CAN207 Continuous and Discrete Time Signals and Systems

Lecture-8

Continuous-Time Fourier Transform

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Room EE322

Content

- 1. Fundamentals of CTFT
  - From CTFS to CTFT
  - Synthesis and Analysis equations
  - Linear and angular frequencies
  - Relationship between FT spectrum and FS coefficients
  - Convergence of CTFT: Dirichlet conditions
  - (Fourier transform of a periodic signal) optional
- 2. Fourier transform pairs
  - $e^{j\omega_0 t}$, $\delta(t)$, constant $x(t) = 1$, impulse train, square wave and sinc() function, etc.
  - Table of CTFT pairs

Recall

- A continuous-time periodic signal $x(t)$
  - $x(t) = x(t+T)$
  - Period T, angular frequency $\omega_0 = \frac{2\pi}{T}$.
- can be represented as sums of complex exponentials:
  - Synthesis equation:

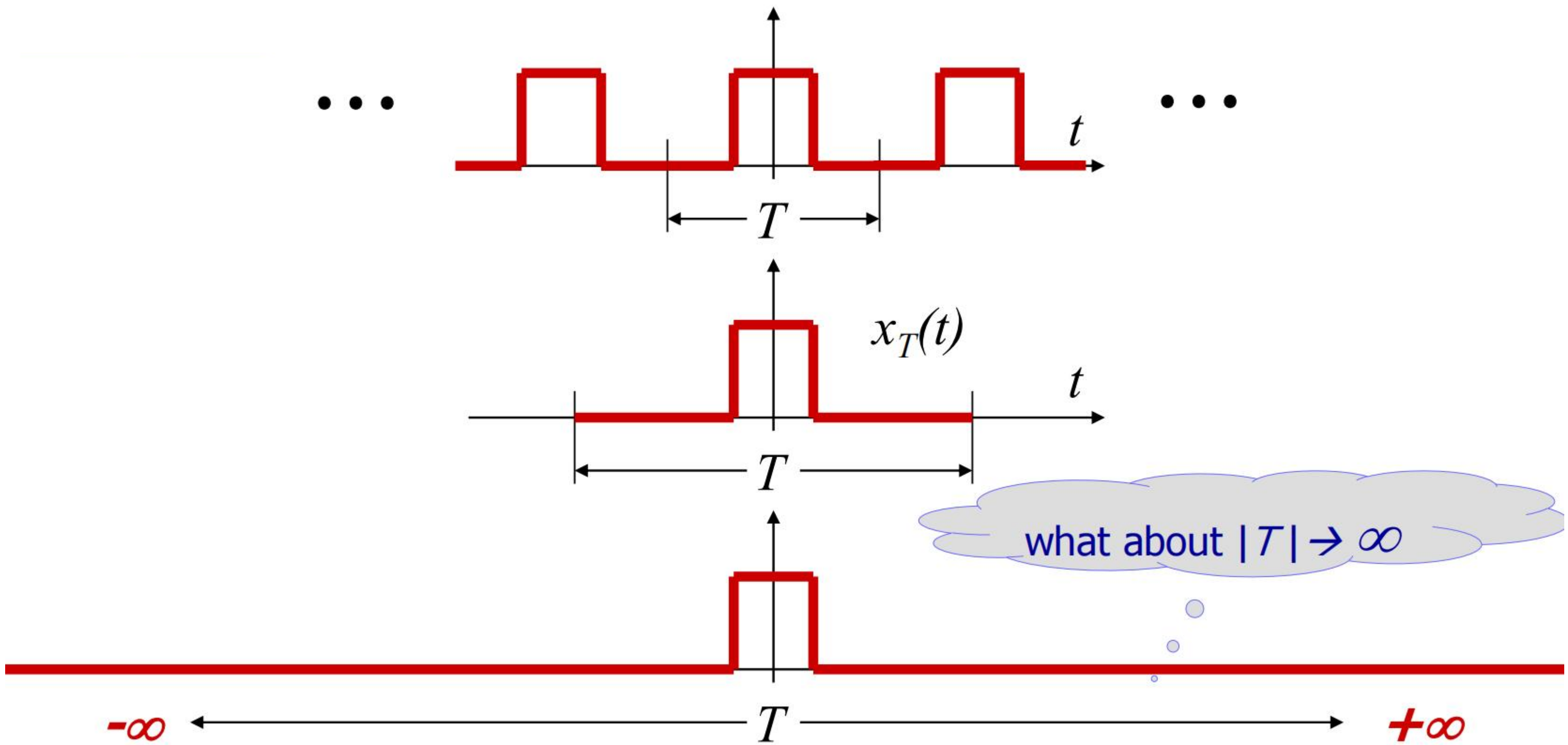
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

  - Analysis equation:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

1.1 From CTFS to CTFT

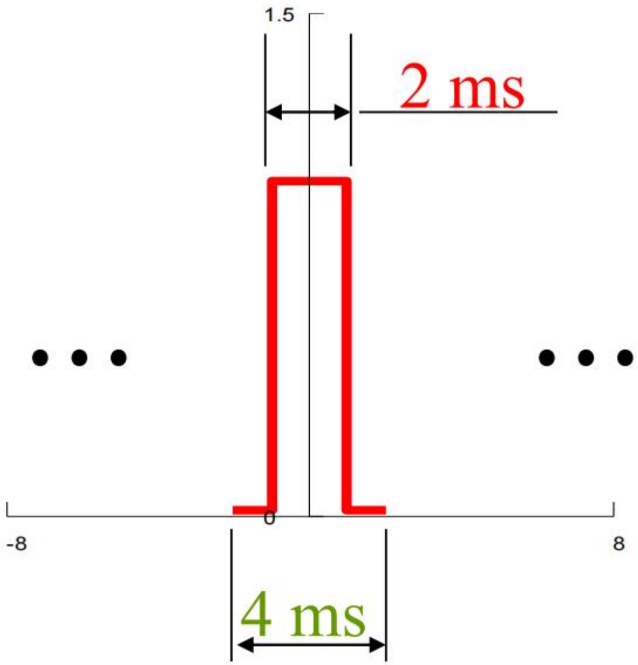
- What will happen to the spectrum of a rectangular pulse when T get increased, while keeping the same width of the pulse?

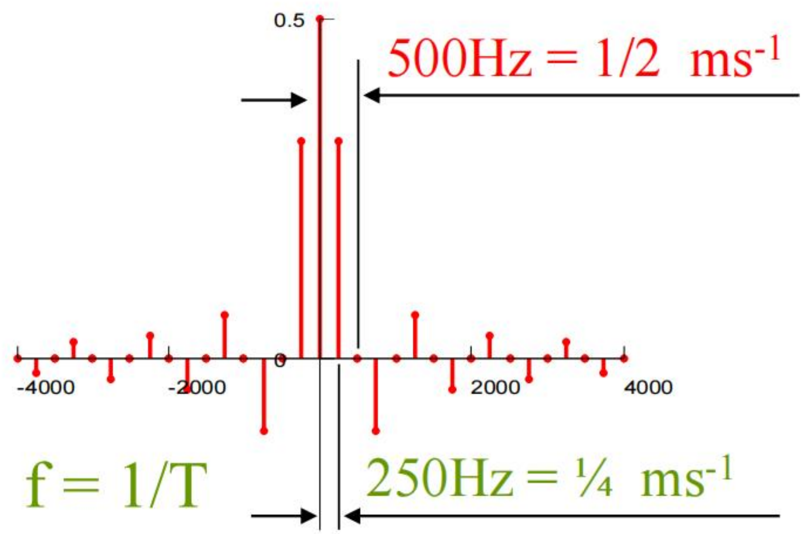


1.1 From CTFS to CTFT

- CTFS of the rectangular pulse signal (T=4ms)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n \frac{t}{T}}$$

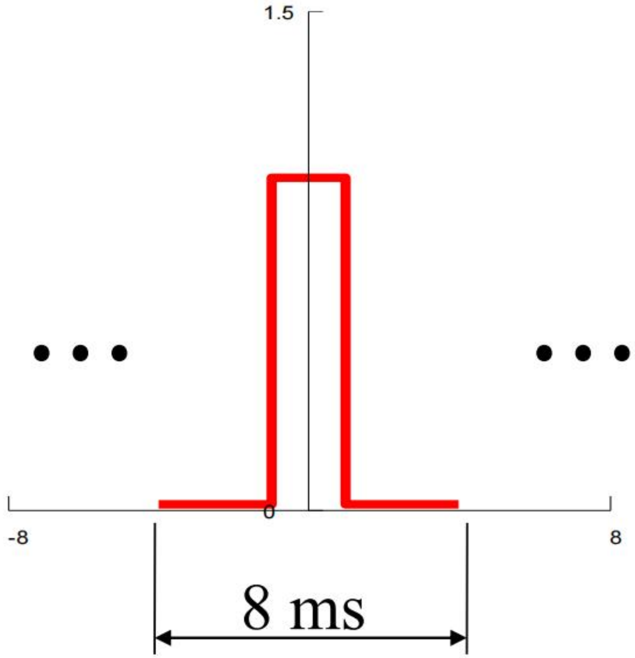


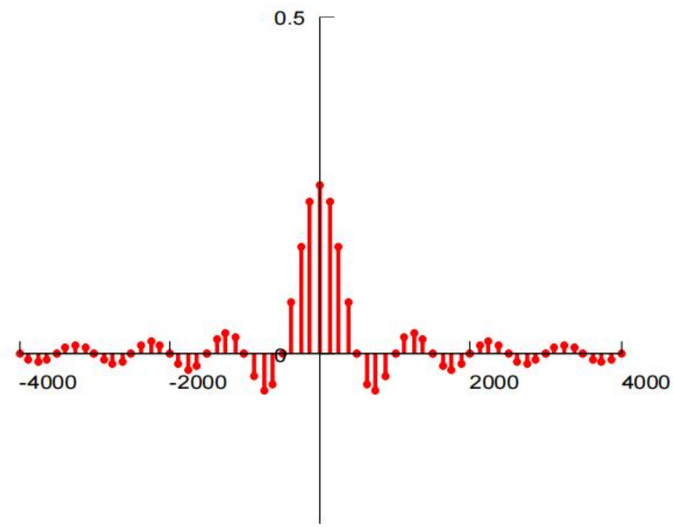


$$C_n = \frac{2}{4} sinc(n\frac{2}{4})$$

1.1 From CTFS to CTFT

- CTFS of the rectangular pulse signal (T=8ms)

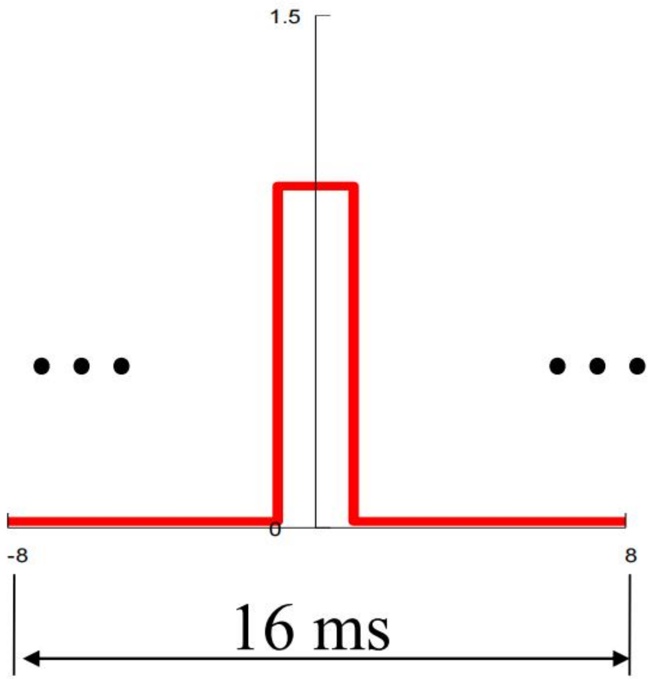


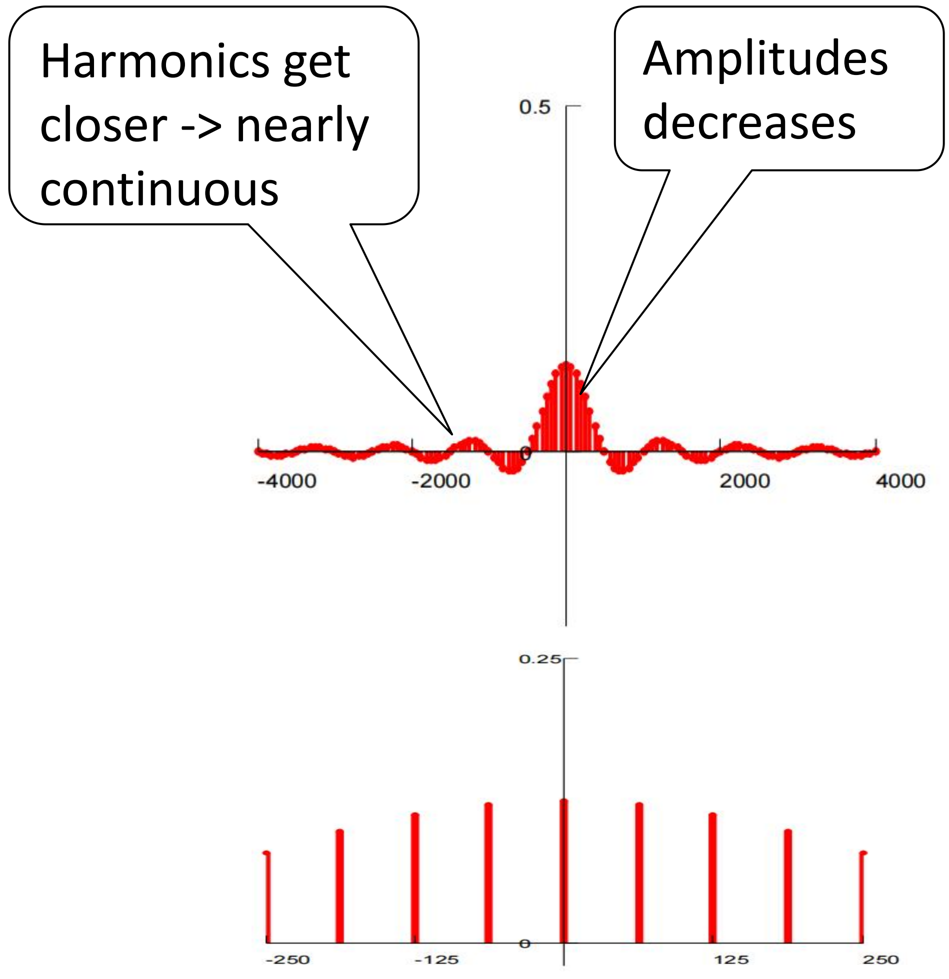


$$C_n = \frac{2}{8} sinc(n\frac{2}{8})$$

1.1 From CTFS to CTFT

- CTFS of the rectangular pulse signal (T=16ms)





1.1 From CTFS to CTFT

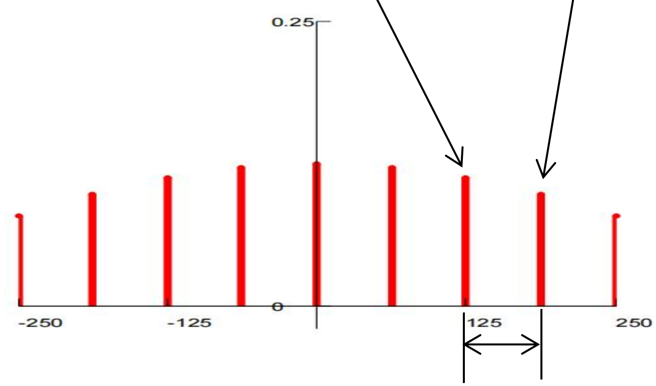
- FS representation of a periodic signal: $x_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t}$

- FS coefficient: $c_n = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-jn\frac{2\pi}{T}t} dt$

  substitue into above equation, get:

$$x_T(t) = \sum_{n=-\infty}^{\infty} \int_{-T/2}^{T/2} x_T(t) e^{-j2\pi t\frac{n}{T}} dt \, e^{j2\pi t\frac{n}{T}} \frac{1}{T}$$

$f_n = \frac{n}{T}$ $\quad \frac{n+1}{T}$



- What will happen when T⟶∞

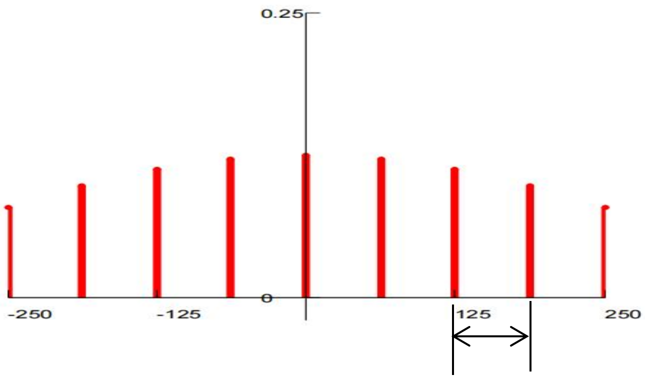
$$\left(\Delta f = \frac{1}{T}\right) \to df$$

$$\left(f_n = \frac{n}{T}\right) \to f$$

In the limit, as T approaches infinity, the spectrum becomes continuous.

$$\Delta f = \frac{n}{T} - \frac{n+1}{T} = \frac{1}{T}$$

1.1 From CTFS to CTFT



$$\Delta f = \frac{n}{T} - \frac{n+1}{T} = \frac{1}{T}$$

when T→ ∞

$$\left(\Delta f = \frac{1}{T}\right) \rightarrow df$$

$$\left(f_n = \frac{n}{T}\right) \rightarrow f$$

$$\boxed{x_T(t)} = \boxed{\sum_{n=-\infty}^{\infty}} \left[\boxed{\int_{-T/2}^{T/2}} x_T(t) e^{-j2\pi t \frac{n}{T}} dt \right] e^{j2\pi t \frac{n}{T}} \frac{1}{T}$$

$$\boxed{x(t)} = \boxed{\int_{-\infty}^{\infty}} \underbrace{\left[\boxed{\int_{-\infty}^{\infty}} x(t) e^{-j2\pi t f} dt \right]}_{\text{depends only on } f} e^{j2\pi t f} df$$

$$\boldsymbol{X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi t f} dt}$$

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1.2 Fourier transform: synthesis and analysis

- Forward Fourier transform (analysis equation)

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi tf}dt$$

- Inverse Fourier transform (synthesis equation)

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi tf}df$$

- Fourier transform pair: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$
  - $x(t)$ exists in the "time domain (TD)"
  - $X(f)$ exists in the "frequency domain (FD)"

$$\begin{tikzcd} x(t) \arrow[r, bend left, "\mathcal{F}"] & X(f) \arrow[l, bend left, "\mathcal{F}^{-1}"] \end{tikzcd}$$

1.3 Linear and angular frequencies

- Linear frequency f
  - Unit: Hz $(1/s)$
  - Fourier transform
    - Forward (analysis)

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi tf}dt$$

    - Inverse (synthesis)

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi tf}df$$

    - FT pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$$

- Angular frequency $\omega = 2\pi f$
  - Unit: rad/s
  - Fourier transform
    - Forward:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

    - Inverse

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

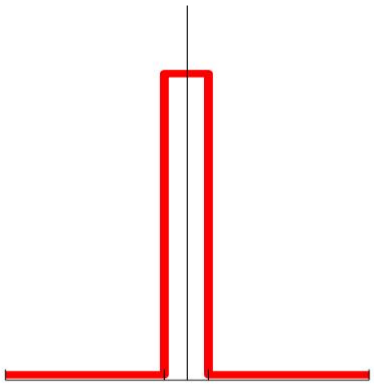
    - FT pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

In our textbook, FT form is denoted as: $X(j\omega)$

Example

- Fourier transform of the rectangular pulse signal



$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}dt$$

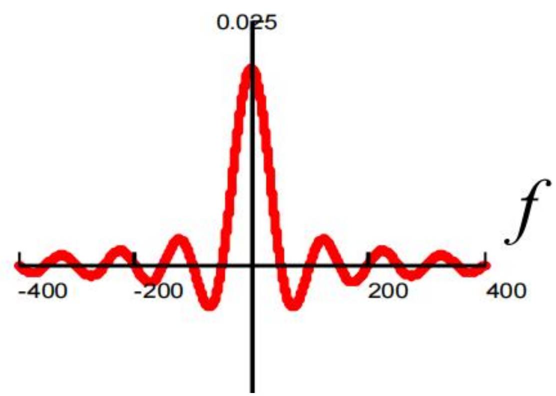
$$= \frac{e^{-j2\pi ft}}{-j2\pi f}\Bigg|_{-\tau/2}^{\tau/2} = \frac{e^{-j\pi f\tau} - e^{j\pi f\tau}}{-j2\pi f}$$

$$x(t) = rect\left(\frac{t}{\tau}\right)$$

$$= \begin{cases} 1, & |t| \leq - \tau/2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \frac{sin(\pi f\tau)}{\pi f}$$

$$= \tau\,\mathrm{sinc}(f\tau)$$



1.4 Relationship between $X(\omega)$ and a_k

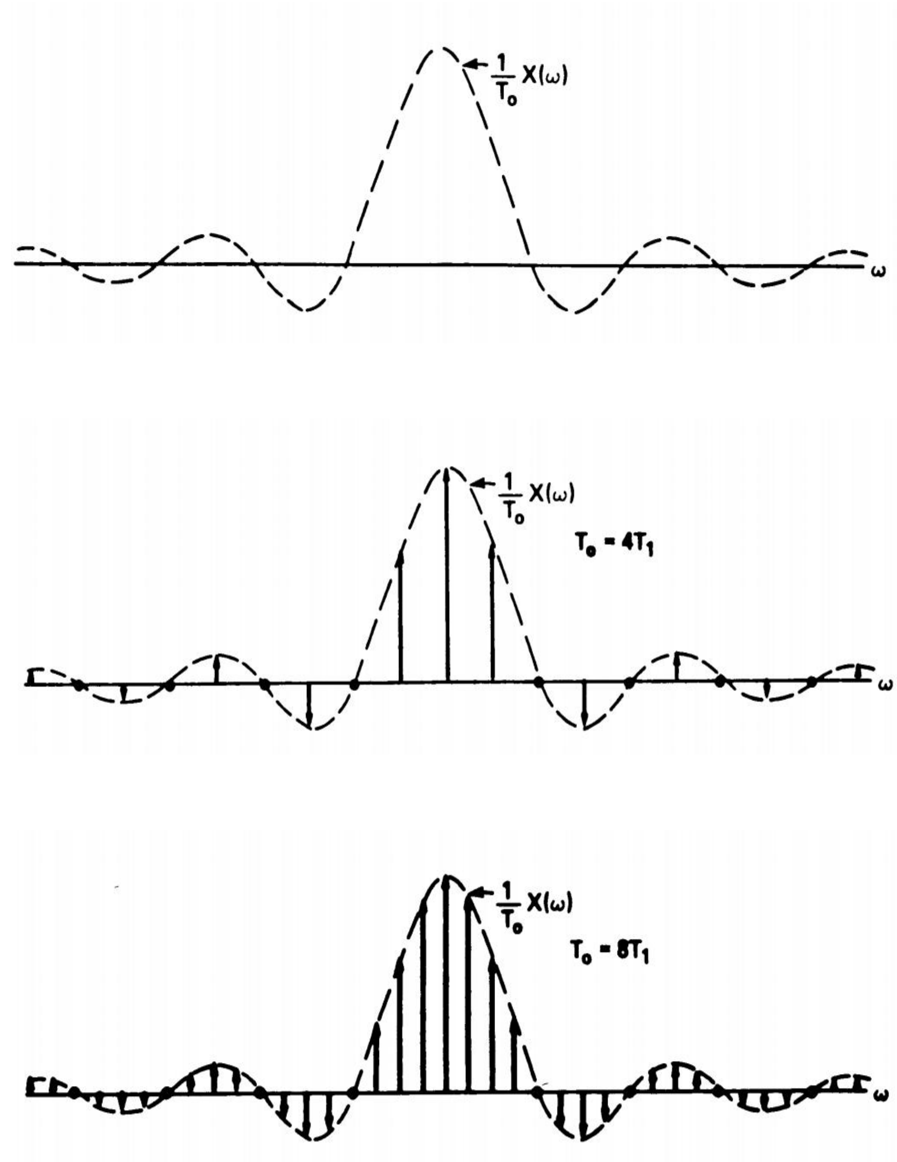
- FS spectrum (coefficients a_k):

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

- FT spectrum:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

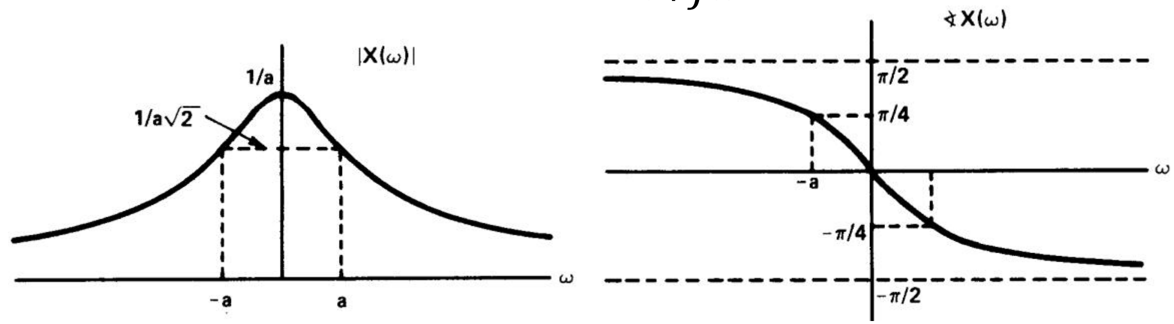
- Then: $Ta_k = X(\omega)|_{\omega=k\omega_0}$
  - $X(\omega)$ is the envolope of Ta_k
  - a_k are the samples of $\frac{1}{T}X(\omega)$



1.5 Meaning of the FT $X(\omega)$

- $X(\omega)$ contains equivalent information to that in $x(t)$.
  - It is widely used to study linear systems.
- Allows to generalize the concept of fourier series to finite duration and non-periodic signals.
- Introduces the concept of *continuous frequency*.
- $X(\omega)$ is called the spectrum of $x(t)$.
  - Magnitude $|X(\omega)|$
  - Phase $\measuredangle X(\omega)$

Example: $e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega} \quad a > 0$



1.6 Convergence of FT: Dirichlet conditions

- If $\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$ goes to infinity then we don't have a valid Fourier transform for $x(t)$.
- If $x(t)$ is a finite-energy signal, i.e. $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$, then $X(\omega)$ is finite.
- Dirichlet conditions that ensures $\mathcal{F}^{-1}\{\mathcal{F}[x(t)]\} = x(t)$ except at discontinuities:
  - Signal $x(t)$ is integrable: $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$;
  - $x(t)$ has a finite maxima and minima within any finite interval;
  - $x(t)$ has a finite number of discontinuities within any interval, and all discontinuities are finite.

1.7 Fourier transform of a periodic signal

- For periodic signal $\tilde{x}(t)$

$\tilde{x}(t) \overset{FS}{\Longleftrightarrow} a_k$ Fourier series coefficients

$\tilde{x}(t) \overset{CTFT}{\Longleftrightarrow} \widetilde{X}(\omega)$ Fourier transform

$$\boxed{\widetilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - \omega_0)}$$

Calculate inverse CTFT of $\widetilde{X}(\omega)$:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{X}(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} 2\pi a_k \underbrace{\int_{-\infty}^{+\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega}_{\boldsymbol{e^{jk\omega_0 t}}} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \tilde{x}(t)$$

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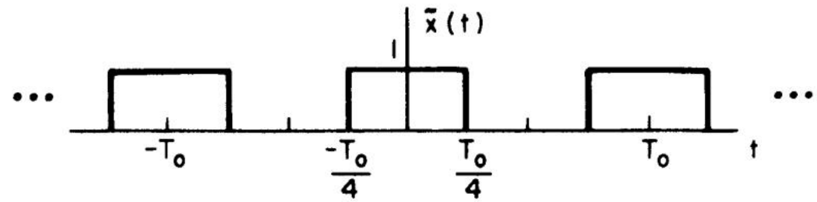
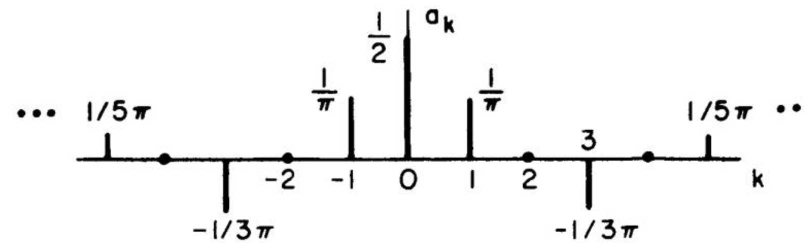
1.7 Fourier transform of a periodic signal

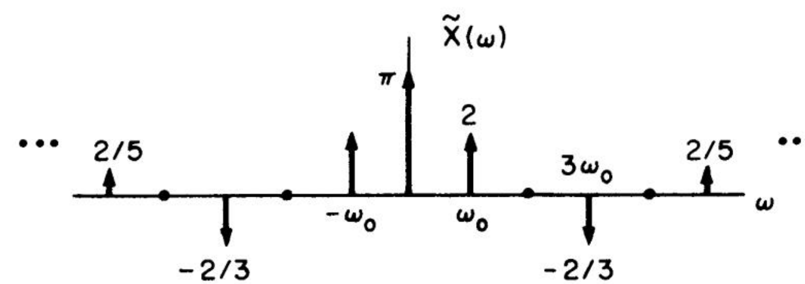
- Example: Symmetric square wave



discrete-frequency sequence a_k



continuous-freq. impulse train



1.7 Fourier transform of a periodic signal

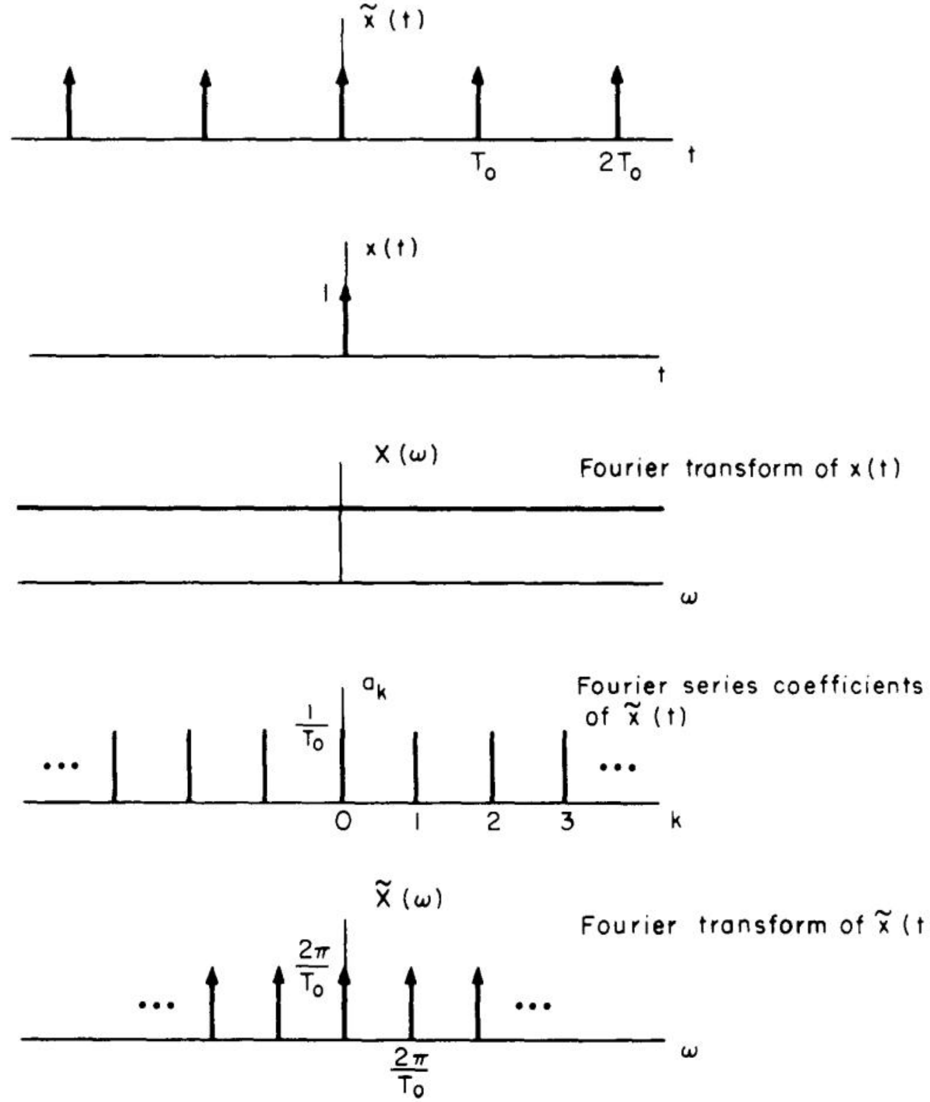
- $\tilde{x}(t)$ is periodic
- $x(t)$ represents one period

- Fourier series of $\tilde{x}(t)$

$$a_k = \frac{1}{T} X(\omega)|_{\omega=k\omega_0}$$

- Fourier transform of $\tilde{x}(t)$

$$\tilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - \omega_0)$$



Quiz 1

- Find the CTFT of the following signals:
  1. $x(t) = e^{-a|t|}, \ \ a > 0$
  2. $x(t) = e^{-2(t-1)}u(t-1)$

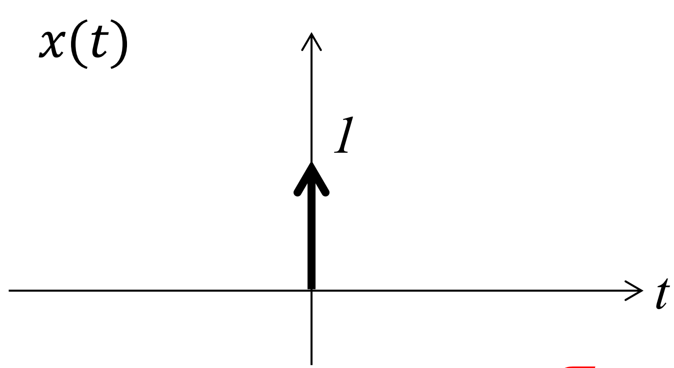
2. Elementary CTFT pairs

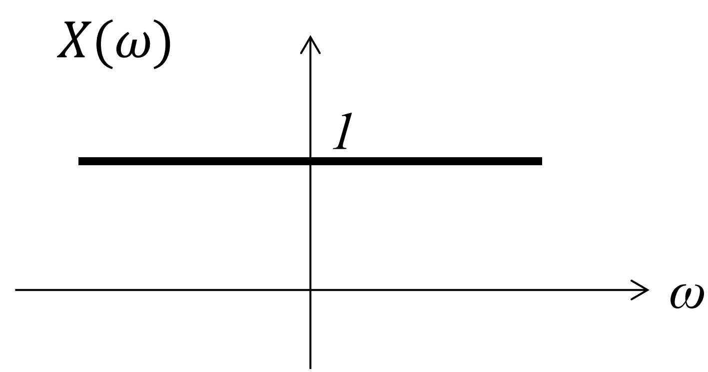
- Eg.1 Calculate the CTFT of a constant function

$$x(t) = \delta(t)$$

- Solution:

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega 0}\int_{-\infty}^{\infty} \delta(t)dt = 1$$





- CTFT pair: $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$

2. Elementary CTFT pairs

- Eg.2 Calculate the CTFT of a constant function

$$x(t) = \delta(t - t_0)$$

- Solution:

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}dt = e^{-j\omega t_0}\int_{-\infty}^{\infty} \delta(t)dt = e^{-j\omega t_0}$$

- CTFT pair: $\delta(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}$

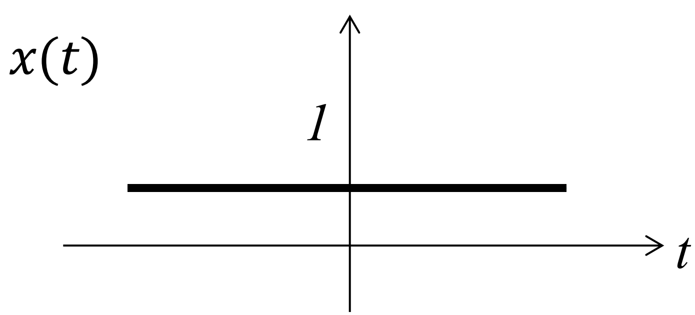
2. Elementary CTFT pairs

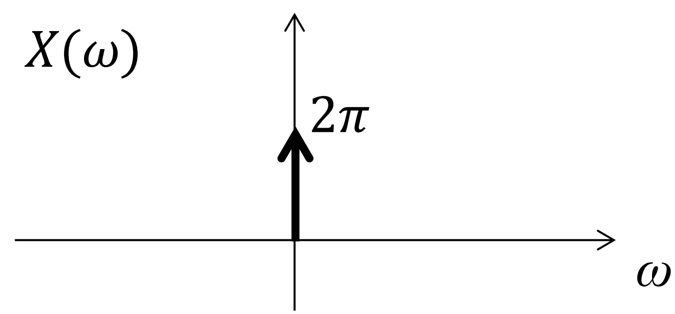
- Eg.3 The CTFT of an aperiodic function $x(t)$ is given by

$$X(\omega) = 2\pi\delta(\omega)$$

- Determine the aperiodic function $x(t)$.
- Solution:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} \delta(\omega)d\omega = 1$$





- CTFT pair: $1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega)$

2. Elementary CTFT pairs

- Eg.4 The CTFT of an aperiodic function $x(t)$ is given by

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

- Determine the aperiodic function $x(t)$.
- Solution:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega$$

$$= e^{j\omega_0 t}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)d\omega = e^{j\omega_0 t}$$

- CTFT pair: $e^{j\omega_0 t}\overset{\mathcal{F}}{\longleftrightarrow}2\pi\delta(\omega - \omega_0)$

2. Elementary CTFT pairs

- Eg.5 Calculate the CTFT of sinusoidal signals

$$x_1(t) = \cos \omega_0 t$$
$$x_2(t) = \sin \omega_0 t$$

2. Elementary CTFT pairs

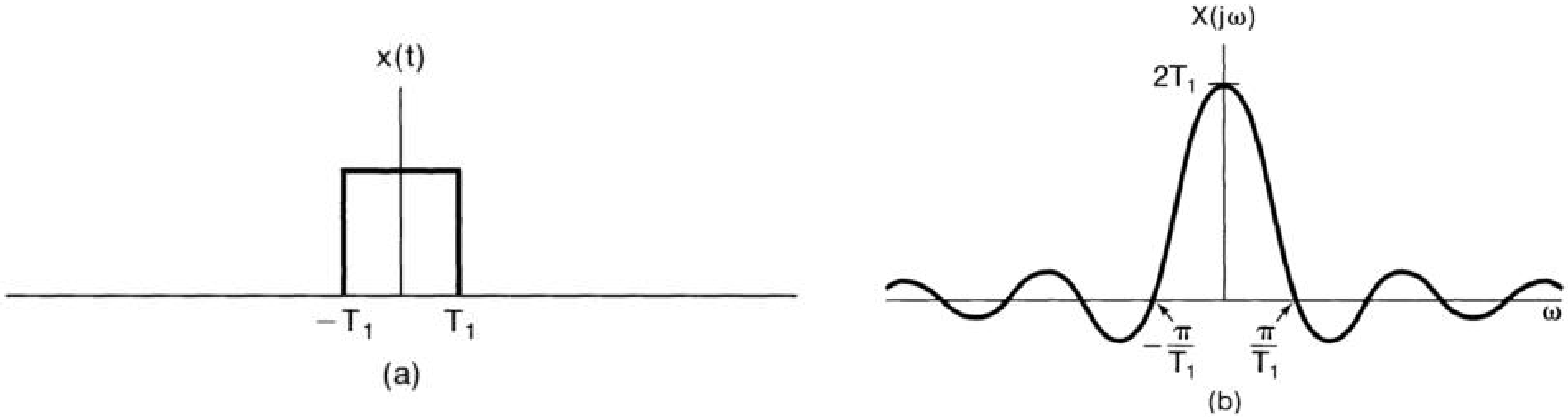
- Eg.6 Calculate the CTFT of a linear combination of complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

2. Elementary CTFT pairs

- Eg.7 Calculate the CTFT of a square wave:

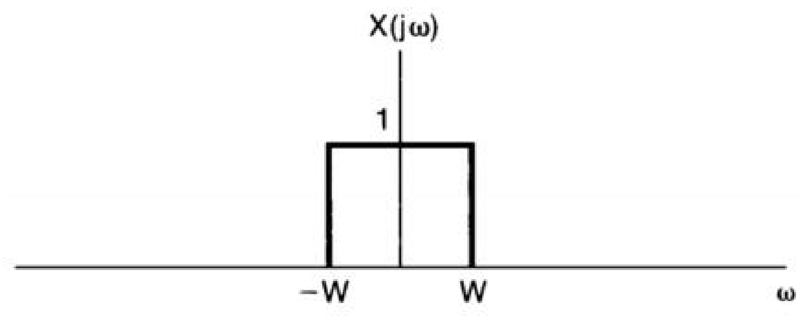
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



2. Elementary CTFT pairs

- Eg.8 Calculate the Inverse CTFT of a sinc function:

$$X(\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



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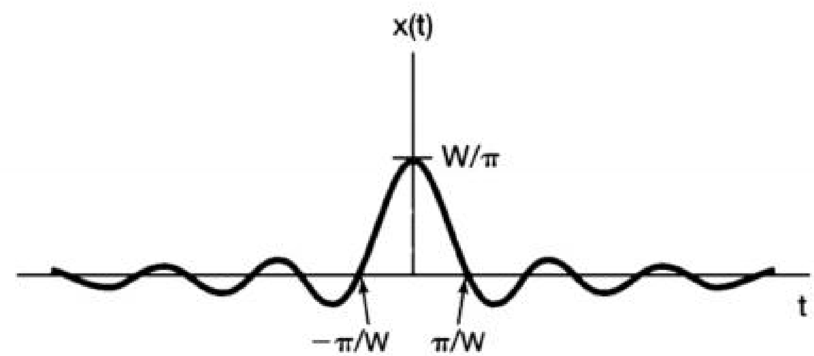
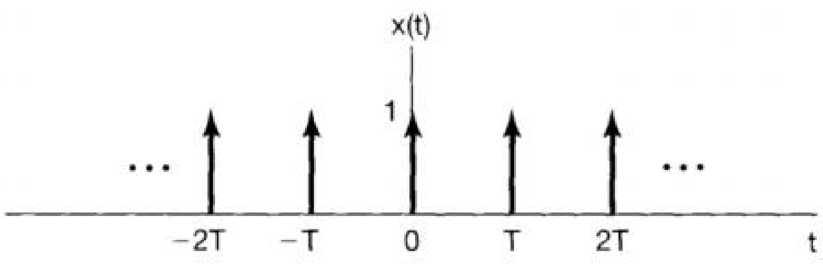


(b)

2. Elementary CTFT pairs

- Eg.9 Calculate the CTFT of an impulse train:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



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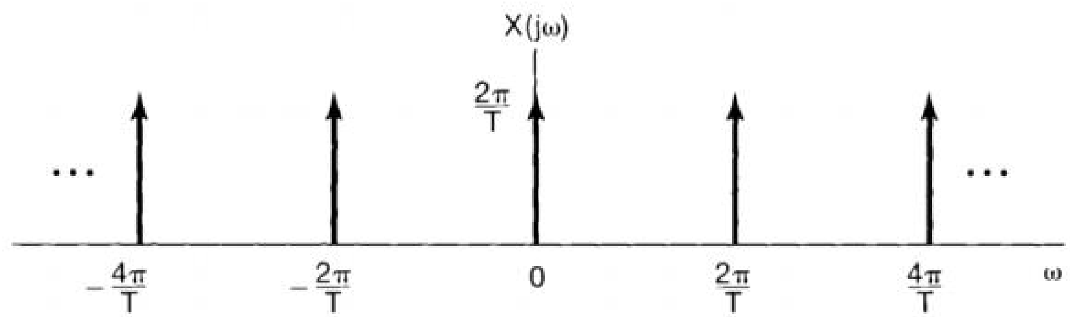
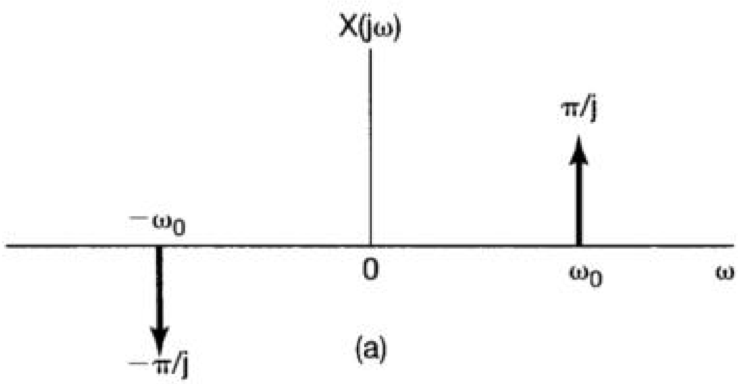
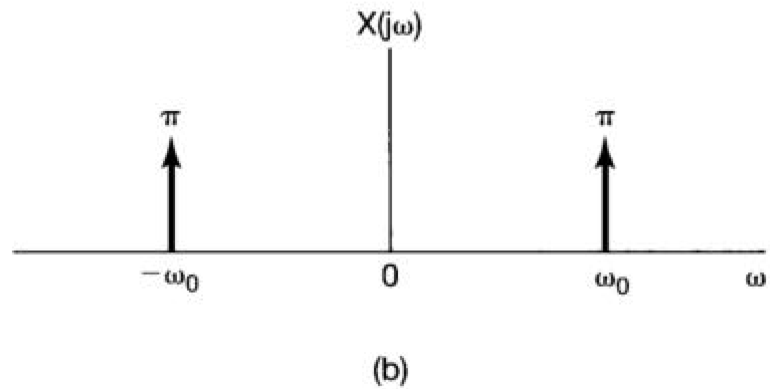


2. Elementary CTFT pairs

- Eg.10 Calculate the CTFT of:

$$x(t) = sin\omega_0 t \quad and \quad x(t) = cos\omega_0 t$$





Quiz 2

- Find the CTFT of the following signals:

1. $x(t) = \frac{d}{dt}[u(-2-t) + u(t-2)]$
2. $x(t) = \sin(2\pi t + \frac{\pi}{4})$

Next ...

- Continuous-Time Fourier Transform
  - 3. Properties of CTFT
    - Linearity, time and frequency scaling, time and frequency shifting, conjugation and symmetry, duality, Parseval's relation, convolution and multiplication properties
  - 4. System characterization
    - Frequency response of a system
    - Impulse response VS frequency response
    - LCCDE VS frequency response

List of Abbreviations

- CT - Continuous Time
- DT - Discrete Time
- TD - Time Domain
- FD - Frequency Domain
- FS(CTFS) - Fourier Series
- FT(CTFT) - Continuous Time Fourier Transform
- LCCDE - Linear Constant Coefficient Differential Equation