CAN207 Continuous and Discrete Time Signals and Systems

Lecture 20

Z-Transform_Part 2

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Content

- 4. Properties of z-transform
  - Linearity, time-shifting, time-reversal, time-scaling, z-domain scaling, z-domain differentiation, time-difference, time-accumulation, conjugation, time convolution.
  - Comparing with DTFT and Laplace transform
- 5. Inverse z-Transform
  - Table Look-up
  - Long Division (Power series expansion)
  - Partial Fraction Expansion
    - ROC determination

4.1 Properties - Linearity

- Given the transform pairs:

$$x_1[n] \xleftrightarrow{z} X_1(z) \quad \text{and} \quad x_2[n] \xleftrightarrow{z} X_2(z)$$

- It can be shown that the following relationship holds:

$$\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{z} \alpha X_1(z) + \beta X_2(z)$$

  – ROC is the overlapping region of $X_1(z)$ and $X_2(z)$.

- Example: determine the z-transform of the following signals:
  - $x[n] = 3\left(\frac{1}{2}\right)^n u[n] - 5\left(\frac{1}{3}\right)^n u[n]$
  - $x[n] = \cos(\omega_0 n)\, u[n]$
  - $x[n] = b^{|n|}$

4.2 Properties - Time-shifting

- Given the transform pair $x[n] \overset{z}{\longleftrightarrow} X(z)$,
- The following is also valid:

$$x[n-k] \overset{z}{\longleftrightarrow} z^{-k} X(z)$$

  – ROC is the same as $X(z)$ with some possible exceptions:
    - left-shfting to cause negative indexed samples: excluding $|z| \rightarrow \infty$;
    - right-shfting to cause positive indexed samples: excluding $|z| \rightarrow 0$.

- Example: determine the z-trans of:
  – $x[n] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & n < 0 \text{ or } n > N-1 \end{cases}$

4.3 Properties - Time-reversal

- Given the transform pair $x[n] \overset{z}{\longleftrightarrow} X(z)$,
- The following is also valid:

$$x[-n] \overset{z}{\longleftrightarrow} X(z^{-1})$$

  - Let the ROC of the original transform $X(z)$ be $r_1 < |z| < r_2$,
  - The the ROC of $X(z^{-1})$ is adjusted to be $\frac{1}{r_2} < |z| < \frac{1}{r_1}$.

- Example: Anti-causal exponential signal revisited. Using the time-reversal and time-shifting properties to find the z-transform of:

$$x[n] = -a^n u[-n-1]$$

4.4 Properties - z-domain scaling

- Given the transform pair $x[n] \overset{z}{\longleftrightarrow} X(z)$,
- The following is also valid:

$$a^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{a}\right)$$

  – Let the ROC of the original transform $X(z)$ be $r_1 < |z| < r_2$,

  – The the ROC of $X\left(\frac{z}{a}\right)$ is adjusted to be: $|a|r_1 < |z| < |a|r_2$.

- Example: determine the z-transform of the following signal:

$$x[n] = r^n \cos(\omega_0 n)\, u[n]$$

4.5 Properties - Time scaling

- There are two types of scaling in the DT domain: decimation and interpolation.
  - **Decimation**: Because of the irreversible nature of the decimation operation, the z-transform of x[n] and its decimated sequence y[n] = x[mn] are not related to each other.
  - **Interpolation**: the interpolation of x[n] is defined as follows:

$$x^{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of interger } m \\ 0, & \text{otherwise} \end{cases}$$

Given the transform pair $x[n] \overset{z}{\longleftrightarrow} X(z)$, then the z-transform is:

$$X^{(m)}(z) = X(z^m)$$

- Let the ROC of the original transform $X(z)$ be $r_1 < |z| < r_2$,
- The the ROC of $X(z^m)$ is adjusted to be $r_1^{1/m} < |z| < r_2^{1/m}$.

4.6 Properties - z-domain Differentiation

- Given the transform pair $x[n] \overset{z}{\longleftrightarrow} X(z)$,
- The following is also valid:

$$nx[n] \overset{z}{\longleftrightarrow} (-z)\frac{dX(z)}{dz}$$

  – ROC is the same as $X(z)$

- Example: determine the z-transform of the following signals:
  – $x[n] = na^n u[n]$
  – $x[n] = nu[n]$
  – $x[n] = n(n+2)u[n]$

4.7 Properties - Time-differencing

- Given the transform pair $x[n] \overset{z}{\longleftrightarrow} X(z)$ with ROC of R,
- The following is also valid:

$$x[n] - x[n-1] \overset{z}{\longleftrightarrow} (1 - z^{-1})X(z)$$

  – ROC is at least R with the possible deletion of $z = 0$.

- Example: Based on the z-transform pair

$$u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$

Calculate the z-transform of the impulse function $x[n] = \delta[n]$ using the time differencing property.

4.7 Properties - Time accumulation

- Given the transform pair $x[n] \overset{z}{\longleftrightarrow} X(z)$ with ROC = R,
- The following is also valid:

$$\sum_{k=-\infty}^{n} x[k] \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}} X(z)$$

  – The ROC is $R \cap |z| > 1$.

- Example: Calculate the z-transform of the function $nu[n]$ using the time-accumulation property.

4.9 Properties - Conjugation

- Given the transform pair $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$,
- The following is also valid:

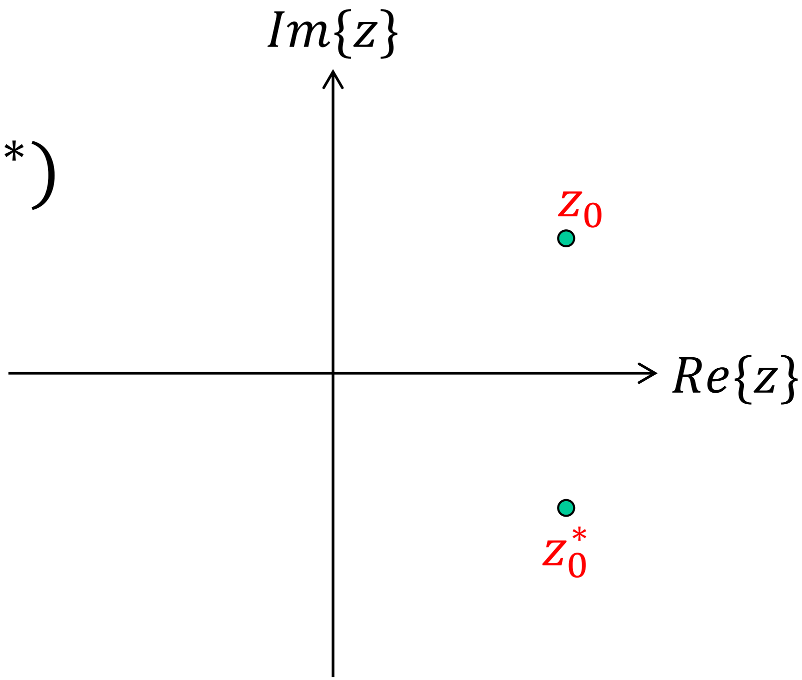
$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$$

  – ROC is the same as $X(z)$.

- If $x[n]$ is real, then

$$X(z) = X^*(z^*)$$

  – Application: in the study of zero-pole locations.
  – If $X(z)$ has a pole (or zero) at $z = z_0$, it must also have a pole (or zero) at the complex conjugate point $z = z_0^*$.



4.10 Properties - Time convolution

- Given the transform pairs:

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z) \quad \text{and} \quad x_2[n] \overset{z}{\longleftrightarrow} X_2(z)$$

- The following is also valid:

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z)$$

  - ROC is **at least** the overlap of the two transforms;
  - It may be greater than the overlap due to the cancellation of some poles in the product of z-transforms.

- Most frequently used in finding the output signal of a system with known impulse response and input signal.

4.10 Properties - Time convolution

- Example: Consider a LTID system described by the impulse response

$$h[n] = 0.9^n u[n]$$

driven by the input signal

$$x[n] = u[n] - u[n-7]$$

Compute the z-transform of the output signal Y(z).

Quiz 1

- Determine the z-transform for each of the following sequences. Sketch the pole-zero plot and indicate the region of convergence. Indicate whether or not the DTFT of the sequence exists.

  1. $\delta[n-2]$
  2. $\left(-\frac{1}{3}\right)^n u[-n-2]$
  3. $2^n u[-n] + \left(\frac{1}{4}\right)^n u[n-1]$
  4. $|n|\left(\frac{1}{2}\right)^{|n|}$

5.1 Methods of Inverse z-transform

- Inverse z-transform is the problem of finding $x[n]$ from the knowledge of $X(z)$.

$$x[n] \overset{z}{\longleftrightarrow} X(z)$$

- Commonly used methods:
  - (i) table look-up method;
  - (ii) inversion formula method (not required);
  - (iii) power series method (long division);
  - (iv) partial fraction expansion (PFE) method.

5.2 Table look-up method

- In this method, the z-transform function X(z) is matched with one of the entries in the table of commonly used pairs.
- As the transform pairs are unique, the inverse transform is readily obtained from the time-domain entry.
- Example: inverse z-trans of

$$X(z) = \frac{1}{1 - 0.3z^{-1}}, ROC: |z| < 0.3$$

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

5.3 Inversion formula method (not required)

- To find the inverse formula (synthesize equation of z-trans.):
  - Since $X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\}$
  - Applying the inverse DTFT to both sides, yields:

$$x[n] = r^n \mathcal{F}^{-1}\{X(re^{j\omega})\} = \frac{r^n}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

  - Moving the r^n term inside the integral and combining with $e^{j\omega n}$:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^n d\omega$$

    - That is, we can recover $x[n]$ from its z-transform evaluated along a contour $z = re^{j\omega}$ in the ROC, with *r* fixed and ω varying over a 2π interval.
  - Change the variable of integration from ω to z
    - with $z = re^{j\omega}$ and r fixed (changing ω), $dz = jre^{j\omega}d\omega = jzd\omega$

5.3 Inversion formula method (not required)

– Therefore, substitute $d\omega = \frac{1}{jz}dz$ back to the equation, get

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

- This is the inverse z-transform definition equation
- Also called the synthesize equation

– In this method, the inverse z-transform is calculated directly by solving the complex contour integral.

- This approach involves contour integration, which is beyond the scope of this module, and not often used.

5.4 Power series method (Long division)

- In some cases, we may be interested in determining only a few values of $x[n]$ for $n \geq 0$, rather than the complete analytical solution, then we can use the power series method, also referred to as ***long division***.
  - simple to use (especially in computer)
  - not full (analytical) solution

- When the transform X(z) is expanded as follows:

$$X(z) = a + bz^{-1} + cz^{-2} + dz^{-3} + \ldots$$

- The corresponding time-domain sequence should be:

$$x[n] = a\delta[n] + b\delta[n-1] + c\delta[n-2] + d\delta[n-3] + \ldots$$

5.4 Power series method (Long division)

- Examples: Calculate the first four non-zero values of the following right-sided sequences using the long division:

1. $X_1(z) = \frac{z}{z^2-3z+2}$
2. $X_2(z) = \frac{1}{1-az^{-1}}$
3. $X_3(z) = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

5.4 Examples:

1. Answer: the long division is shown as:

$$
\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} \\
\hline
z^2 - 3z + 2 & z \\
 & \underline{z - 3 + 2z^{-1}} \\
 & \quad 3 - 2z^{-1} \\
 & \quad \underline{3 - 9z^{-1} + 6z^{-2}} \\
 & \qquad 7z^{-1} - 6z^{-2} \\
 & \qquad \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\
 & \qquad\qquad 15z^{-2} - 14z^{-3} \\
 & \qquad\qquad \underline{15z^{-2} - 45z^{-3} + 30z^{-4}}.
\end{array}
$$

– That means

$$X_1(z) = 0z^0 + z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} + \ldots$$

– So the time domain sequence is:

$$x[n] = \delta[n-1] + 3\delta[n-2] + 7\delta[n-3] + 15\delta[n-4] + \ldots$$

$$\text{i.e.}\ \{x[n]\} = \{\underline{\mathbf{0}}, 1, 3, 7, 15, \ldots\}$$

5.5 Partial Fraction Expansion Method

- Recall the two most useful z-transform pairs:

$$\mathcal{Z}\{a^n u[n]\} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad ROC: |z| > |a|$$

$$\mathcal{Z}\{-a^n u[-n-1]\} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad ROC: |z| < |a|$$

  – they are used as the basis for determining the inverse z-transform of rational fractions using the partial fraction expansion.

- Consider a transform $X(z)$ given with its denominator factored out as

$$X(z) = \frac{N(z)}{(z-z_1)(z-z_2)\ldots(z-z_N)}$$

  – Noted that $X(z)$ is a proper fraction, i.e. the order of $N(z)$ is less than N.

- Expanding the transform into partial fractions in the form

$$X(z) = \frac{k_1 z}{z-z_1} + \frac{k_2 z}{z-z_2} + \ldots + \frac{k_N z}{z-z_N}$$

5.5 Partial Fraction Expansion Method

- Let individual terms in the partial fraction expansion be

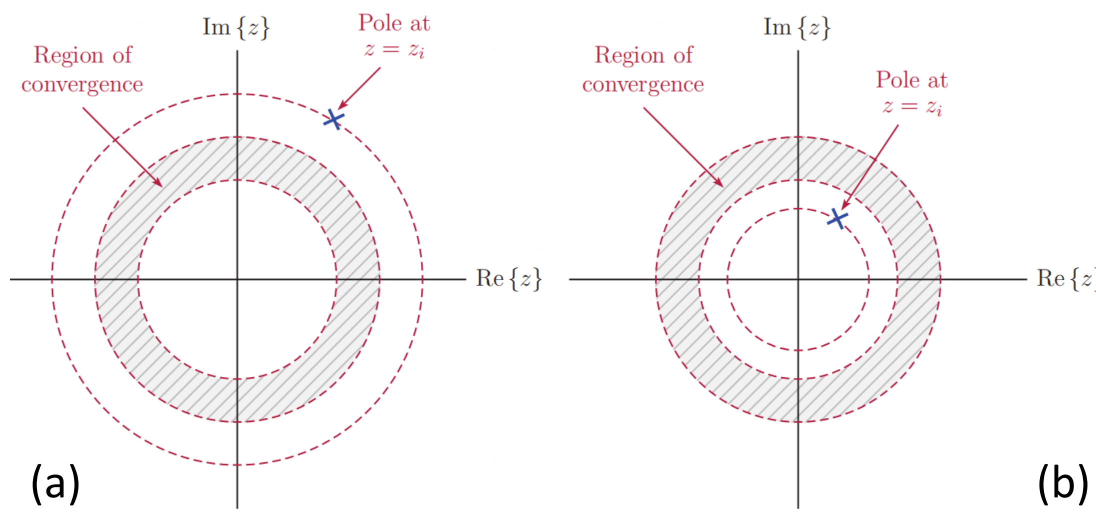
$$X_i(z) = \frac{k_i z}{z - z_i}, \qquad \text{for } i = 1, 2, \ldots, N$$

- Each one has its inverse transform like:

$$x_i[n] = \mathcal{Z}^{-1}\{X_i(z)\} = \mathcal{Z}^{-1}\left\{\frac{k_i z}{z - z_i}\right\} = \begin{cases} a^n u[n], & ROC\colon\ |z| > |a| \\ -a^n u[-n-1], & ROC\colon\ |z| < |a| \end{cases}$$

  – These decisions must be made by looking at the ROC for $X(z)$ and reasoning what the contribution from the ROC of each individual term $X_i(z)$ must be in order to get the overlap that we have.

- Each $X_i(z)$ has only one pole at z_i, so:
  – (a) $|z| < |z_i|$
  – (b) $|z| > |z_i|$



5.5 Partial Fraction Expansion Method

- In LTID signals and systems analysis, the z-transform of a function $x[n]$ generally takes the following rational form:

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_M z^{-M}+b_{M-1}z^{-(M-1)}+\ldots+b_1 z^{-1}+b_0}{a_N z^{-N}+a_{N-1}z^{-(N-1)}+\ldots+a_1 z^{-1}+a_0}$$

- If $M \geq N$, then $X(z)$ can be re-expressed through long division:

$$X(z)=\sum_{l=0}^{M-N} c_l z^{-l}+\frac{P_1(z)}{D(z)}$$

  – where the degree of $P_1(z)$ is less than N.
  – The rational fraction $P_1(z)/D(z)$ is then called a ***proper polynomial***.
  – Example:

$$H(z)=\frac{2+0.8z^{-1}+0.5z^{-2}+0.3z^{-3}}{1+0.8z^{-1}+0.2z^{-2}} \Rightarrow H(z)=-3.5+1.5z^{-1}+\frac{5.5+2.1z^{-1}}{1+0.8z^{-1}+0.2z^{-2}}$$

$$h[n]=-3.5\delta[n]+1.5\delta[n-1]+\ldots$$

5.5 Partial Fraction Expansion Method

- ***Simple Poles***: In most practical cases, the rational z-transform of interest $X(z)$ is a ***proper fraction*** with ***simple poles***, then it can be written in the following form

$$X(z) = \frac{k_1 z}{z - z_1} + \frac{k_2 z}{z - z_2} + \ldots + \frac{k_N z}{z - z_N}$$

- That's equivalent to

$$\frac{X(z)}{z} = \frac{k_1}{z - z_1} + \frac{k_2}{z - z_2} + \ldots + \frac{k_N}{z - z_N}$$

- The coefficients k_i could be obtained from:

$$k_i = \left[(z - z_i)\frac{X(z)}{z}\right]_{z=z_i}$$

  – if no roots are repeated, i.e. simple poles.

5.5 Partial Fraction Expansion Method

- Examples:

1. $X_1(z) = \frac{z}{z^2-3z+2}, \quad ROC: 1 < |z| < 2$
2. $X_3(z) = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}, \quad ROC: |z| > 5$

5.5 Partial Fraction Expansion Method

- ***Multiple Poles***: If the z-domain function contains an m-multiple pole, that is, a term as the following is included

$$\frac{X(z)}{z} = \frac{P_2(z)}{(z - z_i)^m}$$

- This term is expanded as:

$$\frac{X(z)}{z} = \frac{A_1}{z - z_i} + \frac{A_2}{(z - z_i)^2} + \ldots + \frac{A_m}{(z - z_i)^m}$$

- where each coefficient can be computed by taking consecutive derivatives and evaluating the function at the pole

$$A_{m-i} = \frac{1}{(i)!} \frac{d^i\left((z - z_i)^m \frac{X(z)}{z}\right)}{dz^i}\Bigg|_{z=z_i} = \frac{1}{(i)!} \frac{d^i P_2(z)}{dz^i}\Bigg|_{z=z_i}$$

Next ...

- Z-transform Part 3
  - 6. Geometric Evaluation of DTFT based on z-transform
  - 7. Unilateral z-transform
  - 8. Analysis of LTID systems using z-transform