CAN207 Continuous and Discrete Time Signals and Systems

Lecture-2

Mathematical Review

Zhao Wang

Zhao.wang@xjtlu.edu.cn

Room EE322

Outline

- 1. Fundamentals
  - Trigonometry
  - Exponential and logarithmic identities
- 2. Calculus
  - Derivatives and integrals
- 3. Summation formulas
- 4. Complex numbers & Euler's formula
- 5. Orthogonality
- 6. Partial fraction expansion

Mathematical Review 1 - Fundamentals

- 1. Trigonometry
  - Basic and Pythagorean Identities
    - $sin(-t) =- sin(t); \quad cos(t) = cos(t); \quad tan(-t) =- tan(t)$
    - $tan(t) = \frac{sin(t)}{cos(t)}; \quad sin^2(t) + cos^2(t) = 1$
  - Double-Angle and Half-Angle Identities
    - $sin(2t) = 2sin(t)cos(t); \quad cos(2t) = cos^2(t) - sin^2(t)$
    - $sin(\frac{t}{2}) = \pm\sqrt{\frac{1-cos(t)}{2}}; \quad cos(\frac{t}{2}) = \pm\sqrt{\frac{1+cos(t)}{2}}$
  - Sum and -Difference Identities
    - $sin(\alpha \pm \beta) = sin(\alpha)cos(\beta) \pm cos(\alpha)sin(\beta)$
    - $cos(\alpha \pm \beta) = cos(\alpha)cos(\beta) \mp sin(\alpha)sin(\beta)$
  - Product Identities
    - $sin(\alpha)cos(\beta) = \frac{1}{2}[sin(\alpha + \beta) + sin(\alpha - \beta)]$
    - $sin(\alpha)sin(\beta) = \frac{1}{2}[cos(\alpha - \beta) - cos(\alpha + \beta)]$

Mathematical Review 1 - Fundamentals

- 2. Exponent identities
  - $a^{x+y} = a^x a^y; \quad \frac{a^x}{a^y} = a^{x-y}$
  - $(a^x)^y = a^{yx} = a^{xy}$
  - $\ln e^x = x$

https://www.purplemath.com/modules/simpexpo.htm

- 3. Logarithmic identities
  - $\log_{10} x = \log x$ (common logarithm)
  - $\log_e x = \ln x$ (natural logarithm)
  - $\log_n x = \log x / \log n$
  - $\log xy = \log x + \log y$
  - $\log \frac{x}{y} = \log x - \log y$
  - $\log x^n = n \log x$

https://www.purplemath.com/modules/logrules.htm

Mathematical Review 2 - Derivatives

- 1. Fundamental

https://www.derivative-calculator.net/

  - $f(x) = x^a \rightarrow f'(x) = a \cdot x^{a-1}$
  - $f(x) = e^x \rightarrow f'(x) = e^x$
  - $f(x) = ln(x) \rightarrow f'(x) = \frac{1}{x}$
  - $f(x) = sin(x) \rightarrow f'(x) = cos(x)$
  - $f(x) = cos(x) \rightarrow f'(x) =- sin(x)$

- 2. Calculation rules
  - $F(x) = f(x) \cdot g(x) \rightarrow F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
  - $F(x) = \frac{f(x)}{g(x)} \rightarrow F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$
  - $F(x) = f[g(x)] \rightarrow F'(x) = f'[g(x)] \cdot g'(x)$
    - or written as $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, where $y = f(u)$ and $u = g(x)$

Mathematical Review 2 - Integral

- 1. Indefinite integrals:

https://www.integral-calculator.com/

  - $\int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$
  - $\int x^{\alpha} dx = \frac{1}{\alpha+1} x^{\alpha+1} + C$
  - $\int a^{x} dx = \frac{a^{x}}{ln\, a} + C \quad (a > 0, a \neq 1)$
  - $\int \frac{1}{x} dx = ln\, |x| + C$
  - $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} arctan\, \frac{x}{a} + C$
  - $\int \frac{1}{\sqrt{a^2 - x^2}} dx = arcsin\, \frac{x}{a} + C$
  - $\int \cos ax\, dx = \frac{1}{a} \sin ax + C$
  - $\int \sin ax\, dx = -\frac{1}{a} \cos ax + C$
  - $\int e^{x} sin\, x\, dx = \frac{e^{x}(sin\, x - cos\, x)}{2}$; $\int e^{x} cos\, x\, dx = \frac{e^{x}(sin\, x + cos\, x)}{2}$

Mathematical Review 2 - Integral

- 2. Newton-Leibniz Formula (定积分):

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$$

- 3. Integral with Variable Upper Limit (变上限积分):

$$\left(\int_a^t f(\tau)d\tau\right)' = \frac{d}{dt}\left(\int_a^t f(\tau)d\tau\right) = f(t)$$

- 4. Integration by Substitution (换元法):
  - $x = \varphi(t), with\ \varphi(\alpha) = a, \varphi(\beta) = b$

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

- 5. Integration by Parts (分部积分法):

$$\int u(t)dv(t) = \int u(t)v'(t)dt$$

$$= u(t)v(t) - \int v(t)\, du(t) = u(t)v(t) - \int u'(t)v(t)dt$$

Mathematical Review 3 – Summation formula

- Finite sum formula

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & \text{for any complex number } a \neq 1 \end{cases}$$

- Infinite sum formula

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

$$\sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2}, \quad |a| < 1$$

Quiz 1 Integral and Summation

- Q1. Evaluate the following definite integrals

$$\int_{2}^{\infty} e^{-3t} dt$$

- Q2. Evaluate $\int e^x \sin x \, dx$.

- Q3. Show (prove) if $|a| < 1$, then

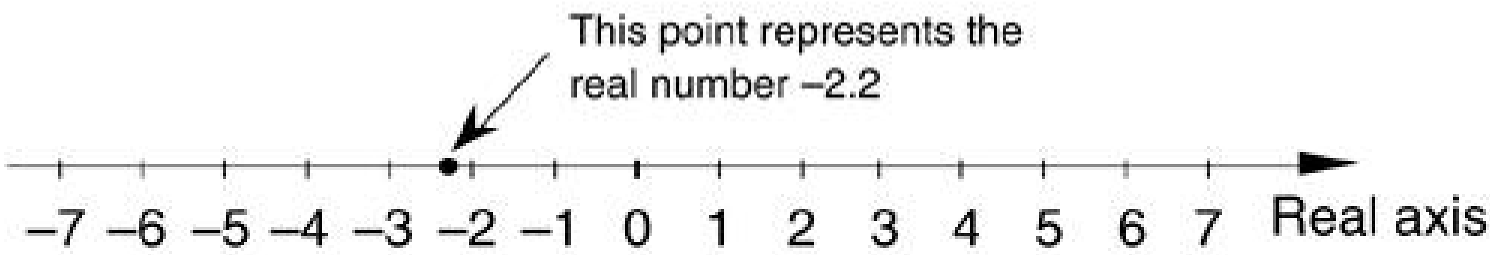
$$\sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2}$$

Mathematical Review 4 – Complex numbers

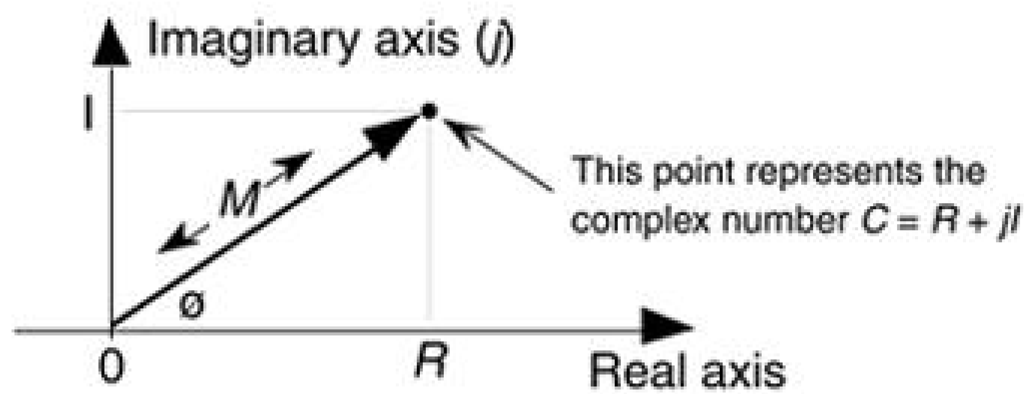
- Real VS. Complex
- Representation of complex numbers
- Euler's formula
- Operations / calculations of complex numbers
  - Addition and subtraction
  - Multiplication and division
  - Conjugate
  - Raising to power and taking roots
  - Logarithms

Graphical representation

- Real numbers: all real numbers correspond to all of the points on the real axis line.



- Complex numbers: a complex number can be treated as a point on a complex plane



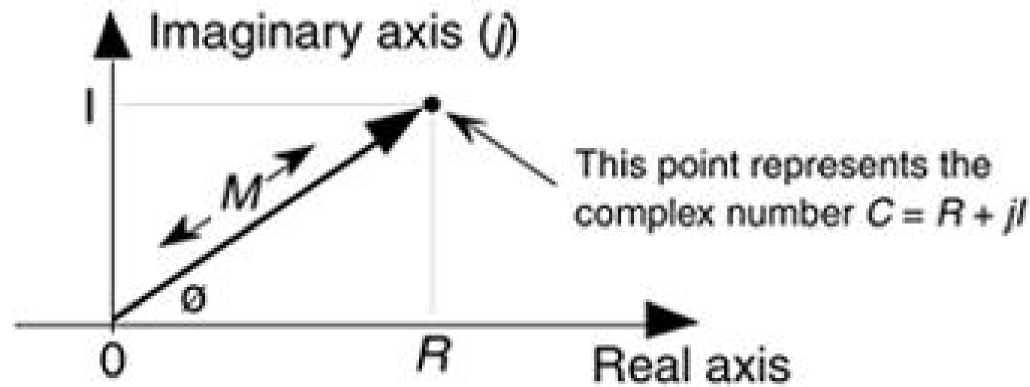
Arithmetic representation

- 1. Rectangular form:

$$C = R + jI$$

- 2. Exponential form:

$$C = Me^{j\theta}$$



- 3. Polar form:

$$C = M\angle\theta$$

- 4. Trigonometric form:

$$C = Mcos\theta + jMsin\theta$$

Imaginary unit 'i' or 'j':

$$i = j = \sqrt{-1}$$

Magnitude M

$$M = |C| = \sqrt{R^2 + I^2}$$

Phase θ

$$\theta = \arctan(I/R)$$

Euler's Formula

- Euler's formula:

$$\boxed{e^{j\theta} = cos\theta + jsin\theta}$$

- Euler's identity:

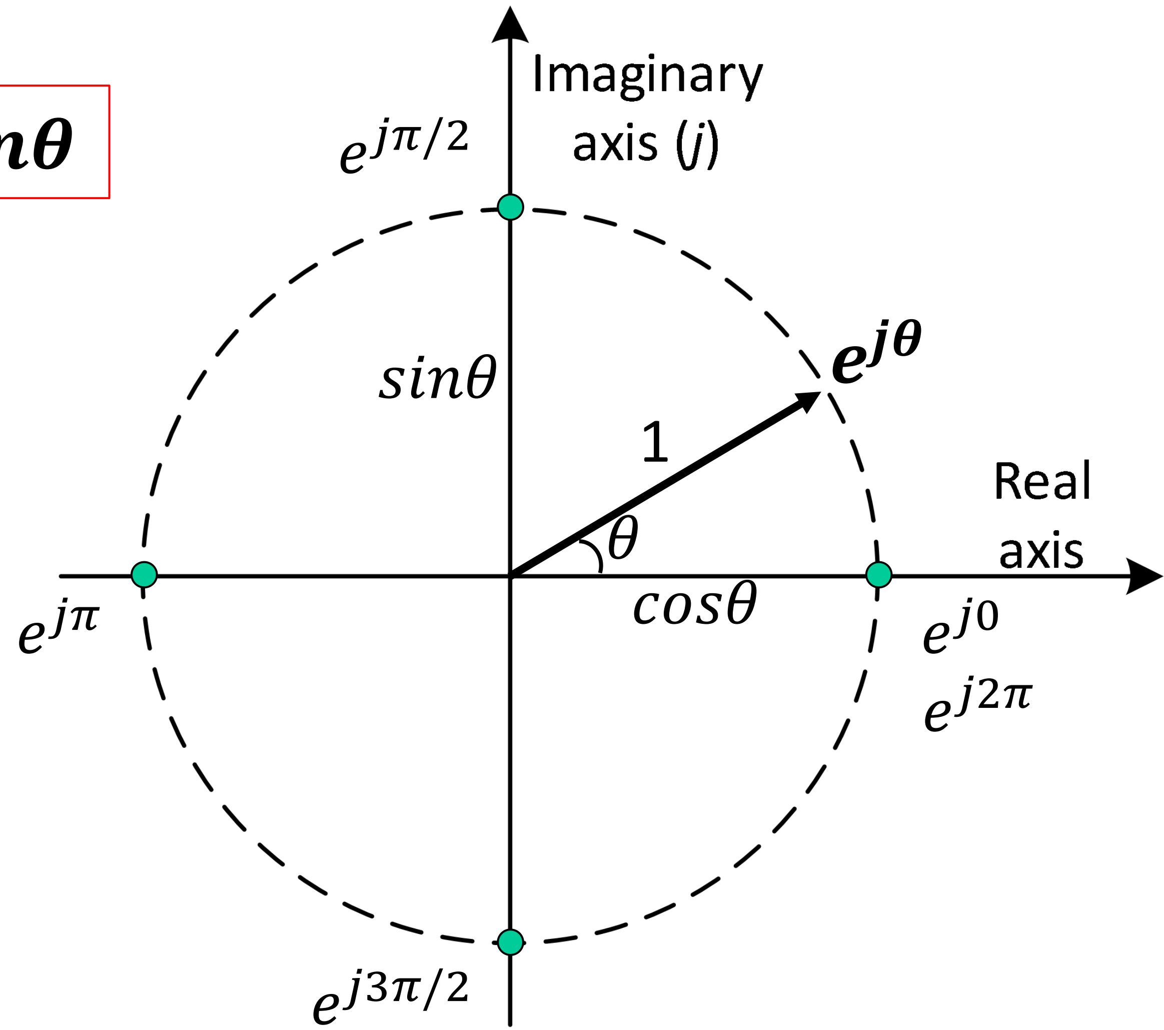
$$\boxed{e^{j\pi} + 1 = 0}$$

- A complex number C can be represented by

$$\boxed{C = Me^{j\theta} = Me^{j(\theta+2n\pi)}}$$

- If the angle $\theta = \omega t$, then we have:

$$\boxed{C = Me^{j\theta} = Me^{j\omega t}}$$



Arithmetic operations

- Addition (using rectangular form):
  - $C_1 + C_2 = (R_1 + jI_1) + (R_2 + jI_2) = (R_1 + R_2) + j(I_1 + I_2)$
- Subtraction (using rectangular form):
  - $C_1 - C_2 = (R_1 - jI_1) + (R_2 - jI_2) = (R_1 - R_2) + j(I_1 - I_2)$
- Multiply two complex numbers (using rectangular form)
  - $C_1C_2 = (R_1 + jI_1)(R_2 + jI_2) = (R_1R_2 - I_1I_2) + j(R_1I_2 + I_1R_2)$
- Multiply two complex numbers (using exponential form)
  - $C_1C_2 = M_1e^{j\theta_1}M_2e^{j\theta_2} = M_1M_2e^{j(\theta_1+\theta_2)}$
- Scaling (using rectangular form)
  - $KC = K(R + jI) = KR + jKI$

Casio FS82ES

- Scaling (using exponential form)
  - $KC = KMe^{j\theta}$

Arithmetic operations

- Conjugation 共轭 (in rectangular and exponential forms)
  - $C^* = R - jI = Me^{-j\theta}$
  - Characteristics of conjugate: If $C = C_1C_2$, then its conjugate C^* is:
    - $C^* = (C_1C_2)^* = M_1M_2e^{-j(\theta_1+\theta_2)} = M_1e^{-j\theta_1}M_2e^{-j\theta_2} = C_1^*C_2^*$
  - The product CC^* is:
    - $CC^* = Me^{j\theta}Me^{-j\theta} = M^2e^{-j0} = M^2$
- Division of two complex numbers (in exponential form)
  - $\frac{C_1}{C_2} = \frac{M_1e^{j\theta_1}}{M_2e^{j\theta_2}} = \frac{M_1}{M_2}e^{j(\theta_1-\theta_2)} = \frac{M_1}{M_2}\angle\theta_1 - \theta_2$
- Inverse of a complex number 倒数 (in exponential form)
  - $\frac{1}{C_2} = \frac{1}{M_2e^{j\theta_2}} = \frac{1}{M_2}e^{-j\theta_2} = \frac{1}{M_2}\angle - \theta_2$

Arithmetic operations

- The k^{th} power of a complex number $C = Me^{j\theta}$

  $C^k = (Me^{j\theta})^k = M^k e^{jk\theta}$

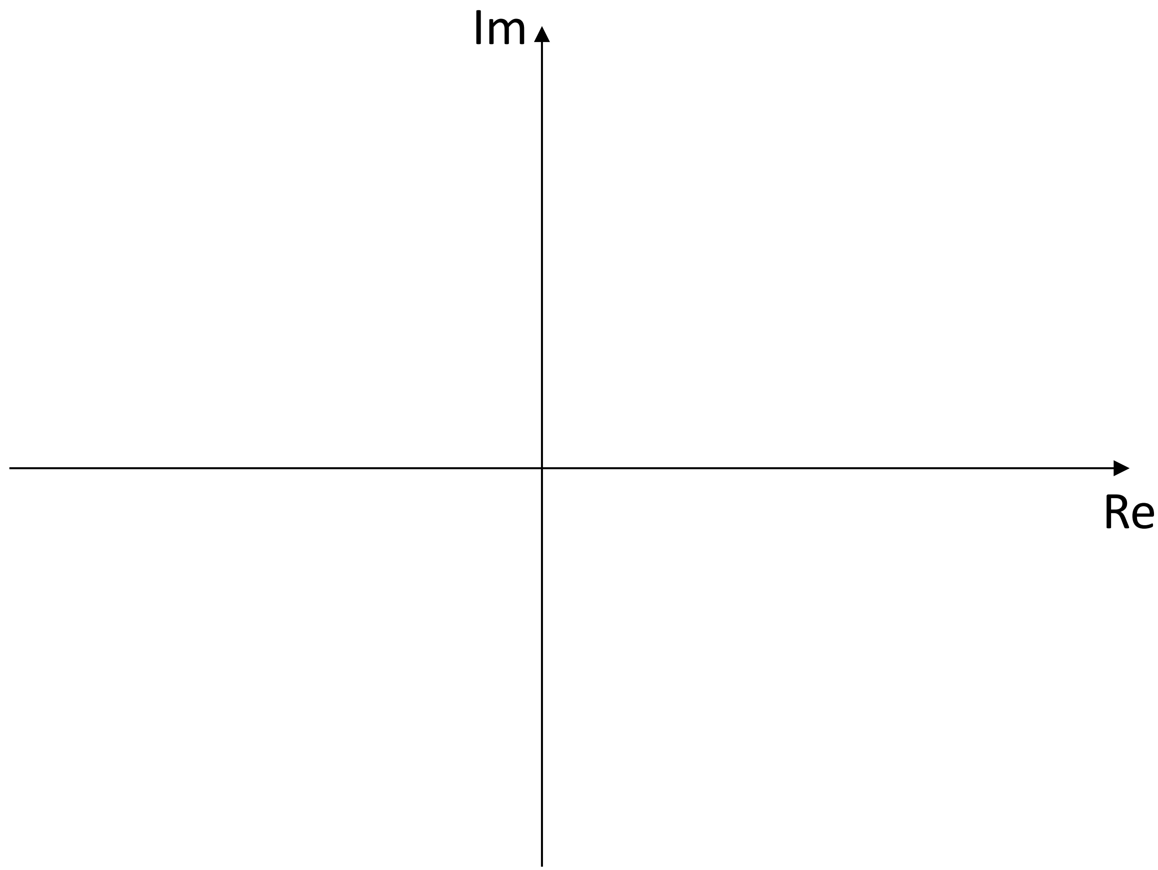
- The k^{th} root of a complex number $C = Me^{j\theta}$
  - Since $C = Me^{j\theta} = Me^{j(\theta+2n\pi)} = Me^{j(\theta+n360°)}$
  - Its roots are:

    $$\sqrt[k]{C} = \sqrt[k]{Me^{j(\theta+n360°)}} = \sqrt[k]{M}e^{j\frac{\theta+n360°}{k}}$$

  - The value of n can be 0, 1, 2, 3, …, k-1.

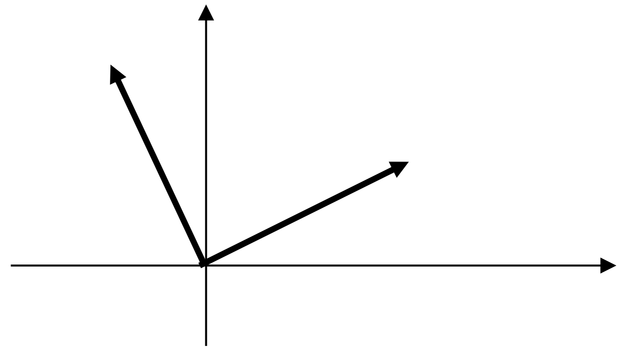
Quiz 2

- For $\boldsymbol{z} = re^{j\theta}$, with $r = 2$ and $\theta = \frac{\pi}{4}$, plot the following functions of z in the complex plane (also called the z-plane)
  - z^*;
  - z^2;
  - jz;
  - zz^*;
  - z/z^*;
  - $1/z$;
  - z^(1/4).



Mathematical Review 5 – Orthogonality

- 1. Fundamental concept
  - In Geometry
    - Orthogonal VS Perpendicular



  - Inner product
    - 2D: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\varphi = 0$
    - Multiple dimensional

$$\begin{aligned}\vec{a} &= \{a_1, a_2, \ldots, a_n\} \\ \vec{b} &= \{b_1, b_2, \ldots, b_n\}\end{aligned} \Longrightarrow \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = \sum_{i=1}^{n} a_i b_i = 0$$

    - Continuous (functions)

$$\begin{aligned}a &= f(x) \\ b &= g(x)\end{aligned} \Longrightarrow a \cdot b = \int_{x_1}^{x_2} f(x) g(x) dx = 0$$

Orthogonality for Trigonometric functions

- 2. Sinusoidal function set

  $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ \ldots\ldots, \sin nx, \cos nx\}$

  – When $n \neq m$
    - $\int_{-\pi}^{\pi} \sin nx \cos mx \, dx$
    - $\int_{-\pi}^{\pi} \cos nx \cos mx \, dx$
    - ……
  – When $n = m$
    - $\int_{-\pi}^{\pi} \sin nx \cos mx \, dx$
    - $\int_{-\pi}^{\pi} \cos nx \cos mx \, dx$
    - ……

Orthogonality for Trigonometric functions

- Consider the two real-valued basis function sets:

$$\phi_k = \cos k\omega_0 t\,, \quad k = -\infty, \ldots, \infty$$

$$\psi_k = \sin k\omega_0 t\,, \quad k = -\infty, \ldots, \infty$$

  - ω_0 is the fundamental frequency in rad/s; Period is $T_0 = \frac{2\pi}{\omega_0}$.

- Each of the two sets is orthogonal within itself:

$$\int_0^{T_0} \phi_m(t)\,\phi_n(t)dt = \begin{cases} 0, & n \neq m \\ T_0/2, & n = m \end{cases}$$

$$\int_0^{T_0} \psi_m(t)\,\psi_n(t)dt = \begin{cases} 0, & n \neq m \\ T_0/2, & n = m \end{cases}$$

- And the two sets are orthogonal to each other:

$$\int_0^{T_0} \phi_m(t)\,\psi_n(t)dt = 0, \ \text{ for any } m,\ n$$

Orthogonality for Exponential functions

- 3. Exponential functions
  - The set of complex periodic basis functions is expressed as: $\phi_k = e^{jk\omega_0 t}$, k is integer
    - with ω_0 as the fundamental frequency
    - $T_0 = \frac{2\pi}{\omega_0}$ as the corresponding period.
  - The basis function set is orthogonal in the sense

$$\int_0^{T_0} \phi_m(t)\, \phi_n^*(t) dt = \begin{cases} 0, & n \neq m \\ T_0, & n = m \end{cases}$$

$$\Rightarrow \quad \int_0^{T_0} e^{jm\omega_0 t}\, e^{-jn\omega_0 t} dt = \begin{cases} 0, & n \neq m \\ T_0, & n = m \end{cases}$$

Quiz 3

- 1. Prove $\int_0^T \sin nx \sin mx \, dx = \begin{cases} 0, & n \neq m \\ \frac{T}{2}, & n = m \end{cases}$.

- 2. Consider the set of discrete-time complex period basis functions

$$W_N^k = e^{-j\frac{2\pi}{N}k}, \quad k = 0,1,\ldots,N-1$$

  – Where N and k are both integers;

  – Prove the orthogonality in the sense:

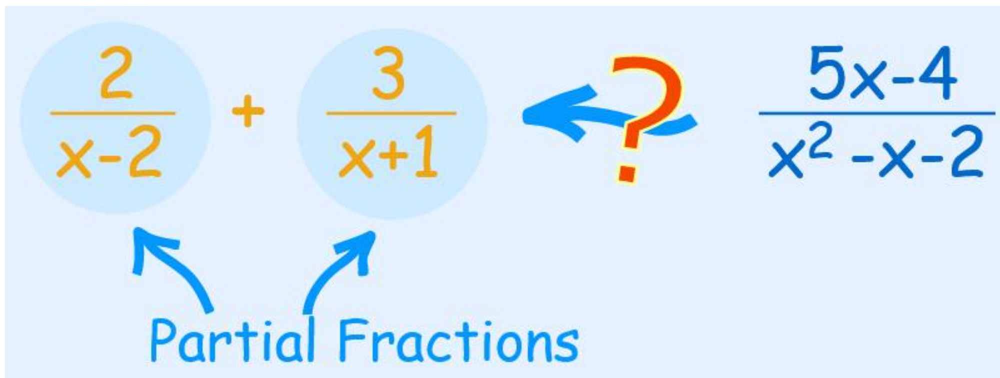
$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}mn} = \begin{cases} 0, & k \neq m \\ N, & k = m \end{cases}$$

Mathematical Review 6 – Partial Fraction Expansion

- What are “partial fractions”?

$$\frac{2}{x-2} + \frac{3}{x+1} \rightarrow \frac{5x-4}{x^2-x-2}$$

  - But how do we go in the opposite direction?



  - It’s called the “Partial Fraction Expansion” 部分分式分解, i.e. PFE.

- Since the partial fractions are each simpler, PFE is broadly used in the inverse transforms, especially Laplace and z transforms.

Partial Fraction Expansion - procedure

- Step 1: Factorise the bottom (denominator)

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)}$$

- Step 2: Write one partial fraction for each of those factors

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A_1}{x-2} + \frac{A_2}{x+1}$$

- Step 3: Reduce the fractions to a common denominator

$$5x-4 = A_1(x+1) + A_2(x-2)$$

- Step 4: Solve for A_1 and A_2

Root for $(x+1)$ is $x = -1$

$$5(-1) - 4 = A_1(-1+1) + A_2(-1-2)$$
$$-9 = 0 + A_2(-3)$$
$$A_2 = 3$$

Root for $(x-2)$ is $x = 2$

$$5(2) - 4 = A_1(2+1) + A_2(2-2)$$
$$6 = A_1(3) + 0$$
$$A_1 = 2$$

$$\boxed{\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}}$$

Important facts to know …

- 1. Proper Rational Expressions
  - It only works for Proper Rational Expressions, where the degree of the top is less than the bottom.
  - If your expression is Improper, then do **polynomial long division** first.

- 2. Factors with Exponents (Optional)
  - When there is a factor with an exponent, like $(x-2)^3$, you need a partial fraction for each exponent from 1 up.

$$\frac{1}{(x-2)^3} \Longrightarrow \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3}$$

Quiz 4

- 1. Find the partial fraction expansion of:

$$\frac{x^2 + 15}{(x+3)(x+5)}$$

- 2. Find the partial fraction expansion of:

$$X(s) = \frac{s+1}{(s+2)^2 + 9}$$

Next …

- Introduction to Signals
  - 1. Signal representation
  - 2. Signal classification (properties)
  - 3. Signal operations (time-domain transformation)
  - 4. Elementary signals and sequences