MTH102 Engineering Mathematics II

Lesson 3: Conditional probability

Term: 2024

Outline

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1 Conditional probability

2 Multiplication rule

3 Law of total probability

Example 1

Consider the distribution of pass/fail in a course by students' gender.

	Pass	Fail	Total
Male	60	30	90
Female	9	1	10
Total	69	31	100

- The probability of picking a student that is male and passed is 60%.
- The probability of picking a student that is female and passed is 9%.
- Can we say that the boys are way better than the girls in the exam?

Sol: Let $P=\{Pass\}$, $E=\{Fail\}$, $M=\{Male\}$, $F=\{Female\}$

$S=\{$All the 100 students in a course$\}$

① $P(M\cap P)=0.6$ ② $P(F\cap P)=0.09$

Example 2

A couple have two children. Consider the gender of the children, then the sample space is

$$S = \{(B, B), (B, G), (G, B), (G, G)\}.$$

The probability of the event A that this couple have two girls is

$$P(A) = \frac{1}{4}.$$

Now knowing that there is at least one girl in this family, called event B, then the sample space is reduced to

$$B = \{(B, G), (G, B), (G, G)\}.$$

And the probability that this couple have two girls is $\frac{1}{3}$.

Conditional probability: definition

Let A and B be events of a sample space S and $P(B) \neq 0$. The probability that an event A occurs given that an event B occurs is called the *conditional probability of* A *given* B and it is denoted by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Similarly if $P(A) \neq 0$, the conditional probability of B given A is defined as

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Example 2

A couple have two children. Consider the gender of the children, then the sample space is

$$S = \{(B, B), (B, G), (G, B), (G, G)\}.$$

The conditional probability of the event A that this couple have two girls given that there is at least one girl in this family, called event B, is then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Example 3

Two dice are rolled. Let X be the number on die 1, and Y be the number on die 2. Find $P(X = 2|X + Y \leq 5)$.

	1	2	3	4	5	6	
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	4
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	3
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	2
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	1
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	

Sol: $\mathbb{P}(X=2|X+Y\leq 5) = \dfrac{\mathbb{P}(\{X=2\}\cap\{X+Y\leq 5\})}{\mathbb{P}(X+Y\leq 5)} = \dfrac{3/36}{10/36} = \dfrac{3}{10}$

Go back to Example 1

Consider the distribution of pass/fail in a course by students' gender.

	Pass	Fail	Total
Male	60	30	90
Female	9	1	10
Total	69	31	100

Recall: $P=\{\text{Pass}\}$, $E=\{\text{Fail}\}$, $M=\{\text{Male}\}$, $F=\{\text{Female}\}$, $S=\{\text{All the 100 students in course}\}$

- The probability of picking a student that is male and passed is 60%.
- The probability of picking a student that is female and passed is 9%.
- Can we say that the boys are way better than the girls in the exam?

Sol: $\mathbb{P}(P|M) = \dfrac{\mathbb{P}(M \cap P)}{\mathbb{P}(M)} = \dfrac{60/100}{90/100} = \dfrac{2}{3}$

$\mathbb{P}(P|F) = \mathbb{P}(P \cap F)/\mathbb{P}(F) = \dfrac{9/100}{10/100} = \dfrac{9}{10}$

Since $\frac{9}{10} > \frac{2}{3}$, $\Rightarrow$ we have girls are better than boys in the exam.

Example 4

Example set of 52 playing cards; 13 of each suit: clubs, diamonds, hearts, and spades

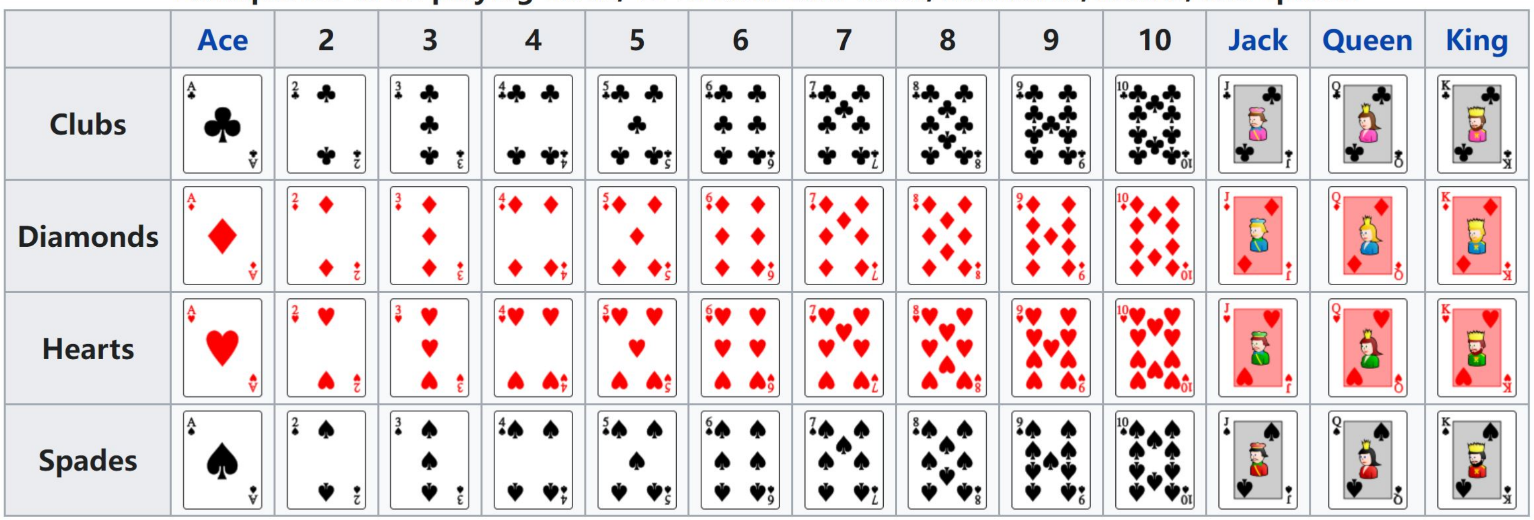


Figure: A deck of 52 cards.

Example 4

Two cards are randomly selected, without replacement from an ordinary deck of 52 cards. Let A be the event that both cards are aces, let B be the event that the ace of spades is chosen, and let C be the event that at least one ace is chosen. Find

(a) $P(A|B)$

(b) $P(A|C)$

Sol: Sample space $S = \{$All combinations of 2 cards selected among 52$\}$, So $|S| = \binom{52}{2}$

(a) $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \dfrac{\mathbb{P}(\{\text{1 ace of spades and a different ace are chosen}\})}{\mathbb{P}(\{\text{an ace of spade is chosen}\})} = \dfrac{\binom{3}{1}/\binom{52}{2}}{\binom{51}{1}/\binom{52}{2}} = \dfrac{3}{51}$

(b) $\mathbb{P}(A|C) = \dfrac{\mathbb{P}(A\cap C)}{\mathbb{P}(C)} = \dfrac{\mathbb{P}(A)}{\mathbb{P}(C)} = \dfrac{\binom{4}{2}/\binom{52}{2}}{1-\mathbb{P}(\text{no ace is chosen})} = \dfrac{\binom{4}{2}/\binom{52}{2}}{1-\binom{52-4}{2}/\binom{52}{2}} = \dfrac{1}{33}$

Since $A \subset C$

Exercise

For a species of animal, the probability that it can survive for more than 10 years is 0.8 and the probability that it can survive for more than 12 year is 0.56. The scientists have found an individual of this species which is 10 years old, what is the probability that it can survive for more than 12 year?

Sol: Let X be the lifetime.

$$\mathbb{P}(X>12 \mid X>10) = \frac{\mathbb{P}(\{X>12\}\cap\{X>10\})}{\mathbb{P}(X>10)} = \frac{\mathbb{P}(X>12)}{\mathbb{P}(X>10)} = \frac{0.56}{0.8} = \frac{7}{10}.$$

Outline

The multiplication rule

If A and B are events of a sample space S with $P(A) \neq 0$ and $P(B) \neq 0$, then

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B).$$

The proof follows directly by the definition of conditional probability.

Example 5

Celine is undecided as to whether to take a French course or a Spanish course. She estimates that her probability of receiving an A grade would be $\frac{1}{2}$ in a French course and $\frac{2}{3}$ in a Spanish course. If she decides to base her decision on the flip of a fair coin, what is the probability that she gets an A in Spanish?

Sol: Let $A = \{\text{Grade A}\}$, $F = \{\text{French course}\}$, $S = \{\text{Spanish course}\}$

Given $\mathbb{P}(A \mid F) = \frac{1}{2}$, $\mathbb{P}(A \mid S) = \frac{2}{3}$, $\mathbb{P}(F) = \mathbb{P}(S) = \frac{1}{2}$

$\mathbb{P}(\{\text{She gets A in Spanish}\}) = \mathbb{P}(A \cap S) = \mathbb{P}(A \mid S)\,\mathbb{P}(S)$ by mutiplication rule

$= \frac{2}{3} \times \frac{1}{2}$

$= \frac{1}{3}$

Example 6

There are 6 white balls and 4 black balls in a bag. If we take one ball and then another without replacement, what is the probability that

(a) the two balls are both white?

(b) the first ball is white and the second ball is black?

Sol: Let $W_i = \{$the i-th ball drawn is white$\}$, $B_i = \{$the i-th ball drawn is black$\}$, $i = 1, 2$. Then by multiplication rule,

(a) $\mathbb{P}(W_1 \cap W_2) = \mathbb{P}(W_2 | W_1)\,\mathbb{P}(W_1) = \frac{5}{9} \times \frac{6}{10} = \frac{1}{3}$

(b) $\mathbb{P}(W_1 \cap B_2) = \mathbb{P}(B_2 | W_1)\,\mathbb{P}(W_1) = \frac{4}{9} \times \frac{6}{10} = \frac{4}{15}$

The multiplication rule: the general case

If $A_1, A_2, \ldots, A_n$ are events of a sample space S with $P(A_i) \neq 0$ for $i = 1, \ldots, n$, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\cdots P(A_n|A_1\cdots A_{n-1}).$$

Here

$$A_1 \cdots A_{n-1} = A_1 \cap \cdots \cap A_{n-1}.$$

Example 7

(Pólya's urn model)

There is one white ball and one black ball in a box. We take a ball out of the box. If it is a white ball, we put it back and add another white ball into the box. We repeat this procedure until we finally get the black ball. What is the probability that we still do not get the black ball after 10 times?

Sol: Set $W_i = \{$get white ball at the i-th drawn$\}$. From multiplication rule,

$\mathbb{P}(W_1) = \mathbb{P}($get white ball at the 1st drawn$) = \frac{1}{2}$

$\mathbb{P}(W_1 \cap W_2) = \mathbb{P}(W_2 | W_1)\mathbb{P}(W_1) = \mathbb{P}($draw 1 white ball among 2 white and 1 black balls$)\ \mathbb{P}(W_1)$

$= \frac{2}{2+1} \times \frac{1}{2} = \frac{1}{3}$

$$\mathbb{P}(W_1 \cap W_2 \cap W_3) = \mathbb{P}(W_3 \mid W_2 W_1)\,\mathbb{P}(W_2 \mid W_1)\,\mathbb{P}(W_1)$$

$= \mathbb{P}(\text{draw 1 white ball among 3 white and 1 black balls}) \times \mathbb{P}(W_2 \mid W_1)\,\mathbb{P}(W_1)$

$$= \frac{3}{3+1} \times \frac{2}{2+1} \times \frac{1}{1+1}$$

$\vdots$

$$\mathbb{P}(W_1 \cap W_2 \cdots \cap W_{10}) = \mathbb{P}(W_{10} \mid W_9 W_8 \cdots W_1)\,\mathbb{P}(W_9 \mid W_8 W_7 \cdots W_1) \cdots \mathbb{P}(W_2 \mid W_1)\,\mathbb{P}(W_1)$$

$= \mathbb{P}(\text{draw 1 white ball among 10 white and 1 black balls}) \times \mathbb{P}(W_9 \mid W_8 \cdots W_1) \cdots \mathbb{P}(W_2 \mid W_1)\,\mathbb{P}(W_1)$

$$= \frac{10}{10+1} \times \frac{\cancel{9}}{\cancel{9+1}} \times \frac{\cancel{8}}{\cancel{8+1}} \times \cdots \times \frac{\cancel{3}}{\cancel{3+1}} \times \frac{\cancel{2}}{\cancel{2+1}} \times \frac{1}{\cancel{1+1}}$$

$$= \frac{1}{11}$$

In conclusion, the probability that we still do not get the black ball after 10 times is $1/11$.

Outline

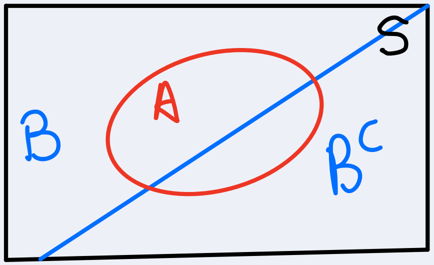
Basic propositions of probability

Law of total probability

Proposition

For any event A and B, it holds that

$$P(A) = P(A \cap B) + P(A \cap B^c).$$



Proof.

Since $S = B \cup B^c$,

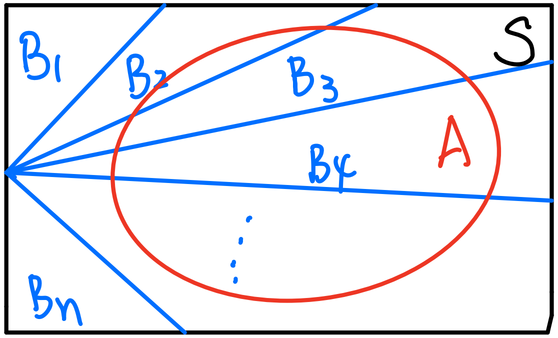
$$A = A \cap S = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c).$$

Note that $B \cap B^c = \emptyset$, we have thus $(A \cap B) \cap (A \cap B^c) = \emptyset$. Therefore by Axiom 3,

$$P(A) = P(A \cap B) + P(A \cap B^c).$$

□

Law of total probability: the general case



Let the events $B_1, B_2, \ldots, B_n$ be mutually exclusive and

$$S = B_1 \cup B_2 \cup \cdots \cup B_n.$$

We assume that $P(B_i) > 0$ for $i = 1, 2, \ldots, n$. Then for any event A, A can be represented as the union of mutually exclusive events, i.e.

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n).$$

Therefore,

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i),$$

which is called **law of total probability**.

Example 8

A visitor is coming for a seminar in XJTLU. The probability that he comes by train, car and flight is respectively 0.5, 0.3 and 0.2. And the probability of delay by these three means of transport is respectively 0.1, 0.3 and 0.4. What is the probability that the visitor is late for the seminar?

Sol : Let $T = \{\text{Coming by train}\}$, $C = \{\text{Coming by car}\}$, $L = \{\text{delay}\}$, $F = \{\text{Coming by flight}\}$. Then $\mathbb{P}(T) = 0.5$, $\mathbb{P}(C) = 0.3$, $\mathbb{P}(F) = 0.2$, $\mathbb{P}(L|T) = 0.1$, $\mathbb{P}(L|C) = 0.3$, $\mathbb{P}(L|F) = 0.4$

$S = T \cup C \cup F$ and T, C, F are multually exclusive events.

Then by law of total proba.,

$$\mathbb{P}(L) = \mathbb{P}(L|T)\mathbb{P}(T) + \mathbb{P}(L|C)\mathbb{P}(C) + \mathbb{P}(L|F)\mathbb{P}(F) = 0.1 \times 0.5 + 0.3 \times 0.3 + 0.4 \times 0.2 = 0.22$$

Exercise

In a grocery shop, champagne glasses are sold in boxes with 6 glasses in each box. Assume that in each box, the respective probabilities of having 0, 1, and 2 defective glass(es) are 0.8, 0.1 and 0.1. A customer takes a box at random, and checks randomly 3 glasses in the box. If there is no defective glass in the 3 glasses he has checked, then he will buy this box. What is the probability that he will buy this box?

Sol: Let $B = \{\text{Buy the box of glasses}\}$, $D_1 = \{\text{there is 1 defective glass}\}$, $D_0 = \{\text{there is 0 defective glass}\}$

$D_2 = \{\text{there are 2 defective glasses}\}$.

Then $\mathbb{P}(D_1) = 0.1$

$\mathbb{P}(D_0) = 0.8$

and $\mathbb{P}(D_2) = 0.1$

Since $B = \left(D_1 \cap \{\text{The defective one is not tested}\}\right) \cup \left(D_2 \cap \{\text{the 2 defective ones are not tested}\}\right) \cup D_0$ and B is union of mutually exclusive events. From Axiom 3 and chain's rule.

$\mathbb{P}(B) = \mathbb{P}(D_1)\,\mathbb{P}(\text{the defective one is not tested} \mid D_1) + \mathbb{P}(D_2) \times \mathbb{P}(\text{the 2 defective ones are not tested} \mid D_2) + \mathbb{P}(D_0)$

$$= 0.1 \times \frac{\binom{5}{3}}{\binom{6}{3}} + 0.1 \times \frac{\binom{4}{3}}{\binom{6}{3}} + 0.8$$

$$= 0.1 \times \left(\frac{1}{2} + \frac{1}{5}\right) + 0.8 = 0.87$$

The probability that he will buy the box is 0.87.