CAN209 Advanced Electrical Circuits and Electromagnetics

Lecture 8-1 Time Harmonic Fields

& Complex Power

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OUTLINE

- Review-Complex Numbers
- Time-harmonic Fields (Sinusoidal Fields)
- Power of EM Fields and Waves
  - ✓ EM Power Flow
  - ✓ Poynting Vector
  - ✓ Time-averaged Poynting Vector

1.1 NOTATIONS

Two forms

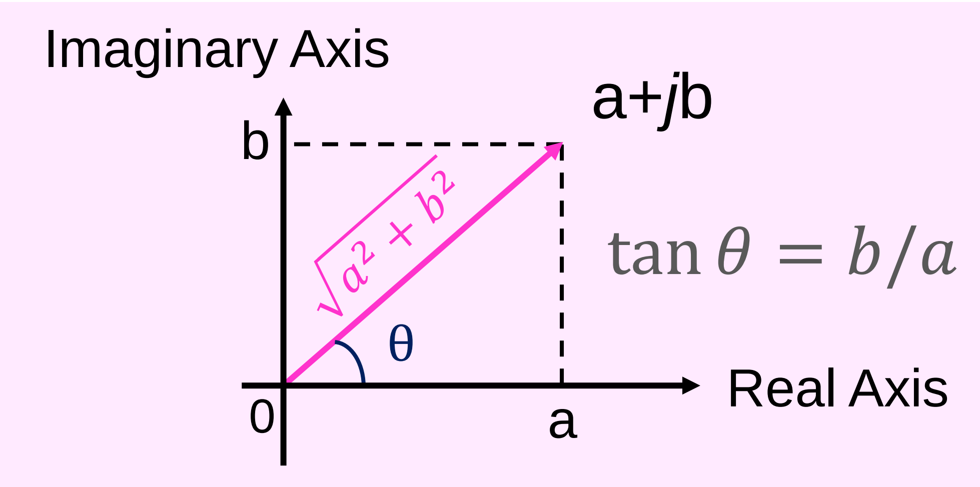
- Cartesian/rectangular form

$$\vec{A} = a + jb$$

Real part — Imaginary number — Imaginary part

$$a = \left|\vec{A}\right| cos\theta$$

$$b = \left|\vec{A}\right| sin\theta$$



- Exponential/Polar form

$$\left|\vec{A}\right| = \sqrt{a^2 + b^2}$$

$$\vec{A} = \left|\vec{A}\right| e^{j\theta} = \left|\vec{A}\right| \angle\theta$$

Exponential form — Polar form

1.2 CONVERT CARTESIAN↔POLAR FORM

- Polar ⟶ Cartesian form

$$\because e^{\pm j\theta} = \cos\theta \pm j\sin\theta \quad \text{(Euler's identity)}$$

$$\vec{A} = |\vec{A}|e^{j\theta} = |\vec{A}|(\cos\theta + j\sin\theta)$$

$$= |\vec{A}|\cos\theta + j|\vec{A}|\sin\theta$$

$$= a + jb$$

$$\begin{cases} a = |\vec{A}|cos\theta = |\vec{A}|Re\{e^{j\theta}\} \\ b = |\vec{A}|sin\theta = |\vec{A}|Im\{e^{j\theta}\} \end{cases}$$

- Cartesian ⟶ Polar form

$$\vec{A} = a + jb = \sqrt{a^2 + b^2}e^{j\theta} = |\vec{A}|e^{j\theta}$$

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RECALL…

Previously, field quantities are expressed as functions of time and position. In real applications, time signals can be expressed as **superimposition** of sinusoidal waveforms.

Laws	Time-varient
Gauss's law: electric fields	$\nabla \cdot \vec{E}(\vec{r},t) = \dfrac{\rho(\vec{r},t)}{\varepsilon_0}$
Faraday's law	$\nabla \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$
Gauss's law: magnetic fields	$\nabla \cdot \vec{B}(\vec{r},t) = 0$
Maxwell-Ampere’s law	$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \dfrac{\partial \vec{D}(\vec{r},t)}{\partial t}$

2.1 WHAT IS A SINUSOIDAL FIELD?

If sources are sinusoidal and the medium is linear, then the fields everywhere are sinusoidal as well. The field at each point is characterized by its amplitude and phase (Phasor).

For example: $f(x,t) = 5\cos[\omega t + x]$

It can be represented as:

$$f(x,t) = 5Re\{e^{j[\omega t + x]}\} = 5Re\{e^{j\omega t}e^{jx}\}$$

$$= Re\{5e^{jx}e^{j\omega t}\}$$

$$= Re\{F(x)e^{j\omega t}\} \quad where\ F(x) = 5e^{jx}$$

For a given sinusoidal E-field:

$$\vec{E}(x,y,z,t) = E_x(x,y,z,t)\hat{x} + E_y(x,y,z,t)\hat{y} + E_z(x,y,z,t)\hat{z}$$

How do we represent it in its phasor form?

2.2 DEFINITION

Time-harmonic (sinusoidal) fields: The excitation source varies sinusoidally in time with a single frequency.

For time-harmonic fields we can employ ***phasor analysis*** to obtain the single-frequency (monochromatic) steady-state response.

When fields are examined in this manner, there is no loss in generality as:

(a) any time-varying periodic function can be represented by a Fourier series in terms of sinusoidal functions;

(b) the principle of superposition can be applied under linear conditions.

In other words, we can obtain the complete response of time-varying periodic fields by using linear combinations of monochromatic responses.

2.2 DEFINITION

Basic Idea:

If the time-variation of fields is known a-priori to be sinusoidal (i.e., the fields are known to be time-harmonic) then, to simplify the math, one may not carry around the time dependence explicitly in calculations.

Let's see how the complex notation can be used to factor out the sinusoidal time dependence for the given sinusoidal electric field:

$$\vec{E}(x,y,z,t) = E_x(x,y,z,t)\hat{x} + E_y(x,y,z,t)\hat{y} + E_z(x,y,z,t)\hat{z}$$

$$= E_{x0}(r)\cos[\omega t + \alpha(r)]\,\hat{x} + E_{y0}(r)\cos[\omega t + \beta(r)]\,\hat{y}$$

$$+E_{z0}(r)\cos[\omega t + \gamma(r)]\,\hat{z}$$

$$\vec{E}(x,y,z,t) = E_x(x,y,z,t)\hat{x} + E_y(x,y,z,t)\hat{y} + E_z(x,y,z,t)\hat{z}$$

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$$+E_{z0}(r)\cos[\omega t + \gamma(r)]\,\hat{z}$$

where

$$E_x(x,y,z,t) = E_x(r,t) = E_{x0}(r)\cos[\omega t + \alpha(r)] = Re\left[E_{x0}(r)e^{j\alpha(r)}e^{j\omega t}\right] = Re\left[\tilde{E}_x(r)e^{j\omega t}\right]$$

$$E_y(x,y,z,t) = E_y(r,t) = E_{y0}(r)\cos[\omega t + \beta(r)] = Re\left[E_{y0}(r)e^{j\beta(r)}e^{j\omega t}\right] = Re\left[\tilde{E}_y(r)e^{j\omega t}\right]$$

$$E_z(x,y,z,t) = E_z(r,t) = E_{z0}(r)\cos[\omega t + \gamma(r)] = Re\left[E_{z0}(r)e^{j\gamma(r)}e^{j\omega t}\right] = Re\left[\tilde{E}_z(r)e^{j\omega t}\right]$$

Complex time-independent vector phasor

$$\therefore \vec{E}(r,t) = Re\left\{\left[\tilde{E}_x(r)\hat{x} + \tilde{E}_y(r)\hat{y} + \tilde{E}_z(r)\hat{z}\right]e^{j\omega t}\right\} = Re\left[\tilde{E}(r)e^{j\omega t}\right]$$

where $\tilde{E}(r) = \tilde{E}_x(r)\hat{x} + \tilde{E}_y(r)\hat{y} + \tilde{E}_z(r)\hat{z}$

2.3 REPRESENTATION

All time-harmonic fields (not just plane waves) can be written in the form:

$$\vec{E}(r,t) = Re[\tilde{E}(r)e^{j\omega t}]$$

where $$\tilde{E}(r) = \tilde{E}_x(r)\hat{x} + \tilde{E}_y(r)\hat{y} + \tilde{E}_z(r)\hat{z}$$

For such a vector phasor field, the time rate of the electric field change is:

$$\frac{\partial \vec{E}(r,t)}{\partial t} = Re[j\omega\tilde{E}(r)e^{j\omega t}]$$

It is convenient to use the **phasor notation** to express fields in the frequency domain, just as in AC circuits.

2.4 MAXWELL'S EQUATIONS (PHASOR FORM)

Maxwell's equations for the time-harmonic case are obtained by replacing each time vector by its corresponding phasor vector and replacing

$$\frac{\partial}{\partial t} \rightarrow j\omega$$

For example:

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$$

$$\because \frac{\partial \vec{B}(\vec{r},t)}{\partial t} \rightarrow j\omega \vec{B}(\vec{r},t)$$

$$\therefore \nabla \times \vec{E}(\vec{r},t) = -j\omega \vec{B}(\vec{r},t)$$

2.4 MAXWELL'S EQUATIONS (PHASOR FORM)

Law	Integral	Differential
Gauss's law for $\vec{E}$	$\oiint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$	$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$
Faraday's law	$\oint_C \vec{E} \cdot d\vec{l} = -j\omega \iint_S \vec{B} \cdot d\vec{s}$	$\nabla \times \vec{E} = -j\omega\vec{B}$
Gauss's law for $\vec{H}$	$\oiint_S \vec{B} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{B} = 0$
Generalised Ampere's law	$\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = I + j\omega \iint_S \vec{D} \cdot d\vec{s}$	$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$

QUIZ 2.1

The electric field in a source-free dielectric region is given as (C is a constant):

$$\vec{E} = C\cos(\omega t\text{-}kz)\,\hat{y} \quad \text{V/m}$$

Determine the expressions of the electric and magnetic field intensity in their **phasor form**.

QUIZ 2.2

The electric field in a source-free dielectric region is given as (C is a constant):

$$\vec{E} = C \sin(\alpha x)\cos(\omega t\text{-}kz)\ \hat{y} \quad \text{V/m}$$

Determine the expressions of the electric and magnetic field intensity in their **phasor form**.

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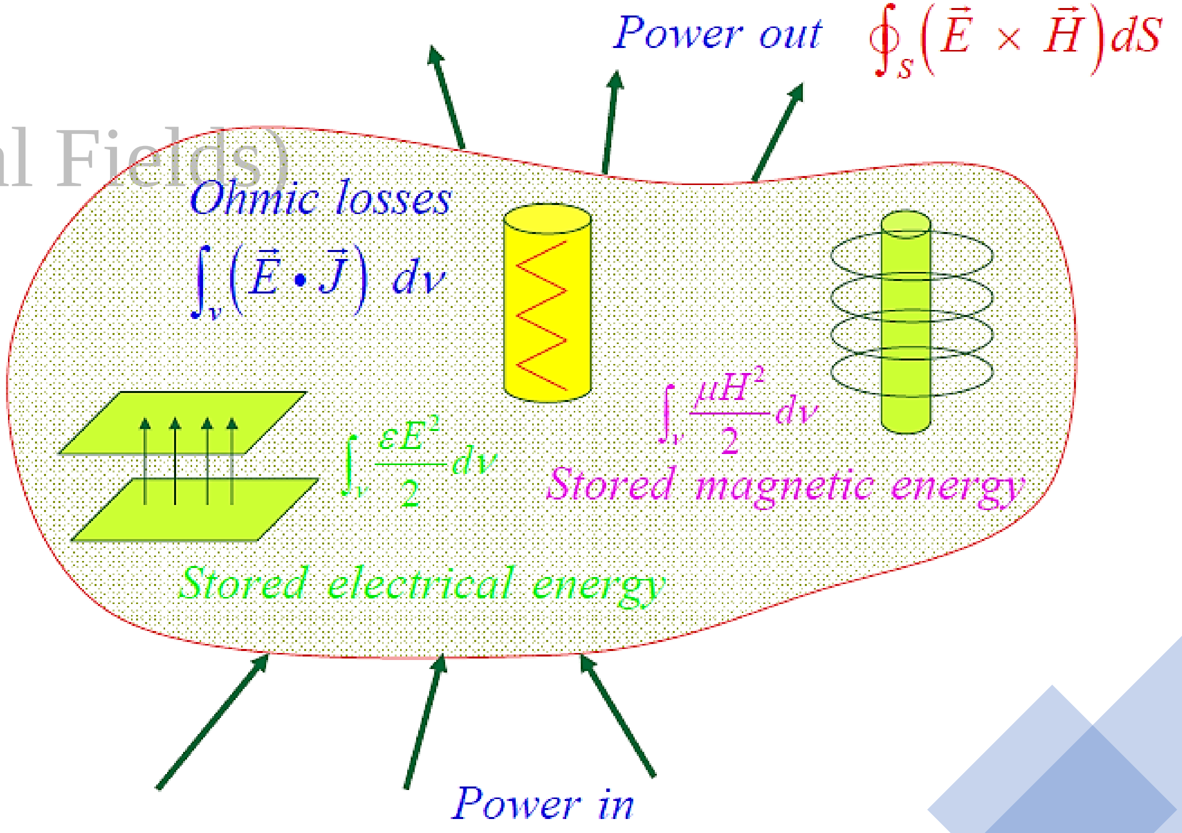
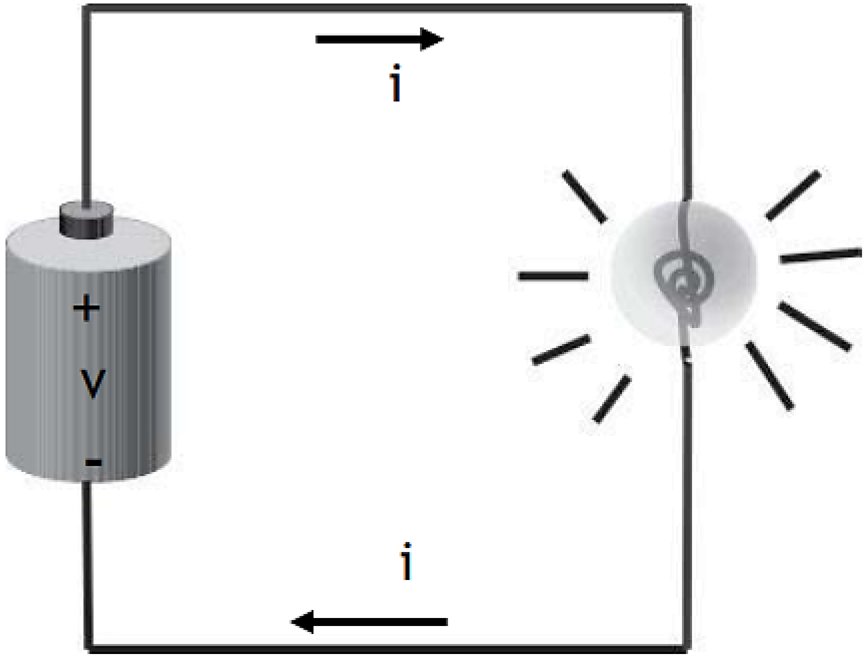


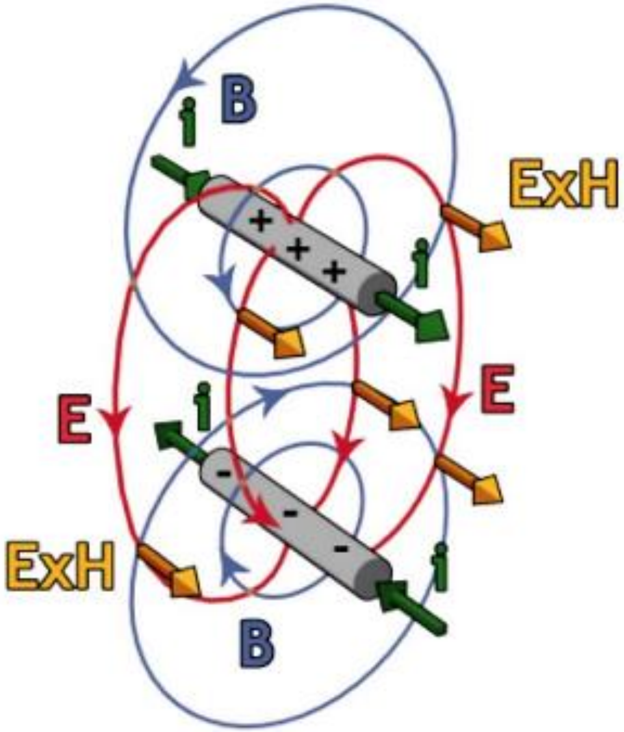
3.1 EM POWER FLOW

Q: How does power flow from the battery to the light bulb?



The wires serve only to guide the fields.

A: Through the EM fields which are guided by the wires!



3.2 POYNTING THEOREM

Differential form:

$$\nabla \cdot (\vec{E} \times \vec{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{J} \cdot \vec{E}$$

Power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the ohmic losses.

Poynting vector: $\vec{S} = \vec{E} \times \vec{H}$

- the **instantaneous** flow of power per unit area (W/m^2)
- the **instantaneous** power density
- points in the direction of power flow

Integral form:

$$\oint_{Area} (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\int_V \frac{d}{dt}\left(\frac{1}{2}\vec{B} \cdot \vec{H}\right) dv - \int_V \frac{d}{dt}\left(\frac{1}{2}\vec{D} \cdot \vec{E}\right) dv - \int_V \vec{J} \cdot \vec{E}\, dv$$

total power flowing out of the volume

$$= -\int_V \frac{d}{dt}\left(\frac{\mu}{2}|\vec{H}|^2 + \frac{\varepsilon}{2}|\vec{E}|^2\right) dv - \int_V \vec{J} \cdot \vec{E}\, dv$$

$$= -\frac{d}{dt}\int_V (w_m + w_e)\, dv - \int_V \vec{J} \cdot \vec{E}\, dv$$

- time rate of increase in energy stored in the H-field instantaneously

- time rate of increase in energy stored in the E-field instantaneously

- total instantaneous power dissipated in the volume

$$w_m = \frac{1}{2}\vec{B} \cdot \vec{H} = \frac{\mu}{2}|\vec{H}|^2 \qquad w_e = \frac{1}{2}\vec{D} \cdot \vec{E} = \frac{\varepsilon}{2}|\vec{E}|^2$$

3.3 AVERAGE POWER DENSITY

$\boldsymbol{A} = A\angle\theta \quad \boldsymbol{A}^* = A\angle-\theta$

In a **lossy** dielectric, if $\vec{E}$ and $\vec{H}$ fields are given by:

$$E_x(z,t) = E_{max}e^{-\alpha z}\cos(\omega t - \beta z + \theta_1)$$

$$H_y(z,t) = H_{max}e^{-\alpha z}\cos(\omega t - \beta z + \theta_2)$$

$$\therefore \vec{S} = \vec{E} \times \vec{H} = E_{max}H_{max}e^{-2\alpha z}\cos(\omega t - \beta z + \theta_1)\cos(\omega t - \beta z + \theta_2)\,\hat{a}_z$$

$$= \frac{1}{2}E_{max}H_{max}e^{-2\alpha z}\{\cos(2\omega t - 2\beta z + \theta_1 + \theta_2) + \cos(\theta_1 - \theta_2)\}\hat{a}_z$$

The **time-average power density** is obtained by integrate S_z over one cycle and divide by period T :

$$\therefore \langle S_z \rangle = \frac{1}{T}\frac{1}{2}E_{max}H_{max}e^{-2\alpha z}\int_0^T \left[\cos(2\omega t - 2\beta z + \theta_1 + \theta_2) + \cos(\theta_\eta)\right] dt$$

$$= \frac{1}{2}E_{max}H_{max}e^{-2\alpha z}\cos(\theta_\eta)$$ Time domain

Time-averaged Poynting vector:

$$\langle \vec{S} \rangle = \frac{1}{2}Re\{\vec{E} \times \vec{H}^*\}$$ Phasor form

QUIZ 3.1

Compute the average power density of a uniform sinusoidal plane wave propagating in **free space** which has the following expression for the instantaneous electric field:

$$\vec{E} = 94.25\cos(\omega t + 6z)\,\hat{a}_x \quad V/m$$

SUMMARY

- A time-harmonic field is one that varies **periodically** or **sinusoidally** with time.
- Wireless applications are possible because electromagnetic fields can propagate in free space without any guiding structures.
- Plane waves are one example of time-harmonic fields.
- When the electric (E) and magnetic (H) field vectors of a wave are in planes perpendicular to the direction of propagation, say the z-direction, this is called a plane wave.
- Plane waves are good approximations of electromagnetic waves in engineering problems after they propagate a short distance from the source.

NEXT…

Wave Propagation along Transmission Lines