CAN209 Advanced Electrical Circuits and Electromagnetics

Lecture 11 Transient Response of 2nd-Order Circuits (Natural Response)

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OUTLINE

- Concepts and Applications
- Analysis of Parallel RLC Circuits
  - ✓ Determine a Response Form
  - ✓ Find General Solutions
- Analysis of Series RLC Circuits
  - ✓ Determine a Response Form
  - ✓ Find General Solutions

2nd-order linear differential equations – General Solutions 二阶线性微分方程解的结构

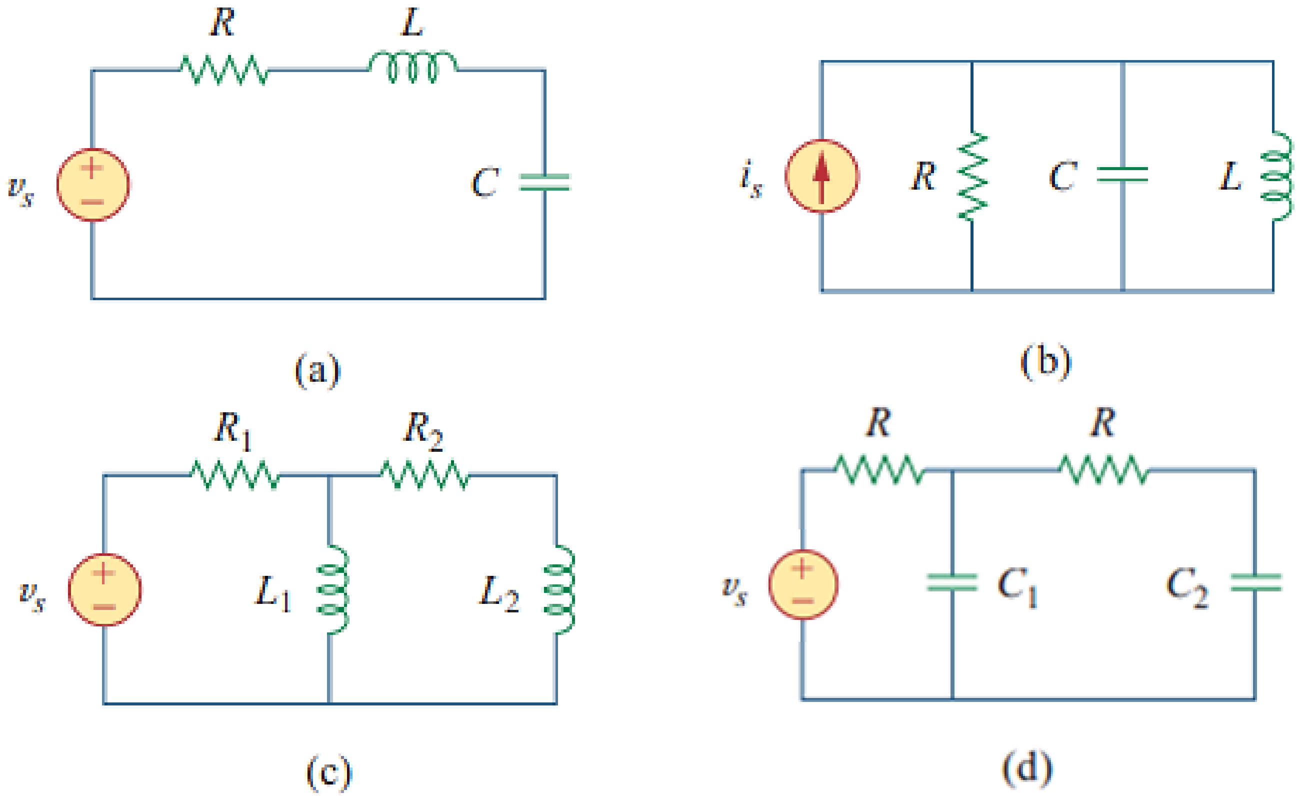
1.1 BASIC CONCEPTS

A second-order circuit is characterised by a second-order differential equation. It consists of resistors and **TWO equivalent** energy storage elements.

Examples:

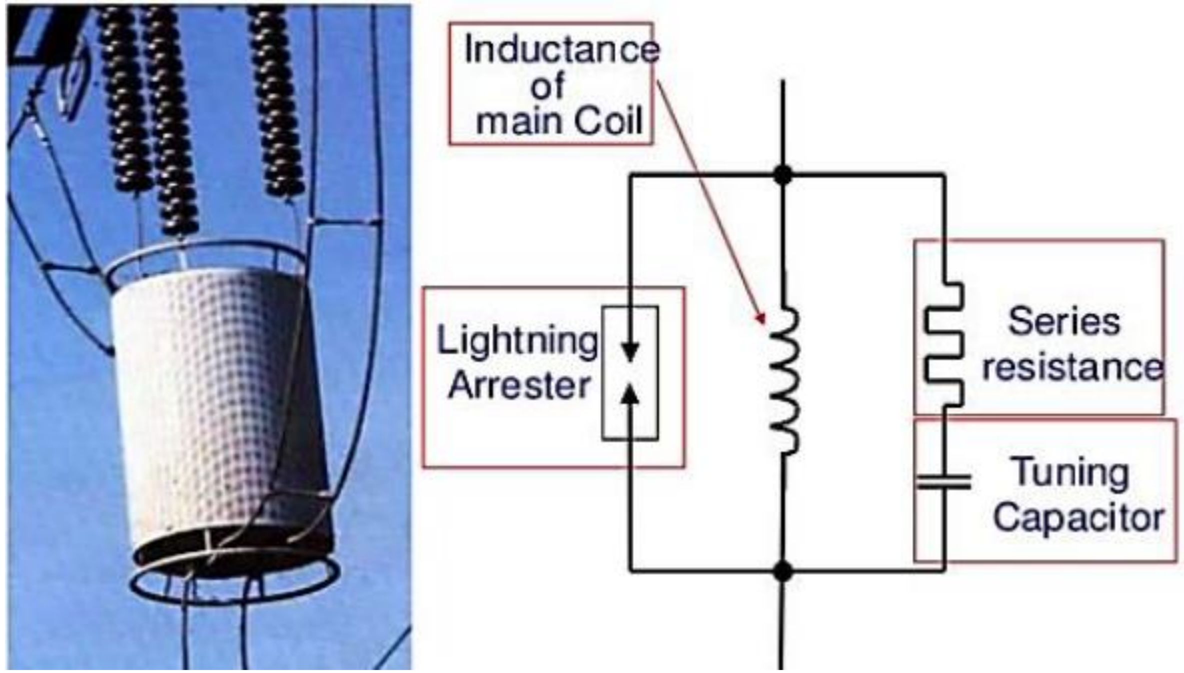
- ✓ Series *RLC* circuit
- ✓ Parallel *RLC* circuit
- ✓ *RLL* circuit
- ✓ *RCC* circuit

voltage source → s.c.
current source → o.c.



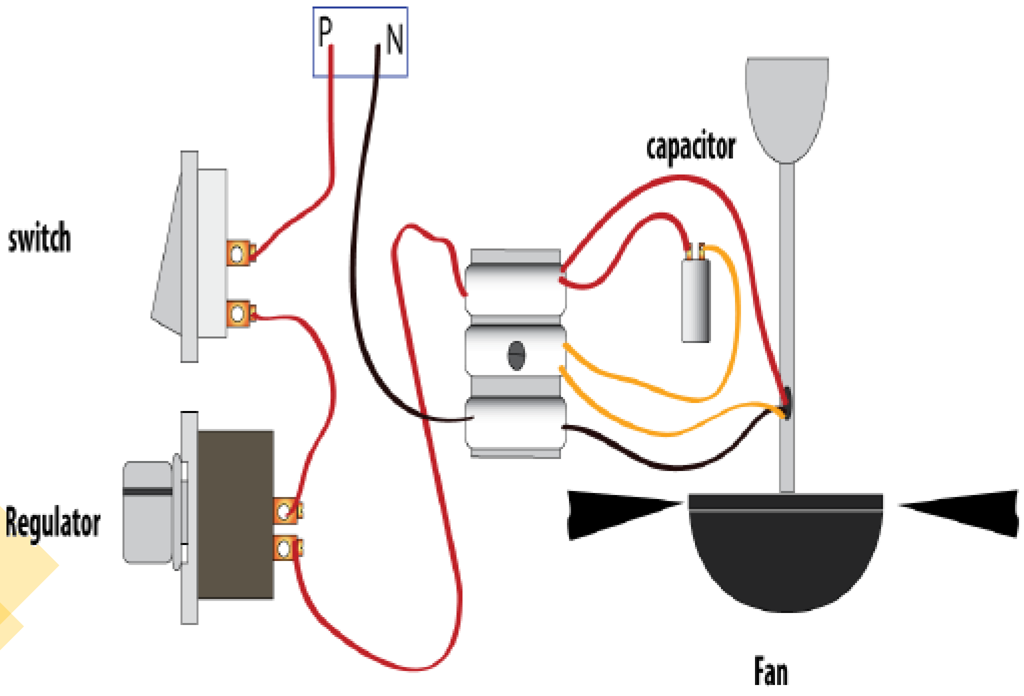
1.2 INSIGHT & REAL-LIFE APPLICATIONS

LCR Digital Metres

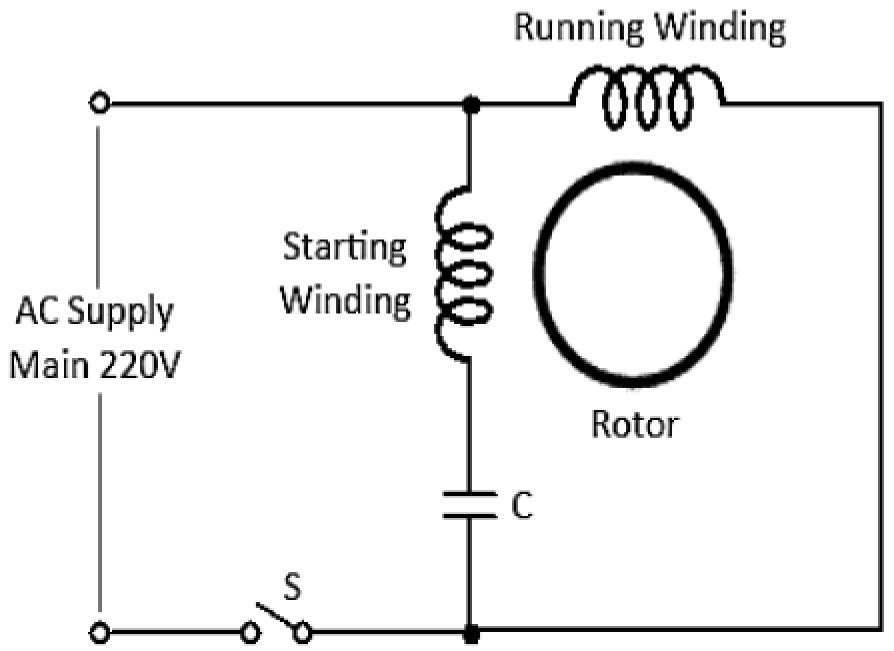


Line Trap Circuit

1.2 INSIGHT & REAL-LIFE APPLICATIONS



Electric Ceiling Fan



Ceiling Fan Wiring Diagram

1.3 GENERAL IDEA

Last week, we looked at the transient response of **first**-order circuits. General procedure to solve this kind of circuits is:

- Apply KVL or KCL
  - ✓ Find the governing first-order differential equation
- Solve the ODE
  - ✓ Determine the complete expression

For transient response of second-order circuits, general procedure is similar:

- Apply KVL or KCL
  - ✓ Find the governing second-order differential equation
- Solve the ODE
  - ✓ Determine the complete expression

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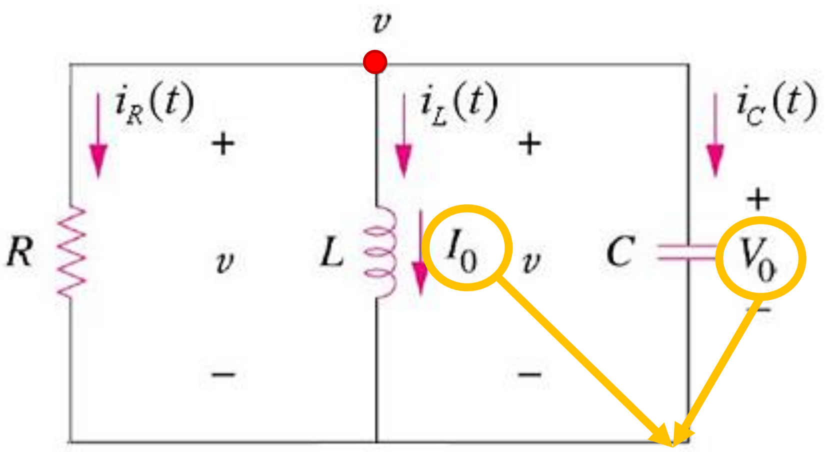
2nd-order linear differential equations – General Solutions
二阶线性微分方程解的结构

2.1 METHOD 1

Task: given the initial energy stored in the inductor or capacitor, find $v(t)$ for $t \geq 0$.



Initial Condition

$$\begin{cases} i_R(t) = \dfrac{v}{R} \\ i_C(t) = C\dfrac{dv}{dt} \\ i_L(t) = \dfrac{1}{L}\displaystyle\int_0^t v\,dt' + i(t=0) \end{cases}$$

Get the ODE

Apply KCL at the node:

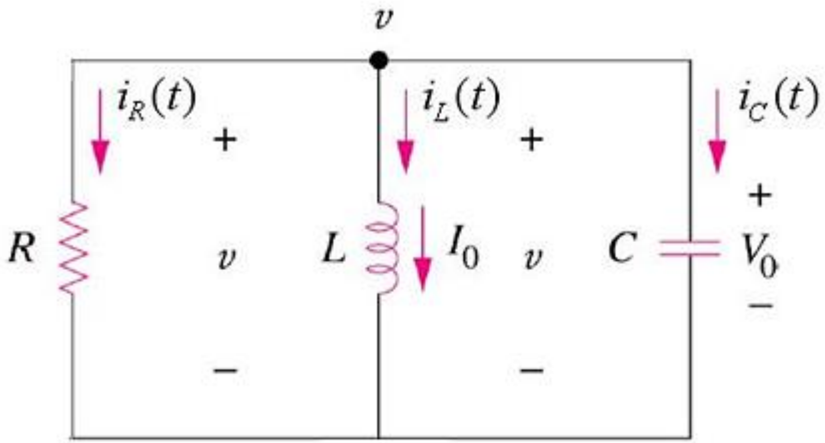
$$i_C(t) + i_R(t) + i_L(t) = 0$$

$$\therefore C\frac{dv}{dt} + \frac{v}{R} + \frac{1}{L}\int_0^t v\,dt' + I_0 = 0$$

Take derivative of both sides & divide by C:

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

CONTINUE...



$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

This equation is:

- ✓ Homogeneous 2nd ODE
- ✓ Constant coefficients

Solve the ODE

The ***characteristic equation*** is

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Define:

ω_0 as the resonant (natural) frequency (rad/s)

α as the neper frequency (rad/s) (exponential damping coefficient)

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \alpha = \frac{1}{2RC}$$

The characteristic eq. becomes:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\therefore s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad and \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

s_1 and s_2 are called complex frequencies.

CONTINUE...

The value of the term $\sqrt{\alpha^2 - \omega_0^2}$ determines the behavior of the response.

- Over Damped $\longrightarrow \alpha > \omega_0$:

  s_1 & s_2 are two unequal real numbers

  Response: $v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

- Critical Damped $\longrightarrow \alpha = \omega_0$:

  s_1 & s_2 are two equal real numbers

  Response: $v_C(t) = e^{-\alpha t}(A_1 t + A_2)$

- Under Damped $\longrightarrow \alpha < \omega_0$:

  s_1 & s_2 are two complex numbers $\begin{cases} s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2} \\ s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2} \end{cases}$

  Response:

  $v_C(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$ $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

  natural resonant (damped) frequency

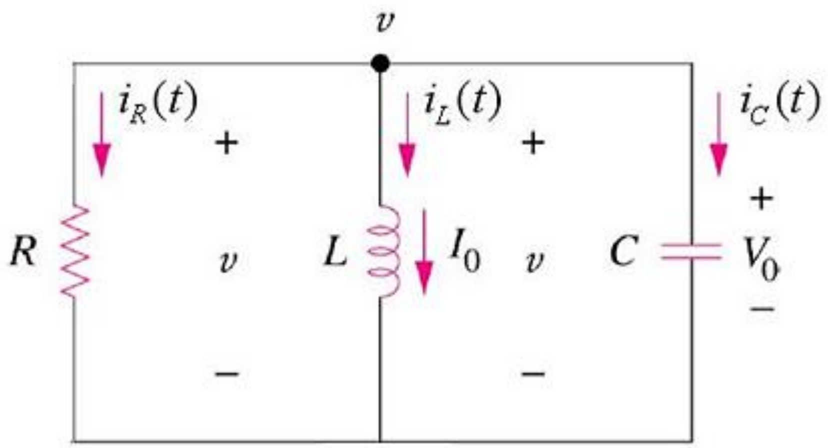
A_1 and A_2 : constant determined by initial conditions

B_1 and B_2: determined by initial conditions

CONTINUE...

Find the Coefficients

- ✓ Polarity of voltage across the C, and the direction of the current through the L.
- ✓ The capacitor's voltage is always continuous, and the inductor's current is always continuous.



Normally start from finding variables that **cannot** change abruptly:

$$v_C(t = 0^+) = v_C(t = 0^-)$$
$$i_L(t = 0^+) = i_L(t = 0^-)$$

CONTINUE...

Find the Coefficients

- Over Damped $\rightarrow \alpha > \omega_0$:

$$v_C(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

$$\begin{cases} v_C(0^+) = A_1 + A_2 \\ \dfrac{dv_C(0^+)}{dt} = A_1 s_1 + A_2 s_2 \end{cases}$$

- Critical Damped $\rightarrow \alpha = \omega_0$:

$$v_C(t) = e^{-\alpha t}(A_1 t + A_2)$$

$$\begin{cases} v_C(0^+) = A_2 \\ \dfrac{dv_C(0^+)}{dt} = A_1 - A_2 \alpha \end{cases}$$

- Under Damped $\rightarrow \alpha < \omega_0$:

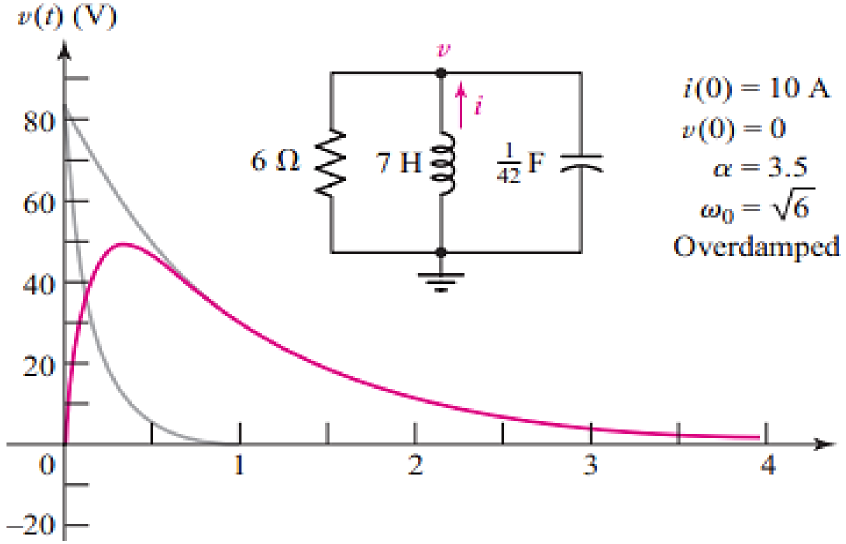
$$v_C(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

$$\begin{cases} v_C(0^+) = B_1 \\ \dfrac{dv_C(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 \end{cases}$$

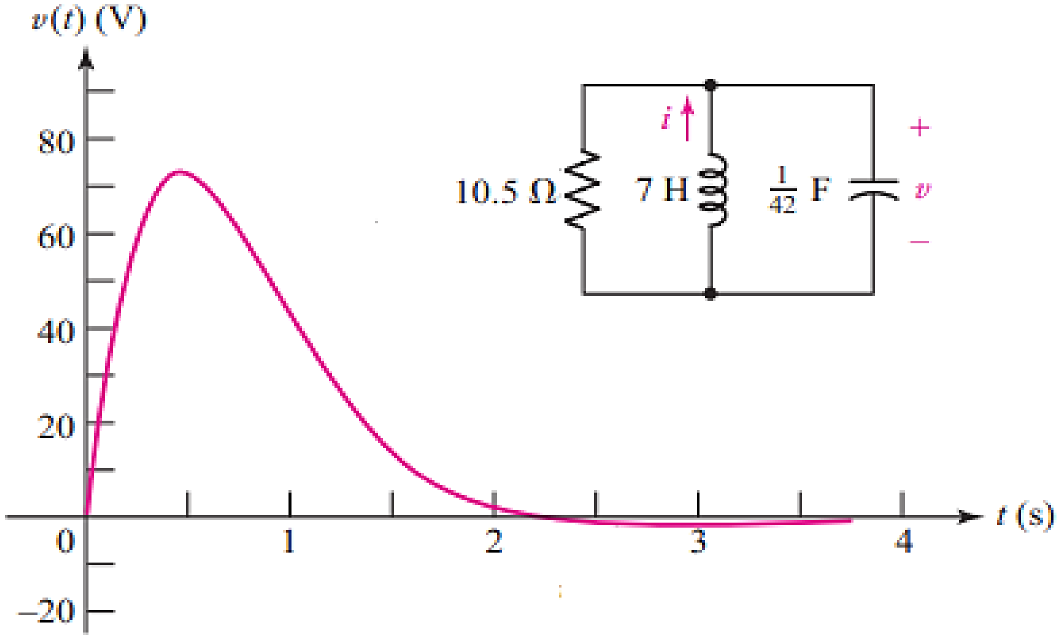
2.2 RULES FOR CIRCUIT DESIGNERS

- ✓ If one desires the circuit **reaches the final value** as **fast** as possible while the minor oscillation is of less concern, choosing *R*, *L*, *C* values to satisfy under-damped condition.
- ✓ If one concerns that the response **not exceed its final value** to **prevent potential damage**, designing the system to be over-damped at the cost of slower response.

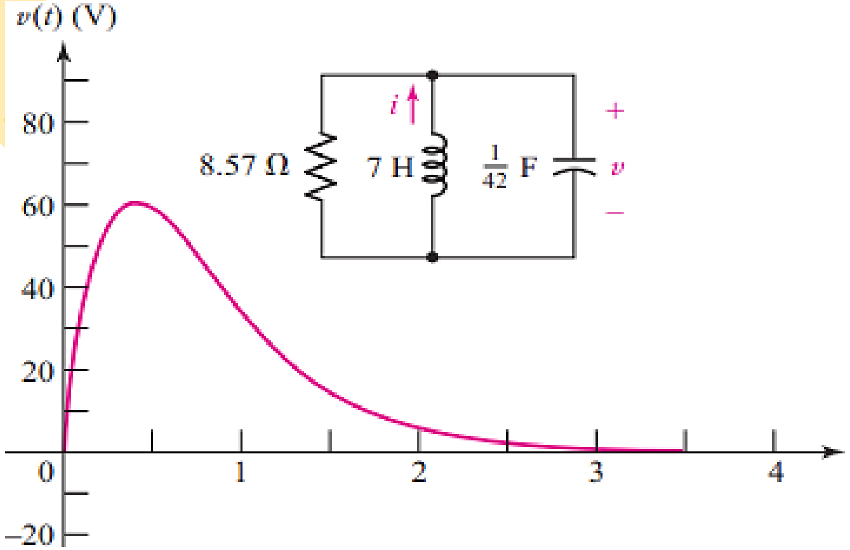
Over Damped



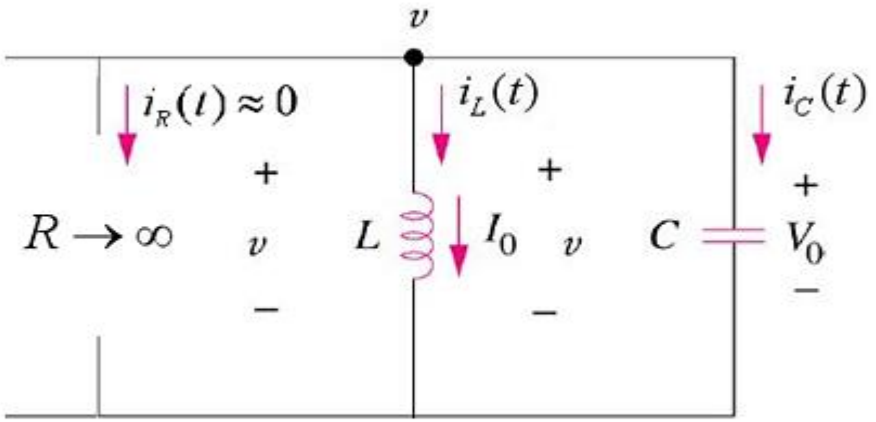
Under Damped

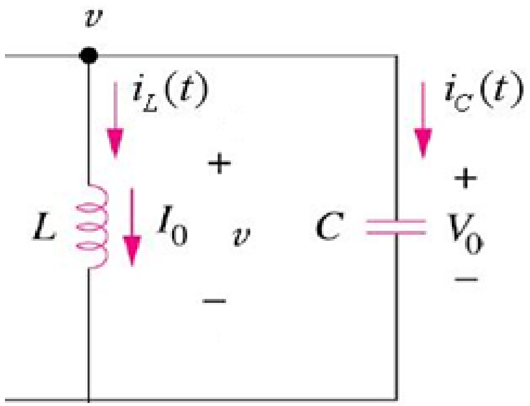


Critical Damped



2.3 ROLE OF THE RESISTOR





$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \alpha = \frac{1}{2RC} \to 0$$

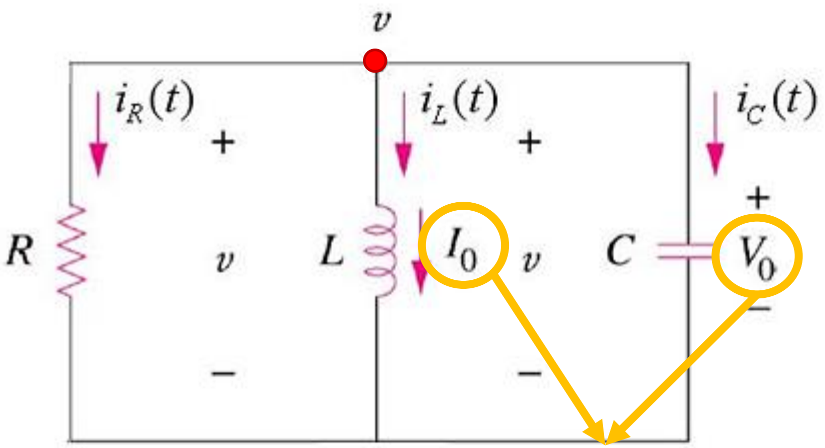
$$\alpha << \omega_0 \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2} \approx \omega_0$$

$$\therefore v_C(t) = B_1 \cos \omega_0 t + B_2 \sin \omega_0 t$$

A parallel *RLC* circuit can be made to have an effective value of *R* to be big enough that a natural ***undamped*** sinusoidal response can be maintained for years without supplying any additional energy.

2.4 METHOD 2

Alternatively, we can start with finding an ODE of inductor's current:



Initial Condition

$$\begin{cases} i_R(t) = \dfrac{v}{R} = \dfrac{L}{R}\dfrac{di_L}{dt} \\ i_C(t) = C\dfrac{dv}{dt} = LC\dfrac{d^2 i_L}{dt^2} \\ v_L(t) = L\dfrac{di_L}{dt} \end{cases}$$

Get the ODE

Applying KCL at the node:

$$i_C(t) + i_R(t) + i_L(t) = 0$$

$$\therefore LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = 0$$

Divide by LC on both sides:

$$\boxed{\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = 0}$$

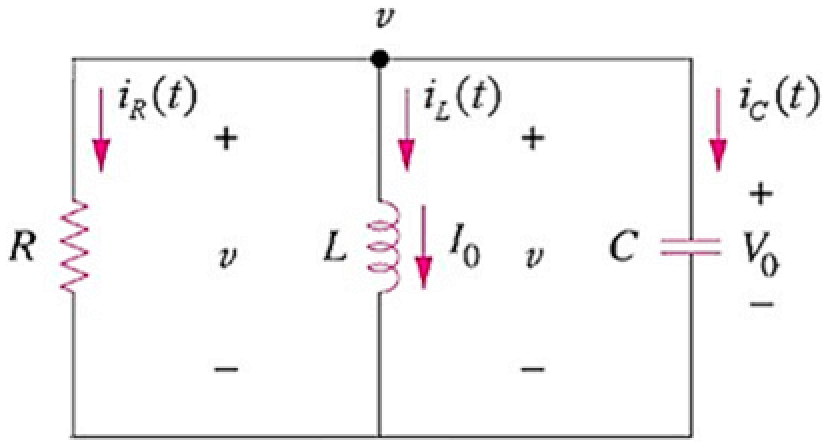
QUIZ 1

The circuit below has the following parameters:

$$R = 500\ \Omega,\ C = 1\mu\text{F},\ L = 0.2\ \text{H}.$$

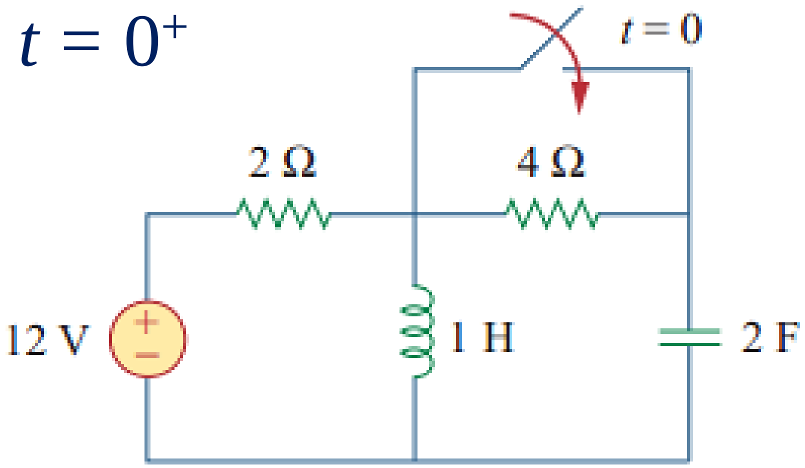
The initial conditions are $i_L(0) = 50$ mA and $v(0) = 0$V.

Determine expressions of $i_L(t)$, $i_R(t)$ and $v_c(t)$ for $t \geq 0$.



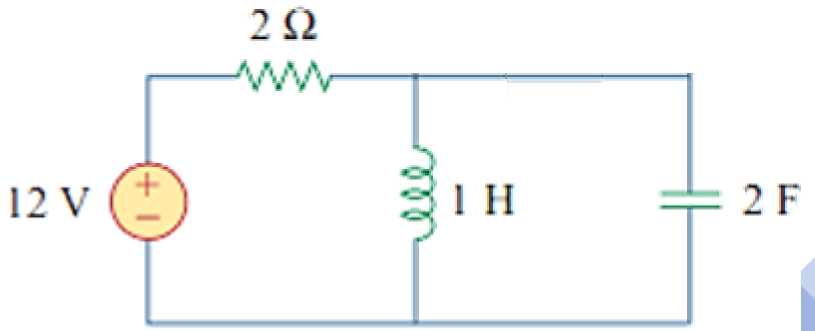
QUIZ 2

1. For the circuit as shown, the capacitor voltage at $t = 0^+$ (just after the switch is closed) is



(a) 0 V (b) 4 V (c) 8 V (d) 12 V

2. For the same circuit, the inductor voltage at $t = 0^+$ (just after the switch is closed) is



(a) 0 V (b) 4 V (c) 8 V (d) 12 V

QUIZ 3

1. If the roots of the characteristic equation of an RLC circuit are -2 and -3, the response is:

(a) $Ae^{-2t} + Be^{-3t}$

(b) $e^{-3t}(At + B)$

(c) $e^{-3t}(A\cos 2\,t + B\sin 3\,t)$

(d) $Ae^{-2t} + Bte^{-3t}$

2. A parallel RLC circuit has L = 4 H and C = 0.25 F. The value of R that will produce under damping factor is:

(a) 0.5 Ω (b) 1 Ω (c) 2 Ω (d) 4 Ω

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2nd-order linear differential equations – General Solutions
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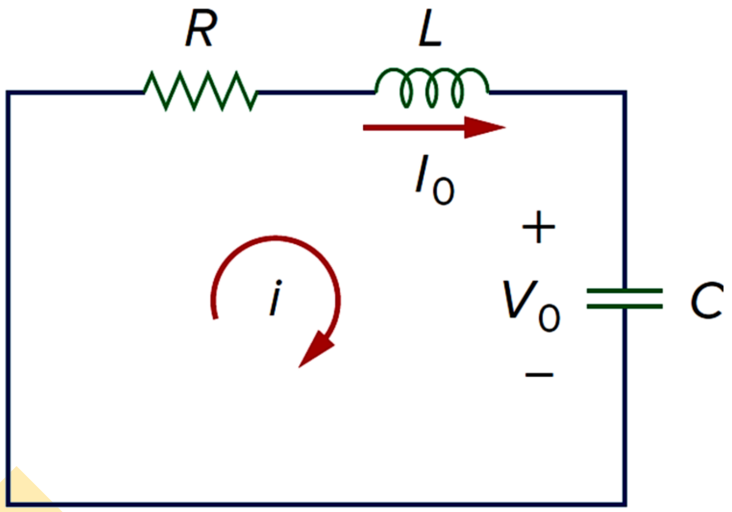
3.1 SERIES RLC CIRCUIT

Problem: given the initial energy stored in the inductor and/or capacitor, find v(t) for $t \geq 0$.



$$\begin{cases} i_R = i_L = i_C(t) = C\dfrac{dv_C}{dt} \\ v_R = Ri_R = RC\dfrac{dv_C}{dt} \\ v_L(t) = L\dfrac{di_L}{dt} = LC\dfrac{d^2v_C}{dt^2} \end{cases}$$

Get the ODE

Apply KVL:

$$v_C(t) + v_R(t) + v_L(t) = 0$$

$$\therefore\ LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = 0$$

Divide by LC on both sides:

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_c = 0$$

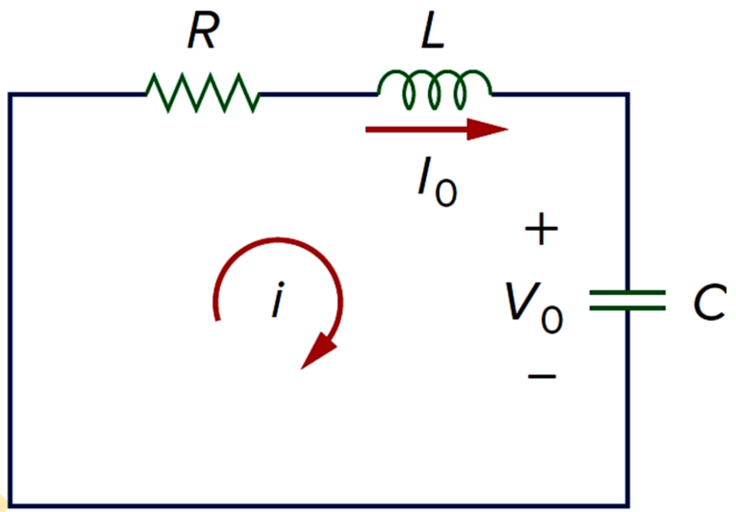
Alternatively:

$$v_C(t) = \frac{q(t)}{C} = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$\Downarrow$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

CONTINUE…



$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC} v_c = 0$$

This equation is:

- ✓ Homogeneous 2nd ODE
- ✓Constant coefficients

Solve the ODE

The ***characteristic equation*** is

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Define:

ω_0 as the resonant (natural) frequency (rad/s)

α as the neper frequency (rad/s)
(exponential damping coefficient)

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \alpha = \frac{R}{2L}$$

The characteristic eq. becomes:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\therefore s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad and \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

s_1 and s_2 are called complex frequencies.

CONTINUE…

The value of the term $\sqrt{\alpha^2 - \omega_0^2}$ determines the behaviour of the response:

- Over Damped $\longrightarrow \alpha > \omega_0$:

  s_1 & s_2 are two unequal real numbers

  Response: $v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

- Critical Damped $\longrightarrow \alpha = \omega_0$:

  s_1 & s_2 are two equal real numbers

  Response: $v_C(t) = e^{-\alpha t}(A_1 t + A_2)$

- Under Damped $\longrightarrow \alpha < \omega_0$:

  s_1 & s_2 are two complex numbers $\begin{cases} s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2} \\ s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2} \end{cases}$

  Response:

  $$v_C(t) = e^{-\alpha t}(B_1 \cos \omega_d\, t + B_2 \sin \omega_d\, t)$$

  $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

  natural resonant (damped) frequency

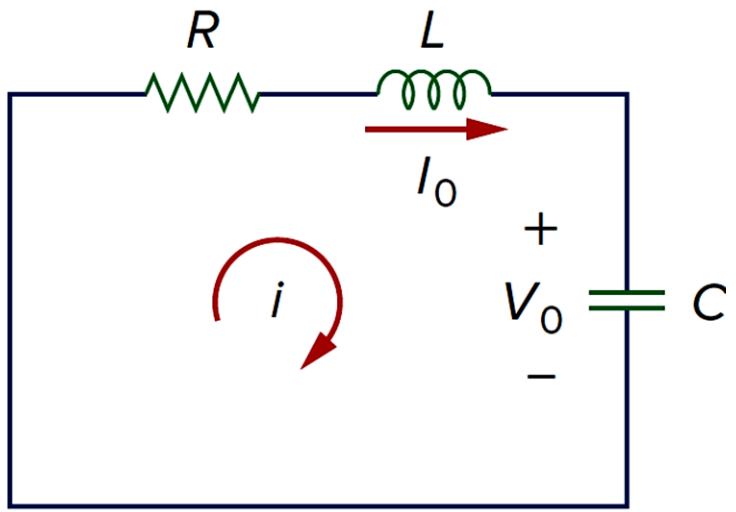
A_1 and A_2 : constant determined by initial conditions

B_1 and B_2: determined by initial conditions

CONTINUE…

Find the Coefficients

- ✓ Polarity of voltage across the C, and the direction of the current through the L.
- ✓ The capacitor voltage is always continuous, and the inductor current is always continuous.



Normally start from finding variables that **cannot** change abruptly.

$$v_C(t = 0^+) = v_C(t = 0^-)$$
$$i_L(t = 0^+) = i_L(t = 0^-)$$

CONTINUE…

Find the Coefficients

- Over Damped $\rightarrow \alpha > \omega_0$:

$$v_C(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

$$\begin{cases} v_C(0^+) = A_1 + A_2 \\ \dfrac{dv_C(0^+)}{dt} = A_1 s_1 + A_2 s_2 \end{cases}$$

- Critical Damped $\rightarrow \alpha = \omega_0$:

$$v_C(t) = e^{-\alpha t}(A_1 t + A_2)$$

$$\begin{cases} v_C(0^+) = A_2 \\ \dfrac{dv_C(0^+)}{dt} = A_1 - A_2 \alpha \end{cases}$$

- Under Damped $\rightarrow \alpha < \omega_0$:

$$v_C(t) = e^{-\alpha t}(B_1 \cos \omega_d\, t + B_2 \sin \omega_d\, t)$$

$$\begin{cases} v_C(0^+) = B_1 \\ \dfrac{dv_C(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 \end{cases}$$

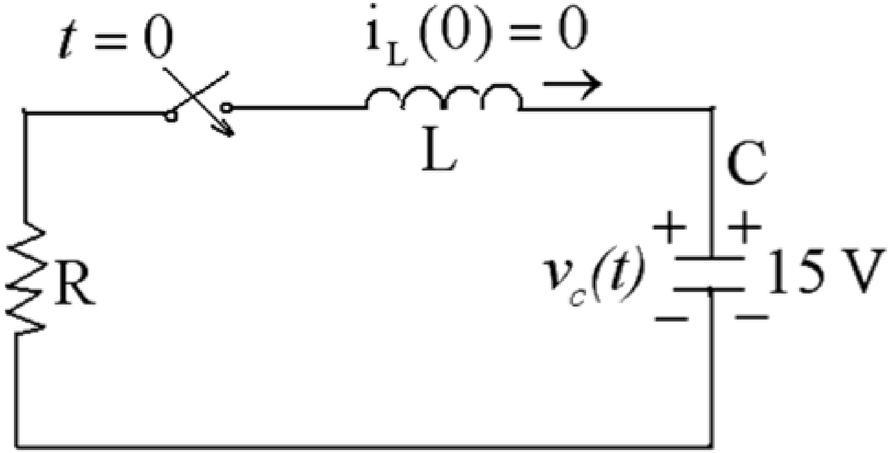
QUIZ 4

The circuit below has the following parameters:

$$R = 8.5\ \mathrm{k\Omega},\ C = 0.25\ \mu\mathrm{F},\ L = 1\ \mathrm{H}$$

The switch has been open for a long time and is closed at $t = 0$. The initial conditions are $i_L(0) = 0$ and $v_c(0) = 15\mathrm{V}$.

Find the capacitor's voltage for $t \geq 0$.



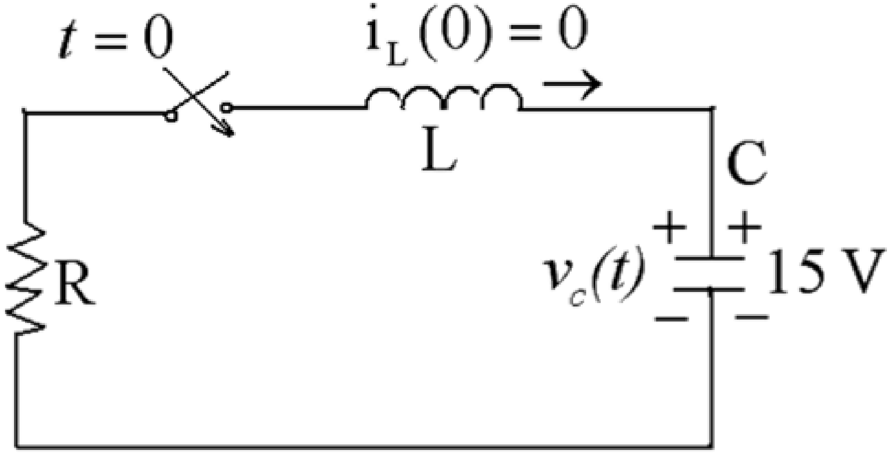
QUIZ 5

The circuit below has the following parameters:

$$R = 4\ \text{k}\Omega,\ C = 0.25\ \mu\text{F},\ L = 1\ \text{H}$$

The switch has been open for a long time and is closed at $t = 0$. The initial conditions are $i_L(0) = 0$ and $v_c(0) = 15\text{V}$.

Find the capacitor's voltage for $t \geq 0$.



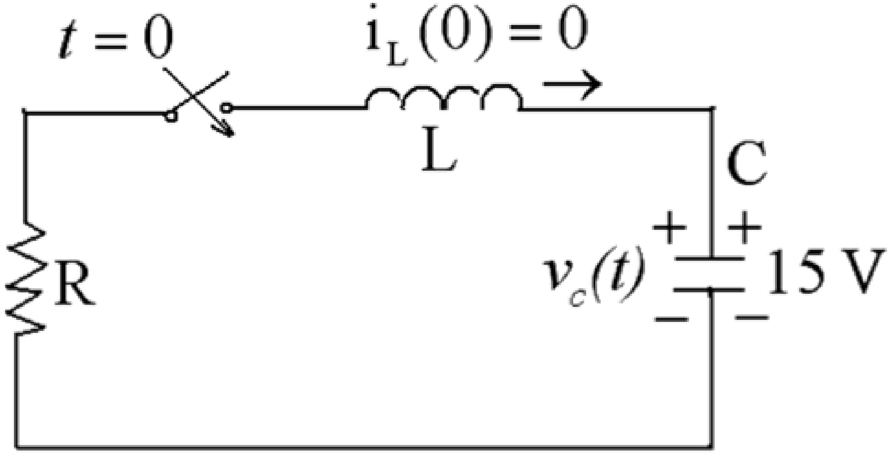
QUIZ 6

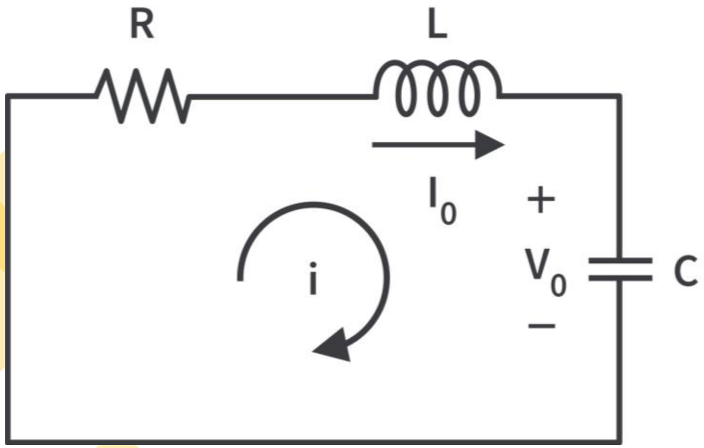
The circuit below has the following parameters:

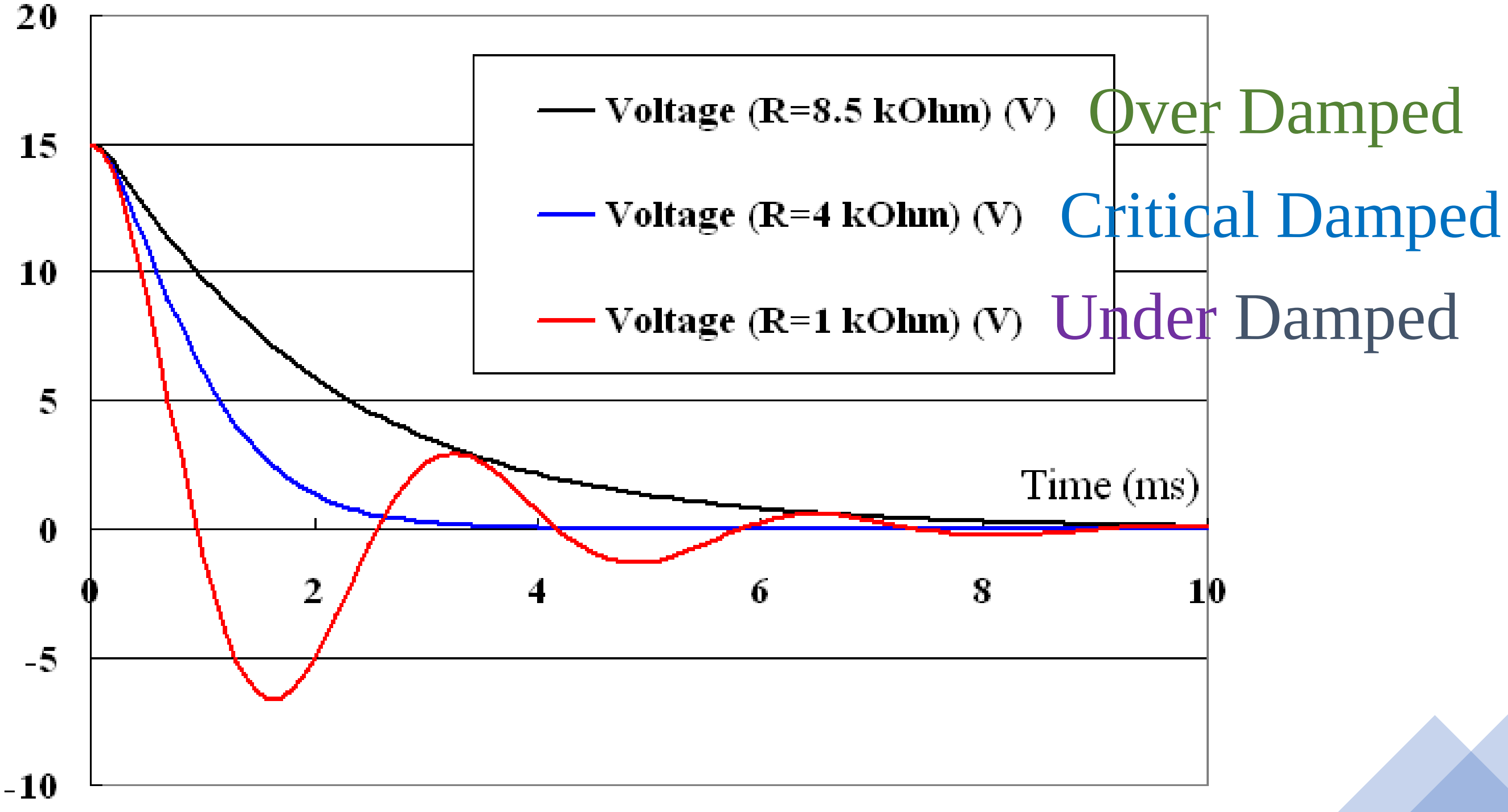
$$R = 1\ \text{k}\Omega,\ C = 0.25\ \mu\text{F},\ L = 1\ \text{H}$$

The switch has been open for a long time and is closed at $t = 0$. The initial conditions are $i_L(0) = 0$ and $v_c(0) = 15\text{V}$.

Find the capacitor's voltage for $t \geq 0$.




R
L
I_0
+
V_0
−
C
i


20
15
10
5
0
-5
-10
0
2
4
6
8
10
Time (ms)
Voltage (R=8.5 kOhm) (V)
Voltage (R=4 kOhm) (V)
Voltage (R=1 kOhm) (V)
Over Damped
Critical Damped
Under Damped

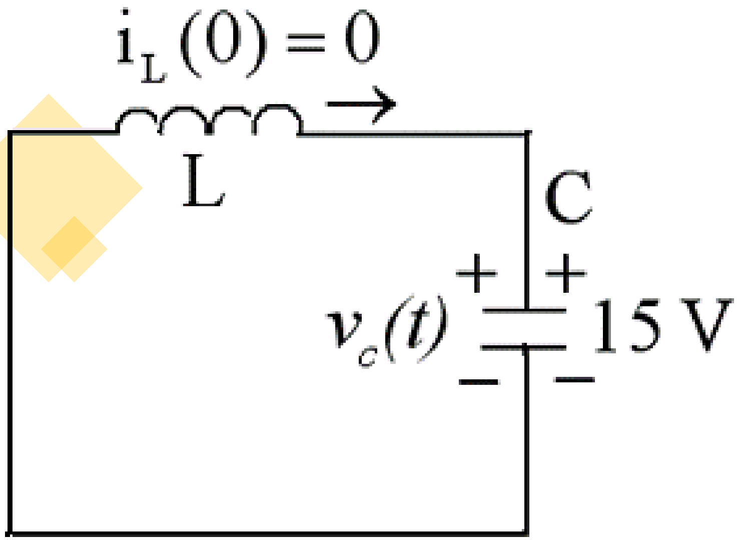
QUIZ 7

The circuit below has the following parameters:

$$C = 0.25\ \mu\text{F},\ L = 1\ \text{H}$$

The switch has been open for a long time and is closed at $t = 0$. The initial conditions are $i_L(0) = 0$ and $v_c(0) = 15\text{V}$.

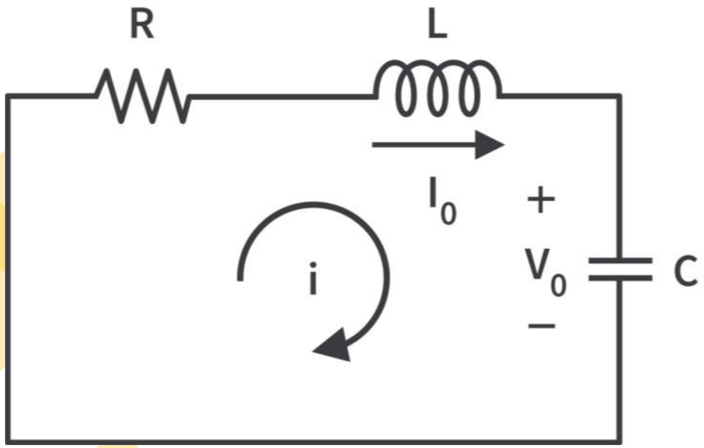
Find the capacitor voltage for $t \geq 0$.

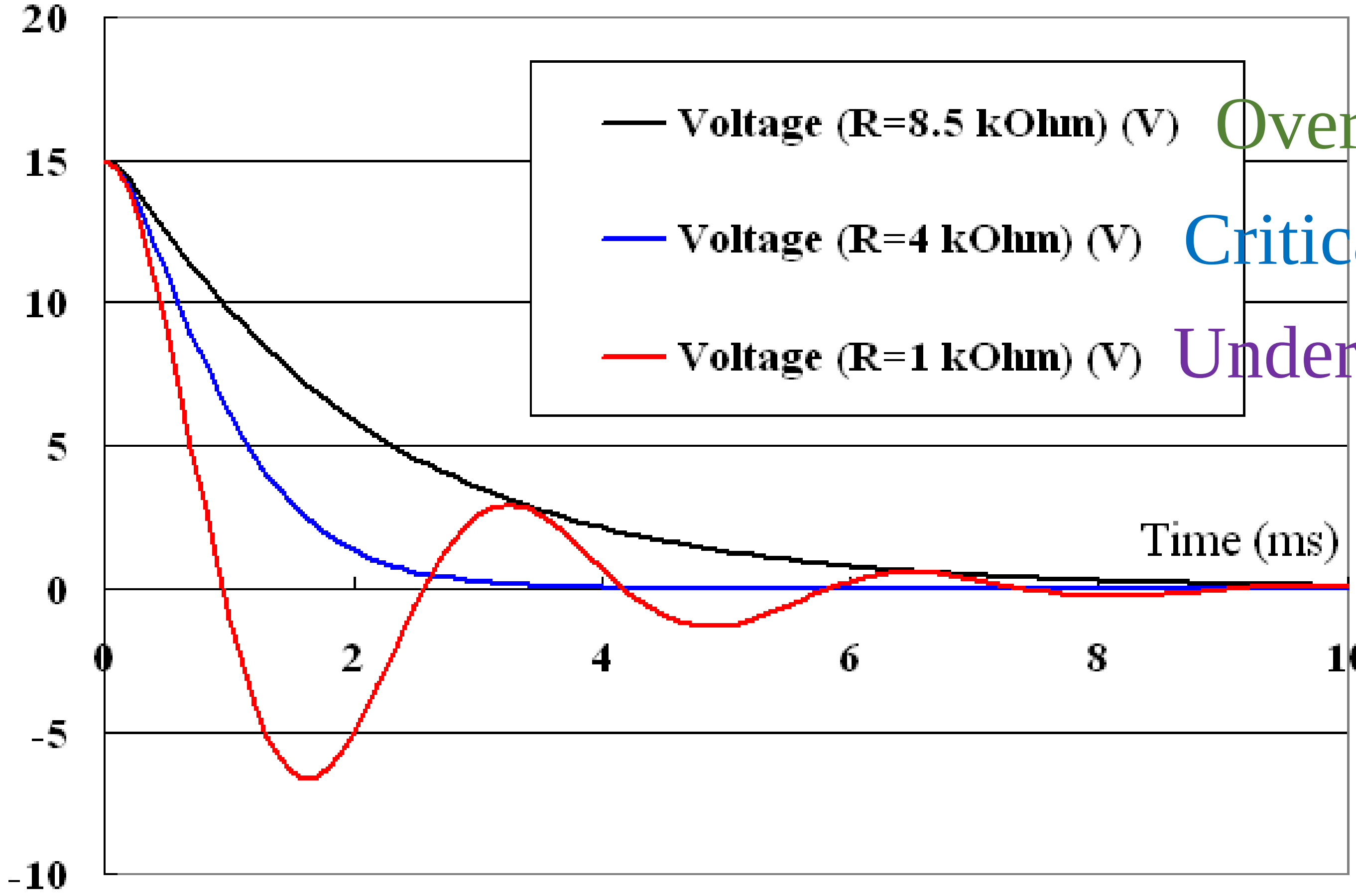


SUMMARY

- ✓ The damping effect is due to the presence of resistance R.
- ✓ The damping factor α determines the rate at which the response is damped.
- ✓ If $R = 0$, then $\alpha = 0$ and we have an LC circuit with $\frac{1}{\sqrt{LC}}$ as the undamped natural frequency. The response in such a case is undamped and purely oscillatory. This circuit is said to be lossless because the dissipating or damping element (R) is absent.
- ✓ The over-damped has the longest settling time because it takes the longest time to dissipate the initial stored energy.
- ✓ If we desire the fastest response without oscillation or ringing, the critical-damped circuit is the right choice.

SUMMARY





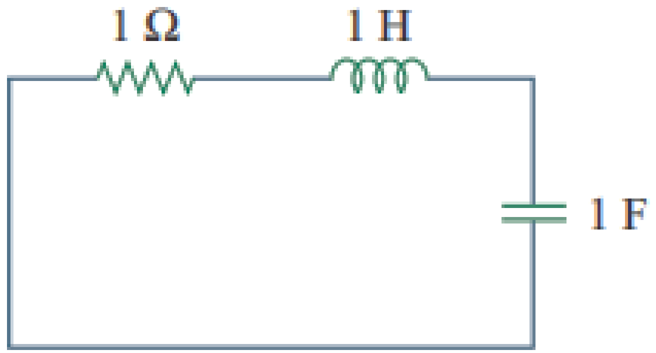
Over Damped

Critical Damped

Under Damped

QUIZ 8

1. Refer to the given series *RLC* circuit, what kind of natural response will it produce?



(a) under damped

(b) over damped

(c) critical damped

(d) un-damped

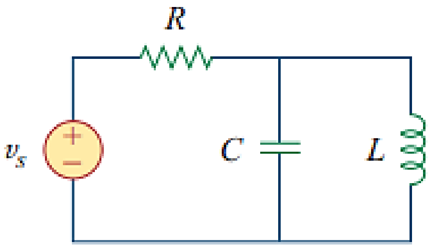
2. Match the circuits with the following items:

(1) 1st-order circuit

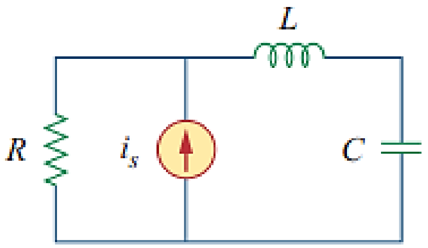
(2) 2nd-order series circuit

(3) 2nd-order parallel circuit

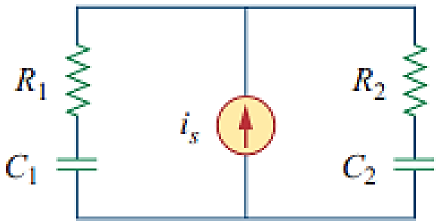
(4) None of the above



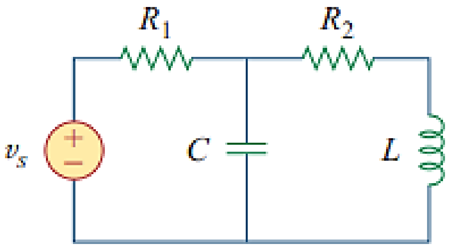
A



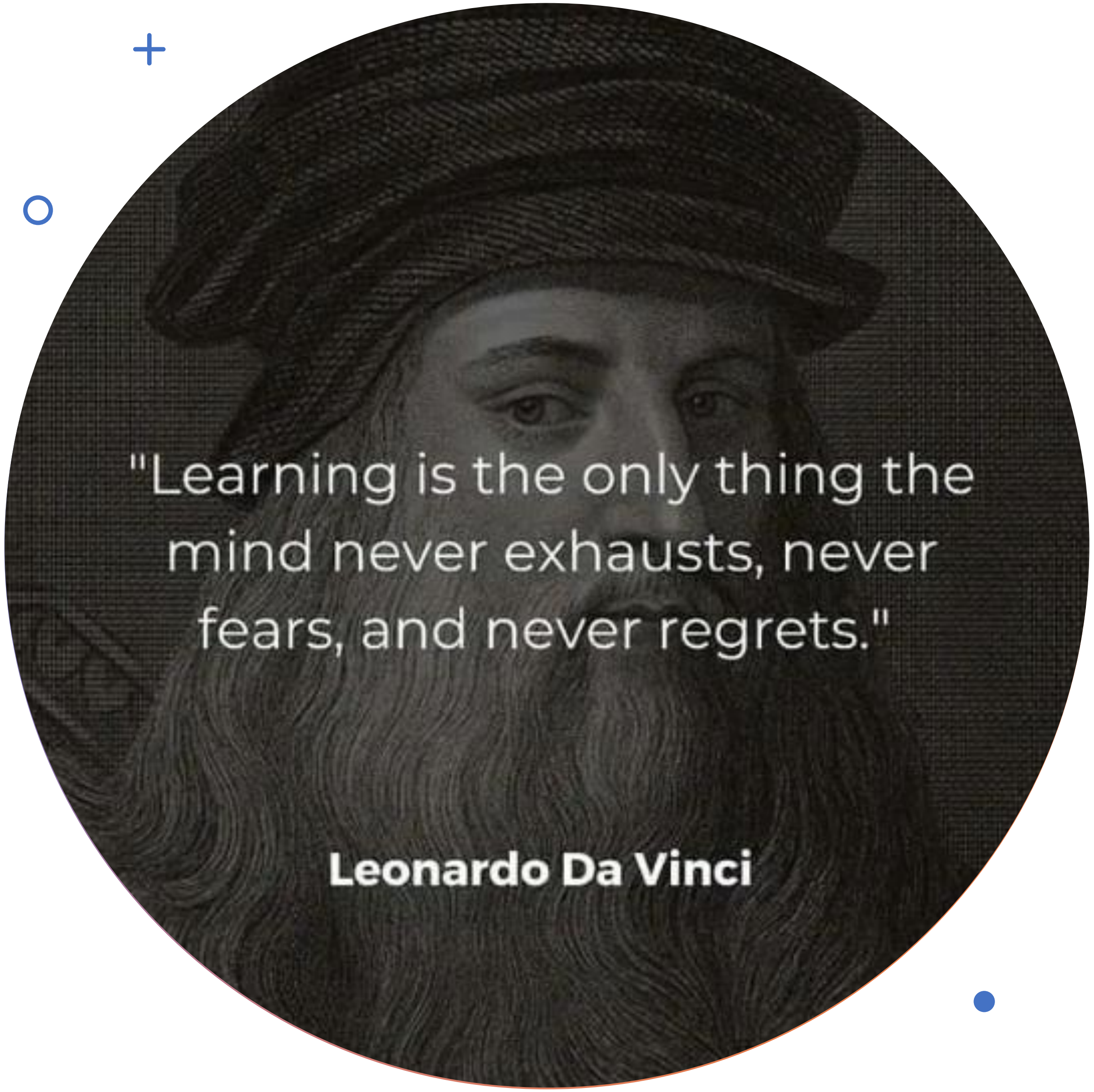
B



C



D



NEXT…

Transient Response of 2nd-Order Circuits

(Step Response)