Circuit Analysis in the *s*-Domain

EEE103 ELECTRICAL CIRCUITS I (Part 3)

Week 11&12

S1, 2023/24

Dr. Chee Shen LIM

Figure reference: some figures are obtained from McGraw Hill's Engineering Circuit Analysis (main text book); some are from own drawings.

Week 11

Content

Week 11

- The intuitive concept of "complex frequency"
- One-sided and two-sided Laplace Transform
- Transform pairs, table
- Inverse Laplace Transform
    - Rational and irrational function
    - Partial fraction
- Initial Value Theorem and Final Value Theorem

Week 12

- Relevance of the transformation concept
- Circuit analysis in the *s*-domain
    - *R*, *L*, *C* in the s-domain
    - KCL, KVL
    - Nodal and mesh analyses, source transformation, Thévenin
- Transfer function (introduction)

Introduction

- In Part 2's Week 8 content "Basic RL and RC circuits", you learnt about obtaining the transients of a relatively simpler circuit due to DC switched inputs.

- In Part 3's week 9 content "Sinusoidal Steady-state Analysis", you learnt about obtaining the steady-state behavior of a relatively simpler circuit due to well behaved AC inputs.

- In reality, not all inputs are "well behaved", and sometimes we will need to deal with both steady-state and transient responses caused by "complicated" AC inputs.
    - We need a more "powerful"/advanced tool!

- This chapter will expose you to Laplace transform in circuit analysis.
    - In simple words, the subject is about representing the time-domain system (circuit) and variables into another domain and solve for the solutions.

- Regardless of analytical tools, you should be familiar with **fundamental circuit theories and techniques** like KVL, KCL, mesh and nodal analyses, Thevenin/Norton equivalent circuit, principle of superposition, series/parallel/delta-wye sources/loads, impedances, etc.

Introducing the Concept of "Complex Frequency"

- Consider an exponentially damped sinusoidal function, e.g., voltage:

$$v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)$$

- Three "cases" of function (in this case, voltage v) can be derived:
  - when $\sigma = \omega = 0$, v is a dc voltage, i.e., $v(t) = V_m \cos(\theta) = V_o$
  - when $\omega = 0$, v is an exponential voltage, i.e., $v(t) = V_o e^{\sigma t}$
  - when $\sigma = 0$, v is a sinusoidal voltage, i.e., $v(t) = V_m \cos(\omega t + \theta)$

Introducing the Concept of "Complex Frequency"

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- Previous expressions can be written in the general form of

$$f(t) = \mathbf{K}e^{\mathbf{s}t}$$

where $\mathbf{K}$ and $\mathbf{s}$ are complex constants that are independent of time. $f(t)$ is characterized by the complex frequency $\mathbf{s}$.

- Rewriting the three cases:
  - dc voltage: $v(t) = V_0 e^{(0)t}$ → $\mathbf{s} = 0$
  - exponential voltage: $v(t) = V_0 e^{\sigma t}$ → $\mathbf{s} = \sigma + j0$
  - sinusoidal voltage:

$$v(t) = V_m \cos(\omega t + \theta) = \frac{1}{2} V_m \left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right]$$

$$= \left(\frac{1}{2} V_m e^{j\theta}\right) e^{j\omega t} + \left(\frac{1}{2} V_m e^{-j\theta}\right) e^{-j\omega t}$$

$$= \mathbf{K}_1 e^{\mathbf{s}_1 t} + \mathbf{K}_2 e^{\mathbf{s}_2 t}$$ → "There are 2 complex freq. components"

A sinusoidal waveform has two complex frequencies: $\mathbf{s}_1 = j\omega$, $\mathbf{s}_2 = \mathbf{s}_1^* = -j\omega$; $\mathbf{K}_1 = \frac{1}{2} V_m e^{j\theta}$, $\mathbf{K}_2 = \mathbf{K}_1^* = \frac{1}{2} V_m e^{-j\theta}$

Exponentially Damped Sinusoidal Case

- A more generalized form – an exponentially damped sinusoid - can be deduced:

$$v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)$$
$$= \frac{1}{2} V_m e^{\sigma t} \left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)} \right]$$
$$= \left(\frac{1}{2} V_m e^{j\theta} \right) e^{(\sigma + j\omega)t} + \left(\frac{1}{2} V_m e^{-j\theta} \right) e^{(\sigma - j\omega)t}$$

- The damped sinusoidal waveform has two complex frequencies:

$$\boxed{v(t) = \mathbf{K}_1 e^{\mathbf{s}_1 t} + \mathbf{K}_2 e^{\mathbf{s}_2 t}}$$

$$\mathbf{s}_1 = \sigma + j\omega \qquad \mathbf{K}_1 = \frac{1}{2} V_m e^{j\theta}$$

$$\mathbf{s}_2 = \mathbf{s}_1^* = \sigma - j\omega \qquad \mathbf{K}_2 = \mathbf{K}_1^* = \frac{1}{2} V_m e^{-j\theta}$$

Quick reflection – relate to real world signals

Improve the description, 2023-11-27

- Given a **real-world** signal $f(t)$ [Note: here, f is not frequency!]:

$$f(t) = \mathbf{K}e^{\mathbf{s}t}$$

- Question: if $\mathbf{s}$ is a real number, e.g., $\mathbf{s} = 5 + j0$, should $\mathbf{K}$ be a real number or complex number?
  - $\mathbf{K}$ must be a real number for the signal f to be a "real physical" signal.
- Question: if $\mathbf{s}$ is an imaginary number, e.g., $\mathbf{s} = 0 + j10$, can **K** be a real number?
  - No. If $\mathbf{s} = j10$, $f(t)$ will become a "complex" function when **K** is a real number, not a "real physical" signal (see next page).

Quick reflection – relate to real world signals

- Given another **real-world** signal $f(t)$:

$$f(t) = \mathbf{K}_1 e^{\mathbf{s}_1 t} + \mathbf{K}_2 e^{\mathbf{s}_2 t}$$

- Question: if $\mathbf{s}_1 = j10 = \mathbf{s}_2^*$, and $\mathbf{K}_1 = 6 - j8 = \mathbf{K}_2^*$, what signal do we get?
  - $\mathbf{K}_1 = 10e^{-j(53.1^o\pi/180^o)} = 10\angle -53.1^o$
  - $\mathbf{K}_2 = 10e^{j(53.1^o\pi/180^o)} = 10\angle 53.1^o$

$$10\cos(10t - 53.1^o) \equiv 10\cos\left(10t - \frac{53.1^o \times \pi}{180^o}\right)$$

$$\begin{aligned} f(t) &= 10e^{-j(53.1^o\times\pi/180^o)}e^{j10t} + 10e^{j(53.1^o\times\pi/180^o)}e^{-j10t} \\ &= 10e^{j(10t-53.1^o\times\pi/180^o)} + 10e^{-j(10t-53.1^o\times\pi/180^o)} \\ &= 10\cos(10t - 53.1^o) + j10\sin(10t - 53.1^o) \\ &+ 10\cos(10t - 53.1^o) - j10\sin(10t - 53.1^o) \\ &= 20\cos(10t - 53.1^o) \end{aligned}$$

- We have....a **real physical signal**:

The Two-sided Laplace Transform

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- The two-sided Laplace transform of a function $f(t)$ is defined as

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) \cdot dt$$

where $F(s)$ is the frequency-domain representation of the time-domain waveform $f(t)$.

  - **NOTE:** from now onwards, including <u>handwriting</u>, we will simply write "$\mathbf{F}$" as "F", and "$\mathbf{s}$" as "s". This notation is also adopted in some reference books.

- Correspondingly, there is also an inversed two-sided Laplace transform

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{st} F(s) \cdot ds$$

- In all circuit applications where there is a known time when all signals "appear", we do not make use of the two-sided/bilateral form, but use the one-sided Laplace transform. This is explained next.

The One-sided Laplace Transform

- For time functions that do not exist for $t < 0$ (or whose behavior is of no interest to us), the time-domain description can be thought of as $v(t)u(t)$, where $u(t) = 1$ for $t \geq 0$ but 0 for $t < 0$.

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)u(t) \cdot dt \quad \rightarrow \quad F(s) = \int_{0^-}^{\infty} e^{-st} f(t) \cdot dt$$

"Laplace Transform, or Two-sided Laplace Transform"

"One-sided Laplace Transform"

- This leads to the One-sided Laplace Transform, which from now on will be called simply as "Laplace Transform".
- In short, it means:

$$F(s) = \mathcal{L}\{f(t)\}$$

Example 11.1: LT of a step function

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- Compute the Laplace transform of the function $f(t) = 2u(t)$

Solution:

Step 1: Apply the first-principle transformation

$$F(s) = \int_{0^-}^{\infty} e^{-st} f(t) \cdot dt = \int_{0^-}^{\infty} e^{-st}\, 2u(t) \cdot dt$$

Step 2: Solve it mathematically

$$F(s) = 2\int_{0}^{\infty} e^{-st} \cdot dt = -\frac{2}{s}\left[e^{-st}\right]_0^{\infty} = -\frac{2}{s}\left(0 - e^0\right)$$

$$F(s) = \frac{2}{s}$$

Inverse Laplace Transform

- The inverse Laplace Transform of one-sided transform (and also two-sided transform) is

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{st}\, F(s) \cdot ds$$

- In short, it means:

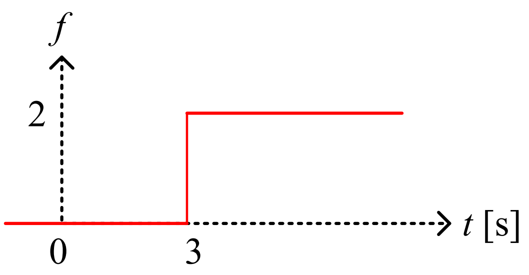
$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

- **Question:** How do you feel about applying those first-principle transformations involving integrals over the limits of "infinity" (and complex number!) in actual circuit analysis?
  - **Good news!** when applying this technique, we never really need to do the integral-based transformation but rely on the use *Transformation table*.

Example 11.2: LT of a delayed step function

- Compute the Laplace transform of the function

$$f(t) = 2u(t-3)$$



Solution:

Step 1: Apply the first-principle transformation

$$F(s) = \int_{0^-}^{\infty} e^{-st} f(t) \cdot dt = \int_{0^-}^{\infty} e^{-st} 2u(t-3) \cdot dt$$

Step 2: Solve mathematically

$$F(s) = 2\int_{3}^{\infty} e^{-st} \cdot dt = -\frac{2}{s}\left[e^{-st}\right]_3^{\infty} = -\frac{2}{s}\left(0 - e^{-3s}\right)$$

$$F(s) = \frac{2}{s} e^{-3s}$$

Intuitive observation: A "delayed" step of 3 seconds results in an additional term of e^{-3s} as compared to Example 12.1

Good news: in actual application to circuit problems, we almost never need to use this basic transformation steps, but rely almost entirely on *Laplace Transform Table/Pairs*.

Laplace Transform of Simple Time Functions

- In circuit analyses, for every time-domain waveform, there exists a one-to-one correspondence of *s*-domain function.
- Common simple time functions that we should know (not just electric circuits, but in most engineering areas):
  - Unit step
  - Unit impulse
  - Sinusoidal function
  - Exponential function
  - Ramp function

Unit step $\boldsymbol{u(t)}$

- The unit step is defined as

$$u(t) = \begin{cases} 1, \text{for } t \geq 0 \\ 0, \text{for } t < 0 \end{cases}$$

- Apply the basic integral transformation:

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} e^{-st} u(t) \cdot dt = \int_{0}^{\infty} e^{-st} \cdot dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

for $\mathrm{Re}(s) > 0$.

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}}$$

Laplace Transform of Simple Time Functions

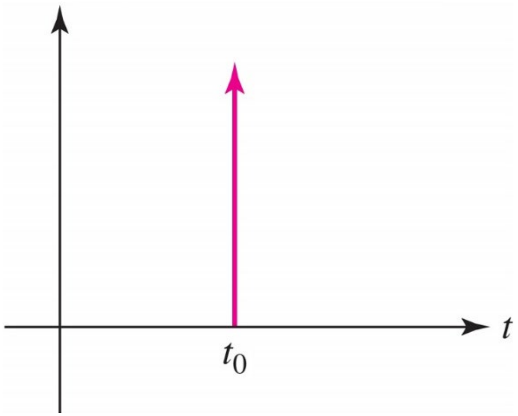
Unit Impulse $\boldsymbol{\delta(t-t_0)}$, $\boldsymbol{\delta(t)}$

Useful to know: $\delta(t) = \dfrac{du(t)}{dt}$

- The unit impulse at time $t = t_0$ is defined as (ε very small)

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t-t_0)\cdot dt = 1$$

$$\delta(t-t_0) = 0,\ t \neq t_0$$



- Apply the basic integral transformation:

$$\mathcal{L}\{\delta(t-t_0)\} = \int_{0^-}^{\infty} e^{-st}\delta(t-t_0)\cdot dt = e^{-st_0}$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

- If $t_0 = 0$ (which means the impulse occurs at $t = 0$), we get:

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

Sinusoidal function

$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$ $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$

- Cosine function:

$$\boxed{\mathcal{L}\{\cos(\omega t)\cdot u(t)\} = \frac{s}{s^2+\omega^2}}$$

NOTE: Derivation is not required in exam. You must know how to know the result.

$$\mathcal{L}\{\cos(\omega t)\cdot u(t)\} = \int_{0^-}^{\infty} e^{-st}\frac{1}{2}\left(e^{j\omega t}+e^{-j\omega t}\right)\cdot dt$$

$$= \frac{1}{2}\left[\int_{0^-}^{\infty} e^{-(s-j\omega)t}\cdot dt + \int_{0^-}^{\infty} e^{-(s+j\omega)t}\cdot dt\right] = \frac{1}{2}\left(\frac{1}{s-j\omega}+\frac{1}{s+j\omega}\right)$$

$$= \frac{1}{2}\left(\frac{2s}{s^2+\omega^2}\right)$$

- Sine function:

$$\boxed{\mathcal{L}\{\sin(\omega t)\cdot u(t)\} = \frac{\omega}{s^2+\omega^2}}$$

- There are many more............
- **Good news:** Derivation of Laplace transformation are not required in EEE103 (and most undergraduate engineering studies). You should know the Laplace Transform table, and how to use them.

Other Important Laplace Transform Pairs

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- **Decaying exponential $e^{-\alpha t}u(t)$:**

$$\boxed{\mathcal{L}\{e^{-\alpha t}u(t)\} = \frac{1}{s+\alpha}}$$

$$\int_{0^-}^{\infty} e^{-st}e^{-\alpha t}u(t)\cdot dt = \frac{1}{s+\alpha}$$

- **Unit ramp function $tu(t)$:**

$$\boxed{\mathcal{L}\{tu(t)\} = \frac{1}{s^2}}$$

- **Ramp exponential function $te^{-\alpha t}u(t)$:**

$$\boxed{\mathcal{L}\{te^{-\alpha t}u(t)\} = \frac{1}{(s+\alpha)^2}}$$

More in Laplace Transform Table

Linearity Theorem

- **Summation of two functions:**

$$
\begin{aligned}
&\mathcal{L}\{f_1(t) + f_2(t)\} \\
&= \int_{0^-}^{\infty} e^{-st}[f_1(t) + f_2(t)] \cdot dt \\
&= \int_{0^-}^{\infty} e^{-st} f_1(t) \cdot dt + \int_{0^-}^{\infty} e^{-st} f_2(t) \cdot dt \\
&= F_1(s) + F_2(s)
\end{aligned}
$$

$$\boxed{\mathcal{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)}$$

- **Constant proportionality:**

$$\mathcal{L}\{kv(t)\} = k\mathcal{L}\{v(t)\} = kV(s)$$

$$\boxed{\mathcal{L}\{kv(t)\} = kV(s)}$$

Summary of Transform Pairs

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$	$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
$\delta(t)$	1	$\frac{1}{\beta - \alpha}(e^{-\alpha t} - e^{-\beta t})u(t)$	$\frac{1}{(s+\alpha)(s+\beta)}$
$u(t)$	$\frac{1}{s}$	$\sin \omega t\, u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$tu(t)$	$\frac{1}{s^2}$	$\cos \omega t\, u(t)$	$\frac{s}{s^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}u(t), n = 1, 2, \ldots$	$\frac{1}{s^n}$	$\sin(\omega t + \theta)\, u(t)$	$\frac{s \sin\theta + \omega\cos\theta}{s^2 + \omega^2}$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\cos(\omega t + \theta)\, u(t)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2 + \omega^2}$
$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$	$e^{-\alpha t}\sin \omega t\, u(t)$	$\frac{\omega}{(s+\alpha)^2 + \omega^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t),\ n = 1,2,\ldots$	$\frac{1}{(s+\alpha)^n}$	$e^{-\alpha t}\cos \omega t\, u(t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \omega^2}$

Example 11.3: Applying LT

(a) Determine $V(s)$ or $I(s)$ of the following functions:

- $v_1(t) = tu(t)$
- $v_2(t) = 4\delta(t-2) - 3tu(t)$
- $v_3(t) = (t-1)u(t-1)$
- $v_4(t) = sin(2t)u(t)$
- $i_5(t) = cos(100t)u(t)$

Answer (use first-principle transformation or the Transform Table):

- $V_1(s) = \frac{1}{s^2}$
- $V_2(s) = 4e^{-2s} - \frac{3}{s^2}$
- $V_3(s) = \frac{e^{-s}}{s^2}$
- $V_4(s) = \frac{2}{s^2+4}$
- $I_5(s) = \frac{s}{s^2+10000}$

Example 11.3: Applying LT

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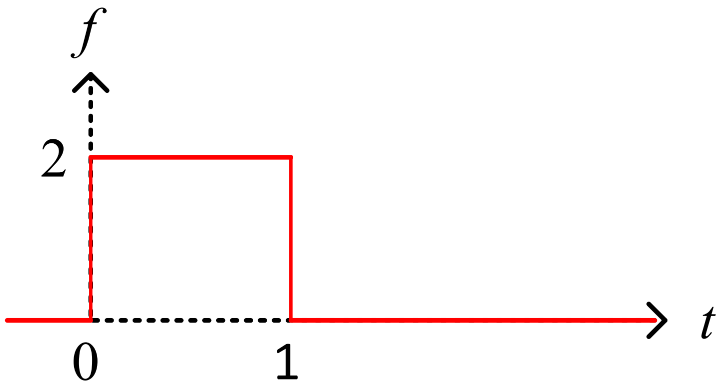
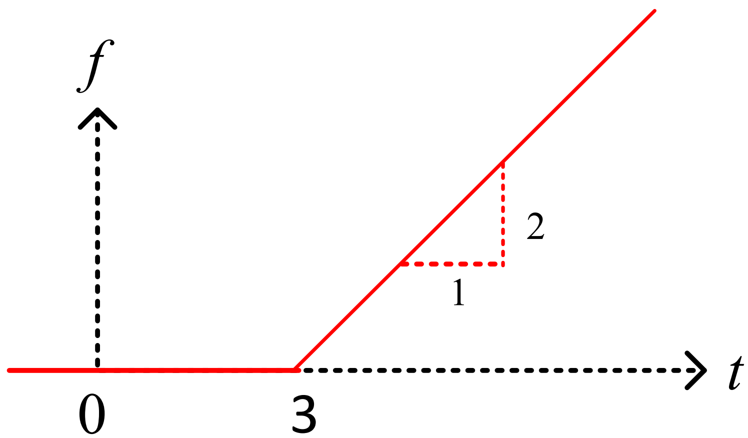
Corrected on 2023-11-27

(b) Define $F(s)$ of the following $f(t)$ waveforms:



$$f(t) = 2u(t) - 2u(t-1)$$

$$F(s) = \frac{2}{s} - \frac{2e^{-s}}{s} = \frac{2(1-e^{-s})}{s}$$



$$f(t) = 2(t-3)u(t-3)$$

$$F(s) = \frac{2e^{-3s}}{s^2}$$

Inverse Transform Techniques for Rational Functions

- Usually towards the last steps operating in s-domain, we will come across the function/expression in a ratio of two *s* polynomial, e.g.:

$$V(s) = \frac{N(s)}{D(s)}$$

- Solutions of s for $N(s) = 0$ are known as **zeros**.
- Solutions of s for $D(s) = 0$ are known as **poles**.
- The first-principle way to "transform" this back to time domain is through the "inverse Laplace transform". Fortunately, in practice, this is almost never needed. Instead, we will make use of the "partial fraction" technique.

Example 11.4: Inverse LT of a **Rational** Function

- Calculate the inverse transform of

$$F(s) = \frac{7}{s} + \frac{31}{(s+17)}$$

Solution:

- **Analysis:** use linearity properties and transform pairs.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{7}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{31}{s+17}\right\} \\ &= 7\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 31\mathcal{L}^{-1}\left\{\frac{1}{s+17}\right\} \\ &= 7u(t) + 31e^{-17t}u(t) \end{aligned}$$

- **Answer:**

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = (7 + 31e^{-17t})u(t)$$

Rational Function – Condition to use Partial Fraction

- For "partial fraction" technique, a condition to be fulfilled before deploying this technique**: the function should be rationale** – it means the degree of numerator *N*(*s*) must be less than that of denominator *D*(*s*).
  - If the function is **irrational**, use long division to get "constant + rational function" (see Example 11.5).

Rational function

$$V_1(s) = \frac{1}{s^2 + 2s + 2}$$

Not a rational function

$$I_1(s) = \frac{s+2}{s+1}$$

Rational function

$$V_1(s) = \frac{s+2}{(s+1)(s+3)}$$

Example 11.5: Inverse LT of **Irrational** Function

- Calculate the inverse transform of

$$F(s) = \frac{2(s+2)}{s}$$

Solution:

- **Analysis:** long division, then use linearity properties and transform pairs.

$$F(s) = 2 + \frac{4}{s} = 2(1) + 4\left(\frac{1}{s}\right)$$

- **Answer:**

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2\delta(t) + 4u(t)$$

Example 11.6: Case I - Distinct Poles

- **Case I:** Distinct Poles (means poles are all different).

$$V(s) = \frac{1}{(s+\alpha)(s+\beta)}$$

$$V(s) = \frac{A}{s+\alpha} + \frac{B}{s+\beta}$$

Note:
1) α, β can be real number or complex number! See next page for examples
2) There can be more than 2 roots/poles.....

$$A = \left.\frac{1}{s+\beta}\right|_{s=-\alpha} = \frac{1}{\beta-\alpha} \text{ and } B = \left.\frac{1}{s+\alpha}\right|_{s=-\beta} = \frac{1}{\alpha-\beta}$$

$$V(s) = \frac{1/(\beta-\alpha)}{s+\alpha} + \frac{1/(\alpha-\beta)}{s+\beta}$$

$$v(t) = \frac{1}{\beta-\alpha}\left(e^{-\alpha t} - e^{-\beta t}\right)u(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+\alpha)(s+\beta)}\right\} = \frac{1}{\beta-\alpha}\left(e^{-\alpha t} - e^{-\beta t}\right)u(t)$$

Case II: Repeated poles

- Occasionally, the transfer function or the output response in *s* terms may have repeated poles.
- Consider the function: $$\boldsymbol{V}(s) = \frac{\boldsymbol{N}(s)}{(s-p)^n}$$
- We can expand it into: $$\boldsymbol{V}(s) = \frac{a_n}{(s-p)^n} + \frac{a_{n-1}}{(s-p)^{n-1}} + \cdots + \frac{a_1}{(s-p)^1}$$

where the constants "$a_{n, (n-1), \ldots 1}$" obtained through:

$$a_n = (s-p)^n \boldsymbol{V}(s)\Big|_{s=p}$$

$$a_{n-1} = \frac{d}{ds}(s-p)^n \boldsymbol{V}(s)\Big|_{s=p}$$

$$a_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}(s-p)^n \boldsymbol{V}(s)\Big|_{s=p}$$

...

$$a_{n-k} = \frac{1}{k!}\frac{d^k}{ds^k}(s-p)^n \boldsymbol{V}(s)\Big|_{s=p}$$

Example 11.7: Case II – Repeated Poles

- Case II: Repeated Poles, for example:

$$V(s) = \frac{2}{s^3 + 12s^2 + 36s}$$

$$V(s) = \frac{2}{s(s+6)^2}$$

$$V(s) = \frac{A}{s} + \frac{B}{(s+6)^2} + \frac{C}{s+6} \quad \text{or} \quad V(s) = \frac{D}{s} + \frac{Es+F}{(s+6)^2}$$

<u>Step 1</u>: Partial fraction

$$A = \left.\frac{2}{(s+6)^2}\right|_{s=0} = \frac{1}{18}$$

$$B = \left.\frac{2}{s}\right|_{s=-6} = -\frac{1}{3}$$

$$C = \left.\frac{d}{ds}\left(\frac{2}{s}\right)\right|_{s=-6} = -\left.\frac{2}{s^2}\right|_{s=-6} = -\frac{1}{18}$$

$$V(s) = \frac{1/18}{s} - \frac{1/3}{(s+6)^2} - \frac{1/18}{s+6}$$

<u>Step 2</u>: Get the time-domain func. v(t)

– Transform Table:

$$\mathcal{L}^{-1}\left\{\frac{1/18}{s}\right\} = \frac{1}{18}u(t)$$

$$\mathcal{L}^{-1}\left\{-\frac{1/18}{(s+6)}\right\} = -\frac{1}{18}e^{-6t}u(t)$$

$$\mathcal{L}^{-1}\left\{-\frac{1/3}{(s+6)^2}\right\} = -\frac{1}{3}te^{-6t}u(t)$$

$$v(t) = \frac{1}{18}[1 - (1+6t)e^{-6t}]u(t)$$

Time Differentiation

$$\int U \cdot dV = UV - \int V \cdot dU$$

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- Time differentiation is very commonly seen in circuit analysis, e.g., $\frac{dv(t)}{dt}$ or $\frac{di(t)}{dt}$.
- Taking $\frac{di}{dt}$ as an example, its Laplace Transform is

$$\mathcal{L}\left\{\frac{di}{dt}\right\} = \int_{0^-}^{\infty} e^{-st}\frac{di}{dt} \cdot dt$$

NOTE: Derivation is not required in exam. You should know how to use the result.

- Using "integral by part":

$$U = e^{-st}, \qquad dV = \frac{di}{dt} \cdot dt$$

$$dU = -se^{-st}dt, \qquad V = i$$

$$\mathcal{L}\left\{\frac{di}{dt}\right\} = i(t)e^{-st}\Big|_{0^-}^{\infty} - \left[\int_{0^-}^{\infty} i(t)(-se^{-st}) \cdot dt\right] = i(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} e^{-st}i(t) \cdot dt$$

$$= 0 - i(0^-) + sI(s)$$

- Using *i*(*t*) and *v*(*t*) as examples, the Laplace Transform for "Time differentiation" are:

$$\boxed{\mathcal{L}\left\{\frac{di}{dt}\right\} = sI(s) - i(0^-)} \qquad \boxed{\mathcal{L}\left\{\frac{dv}{dt}\right\} = sV(s) - v(0^-)}$$

Time-Integration

$\int U \cdot dV = UV - \int V \cdot dU$

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- Time integration is very commonly seen in circuit analysis, e.g., $\int i(t) \cdot dt$ and $\int v(t) \cdot dt$.
- Taking $\int_{0^-}^{t} i(x) \cdot dx$ as an example, its Laplace Transform is

$$\mathcal{L}\left\{\int_{0^-}^{t} i(x) \cdot dx\right\} = \int_{0^-}^{\infty} e^{-st}\left[\int_{0^-}^{t} i(x) \cdot dx\right] \cdot dt$$

NOTE: Derivation is not required in exam. You should know how to use the result.

- Using "integral by part":

$$U = \int_{0^-}^{t} i(x) \cdot dx, \qquad dV = e^{-st} \cdot dt$$

$$dU = i(t) \cdot dt, \qquad V = -\frac{1}{s}e^{-st}$$

$$\mathcal{L}\left\{\frac{di}{dt}\right\} = \left[\left(\int_{0^-}^{t} i(x) \cdot dx\right)\left(-\frac{1}{s}e^{-st}\right)\right]_{0^-}^{\infty} - \int_{0^-}^{\infty}\left(-\frac{1}{s}e^{-st}\right) i(t) \cdot dt$$

$$= \left[-\frac{1}{s}e^{-st}\int_{0^-}^{t} i(x) \cdot dx\right]_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} e^{-st}\, i(t) \cdot dt$$

$$= 0 + \frac{I(s)}{s}$$

- Using *i*(*t*) and *v*(*t*) as examples, the Laplace Transform for "Time integration" are:

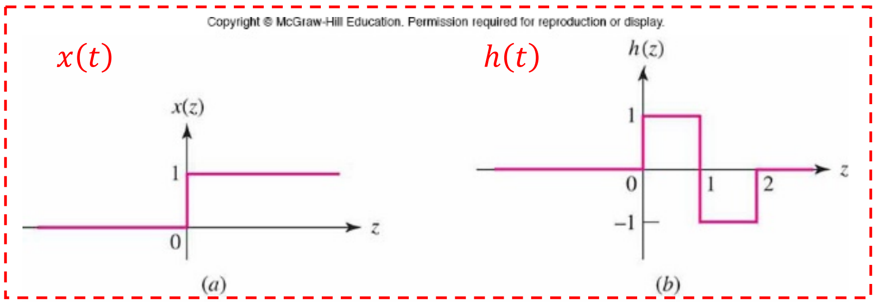
$$\boxed{\mathcal{L}\left\{\int_{0^-}^{t} i(x) \cdot dx\right\} = \frac{I\,(s)}{s}} \qquad \boxed{\mathcal{L}\left\{\int_{0^-}^{t} v(x) \cdot dx\right\} = \frac{V\,(s)}{s}}$$

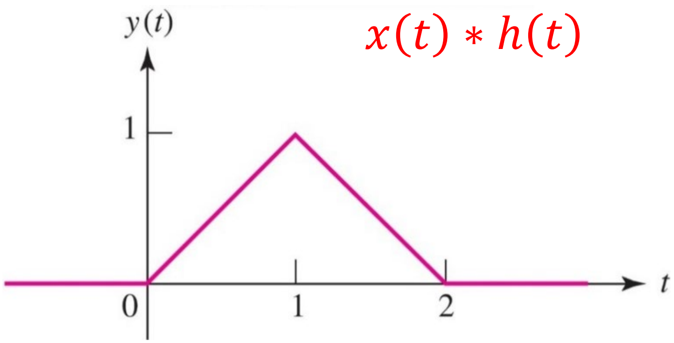
Convolution (1)

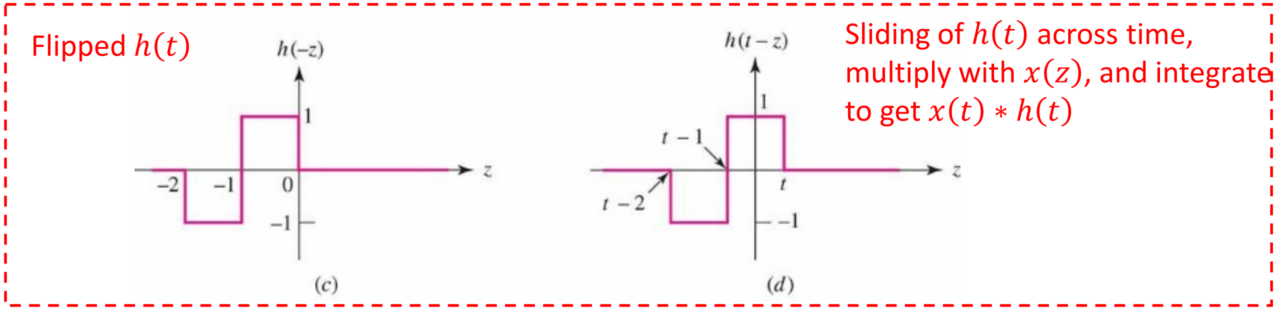
- If we define the impulse response of a system *N* as *h*(*t*), then the output *y*(*t*) is related to the input *x*(*t*) via the convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(z)h(t-z) \cdot dz = \int_{-\infty}^{\infty} x(t-z)h(z) \cdot dz$$

- Given the $x(t)$ and $h(t)$, by graphically flip/slide/integrate, we can establish $y(t)$ below.





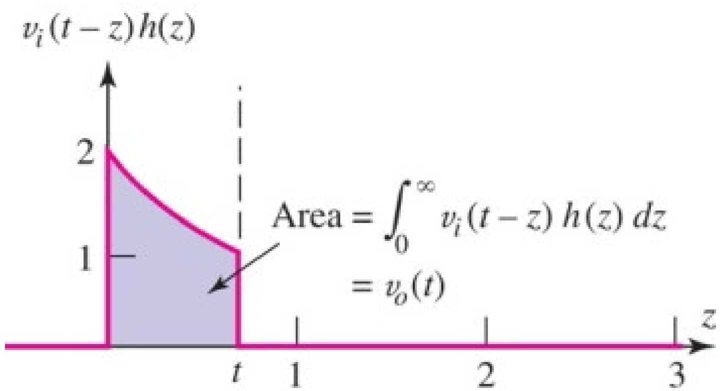


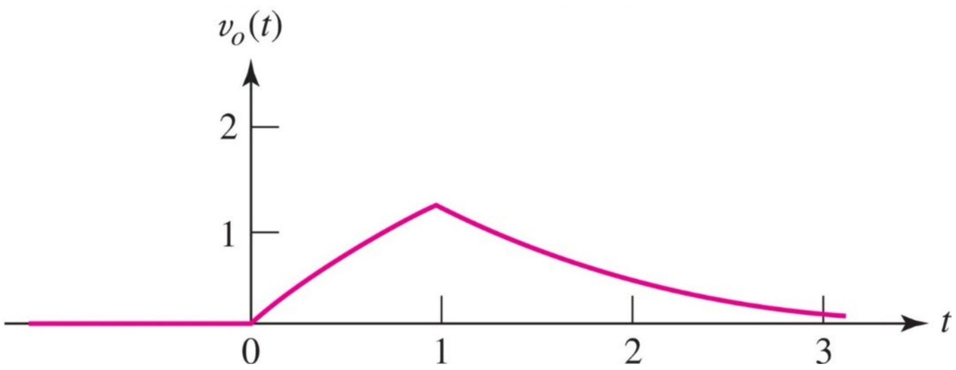
Convolution (2)

- For example, if $x(t) = v_i(t) = u(t) - u(t-1)$ and $h(t) = 2e^{-t}u(t)$, then by flip/slide/integrate, we find can sketch the $v_o(t)$ as follows:

NOTE: This page is mainly "for your information". Not required in exam.

$$v_o(t) = x(t) * h(t)$$





- **Highlight:** Convolution in the time domain is "multiplication" in the frequency domain (proof in Textbook, 10th ed., pg. 591):

$$\boxed{\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s) \cdot F_2(s)}$$

Time-Shift Theorem

- We have seen some hints about how the delay in time is represented in the s-domain. Here is a formal derivation to prove it:
- Give a delayed signal $f(t-a)u(t-a)$, means time delay of a seconds, apply the first-principle Laplace transformation:

$$\mathcal{L}[f(t-a)u(t-a)] = \int_{0^-}^{\infty} e^{-st} f(t-a)u(t-a) \cdot dt$$
$$= \int_{a^-}^{\infty} e^{-st} f(t-a) \cdot dt$$

With a change of variable: $\tau = t - a$, means $d\tau = dt$

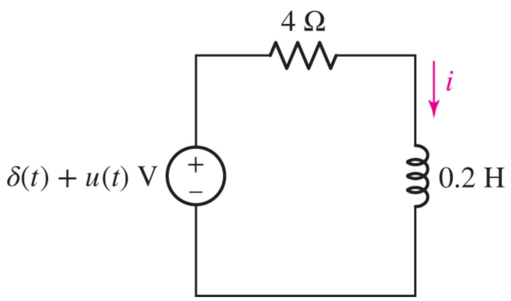
$$\mathcal{L}[f(t-a)u(t-a)] = \int_{0^-}^{\infty} e^{-s(\tau+a)} f(\tau) \cdot d\tau = e^{-as} \int_{0^-}^{\infty} e^{-s\tau} f(\tau) \cdot d\tau$$

$$\boxed{\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)}$$

Example 11.8: RL circuit (from rest)

- Use Laplace Transform to find $i(t)$ in the circuit.

Initial circuit/inductor current before imposing any input voltage, $i(0^-)$, is zero.



$R = 4\ \Omega$, $L = 0.2$ H

Solution:

- <u>Step 1</u>: transform input voltage, form circuit equation.

$$v_s(t) = \delta(t) + u(t)$$

$$V_s(s) = \mathcal{L}\{v_s(t)\} = 1 + \frac{1}{s}$$

$$v_s = Ri + L\frac{di}{dt}$$

$$\begin{aligned} V_s(s) &= RI(s) + L[sI(s) - i(0^-)] \\ &= RI(s) + sLI(s) \end{aligned}$$

- <u>Step 2</u>: obtain $I(s)$.

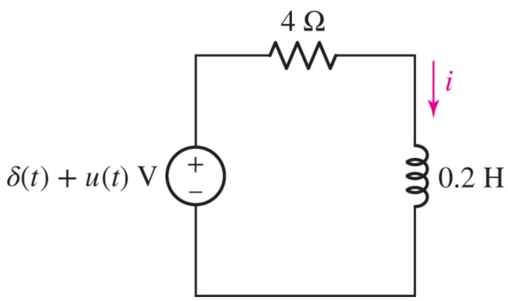
$$1 + \frac{1}{s} = 4I(s) + s(0.2)I(s)$$

$$\frac{5(s+1)}{s} = 20I(s) + sI(s)$$

$$I(s) = \frac{5(s+1)}{s(s+20)}$$

- Step 3: obtain $i(t)$ from $I(s)$.

$$I(s) = \frac{5(s+1)}{s(s+20)} = \frac{A}{s} + \frac{B}{s+20}$$



$R = 4\ \Omega,\ L = 0.2\ \mathrm{H}$

- Apply partial fraction technique:

$$A = \left.\frac{5(s+1)}{(s+20)}\right|_{s=0} = 0.25$$

$$B = \left.\frac{5(s+1)}{s}\right|_{s=-20} = 4.75$$

- We get

$$I(s) = \frac{0.25}{s} + \frac{4.75}{s+20}$$

$$i(t) = 0.25u(t) + 4.75e^{-20t}u(t) = (0.25 + 4.75e^{-20t})u(t)$$

Summary of Transform Operations

Operation	$f(t)$	F(s)
Addition	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(\mathbf{s}) \pm \mathbf{F}_2(\mathbf{s})$
Scalar multiplication	$kf(t)$	$k\mathbf{F}(\mathbf{s})$
Time differentiation	$\frac{df}{dt}$	$\mathbf{sF}(\mathbf{s}) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$\mathbf{s}^2\mathbf{F}(\mathbf{s}) - \mathbf{s}f(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$\mathbf{s}^3\mathbf{F}(\mathbf{s}) - \mathbf{s}^2f(0^-) - \mathbf{s}f'(0^-) - f''(0^-)$
Time integration	$\int_{0^-}^{t} f(t)dt$	$\frac{1}{\mathbf{s}}\mathbf{F}(\mathbf{s})$
	$\int_{-\infty}^{t} f(t)dt$	$\frac{1}{\mathbf{s}}F(\mathbf{s}) + \frac{1}{\mathbf{s}}\int_{-\infty}^{0^-} f(t)dt$
Convolution	$f_1(t) * f_2(t)$	$\mathbf{F}_1(\mathbf{s})\mathbf{F}_2(\mathbf{s})$
Time shift	$f(t-a)u(t-a),\ a \geq 0$	$e^{-a\mathbf{s}}\mathbf{F}(\mathbf{s})$
Frequency shift	$f(t)e^{-at}$	$\mathbf{F}(\mathbf{s} + a)$
Frequency differentiation	$tf(t)$	$-\frac{d\mathbf{F}(\mathbf{s})}{d\mathbf{s}}$
Frequency integration	$\frac{f(t)}{t}$	$\int_{\mathbf{s}}^{\infty} \mathbf{F}(\mathbf{s})\, d\mathbf{s}$
Scaling	$f(at),\ a \geq 0$	$\frac{1}{a}\mathbf{F}\left(\frac{\mathbf{s}}{a}\right)$
Initial value	$f(0^+)$	$\lim_{\mathbf{s}\to\infty} \mathbf{sF}(\mathbf{s})$
Final value	$f(\infty)$	$\lim_{\mathbf{s}\to 0} \mathbf{sF}(\mathbf{s})$, all poles of $\mathbf{sF}(\mathbf{s})$ in LHP
Time periodicity	$f(t) = f(t + nT)$, $n = 1, 2, \ldots$	$\frac{1}{1 - e^{-T\mathbf{s}}}\mathbf{F}_1(\mathbf{s})$, where $\mathbf{F}_1(\mathbf{s}) = \int_{0^-}^{T} f(t)e^{-\mathbf{s}t}\, dt$

Initial Value Theorem and Final Value Theorem

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- In some circuit problems, the main interest of is about knowing the initial and steady-state final values, e.g., voltage/current, of the circuit.
- Instead of going through the full process, there are two handy/useful theorems that operate primarily only in the *s*-domain:
  - Initial Value Theorem (IVT):

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} [sF(s)]$$

Note: Derivation of these theorems are not needed in EEE103, but you can still read them in the Textbook.

  - Final Value Theorem (FVT):

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} [sF(s)]$$

- **NOTE:** FVT cannot be applied to the system with poles on the imaginary axis or the right half plane (i.e., poles of zero or positive real parts). These systems have perpetual oscillation or be infinite, hence has no well-defined final value. **Don't worry about this – you will understand more about this in higher level learning.**

Example 11.9: Use of IVT or FVT

Use the final-value theorem to determine $f(\infty)$ for the function

$$f(t) = (1 - e^{-at})u(t)$$

where $a > 0$.

Solution:

$$F(s) = \frac{1}{s} - \frac{1}{s+a} = \frac{a}{s(s+a)}$$

FVT: $$\lim_{s \to 0} sF(s) = \lim_{s \to 0} \left[\frac{as}{s(s+a)}\right] = \lim_{s \to 0} \left[\frac{a}{s+a}\right] = 1$$

Checking: $$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} (1 - e^{-at})u(t) = 1$$

Example 11.10: Two distinct poles

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

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Real roots/poles

$$V_1(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \left[(s+1)\frac{1}{(s+1)(s+2)}\right]\bigg|_{s=-1} = \frac{1}{(s+2)}\bigg|_{s=-1} = 1$$

$$B = \left[(s+2)\frac{1}{(s+1)(s+2)}\right]\bigg|_{s=-2} = \frac{1}{(s+1)}\bigg|_{s=-2} = -1$$

$$V_1(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$v_1(t) = (e^{-t} - e^{-2t})u(t)$$

This example further illustrates Example 11.6 in page 27 of this ppt..

Complex roots/poles

$$V_2(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1-j)(s+1+j)} = \frac{A}{(s+1-j)} + \frac{B}{(s+1+j)}$$

$$A = \left[(s+1-j)\frac{1}{(s+1-j)(s+1+j)}\right]\bigg|_{s=-(1-j)} = \frac{1}{(s+1+j)}\bigg|_{s=-(1-j)} = \frac{1}{2j}$$

$$B = \left[(s+1+j)\frac{1}{(s+1-j)(s+1+j)}\right]\bigg|_{s=-(1+j)} = \frac{1}{(s+1-j)}\bigg|_{s=-(1+j)} = -\frac{1}{2j}$$

$$V_2(s) = \frac{1/2j}{s+1-j} - \frac{1/2j}{s+1+j}$$

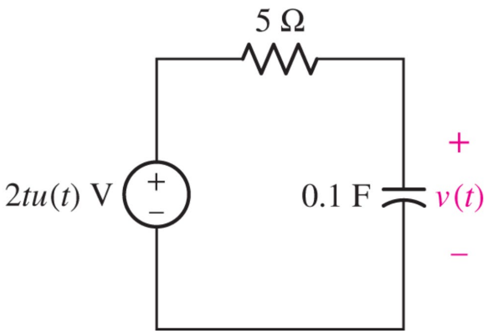
$$v_2(t) = \frac{1}{j2}\left(e^{-(1-j)t} - e^{-(1+j)t}\right)u(t)$$

$$= e^{-t}\left(\frac{e^{jt} - e^{-jt}}{j2}\right)u(t) = e^{-t}\sin t \cdot u(t)$$

Exercise 11.1: RC circuit (from rest)

- Use Laplace Transform to find $v(t)$ in the circuit.

Initial capacitor voltage before imposing any input volt $v(0^-)$, is zero.



$R = 5\ \Omega,\ C = 0.1\ \text{F}$

Solution:

- **Step 1**: transform input voltage, form circuit equation.

$$v_s(t) = 2tu(t)$$
$$V_s(s) = \frac{2}{s^2}$$

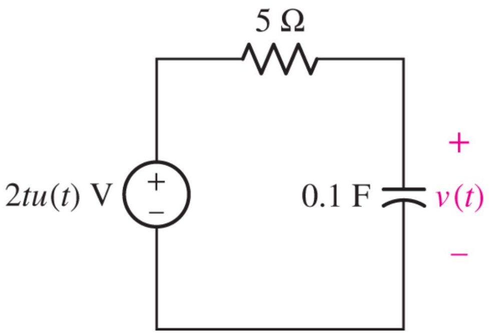
$$\frac{v_s - v}{R} = C\frac{dv}{dt}$$
$$\frac{V_s - V}{R} = C[sV - v(0^-)]$$
$$V_s = (1 + sRC)V$$

- **Step 2**: obtain $I(s)$.

$$\frac{2}{s^2} = (1 + s(5)(0.1))V$$
$$\frac{4}{s^2} = (2 + s)V$$
$$V(s) = \frac{4}{s^2(s+2)}$$

- **Step 3**: obtain $v(t)$ from $V(s)$.

$$V(s) = \frac{4}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2}$$



$R = 5\ \Omega,\ C = 0.1\ \mathrm{F}$

- Apply partial fraction technique:

$$A = \left.\frac{4}{(s+2)}\right|_{s=0} = 2$$

$$B = \left.\frac{d}{ds}\frac{4}{(s+2)}\right|_{s=0} = -\left.\frac{4}{(s+2)^2}\right|_{s=0} = -1$$

$$C = \left.\frac{4}{s^2}\right|_{s=-2} = 1$$

- We get $V(s) = \frac{2}{s^2} - \frac{1}{s} + \frac{1}{s+2}$
- Inverse transform to time-domain:

$$v(t) = 2tu(t) - u(t) + e^{-2t}u(t) = [(2t-1) + e^{-2t}]u(t)$$

Exercise 11.2: IVT and FVT

- Given $F(s)=\dfrac{1}{s(s+2)}$

- Find the initial value using IVT:

$$\lim_{s\to\infty} sF(s)=\lim_{s\to\infty} s\left[\frac{1}{s(s+2)}\right]$$
$$=\lim_{s\to\infty}\left[\frac{1}{s+2}\right]$$
$$=0$$

$$f(t)=0.5(1-e^{-2t})u(t)$$
$$\lim_{t\to 0} f(t)=0$$

- Find the final value using FVT:

$$\lim_{s\to 0} sF(s)=\lim_{s\to 0} s\left[\frac{1}{s(s+2)}\right]$$
$$=0.5$$

$$f(t)=0.5(1-e^{-2t})u(t)$$
$$\lim_{t\to\infty} f(t)=0.5$$

In-Class Quiz 3

- **Week 11**

(1) Please proceed to your tutorial session for In-class Quiz 3. Please only go to your own room!

Group 2	(Tutorial*)	(SA136*, SB152*, SB120*)	(continue to 12noon-1pm)

***NOTES:** The tutorial rooms are allocated according to your programmes. Please attend to the assigned session BUT NOT other rooms to avoid overcrowding. **Attendance of tutorials will be taken.**

SA136 – ~~CST and DMT students~~ CST and **EE** students (updated on 13th Sept. 2023)

SB152 – ~~EE and EST students~~ **DMT** and EST students (updated on 13th Sept. 2023)

SB120 – MRS and TE students

(2) In-class Quiz 3 (10 questions, mix MCQ + fill-in-the-blank questions, 40 mins)

Week 12

Continue from Week 11's Chapter on

"Circuit Analysis in the *s*-Domain"

EEE103 ELECTRICAL CIRCUITS I (Part 3)
Week 11&12
S1, 2023/24

Dr. Chee Shen LIM

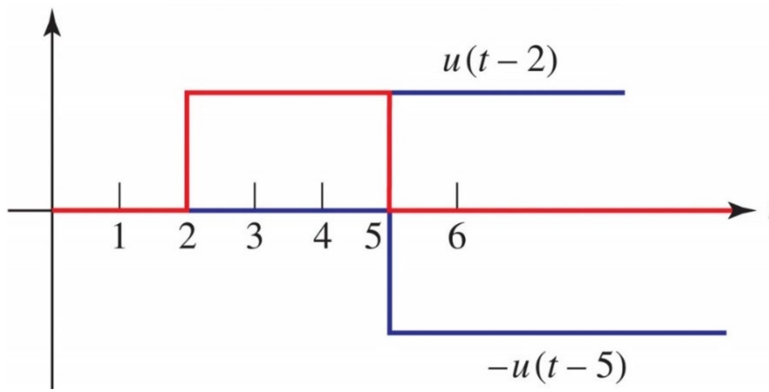
Circuit Analysis in the *s*-domain

- Irregular waveform (pulse, exponentially decayed sinusoidal, multiple frequencies, etc.) can be represented rather conveniently in the *s*-domain.
- We will now see the how s-domain is useful in dealing with irregular waveforms.
- Then, we will move on to applying s-domain to previously learnt circuit analysis techniques, namely
  - Nodal and mesh analyses
  - Source transformation
  - Thévenin/Norton equivalent circuit
  - Etc.

Example 12.1: Pulse waveform

- Obtain the Laplace transform of the rectangular pulse of the time-domain equation:

$$v(t) = u(t-2) - u(t-5)$$



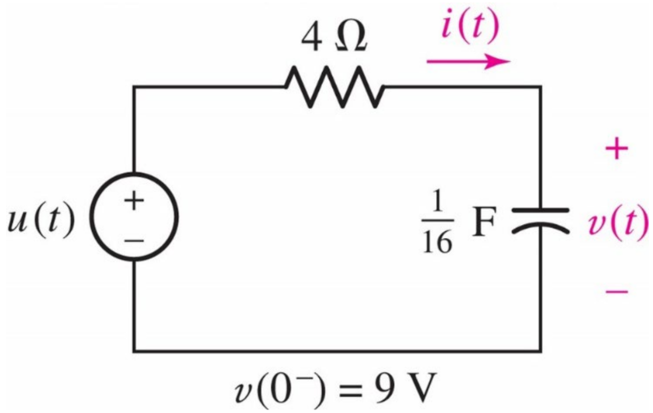
Solution:

$$V(s) = \mathcal{L}[u(t-2) - u(t-5)]$$

$$V(s) = \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-5s} = \frac{e^{-2s} - e^{-5s}}{s}$$

Example 12.2: Using LT to Solve Simple Circuit Problem

- Given that $u(t)$ is a unit step input at time zero, determine $i(t)$ and $v(t)$ for $t > 0$ in the series RC circuit shown:



$$u = Ri + \frac{1}{C}\int i \cdot dt$$

Analysis:

- Input is a step DC voltage source of magnitude 1.
- Initial capacitor voltage is given, 9 V.
- Use KVL, then transform to s-domain, solve, and inverse transform back to time domain

Method 1: start with KCL

Solution: Write KCL equation at the node before the capacitor

Circuit voltage $v(t)$ expression

$$\frac{u(t)-v(t)}{R}=C\frac{dv(t)}{dt}$$

$$\frac{U(s)-V(s)}{R}=C[sV(s)-v(0^-)]$$

$$V(s)=\frac{U(s)+RCv(0^-)}{1+sRC}=\frac{1/s+9RC}{1+sRC}=\frac{4+9s}{s^2+4s}=\frac{1}{s}+\frac{8}{s+4}$$

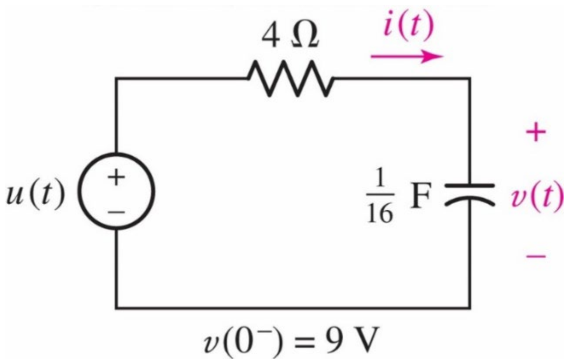
$$v(t)=(1+8e^{-4t})u(t)\ \text{V}$$

Circuit current $i(t)$ expression - extra

$$i(t)=C\frac{dv(t)}{dt}$$

$$i(t)=\frac{1}{16}\frac{d}{dt}[1+8e^{-4t}]$$

$$i(t)=-2e^{-4t}u(t)\ \text{A}$$



Method 2: start with KVL

Solution: write KVL equation around the loop

<u>Circuit current $i(t)$ expression</u>

$$u(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(x)\cdot dx$$

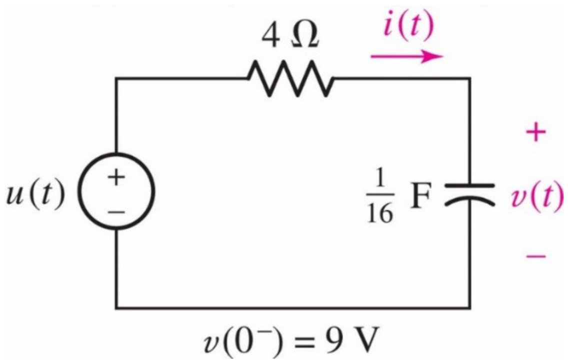
$$u(t) = Ri(t) + \left[\frac{1}{C}\int_{-\infty}^{0^-} i(x)\cdot dx + \frac{1}{C}\int_{0^-}^{t} i(x)\cdot dx\right] = Ri(t) + \left[v(0^-) + \frac{1}{C}\int_{0^-}^{t} i(x)\cdot dx\right]$$

$$u(t) = 4i(t) + \left[9u(t) + \frac{1}{(1/16)}\int_{0^-}^{t} i(x)\cdot dx\right]$$

$$\frac{1}{s} = 4I(s) + \frac{9}{s} + \frac{16I(s)}{s}$$

$$I(s) = \frac{1/s - 9/s}{4 + 16/s} = \frac{-8}{4s+16} = -\frac{2}{s+4}$$

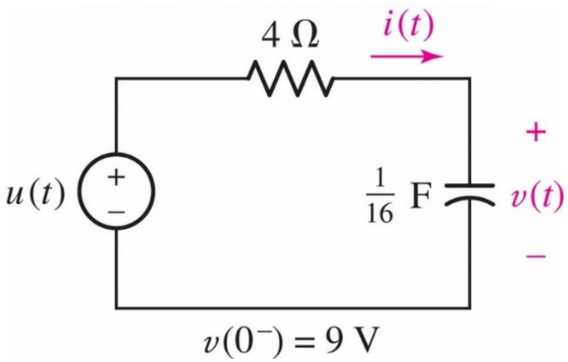
$$i(t) = -2e^{-4t}u(t)\ \mathrm{A}$$



Capacitor voltage $v(t)$ expression

$$i(t) = C\frac{dv(t)}{dt} = -2e^{-4t}\ \mathrm{A}$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\cdot dt$$
$$= \frac{1}{C}\int_{0^-}^{t} -2e^{-4t}dt + \frac{1}{C}\int_{-\infty}^{0^-} -2e^{-4t}dt$$
$$= \frac{1}{C}\int_{0^-}^{t} -2e^{-4t}dt + v(0^-)$$
$$=16\left[\frac{1}{2}e^{-4t}\right]_{0^-}^{t} + 9$$
$$= (8e^{-4t} - 8 + 9)$$
$$= (1 + 8e^{-4t})u(t)\ \mathrm{V}$$



Resistors in the Frequency Domain

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- Ohm's law specifies that

$$v(t) = Ri(t)$$

- Taking the Laplace transform of both sides

$$\boxed{V(s) = RI(s)}$$

- The impedance $Z(s)$ is defined as

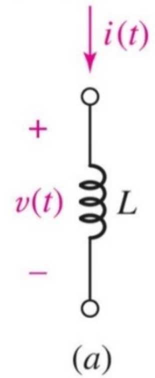
$$Z(s) = \frac{V(s)}{I(s)} = R$$

- The admittance

$$Y(s) = \frac{I(s)}{V(s)} = \frac{1}{R}$$

Inductors in the s-domain

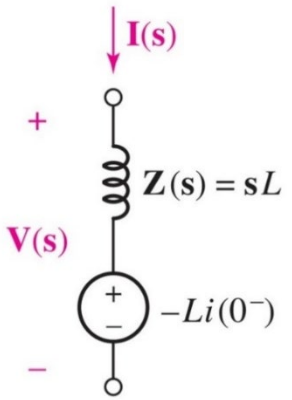
- Given a simple inductor, in figure *(a)*, we have $v(t) = L\frac{di(t)}{dt}$. its Laplace transform gives

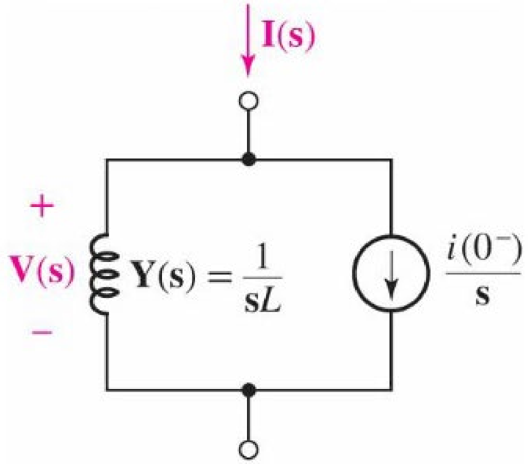


$$V(s) = L[sI(s) - i(0^{-})]$$

$$\boxed{V(s) = sLI(s) - Li(0^{-})}$$

- If initial condition **is zero**, the impedance is $Z(s) = sL$.
- If the initial condition is **NOT zero**, the initial conditions can be modeled as (either one)
  - **a voltage source in series**
    - **Preferred, because of "consistency in current…"**
  - a current source in parallel





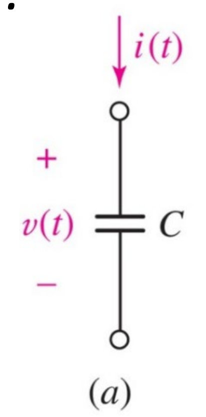
$$\frac{V(s)}{sL} = I(s) - \frac{i(0^{-})}{s}$$

Capacitors in the s-domain

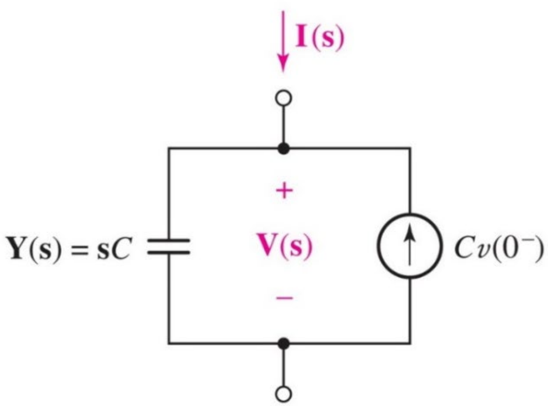
- Given a simple capacitor, in figure (a), we have $i(t) = C\frac{dv(t)}{dt}$.

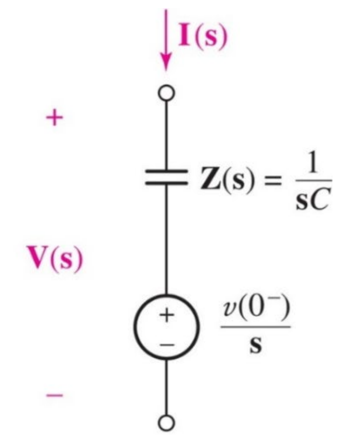
$$I(s) = C[sV(s) - v(0^-)]$$

$$\boxed{I(s) = sCV(s) - Cv(0^-)}$$



- If initial condition **is zero**, the impedance is $Z(s) = \frac{1}{sC}$.
- If the initial condition is **NOT zero**, then the initial condition can be modelled as (either one):
  - a voltage source in series
  - **a current source in parallel**
    - **Preferred, because of "consistency in voltage…"**





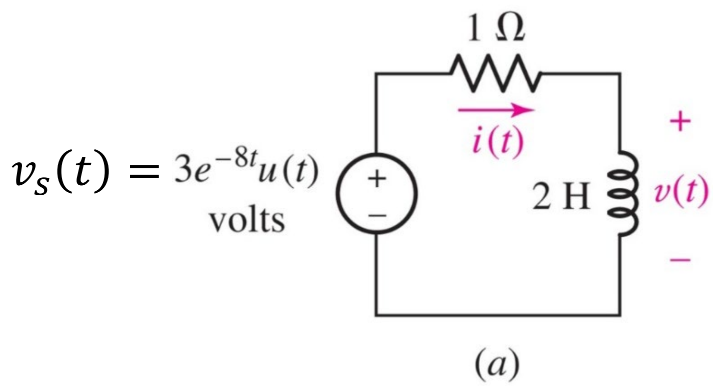
$$\frac{I(s)}{sC} + \frac{v(0^-)}{s} = V(s)$$

Example 12.3: Inductors

$V_L(s) = sLI_L(s) - Li_L(0^-)$

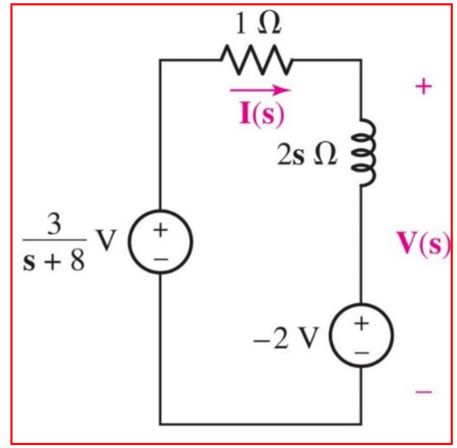
- Refer to figure (a), find the voltage $v(t)$ given that the initial current $i(0^-) = 1$ A.

$$v_s(t) = 3e^{-8t}u(t)$$



(a)

Take note on how to set/label the "signs" of *v* and *i* across *L*



Solution:

Step 1: Move to s-domain, get $V_s(s)$.

Step 2: Write KVL (of the transformed circuit), and get $I(s)$

$$\frac{3}{s+8} - (-2) = RI(s) + sLI(s)$$

$$\frac{3}{s+8} + 2 = (1)I(s) + (2)sI(s)$$

$$I(s) = \frac{2s+19}{(1+2s)(s+8)}$$

Step 3: Find $V(s)$

$$V(s) = L[sI(s) - i(0^-)] = \frac{2s(2s+19)}{(1+2s)(s+8)} - 2 = \frac{2s-8}{(s+0.5)(s+8)}$$

Step 4: Solve through partial fraction

$$V(s) = \frac{A}{s+0.5} + \frac{B}{s+8} = -\frac{1.2}{s+0.5} + \frac{3.2}{s+8}$$

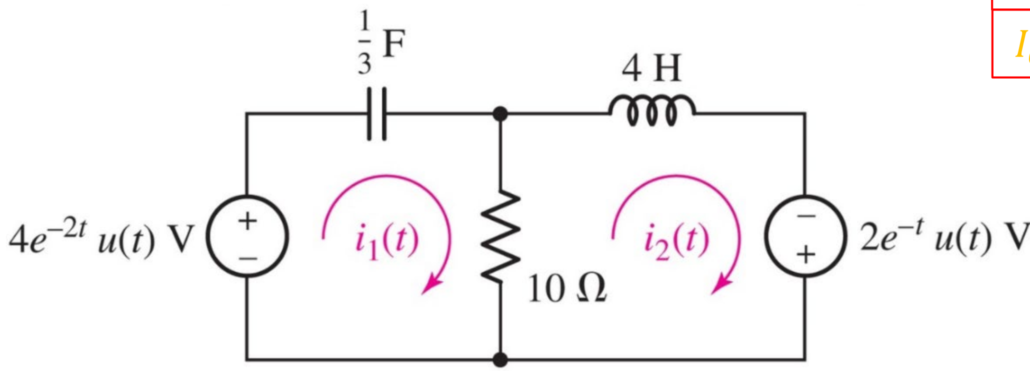
$$v(t) = [3.2e^{-8t} - 1.2e^{-0.5t}]u(t)\ \mathrm{V}$$

Example 12.4: Mesh Analysis in the *s*-domain

- Determine the mesh currents $i_1(t)$ and $i_2(t)$. Assume no energy initially stored in the circuit.

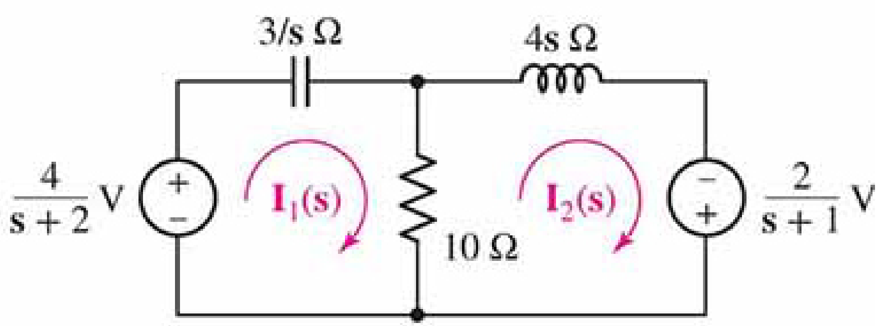
$$V_L(s) = sLI_L(s) - Li_L(0^-)$$

$$I_C(s) = sCV_C(s) - Cv_C(0^-)$$

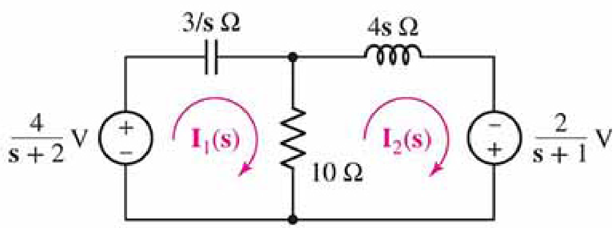


Solution:

Step 1: draw the equivalent circuit in *s*-domain. Consider the initial condition in the process (in this example, initial conditions are zero).



Step 2: write the two mesh equations



Mesh 1: $\frac{4}{s+8} = \frac{3}{s}I_1 + 10(I_1 - I_2)$

Mesh 2: $\frac{2}{s+1} = 10(I_2 - I_1) + 4sI_2$

Step 3: solve the simultaneous equations of I_1 and I_2, then inverse transform to get i_1 and i_2.

Manipulate them further:

$$4s = (3 + 10s)(s + 8)I_1 - 10s(s + 8)I_2$$

$$I_2 = \frac{(3 + 10s)(s + 8)I_1 - 4s}{10s(s + 8)}$$

...

... (some problems can be quite complicated to handle manually)

$$I_1 = \frac{2s(4s^2 + 19s + 20)}{20s^4 + 66s^3 + 73s^2 + 57s + 30}$$

$$I_2 = \frac{30s^2 + 43s + 6}{20s^4 + 66s^3 + 73s^2 + 57s + 30}$$

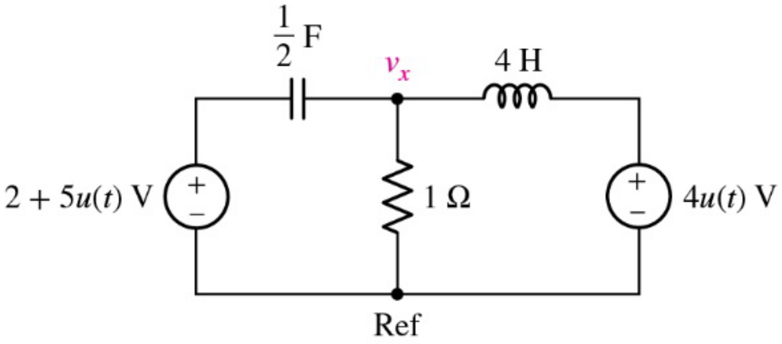
Answer: (use a computer software to calculate)

$$i_1(t) = -96.39e^{-2t} - 344.8e^{-t} + 841.2e^{-0.15t}\cos(0.8529t) + 197.7e^{-0.15t}\sin(0.8529t)\ \text{mA}$$

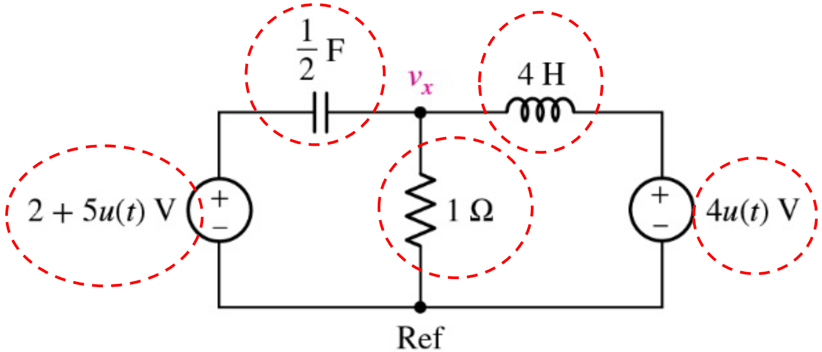
$$i_2(t) = -481.9e^{-2t} - 241.4e^{-t} + 723.3e^{-0.15t}\cos(0.8529t) + 472.8e^{-0.15t}\sin(0.8529t)\ \text{mA}$$

Example 12.5: Nodal Analysis in the *s*-domain

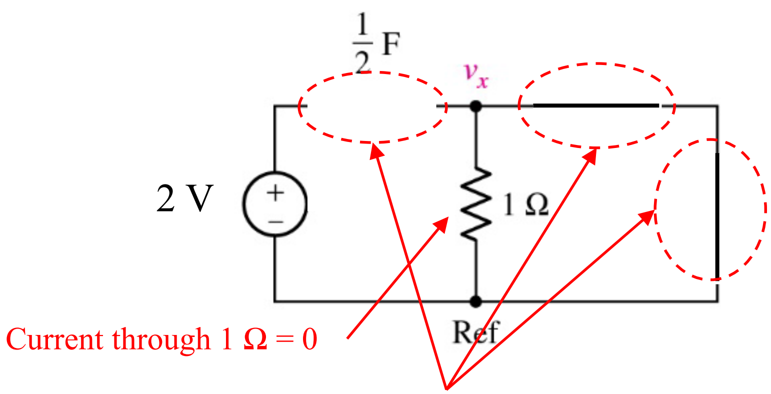
- The capacitor has an initial voltage of 2V (positive end at the source side), determine the node voltage $v_x(t)$ expression in the circuit.



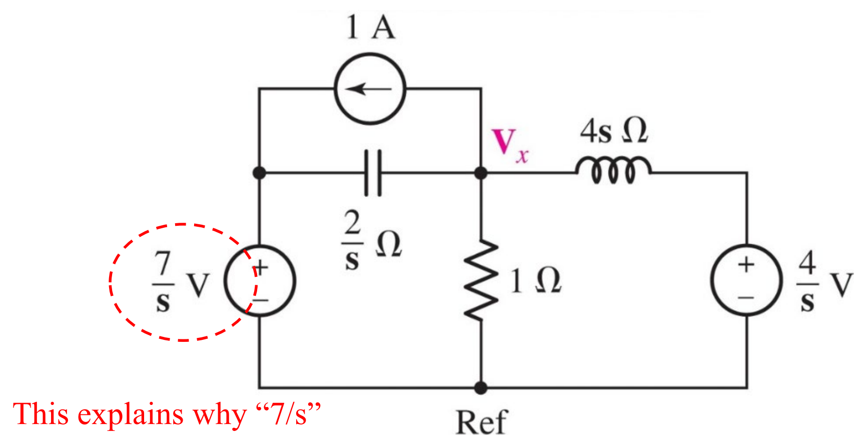
Example 12.5: Interpret the circuit operating conditions



At time before 0 s, i.e., $t \leq 0$ s: *DC circuit behavior*



At time after 0 s, i.e., $t > 0$ s:



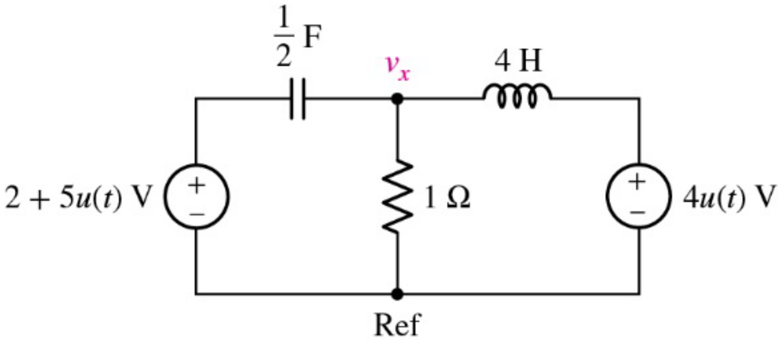
- What behavior shall depend on the circuit elements (L or C; current or voltage source)

Example 13.5 (continue)

$V_L(s) = sLI_L(s) - Li_L(0^-)$

$I_C(s) = sCV_C(s) - Cv_C(0^-)$

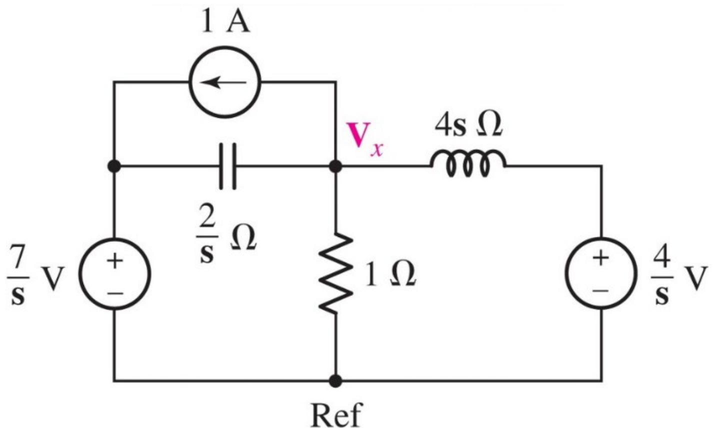
- The capacitor has an initial voltage of 2V (positive end at the source side), determine the node voltage $v_x(t)$ expression in the circuit.



Solution:

Analysis: Two independent sources, initial capacitor voltage. Probably most convenient to get v_x through nodal analysis.

Step 1: Convert to s-domain



Step 2: Write nodal equation.

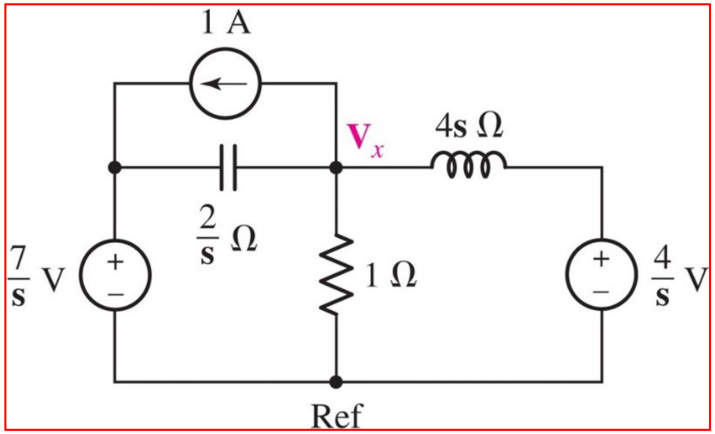
Node x: $0 = 1 + \frac{V_x - 7/s}{2/s} + \frac{V_x}{1} + \frac{V_x - 4/s}{4s}$

Step 3: Solve for $v_x(t)$

$$V_x = \frac{10s^2 + 4}{s(2s^2 + 4s + 1)} = \frac{5s^2 + 2}{s(s + 1.707)(s + 0.2929)}$$

$$= \frac{4}{s} + \frac{6.864}{s + 1.707} - \frac{5.864}{s + 0.2929}$$

$$v_x = [4 + 6.864e^{-1.707t} - 5.864e^{-0.2929t}]u(t)$$

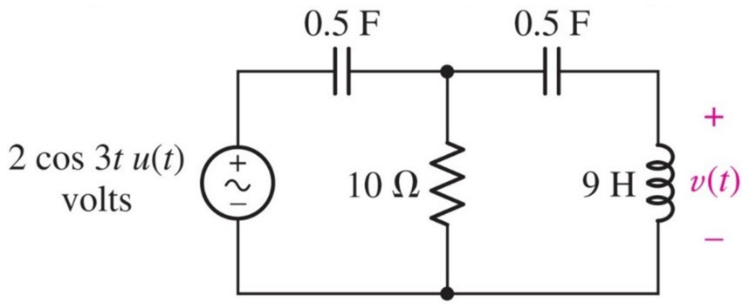


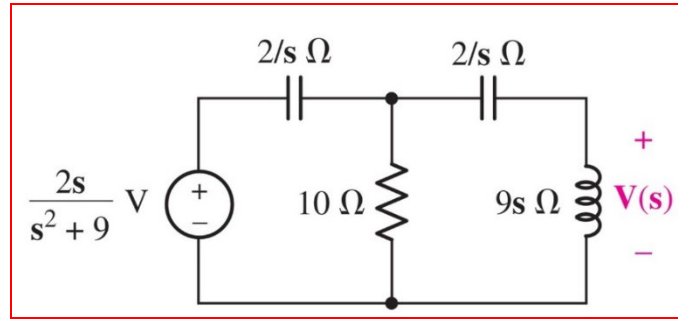
Answer:

$v_x(t) = [4 + 6.864e^{-1.707t} - 5.864e^{-0.2929t}]u(t)$ V

Example 12.6: Source Trans. in the *s*-domain

- Assume zero initial condition, find the expression of inductor voltage $v(t)$ using Norton equivalent circuit transformation (and a computer).





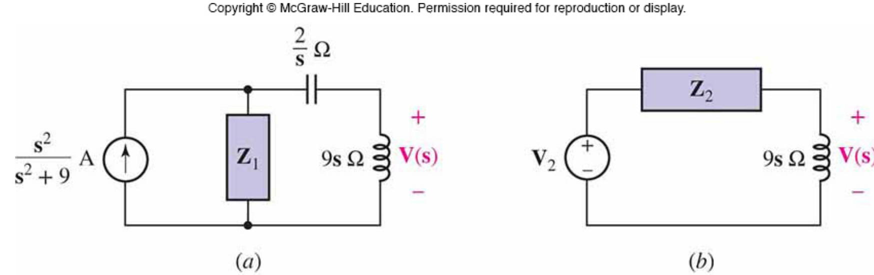
Solution:

Analysis: AC independent source, zero initial capacitor voltages. Question asks for Norton. If not specified, can also use Thévenin, or probably also mesh analysis.

Step 1: Convert to *s*-domain.

Step 2: Norton equivalent circuit, then further manipulate to get $V(s)$.

Step 3: Inverse transform to get $v(t)$.



Answer:

$$v(t) = \begin{bmatrix} 5.590\times10^{-5}e^{-0.1023t} + 2.098\cos\left(3t+3.912^\circ\right) \\ +0.1017e^{-0.04885t}\cos\left(0.6573t+157.9^\circ\right) \end{bmatrix} u(t)$$

You may refer to the detailed solution in Example 14.16 of the textbook, on pg. 580.

Example 12.7: Find Thévenin Equiv. in the *s*-domain

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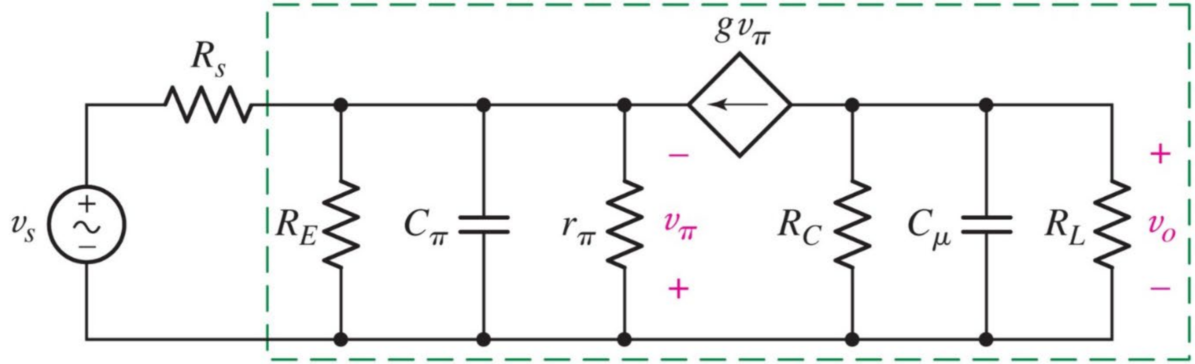
- This is a "hybrid π" model for a common base amplifier. Determine the frequency-domain Thévenin equivalent of the highlighted network:



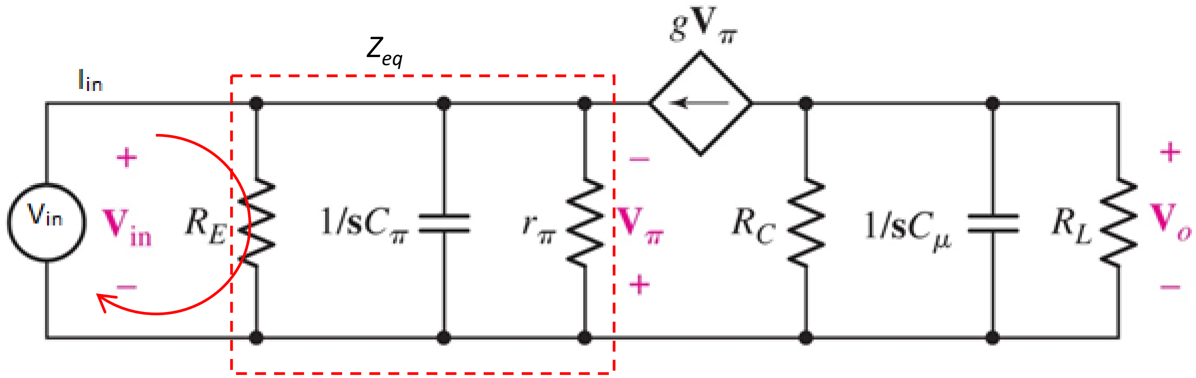
Analysis:

- V_s is an AC source. This AC source "sees" the green network as "a load", with impedance known commonly as *input impedance*.

Method 1: To find input impedance, we can insert a test voltage source V_{in}, form equations using KVL/KCL/others, and find Z_{in} (=V_{in}/I_{in}).

Method 2: To find input impedance, we can insert a test current source I_{in} form equations using KVL/KCL/others, and find Z_{in} (=V_{in}/I_{in}).

Example 12.8: Find Thévenin Equiv. in the *s*-domain (solve through Method 1)



$$Z_{eq} = R_E // \frac{1}{sC_\pi} // r_\pi = \frac{\frac{R_E}{sR_E C_\pi + 1} r_\pi}{\frac{R_E}{sR_E C_\pi + 1} + r_\pi} = \frac{R_E r_\pi}{r_\pi + R_E + sR_E r_\pi C_\pi}$$

Solution:

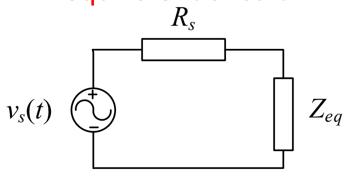
Step 1: Transform the circuit into *s* domain.

Step 2: Insert test voltage source (and therefore also the current)

Step 3: KVL of loop, and solve:

$$V_{in} = (I_{in} + gV_\pi)Z_{eq}$$
$$= (I_{in} - gV_{in})Z_{eq}$$
$$(1 + gZ_{eq})V_{in} = I_{in}Z_{eq}$$
$$Z_{in} = \frac{V_{in}}{I_{in}} = \frac{Z_{eq}}{1 + gZ_{eq}} = \frac{R_E r_\pi}{r_\pi + R_E + sR_E r_\pi C_\pi + gR_E r_\pi}$$

Being slightly different from the usual "Thevenin equi." question – the objective of this question is to get this equivalent circuit.



Example 12.9: Thévenin Equiv. in the *s*-domain (solve through Method 2, and for convenience, set source to 1 A)

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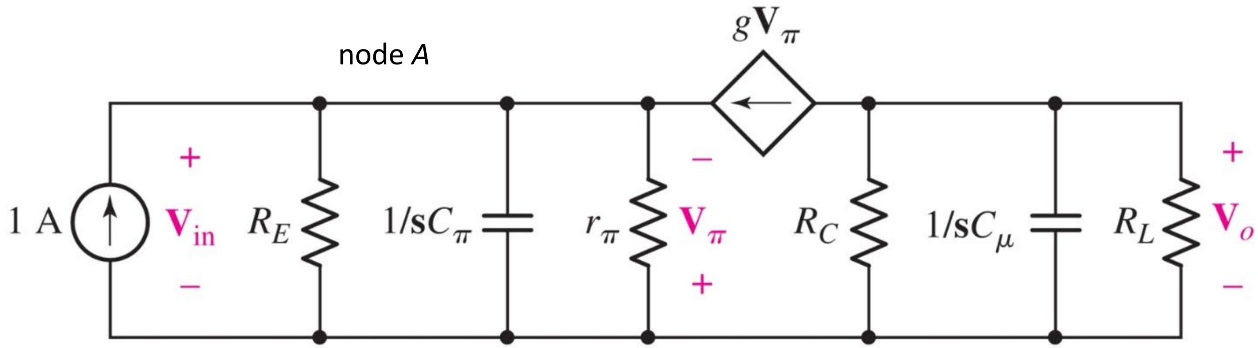
Note: See Textbook example 14.17 for another different way of interpreting the problem.



$$Z_{eq} = R_E // \frac{1}{sC_\pi} // r_\pi$$
$$= \frac{R_E r_\pi}{r_\pi + R_E + sR_E r_\pi C_\pi}$$

Solution:

Step 1: Transform the circuit into *s* domain.

Step 2: Insert test current source (and therefore also the voltage label; optionally, set current source as 1 A)

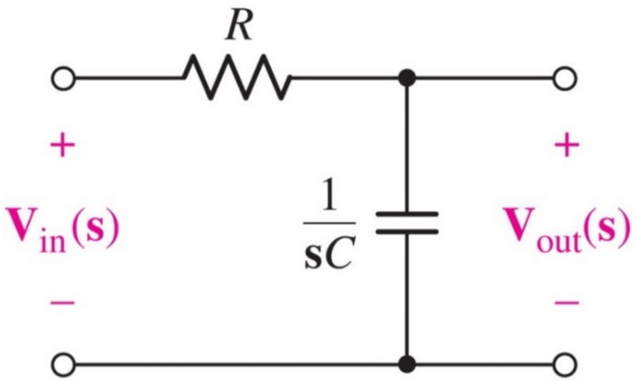
Step 3: KCL at node A, and solve:

$$1 + gV_\pi = \frac{V_{in}}{Z_{eq}}$$
$$Z_{eq}(1 - gV_{in}) = V_{in}$$
$$V_{in} = \frac{Z_{eq}}{1 + gZ_{eq}}$$
$$(1)Z_{in} = V_{in} = \frac{Z_{eq}}{1 + gZ_{eq}} = \frac{R_E r_\pi}{r_\pi + R_E + sR_E r_\pi C_\pi + gR_E r_\pi}$$

Transfer Function

- By definition, a transfer function (TF) is the ratio of the output to the input of a system in the frequency domain.
  - Clearly, it depends on the “output” and “input” of interest (voltage and current at different part of the circuit), we can derive more than one transfer functions. [“Interestingly”, their denominator will be the same – a deeper concept you will learn in more advanced modules]
  - Roots of the numerator → zeros; roots of the denominator→ poles
- For any system (not just circuits), we can analyze the system’s transfer function to tell how the system will behave system.

- **Example of TF:** An RC circuit, with source voltage as the input and capacitor’s voltage as the output.



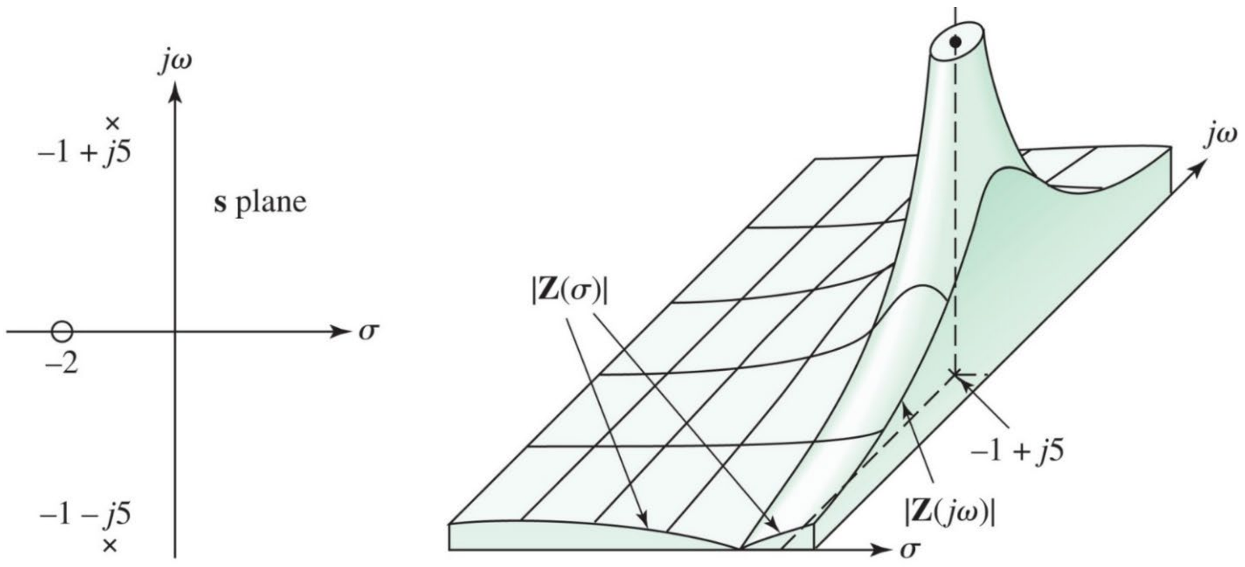
$$\boxed{TF = \frac{Output(s)}{Input(s)} = \frac{\text{Numerator}(s)}{\text{Denominator}(s)}}$$

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + sRC} = \frac{1/RC}{s + 1/RC}$$

Visualizing Laplace: the *s* Plane

- Given any *s*-domain transfer function, we can visualize in the s-plane (figure (a)).
- A less common way of "visualizing" the magnitude of the TF through pole-zero constellation (figure (b)) .
- For example,

$$Z(s) = k\frac{s+2}{s^2+2s+26} = \frac{k(s+2)}{(s+1-j5)(s+1+j5)}$$

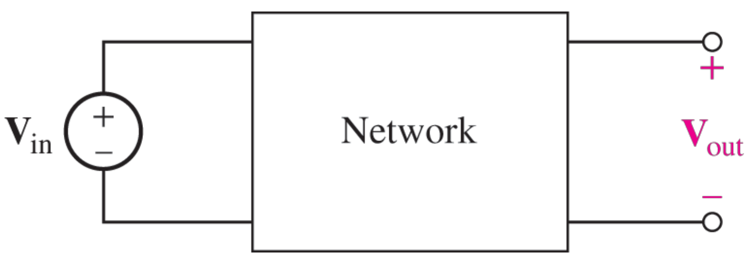


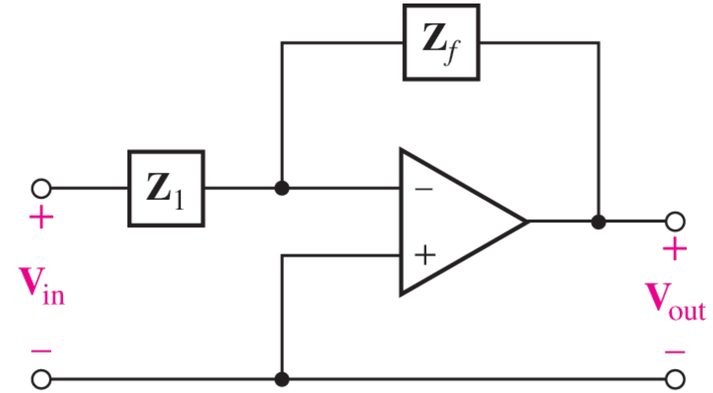
(a) (b)

Defining or Synthesizing a Transfer Function

- We can use op-amp circuits, e.g., an inverting amplifier with different impedance design, to synthesize a desired Transfer Function.

- Consider an inverting op-amp circuit:





Knowing the basic property of the inverting op-amp, write KCL at the inverting input node:

$$\frac{V_{in}(s)}{Z_1(s)} + \frac{V_{out}(s)}{Z_f(s)} = 0$$

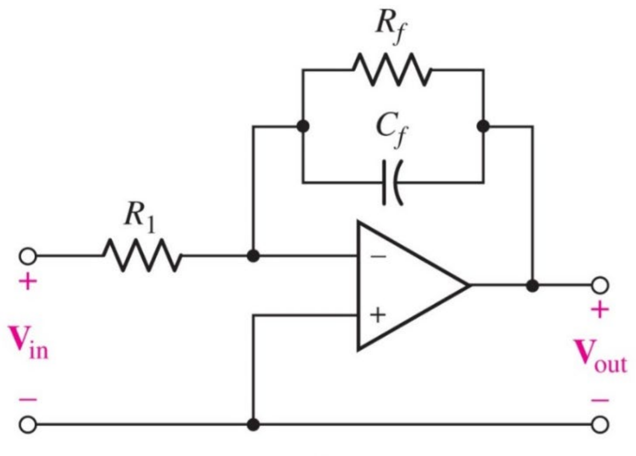
$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = -\frac{Z_f(s)}{Z_1(s)}$$

Implementing a Pole and a Zero

(a) Single pole, with negative (-) gain

$$\frac{V_{out}}{V_{in}} = -\frac{Z_f}{Z_1} = -\frac{R_f || \frac{1}{sC_f}}{R_1}$$

$$H(s) = -\frac{1/R_1C_f}{s + 1/R_fC_f}$$

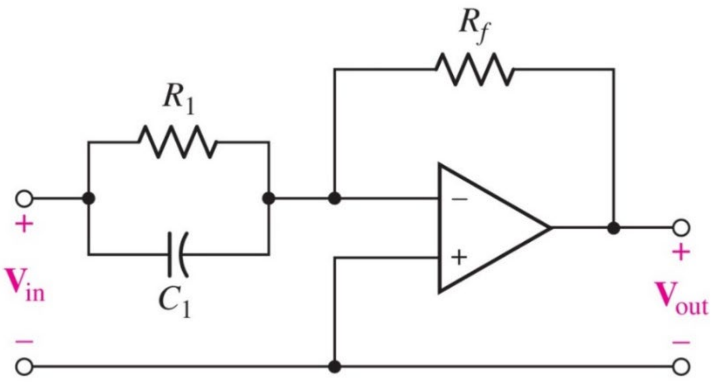


(a)

(b) Single zero, with negative (-) gain

$$\frac{V_{out}}{V_{in}} = -\frac{Z_f}{Z_1} = -\frac{R_f}{R_1 || \frac{1}{sC_1}}$$

$$H(s) = -R_fC_1\left(s + \frac{1}{R_1C_1}\right)$$



(b)

Example 12.10: Designing a Circuit to Achieve a Particular $H(s)$

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- Synthesize a circuit that will yield the transfer function

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{10(s+2)}{s+5}$$

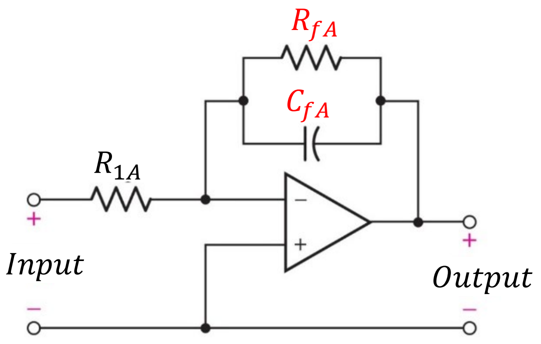
Solution:

- **Analysis:**
  - synthesize the -5 pole using inverting op-amp with R//C at the feedback path;
  - synthesize the -2 zero using another inverting op-amp with R//C at input path;
  - set the final gain as 10.

Example 12.10 Solution steps

- **Step 1:** "-5" pole

$$H_A(s) = -\frac{R_{fA} || \frac{1}{sC_{fA}}}{R_{1A}} = -\frac{1/R_{1A}C_{fA}}{s + 1/R_{fA}C_{fA}} = \frac{"XXX"}{s+5}$$



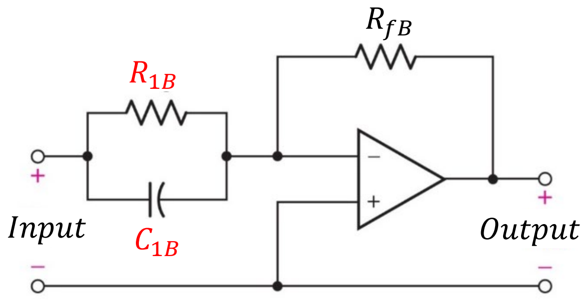
Decide a practical value of R_{fA}, e.g., 100 kΩ, $C_{fA} = 2\ \mu\text{F}$.

$$\frac{1}{R_{fA}C_{fA}} = \frac{1}{(100\text{k})(2\mu)} = 5$$

- **Step 2:** "-2" zero

$$H_B(s) = -\frac{R_{fB}}{R_{1B} || \frac{1}{sC_{1B}}} = -R_{fB}C_{1B}\left(s + \frac{1}{R_{1B}C_{1B}}\right)$$

$$= "XXX"(s+2)$$



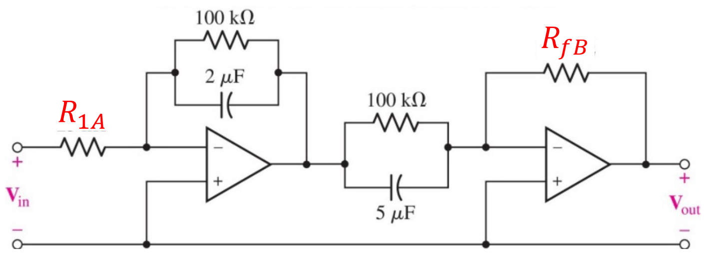
Decide a practical value of R_{1B}, e.g., 100 kΩ, $C_{1B} =$ 5 μF.

$$\frac{1}{R_{1B}C_{1B}} = \frac{1}{(100\text{k})(5\mu)} = 2$$

- **Step 3:** match the gain

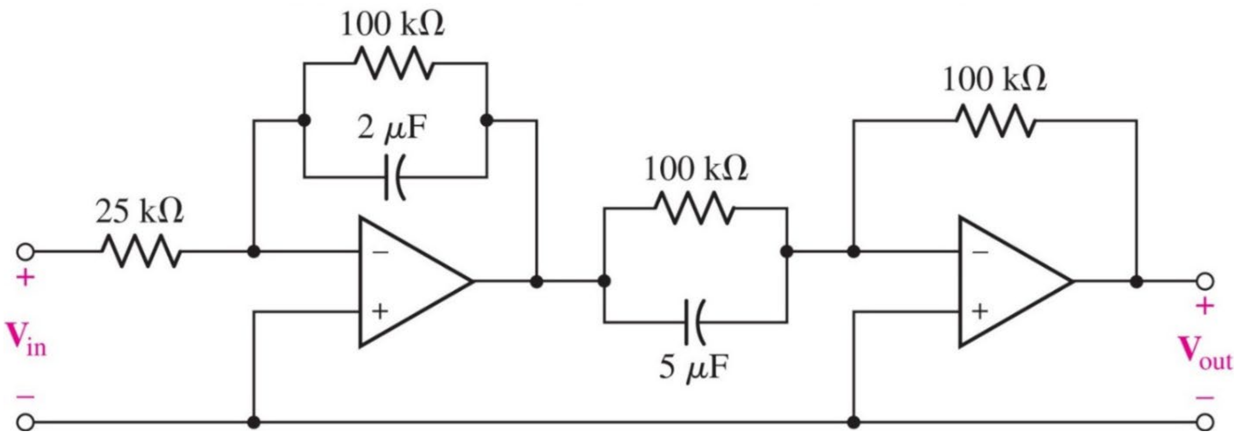
$$H = H_A H_B = \frac{1/R_{1A}C_{fA}}{s + 1/R_{fA}C_{fA}} R_{fB}C_{1B}\left(s + \frac{1}{R_{1B}C_{1B}}\right)$$

$$H = \frac{R_{fB}(5\ \mu)}{R_{1A}(2\ \mu)}\left(\frac{s+2}{s+5}\right) = \frac{10(s+2)}{s+5}$$



Decide a practical value of R_{fB}, e.g., $100\ \text{k}\Omega$, $R_{1A} = 25\ \text{k}\Omega$.

Answer:



Tutorial, and some selected questions from the Textbook for self-practices

- **Week 12**

(1) Please proceed to your tutorial session for some tutorial exercises.

Group 2	(Tutorial*)	(SA136*, SB152*, SB120*)	(continue to 12noon-1pm)

***NOTES:** The tutorial rooms are allocated according to your programmes. Please attend to the assigned session BUT NOT other rooms to avoid overcrowding. **Attendance of tutorials will be taken.**

SA136 – ~~CST and DMT students~~ CST and **EE** students (updated on 13^{th} Sept. 2023)

SB152 – ~~EE and EST students~~ **DMT** and EST students (updated on 13^{th} Sept. 2023)

SB120 – MRS and TE students

(2) After your tutorial, you can also self-practice some questions. For example:

Engineering Circuit Analysis, 9^{th} ed. or 10^{th} ed., Chapter 14

Pg. 606 – 614: 17, 27, 42, 58, 61.

- If you have extra time, others too (but be selective; you don't have time for all questions!).
- These questions will be displayed in the tutorial class, and sample solutions will be uploaded to LMC->Tutorials folder for your self-checking.