CAN207 Continuous and Discrete Time Signals and Systems

Lecture-6

LTI Systems & Convolution

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Room EE322

Content

- 1. Importance of LTI systems
  - Strategy of analysis
  - Introduction of “convolution”
  - Impulse response of a system
- 2. Calculation of convolution
  - Conv. sum for DT systems
  - Conv. integral for CT systems
- 3. Convolution properties
- 4. Properties of LTI systems

1.1 Review: system properties

- Here listed 6 basic properties of systems:
  - 1. Memory;
  - 2. Invertibility;
  - 3. Causality;
  - 4. Stability;
  - 5. Linearity;

    For CT system: $\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t)$

    For DT system: $\alpha x_1[n] + \beta x_2[n] \rightarrow \alpha y_1[n] + \beta y_2[n]$
  - 6. Time invariance.

    For CT system: $x(t - t_0) \rightarrow y(t - t_0)$

    For DT system: $x[n - n_0] \rightarrow y[n - n_0]$

LTIC

LTID

1.2 Strategy of analysing LTI systems

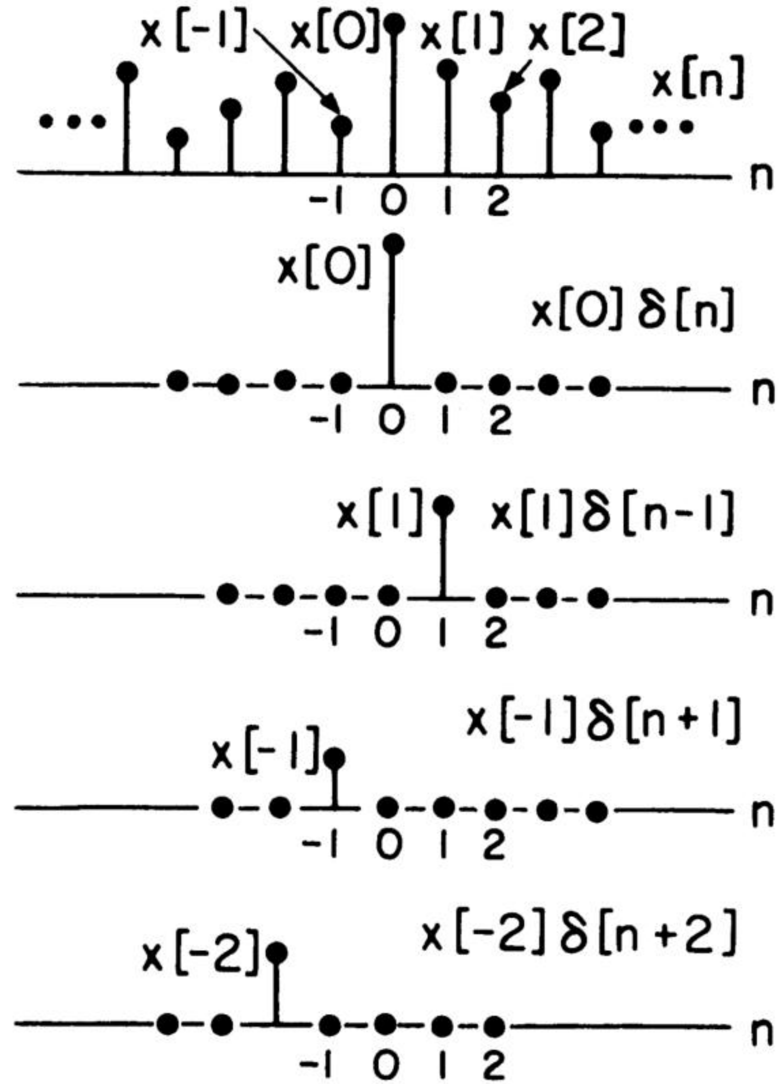
- LTI systems possess the ***superposition property***.
  - Input (linearly combined) → Output (linearly combined)
- Strategy:
  - Decompose input signal into a linear combination of *basic signals*;
  - Choose basic signals so that responses are easy to compute.
- Basic signals?

delayed impulses	⟷	convolution	in Time Domain
complex exponentials	⟷	Fourier analyses	in Frequency Domain

1.3 LTID system's basic signal - δ[n]

- A DT signal x[n] can be considered as a sequence of impulses δ[n-k], each one has a weighting function x[k].

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[-1]\delta[n+1] + \ldots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



A general discrete time signal expressed as a superposition of weighted, delayed unit impulses.

1.3 LTID impulse response h[n]

- Input signal can be expressed as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The LTID system response to the basic signals:
  - Each individual sequence value can be viewed as triggering a response;
  - Basic on the linearity: all the responses are added to form the total output.

$$\longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\underbrace{h_k[n]}_{\delta[n-k] \to h_k[n]}$$

  - Based on the time-invariance: $h_k[n] = h[n-k]$

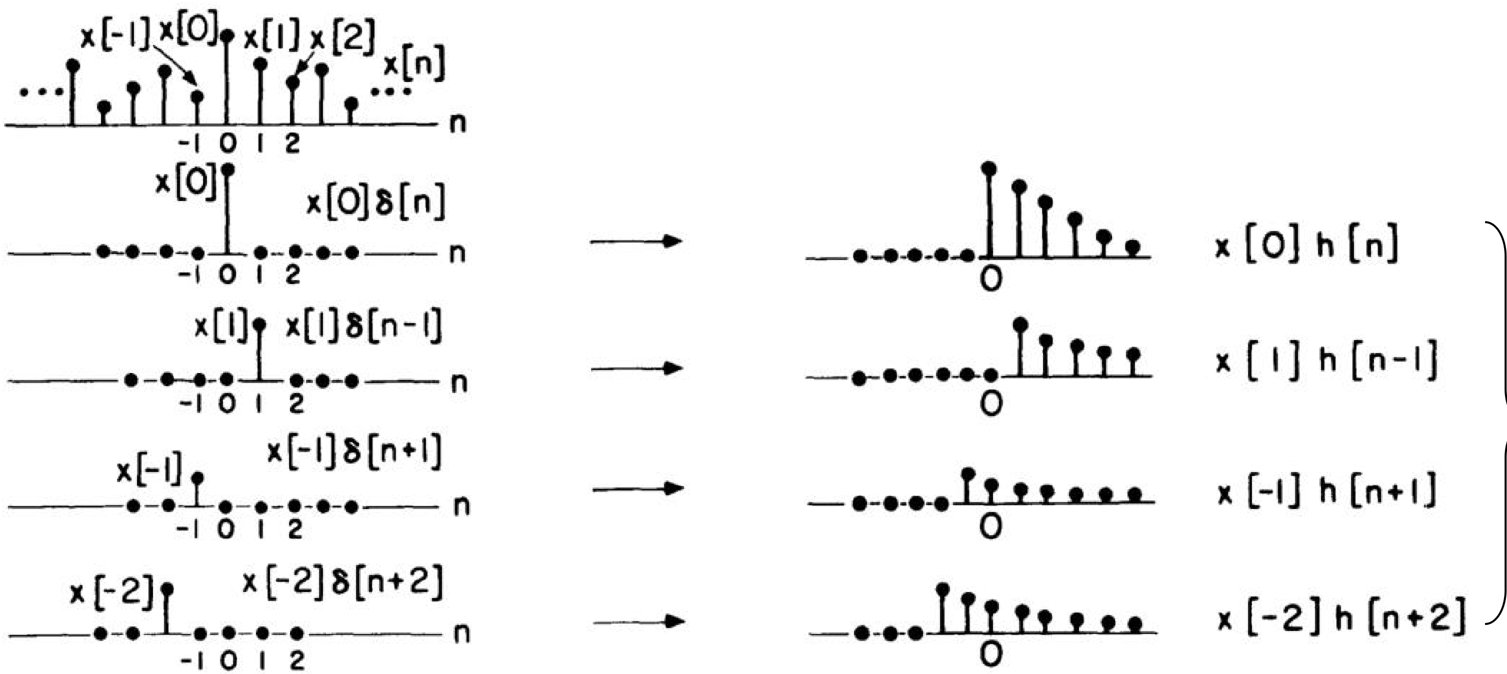
- LTID system's overal response to x[n] is:

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]}$$

Convolution Sum

1.3 LTID: graphical explaination

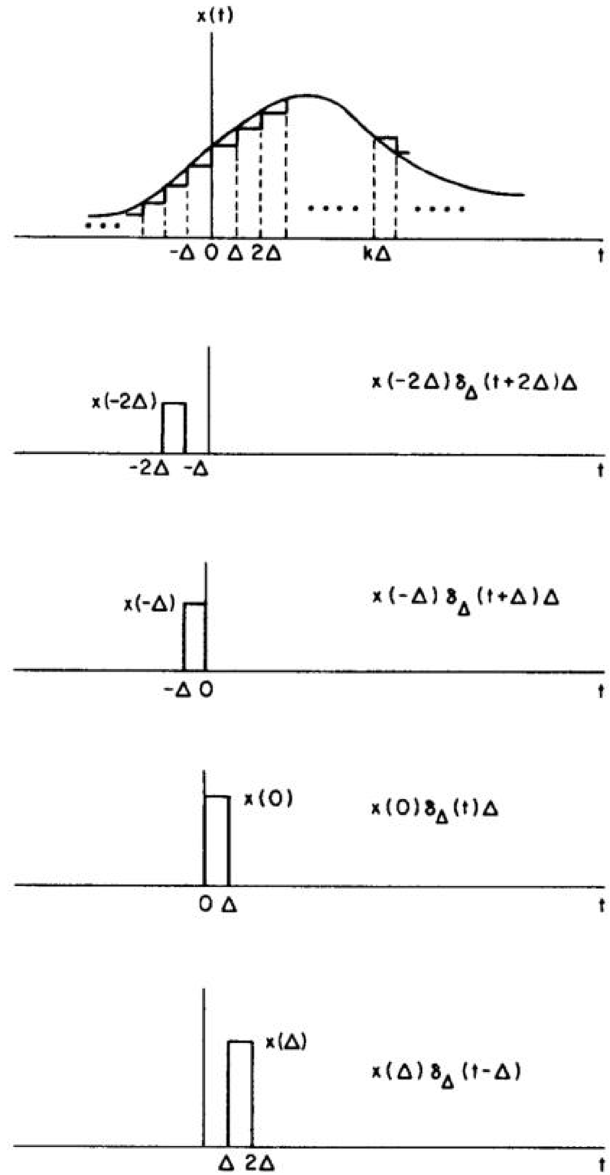
- The convolution sum for linear, time-invariant discrete-time (LTID) systems expressing the system output as a ***weighted sum of delayed unit impulse responses***.



$$\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

1.4 LTIC system's basic signal - $\delta(t)$

- A CT signal x(t) can be considered as a combination of staircases, each one with a width of Δ and weighted by $\delta_\Delta(t)$.



$$x(t) \cong x(0)\delta_\Delta(t)\Delta + x(\Delta)\delta_\Delta(t-\Delta)\Delta + x(-\Delta)\delta_\Delta(t+\Delta)\Delta + \dots$$

$$\cong \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

sifting equation

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta = \boxed{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

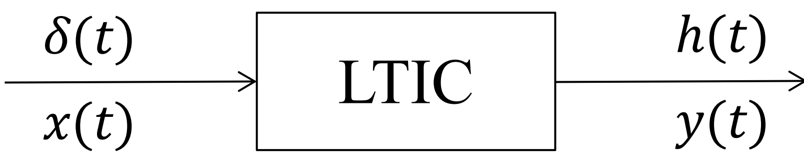
When $\Delta \to 0$,
$k\Delta \to \tau$,
$\delta_\Delta(t) \to \delta(t)$
$\Delta \to d\tau$

1.4 LTIC impulse response h(t)

- Input signal can be expressed as:

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

- The LTIC system response to the basic signals:
  - Basic on the linearity: all the responses are added to form the total output.



$$y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\underline{h_{k\Delta}(t)}\Delta$$

$\delta(t-k\Delta) \to h_{k\Delta}(t)$

  - Based on the time-invariance: $h_{k\Delta}(t) = h(t-k\Delta)$
  - Replace $k\Delta$ by τ, we have $h_{k\Delta}(t) = h(t-\tau)$, $x(k\Delta) = x(\tau)$

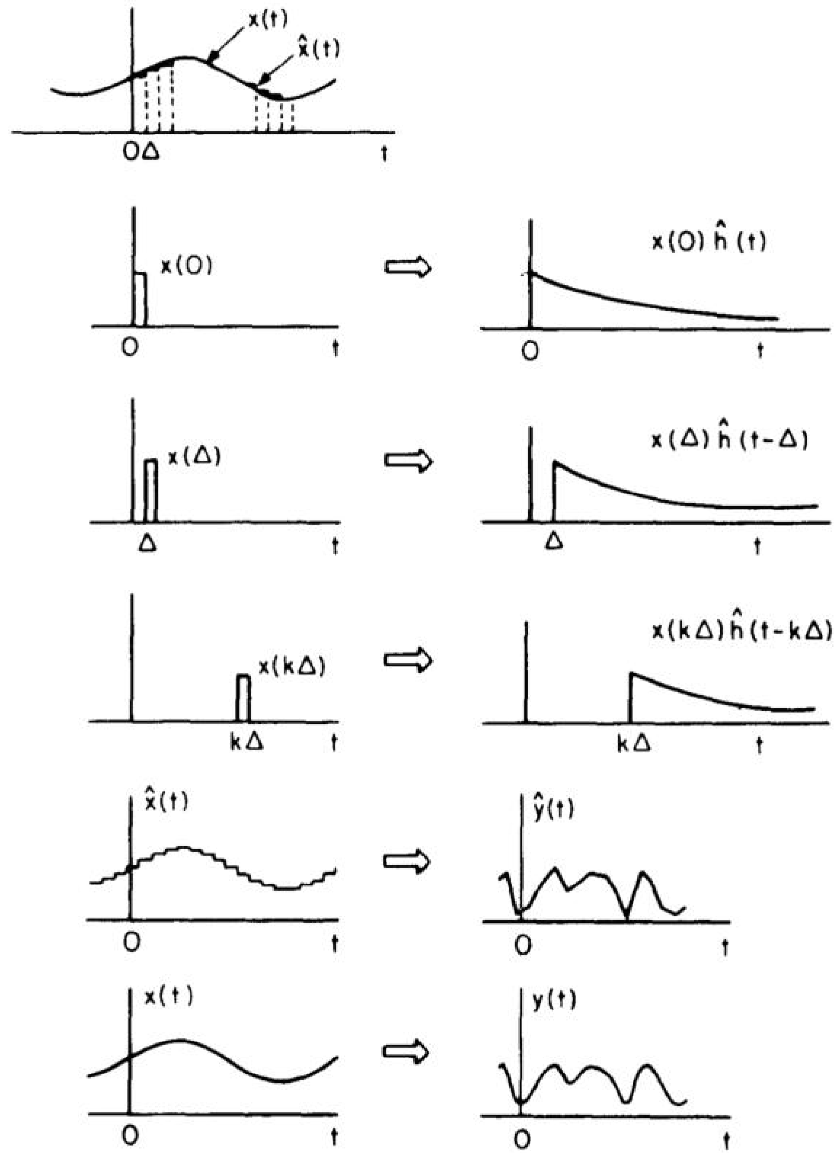
- LTIC system's overal response to x(t) is:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)}$$

Convolution Integral

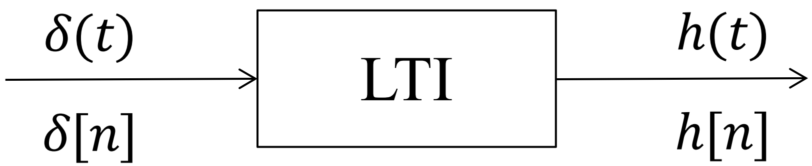
1.4 LTIC: graphical explaination

- Interpretation of the convolution integral as a superposition of the responses from each of the rectangular pulses in the representation of the input.



1.5 Impulse Response

- The impulse response of an LTI system is the output of the system when a unit impulse is applied at the input:



$$\delta(t) \rightarrow h(t)$$
$$\delta[n] \rightarrow h[n]$$

  - We often use ***h(t)*** or ***h[n]*** to represent the system's impulse response.

- Because the system is LTI, it satisfies the linearity and the time-shifting properties:

$$\alpha\delta(t - t_0) \rightarrow \alpha h(t - t_0)$$
$$\alpha\delta[n - n_0] \rightarrow \alpha h[n - n_0]$$

Quiz 1

- 1. Calculate the impulse response of the following systems:

$$y(t) = x(t-1) + 2x(t-3)$$

- 2. The impulse response of an LTIC system is given by $h(t) = e^{-3t}u(t)$. Determine the output of the system for the input signal $x(t) = \delta(t+1) + 3\delta(t-2) + 2\delta(t-6)$.

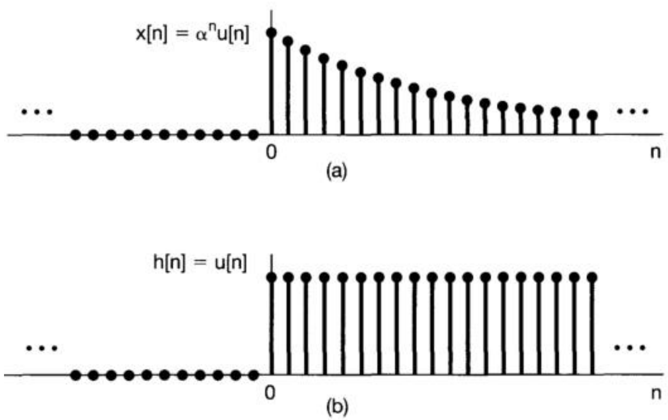
2.1 Calculate convolution sum for DT sys.

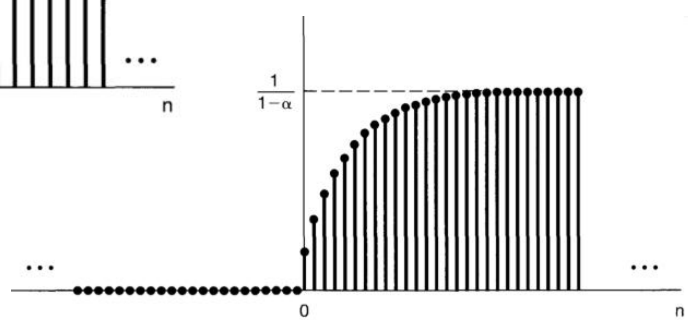
- Example 1. Consider an input x[n] and a unit impulse response h[n] given by

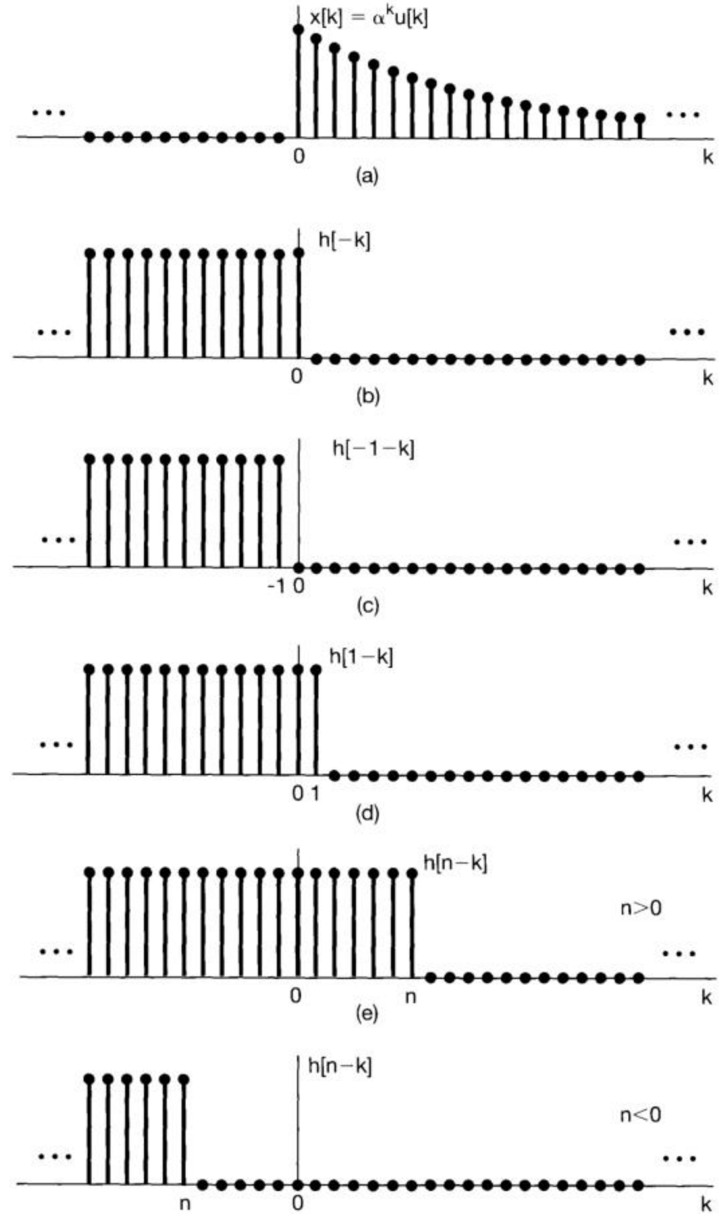
$$x[n] = \alpha^n u[n]$$

$$h[n] = u[n]$$

- Calculate $x[n] * h[n]$





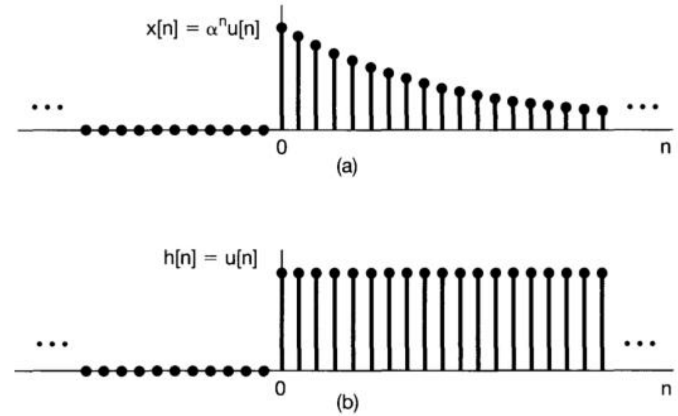


2.2 Graphical method

- Example: Consider an input x[n] and a unit impulse response h[n] given by

$$x[n] = \alpha^n u[n]$$

$$h[n] = u[n]$$



- Interval 1: $n < 0$
- Interval 2: $n \geq 0$

2.2 Graphical method

- Summary of the graphical method:
  - 1. Fix x[k];
  - 2. Time reversal h[k] → h[-k];
  - 3. Time shifting h[n-k];
  - 4. Multiply x[k]h[n-k];
  - 5. According to different *n*, have multiple intervals with different upper and lower limits for summation;
  - 6. Sum with appropriate upper and lower limits for each interval.

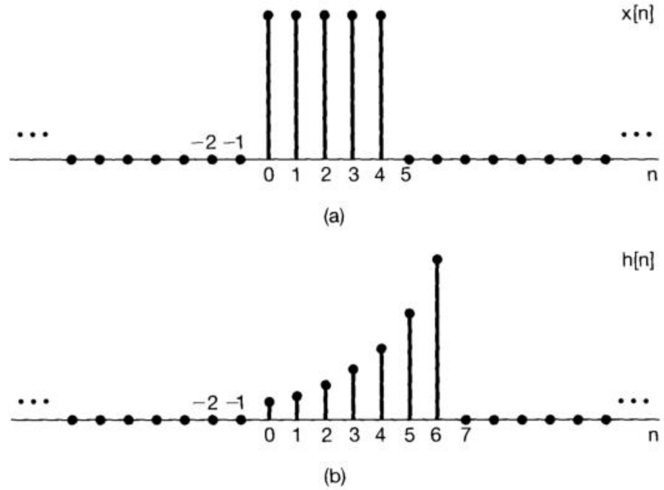
2.3 Long Multiplication method

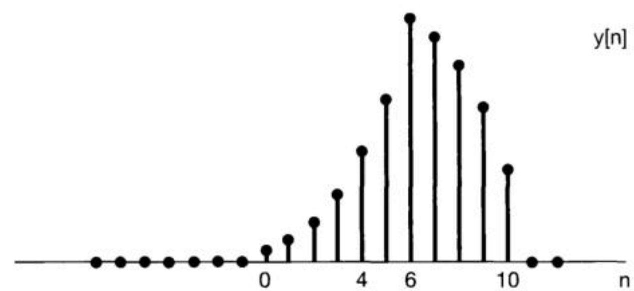
- Example 2. Change both input to finite length given by

$$x[n] = u[n] - u[n-5]$$

$$h[n] = \alpha^n(u[n] - u[n-7])$$

- Calculate $x[n] * h[n]$





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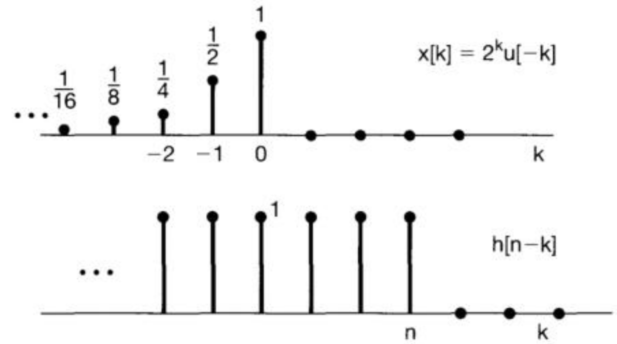
Quiz 2

- Consider an LTI system with input x[n] and unit impulse response h[n] specified as follows: $x[n] = 2^n u[-n]$

$$h[n] = u[n]$$



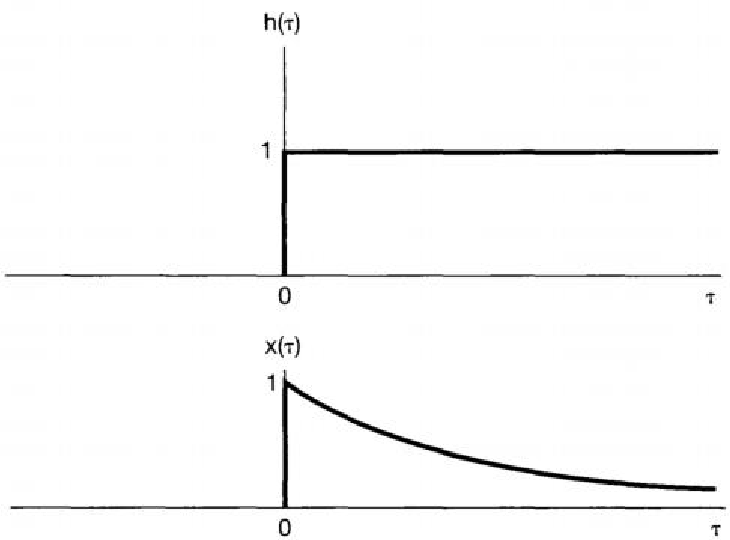
- Find the output signal y[n].

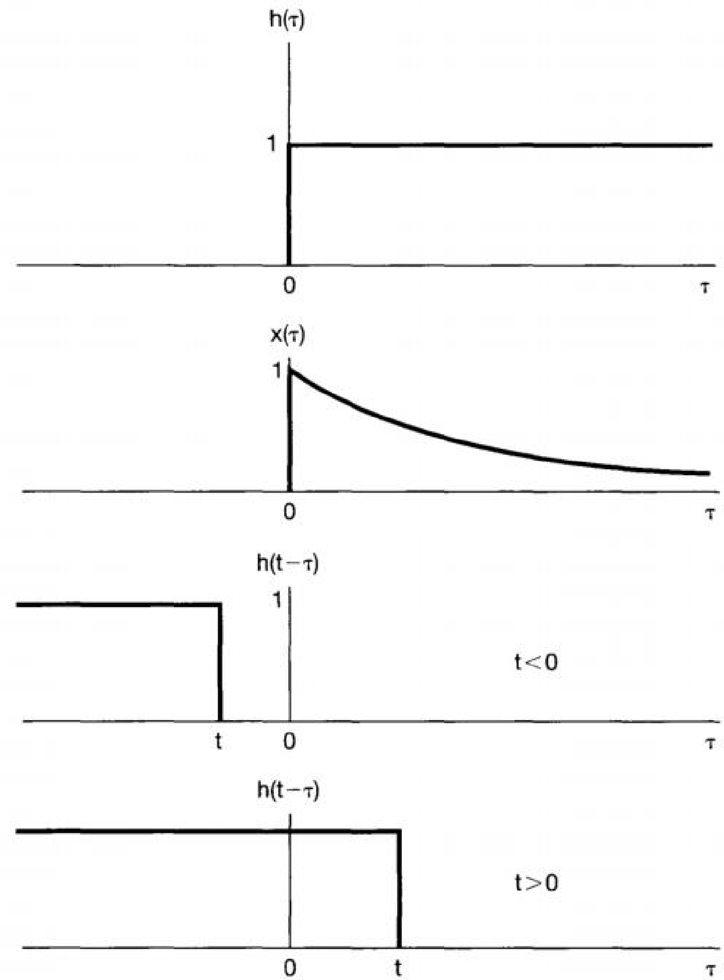
2.4 Calculate convolution integral for CT sys.

- Example 1. Let x(t) be the input to an LTI system with unit impulse response h(t) where

$$x(t) = e^{-\alpha t}u(t), \ \ \alpha > 0$$

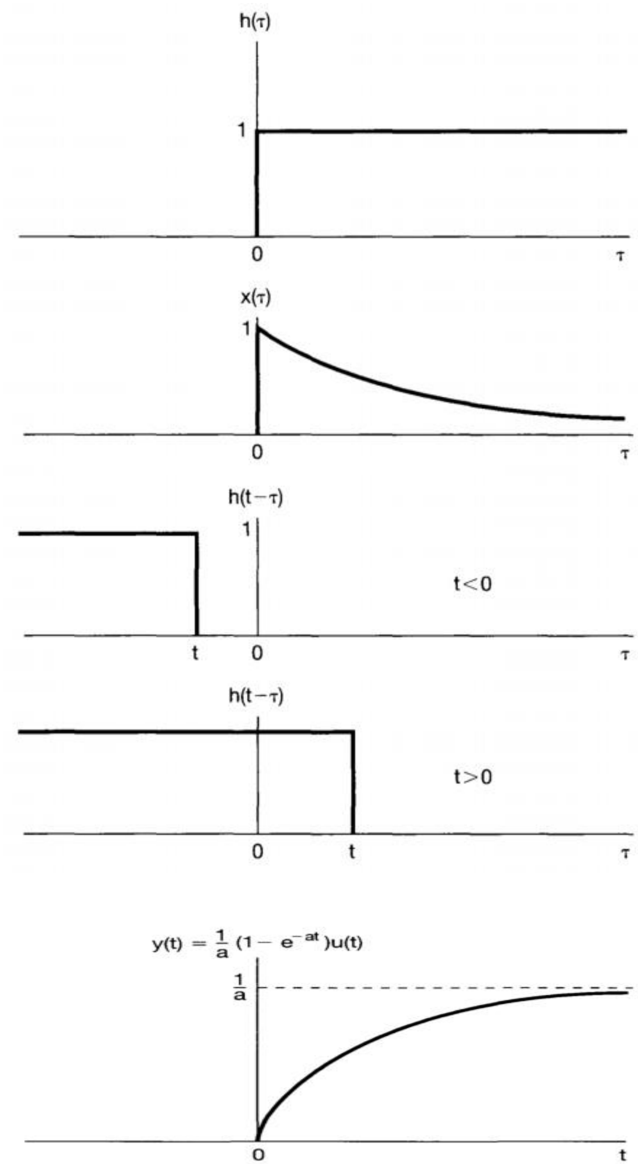
$$h(t) = u(t)$$





2.3 Calculate convolution integral for CT sys.

- Solve:
  - when $t < 0$:
  - when $t > 0$:
  - Thus, for all t, y(t) is



Quiz 3

- Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

- Find the output signal y(t) in figure and expression.

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$

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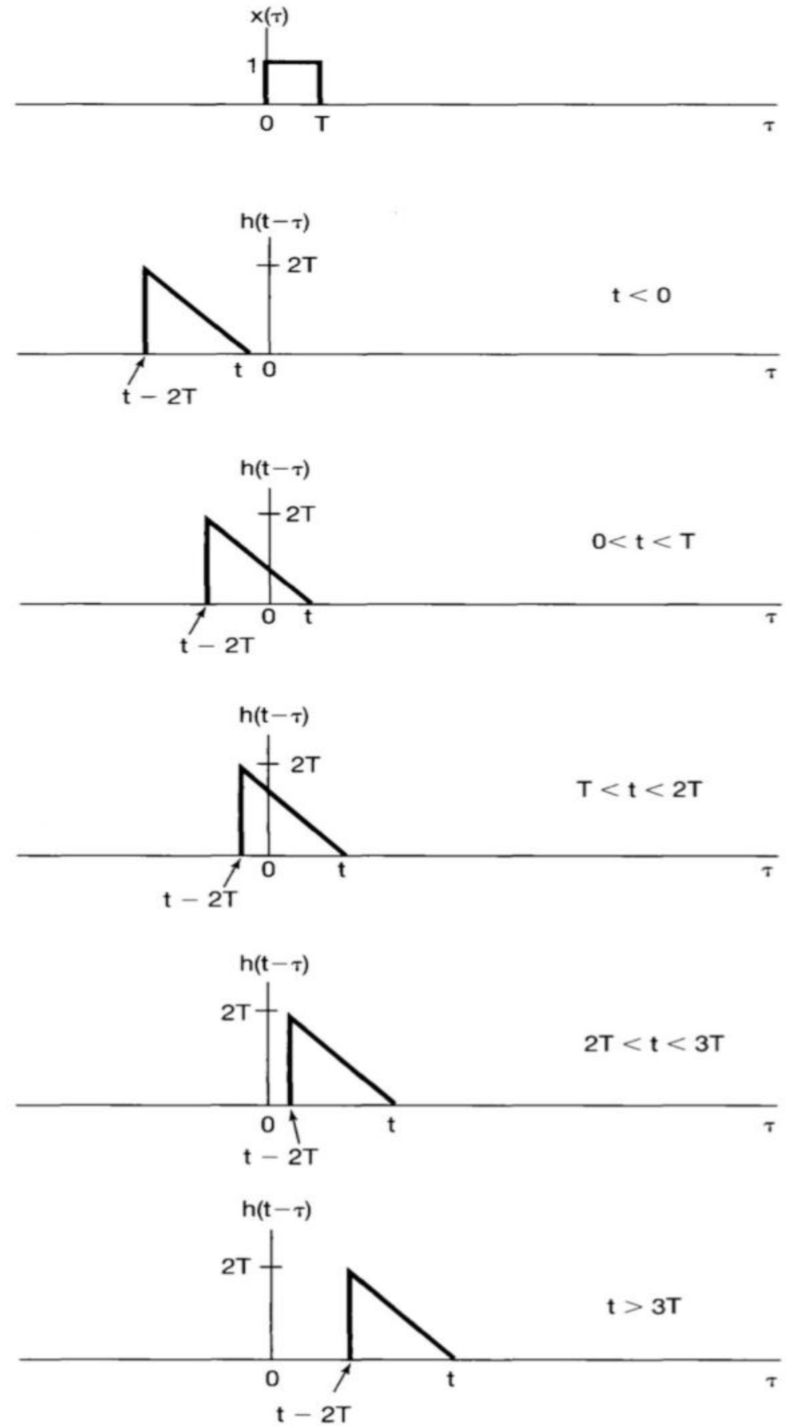

x(τ)
1
0
T
τ
h(t − τ)
2T
t < 0
t − 2T
t
0
h(t − τ)
2T
0< t < T
0
t
t − 2T
h(t − τ)
2T
T < t < 2T
0
t
t − 2T
h(t − τ)
2T
2T < t < 3T
0
t
t − 2T
h(t − τ)
2T
t > 3T
0
t
t − 2T

3. Properties of Convolution

- 1. ***Commutative*** property

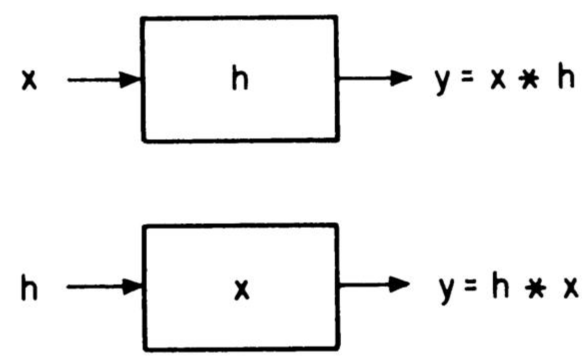
$$x[n] * h[n] = h[n] * x[n]$$
$$x(t) * h(t) = h(t) * x(t)$$

- Proof:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \xRightarrow{m=n-k} \sum_{m=+\infty}^{-\infty} x[n-m]h[m] = h[n] * x[n]$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \xRightarrow{r=t-\tau} \int_{+\infty}^{-\infty} x(t-r)h(\tau)d(-r) = h(t) * x(t)$$

- Meaning: In LTI systems, the outputs are the same if input and impulse response interchanged.



3. Properties of Convolution

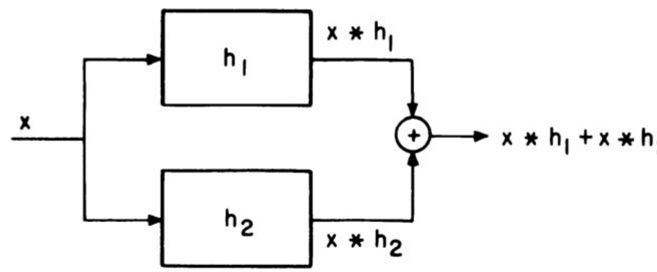
- 2. ***Distributive*** property

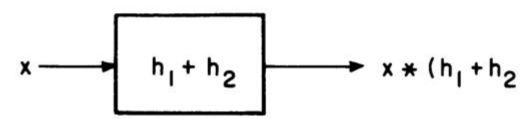
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$
$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

- Proof:

$$\sum_k x[k](h_1[n-k] + h_2[n-k]) = \sum_k x[k]h_1[n-k] + \sum_k x[k]h_2[n-k]$$
$$= x[n] * h_1[n] + x[n] * h_2[n]$$

- Meaning: The overall system impulse response equals the summation of the impulse responses of two parallel sub-systems.





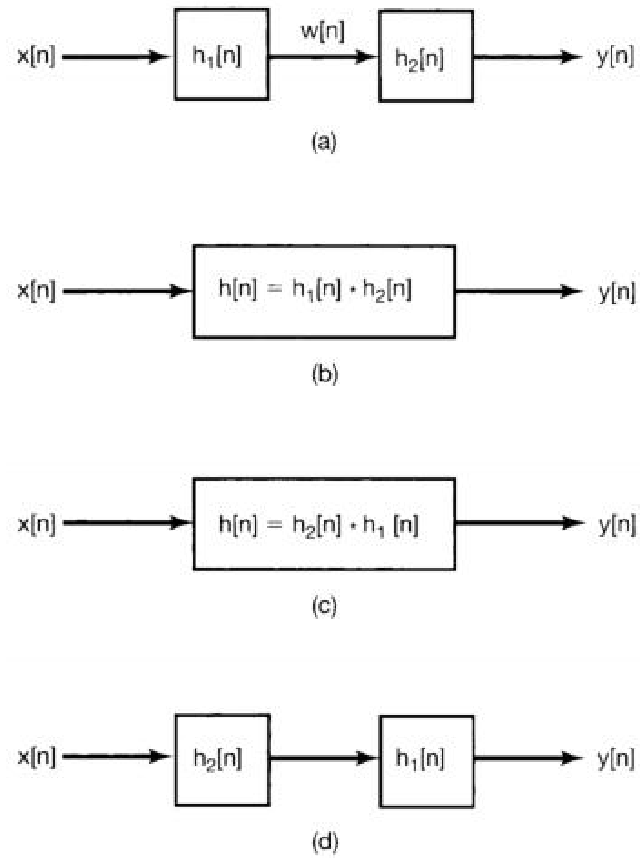
3. Properties of Convolution

- ***3. Associative property***

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$

$$(x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t))$$

- According to the associative property, the series interconnection of the two systems in (a) is equivalent to the single system in (b), whose impulse response is the convolution of two sub-systems.
- By using the commutative property, $h_1[n]$ and $h_2[n]$ could be in either order, i.e. two sub-systems are interchangable.



3. Properties of Convolution

- ***4. Shifting property***
  - if $x_1(t) * x_2(t) = g(t)$ then
  $$x_1(t - T_1) * x_2(t - T_2) = g(t - T_1 - T_2)$$
  - if $x_1[n] * x_2[n] = g[n]$ then
  $$x_1[n - N_1] * x_2[n - N_2] = g[n - N_1 - N_2]$$

- ***5. Duration of convolution***
  - Let the non-zero durations (or widths) of $x_1(t)$ and $x_2(t)$ be denoted by T_1 and T_2. The duration of the convolution integral is T_1+T_2.
  - For DT signals $x_1[n]$ and $x_2[n]$ with length N_1 and N_2, the length of convolution sum is $N_1 + N_2 -1$.

3. Properties of Convolution

- ***6. Convolution with impulse function***

- CT: Conv. integral:

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$$\int_{-\infty}^{\infty} x(t-\tau)\delta(\tau-t_0)d\tau$$

$$= \int_{-\infty}^{\infty} x(t-t_0)\delta(\tau-t_0)d\tau$$

$$= x(t-t_0)\cancel{\int_{-\infty}^{\infty} \delta(\tau-t_0)d\tau} = x(t-t_0)$$

- DT: Conv. sum:

$$x[n] * \delta[n-n_0] = x[n-n_0]$$

$$\sum_k x[n-k]\delta[k-n_0]$$

$$= \sum_k x[n-n_0]\delta[k-n_0]$$

$$= x[n-n_0]\cancel{\sum_k \delta[k-n_0]} = x[n-n_0]$$

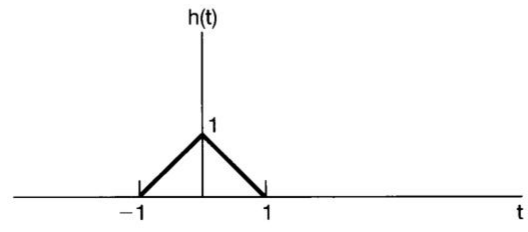
- In other words, convolving a signal with a unit impulse function whose origin is at $t = t_0$ shifts the signal to the origin of the unit impulse function.

Quiz 4

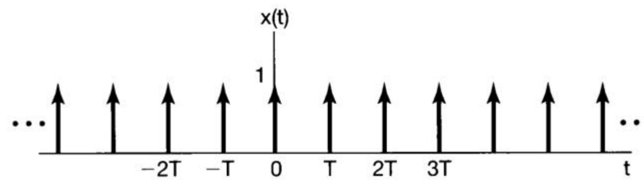
- Let h(t) be the triangular pulse shown in figure (a) and let x(t) be the impulse train depicted in figure (b). That is,

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

- Determine and sketch $y(t) = x(t) * h(t)$ for the following values of T:
  - (a) T=4;
  - (b) T=2;
  - (c) T=3/2;
  - (d) T=1.



(a)



(b)

Quiz 5

- Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 2, & 2 < t < 3 \\ 0, & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} t - 1, & 1 < t < 3 \\ 0, & \text{otherwise} \end{cases}$$

- Find the output signal y(t) in figure and expression.

4. Properties of LTI systems

- 1. Memoryless LTI system:
  - A CT system is said to be memoryless if its output y(t) at time $t = t_0$ depends only on the value of the applied input signal x(t) at the same time instant $t = t_0$. In other words, a memoryless LTIC system typically has an input–output relationship of the form

$$y(t) = Kx(t)$$

  - where K is a constant. Substituting $x(t) = \delta(t)$, the impulse response h(t) of a memoryless system can be obtained as follows:

$$h(t) = K\delta(t)$$

An LTIC system will be **memoryless** if and only if its impulse response $h(t) = 0$ for $t \neq 0$.

An LTID system will be **memoryless** if and only if its impulse response $h[n] = 0$ for $n \neq 0$.

4. Properties of LTI systems

- 2. Causal LTI system:
  - A CT system is said to be causal if the output at time $t = t_0$ depends only on the value of the applied input signal x(t) at and before the time instant $t = t_0$. The output of an LTIC system at time $t = t_0$ is given by

$$y(t_0) = \int_{-\infty}^{\infty} x(\tau)h(t_0 - \tau)d\tau$$

  - In a causal system, output $y(t_0)$ must not depend on $x(\tau)$ for $\tau > t_0$. This condition is only satisfied if the time-shifted and reflected impulse response $h(t_0 - \tau) = 0$ for $\tau > t_0$.
  - Choosing $t_0 = 0$, the causality condition reduces to $h(-\tau) = 0$ for $\tau > 0$, which is equivalent to stating that $h(\tau) = 0$ for $\tau < 0$.

An LTIC system will be **causal** if and only if its impulse response $h(t) = 0$ for $t < 0$.

An LTID system will be **causal** if and only if its impulse response $h[n] = 0$ for $n < 0$.

4. Properties of LTI systems

- 3. Stable LTI system:
  - A CT system is BIBO stable if an arbitrary bounded input signal produces a bounded output signal. Consider a bounded signal x(t) with $|x(t)| < B_x$ for all t, applied as input to an LTIC system with impulse response h(t). The magnitude of output y(t) is given by

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau$$

  - Using the Schwartz inequality, we can say that the output is bounded

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau \leq B_x \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

  - Therefore, for the output y(t) to be bounded, the integral of h(τ) within the limits [−∞, ∞] should also be bounded.

If the impulse response of an LTIC system is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

the system is BIBO stable.

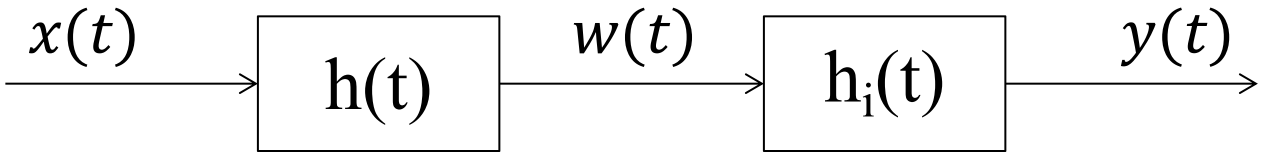
If the impulse response of an LTID system is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

the system is BIBO stable.

4. Properties of LTI systems

- 4. Invertible LTIC system:



  - The output w(t) of the system for an input signal x(t) is given by w(t) = x(t) * h(t). For the system to be invertible, we cascade a second system with impulse response hi(t) in series with the original system. The output of the second system is given by y(t) = w(t) * hi(t).
  - For the second system to be an inverse of the original system, output y(t) should be the same as x(t).
  - Substituting w(t) = x(t) * h(t) in the above expression results in the following condition for invertibility:

    x(t) = [x(t) * h(t)] * hi(t) = x(t) * [h(t) * hi(t)].

  - The above equation is true if and only if $\boldsymbol{h(t) * h_i(t) = \delta(t)}$.
  - The existence of hi(t) determines whether an LTIC system is invertible.

Quiz 6

- Determine if systems with the following impulse responses:
  - (i) $h(t) = \delta(t) - \delta(t - 2)$;
  - (ii) $h(t) = 2\,\mathrm{rect}(t/2)$;
  - (iii) $h(t) = 2\exp(-4t)\,u(t)$;
  - (iv) $h(t) = [1-\exp(-4t)]\,u(t)$;
- are memoryless, causal, and stable.

Next ...

- LTI systems described by ***differential*** and ***difference*** equations
  - – LCCDE
  - – Block diagram representation
  - – Zero-input and zero-state solutions