CAN102 Electromagnetism and Electromechanics

Lecture-6 Static Electric Fields IV

(Curl, Materials in E-field and Boundary Conditions)

Zhao Wang

Zhao.wang@xjtlu.edu.cn

Room EE322

Outline

- Maxwell's equation II - Electric field loop theorem
  - Curl
  - Stoke's Theorem
  - Integral and Differential forms
- Conductors and Dielectrics
  - Ideal conductors
  - Electric Equilibrium
  - Dielectrics and Permittivity
- Boundary Conditions
  - Tangential and normal components of E-field

1.1 Circulation

- The line integral of some vector field $\boldsymbol{F}(x, y, z)$ taken around a closed path (curve C) is called ***circulation***:

$$\Gamma = \oint_C \boldsymbol{F} \cdot d\boldsymbol{l}$$

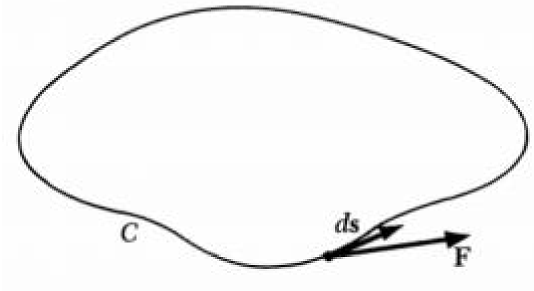
  – Curve C: can be visualized as the boudary of some surface S which spans it;
  – $d\boldsymbol{l}$ is the line element of path, an infinitesimal vector locally tangent to C.

- Bridge C with a new path B => two loops C_1 and C_2:

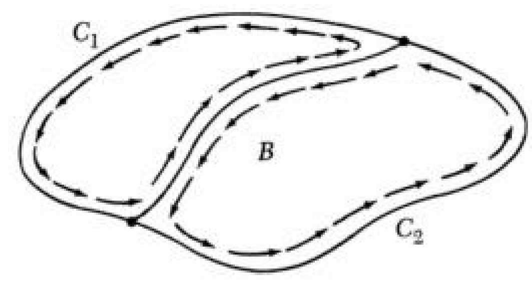
$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s_1} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s_2} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{l}$$

- Furthermore, subdivision into many loops:

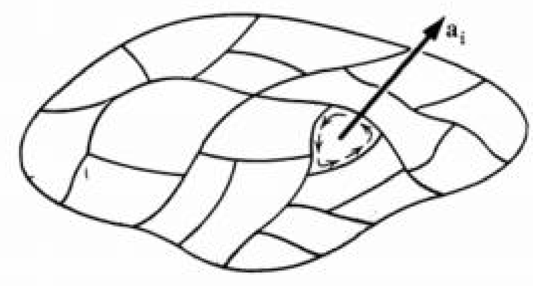
$$\Gamma = \oint_C \boldsymbol{F} \cdot d\boldsymbol{l} = \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N}} \int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{l_i} = \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N}} \Gamma_{\boldsymbol{i}}$$



(a)



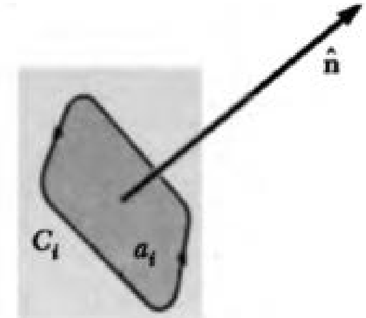
(b)



(c)

1.2 Curl

- Make the small loops approach to a point, the limit of *the ratio of circulation to patch area* is defined as the ***curl***:



$$curl(\boldsymbol{F}) = \lim_{a_i \to 0} \frac{\Gamma_{\boldsymbol{i}}}{a_i}$$

- In Cartesian CS:

$$curl(\boldsymbol{F}) = \hat{\boldsymbol{x}}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\boldsymbol{y}}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{\boldsymbol{z}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

- Use the symbol $\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}}$

$$curl(\boldsymbol{F}) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \nabla \times \boldsymbol{F}$$

Quiz 1

- 1. Prove that the curl of a gradient is always zero.

$$\nabla \times (\nabla f) \equiv 0$$

  – f is a scalar field.

Electric field is the gradient of V, so the curl of electric field holds the property:

$$\nabla \times \boldsymbol{E} = 0$$

- 2. Prove that the divergence of a curl is always zero.

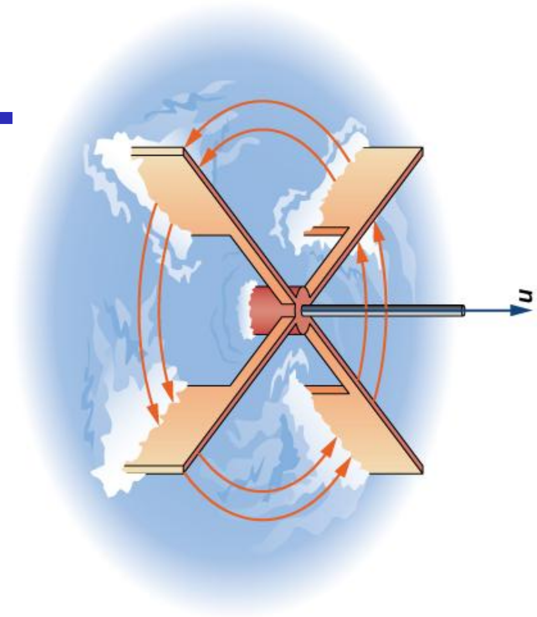
$$\nabla \cdot (\nabla \times \boldsymbol{F}) \equiv 0$$

  – $\boldsymbol{F}$ is a vector field.

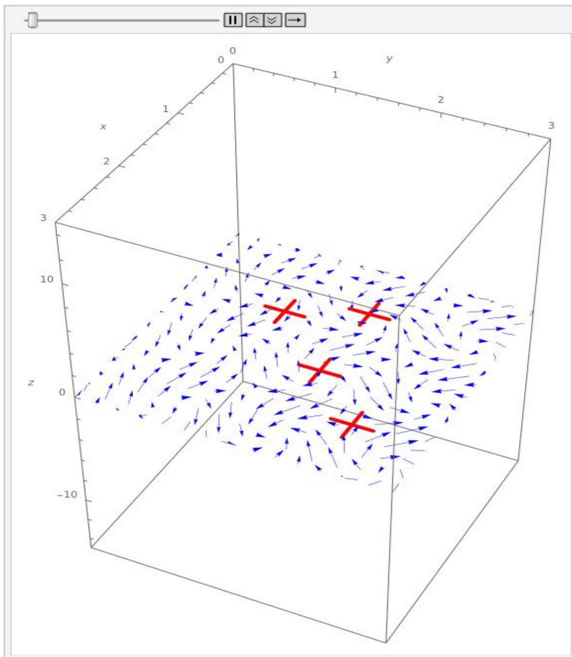
1.2 Curl in three CSs

- Curl in different coordinate systems:
  - Cartesian: $\nabla \times \vec{A}(\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$
  - Cylindrical: $\nabla \times \vec{A}(\vec{r}) = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$
  - Spherical: $\nabla \times \vec{A}(\vec{r}) = \frac{1}{R^2 sin\theta}\begin{vmatrix} \hat{R} & R\hat{\theta} & Rsin\theta\hat{\varphi} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_R & RA_\theta & Rsin\theta A_\varphi \end{vmatrix}$

1.3 Physical Meaning of the Curl



- The curl of a vector allows us to measure the spinning action present in a vector field.
  - Value: the maximum total circulation of per unit area = how strong the rotation is
  - Direction: normal to the area (right-hand rule)



- The curl of a vector field could be interpreted as the angular velocity at any point contained within the given vector field.
- The curl of a vector field is zero => the vector field is said to be ***irrotational***.

$$\nabla \times \boldsymbol{F} = 0$$

1.4 Stoke's Theorem

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{l} = \sum_{i=1}^{N} \Gamma_i = \sum_{i=1}^{N} a_i \boxed{\left(\frac{\Gamma_i}{a_i}\right)}$$

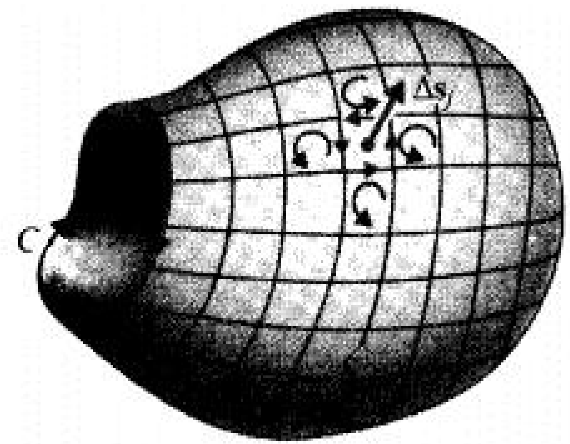
$$(\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}_{\boldsymbol{i}}$$

- So overally:

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{l} = \sum_{i=1}^{N} a_i \left(\frac{\Gamma_i}{a_i}\right) = \sum_{i=1}^{N} (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}_{\boldsymbol{i}} a_i \rightarrow \iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{s}$$

- This is the Stoke's Theorem:

$$\boxed{\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{s}}$$



  – The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

1.5 Electric-field Loop Theorem - Differential Form

- Considering the circulation:

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

- Using the Stoke's Theorem:

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = \iint_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{s} = 0$$

This is true for any surface S bounded by a closed loop C. So, the two integrands must be equal.

- Thus, at any point in space, we have:

$$\boxed{\nabla \times \boldsymbol{E} = 0}$$

Summary: Special Vector Fields

- Conservative Field (保守场): no work is done around a closed path (i.e. the every is conserved)

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

  - In this case, the work done is only related to the initial and final points, not the paths.
  - The field could be expressed as the gradient of a scalar field (有势场).

- Irrotational Field (无旋场): no rotational source of the field

$$\nabla \times \boldsymbol{E} = 0$$

  - According to Stoke's theorem, conservative field = irrotational field.

Summary: Maxwell's Equations for Static Fields

Law	Integral	Differential	Physical meaning
Gauss's law for **E-field**	$\oiint_S \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon}$	$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon}$	Electric flux through a closed surface is proportional to the charges enclosed
E-field Loop Theorem	$\oint_C \vec{E}\cdot d\vec{l} = 0$	$\nabla\times\vec{E} = 0$	Work done by moving a charge in the E-field along a closed loop is 0
Gauss's law for **H-field**	$\oiint_S \vec{B}\cdot d\vec{s} = 0$	$\nabla\cdot\vec{B} = 0$	The total magnetic flux through a closed surface is 0
H-field Loop Theorem	$\oint_C \vec{H}\cdot d\vec{l} = \mathrm{I}$	$\nabla\times\vec{H} = \vec{J}$	The H-field produced by an electric current is proportional to the current

Quiz 2

- 1. Determine whether the vector field **A** is conservative:

$$\boldsymbol{A} = (x^2 y, 2xyz, xy^2)$$

- 2. Find the rotational source of the field **B** at $\left(\frac{\pi}{2}, 0, \frac{\pi}{2}\right)$

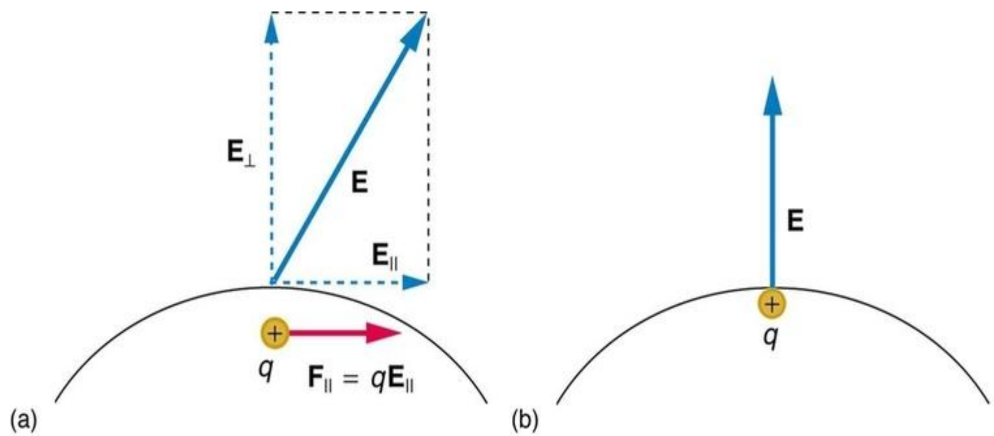
$$\boldsymbol{B} = (\sin x \sin z, \cos y \cos z, \sin x \cos y)$$

2.1 Materials Types **free electrons**

- According to their electrical properties, classify materials into:
  - ***Conductors***: a material that possesses a relatively large number of **free electrons**. The electrons can drift freely in the conductor. Its ability of conducting electric current is described by the conductivity. (Au, Ag, Cu, Al,…..) => Large conductivity σ
  - ***Dielectrics*** (Insulators): a material **without** free electrons in its lattice structure. In an ideal dielectric, positive and negative charges are so sternly bound that they are inseparable. For perfect dielectrics, it has zero conductivity (Rubber, plastic, glass,….) => Tiny conductivity σ
  - ***Semiconductors***: In some special materials such as silicon and germanium, a small fraction of the total number of valence electrons are free to move about randomly with the space lattice. NOT interested in this module.

2.2 Electrostatic Equilibrium

- When excess charge is placed on a conductor or the conductor is put into a static electric field, charges in the conductor quickly respond (redistribute themselves) to reach a steady state called ***electrostatic equilibrium*** (静电平衡).
- Properties:
  1. The **static** electric field intensity at the surface of a conductor is everywhere directed **normal** to that surface.

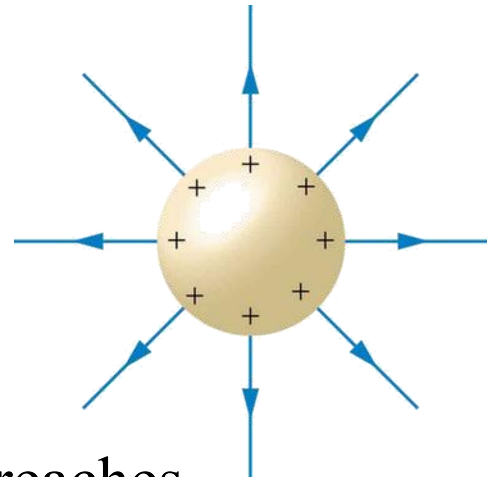


If the E-field is not perpendicular to the boundary, the electron will keep moving => not static

2.2 Electrostatic Equilibrium

- Properties:
  2. The **static** electric field intensity inside a conductor is **zero**.

  Excess charges placed in a conductor repel and move until they are evenly distributed.



  3. Net charge can **only** reside on the surface when reaches electrostatic equilibrium.

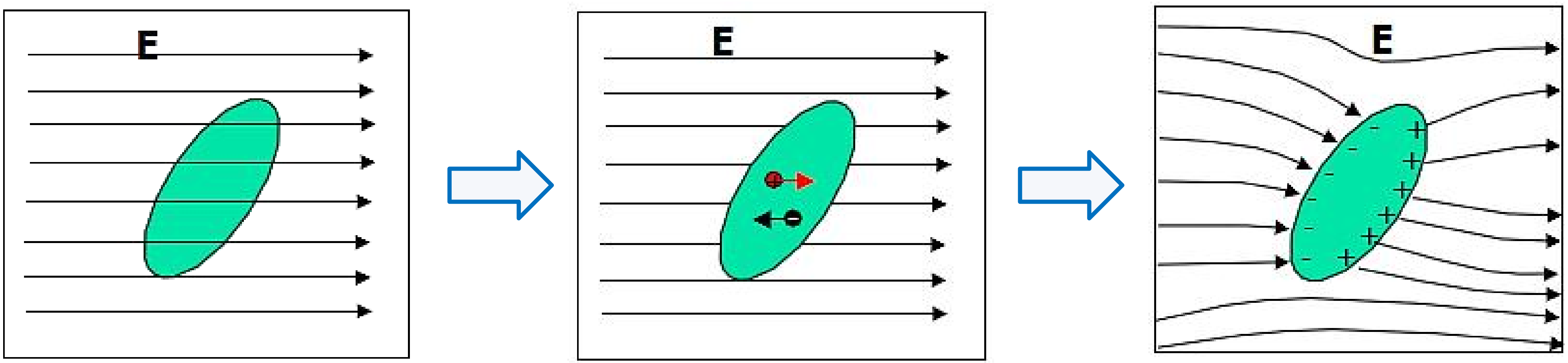
  Excess charge is forced to the surface until the field inside the conductor is zero.

  4. Conductor’s surface is an **equipotential** surface. Actually the whole conductor is an equipotential object.

  If the conductor is not equipotential, i.e. some potential difference exist => there must be electron movement => not “equilibrium”.

2.3 Conductors in Electric field

- When tangential component of electric field intensity $\vec{E}_{tan} \neq 0$, the charges will move from where the potential is higher to where the potential is lower. The moving will stop only when $\vec{E}_{tan} = 0$.



- If there are two points inside the conductor, the potential difference is:

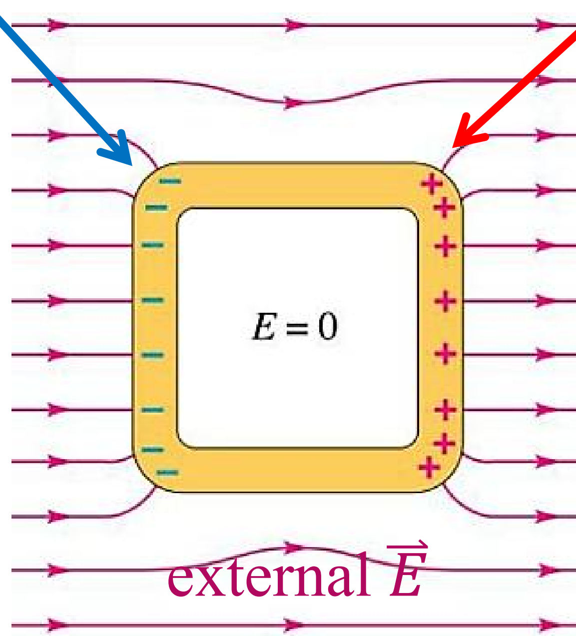
$$V_{AB} = -\int_{B}^{A} \vec{E}_{in} \cdot d\vec{l} = 0$$

2.3 Conductors in Electric field

- If any field line did NOT come it at a 90º angle, a tangential component of the field would be on the surface & electrons would be accelerated within the conductor, move towards the surface, and redistribute on the surface when reaches electrostatic equilibrium.

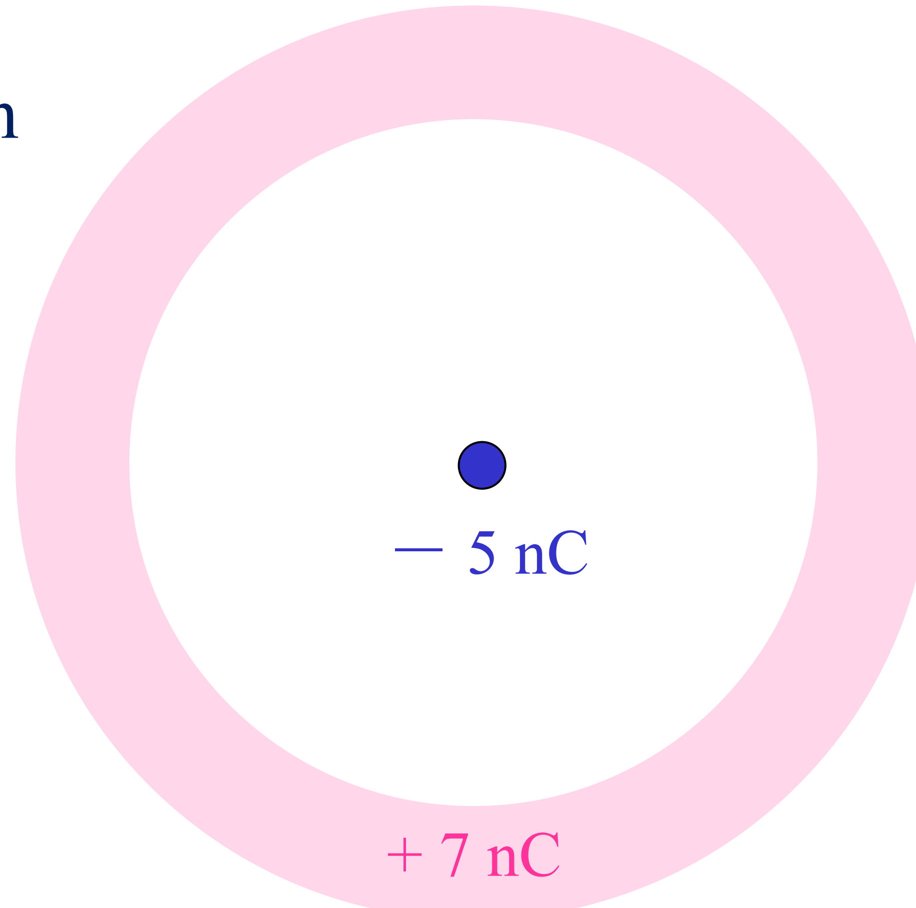
Field pushes electrons towards left side

Net positive charge remains on right side



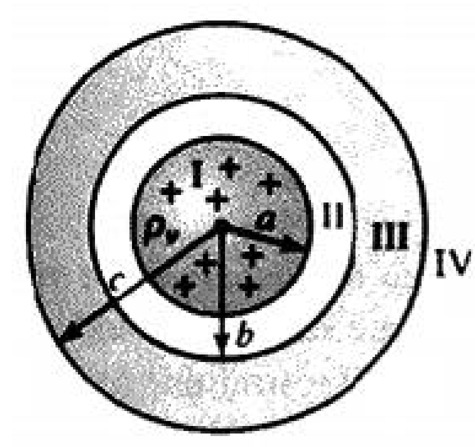
Example

- A solid conductor (pink) with a cavity carries a total charge of +7 nC. Within the cavity, insulated from the conductor, is a point charge of −5 nC.
- How much charge is on each surface (inner and outer) of the conductor?



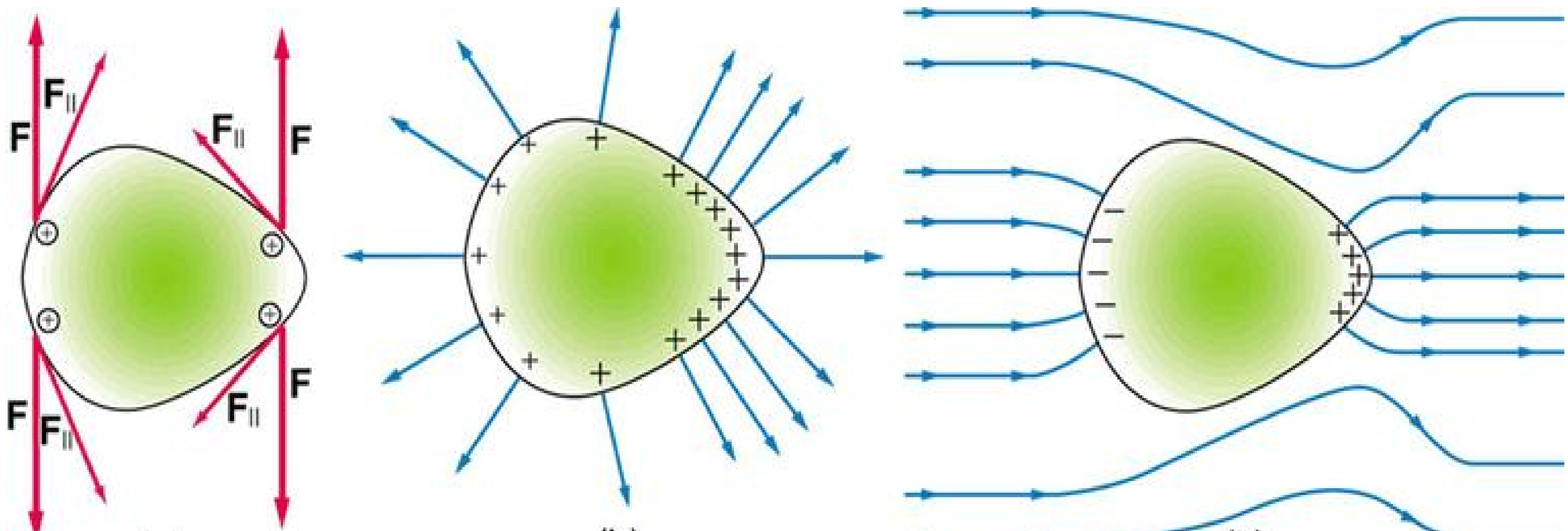
Quiz 3

- Charge is uniformly distributed within a spherical region of radius a. An isolated conducting spherical shell with inner radius b and outer radius c is placed concentrically.
- Determine E everywhere in the region.



2.4 E-fields on Uneven Surfaces

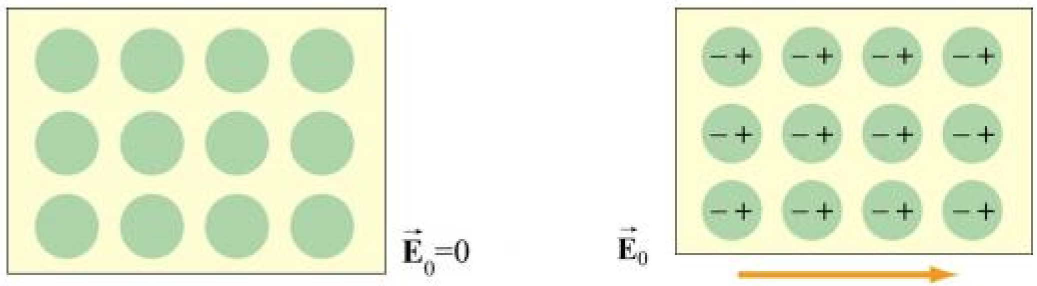
- Excess charges on a nonuniform conductor become concentrated at the sharpest points.
- Additionally, excess charge may move on or off the conductor at the sharpest points.



(a) (b) (c)

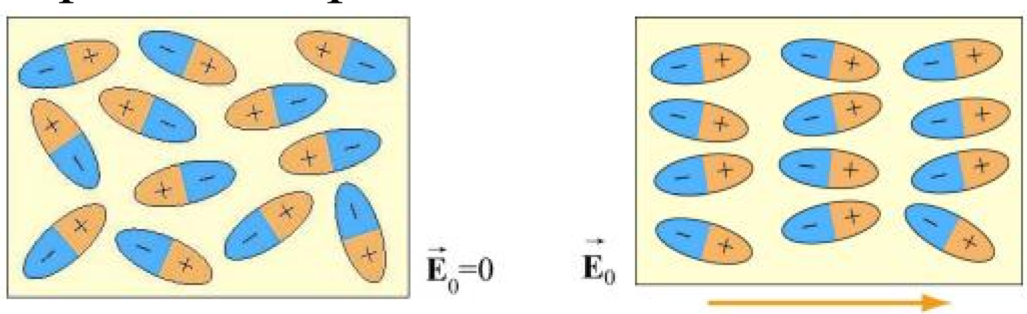
2.5 Dielectrics (介质) / Insulators (绝缘体)

- Under the influence of an electric force, the molecules of a dielectric material experience distortion => being ***polarized***.
  - For non-polar molecules: the center of a positive charge of a molecule no longer coincides with the center of a negative charge



Dipoles

  - For polar molecules, the orientation of polar molecules is random in the absence of an external field. When an external electric field $\underline{\mathbf{E}_0}$ is present, a torque is set up and causes the molecules to align with $\underline{\mathbf{E}_0}$.



Dipoles

2.6 Electric Field in Dielectrics

- The **polarisation** of a dielectric material results in bound charge distributions
  - Different from free charges;
  - Created by separating the charge pairs;
  - Polarization vector $\vec{P} = \lim_{\Delta v \to 0} \frac{\Delta \vec{p}}{\Delta v} = \varepsilon_0 \chi \vec{E}$.
- The electric flux density

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1+\chi)\, \vec{E}$$

$$\boxed{\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}}$$

where ε_0 is the permittivity in vacuum

ε_r is the relative permittivity (unitless)

ε is the permittivity of the medium

2.6 E-field intensity and potential of dielectrics

- In any medium, the electrostatic fields satisfy:

$$\boxed{\begin{gathered}\nabla \times \boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{D} = \rho_v \\ \boldsymbol{D} = \varepsilon \boldsymbol{E}\end{gathered}}$$

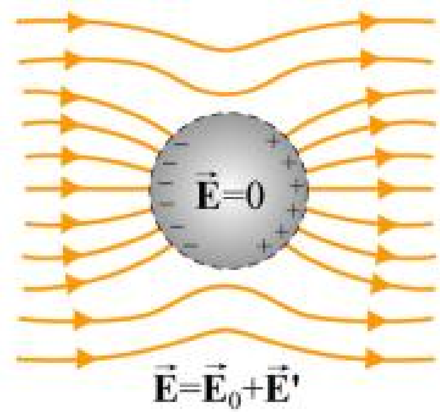
- Recall the E-field Intensity and Potential due to point charge:

In Vacuum (Free space)	In Dielectrics	In Conductors
$\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$	$\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 \varepsilon_r r^2}\hat{r}$	0
$V = \dfrac{Q}{4\pi\varepsilon_0 r}$	$V = \dfrac{Q}{4\pi\varepsilon_0 \varepsilon_r r}$	V_{const}

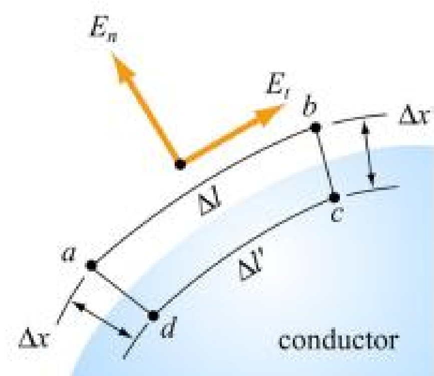
3.1 Boundary Conditions – *E field*

In E-field: conductors

- The basic properties of a conductor:
  - (1) The electric field inside a conductor is zero.
  - (2) Any net charge must reside on the surface of the conductor.
  - (3) The surface of a conductor is an equipotential surface.



  - (4) The tangential component of the electric field on the surface is zero.
  - (5) Just outside the conductor, the electric field is normal to the surface.



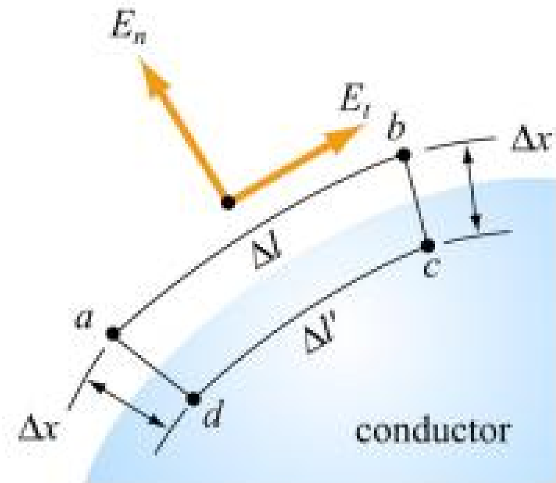
3.1 Boundary Conditions – *E field*

E_t and E_n

- Consider the line integral around a closed path *abcda* in the figure.
  - Since the electric field **E** is conservative, the line integral over the closed path should be 0:

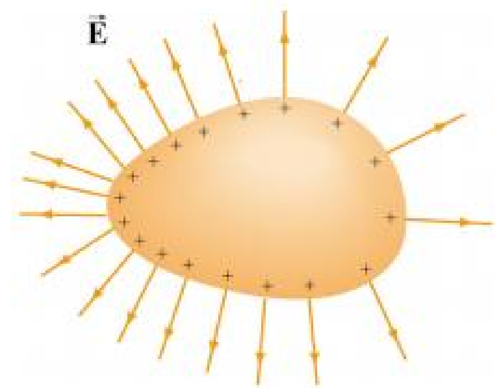
$$E_t(\Delta l) - E_n(\Delta x') + 0(\Delta l') + E_n(\Delta x) = 0$$

  - where E_t and E_n are the tangential and the normal components of the electric field.



  - While approaching to the surface, Δx and $\Delta x'$ → 0. So we must have $E_t\Delta l = 0$. Δl does not need to be 0, so E_t must be 0 for anytime.

  - Since the **E** has no tangential components, $\vec{E} = \hat{n}E_n$, so **E** must be normal to the surface.



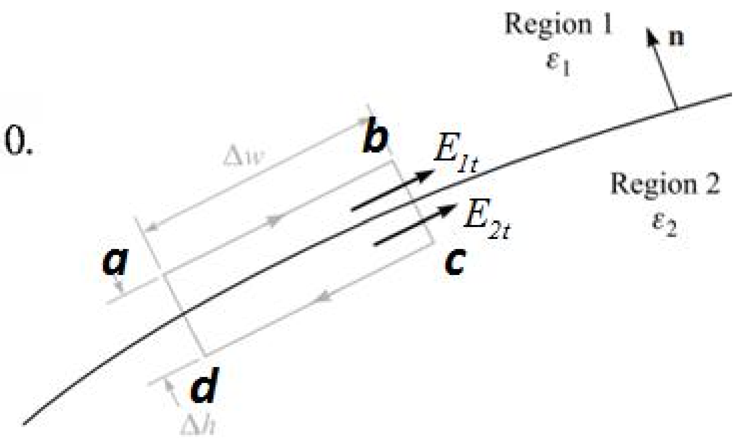
3.2 Boundary conditions – *E field*

In E-field: general materials

- To solve the ***tangential*** component
- Construct a small path *abcda*
  - Sides *ab* and *cd* are parallel to the interface and equal to Δw.
  - Sides *bc* and *da* are penetrating the interface, $bc = da = \Delta h \rightarrow 0$.
  - Therefore:

$$\oint_{abcda} \mathbf{E} \cdot d\boldsymbol{\ell} = \mathbf{E}_1 \cdot \Delta\mathbf{w} + \mathbf{E}_2 \cdot (-\Delta\mathbf{w}) = E_{1t}\Delta w - E_{2t}\Delta w = 0.$$

$$\boxed{E_{1t} = E_{2t}}$$



- If material 2 is conductor

  Inside the conductor, E_{2t}=0, so outside the conductor, E_{1t} also equals to 0.
- If both materials are dielectrics:

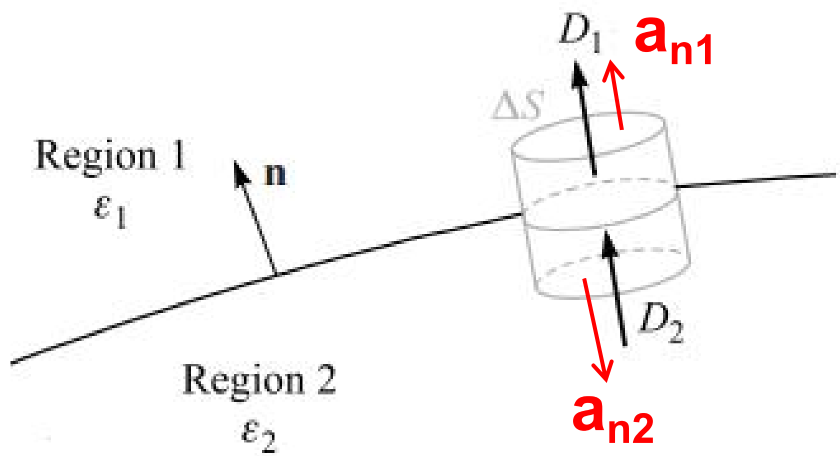
$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

3.2 Boundary conditions – *E field*

In E-field: general materials

- To solve the ***normal*** component
- Construct a small pillbox surface
  - Top and bottom faces are parallel to the interface and equal to *ΔS*.
  - Height $\Delta h \rightarrow 0$.
  - Apply Gauss's Law, get:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = (\mathbf{D}_1 \cdot \mathbf{a}_{n1} + \mathbf{D}_2 \cdot \mathbf{a}_{n2})\Delta S$$
$$= \mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2)\Delta S$$
$$= \rho_s \Delta S,$$



  - Therefore $\boxed{D_{1n} - D_{2n} = \rho_s}$
    - If material 2 is conductor

      Inside the conductor, $D_{2n} = 0$, so outside the conductor, $D_{1n} = \rho_s$.
    - If there is no free charges at the interface, i.e. $\rho_s = 0$, we have

$$D_{1n} = D_{2n} \quad \text{or} \quad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

3.2 Boundary conditions – E field

In E-field: general materials

$$E_{1t} = E_{2t}$$
$$D_{1n} - D_{2n} = \rho_s$$

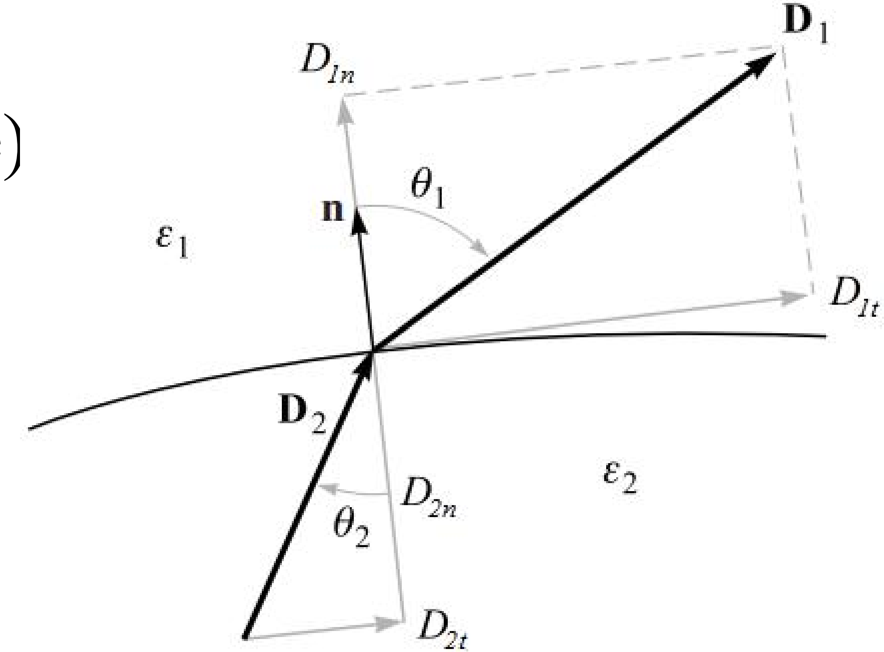
$$\Rightarrow$$

$$\left(\vec{E}_1 - \vec{E}_2\right) \times \hat{n} = 0$$
$$\left(\vec{D}_1 - \vec{D}_2\right) \cdot \hat{n} = \rho_s$$

- The boundary conditions state that:
  - The tangential component of an **E** field is continuous across an interface;
  - The normal component of **D** field is discontinuous across an interface where a surface charge exists – the amount of discontinuity being equal to the surface charge density.

Example: Refraction

- The refraction of the **E** (**D**) field at a dielectric interface (source free)



- Solution:
  - Normal components of **D** are continuous:
    $$D_{1n} = D_1 cos\theta_1 = D_{2n} = D_2 cos\theta_2$$
  - Tangential components of **E** are continuous:
    $$E_{1t} = \frac{D_{1t}}{\varepsilon_1} = \frac{D_1 sin\theta_1}{\varepsilon_1} = E_{2t} = \frac{D_{2t}}{\varepsilon_2} = \frac{D_2 sin\theta_2}{\varepsilon_2}$$
  - Combine the two equations together, remove D_1 and D_2, we get:
    $$\frac{tan\theta_1}{tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

Example: Normal Incident

- Locate a slab of Teflon ($\varepsilon_r = 2.1$) in the region $0 \le x \le a$, and assume free space where $x < 0$ and $x > a$. Outside the Teflon there is a uniform field $\mathbf{E}_{\mathbf{out}} = E_0\mathbf{a}_\mathbf{x}$ V/m
- Find values for **D**, **E**, and **P**.
- Solution:
  - Outside the Teflon slab:

$$\vec{E} = E_0\hat{a}_x$$

$$\vec{D} = \varepsilon E_0\hat{a}_x = \varepsilon_0 E_0\hat{a}_x$$

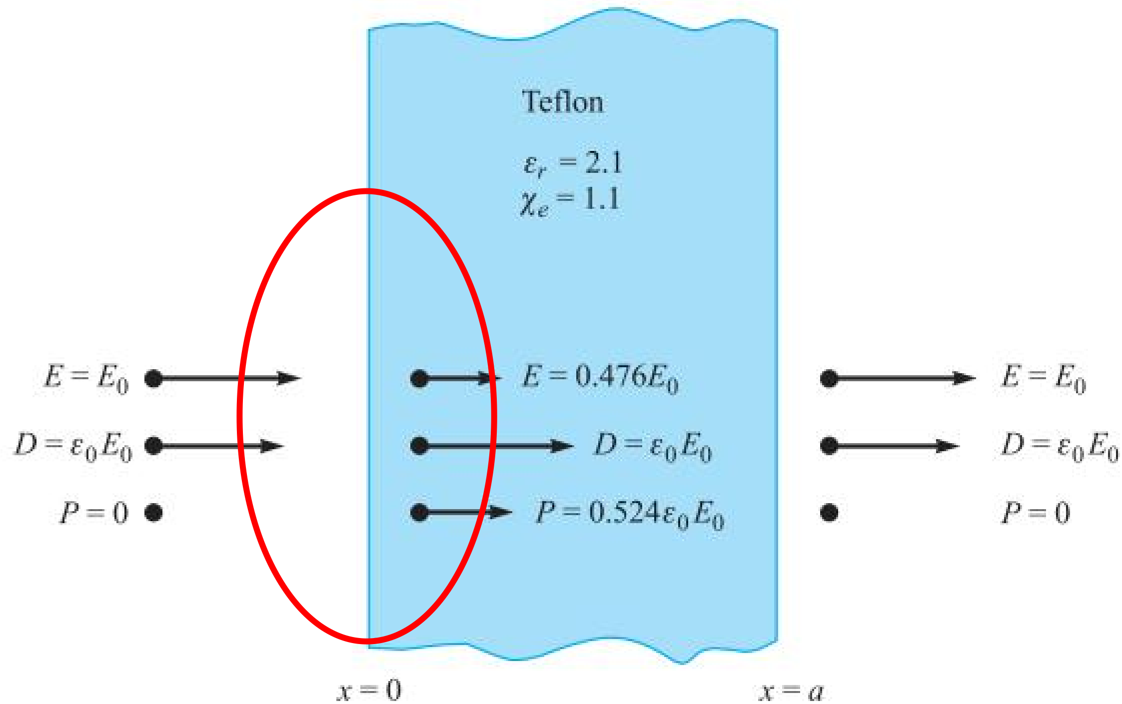
$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = 0$$

  - Inside the Teflon slab:

$$\vec{D}_2 = \vec{D}_1 = \varepsilon E_0\hat{a}_x = \varepsilon_0 E_0\hat{a}_x$$

$$\vec{E}_2 = \frac{\varepsilon_1}{\varepsilon_2}\vec{E}_1 = \frac{1}{2.1}E_0\hat{a}_x = 0.476E_0\hat{a}_x$$

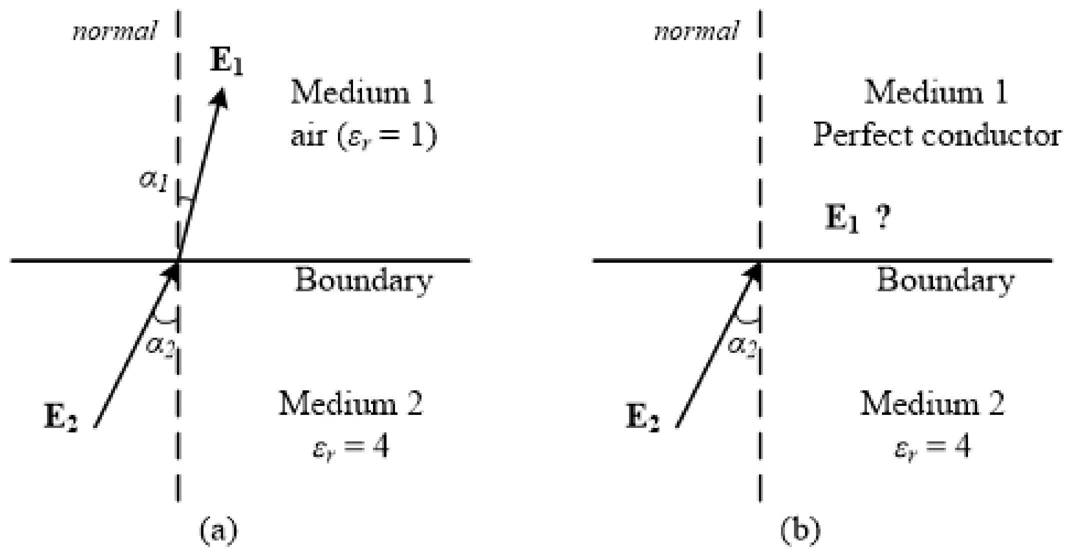
$$\vec{P}_2 = \vec{D}_2 - \varepsilon_0\vec{E}_2 = 0.524\varepsilon_0 E_0\hat{a}_x$$



Normal Incident: no tangential components, so:
$\mathbf{D_1} = \mathbf{D_2}$ and $\varepsilon_1\mathbf{E_1} = \varepsilon_2\mathbf{E_2}$

Quiz 4

- Two isotropic dielectric media are separated by a charge-free plane boundary as shown below. The relative permittivities of the two materials are $\varepsilon_{r1} = 1$ and $\varepsilon_{r2} = 4$.
  - If the angle between the E-field intensity $\mathbf{E_2}$ and normal direction is $\alpha_2 = 30^o$, find the angle between $\mathbf{E_1}$ and the normal direction;
  - If medium 1 is changed to be a perfect conductor as shown in the figure on the right, sketch the electric field lines in medium 1. Explain your answer



Next ...

- Steady current
  - Electric properties
  - As the source of magnetic field

- Resistors and capacitors
  - Structure
  - Calculation