CAN207 Continuous and Discrete Time Signals and Systems

Lecture-9

CTFT properties and Frequency responses

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Content

- 3. Properties of CTFT
  - Linearity, time and frequency scaling, time and frequency shifting, conjugation and symmetry, duality, Parseval's relation, convolution and multiplication properties

- 4. System characterization
  - Frequency response of a system
    - Impulse response VS frequency response
  - Systems in series connection
  - LCCDE VS frequency response

3.1 Linearity

- Fourier transform is a linear operator.
  - For any two signals $x_1(t)$ and $x_2(t)$ with

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$$

$$x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

  - with any two constants α_1 and α_2, it can be shown that

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$$

- Proof:

$$\begin{aligned}
\mathcal{F}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} &= \int_{-\infty}^{\infty} [\alpha_1 x_1(t) + \alpha_2 x_2(t)]\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \alpha_1 x_1(t)\, e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \alpha_2 x_2(t)\, e^{-j\omega t}\, dt \\
&= \alpha_1 \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt + \alpha_2 \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt \\
&= \alpha_1 \mathcal{F}\{x_1(t)\} + \alpha_2 \mathcal{F}\{x_2(t)\}
\end{aligned}$$

3.2 Time Shifting

- For a transform pair

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

  - it can be shown that

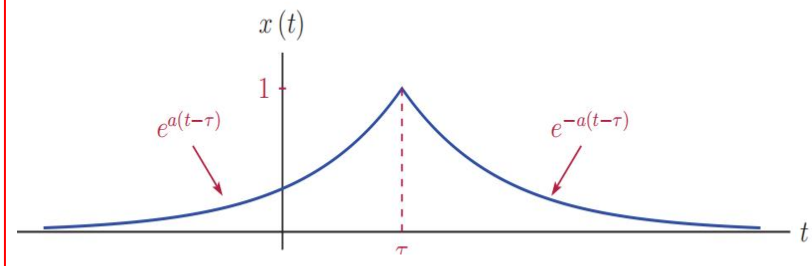
$$x(t-\tau) \xleftrightarrow{\mathcal{F}} X(\omega)\, e^{-j\omega\tau}$$

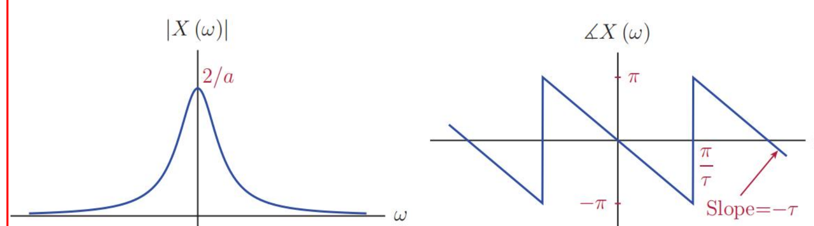
- Proof: $\mathcal{F}\{x(t-\tau)\} = \int_{-\infty}^{\infty} x(t-\tau)\, e^{-j\omega t}\, dt$
  - let $\lambda = t - \tau$, get

$$\begin{aligned}
\mathcal{F}\{x(t-\tau)\} &= \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, e^{-j\omega\tau}\, d\lambda \\
&= e^{-j\omega\tau} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda \\
&= e^{-j\omega\tau}\, X(\omega)
\end{aligned}$$

Example: Time shifting a two-sided exponential signal

$$x(t) = e^{-a|t-\tau|}, \quad a > 0$$





3.2 Time Shifting

$$x\,(t-\tau)\;\overset{\mathcal{F}}{\longleftrightarrow}\;X\,(\omega)\;e^{-j\omega\tau}$$

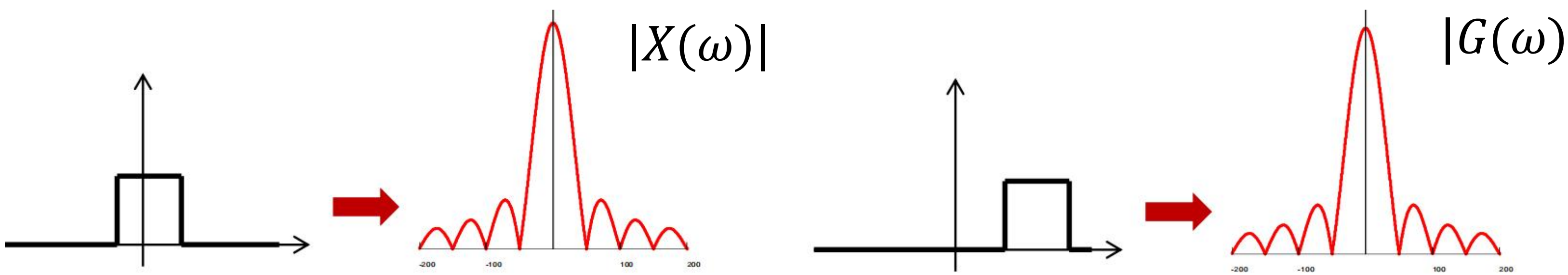
- A time-shifting in time domain is equivalent to a linear phase shift in frequency domain (i.e., multiplying with a complex exponential).

  magnitude $\quad |G(\omega)| = |\mathrm{e}^{-\mathrm{j}\omega t_0} X(\omega)| = |\mathrm{e}^{-\mathrm{j}\omega t_0}||X(\omega)| = |X(\omega)|;$

  phase $\quad <G(\omega) = <\{\mathrm{e}^{-\mathrm{j}\omega t_0} X(\omega)\} = <\mathrm{e}^{-\mathrm{j}\omega t_0} + <X(\omega) = -\omega t_0 + <X(\omega).$

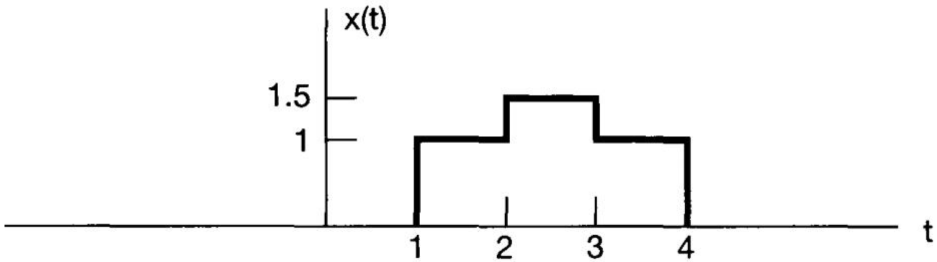
  – The magnitude spectrum depends only on the shape of a signal, in time domain, which is unchanged in a time shift.

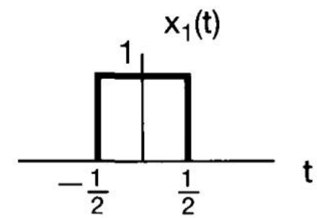
  – In a time shift only the phase spectrum will be changed.



Quiz 1

- To illustrate the usefulness of the linearity and time-shifting properties, let us consider the evaluation of the Fourier transform of the signal x(t) shown below:





- With the knowledge that FT of $x_1(t)$ is $X_1(\omega) = \frac{2\sin(\omega/2)}{\omega}$, find the expression of the FT of $x(t)$.

3.3 Scaling

- For a transform pair

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

  - it can be shown that

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

  - where a is any non-zero and real-valued constant.

- Proof:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

  - let $\lambda = at$, then

$$t = \frac{\lambda}{a} \quad \text{and} \quad dt = \frac{d\lambda}{a}$$

  - If $a > 0$, then the integral limits unchanged under the variable change, so

$$\begin{aligned} \mathcal{F}\{x(at)\} &= \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda/a}\, d\lambda \\ &= \frac{1}{a} X\left(\frac{\omega}{a}\right), \quad a > 0 \end{aligned}$$

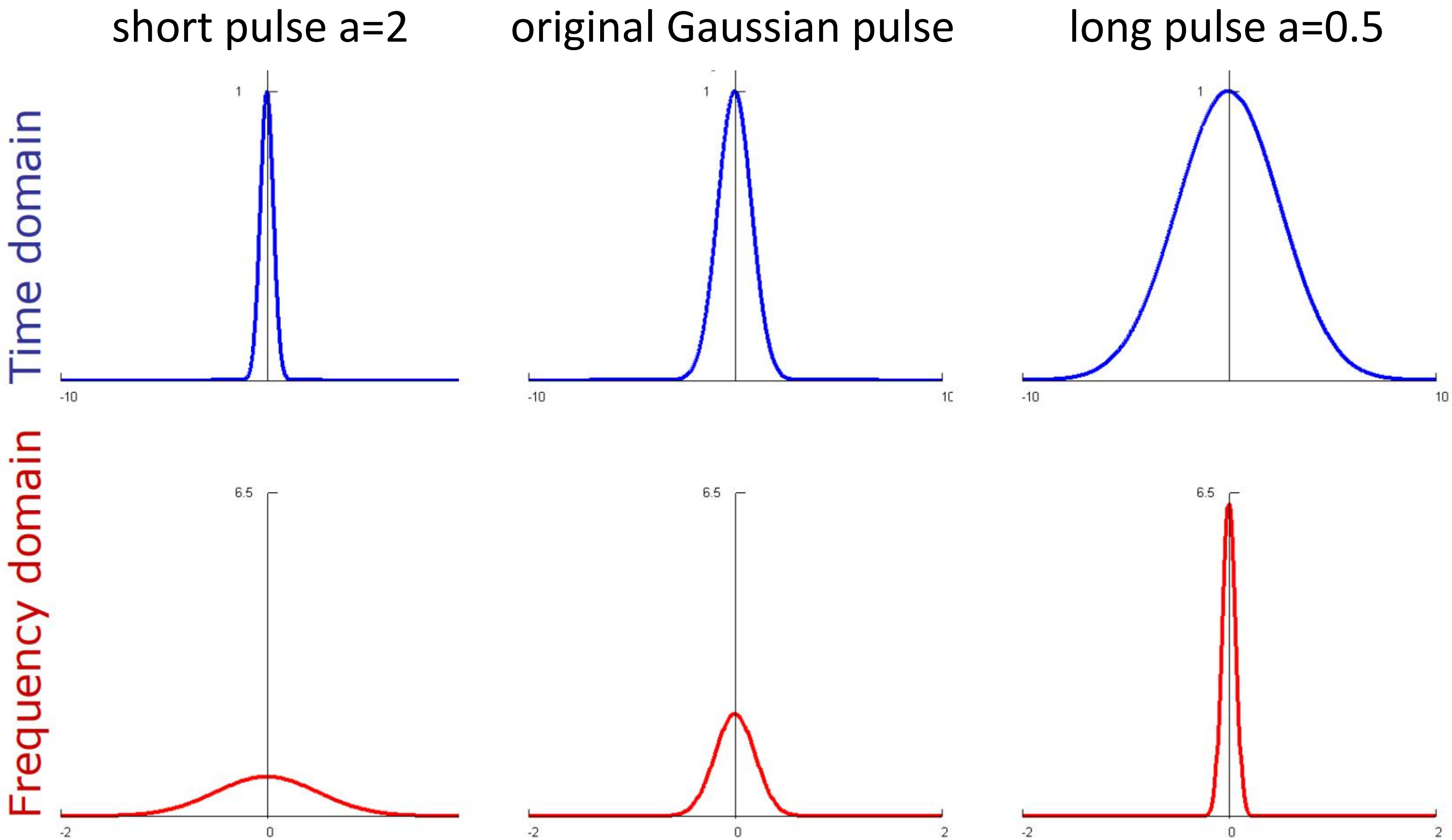
  - If $a < 0$, swapping the lower and upper limits of the integral, so it changes to

$$\begin{aligned} \mathcal{F}\{x(\lambda)\} &= -\frac{1}{a} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda/a}\, d\lambda \\ &= -\frac{1}{a} X\left(\frac{\omega}{a}\right), \quad a < 0 \end{aligned}$$

It's possible to combine them.

3.3 Scaling

- The property suggests that compressing (expanding) the signal in time would expand (compress) the spectrum in frequency.



3.4 Duality

- For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

  - implies that

$$X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$$

- Time domain and frequency domain are symmetric.
  - This property suggests if signal A's frequency spectrum is signal B, then signal B's frequency spectrum takes a form similar to signal A.
- Using linear frequency *f* instead of angular frequency *ω*, there is:

$$X(t) \overset{\mathcal{F}}{\longleftrightarrow} x(-f)$$

- Proof:
  - replace variable *ω* by *λ*:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda$$

  - change *t* to -*ω*, get

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\, e^{-j\lambda\omega}\, d\lambda$$

  - then change *λ* to *t*, it becomes:

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\, e^{-jt\omega}\, dt$$

  - and multiply 2π, that is:

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-jt\omega}\, dt$$

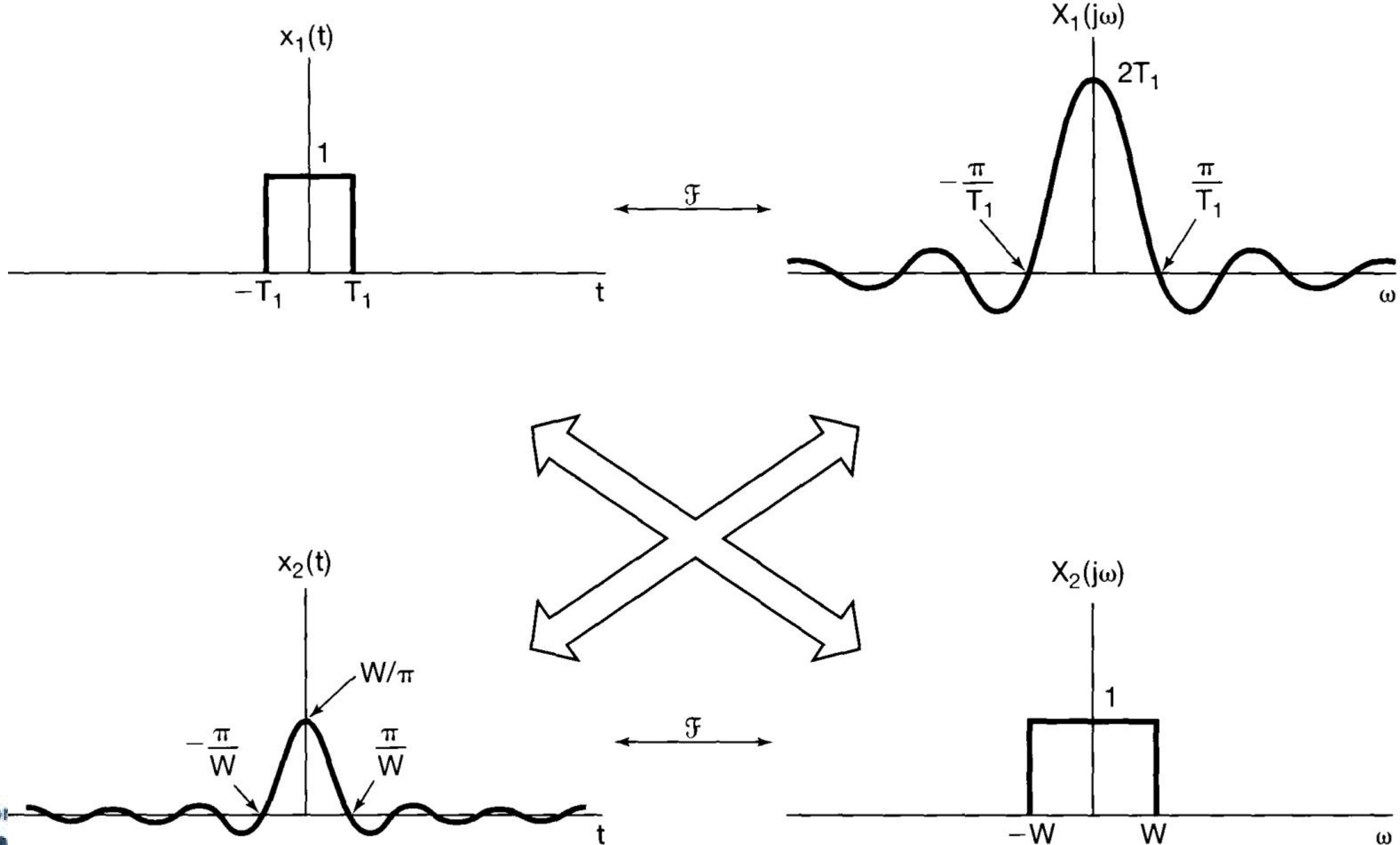
  - which is the inverse FT equation, meaning:

$$X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$$

3.4 Duality

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{\mathfrak{F}}{\longleftrightarrow} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

$$x_2(t) = \frac{\sin Wt}{\pi t} \overset{\mathfrak{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



Quiz 2

- Consider using duality to find the FT of the signal:

$$g(t) = \frac{2}{1 + t^2}$$

- Hint: recall the FT pair $x(t) = e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$.

3.5 Frequency Shifting

- For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

  – it can be shown that

$$x(t)\, e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)$$

- Proof
  – Method 1: directly apply the forward FT equation:

$$\begin{aligned}\mathcal{F}\left\{x(t)\, e^{j\omega_0 t}\right\} &= \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt \\ &= X(\omega-\omega_0)\end{aligned}$$

  – Method 2: using the duality principle in conjunction with the time shifting property

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$x(t-\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)\, e^{-j\omega\tau}$$

$\downarrow$ Apply the duality

$$X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$$

$$X(t)\, e^{-jt\tau} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega-\tau)$$

$\downarrow$ Let $\tilde{x}(t) = X(t)$ and $\tilde{X}(\omega) = 2\pi x(-\omega)$ substitue $\omega_0 = -\tau$

$$\tilde{x}(t) \overset{\mathcal{F}}{\longleftrightarrow} \tilde{X}(\omega)$$

$$\tilde{x}(t)\, e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \tilde{X}(\omega-\omega_0)$$

3.6 Conjugation and Conjugate symmetry

- Conjugation Property
  - if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-\omega)$
- Proof:

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt.$$

  - replacing ω by -ω, get

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j(-\omega)t}dt$$

- Conjugate symmetry:
  - if $x(t)$ is real, then $X(\omega)$ has *conjugate symmetry*:

$$X(-\omega) = X^*(\omega)$$

  - i.e. Hermitian symmetry
- Proof:
  - take conjugate of $X(\omega)$

$$X^*(\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt = X(-\omega)$$

3.6 Conjugation and Conjugate symmetry

- FT of even signals
  - If the real-valued signal $x(t)$ is an even function of time, the resulting transform $X(\omega)$ is ***real***-valued for all ω.

$$x(-t) = x(t)\,, \quad \text{all } t$$

$$\downarrow$$

$$\text{Im}\{X(\omega)\} = 0\ , \text{all } \omega$$

  - $X(\omega)$ is also a ***real and even*** function of ω.

- FT of odd signals
  - If the real-valued signal $x(t)$ is an odd function of time, the resulting transform $X(\omega)$ is ***imaginary***-valued for all ω.

$$x(-t) = -x(t)\,, \quad \text{all } t$$

$$\downarrow$$

$$\text{Re}\{X(\omega)\} = 0\ , \text{all } \omega$$

  - $X(\omega)$ is an ***imaginary and odd*** function of ω.

3.7 Differentiation

- For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

  – it can be shown that

$$\frac{d^n}{dt^n}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^n \, X(\omega)$$

This is a particularly important property, as it replaces the operation of differentiation in the time domain with that of multiplication by $j\omega$ in the frequency domain.

- Proof:

$$\begin{aligned}\frac{d}{dt}[x(t)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{dt}\left[X(\omega)\, e^{j\omega t}\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega\, X(\omega)]\, e^{j\omega t}\, d\omega \\ &= \mathcal{F}^{-1}\{j\omega\, X(\omega)\}\end{aligned}$$

$$\frac{d}{dt}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} j\omega\, X(\omega)$$

$$\begin{aligned}\frac{d}{dt}\left[\frac{d}{dt}[x(t)]\right] &= \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, X(\omega)\, e^{j\omega t}\, df\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{dt}\left[j\omega\, X(\omega)\, e^{j\omega t}\right] df \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[(j\omega)^2\, X(\omega)\right] e^{j\omega t}\, df \\ &= \mathcal{F}^{-1}\left\{(j\omega)^2\, X(\omega)\right\}\end{aligned}$$

$$\frac{d^2}{dt^2}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^2\, X(\omega)$$

3.7 Differentiation

- Application: electrical circuits with steady sinusoidal source $A\cos(\omega t + \theta)$.
- V-I relationship of three elementary components: R, L and C.
  - Resistor: $v_R = Ri_R$
  - Inductor: $v_L = L\frac{di_L}{dt}$
  - Capacitor: $i_C = C\frac{dv_C}{dt}$
- The KVL or KCL of a circuit should be a differntial equation as shown in Lecture 6, such as:

$$\frac{d^2y}{dt^2} + \frac{1}{RC}\frac{dy}{dt} + \frac{1}{LC}y(t) = \frac{1}{RC}\frac{dv}{dt}.$$

- In phasor form (express all the time-dependent variables, i.e. voltages and currents as $\boldsymbol{A} = A\angle\theta$.
- V-I relationship:
  - Resistor: $\boldsymbol{V_R} = R\boldsymbol{I_R}$
  - Inductor: $\boldsymbol{V_L} = j\omega\boldsymbol{I_L}L$
  - Capacitor: $\boldsymbol{I_C} = j\omega\boldsymbol{V_C}C$
    - more importantly, the integral becomes easier:

$$\boldsymbol{V_C} = \frac{\boldsymbol{I_C}}{j\omega C}$$

- The differential equation changes to:

$$(j\omega)^2\boldsymbol{Y} + \frac{j\omega}{RC}\boldsymbol{Y} + \frac{1}{LC}\boldsymbol{Y} = \frac{j\omega}{RC}\boldsymbol{V}$$

3.8 Integration

- For a transform pair

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

  – it can be shown that

$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \xleftrightarrow{\mathcal{F}} \boxed{\frac{X(\omega)}{j\omega}} + \boxed{\pi\, X(0)\, \delta(\omega)}$$

Since differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain, one might conclude that integration should involve division by $j\omega$ in the frequency domain.

The impulse term on the right-hand side reflects the DC or average value that can result from integration.

3.8 Integration

- Example: find the FT of the unit step $u(t)$.
- Solution:
  - Recall the FT of the unit impulse $\delta(t)$:

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 = G(\omega)$$

  - The unit step u(t) is the integral of $\delta(t)$, so:

$$X(\omega) = \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

  - since $G(\omega) \equiv 1$, so $X(\omega)$ is:

$$X(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

  - Observe that we can apply the differentiation property:

$$\delta(t) = \frac{du(t)}{dt} \overset{\mathfrak{F}}{\longleftrightarrow} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1$$

where the last equality follows from the fact that $\omega\delta(\omega) = 0$

3.9 Parseval's relation

- For a transform pair

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

  – it can be shown that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

- Proof:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\underbrace{\left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]}_{X(j\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

- The relation suggests that one may determine the energy of $x(t)$ from its FT $X(\omega)$;
- As a result, $|X(\omega)|^2$ is referred to as the energy-density spectrum of the signal $x(t)$.
- (extended) the energy-density spectrum $|X(\omega)|^2$ can also be calculated as the Fourier transform of the ***autocorrelation*** of the signal.

3.9 Parseval's relation

- Example: Calculate the energy of the CT signal $e^{-at}u(t)$ in the time and frequency domains.
- Verify that Parseval's relation is valid by comparing the two answers.
- Solution:
  - Time domain: $$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{0}^{\infty} \mathrm{e}^{-2at} \mathrm{d}t = \left[\frac{\mathrm{e}^{-2at}}{-2a}\right]_0^{\infty} = \frac{1}{2a}$$
  - Its FT is: $$\mathrm{e}^{-at}u(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{a+\mathrm{j}\omega}$$
  - Frequency domain: $$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2} \mathrm{d}\omega$$ $$= \frac{1}{2\pi}\left[\frac{1}{a}\tan^{-1}\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty} = \frac{1}{2a}$$

3.10 Convolution Property

- For two transform pairs

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

  – it can be shown that

$$x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)\, X_2(\omega)$$

- Proof:

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda\right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\lambda)\; x_2(t-\lambda)\; e^{-j\omega t}\, dt\right] d\lambda$$

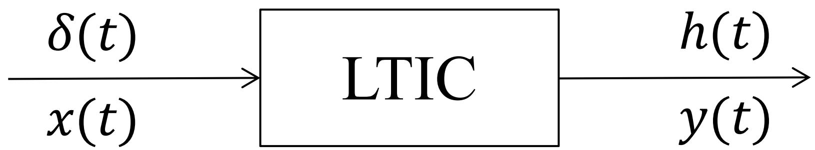
$$= \int_{-\infty}^{\infty} x_1(\lambda) \underbrace{\left[\int_{-\infty}^{\infty} x_2(t-\lambda)\; e^{-j\omega t}\, dt\right]}_{\mathcal{F}\{x_2(t-\lambda)\} = X_2(\omega)\; e^{-j\omega\lambda}} d\lambda = \left[\int_{-\infty}^{\infty} x_1(\lambda)\, e^{-j\omega\lambda}\, d\lambda\right] X_2(\omega)$$

$$= X_1(\omega)\; X_2(\omega)$$

Convolution between two signals in the time domain is equivalent to the multiplication of the CTFTs of the two signals in the frequency domain.

3.10 Convolution Property

- Recall the relationship among the input $x(t)$, output $y(t)$ and the impulse response of a system $h(t)$ (lecture 6):



  - In time domain: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$
  - In Frequency domain, using the convolution property, there is:

$$Y(\omega) = X(\omega)H(\omega)$$

- Example: in response to the input signal $x(t) = e^{-t}u(t)$, find the spectrum of the output from an LTIC system with the impulse response $h(t) = e^{-2t}u(t)$.

3.11 Multiplication Property

- For two transform pairs

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

  – it can be shown that

$$x_1(t)\, x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

- Proof:

$$\frac{1}{2\pi} X_1(\omega) * X_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, X_2(\omega-\lambda)\, d\lambda$$

$$\mathcal{F}^{-1}\left\{\frac{1}{2\pi} X_1(\omega) * X_2(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, X_2(\omega-\lambda)\, d\lambda\right] e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, \underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega-\lambda)\, e^{j\omega t}\, d\omega\right]}_{x_2(t)\, e^{j\lambda t}} d\lambda$$

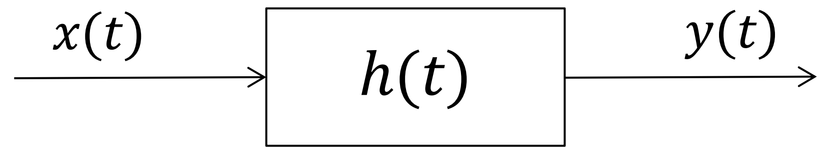
$$= x_2(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, e^{j\lambda t}\, d\lambda\right]$$

$$= x_1(t)\, x_2(t)$$

- Multiplication of the two signals in the time domain is equivalent to the convolution of their CTFTs in the frequency domain.
- Two most important applications of this property:
  - sampling
  - modulation

  will be addressed later.

4.1 Frequency Response



- In time domain, the output signal $y(t)$ can be obtained by taking the convolution of the input signal $x(t)$ and the impulse response of the system $h(t)$:

$$y(t) = x(t) * h(t)$$

- Using the convolution property, the relationship is:

$$Y(\omega) = X(\omega)H(\omega)$$

  - $X(\omega)$ and $Y(\omega)$ are the spectrums (CTFTs) of the input and output signals.
  - $H(\omega) = Y(\omega)/X(\omega)$ defines the operation of the system, called the ***Frequency response*** of the system.
  - There exists such a relationship:

impulse response

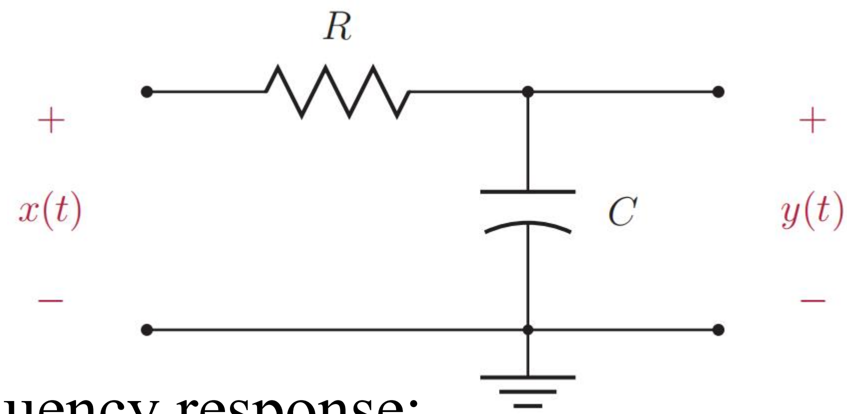
frequency response $$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

  - They form a FT pair as: $h(t) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$

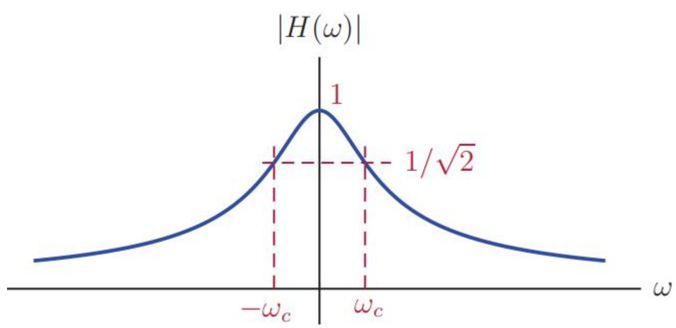
Example

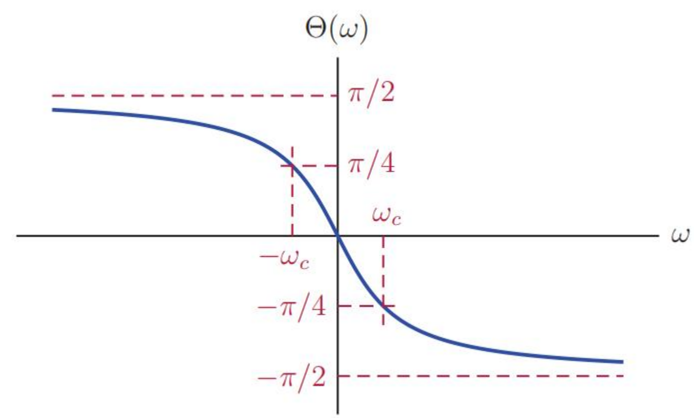
- The impulse response of the RC circuit as shown is

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$



- Find the frequency response of the RC-system using:
  - 1. known impulse response → frequency response;
  - 2. input-output relationship
- Solution: $H(\omega) = \frac{1}{1+j\omega RC}$

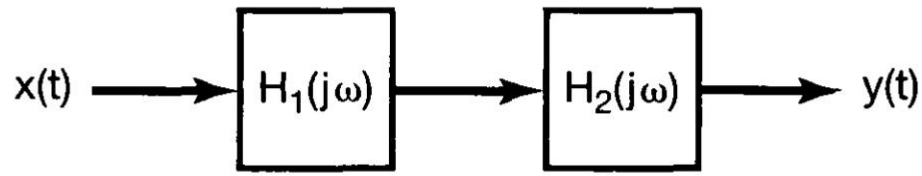




4.2 Systems in Series

- The impulse response $h(t)$ specifies an LTI system, then the frequency response $H(\omega)$ also specifies the system.

$$Y(\omega) = X(\omega)H_1(\omega)H_2(\omega)$$

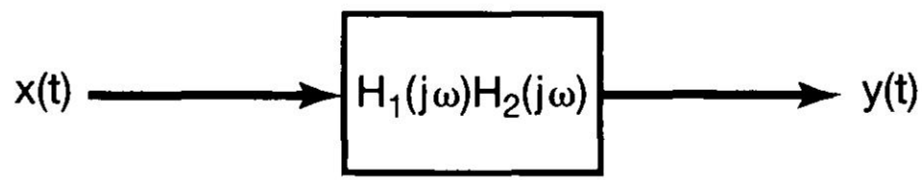


(a)

One observation is that we can treat the cascaded system as one LTI system $H(\omega)$

$$Y(\omega) = X(\omega)(H_1(\omega)H_2(\omega))$$

$$H(\omega) = H_1(\omega)H_2(\omega)$$

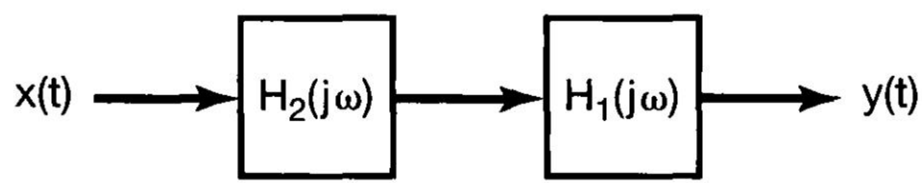


(b)

Another observation is that the order of the two systems does not matter

$$Y(\omega) = X(\omega)H_2(\omega)H_1(\omega)$$

$$H(\omega) = H_2(\omega)H_1(\omega)$$



(c)

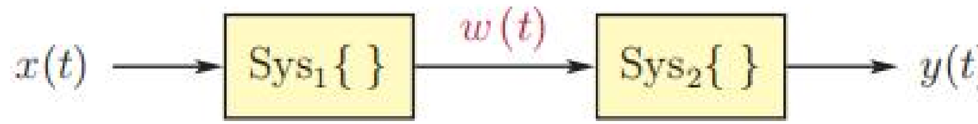
Quiz 3

- Consider the cascade combination of two systems shown in (a):
- Let the input-output relationships of the two subsystems be given as

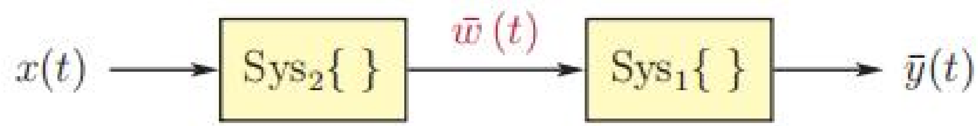
$$\mathrm{Sys}_1\{x(t)\} = 3\,x(t)$$

$$\mathrm{Sys}_2\{w(t)\} = w(t-2)$$

  – Write the relationship between $X(\omega)$ and $Y(\omega)$;
  – Let the order of the two subsystems be changed as shown in (b). Write the relationship between $X(\omega)$ and $\widetilde{Y}(\omega)$.



(a)



(b)

4.3 LCCDE VS Frequency response

- Linear constant-coefficient differential equation:

$$\sum_{k=0}^{n} a_k \frac{\mathrm{d}^k x}{\mathrm{d}t^k} = \sum_{k=0}^{m} b_k \frac{\mathrm{d}^k x}{\mathrm{d}t^k}.$$

- Convert to frequency domain, using $\frac{\mathrm{d}^n x}{\mathrm{d}t^n} \xleftrightarrow{\text{CTFT}} (\mathrm{j}\omega)^n X(\omega)$
- So the LCCDE changes to:

$$\sum_{k=0}^{n} a_k (\mathrm{j}\omega)^k Y(\omega) = \sum_{k=0}^{m} b_k (\mathrm{j}\omega)^k X(\omega)$$

- The <u>frequency response</u> is obtained by:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{n} b_k (\mathrm{j}\omega)^k}{\sum_{k=0}^{m} a_k (\mathrm{j}\omega)^k}$$

spectrum
of signals

Quiz 4

- Consider an LTIC system whose input–output relationship is modeled by the following third-order differential equation:

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + 6\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 11\frac{\mathrm{d}y}{\mathrm{d}t} + 6y(t) = 2\frac{\mathrm{d}x}{\mathrm{d}t} + 3x(t).$$

- Calculate the frequency response H(ω) for the LTIC system.

Next ...

- Laplace transform
  - Derived (extended) from CTFT
  - Forward and inverse s-transform
  - Existence of Laplace transform and Region of convergence
  - Example s-trans pairs
  - Properties