Xi'an Jiaotong-Liverpool University
西交利物浦大學

MEC208 Instrumentation and Control System

2024-25 Semester 2

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MEC208 office hour in Revision Week:

15th, 16th May 2025, 2-4pm (2 days).

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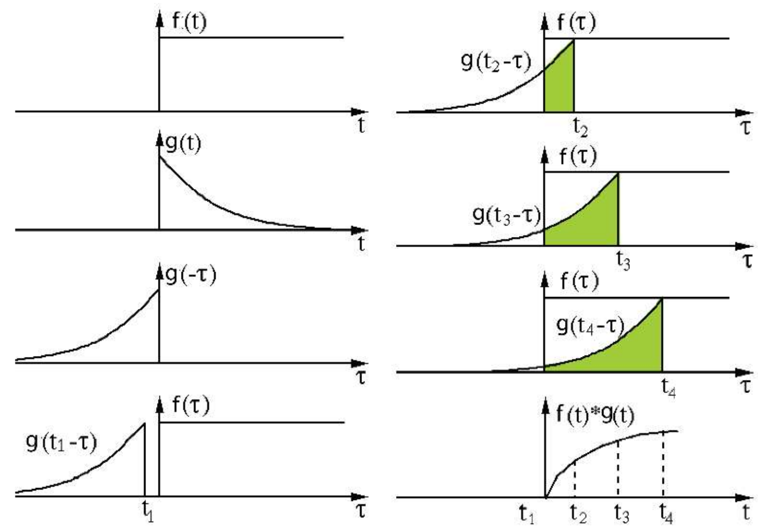
Chapters in Part 3 of MEC208 (CS Lim)

- **Stability of Linear Feedback Systems**
- **The Root Locus Method**
- **Frequency Response Methods – Bode Plot**

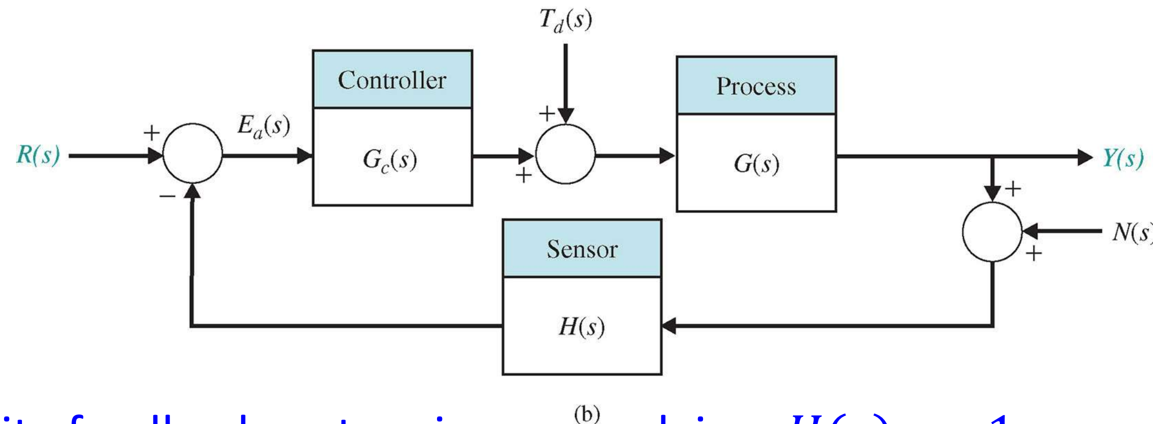
Laplace Transformation

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{\left(s^2+a^2\right)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{\left(s^2+a^2\right)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s\left(s^2+3a^2\right)}{\left(s^2+a^2\right)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			



A general representation of a feedback control system



For simplicity, unity feedback system is assumed, i.e., $H(s) = 1.$

Output: $$Y(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)}R(s) + \frac{G(s)}{1+G_c(s)G(s)}T_d(s) - \frac{G_c(s)G(s)}{1+G_c(s)G(s)}N(s)$$

Tracking error: $$E(s) = R(s) - Y(s)$$

$$E(s) = \frac{1}{1+G_c(s)G(s)}R(s) - \frac{G(s)}{1+G_c(s)G(s)}T_d(s) + \frac{G_c(s)G(s)}{1+G_c(s)G(s)}N(s)$$

Closed-loop transfer function: $$T(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G_c(s)G(s)}$$

Assume $T_d = 0, N = 0.$

Loop transfer function: $$L(s) = G_c(s)G(s)H(s)$$

Characteristic equation: $$\Delta(s) = 1 + G_c(s)G(s) = 0$$

Routh-Hurwitz Criterion

- A BIBO stable system is a dynamic system with a bounded (limited) response to a bounded input.

RHC states that:

"The number of roots of a characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array"

- Additional rules to complement the analysis.

Characteristic eq.: $q(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$

Routh Array

s^n	a_n	a_{n-2}	a_{n-4}	…
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	…
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	…
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	…
…	…	…	…	
s^0	h_{n-1}			

Own calculation

$$b_{n-1} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_{n-1} = -\frac{1}{b_{n-1}}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

…and so on and so forth.

Routh-Hurwitz Stability Criterion – distinct cases

Four distinct (special) cases or configurations of the first column array must be considered, and each must be treated separately and requires suitable modifications of the array calculation procedure:

1) No element in the first column is zero;
2) There is a “zero” in the first column, but some other elements of the row containing the “zero” are non-zero;
3) There is a “zero” in the first column, and the other elements of the row containing the “zero” are also zero; [Means factors such as $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$ are present in the characteristic equation]
4) Extension from Case (3), this fourth rule determines whether there are repeated roots on the $j\omega$-axis; if yes, the system is unstable; if no, determine the stability using the auxiliary polynomial. [Extension from Case 3]

Stability of a state-space/state variable system

- If a system is represented in the state-space model, the transfer function can be obtained through the following method

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(\mathrm{t}) + \mathbf{B}\mathbf{u}(\mathrm{t})$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(\mathrm{t}) + \mathbf{D}\mathbf{u}(\mathrm{t})$$

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

$$\mathbf{G}(s) = \mathbf{C}\frac{adj(s\mathbf{I} - \mathbf{A})}{|s\mathbf{I} - \mathbf{A}|}\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}[adj(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|}$$

Setting the denominator of the transfer function matrix **G**(s) to be zero, we get the **characteristic equation:**

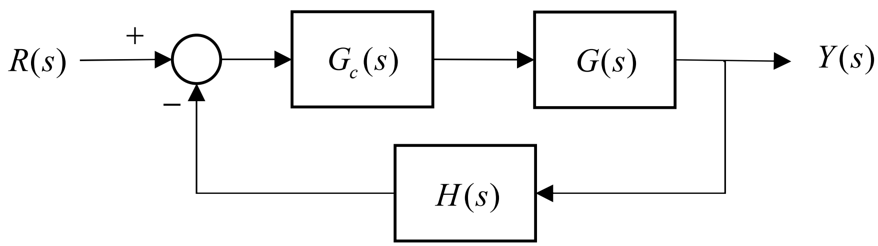
$$\Delta(\boldsymbol{s}) = |s\mathbf{I} - \mathbf{A}| = \mathbf{0}$$

$\boldsymbol{n}$: order of the system
$\boldsymbol{A}$: $\boldsymbol{n} \times \boldsymbol{n}$ matrix
$\boldsymbol{s}\mathbf{I} - \mathbf{A}$: $\boldsymbol{n} \times \boldsymbol{n}$ matrix
$|\boldsymbol{s}\mathbf{I} - \mathbf{A}|$: $\boldsymbol{n}$-th order polynomial

IMPORTANT OBSERVATION:

- *It means one does not need to obtain the full transfer function to proceed with stability analysis (which requires determination of an inverse matrix).*
- *One only needs to extract the characteristic function from the determinant $|s\boldsymbol{I} - \boldsymbol{A}|$.*

Fundamentals of Root Locus Method/Analysis



Closed-loop TF: $T_{CL}(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)H(s)}$

(Open) Loop TF: $T_L(s) = G_c(s)G(s)H(s)$

Characteristic function $\Delta(s) = 1 + G_c(s)G(s)H(s)$

Assume $G_c(s)G(s)H(s) = \frac{kb(s)}{a(s)} = kL(s)$, where $a(s) = \prod_{j=1}^{n}(s + p_j)\,, b(s) = \prod_{i=1}^{m}(s + z_i)\,, and\ 0 < k < \infty$:

$$T_{CL}(s) = \frac{G_c(s)G(s)}{1 + \frac{kb(s)}{a(s)}}$$

$$T_L(s) = \frac{kb(s)}{a(s)}$$

$a(s) = 0 \rightarrow$ OL poles

$b(s) = 0 \rightarrow$ OL zeros

CL system poles: $a(s) + kb(s) = 0$

- By comparing the poles-zeroes of the two systems, we can establish that:
  - when k is zero, the CL poles coincide with the LTF poles (or simply, OL poles).
  - When k is ∞, the CL poles coincide with the LTF zeros (or simply, OL zeros) .
  - So, it is logical to expect that, for $0 < k < \infty$, as k increases from 0 to *inf*, the CL poles are moving from the OL poles towards the OL zeroes. The traces of these CL poles form the loci of the CL characteristic equation's roots, a.k.a. "root locus".

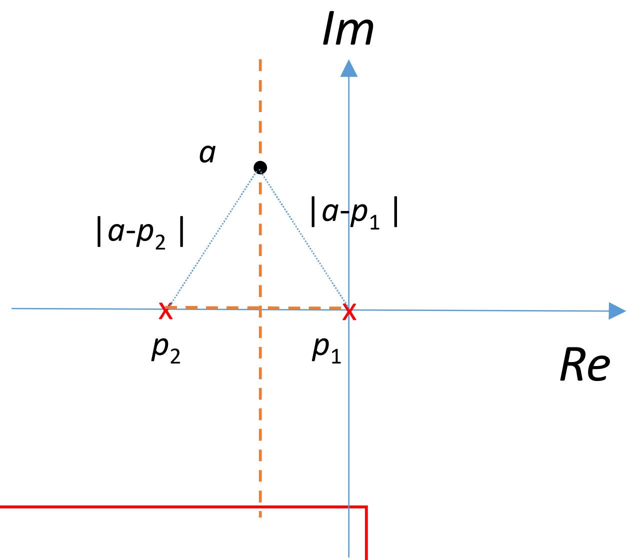
Checkpoints (1)

- ✓ Interpret the problem diligently:
  - Open-loop or closed-loop system?
  - Transfer function or state-variable model?
  - What is the control variable of interest (input, output, error)?
  - With disturbance? With noise? Or none of them?
  - Steady state or transient?
- ✓ Relationship between root locations and stability of a system?
- ✓ How to construct Routh array? (Lecture 15; ; examples and exercises)
- ✓ In RHC, how to use auxiliary equation to find the $j\omega$ roots? (Lecture 16; ; examples and exercises)

Arbitrary points on the locus/loci

Recall $L(s) = \frac{kb(s)}{a(s)} = \frac{k\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$

$$T_{CL} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)}$$



- Let $G_c(s)G(s)H(s) = kL(s)$, where $0 \leq k < \infty$.
- Characteristic equation: $1 + kL(s) = 0$

$$kL(s) = -1$$

- Therefore,

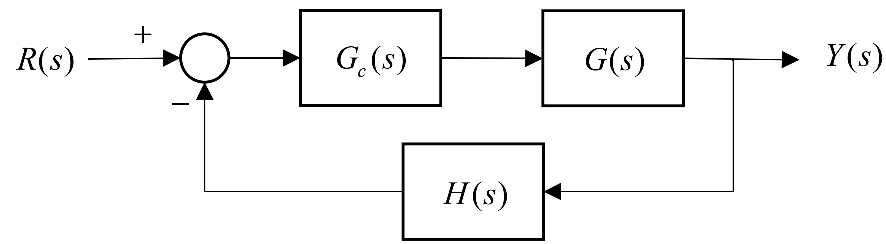
$$\boxed{\begin{array}{c} |kL(s)|=1 \\ \angle kL(s) = 180^o + m360^o, m = 0, \pm 1, \pm 2, \ldots \ (e.g. \pm 180^o, \pm 540^o, \ldots) \end{array}}$$

- Two **very important** insights can be gathered from these two equations:
  - Since $k = \frac{\left|\prod_{all\ j}(s+p_j)\right|}{\left|\prod_{all\ i}(s+z_i)\right|}$, we can determine k for any arbitrary point '*a*' on the root locus. Draw vectors from every OL pole (p_j) and OL zero (z_i) to the point '*a*' and insert these vectors' magnitudes into this k expression.
  - Sum of the angles of the point '*a*' to all the poles minus that to all the zeroes must be equal to 180^o.

Summary: 7+1 rules to remember! *(NOTE: Rules here are slightly different from that in the reference book, but they will produce the same exact RL plot!)*

Prepare the OLTF of the CL system, compute the poles and zeroes, then proceed with the following rules:

with a variable of interest, e.g., "*k*"



- Symmetry (about the real axis)
- Number of loci = number of OL poles
- Rank = number of OL zeros at infinity (i.e., no. of finite *p* – no. of finite *z*)
- Odd number of poles-zeros on the real-axis's RHS = loci exists on the real axis
- Asymptotes angle = $\frac{2q+1}{Rank} \cdot 180^o$, *q*=0,1,2…(*Rank*-1);
  - The only POI of the asymptotes at the real axis = $\frac{Re(\sum p - \sum z)}{Rank}$
- Break-in break-out (a.k.a. break-away break-in) points, $\frac{d}{ds}\left[\frac{-a(s)}{b(s)}\right] = 0$ (provided the transfer function is coprime)
- Angle of departure from a complex OL pole
  - $\angle target_{pole} + sum(\angle p) - sum(\angle z) = 180^o$
  - $\angle p$ or $\angle z$ is the angle of the target pole, referencing to other poles p or zeroes z, measured in the counter-clockwise direction
- Lastly, if the loci do cross the imaginary axis, the crossing points and gain can be found through the CL characteristic equation (or RHC!).

Reference book's RLM (optional, FYR)

Note: The sequence of the following plotting rules (details see Ref. book, Chap. 7) are slightly different from what were presented. Ultimately, they will produce the same Root Locus plot.

Table 7.2 Seven Steps for Sketching a Root Locus

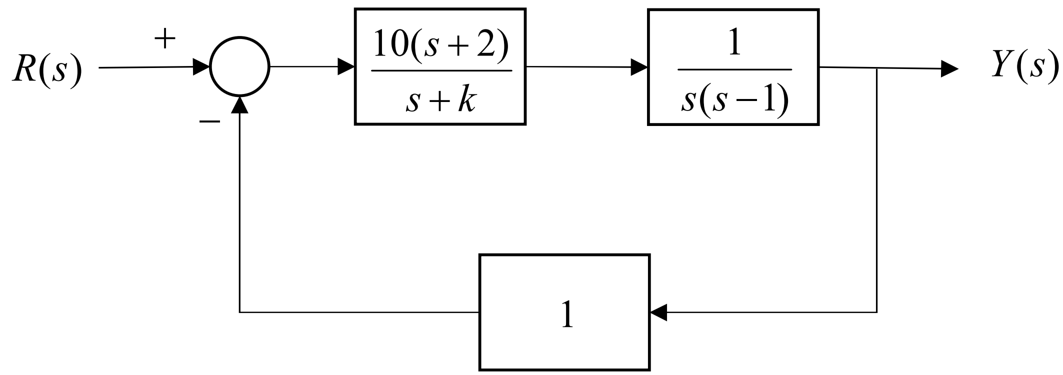
Step	Related Equation or Rule
1. Prepare the root locus sketch.	
(a) Write the characteristic equation so that the parameter of interest, K, appears as a multiplier.	$1 + KP(s) = 0.$
(b) Factor $P(s)$ in terms of n poles and M zeros.	$1 + K\dfrac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0.$
(c) Locate the open-loop poles and zeros of $P(s)$ in the s-plane with selected symbols.	$\times$ = poles, $\bigcirc$ = zeros
	Locus begins at a pole and ends at a zero.
(d) Determine the number of separate loci, SL.	$SL = n$ when $n \geq M$; n = number of finite poles, M = number of finite zeros.
(e) The root loci are symmetrical with respect to the horizontal real axis.	
2. Locate the segments of the real axis that are root loci.	Locus lies to the left of an odd number of poles and zeros.

Table 7.2 Seven Steps for Sketching a Root Locus

Step	Related Equation or Rule
3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A.	$\sigma_A = \dfrac{\sum(-p_j) - \sum(-z_i)}{n - M}$. $\phi_A = \dfrac{2k+1}{n-M} 180°, k = 0, 1, 2, \ldots, (n - M - 1)$.
4. Determine the points at which the locus crosses the imaginary axis (if it does so).	Use Routh–Hurwitz criterion.
5. Determine the breakaway point on the real axis (if any).	a) Set $K = p(s)$. b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s)$.
6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion.	$\underline{/P(s)} = 180° + k360°$ at $s = -p_j$ or $-z_i$.
7. Complete the root locus sketch.	

To tune other controller parameters (e.g., time constant of the first-order pole)

- If the parameter under consideration is not the scalar gain but being the pole/zero location, some manipulation can be done to fit the problem to the form required by RLM.



To find the CL poles, we equate the CL characteristic function to zero:

$$0 = \Delta(s)$$

$$0 = 1 + \frac{10(s+2)}{s+k}\frac{1}{s(s-1)}$$

$$0 = s(s+k)(s-1) + 10(s+2)$$

$$0 = s^2(s-1) + 10(s+2) + ks(s-1)$$

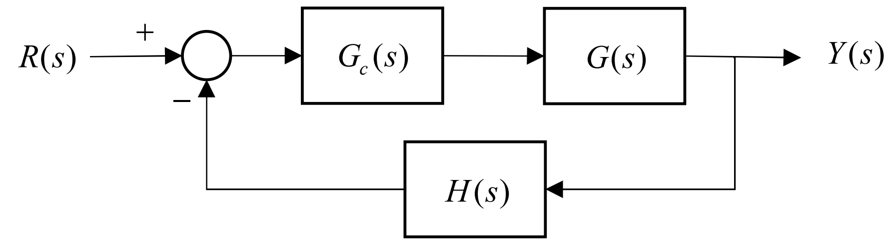
$$0 = 1 + \frac{ks(s-1)}{s^2(s-1)+10(s+2)}$$

This changes the location of controller pole *k* to a new LTF where RLM will be relevant:

$$LTF\,(or\ T_L) = \frac{ks(s-1)}{s^2(s-1)+10(s+2)}$$

PID Controllers

$$\boxed{G_c(s) = K_P + \frac{K_I}{s} + K_D s}$$



$$G_c(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D\left(s^2 + \frac{K_P}{K_D}s + \frac{K_I}{K_D}\right)}{s}$$

A PID controller adds two zeroes and a pole ($s = 0$) to the forward-path of the control system.

Table 7.4 Effect of Increasing the PID Gains K_p, K_D, and K_I on the Step Response

PID Gain	Percent Overshoot	Settling Time	Steady-State Error
Increasing K_P	Increases	Minimal impact	Decreases
Increasing K_I	Increases	Increases	Zero steady-state error
Increasing K_D	Decreases	Decreases	No impact

PID tuning (some can be used for general controllers)

- **Tuning methods:**
    - Manual tuning – Tune K_p to achieve quarter amplitude decay, then manually tune K_i and/or K_d (may use RLM to assist the process of selection). Usually design for "good" step response.
- **Ziegler Nichols:** Find ultimate gain and period, K_u and P_u, then use the table. Designed for "good" disturbance rejection.

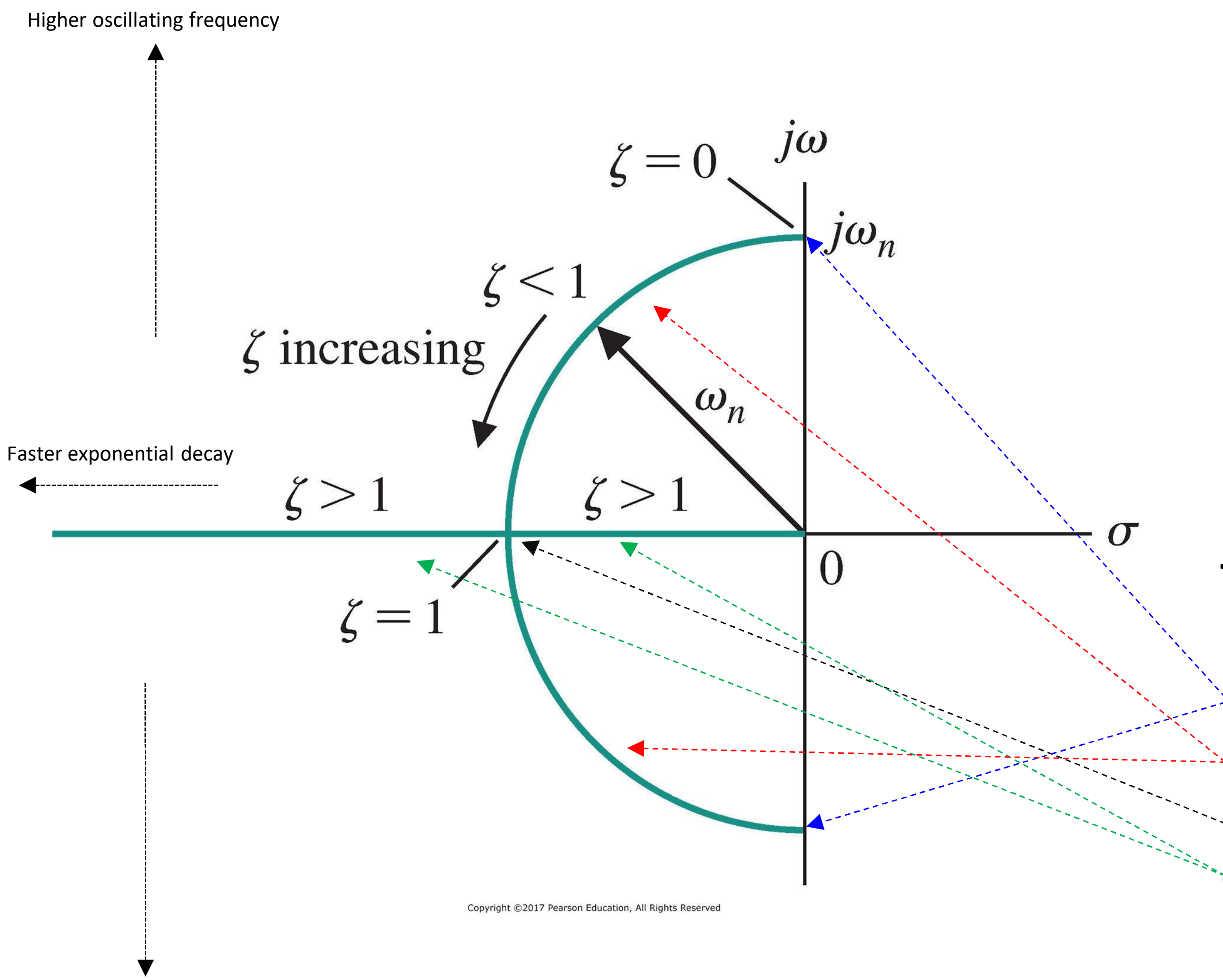
Table 7.7 Ziegler-Nichols PID Tuning Using Ultimate Gain, K_U, and Oscillation Period, P_U

Ziegler-Nichols PID Controller Gain Tuning Using Closed-loop Concepts

Controller Type	K_P	K_I	K_D
Proportional (P) $G_c(s) = K_P$	$0.5K_U$	–	–
Proportional-plus-integral (PI) $G_c(s) = K_P + \frac{K_I}{s}$	$0.45K_U$	$\frac{0.54K_U}{T_U}$	–
Proportional-plus-integral-plus-derivative (PID) $G_c(s) = K_P + \frac{K_I}{s} + K_D s$	$0.6K_U$	$\frac{1.2K_U}{T_U}$	$\frac{0.6K_U T_U}{8}$

- **Analytical (pole placement)**: place the closed-loop poles to the desired locations guided by the transient specifications (steady-state error specification is met through the steady-state error analysis)

Root Locations (2^{nd} order) and transient behaviour



$$T(s) = \frac{X(s)}{Y(s)}$$

$$= \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Poles:

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Transient behaviour due to unit-step response:

- Undamped oscillation
- Underdamped oscillation
- Critically damped oscillation
- Overdamped without any oscillation

Checkpoints (2)

- ✓ Do you already know all the RLM plotting steps? (Lecture 17, 18; examples and exercises)
- ✓ Calculate the gain value on a particular point on the root locus? (Explanation/example/exercise in Lecture 17, 18)
- ✓ Oscillatory performance as CL poles move from real axis to complex plane, and move around in the complex plane?
    - Undamped, underdamped, critically damped, overdamped
    - Fast or slower amplitude decay
    - Lower or higher oscillating frequency
- ✓ Stability analysis using RLM? (Understand this concept from the explanation and examples in Lecture 17, 18)
- ✓ Understand the meaning of *dominant pole location* and *time-domain output*? Can differentiate the concept of "*dominant poles*" and the concept of simplifying TFs when part of the TF meet the "10 times rule of thumb". (Understand this concept from the explanation and examples in Lecture 17-18, 19)

Frequency Response Methods

- The frequency response of a system is defined as the **steady-state response** of the system to a sinusoidal input signal.

$$\boldsymbol{r(t) = A \sin \omega t} \qquad \boldsymbol{R(s) = \frac{A\omega}{s^2 + \omega^2}} \qquad \boldsymbol{Y(s) = T(s)R(s)}$$

for $y(t)$, it can be shown, for $t \to \infty$ (the steady state),

$$y(t) = A|T(j\omega)| \sin(\omega t + \phi)$$

where $\emptyset = \angle T(j\omega)$.

The steady-state response described above is true only for stable systems.

Therefore, the steady-state output signal depends only on the magnitude ($|T(j\omega)|$) and phase ($\emptyset$) of $T(j\omega)$.

Generalisation of Bode Plot

- Consider the following generalised transfer function:

$$G(j\omega) = \frac{k\prod(1+j\omega\tau_z)}{(j\omega)^N\prod(1+j\omega\tau_p)\prod\left(1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

- 5 terms are to be considered:
  - Scalar gain
  - Negative real zeroes
  - Poles (and zeroes) at origin
  - Negative real poles
  - Complex conjugate poles

Know how to manipulate the transfer function. E. g. :

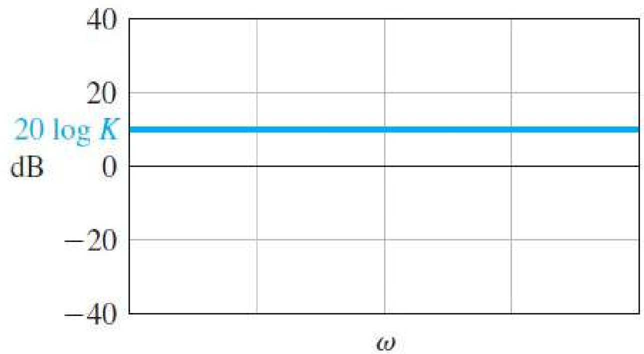
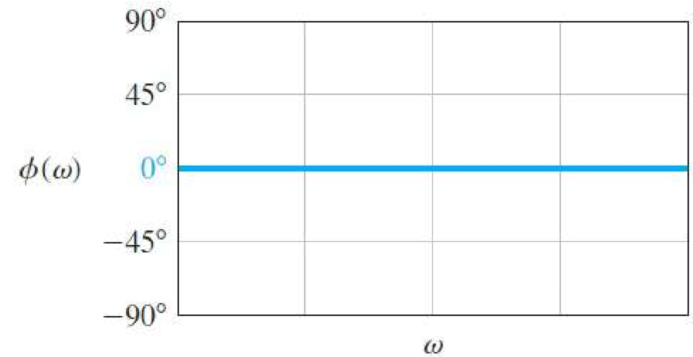
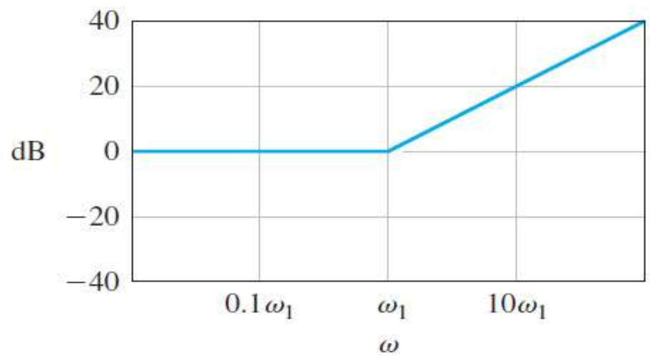
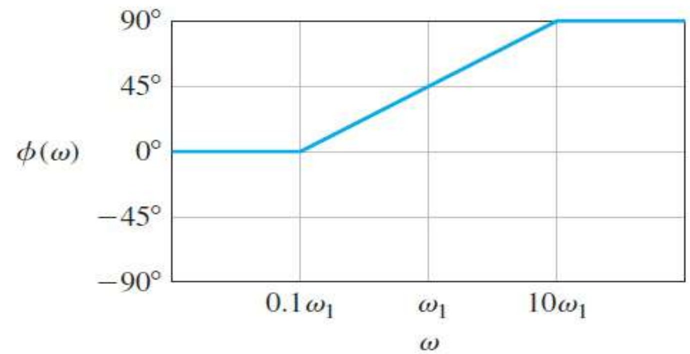
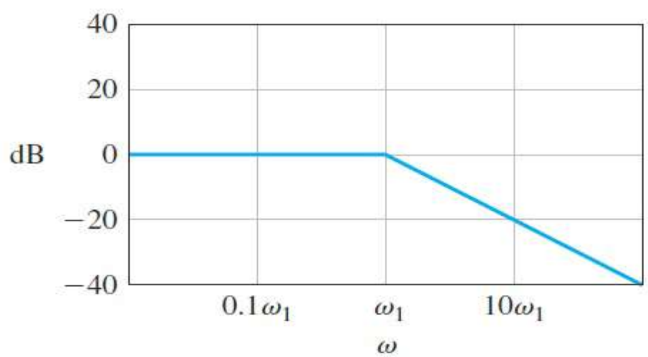
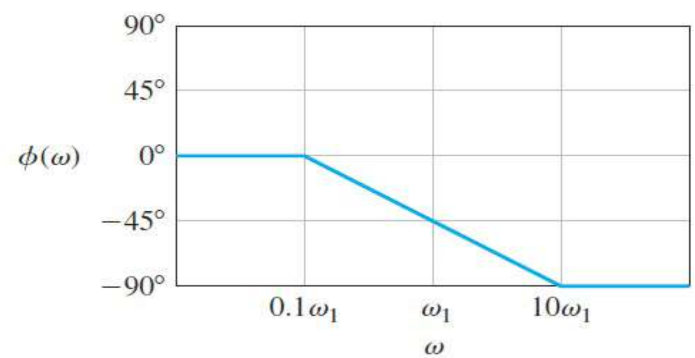
$$L(s) = \frac{k(s+2)}{s(s+3)(s^2+2s+2)}$$

$$L(j\omega) = \frac{k(j\omega+2)}{j\omega(j\omega+3)((j\omega)^2+2j\omega+2)} = \frac{\frac{\cancel{2}k}{3\cdot\cancel{2}}(j\frac{\omega}{2}+1)}{j\omega(j\frac{\omega}{3}+1)\left[1-\left(\frac{\omega}{\sqrt{2}}\right)^2+j\sqrt{2}(\frac{\omega}{\sqrt{2}})\right]}$$

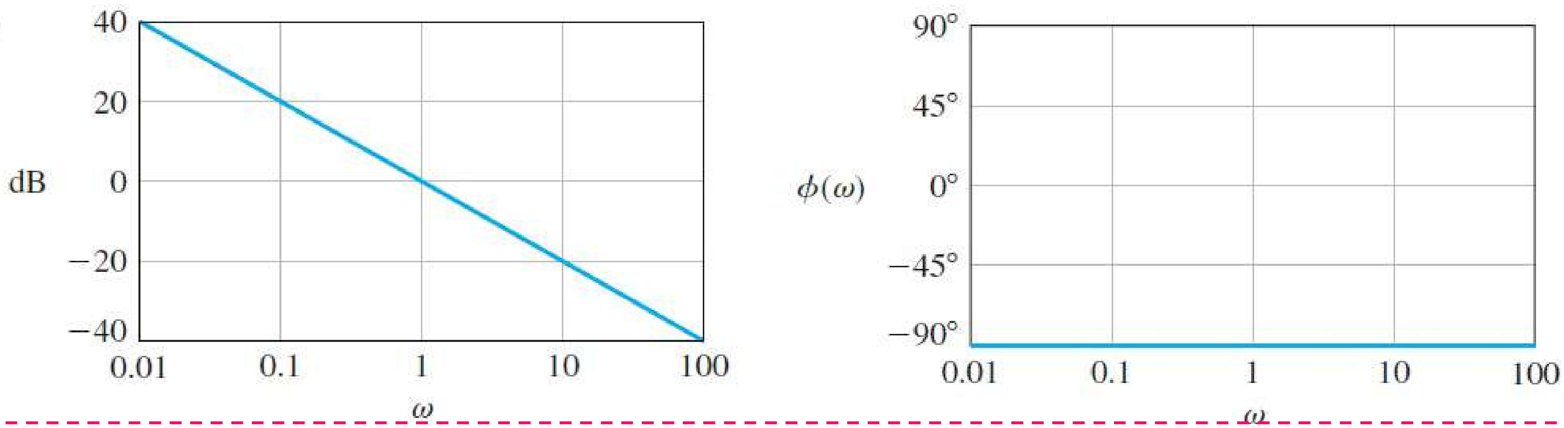
- Use the general forms of asymptotic behavior to obtain the overall frequency response for any system by summing the curves that result for each element.

- **Important**: the transfer function must be in the form of the product of the terms $(j\omega\tau)$, $(1+j\omega\tau)$, $\left(1-\left(\frac{\omega}{\omega_n}\right)^2+j\omega\left(\frac{2\zeta}{\omega_n}\right)\right)$, and/or k.

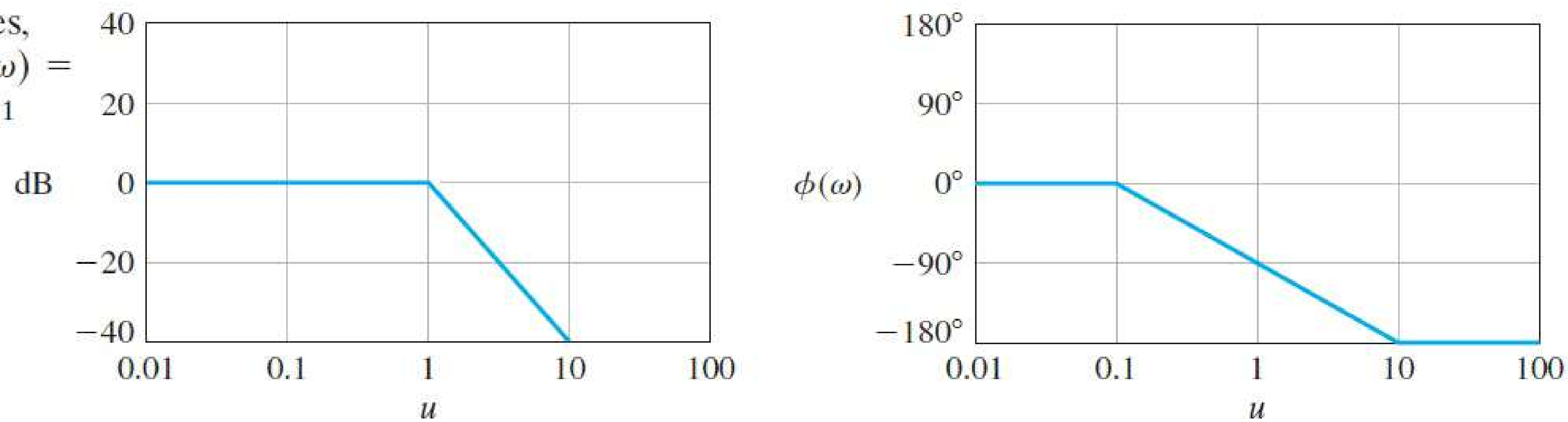
Asymptotic Curves of Bode Plot for Basic Terms of a Transfer Function

Term	Magnitude 20 log|G|	Phase $\phi(\omega)$
1. Gain, $G(j\omega) = K$		
2. Zero, $G(j\omega) = 1 + j\omega/\omega_1$		
3. Pole, $G(j\omega) = (1 + j\omega/\omega_1)^{-1}$		

4. Pole at the origin, $G(j\omega) = 1/j\omega$



5. Two complex poles, $0.1 < \zeta < 1$, $G(j\omega) = (1 + j2\zeta u - u^2)^{-1}$
$u = \omega/\omega_n$



Further illustration in the next page

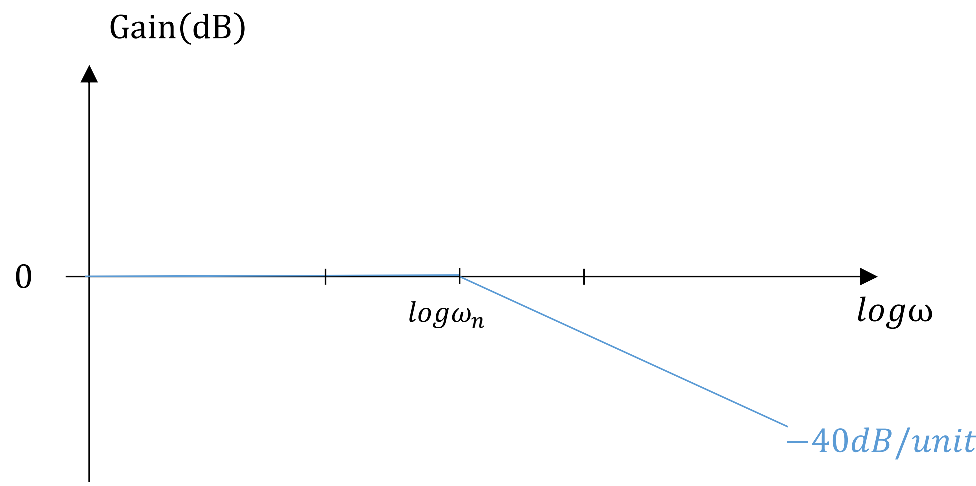
Complex Conjugate Poles

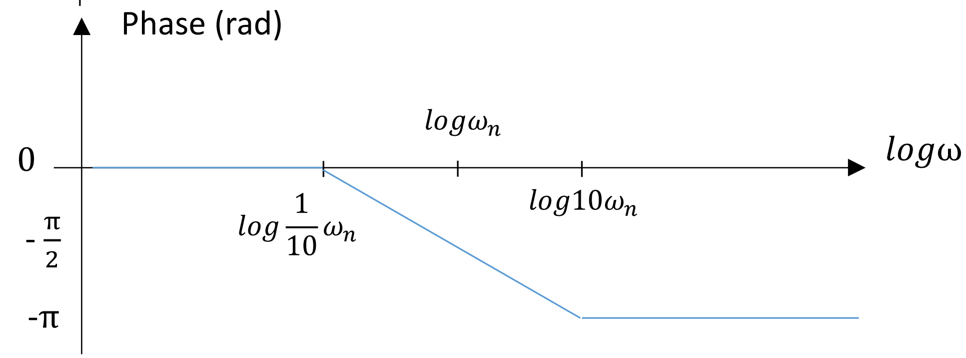
$$G(j\omega) = \frac{k\prod(1+j\omega\tau_z)}{(j\omega)^N\prod(1+j\omega\tau_p)\prod\left(1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

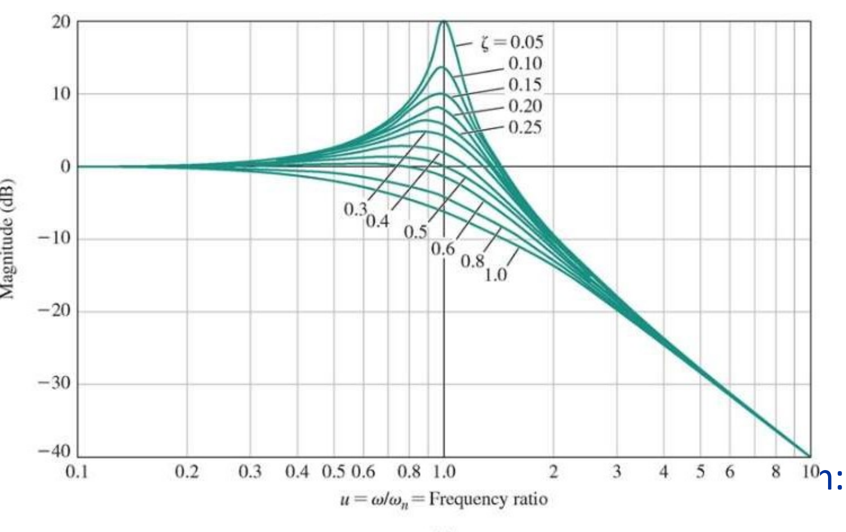
$$\text{Gain(dB)} = -10log_{10}\left(\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right)$$

$$\text{Phase}(rad) = -tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right) for\ all\ \omega$$

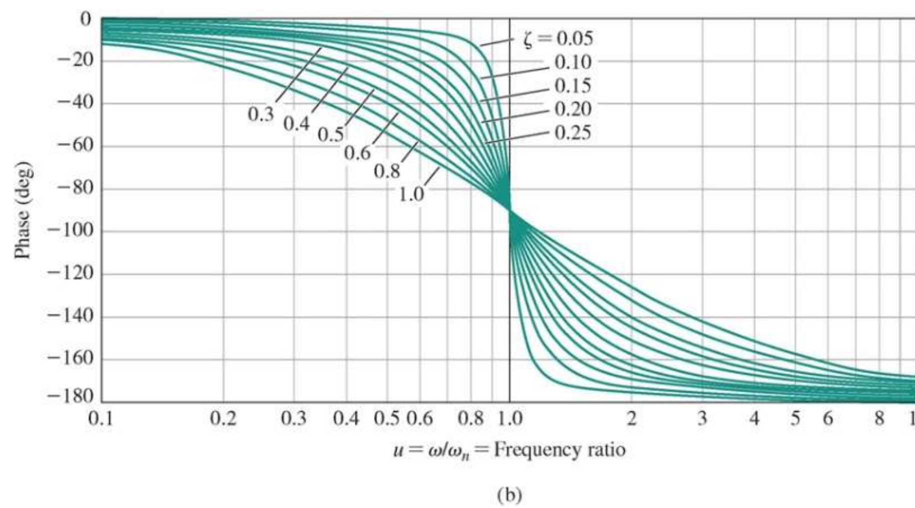
Range of ω	**Gain**	**Phase** (use points $\omega_n/10$, ω_n, and $10\omega_n$)
$\omega \ll \omega_n$	$-10log_{10}(1) = 0dB$	0
$\omega = \omega_n$	$-20log_{10}(2\zeta)$	-90° or $-\frac{\pi}{2}$
$\omega \gg \omega_n$	$-40log_{10}\left(\frac{\omega}{\omega_n}\right) = -40log_{10}(\omega) + 40log_{10}(\omega_n)$	-180° or $-\pi$







(a)



(b)

Sketching a Bode Plot

$$G(j\omega) = \frac{k\prod(1+j\omega\tau_z)}{(j\omega)^N\prod(1+j\omega\tau_p)\prod\left(1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}$$

Consider the system with the following transfer function

$$G(j\omega) = \frac{5(1+j0.1\omega)}{j\omega(1+j0.5\omega)\left(1+j0.6\left(\frac{\omega}{50}\right)+\left(j\frac{\omega}{50}\right)^2\right)}$$

Solutions:

There are five terms in the transfer function:

1. A constant gain $K = 5$
2. A pole at the origin
3. A pole at $\omega = 2$
4. A zero at $\omega = 10$
5. A pair of complex poles at $\omega_n = 50, 2\zeta = 0.6 \rightarrow \zeta = 0.3$

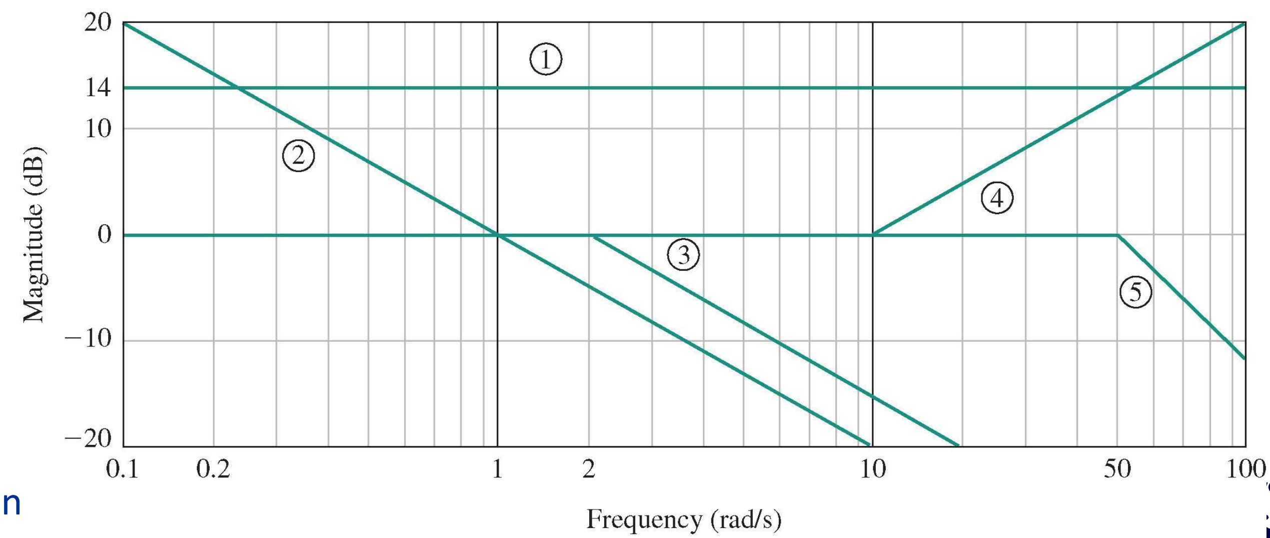
Magnitude Plot

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)[1 + j0.6(\omega/50) + (j\omega/50)^2]}$$

We plot the magnitude characteristic for each individual pole and zero factor and the constant gain:

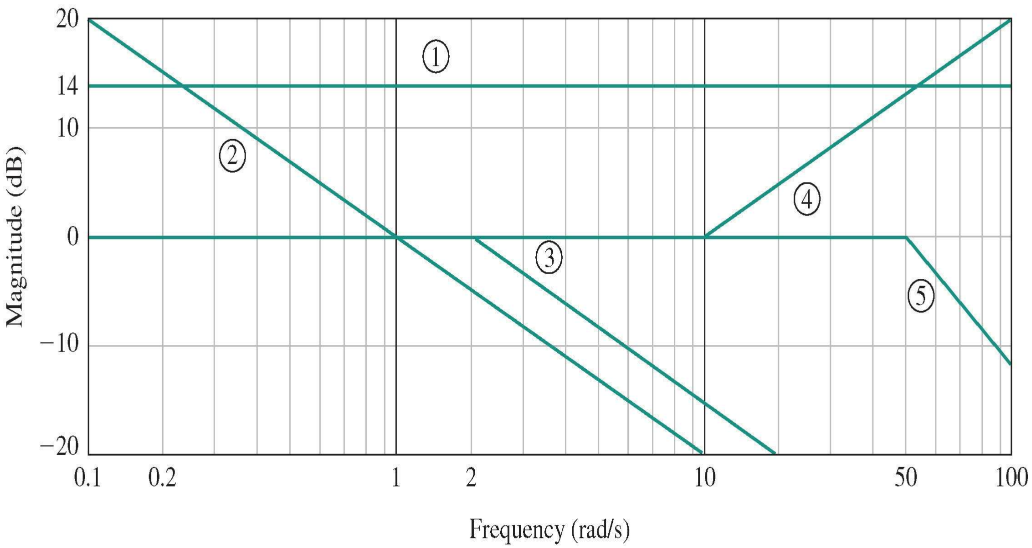
1. The constant gain is $20\log5 = 14\text{dB}$.
2. The magnitude of the pole at the origin extends from zero to infinite frequencies, and has a slope of -20dB/dec intersecting the 0dB line at $\omega = 1\ rad/s$.
3. The asymptotic (linear) approximation of the magnitude of the pole at $\omega = 2\ rad/s$ has a slope of -20dB/dec beyond the corner frequency of $\omega = 2\ rad/s$. The asymptotic magnitude below the corner frequency is 0dB.
4. The asymptotic (linear) approximation of the magnitude of the zero at $\omega = 10\ rad/s$ has a slope of $+20\text{dB/dec}$ beyond the corner frequency at $\omega = 10\ rad/s$. The asymptotic magnitude below the corner frequency is also 0dB.
5. The magnitude slope for the complex poles after the corner frequency, $\omega = \omega_n = 50\,rad/s$, is -40dB/dec. The linear approximation around the corner frequency ω_n must be corrected based on the damping ratio ζ, in the example being 0.3 (approximately, ζ lower than $1/\sqrt{2}$ has a peak; higher than $1/\sqrt{2}$ has none).

Magnitude's asymptotic lines



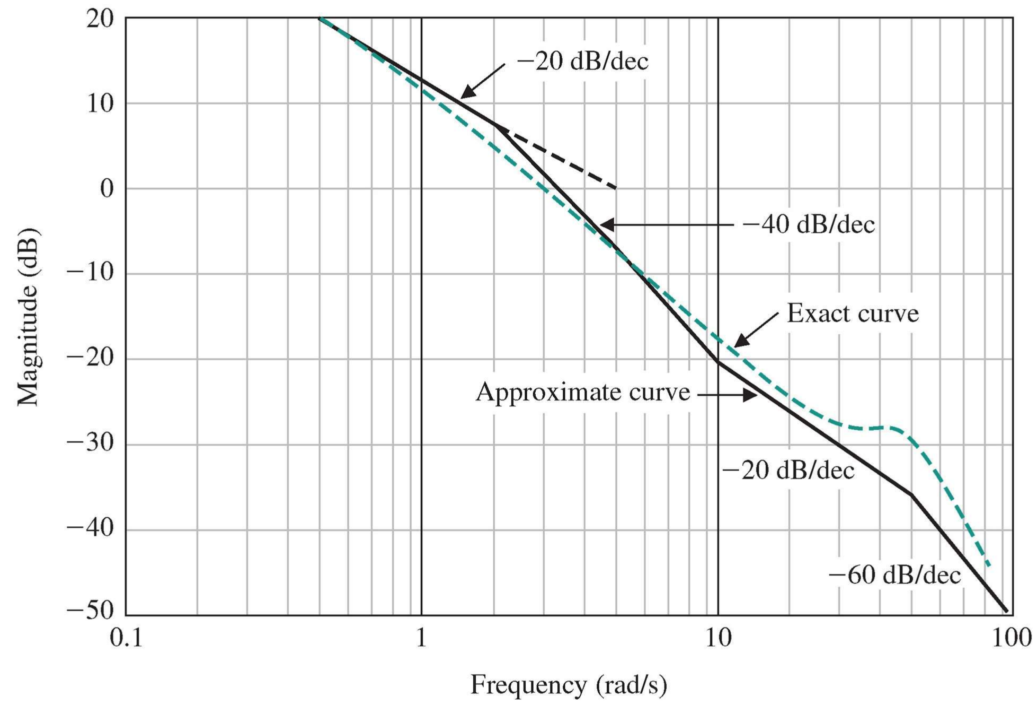
Mag. Plot

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)[1 + j0.6(\omega/50) + (j\omega/50)^2]}$$



Magnitude's asymptotic lines

FIGURE 8.20 Magnitude characteristic.



Sketch of the magnitude characteristic

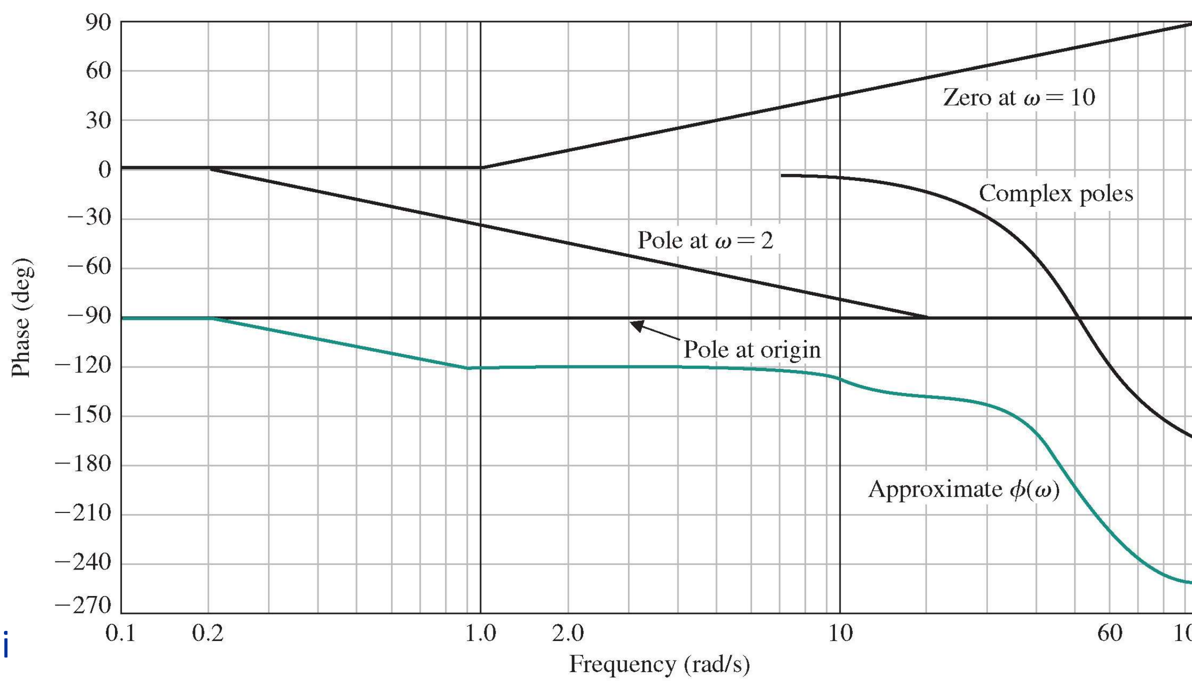
Phase Plot

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)[1 + j0.6(\omega/50)\ + (j\omega/50)^2]}$$

Then, we plot the phase characteristic for each individual pole and zero factor and the constant gain:

1. The phase of the constant gain is 0^o.
2. The phase of the pole at the origin is a constant -90^o.
3. The linear approximation of the phase characteristic for the pole at $\omega = 2\ rad/s$ is three-section straight lines (as shown below) with -45^o at $\omega = 2\ rad/s$.
4. The linear approximation of the phase characteristic for the zero at $\omega = 10\ rad/s$ is three-section straight lines (as shown below) with $+45^o$ at $\omega = 10\ rad/s$.
5. The phase characteristic approximation for the complex poles with the corner frequency $\omega_n = 50\ rad/s$ is the corrected curve based on the damping factorbased on the damping ratio ζ, in the example being 0.3 (reminder: ζ near 1 approximates to the three-section straight lines; ζ near 0 approximates to two-section step without the slope).

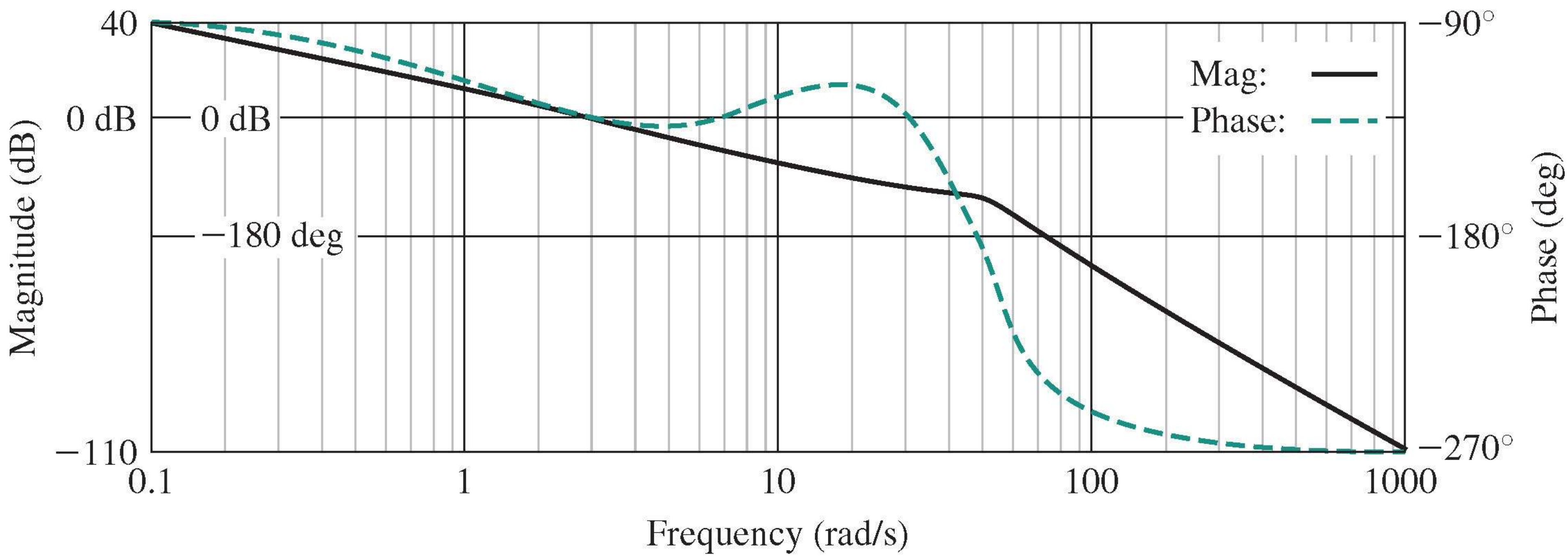
Phase's asymptotic lines



Phase Plot

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)[1 + j0.6(\omega/50) + (j\omega/50)^2]}$$

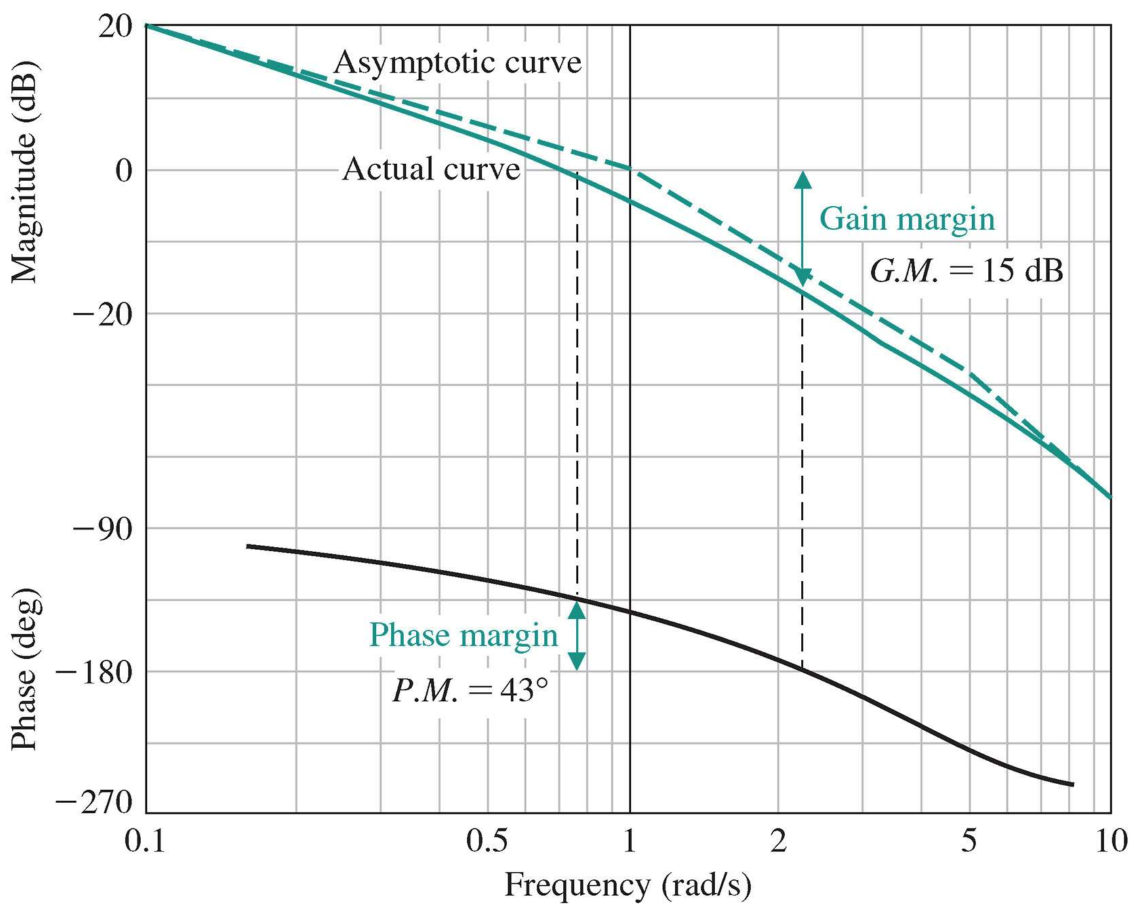
Sketch of the phase characteristic



Gain Margin and Phase Margin

$$L(j\omega) = |L(j\omega)|e^{j\phi(\omega)}$$

$$\boxed{\textbf{Logarithm Gain} = \mathbf{20log_{10}}|\boldsymbol{L(j\omega)}|}$$



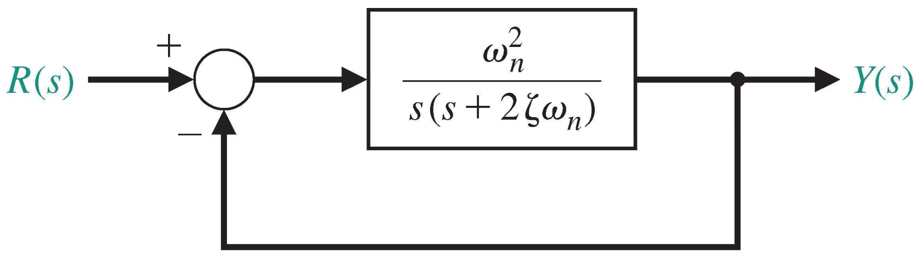
Relative Stability

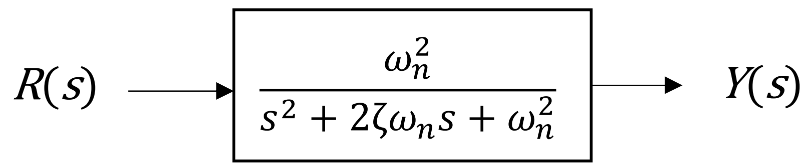
- The gain margin GM is the increase in the gain of the $L(j\omega)$ when phase = $\pm 180^o$ that will result in a marginally stable system;
- The phase margin $\emptyset_{pm}$ is the amount of phase shift of the $L(j\omega)$ at unity magnitude (0 dB) that will result in a marginally stable system.

$$\boxed{\zeta \approx 0.01\emptyset_{PM}}$$

Performance Specifications in Frequency Domain

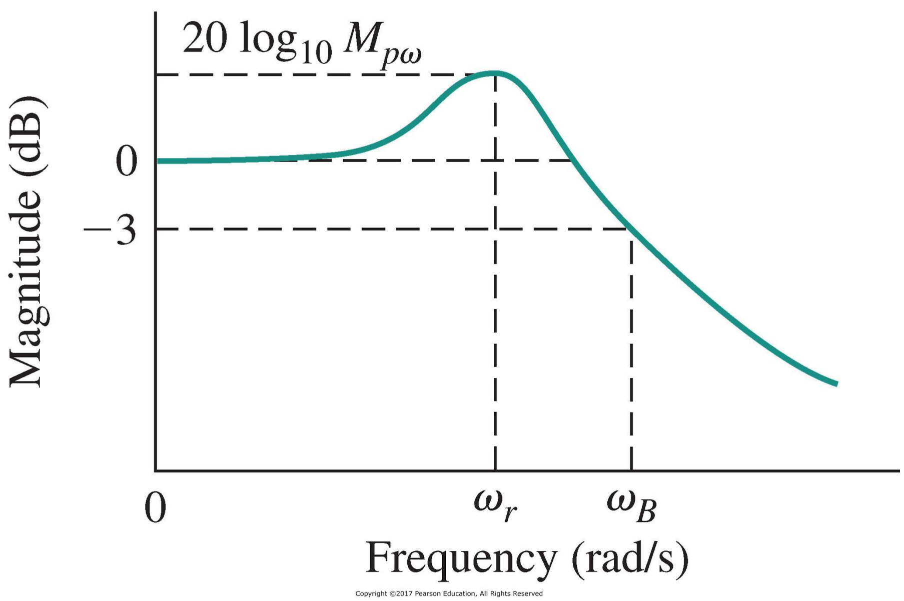
- For a second-order system with a pair of complex poles:





Two main frequency specifications:

1. **Maximum magnitude - $M_{p\omega}$**
2. **Bandwidth - ω_B**



Linear approximation

$$\frac{\omega_B}{\omega_n} \approx -1.19\zeta + 1.85 \qquad \sim 0.2 < \zeta < \sim 0.8$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}, \zeta < 0.707$$

$$M_{p\omega} = |G(j\omega_r)| = \left(2\zeta\sqrt{1-\zeta^2}\right)^{-1}, \zeta < 0.707$$

Maximum magnitude $M_{p\omega}$ ↑ → Damping ratio ζ ↓ → Percent Overshoot P.O. ↑

Bandwidth ω_B ↑ (with a constant ζ) → Natural frequency ω_n ↑ → Settling time T_s ↓

$$G_c(s) = \frac{K(j\omega + z)}{(j\omega + p)} = \frac{K(1 + j\omega\alpha\tau)}{\alpha(1 + j\omega\tau)}$$

Compensators

$$G_c(s) = \frac{K(j\omega + z)}{(j\omega + p)} = \frac{K\alpha(1 + j\omega\tau)}{(1 + j\omega\alpha\tau)}$$

Lead compensator:

$$\tau = \frac{1}{p} \text{ and } \alpha = p/z > 1$$

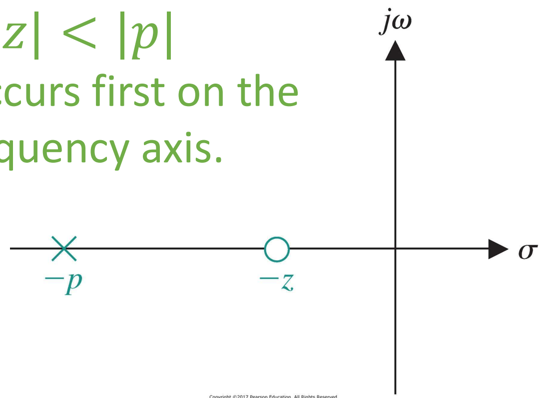
Lag compensator:

$$\tau = \frac{1}{z} \text{ and } \alpha = z/p > 1$$

- Phase – Lead Compensator

$|z| < |p|$

Zero occurs first on the frequency axis.



The maximum phase lead $\emptyset_m$ occurs at $\boldsymbol{\omega_m}$:

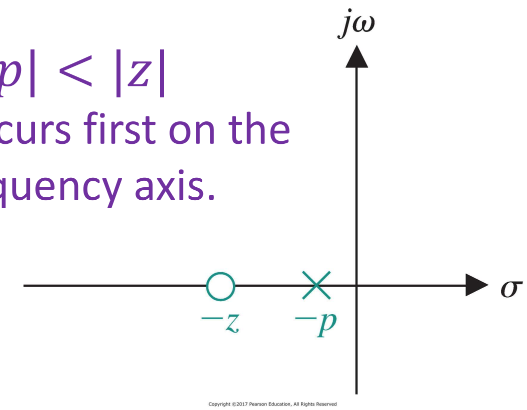
$$\boxed{\omega_m = \sqrt{zp} = \frac{1}{\tau\sqrt{\alpha}}}$$

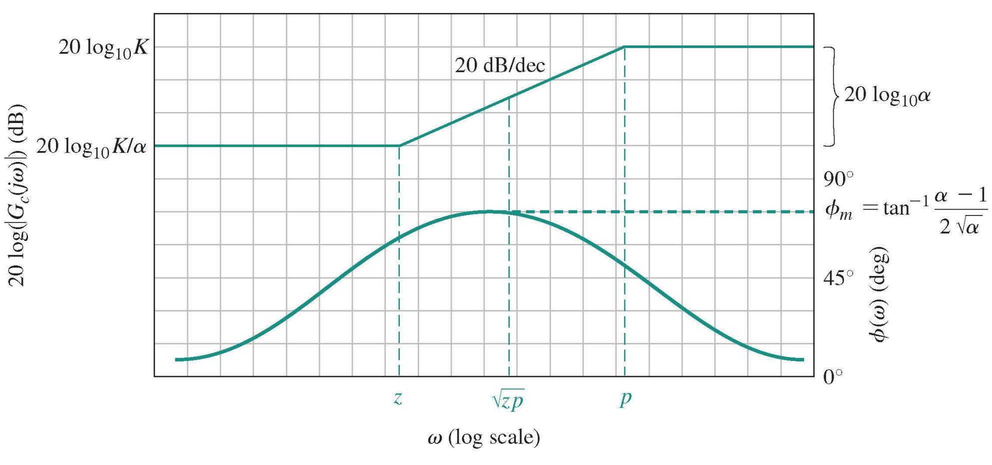
$$\boxed{\sin\phi_m = \frac{\alpha - 1}{\alpha + 1}}$$

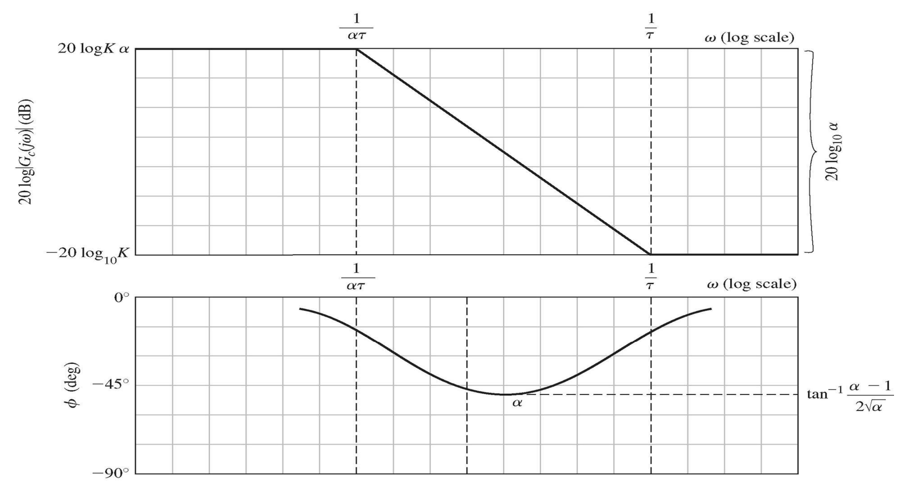
- Phase - Lag Compensator

$|p| < |z|$

Pole occurs first on the frequency axis.







Other checkpoints (3)

- ✓ Clear on all plotting steps for Bode plot? i.e., how to sketch the asymptotes of magnitude plot for a given system? ? (Lecture 20; examples and exercises)
- ✓ How to determine the phase and gain crossover frequency? How to determine gain and phase margins from calculation? (Lecture 20 and 21; examples and exercises)
- ✓ How to calculate magnitude and phase of a transfer function at a given ω (e.g., for a given loop TF, when $\omega = xxx\, rad/s$) ? (Lecture 20 and 21; examples and exercises)
- ✓ How to obtain gain and phase margin from plots? (Lecture 20 and 21 ; examples and exercises)
- ✓ Know how to interpret bode plot? e.g., what happen if the gain/phase added to the loop transfer function is beyond the "gain/phase" margin of the original system, and their relationship with system stability? (Lecture 20 and 21; examples and exercises)
- ✓ Understand the concept of compensator design, their standard forms (Lecture 21)

The END (for Part 3)

- THANK YOU for your attention, especially to those who frequently engaged during and after the lectures.
- For those students who missed some of the classes, I hope you are able to learn and acquire the knowledge and understanding from the teaching materials (ppt, examples and exercises, quizzes, lab guidance and questions, consultation) provided.
- Hope you have enjoyed learning the subject and build up some useful skills based on true understanding of the control system subject.
  - In this GAI-era, your learning will certainly be accelerated if you have good foundation of understanding, good first-hand experiences/skills, and use the GAI tools in the constructive manner.
  - Lecturers use GAI for daily tasks too. I can feel that, when dealing with a more in-depth technical subject, a good foundation of understanding can carry one further as compared to those who don't have. (Just my personal view for now; things will continue to evolve as the technology develops further…)
- Good luck to you in the upcoming exam season.