \Rightarrow \frac{U(s)}{I(s)} = \frac{1}{sC}$$

- Input: $v_1(t)$, output: $v_2(t)$
  - in s domain, input $V_1(s)$, output $V_2(s)$
- Transfer function:

$$V_2(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_1(s) \Rightarrow G(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RCs + 1}$$

$$= \frac{1}{\tau s + 1}$$

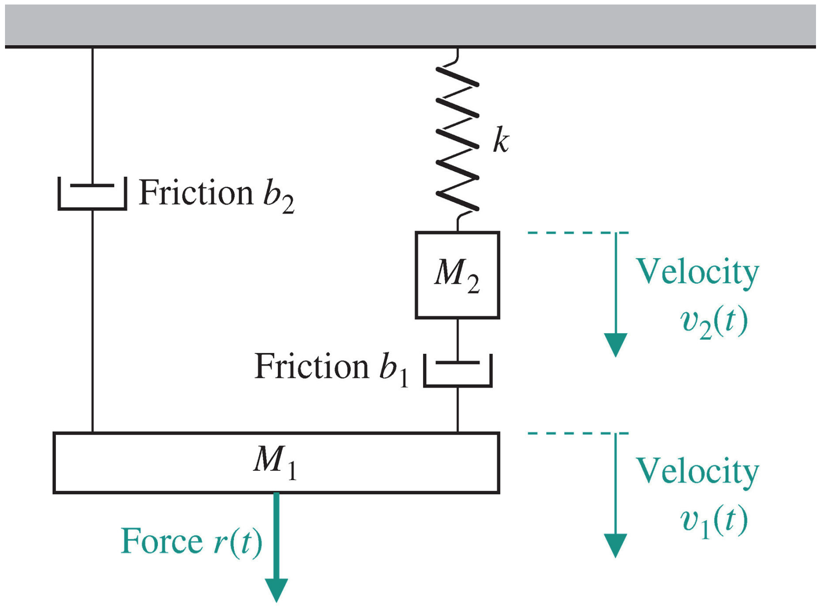
$\tau = RC$: time constant of the network

Transfer function of Mechanical System

Assuming that the initial conditions are zero.

Analyzing M_1 first.

- Based on Newton's Law:
$$F = ma$$
- Mass M_1, acceleration $a = \dot{v}_1(t)$.
- Analyze imposed force:
$$F = r(t) - b_2 v_1(t) - b_1[v_1(t) - v_2(t)]$$



(a) Two-mass mechanical system.

The relation in time domain
$$r(t) - b_2 v_1(t) - b_1[v_1(t) - v_2(t)] = M_1 \dot{v}_1(t)$$
Transform to s domain
$$R(s) - b_2 V_1(s) - b_1[V_1(s) - V_2(s)] = M_1 s V_1(s)$$

$$M_1 s V_1(s) + b_2 V_1(s) + b_1[V_1(s) - V_2(s)] = R(s)$$

Simplified equation:
$$(M_1 s + b_1 + b_2)V_1(s) - b_1 V_2(s) = R(s)$$

Transfer function of Mechanical System

Assuming that the initial conditions are zero.

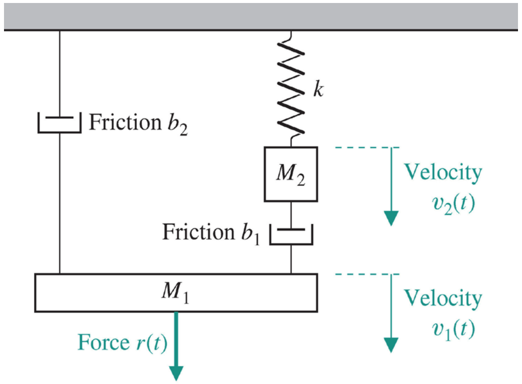
Analyzing M_2.

- Based on Newton's Law:

$$F = ma$$

- Mass M_2, acceleration $a = \dot{v}_2(t)$.
- Analyze imposed force:

$$F = b_1[v_1(t) - v_2(t)] - k\int v_2 dt$$



(a) Two-mass mechanical system.

The relation in time domain

$$b_1[v_1(t) - v_2(t)] - k\int v_2 dt = M_2\dot{v}_2(t)$$

Transform to s domain

$$b_1[V_1(s) - V_2(s)] - \frac{kV_2(s)}{s} = M_2 s V_2(s)$$

$$M_2 s V_2(s) - b_1[V_1(s) - V_2(s)] + \frac{kV_2(s)}{s} = 0$$

Simplified equation:

$$[M_2 s^2 + b_1 s + k]V_2(s) - b_1 s V_1(s) = 0$$

Transfer function of Mechanical System

Assuming that the initial conditions are zero.

Transfer Function:

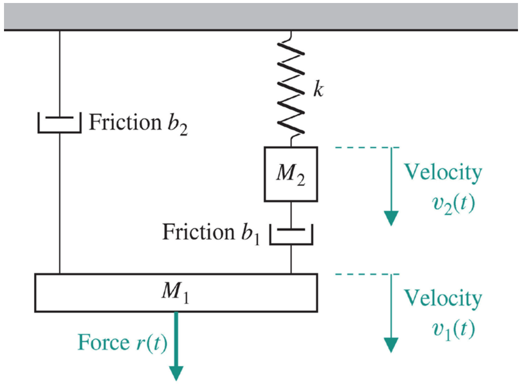
$$(M_1 s + b_1 + b_2)V_1(s) - b_1 V_2(s) = R(s)$$

$$[M_2 s^2 + b_1 s + k]V_2(s) - b_1 s V_1(s) = 0$$

Assume $v_1(t)$ as output, $r(t)$ as input

$$(M_1 s + b_1 + b_2)V_1(s) - b_1 \cdot \frac{b_1 s}{M_2 s^2 + b_1 s + k} V_1(s) = R(s)$$

$$G(s) = \frac{V_1(s)}{R(s)} = \frac{M_2 s^2 + b_1 s + k}{(M_1 s + b_1 + b_2)(M_2 s^2 + b_1 s + k) - b_1^2 s}$$



(a) Two-mass mechanical system.

Transfer function of Mechanical System

Assuming that the initial conditions are zero.

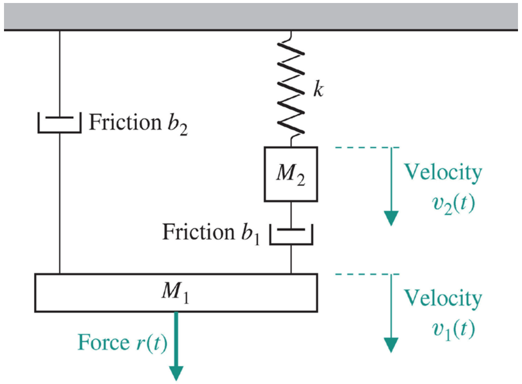
Transfer Function:

$$G(s) = \frac{V_1(s)}{R(s)} = \frac{M_2 s^2 + b_1 s + k}{(M_1 s + b_1 + b_2)(M_2 s^2 + b_1 s + k) - b_1^2 s}$$

What if we set position $x_1(t)$ as output?

$$x_1(t) = \int v_1(t)\mathrm{dt} \Rightarrow X_1(s) = \frac{V_1(s)}{s}$$

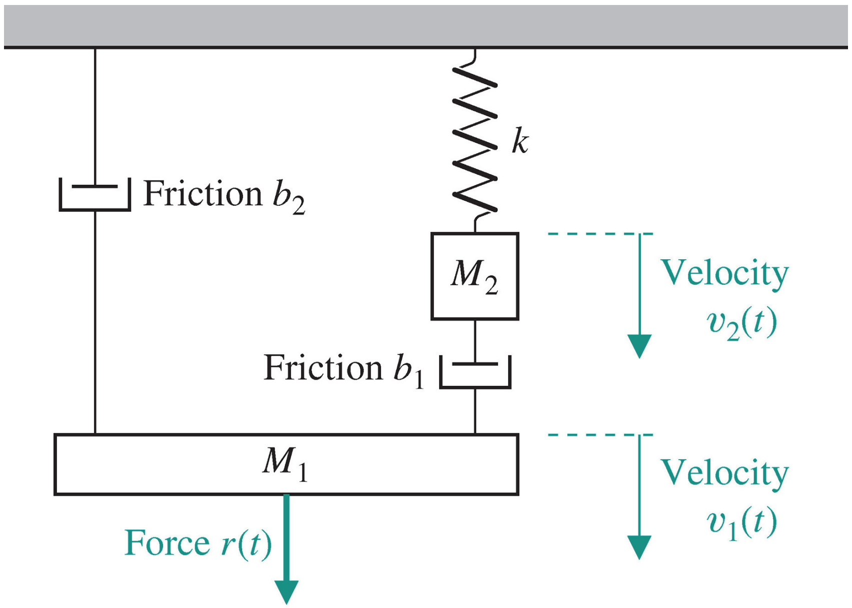
$$G(s) = \frac{X_1(s)}{R(s)} = \frac{M_2 s^2 + b_1 s + k}{s[(M_1 s + b_1 + b_2)(M_2 s^2 + b_1 s + k) - b_1^2 s]}$$



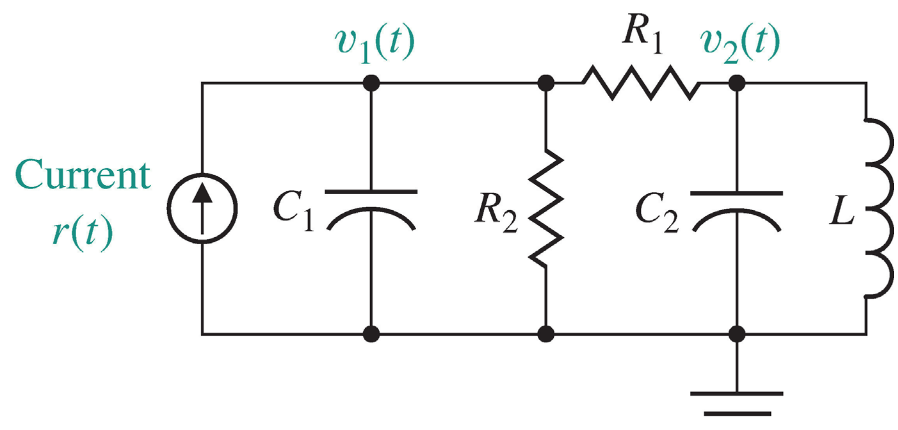
(a) Two-mass mechanical system.

Transfer Function – Mechanical vs Electrical

Mechanical vs. Electrical System: Analogue



(a) Two-mass mechanical system.



(b) Two-node electric circuit

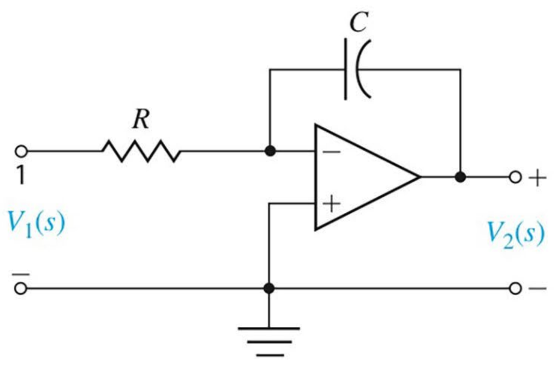
Try to derive the transfer function of the circuit by yourself

TF is the same!

Transfer Function of Op-Amp Circuits

Element or System	$G(s)$

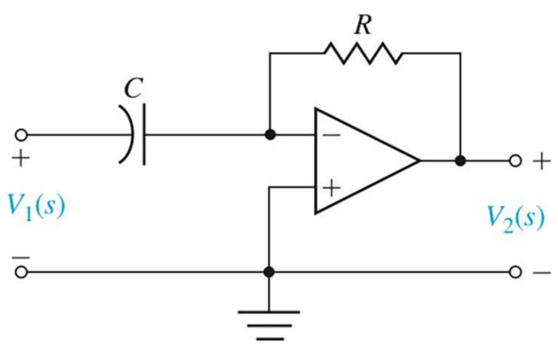
1. Integrating circuit, filter



$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$

$$\dot{v}_2(t) = -\frac{1}{RC} v_1(t)$$

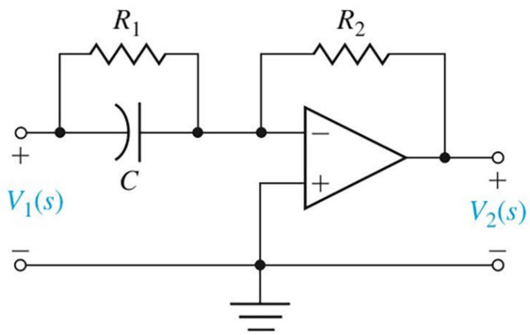
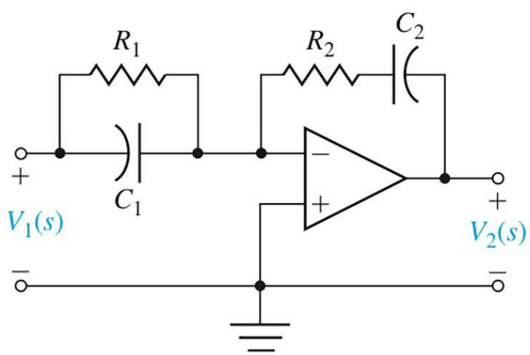
2. Differentiating circuit



$$\frac{V_2(s)}{V_1(s)} = -RCs$$

$$v_2(t) = -RC\dot{v}_1(t)$$

Transfer Function of Op-Amp Circuits

Element or System	$G(s)$
3. Differentiating circuit 	$$\frac{V_2(s)}{V_1(s)} = -\frac{R_2(R_1Cs + 1)}{R_1}$$
4. Integrating filter 	$$\frac{V_2(s)}{V_1(s)} = -\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_2s}$$

Try to analyze the circuit and derive the transfer function

Characteristic Equation

Characteristic equation & Poles

- For systems with transfer function as

$$G(s) = \frac{P(s)}{Q(s)}$$

- The denominator $Q(s)$, when set equal to zero, is called the **characteristic equation**.
  - The roots of this equation determine the character of the time response.
  - Order of characteristic equation is the **order of the system**.
- Roots of $Q(s) = 0$ are called **poles**, roots of $P(s) = 0$ are called **zeros**.

Characteristic Equation

Characteristic equation & Poles

- In example 7.1, transfer function is

$$G(s) = \frac{1}{s^2 + 3s + 2}$$

- Characteristic equation is

$$s^2 + 3s + 2 = 0$$

  - Poles are $s = -1, s = -2.$

- Impulse output is $y(t) = e^{-t} - e^{-2t}$
- Step output is $y(t) = 0.5 - e^{-t} + 0.5e^{-2t}$

Characteristic Equation

Characteristic equation & Poles

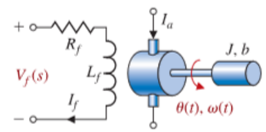
- If a system has **positive poles**, then it will diverge, leading to **unstable system**
- When a system is stable, its poles should all be **negative**
  - The **largest negative pole** is the dominant pole, it determines the main dynamic process of the system.
  - If one pole is **very small** when compared to others, i.e., far away from others, then we may neglect this pole and **reduce the order of the system**.

Transfer Functions for Dynamic Elements

- Table 2.4 in textbook

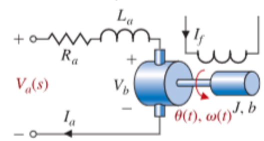
Element or System	G(s)

5. DC motor, field-controlled, rotational actuator



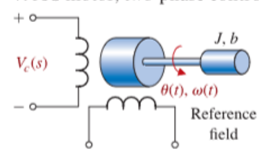
$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js+b)(L_f s + R_f)}$$

6. DC motor, armature-controlled, rotational actuator



$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js+b) + K_b K_m]}$$

7. AC motor, two-phase control field, rotational actuator

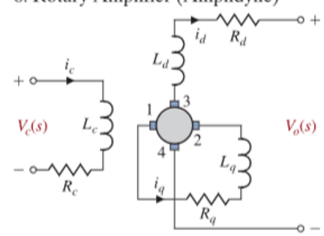


$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

8. Rotary Amplifier (Amplidyne)



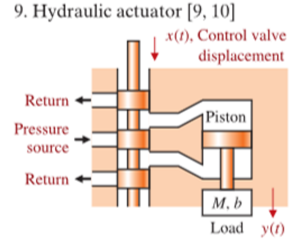
$$\frac{V_o(s)}{V_c(s)} = \frac{K/(R_c R_q)}{(s\tau_c + 1)(s\tau_q + 1)}$$

$$\tau_c = L_c/R_c, \qquad \tau_q = L_q/R_q$$

for the unloaded case, $i_d \approx 0$, $\tau_c \approx \tau_q$, $0.05\ \text{s} < \tau_c < 0.5\ \text{s}$

$$V_q, \ V_{34} = V_d$$

9. Hydraulic actuator [9, 10]



$$\frac{Y(s)}{X(s)} = \frac{K}{s(Ms+B)}$$

$$K = \frac{Ak_x}{k_p}, \qquad B = \left(b + \frac{A^2}{k_p}\right)$$

$$k_x = \left.\frac{\partial g}{\partial x}\right|_{x_0, P_0}, \qquad k_p = \left.\frac{\partial g}{\partial P}\right|_{x_0, P_0},$$

$g = g(x, P)$ = flow

A = area of piston

M = load mass

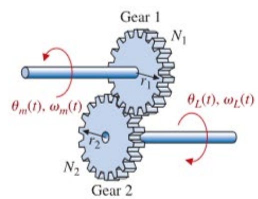
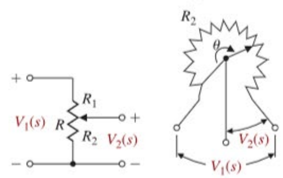
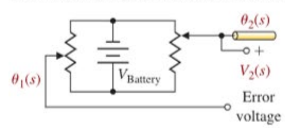
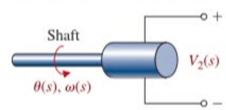
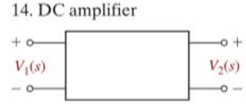
b = load friction

(*continued*)

Transfer Functions for Dynamic Elements

- Table 2.4 in textbook

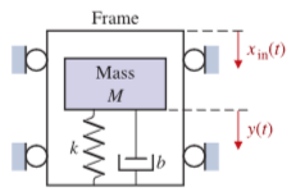
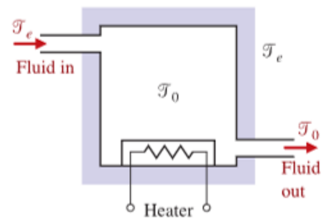
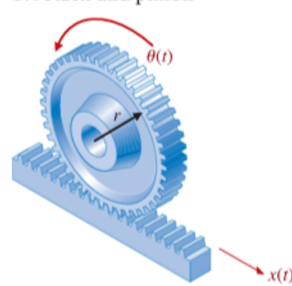
Table 2.4 ***(continued)***

Element or System	$G(s)$
10. Gear train, rotational transformer 	Gear ratio $= n = \dfrac{N_1}{N_2}$ $N_2\theta_L(t) = N_1\theta_m(t), \quad \theta_L(t) = n\theta_m(t)$ $\omega_L(t) = n\omega_m(t)$
11. Potentiometer, voltage control 	$\dfrac{V_2(s)}{V_1(s)} = \dfrac{R_2}{R} = \dfrac{R_2}{R_1 + R_2}$ $\dfrac{R_2}{R} = \dfrac{\theta}{\theta_{max}}$
12. Potentiometer, error detector bridge 	$V_2(s) = k_s(\theta_1(s) - \theta_2(s))$ $V_2(s) = k_s\theta_{error}(s)$ $k_s = \dfrac{V_{Battery}}{\theta_{max}}$
13. Tachometer, velocity sensor 	$V_2(s) = K_t\omega(s) = K_t s\theta(s)$ $K_t =$ constant
14. DC amplifier 	$\dfrac{V_2(s)}{V_1(s)} = \dfrac{k_a}{s\tau + 1}$ $R_o =$ *output resistance* $C_o =$ *output capacitance* $\tau = R_o C_o$, $\tau = 1s$ and is often negligible for controller amplifier

(continued)

Transfer Functions for Dynamic Elements

- Table 2.4 in textbook

Element or System	$G(s)$
15. Accelerometer, acceleration sensor 	$x_o(t) = y(t) - x_{in}(t)$, $\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$ For low-frequency oscillations, where $\omega < \omega_n$, $\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$
16. Thermal heating system 	$\frac{T(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}$, where $T = T_o - T_e =$ *temperature difference* due to thermal process $C_t =$ thermal capacitance $Q =$ fluid flow rate $=$ *constant* $S =$ specific heat of water $R_t =$ thermal resistance of insulation $q(s) =$ transform of rate of heat flow of heating element
17. Rack and pinion 	$x(t) = r\theta(t)$ converts radial motion to linear motion

Outline

Control Systems:

Mathematical Models of Systems

- ❑ Differential Equations of Physical Systems
- ❑ Linear Approximation of Physical Systems
- ❑ The Laplace Transform
- ❑ The Transfer Function of Linear Systems
- ❑ *Block Diagram Models*
- ❑ Signal-Flow Graph Models
- ❑ Simulation Tool

Block Diagram Models

- The dynamic systems are typically represented mathematically by a set of simultaneous **differential equations**; the **Laplace transform** reduces the problem to the solution of a set of linear algebraic equations;

- **Transfer functions** are introduced to represent the relationship between the input and output variables of the system; it is an important relation in control engineering;

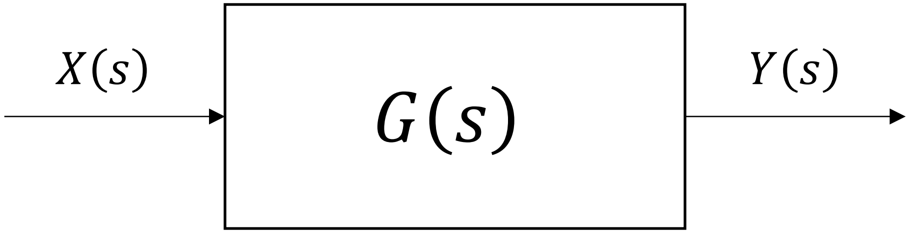
- **Block diagram** is a graphic way to represent this important cause-and-effect relationship. It consists of unidirectional, operational blocks that represent the transfer function of the systems of interest.

Block Diagram Models – Elements

Block diagram consists of three basic elements:

- Function block
  - Inside the function block is transfer function
  - Input signal goes into the function block, and gives the output signal

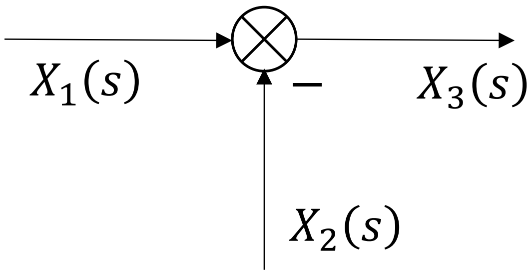
$$G(s) = \frac{Y(s)}{X(s)}$$



Block Diagram Models – Elements

Block diagram consists of three basic elements:

- Summing point
  - ➢If two signals goes into the same box, then a summing point is required
  - ➢The sign attached to the input of summing point indicates the add/minus behavior inside the point

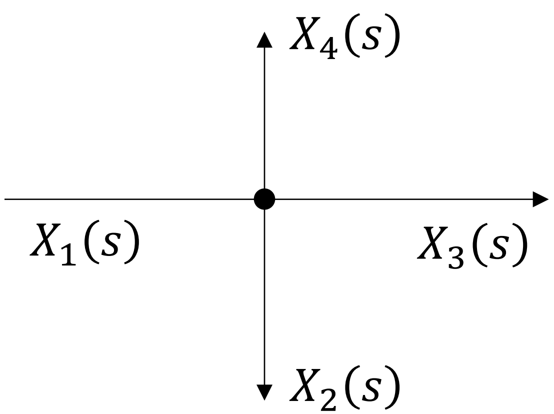


$$X_3(s) = X_1(s) - X_2(s)$$

Block Diagram Models – Elements

Block diagram consists of three basic elements:

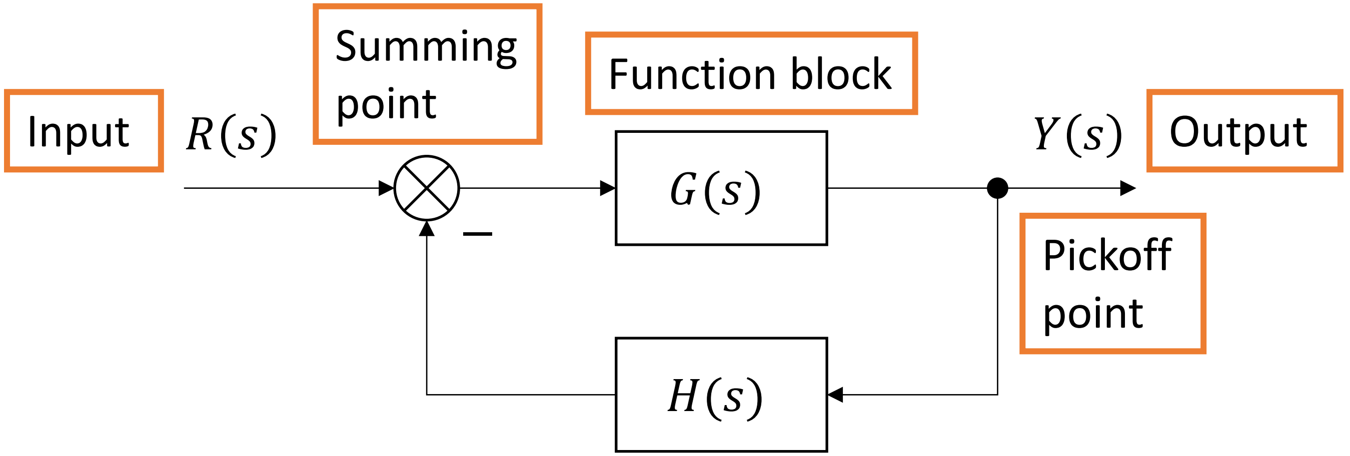
- Pickoff point
  - ➢If one signal is transmitted to multiple destinations, pickoff point is required in block diagram
  - ➢Multiple signals split from one pickoff point is always the same



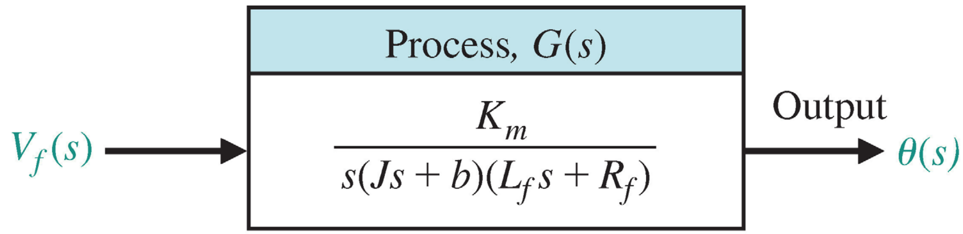
$$X_1(s) = X_2(s) = X_3(s) = X_4(s)$$

Examples

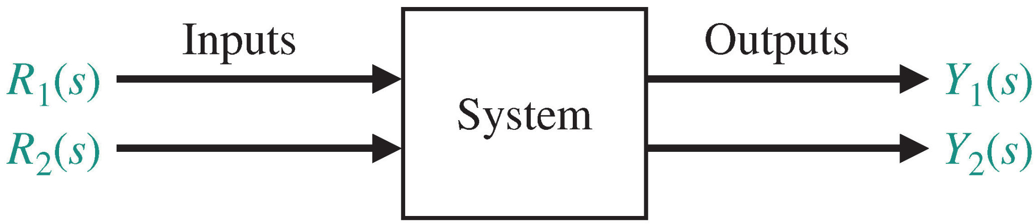
- A simple negative feedback closed-loop system:



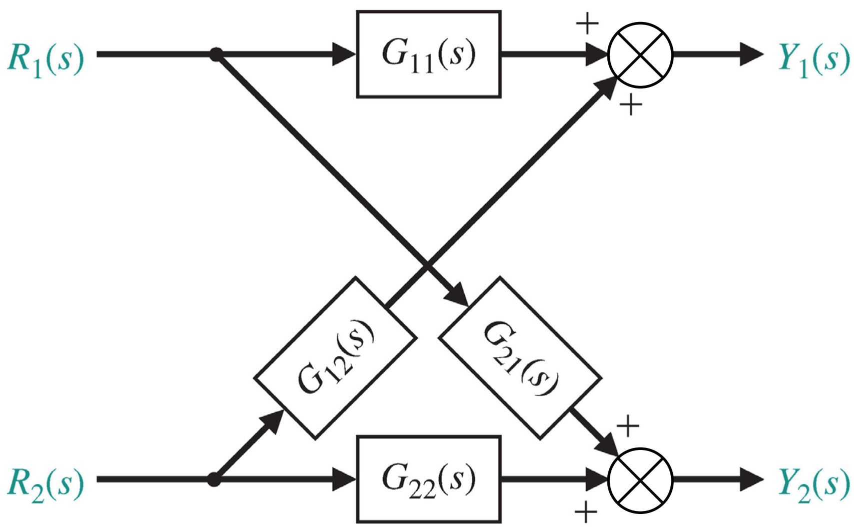
Examples



(a) Block diagram of a DC motor.

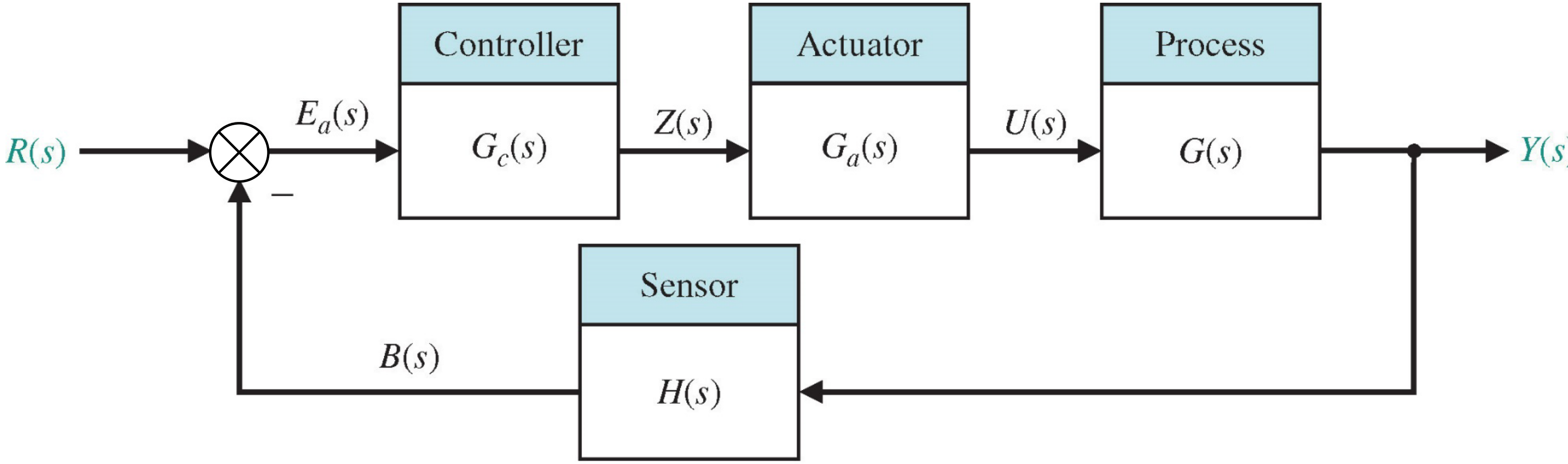


(b) General block representation of two-input, two-output system.



(c) Block diagram of a two-input, two-output interconnected system.

Transfer Function of A Negative Feedback Control System



The error signal or actuating signal $E_a(s)$ is

$$E_a(s) = R(s) - H(s)Y(s)$$

The output signal $Y(s)$ can be represented as

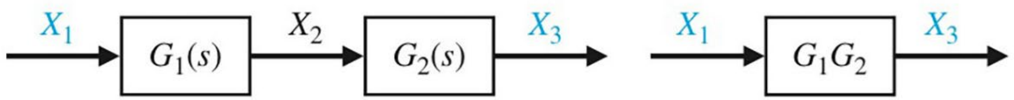
$$Y(s) = E_a(s)G_c(s)G_a(s)G(s)$$

Combining the above two equations, the transfer function of the system is

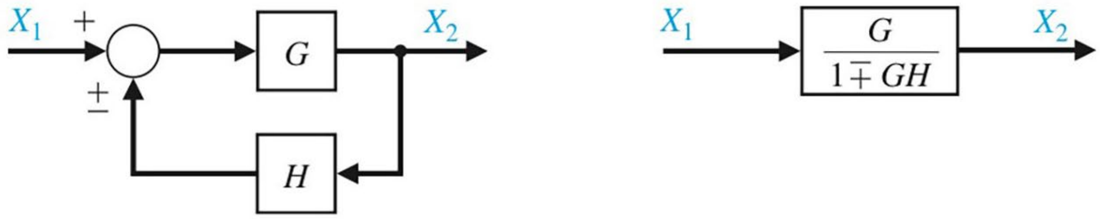
$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G_a(s)G(s)}{1 + G_c(s)G_a(s)G(s)H(s)}$$

Block Diagram Transformations

- Serial connection

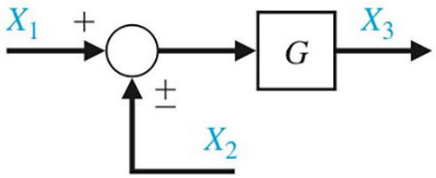


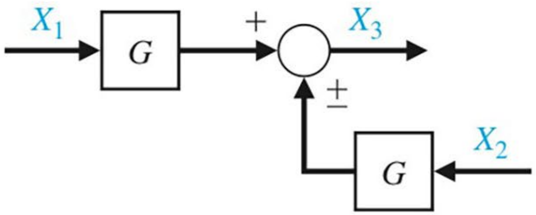
- Feedback loop



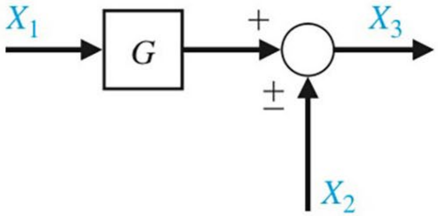
Block Diagram Transformations

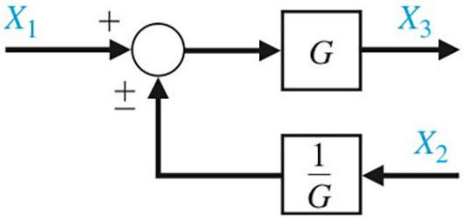
- Move summing point behind a block





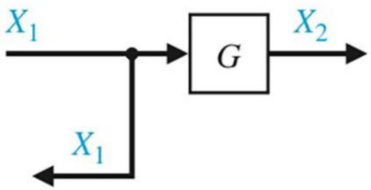
- Move summing point ahead of a block

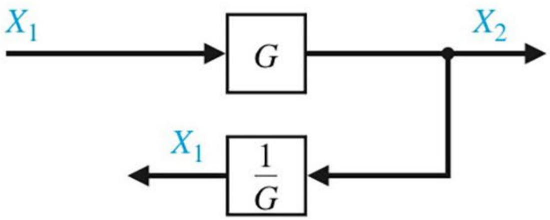




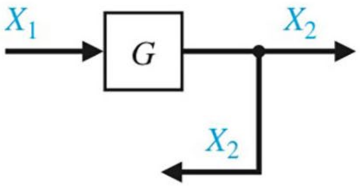
Block Diagram Transformations

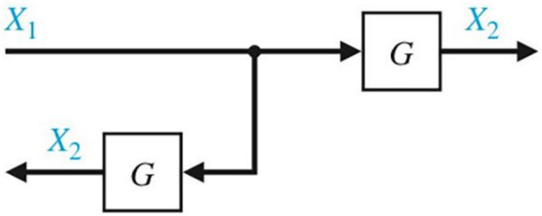
- Move pickoff point behind a block





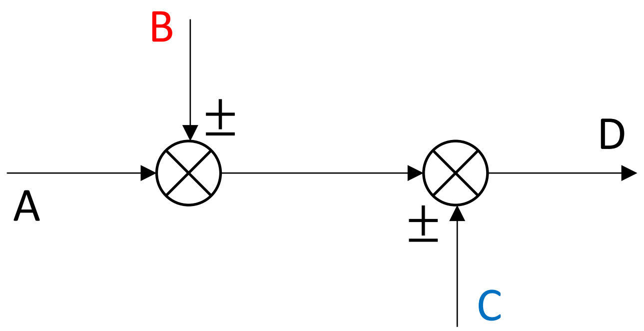
- Move pickoff point ahead a block



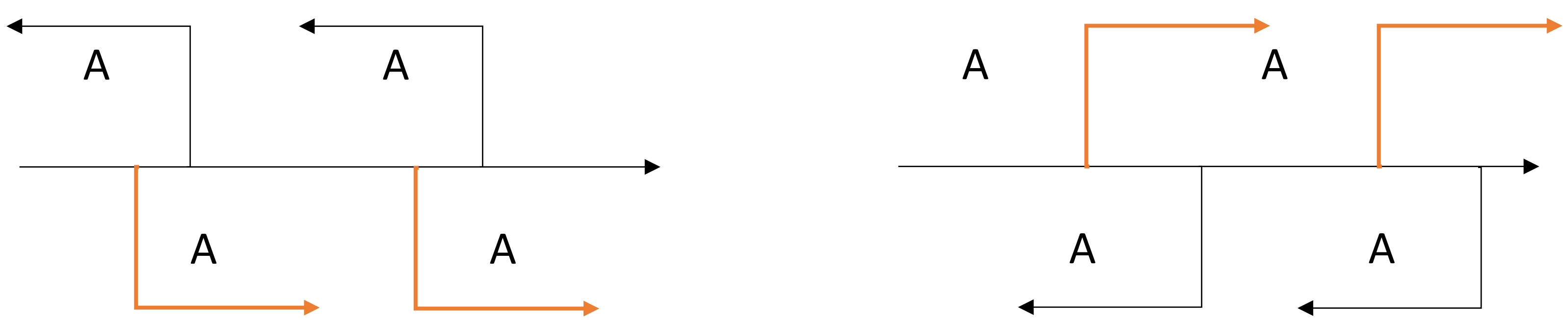


Block Diagram Transformations

- Exchange the position of summing point

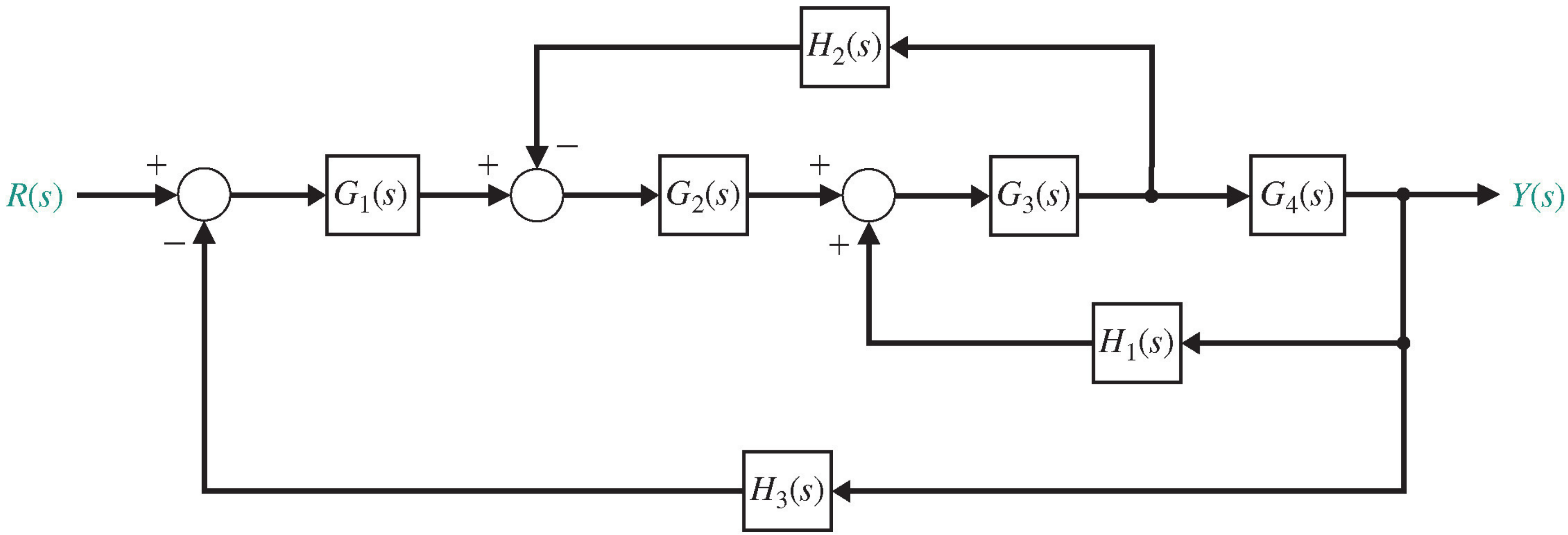


- Exchange the position of pickoff point

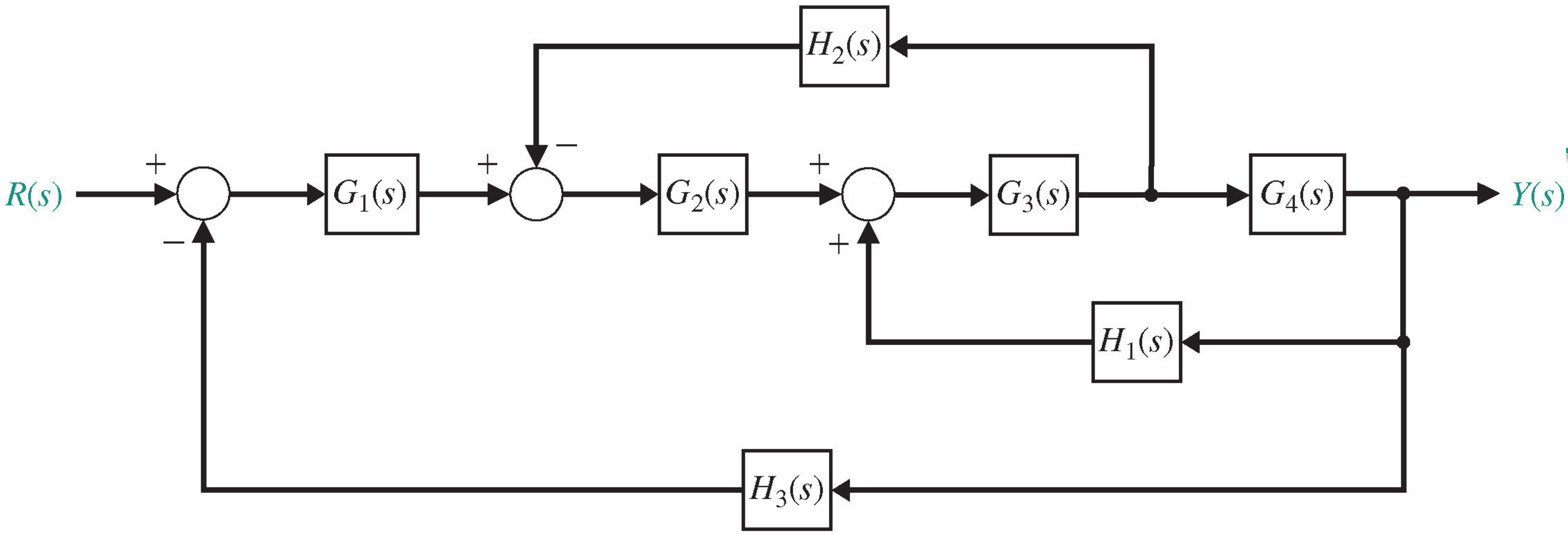


Reduction of the Block Diagram

A multi-loop feedback control system

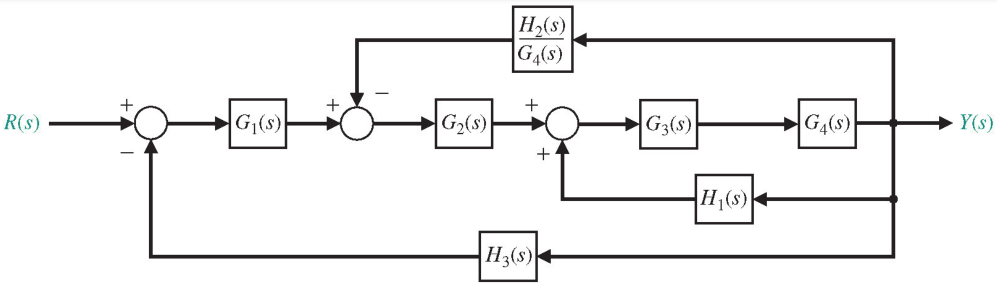


Please note the feedback $H_1G_3G_4$ is a positive feedback; while $G_2G_3H_2$ and $G_1G_2G_3G_4H_3$ are negative feedback loops.

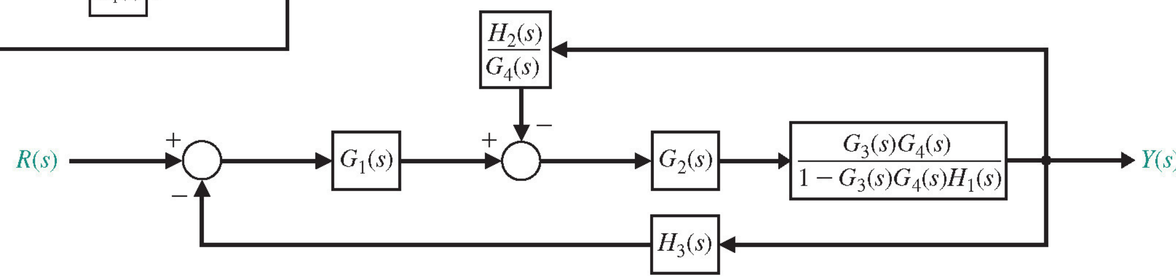


$H_2(s)$
$R(s)$
$G_1(s)$
$G_2(s)$
$G_3(s)$
$G_4(s)$
$Y(s)$
$H_1(s)$
$H_3(s)$
+
−
+
−
+
+

Reduction of the Block Diagram

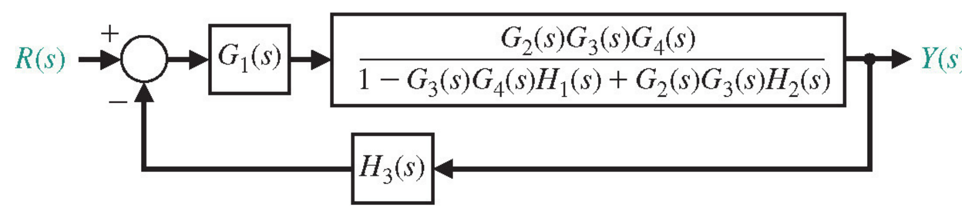


(a)

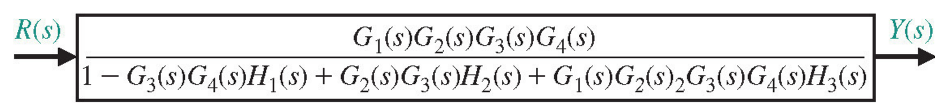


(b)

- Step 1: move the pickoff point between G3 & G4 after G4 (from original diagram to (a));
- Step 2. eliminate loop 1 (G3G4H1) (from (a) to (b));
- Step 3. eliminate loop 2 (from (b) to (c));
- Step 4. eliminate loop 3 (from (c) to (d)).



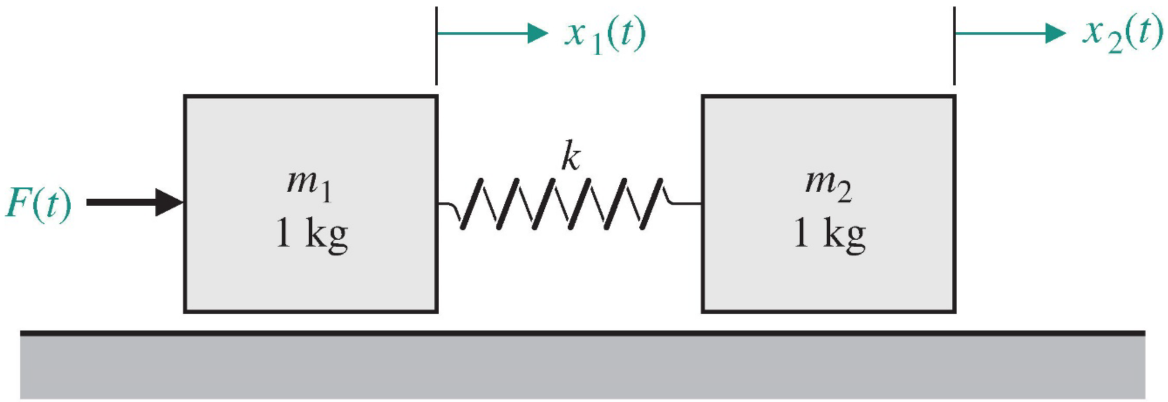
(c)



(d)

Quiz 7.1

Determine the transfer function between $x_2(t)$ and $F(t)$ for the following system (assume k=1):



Quiz 7.2

A system has the following transfer function:

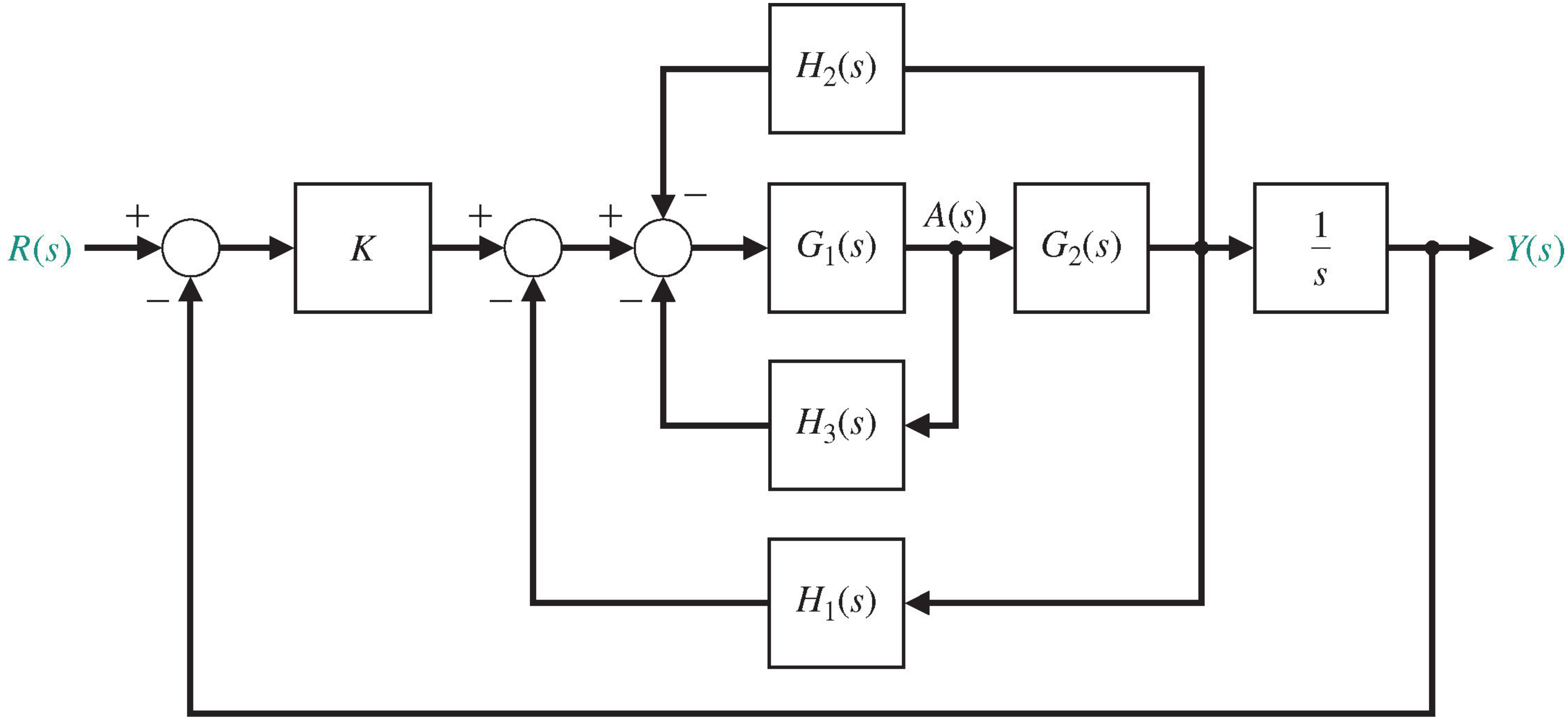
$$\frac{Y(s)}{R(s)} = \frac{4(s+50)}{s^2+30s+200}$$

(a) If $r(t)$ is a unit step input, find the output $y(t)$;

(b) What is the final value of $y(t)$?

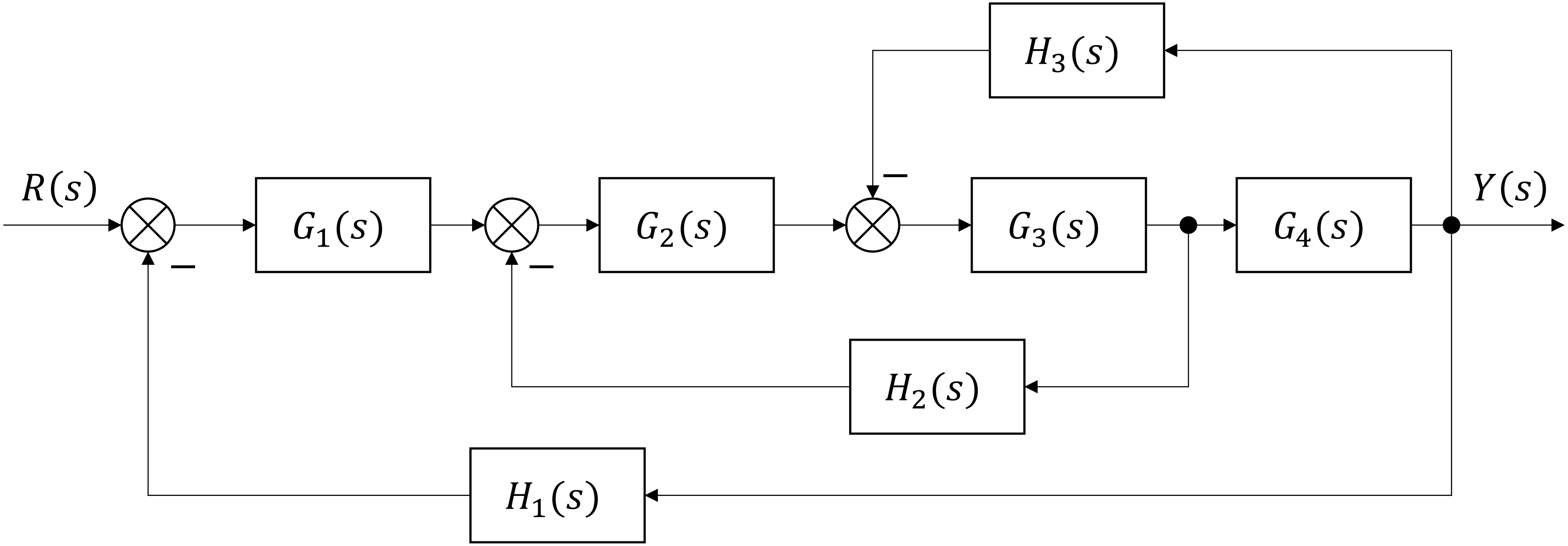
Quiz 7.3

Determine the transfer function for the following system:



Quiz 7.4

Determine the transfer function for the following system:



Outline

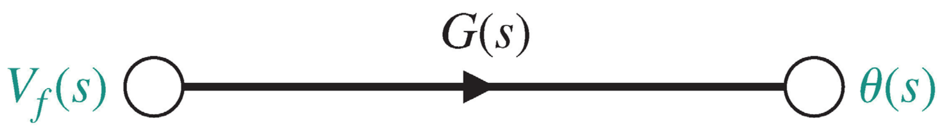
Control Systems:

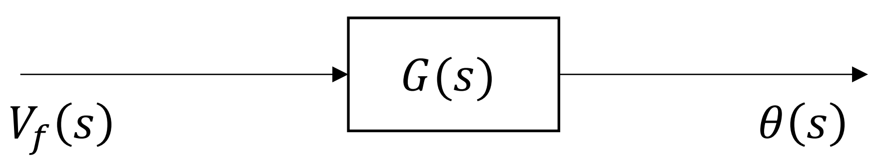
Mathematical Models of Systems

- ❑ Differential Equations of Physical Systems
- ❑ Linear Approximation of Physical Systems
- ❑ The Laplace Transform
- ❑ The Transfer Function of Linear Systems
- ❑ Block Diagram Models
- ❑ *Signal-Flow Graph Models*
- ❑ Simulation Tool

Signal-Flow Graph Models

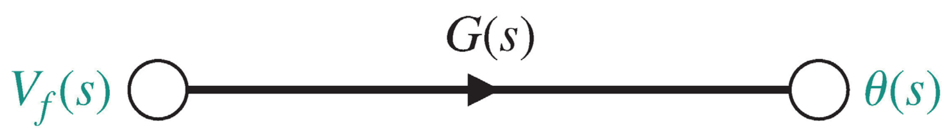
- A **signal-flow graph** is a diagram consisting of **nodes** that are connected by several **directed branches** and is a graphic representation of a set of linear relations;
- Signal-flow graph is particularly useful for feedback control systems because feedback theory is primarily concerned with the flow and processing of signals in the system;

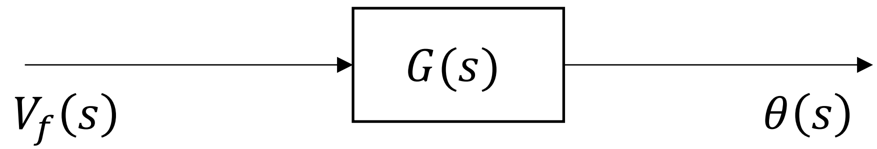




Signal-Flow Graph Models

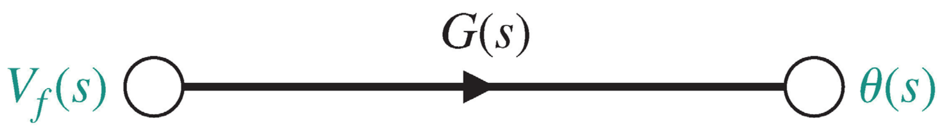
- The basic element of a signal-flow graph is a unidirectional path segment called a **branch**, which relates the dependency of input and an output variable in a manner equivalent to a **block** of a block diagram.
- For complex systems, the block diagram method can become difficult to complete. By using the signal-flow graph model, the reduction procedure (used in the block diagram method) is not necessary to determine the relationship between system variables.





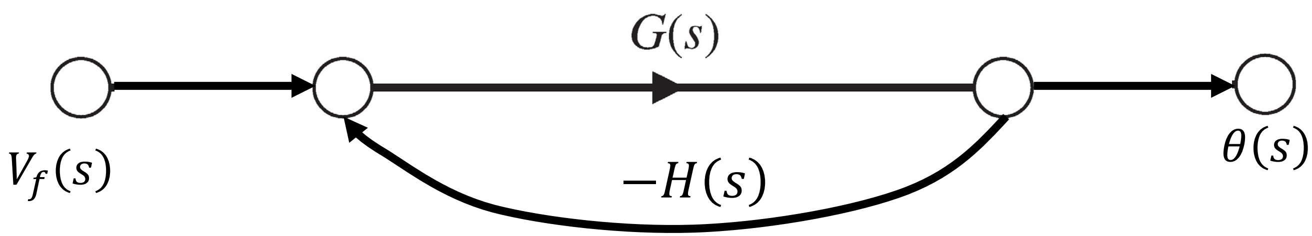
Basic Concepts

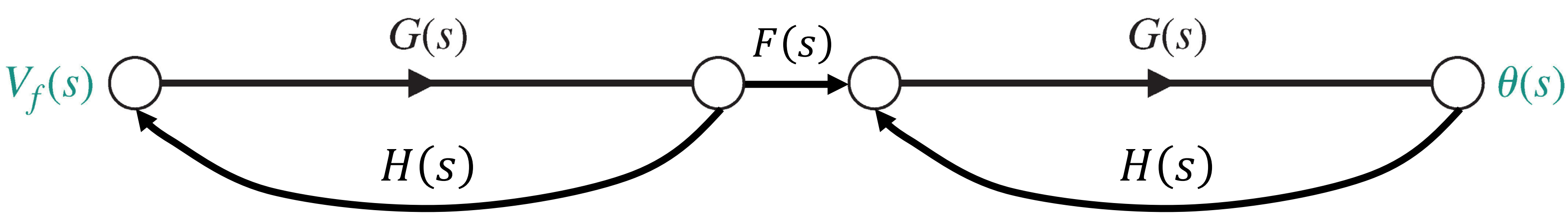
- **Nodes**: are the input and output points or junctions; all branches leaving a node will pass the nodal signal to the output node of each branch (unidirectional); the summation of all signals entering a node is equal to the node variable;
- A **Path**: is a branch or a continuous sequence of branches that can be traversed from one node (signal) to another node (signal);



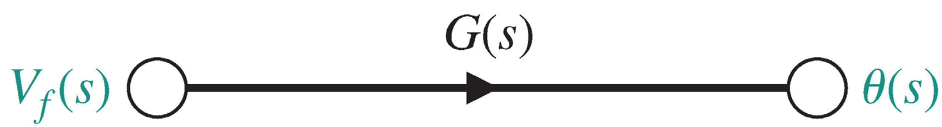
Basic Concepts

- A **Loop**: is a closed path that originates and terminates on the same node, with no node been met twice along the path;
- Two loops are said to be **nontouching** if they don't have a common node; two touching loops share one or more common nodes;

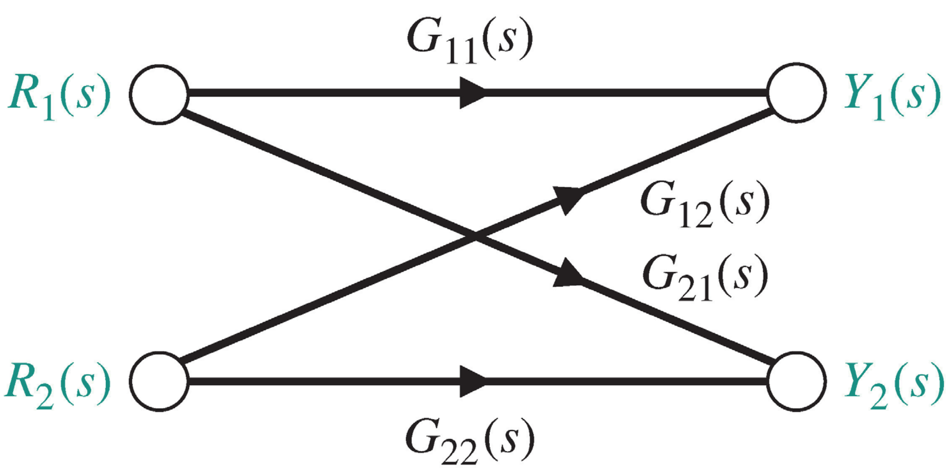




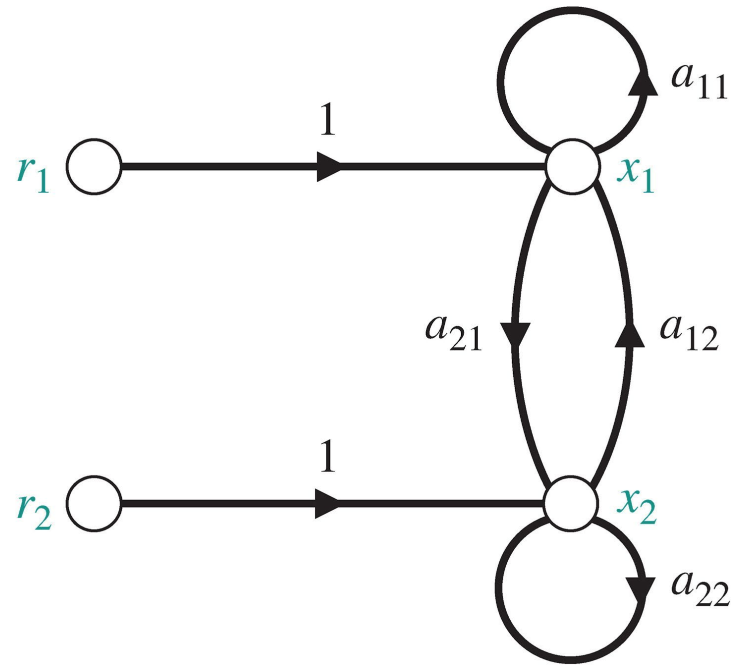
Examples



(a) Signal-flow graph of a DC motor.



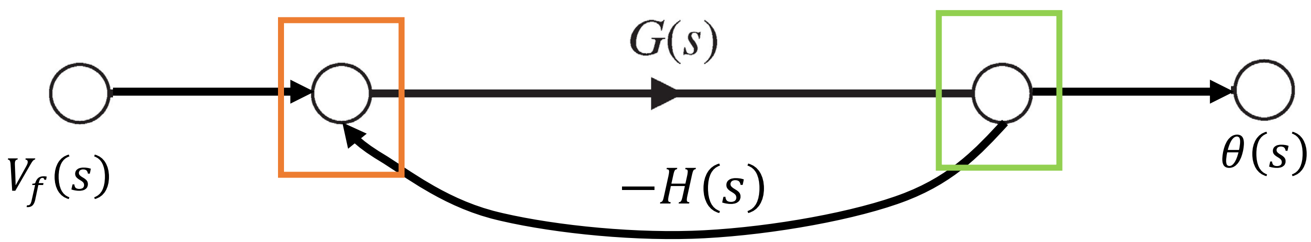
(b) Signal-flow graph of two-input, two-output interconnected system.

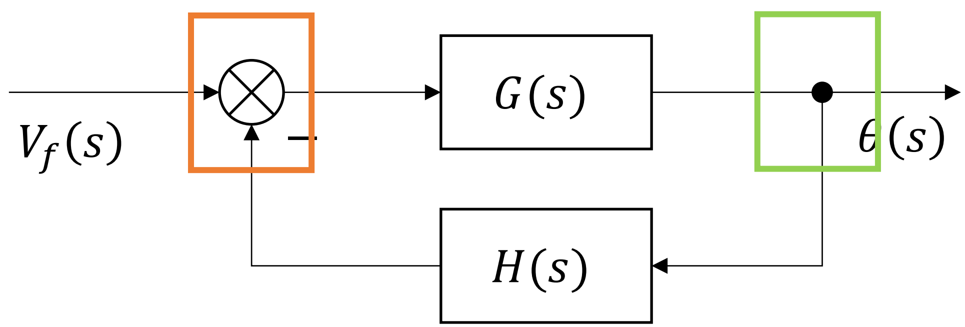


(c) Signal-flow graph of two algebraic equations.

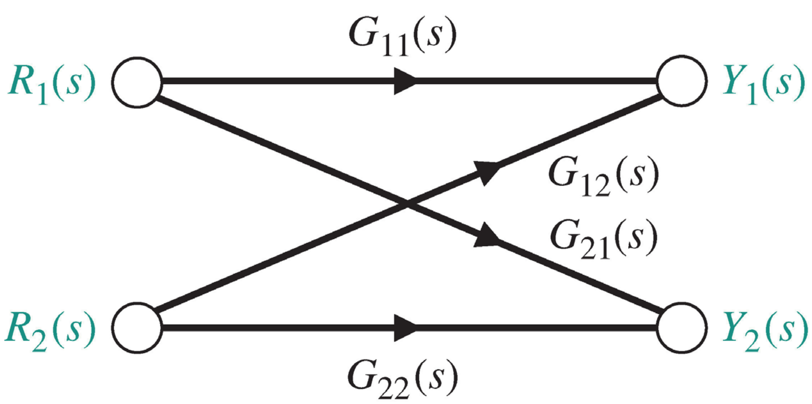
Equivalence to Block Diagram

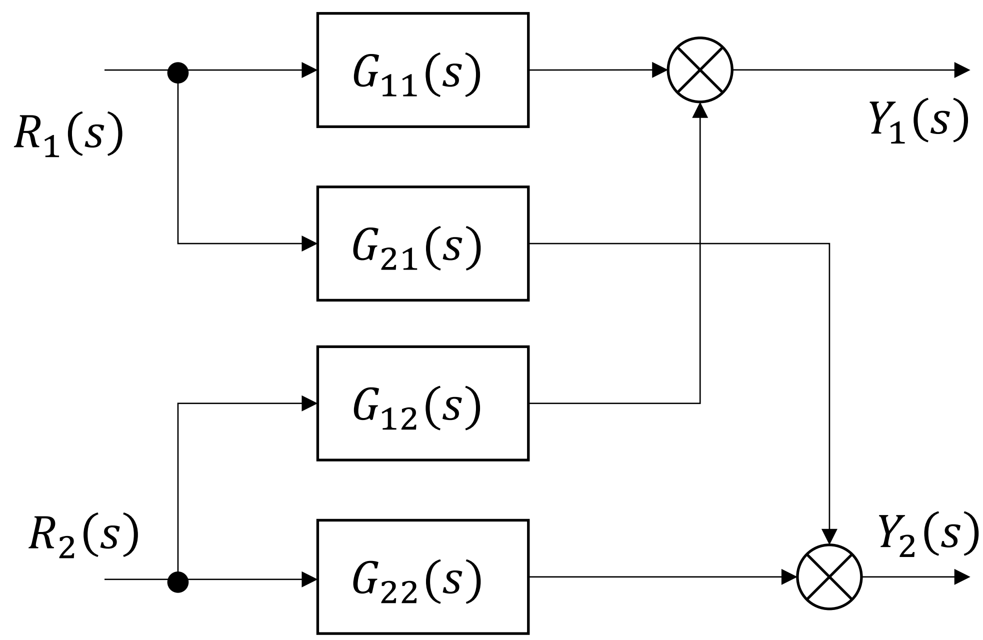
- Signal flow graph and the block diagram are equivalent.
- Below is an example for typical feedback system.
- Signals flow out of a node denote pickoff point; signals flow into a node denote summing point.



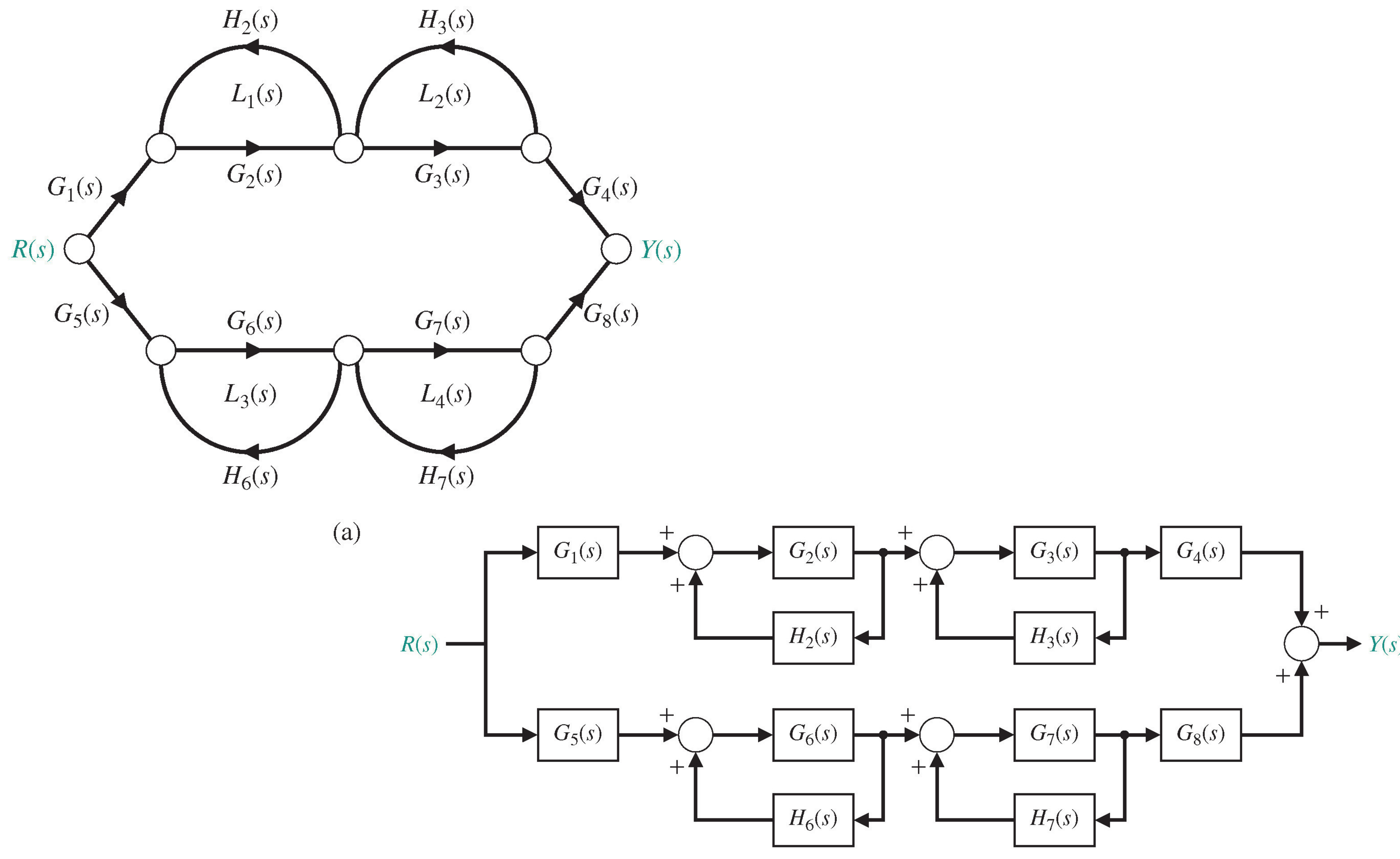


Equivalence to Block Diagram



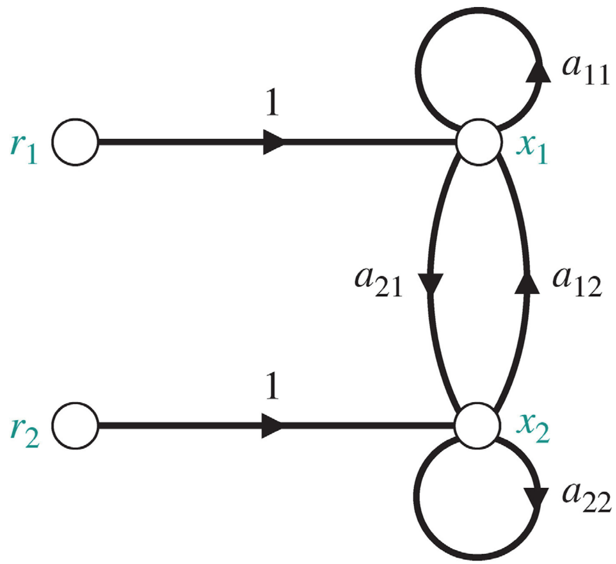


Equivalence to Block Diagram



(b)

Determine Transfer Function from the Graph



$$a_{11}x_1 + a_{12}x_2 + r_1 = x_1$$
$$a_{21}x_1 + a_{22}x_2 + r_2 = x_2.$$

$$x_1(1 - a_{11}) + x_2(-a_{12}) = r_1,$$

$$x_1 = \frac{(1 - a_{22})r_1 + a_{12}r_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{22}}{\Delta}r_1 + \frac{a_{12}}{\Delta}r_2,$$

$$x_2 = \frac{(1 - a_{11})r_2 + a_{21}r_1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{11}}{\Delta}r_2 + \frac{a_{21}}{\Delta}r_1.$$

Where: $\Delta = (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} = 1 - a_{11} - a_{22} + a_{11}a_{22} - a_{12}a_{21}.$

Mason's Signal-flow Gain Formula

In general, the linear dependence $T_{ij}(s)$ between the independent variable x_i (often called the input variable) and a dependent variable x_j (output variable)is given by:

$$T_{ij}(s) = \frac{\sum_{k=1}^{n} P_{ijk}(s)\Delta_{ijk}(s)}{\Delta}$$

The summation is taken over all possible k paths from x_i to x_j.

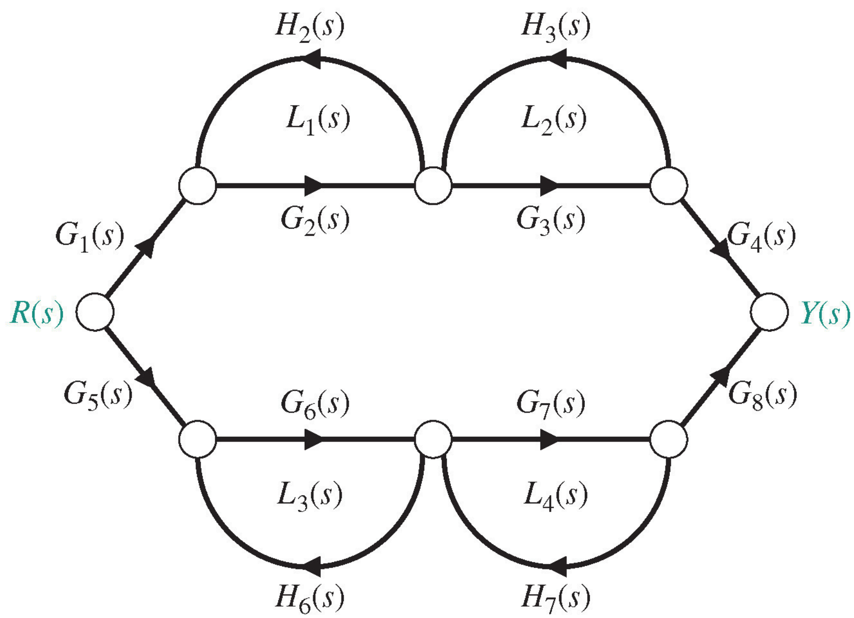
- $\boldsymbol{P_{ijk}}$: is the k^{th} path gain from x_i to x_j, defined as the product of gains of the branches on the k^{th} path, traversed in the direction of the arrows with no node encountered more than once;
- Δ_{ijk}: cofactor, is the determinant with the loops touching the kth path removed.
- Δ: the determinant, is:

$$\Delta = 1 - \sum_{n=1}^{N} L_n + \sum_{\substack{n,\, m \\ \text{nontouching}}} L_n L_m - \sum_{\substack{n,\, m,\, p \\ \text{nontouching}}} L_n L_m L_p + \cdots,$$

Where L_q equals the value of qth loop transmittance, therefore:

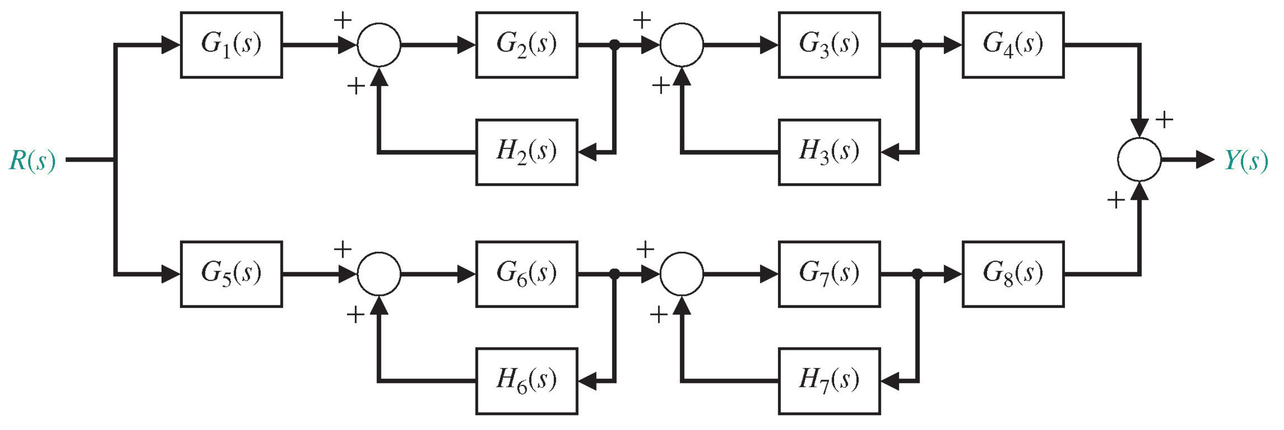
$$\begin{aligned}\Delta = 1 &- (\text{sum of all different loop gains}) \\ &+ (\text{sum of the gain products of all combinations of two nontouching loops}) \\ &- (\text{sum of the gain products of all combinations of three nontouching loops}) \\ &+ \cdots.\end{aligned}$$

Mason's Rule: Application



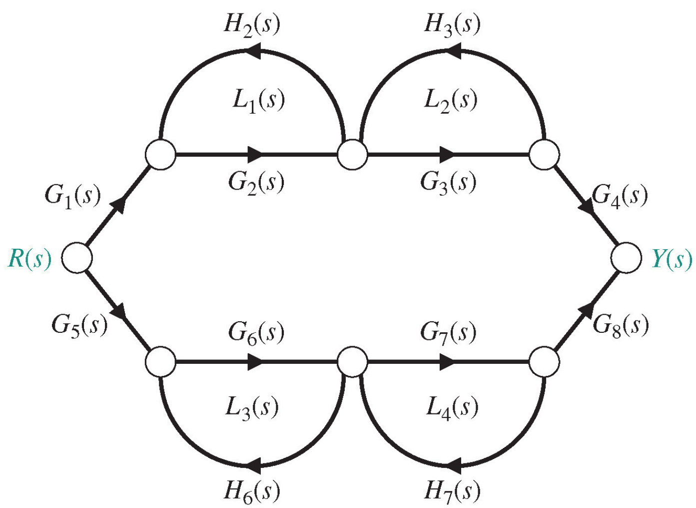
(a)

How to obtain Transfer Function between Input R(s) and Output Y(s)?



(b)

Example 10.1



$$T_{ij}(s) = \frac{\sum_{k=1}^{n} P_{ijk}(s)\Delta_{ijk}(s)}{\Delta}$$

1. How many possible paths? $P_1 = G_1G_2G_3G_4$ (path 1) and $P_2 = G_5G_6G_7G_8$ (path 2).

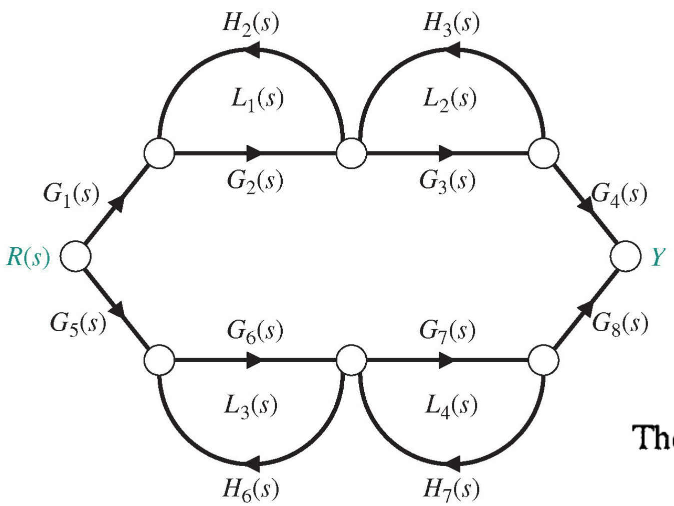
2. How many loops? There are four self-loops:

$$L_1 = G_2H_2, \quad L_2 = H_3G_3, \quad L_3 = G_6H_6, \quad \text{and} \quad L_4 = G_7H_7.$$

3. How many groups of 2 nontouching loops? Loops L_1 and L_2 do not touch L_3 and L_4.

4. How many groups of 3 nontouching loops? None.

Example 10.1



$$T_{ij}(s) = \frac{\sum_{k=1}^{n} P_{ijk}(s)\Delta_{ijk}(s)}{\Delta}$$

$\Delta = 1 -$ (sum of all different loop gains)
$+$ (sum of the gain products of all combinations of two nontouching loops)
$-$ (sum of the gain products of all combinations of three nontouching loops)
$+ \cdots.$

There are four self-loops:

$$L_1 = G_2H_2, \quad L_2 = H_3G_3, \quad L_3 = G_6H_6, \quad \text{and} \quad L_4 = G_7H_7.$$

Loops L_1 and L_2 do not touch L_3 and L_4. Therefore, the determinant is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4).$$

The cofactor of the determinant along path 1 is evaluated by removing the loops that touch path 1 from Δ. Hence, we have

$$L_1 = L_2 = 0 \quad \text{and} \quad \Delta_1 = 1 - (L_3 + L_4).$$

Similarly, the cofactor for path 2 is

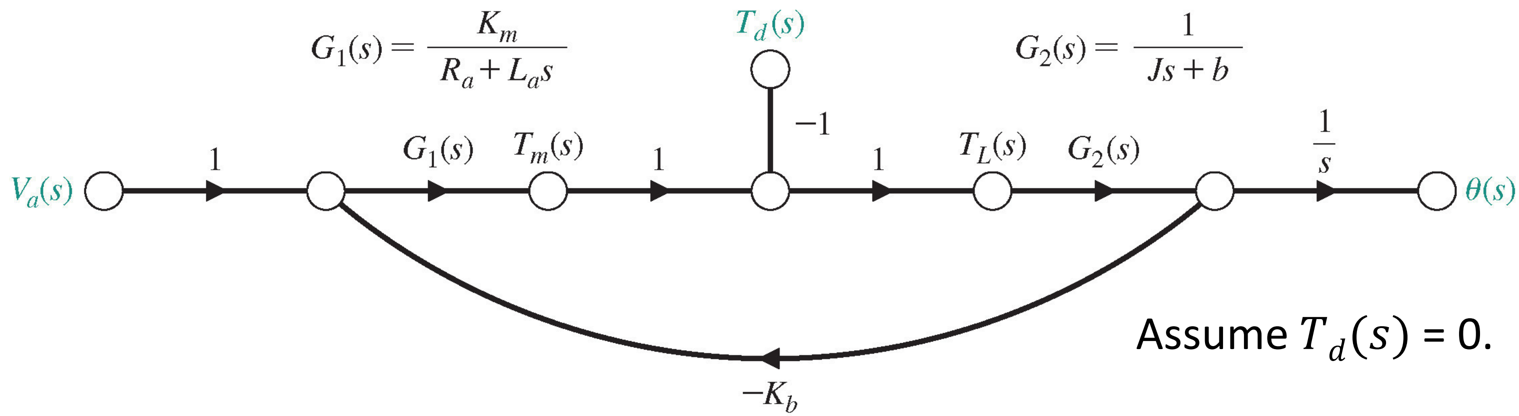
$$\Delta_2 = 1 - (L_1 + L_2).$$

Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3G_4(1 - L_3 - L_4) + G_5G_6G_7G_8(1 - L_1 - L_2)}{1 - L_1 - L_2 - L_3 - L_4 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4}$$

Example 10.2



Signal-flow graph of the armature-controlled DC motor

There is only 1 forward path, which touches one loop:

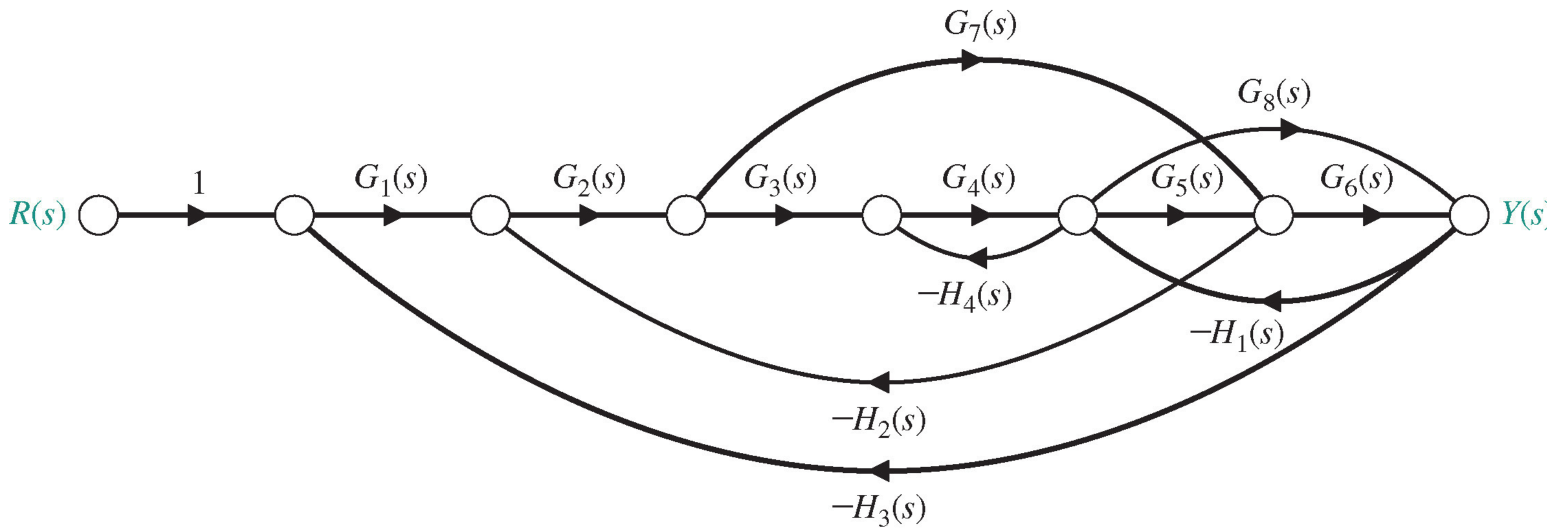
$$P_1(s) = \frac{1}{s} G_1(s) G_2(s) \quad \text{and} \quad L_1(s) = -K_b G_1(s) G_2(s). \qquad \Delta_1 = 1$$

Therefore, the transfer function is

$$T(s) = \frac{P_1(s)}{1 - L_1(s)} = \frac{(1/s) G_1(s) G_2(s)}{1 + K_b G_1(s) G_2(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$$

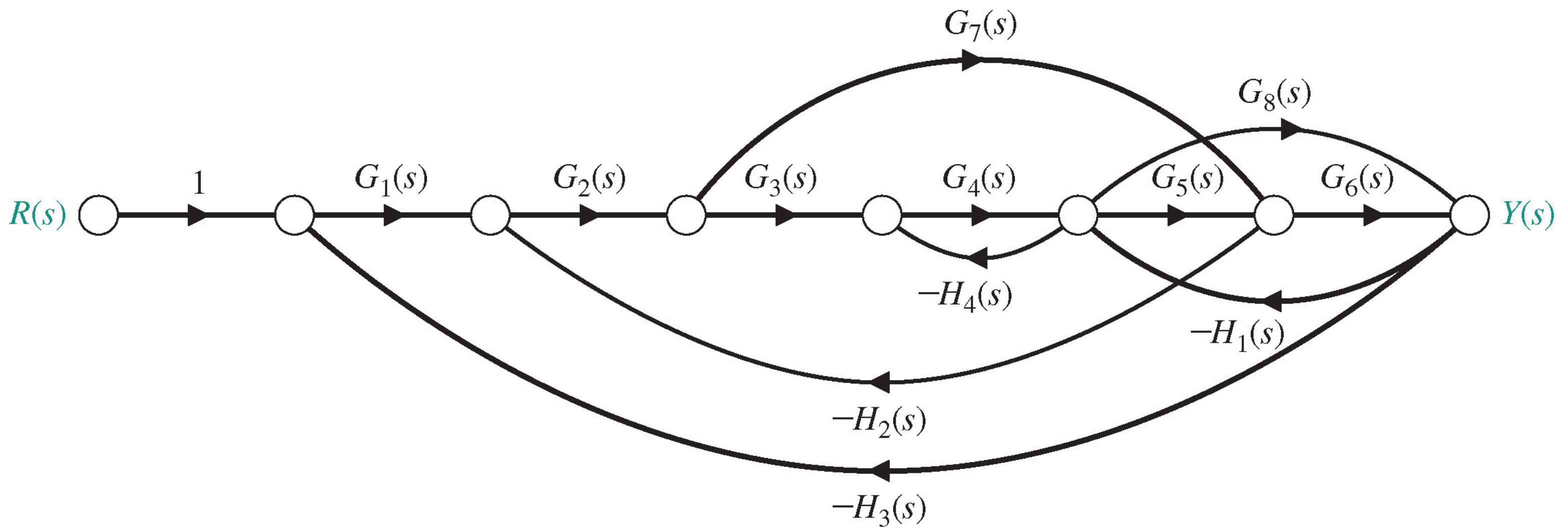
Example 10.3

Consider a reasonably complex system that would be difficult to reduce by block diagram techniques:



Example 10.3

Consider a reasonably complex system that would be difficult to reduce by block diagram techniques:

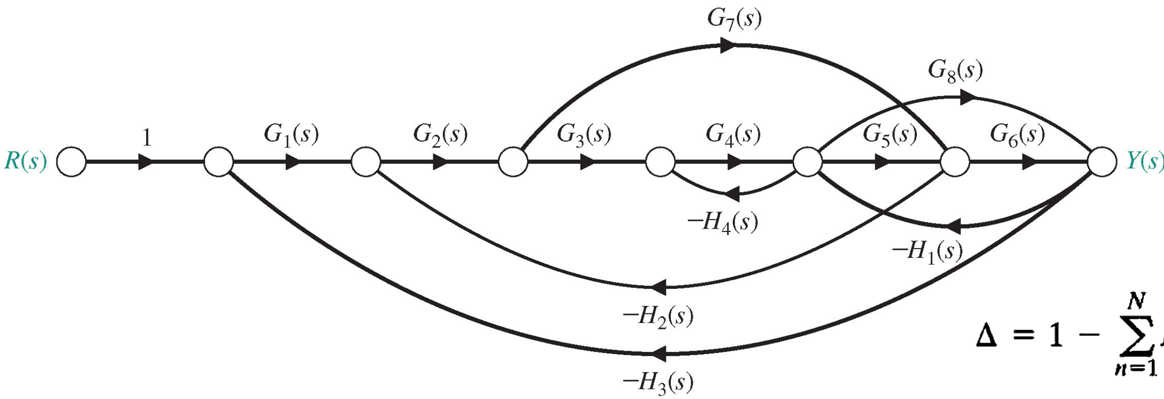


$$P_1 = G_1G_2G_3G_4G_5G_6, \qquad P_2 = G_1G_2G_7G_6, \quad \text{and} \quad P_3 = G_1G_2G_3G_4G_8.$$

The feedback loops are

$$L_1 = -G_2G_3G_4G_5H_2, \qquad L_2 = -G_5G_6H_1, \qquad L_3 = -G_8H_1, \qquad L_4 = -G_7H_2G_2,$$
$$L_5 = -G_4H_4, \qquad L_6 = -G_1G_2G_3G_4G_5G_6H_3, \qquad L_7 = -G_1G_2G_7G_6H_3, \quad \text{and}$$
$$L_8 = -G_1G_2G_3G_4G_8H_3.$$

Example 10.3



$$T_{ij}(s) = \frac{\sum_{k=1}^{n} P_{ijk}(s)\Delta_{ijk}(s)}{\Delta}$$

$$\Delta = 1 - \sum_{n=1}^{N} L_n + \sum_{\substack{n,\,m \\ \text{nontouching}}} L_n L_m - \sum_{\substack{n,\,m,\,p \\ \text{nontouching}}} L_n L_m L_p + \cdots,$$

Loop L_5 does not touch loop L_4 or loop L_7, and loop L_3 does not touch loop L_4; but all other loops touch. Therefore, the determinant is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_5 L_7 + L_5 L_4 + L_3 L_4).$$

The cofactors are

$$\Delta_1 = \Delta_3 = 1 \quad \text{and} \quad \Delta_2 = 1 - L_5 = 1 + G_4 H_4.$$

Finally, the transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{P_1 + P_2\Delta_2 + P_3}{\Delta}.$$

Thank You !