CAN102 Electromagnetism and Electromechanics

Lecture-5 Static Electric Fields III

(Potential, Loop Theorem and Gradient)

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Outline

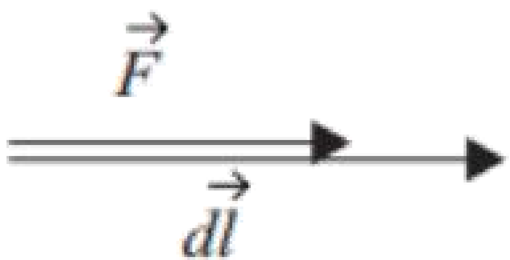
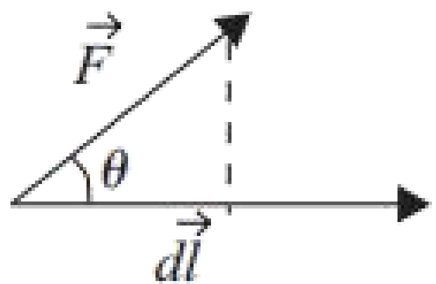
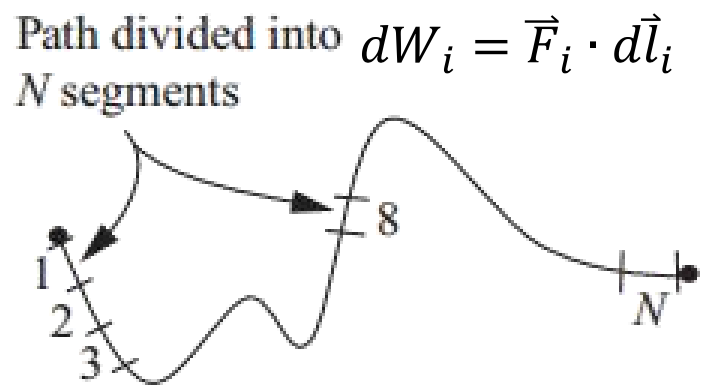
- Electric Potential
  - Work and energy
  - Potential difference and Potential
  - Potential field due to charges
  - Equipotential lines / surfaces

- E-field Loop Theorem
  - Electric field circulation
  - Conservative fields
  - Gradient

- Poisson’s and Laplace’s Equations

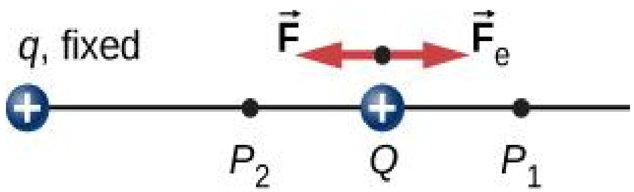
1.1 Path Integral

- Consider the work done by a force (the electric force) as it moves an object (a point charge) along a path.

Straight path Parallel $\vec{F}$ & $\vec{l}$	Straight path $\vec{F}$ & $\vec{l}$ with an angle θ	General case: Path is a curve, $\vec{F}$ varies
		 $dW_i = \vec{F}_i \cdot d\vec{l}_i$
$W = \lvert\vec{F}\rvert\lvert d\vec{l}\rvert$	$W = \vec{F} \cdot d\vec{l} = \lvert\vec{F}\rvert\lvert d\vec{l}\rvert cos\theta$	$W = \sum dW_i = \sum \vec{F}_i \cdot d\vec{l}_i$

$$W = \int \vec{F} \cdot d\vec{l}$$

1.1 Work of moving a charge



- The work done by the applied force ***F*** on the charge Q changes the potential energy of Q.

$$W_{12} = \int_{P_1}^{P_2} \boldsymbol{F} \cdot d\boldsymbol{l}$$

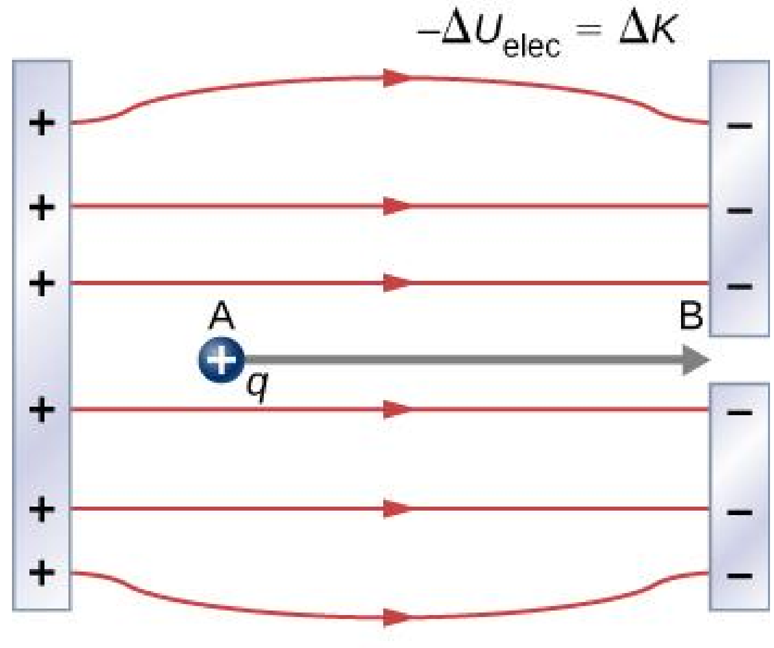
- The applied force ***F*** balances the electric force $\boldsymbol{F_e}$ on Q:

$$\boldsymbol{F} = -\boldsymbol{F_e} = -Q\boldsymbol{E}$$

- The total work for moving a charge is:

$$W_{12} = -Q \int^{P_2} \boldsymbol{E} \cdot d\boldsymbol{l}$$

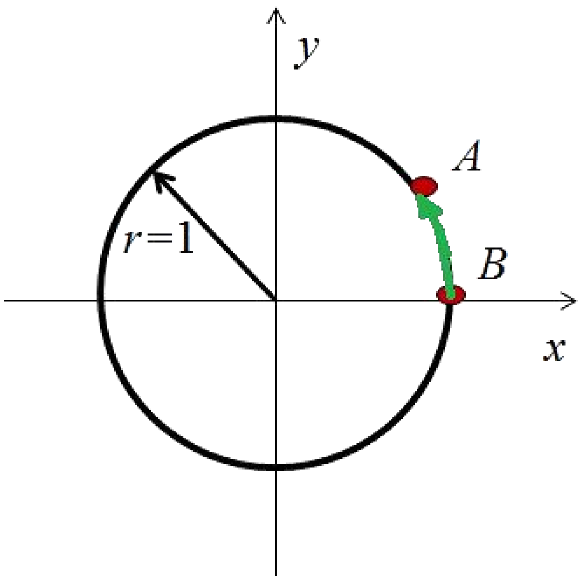
A negative sign is required since we are asking for the work required to move the charge against the field.



A charge accelerated by an electric field is analogous to a mass going down a hill.
In both cases, potential energy decreases as kinetic energy increases.

Case 1: *Work done by moving a charge in an E-field*

- Given that a nonuniform field: $\vec{E} = y\hat{x} + x\hat{y} + 2\hat{z}$
- Determine the work **expended** in carrying $q = 2$ C from point B (1, 0, 1) m to point A (0.8, 0.6, 1) m along the shorter arc of the circle $x^2 + y^2 = 1,\ z = 1$



$$\because d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$\therefore W_{AB} = -2\int_B^A (y\hat{x} + x\hat{y} + 2\hat{z}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$

$$= -2\left(\int_{x=1}^{x=0.8} ydx + \int_{y=0}^{y=0.6} xdy + \cancel{\int_{z=1}^{z=1} 2dz}\right) \quad \mathbf{= 0}$$

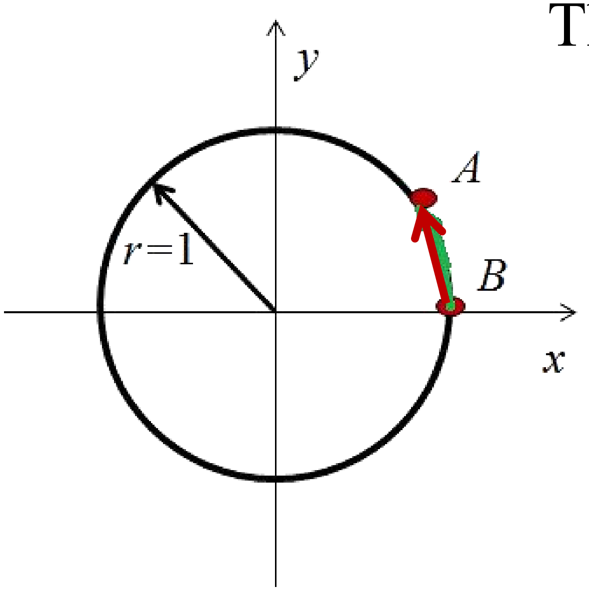
$$= -2\left(\int_{x=1}^{x=0.8} \sqrt{1-x^2}dx + \int_{y=0}^{y=0.6} \sqrt{1-y^2}dy\right)$$

$$= -\left[x\sqrt{1-x^2} + sin^{-1}x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + sin^{-1}y\right]_0^{0.6}$$

$$= -0.96\, J$$

Case 1: Work done by moving a charge in an E-field

- Given that a nonuniform field: $\vec{E} = y\hat{x} + x\hat{y} + 2\hat{z}$
- Determine the work **expended** in carrying $q = 2$ C from point B (1, 0, 1) m to point A (0.8, 0.6, 1) m along the straight line $\overline{BA}$.



The equation of the straight line path from point B to A:

$$y = -3(x - 1)$$

$$\because d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$\therefore W_{AB} = -2\int_B^A (y\hat{x} + x\hat{y} + 2\hat{z}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$

$$= -2\left(\int_{x=1}^{x=0.8} y\,dx + \int_{y=0}^{y=0.6} x\,dy + \underbrace{\cancel{\int_{z=1}^{z=1} 2dz}}_{=0}\right)$$

$$= -2\left(\int_{x=1}^{x=0.8} -3(x-1)\,dx + \int_{y=0}^{y=0.6} \left(1 - \frac{y}{3}\right) dy\right)$$

$$= -0.96\,J$$

1.2 Potential difference and Work

- Define: ***Potential difference*** *V* is the work done (by an external force) in moving a unit positive charge from the initial point to the final point:

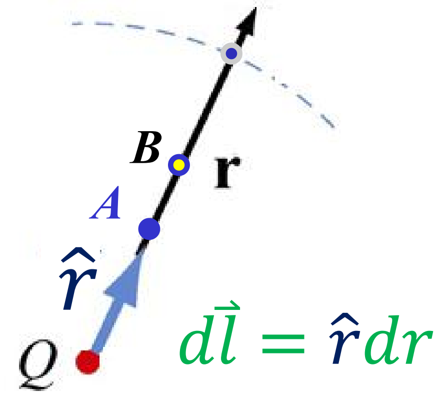
$$V = \frac{W}{q} = -\int_{init}^{final} \boldsymbol{E} \cdot d\boldsymbol{l}$$

  – Units: J/C = V (volt)

  – The potential difference between points A (final point) and B (initial point) is:

$$V_{AB} = -\int_{B}^{A} \boldsymbol{E} \cdot d\boldsymbol{l}$$

Case 2: Potential due to a point charge



- A point charge Q located at the origin of a coordinate system, creates the electric field at a distance r:

$$\vec{E} = \frac{\vec{F}}{q} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

- The ***potential difference*** at distances r_A and r_B is:

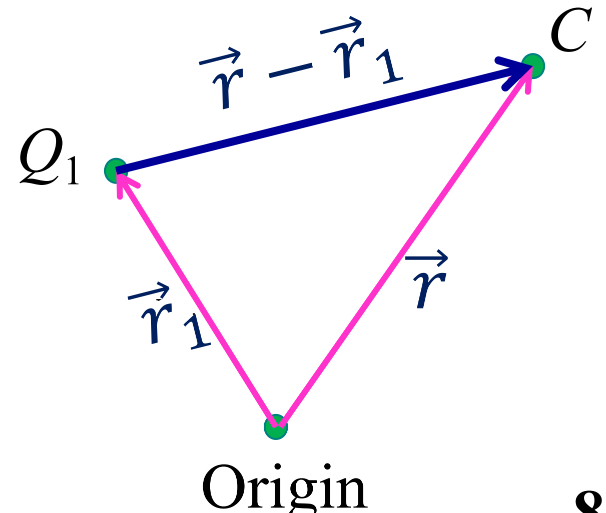
$$V_{AB} = -\int_B^A \vec{E}\cdot d\vec{l} = -\int_B^A \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}\cdot(\hat{r}dr) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B$$

- If distance r_B is infinity, V_A is the ***potential*** at A:

$$V_A = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_A} = k\frac{Q}{r_A}$$

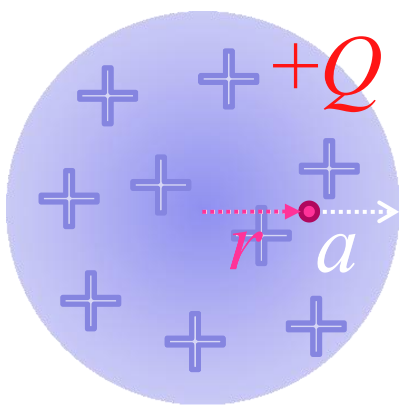
- A point charge Q_1 located at r_1, the potential created by Q_1 at a point C is:

$$V_C = \frac{Q_1}{4\pi\varepsilon_0}\frac{1}{|\vec{r} - \vec{r}_1|} = k\frac{Q_1}{|\vec{r} - \vec{r}_1|}$$



Case 3: Potential of a spherical distribution

- Positive charge $+Q$ **uniformly** distributed throughout **non-conducting solid sphere** of radius a. Calculate the electric potential in free space.



$$r > a: V = V_{\infty \to r} = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l} = -k\int_{\infty}^{r} \frac{Q}{r'^2} dr' = k\frac{Q}{r}$$

$$r < a: V = V_{\infty \to r} = V_{\infty \to a} + V_{a \to r}$$

$$= -kQ\int_{\infty}^{a} \frac{1}{r'^2} dr' - \frac{kQ}{a^3}\int_{a}^{r} r' \, dr'$$

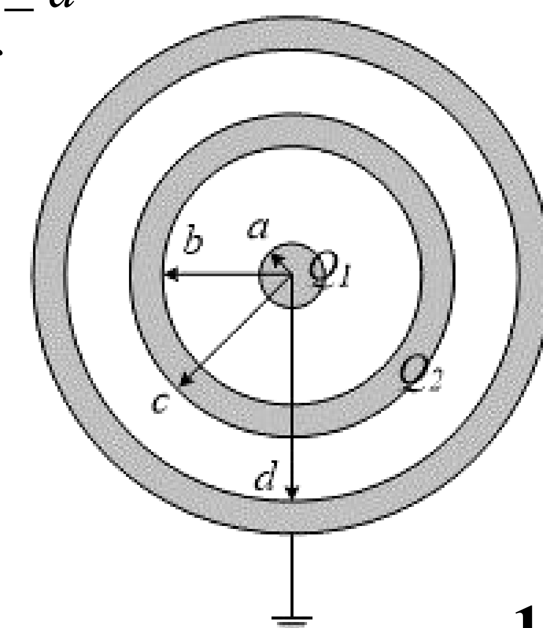
$$= +k\frac{Q}{a} - k\frac{Q}{2a^3}(r^2 - a^2)$$

$$= \frac{kQ}{2a}\left(3 - \frac{r^2}{a^2}\right)$$

$$\boxed{\vec{E} = \begin{cases} \dfrac{kQ}{r^2}\hat{r} & r > a \\ \dfrac{kQr}{a^3}\hat{r} & r < a \end{cases}}$$

Quiz 1

- A system consisting of three concentric spherical *conductors* (the inner conductor is a solid sphere, while the remaining two are spherical shells). The radius of the inner conductor is ***a***. The inner and outer radii of the middle conductor are ***b*** and ***c***. The inner radius of the outer conductor is ***d***. The charges on the inner and middle conductors are Q_1 and Q_2, respectively. The space between the conductors is air-filled.
  a) Determine the electric field intensity in the region $0 < r \leq d$
  b) When the outer conductor is grounded, the potentials of the inner and middle conductors with respect to the ground are $V_1 = 15$ V and $V_2 = 10$ V, respectively. If $a = 1$ mm, $b = 3$ mm, $c = 7$ mm, and $d = 9$ mm, determine the values of Q_1 and Q_2.



1.3 Superposition of Potential

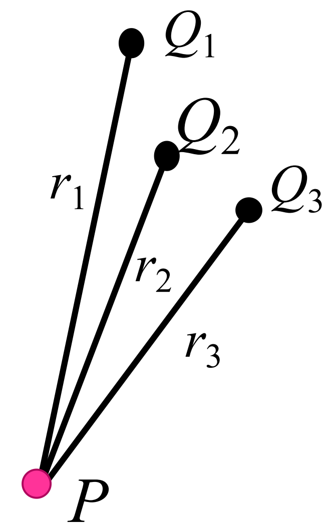
- The total electric potential at a point is the ***algebraic sum*** of the individual potentials at the point.
- For a **zero Reference** at infinity:
  1. The potential arising from a single point charge is the work done in carrying a unit positive charge from infinity to the point at which we desire the potential, and the work is independent of the path chosen between those two points.
  2. The potential field in the presence of a number of point charges is **the sum** of the individual potential fields arising from each charge.

1.3 Superposition of Potential

- Example: For 3 point charges Q_1, Q_2, and Q_3, the total electric potential at the point P is:

$$V_P = k\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3}\right)$$

where r_1 = distance from Q_1 to P
r_2 = distance from Q_2 to P
r_3 = distance from Q_3 to P



- Generally:

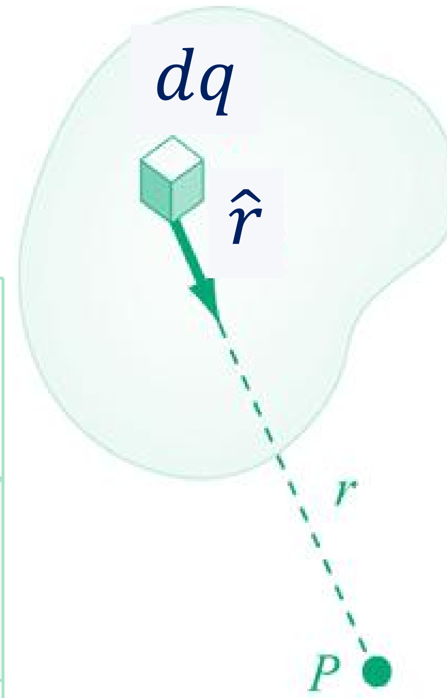
$$V_P = k\sum_{n=1}^{N}\frac{Q_n}{r_n} \Rightarrow V(\boldsymbol{r}) = k\int\frac{Q'}{|\boldsymbol{r}-\boldsymbol{r}'|} = \begin{cases} k\int_L \dfrac{\rho_L(\boldsymbol{r}')d\boldsymbol{l}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \\ k\iint_S \dfrac{\rho_S(\boldsymbol{r}')d\boldsymbol{s}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \\ k\iiint_V \dfrac{\rho_v(\boldsymbol{r}')d\boldsymbol{v}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \end{cases}$$

’ represents source

1.4 Continuous Distribution

- The charge of a point charge is considered to reside at an infinitesimally small point. Charge is usually distributed as a line charge, a surface charge or a volume charge.

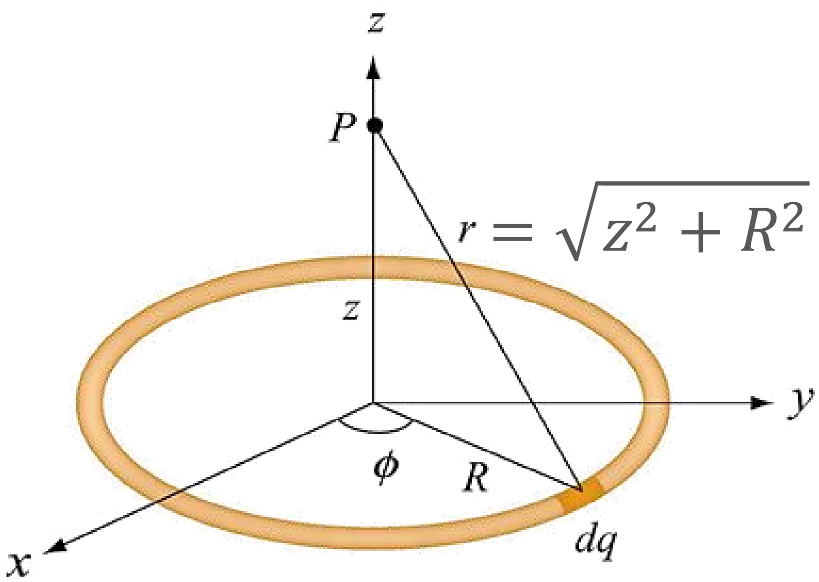
	Total Amount of Charge Q	Electric Potential
Line	$Q = \int_L \rho_l \, dl$	$V = k \int_L \frac{\rho_L}{r} dl$
Surface	$Q = \iint_S \rho_S \, dA$	$V = k \iint_S \frac{\rho_S}{r} dA$
Volume	$Q = \iiint_V \rho_V \, dV$	$V = k \iiint_V \frac{\rho_V}{r} dV$



$$dV = k\frac{dq}{r}$$

Case 4: Potential of Line charge distribution

- A **uniformly** charged ring of a radius R and charge density ρ_L. Calculate the electric potential at a distance z from the central axis in free space.



$$\boxed{dV = k\frac{dQ}{r}}$$

The electric potential at point P:

$$dV = k\frac{dQ}{r} \quad = k\frac{\rho_L R}{\sqrt{z^2+R^2}}d\phi$$

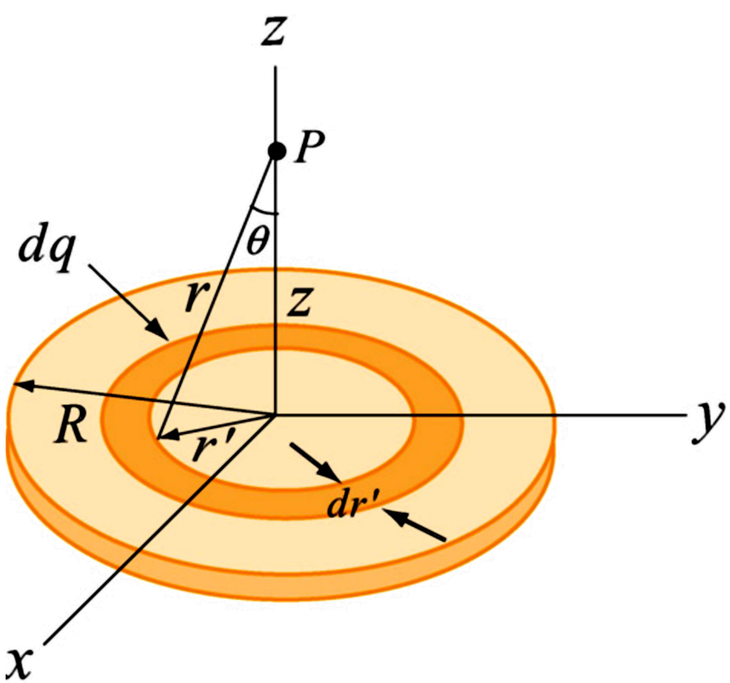
$$\therefore V = \int dV = k\frac{\rho_L R}{\sqrt{z^2+R^2}}\int_0^{2\pi} d\phi$$

$$= k\frac{2\pi\rho_L R}{\sqrt{z^2+R^2}}$$

If $z >> R$: $\quad V \approx k\frac{2\pi\rho_L R}{z} = k\frac{Q}{z}$

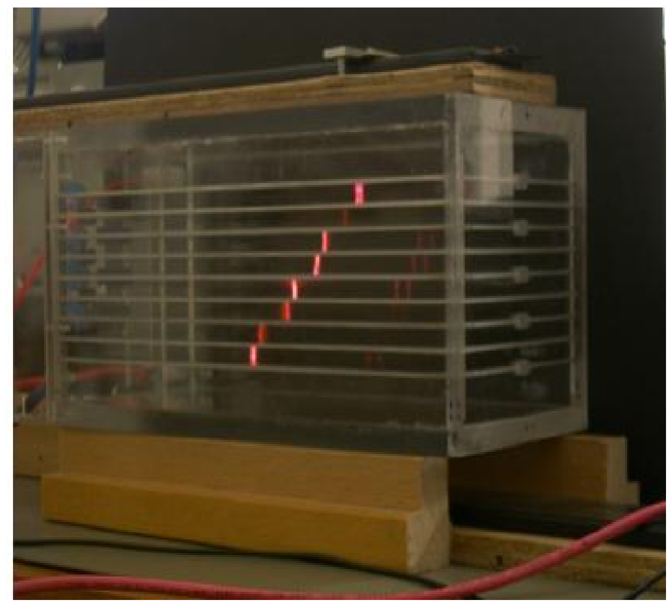
Quiz 2: Surface charge distribution

- A **uniformly** charged disk of a radius R and charge density ρ_S. Calculate the electric potential at a distance z from the central axis in free space.



Quiz 3: Highest Possible Voltage

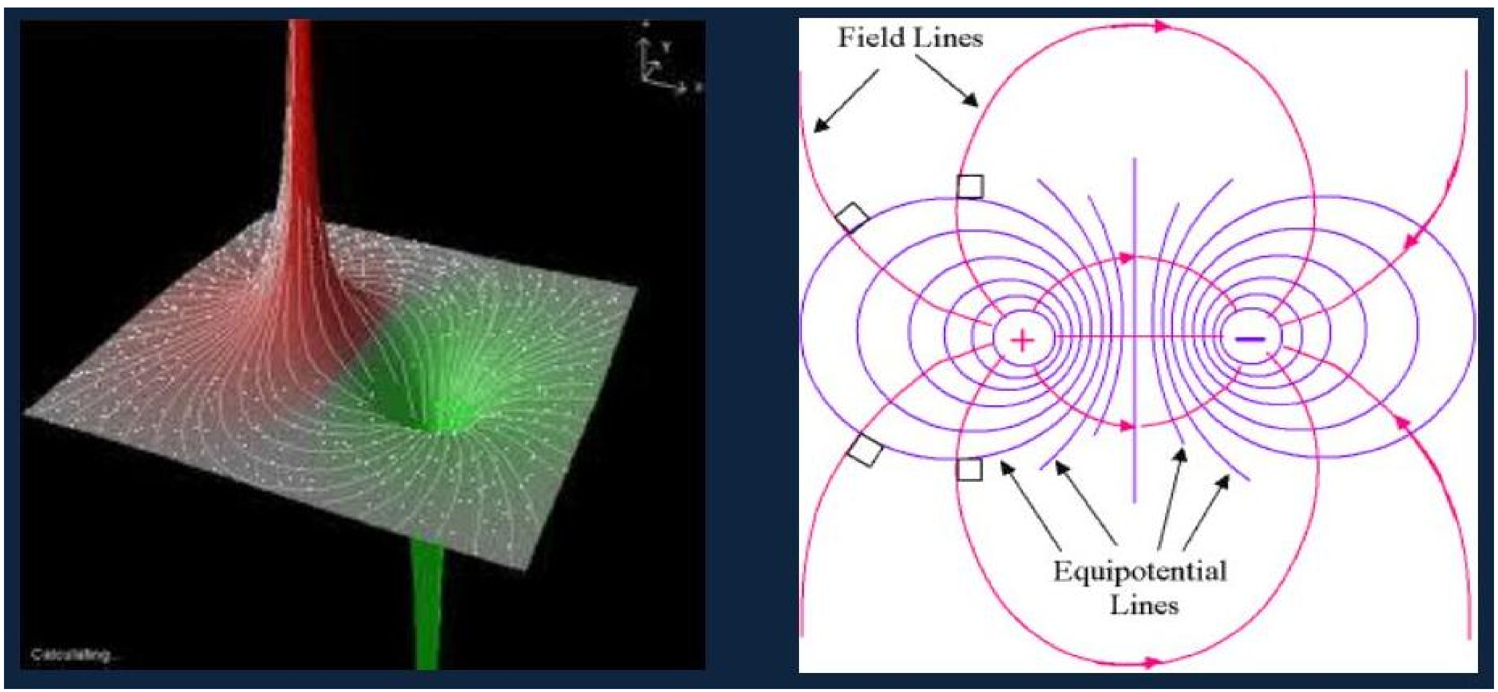
- Dry air can support a maximum electric field strength of about 3.0×10^{6} V/m. Above that value, the field creates enough ionization in the air to make the air a conductor. This allows a discharge or spark that reduces the field.
- What, then, is the maximum voltage between two parallel conducting plates separated by 2.5 cm of dry air?



A spark chamber

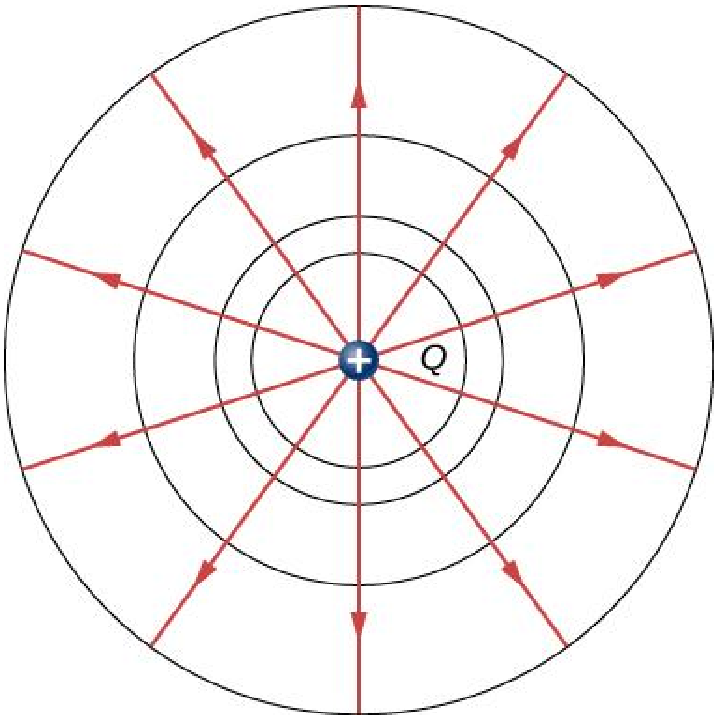
1.5 Equipotential lines / surfaces

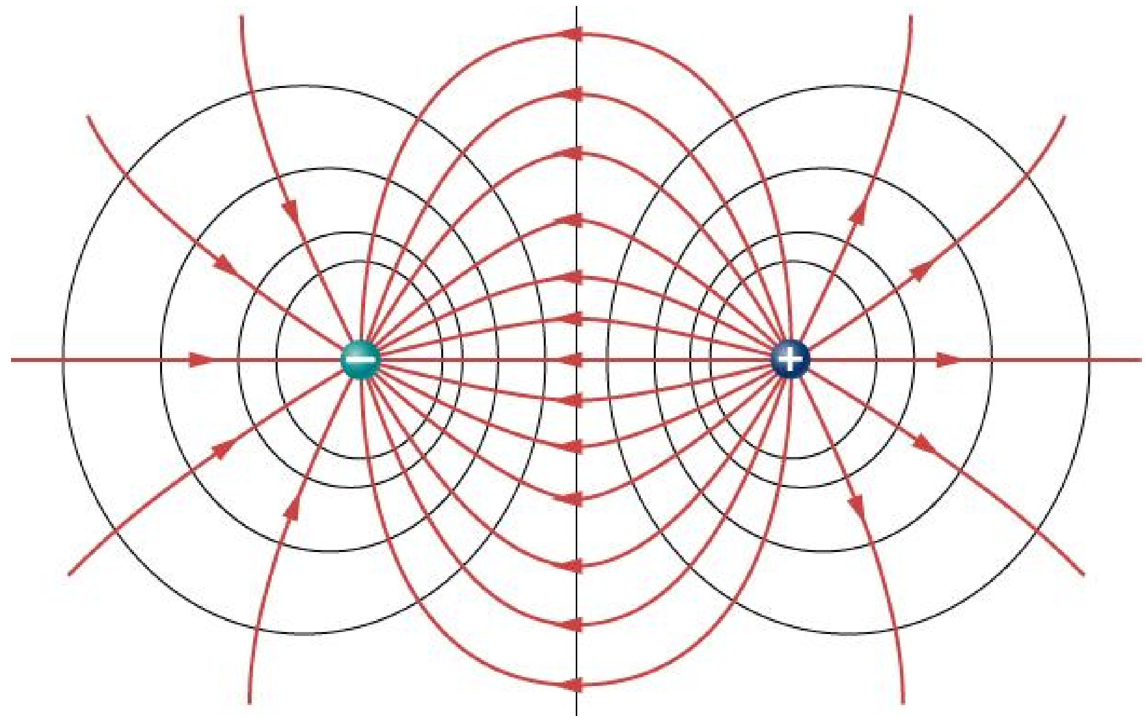
- To represent electric potentials (voltages) pictorially:
  - red arrows: the magnitude and direction of the electric field
  - black lines: places where the electric potential is constant.
    - These are called ***equipotential surfaces*** in three dimensions, or ***equipotential lines*** in two dimensions. The



1.5 Equipotential lines / surfaces

- Examples:



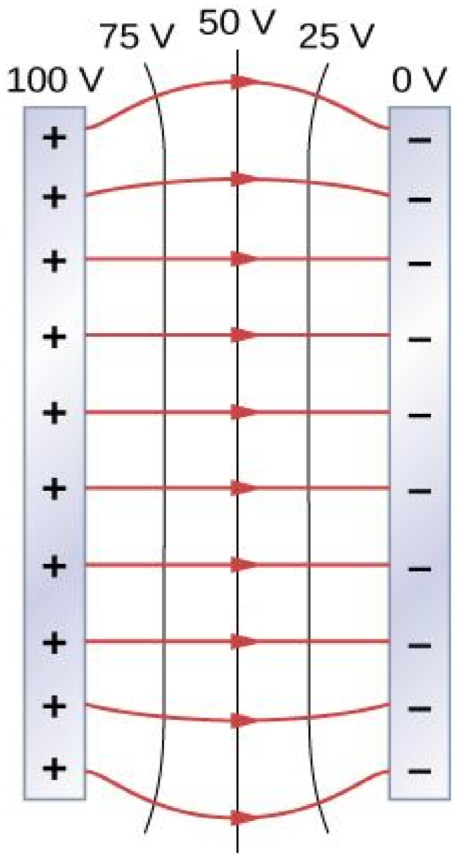


(a) Single Positive charge (b) Pair of opposite charges

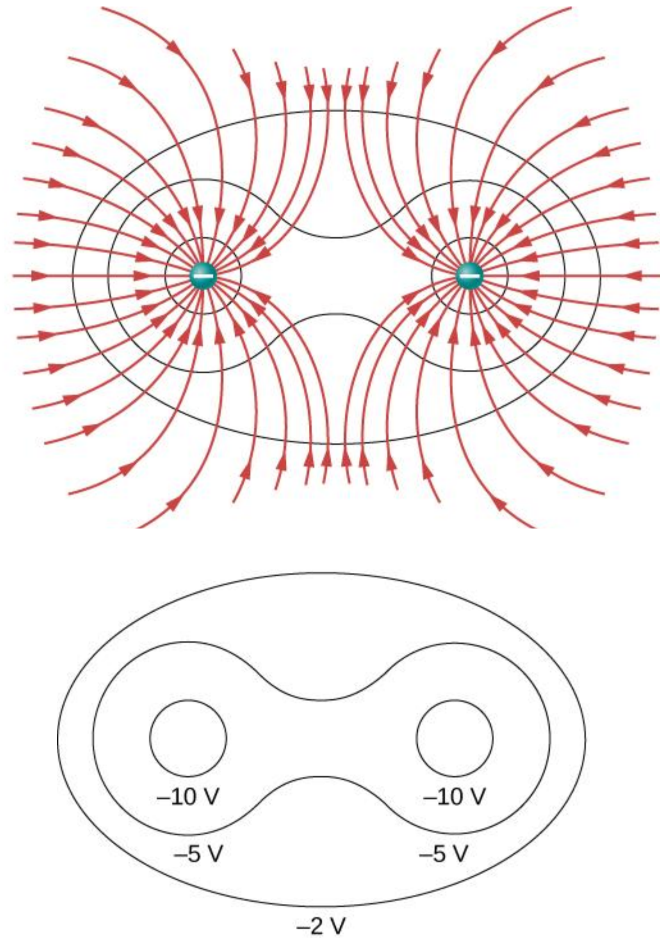
The electric field lines and equipotential lines for

1.5 Equipotential lines / surfaces

- More examples:



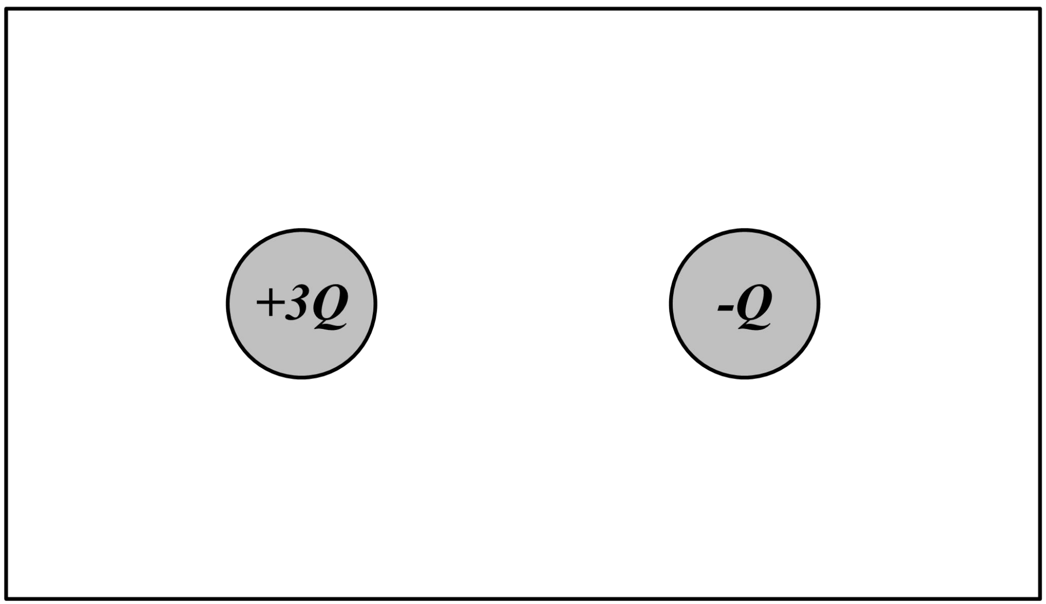
(a) Two oppositely charged metal plates;



(b) Two point charges with the same type.

Quiz 4

- A positive point charge + 3Q and a negative point charge －Q are placed in free space as shown below.
- Draw the ***equipotential lines*** and ***electric field lines*** inside the boxed region.



2.1 E-field Circulation

- Define the ***voltage*** or ***potential difference*** between points A and B as the work per unit of charge required to move the charge from A to B:

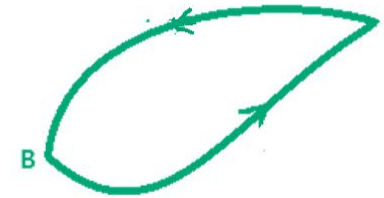
$$V_{AB} = \frac{W_{AB}}{Q} = -\int_{B}^{A} \vec{E} \cdot d\vec{l} = V_A - V_B$$

- The potential of a point, such as point A, should be based on the potential of a **reference point**, usually the potential at infinite distance to the source, set as $V_\infty = 0$:

$$V_A = V_A - V_\infty = -\int_{\infty}^{A} \vec{E} \cdot d\vec{l}$$

2.1 E-field Circulation - Conservative Fields

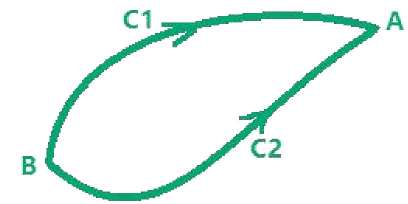
- If we move the charge from point B to point A, then return to B, moving along the path *around*, the net work done is zero, so:



$$\int_B^B \vec{E} \cdot d\vec{l} = \oint_C \vec{E} \cdot d\vec{l} = 0$$

  – The circulation equals to zero means the static E-field is ***conservative***.

- Break the close loop C into two parts, c_1 and c_2:



$$\oint_C \vec{E} \cdot d\vec{l} = \int_{c_1} \vec{E} \cdot d\vec{l} - \int_{c_2} \vec{E} \cdot d\vec{l} = 0$$

  – The voltage between two points is only determined by the relative position of the two points regardless of the path taken.

$$\oint_C \vec{E} \cdot dl = 0$$

Loop Theorem: Work done by moving a charge around a closed loop must be zero. (i.e., the static electric field is **conservative**)

2.2 Gradient - Potential changing rate

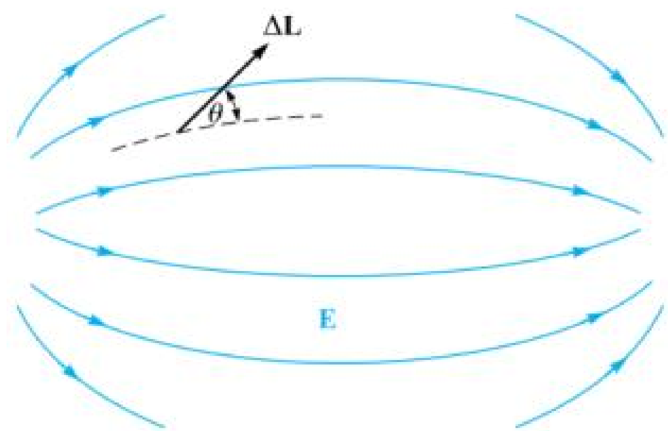
- Considering the general line integral relationship:

$$V = -\int \boldsymbol{E} \cdot d\boldsymbol{L}$$

- Apply it to a very short element of length ΔL along which $\boldsymbol{E}$ is essentially constant, leading to an incremental potential difference $\Delta V = -\boldsymbol{E} \cdot \Delta \boldsymbol{L}$

- Choose an incremental vector element of length $\Delta \boldsymbol{L} = \Delta L \boldsymbol{a}_L$ and multiply its magnitude by the component of $\boldsymbol{E}$ in the direction of $\boldsymbol{a}_L$ to obtain the small potential difference

$$\Delta V = -\boldsymbol{E} \cdot \Delta \boldsymbol{L} = -E \Delta L \cos\theta$$



- Pass to the limit and obtain

$$\frac{dV}{dL} = -E\cos\theta = -\boldsymbol{E} \cdot \boldsymbol{a}_L$$

2.2 Gradient - Maximum changing rate

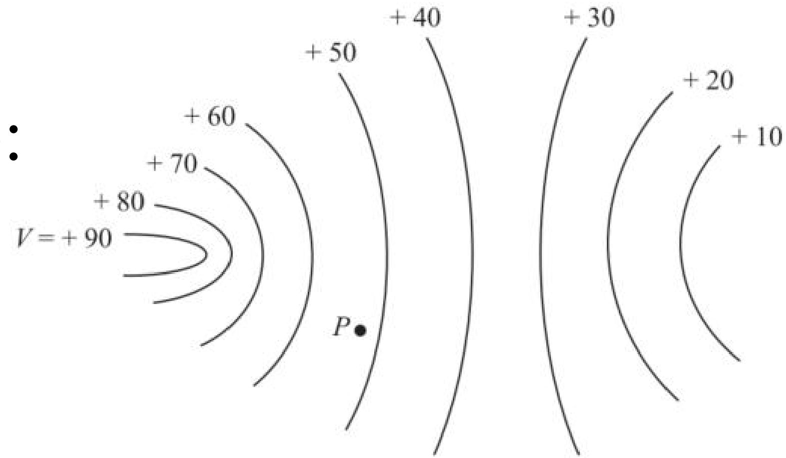
- In which direction should $\Delta \boldsymbol{L}$ be placed to obtain a maximum value of ΔV?
  - When $\cos\theta = -1$, i.e. $\boldsymbol{a_L}$ points in the direction opposite to **E**

$$\left.\frac{dV}{dL}\right|_{max} = E$$

- Therefore, we know:
  1. The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.
  2. This maximum value is obtained when the direction of the distance increment is opposite to E or, in other words, the direction of E is opposite to the direction in which the potential is increasing the most rapidly.

2.2 Gradient - Direction

- Considering the equipotential lines of the field ($\Delta V = 0$ along any line):



- If $\Delta\boldsymbol{L}$ is directed along an equi-potential line:

$$\Delta V = -\boldsymbol{E} \cdot \Delta\boldsymbol{L} = 0$$

  – As neither $\boldsymbol{E}$ nor $\Delta\boldsymbol{L}$ is zero, there must be $\boldsymbol{E} \perp \Delta\boldsymbol{L}$.

  – $\boldsymbol{E}$ is always perpendicular to the equipotential lines / surfaces.

- So the electric field intensity could be expressed as:

$$\boldsymbol{E} = -\left.\frac{dV}{dL}\right|_{max} \boldsymbol{a}_N$$

  – $\boldsymbol{a}_N$ is the unit vector normal to the equipotential surface and directed toward the higher potentials

2.2 Gradient (梯度)

- Definition: the gradient of a scalar field V is:

$$Gradient\ of\ V = grad(V) = \left.\frac{dV}{dL}\right|_{max} \boldsymbol{a_N} = \frac{dV}{dN} \boldsymbol{a_N}$$

  – where $\boldsymbol{a_N}$ is a unit vector normal to the equipotential surface and points in the direction of increasing values of V;

  – $\frac{dV}{dN}$ means $\left.\frac{dV}{dL}\right|_{max}$ occurs in the direction of $\boldsymbol{a_N}$.

- The relationship between V and $\boldsymbol{E}$ is:

$$\boldsymbol{E} = -grad(V)$$

- For dV in Cartesian CS:

$$\left.\begin{aligned} dV &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ dV &= -\boldsymbol{E}\cdot d\boldsymbol{L} = -E_x dx - E_y dy - E_z dz \end{aligned}\right\} \boldsymbol{E} = -\left(\frac{\partial V}{\partial x}\boldsymbol{a_x} + \frac{\partial V}{\partial y}\boldsymbol{a_y} + \frac{\partial V}{\partial z}\boldsymbol{a_z}\right)$$

2.2 Gradient - vector operator ∇

- In Cartesian CS:

$$grad(V) = \frac{\partial V}{\partial x}\boldsymbol{a_x} + \frac{\partial V}{\partial y}\boldsymbol{a_y} + \frac{\partial V}{\partial z}\boldsymbol{a_z}$$

- Define a vector operator "del"

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{a_x} + \frac{\partial}{\partial y}\boldsymbol{a_y} + \frac{\partial}{\partial z}\boldsymbol{a_z}$$

- The the gradient could be expressed as

$$grad(V) = \nabla V$$

- The relationship between V and $\boldsymbol{E}$ is:

$$\boldsymbol{E} = -\nabla V$$

2.2 Gradient - in three CSs

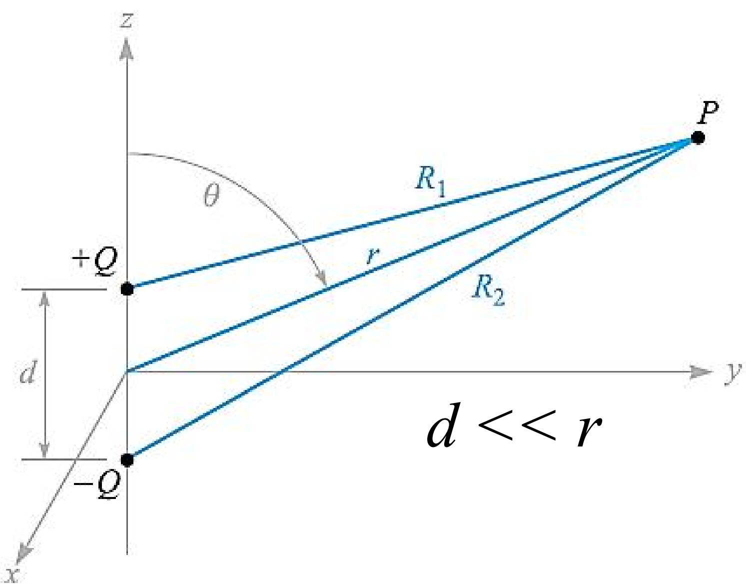
- Gradient in different coordinate systems:
  - Cartesian: $\nabla V = \frac{\partial V}{\partial x}\boldsymbol{a_x} + \frac{\partial V}{\partial y}\boldsymbol{a_y} + \frac{\partial V}{\partial z}\boldsymbol{a_z}$
  - Cylindrical: $\nabla V = \frac{\partial V}{\partial r}\boldsymbol{a_r} + \frac{1}{r}\frac{\partial V}{\partial \varphi}\boldsymbol{a_\varphi} + \frac{\partial V}{\partial z}\boldsymbol{a_z}$
  - Spherical: $\nabla V = \frac{\partial V}{\partial r}\boldsymbol{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\boldsymbol{a_\varphi}$

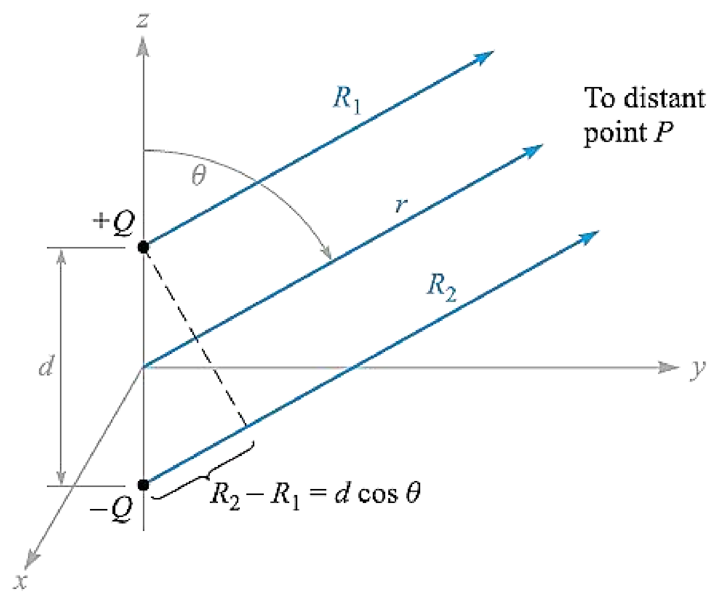
Quiz 5

- Given the potential field, $V = 2x^2y - 5z$ and a point P(−4, 3, 6), find several numerical values at point P:
  a) the potential V
  b) the electric field intensity $\boldsymbol{E}$
  c) the direction of $\boldsymbol{E}$
  d) the electric flux density $\boldsymbol{D}$
  e) the volume charge density ρ_v

Quiz6: Dipole (a pair of different charges)

- A pair of charges of equal magnitude but opposite sign is called an ***electric dipole***. When the charges are symmetrically placed along the z axis, and the point of observation is very far away ($d << r$, d is much smaller compare with r)
  - can approximate r_1 and r_2 as almost parallel
- Find the potential field and electric field generated by this dipole.





3.1 Divergence and Gradient

- Two functions related to the electric field:

$$\boldsymbol{E} = -\nabla V$$

$$\nabla \cdot \boldsymbol{D} = \nabla \cdot \varepsilon \boldsymbol{E} = \rho_v$$

  – Therefore, consider to combine them:

$$\nabla \cdot \varepsilon(-\nabla V) = \rho_v$$

  – In Cartesian CS:

$$\left(\frac{\partial}{\partial x}\boldsymbol{a_x} + \frac{\partial}{\partial y}\boldsymbol{a_y} + \frac{\partial}{\partial z}\boldsymbol{a_z}\right) \cdot \varepsilon\left(-\left(\frac{\partial V}{\partial x}\boldsymbol{a_x} + \frac{\partial V}{\partial y}\boldsymbol{a_y} + \frac{\partial V}{\partial z}\boldsymbol{a_z}\right)\right) = \rho_v$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

  – Define the symbol $\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, then

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$

Poisson's equation

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3.2 Laplace's equation

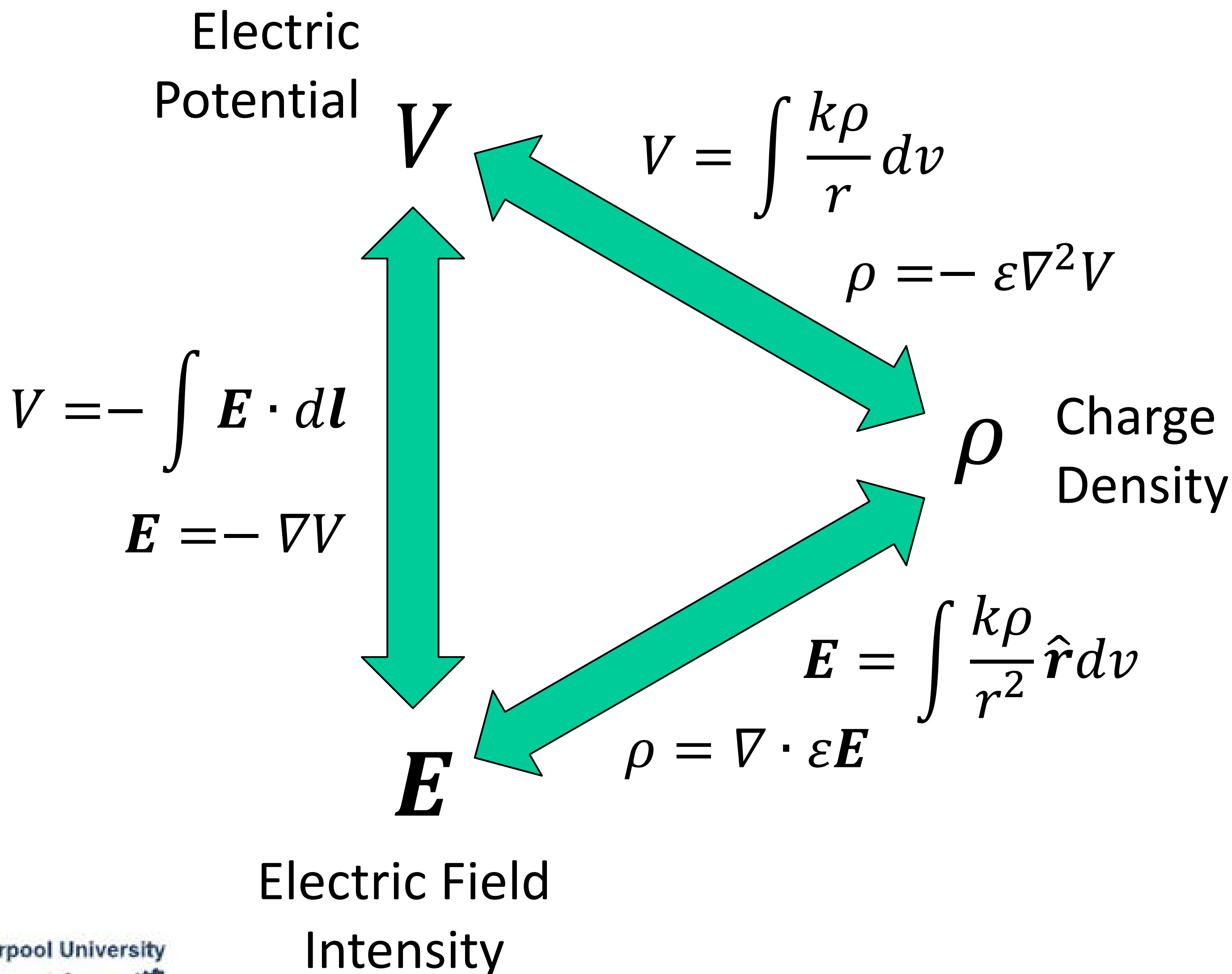
- If $\rho_v = 0$, then the Poisson's equation changes to:

$$\nabla^2 V = 0$$

  - which is Laplace's equation.
  - The symbol ∇^2 is read as "Laplacian of"

- Laplace's equations in three CSs:
  - Cartesian: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$
  - Cylindrical: $\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0$
  - Spherical: $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}$

*The relationship among V, **E** and ρ*



Next ...

- Maxwell’s equation II - Electric field loop theorem
  - Curl
  - Stoke’s Theorem
  - Integral and Differential forms
- Conductors and Dielectrics
  - Ideal conductors
  - Electric Equilibrium
  - Dielectrics and Permittivity
- Boundary Conditions