Xi'an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

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Dr. Bangxiang Chen

Email: Bangxiang.Chen@xjtlu.edu.cn

Office: SC554E

Department of Mechatronics and Robotics

School of Advanced Technology

Quiz 9.1

Consider the system with the mathematical model given by the differential equation:

$$5\frac{d^3y(t)}{dt^3} + 10\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = u(t)$$

Obtain a state variable model of this system.

Select state variables as $x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y}$, then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -2x_3 - x_2 - 0.4x_1 + 0.2u$$

State space model

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.4 & -1 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\boldsymbol{x} + [0]u$$

Lecture 10

Outline

State Variable Models

- ❑ **Introduction**
- ❑ **State Variables**
- ❑ **State-space Modeling**
- ❑ **State Space Representation in Matrix Form**
- ❑ ***Time-domain response (Solution of State-space Models)***
- ❑ **Conversion between State-space Model and Transfer Function**
- ❑ **Analysis of the State-space Models using Matlab**

Solution of State Equations

Consider the first-order differential equation (**one-dimension**):

$$\dot{x} = ax + bu$$

With Laplace transform,

$$sX(s) - x(0) = aX(s) + bU(s)$$

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

Inverse Laplace transform satisfies, note : $\mathcal{L}^{-1}(f \cdot g) = \mathcal{L}^{-1}(f) * \mathcal{L}^{-1}(g)$:

$$x(t) = e^{at}x(0) + be^{at} * u(t)$$

Therefore:

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau$$

Solutions for Matrix Form

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s),$$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

And we let $[s\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{\Phi}(s)$, it is the Laplace form of

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$$

Assume $\mathbf{U}(s) = 0$, then

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0)$$

$\mathbf{\Phi}(s)$ describes the unforced response of the system.

Solutions for Matrix Form

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s),$$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

The matrix exponential function of $e^{\mathbf{A}t}$ is defined as:

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^k t^k}{k!} + \ \ldots$$

Then:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

Solution to the matrix:

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\,\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

Fundamental/State Transition Matrix

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$$ $\boxed{\text{State Transition Matrix}}$

The solution to the unforced system (when u(t)=0):

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{bmatrix} \emptyset_{11} & \dots & \emptyset_{1n} \\ \emptyset_{12} & \dots & \emptyset_{2n} \\ \vdots & \ddots & \vdots \\ \emptyset_{n1} & \dots & \emptyset_{nn} \end{bmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix}$$

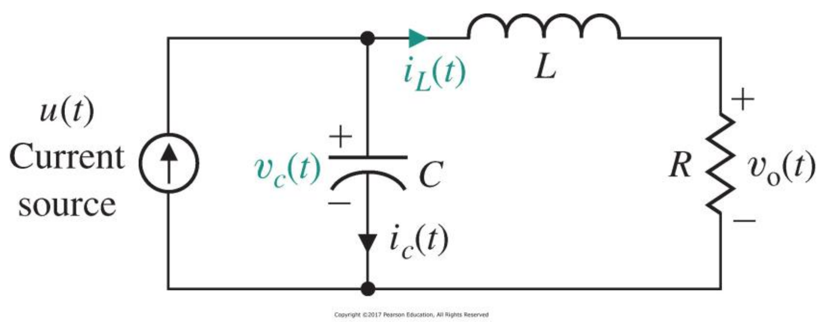
– $\emptyset_{ij}$ is the response of the i^{th} state variable due to an initial condition on the j^{th} state variable when there are zero initial conditions on all the other variables.

Example 9.5: Time Response of Unforced System

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$

$$\boxed{\begin{gathered}[s\mathbf{I}-\mathbf{A}]^{-1} = \mathbf{\Phi}(s) \\ \mathbf{\Phi}(t) = \exp(\mathbf{A}t) = \mathcal{L}^{-1}(\mathbf{\Phi}(s))\end{gathered}}$$

--- ***fundamental/state transition matrix***



$$\mathbf{A} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [0 \quad R], \qquad \mathbf{D} = [0]$$

Assume C = ½, L = 1, R = 3, then:

$$\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \longrightarrow s\mathbf{I}-\mathbf{A} = \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}$$

Inverse of a 2*2 matrix

$$\mathbf{\Phi}(s) = [s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{-2}{s^2+3s+2} \\ \frac{1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

Example 9.5: Time Response of Unforced System

$$\mathbf{\Phi}(t)=\mathcal{L}^{-1}(\mathbf{\Phi}(s)) = \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{-2}{s^2+3s+2} \\ \frac{1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \left(\frac{2}{s+1}+\frac{-1}{s+2}\right) & \left(\frac{-2}{s+1}+\frac{2}{s+2}\right) \\ \left(\frac{1}{s+1}+\frac{-1}{s+2}\right) & \left(\frac{-1}{s+1}+\frac{2}{s+2}\right) \end{bmatrix}$$

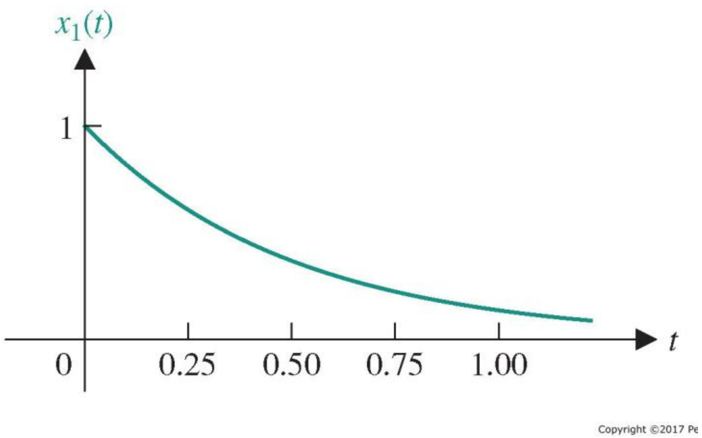
$$= \begin{bmatrix} (2e^{-t}-e^{-2t}) & (-2e^{-t}+2e^{-2t}) \\ (e^{-t}-e^{-2t}) & (-e^{-t}+2e^{-2t}) \end{bmatrix}$$

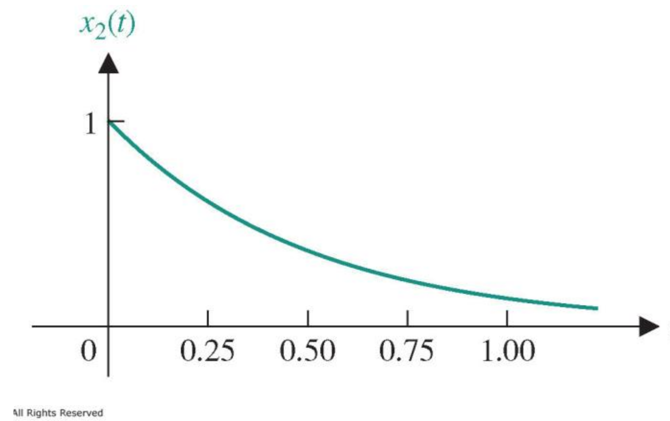
Assume $x_1(0) = x_2(0) = 1$: $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} (2e^{-t}-e^{-2t}) & (-2e^{-t}+2e^{-2t}) \\ (e^{-t}-e^{-2t}) & (-e^{-t}+2e^{-2t}) \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$

Example 9.5: Time Response of Unforced System

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$





How about $y(t)$?

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Example 9.6: Time Response of Forced Systems

Consider the same system in example 9.5, compute the step response.

The response with control is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

If we define $\tau' = t - \tau, \tau \in [0, t]$, then

$$\tau' = t - \tau \in [0, t]$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(\tau')\mathbf{B}\mathbf{u}(t-\tau')\, d\tau'$$

$$= \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(\tau)\mathbf{B}\mathbf{u}(t-\tau)\, d\tau$$

Sometimes this is more convenient

Example 9.6: Time Response of Forced Systems

The state transition matrix is the compute before as

$$\mathbf{\Phi}(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

When $u(t) = 1(t)$, the response is written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(\tau)\mathbf{B}\, d\tau$$

While the first term is the same, the second term is

$$\int_0^t \mathbf{\Phi}(t)\mathbf{B}\, d\tau = \int_0^t \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} d\tau$$

Example 9.6: Time Response of Forced Systems

$$\int_0^t \mathbf{\Phi}(t)\mathbf{B}\,d\tau = \int_0^t \begin{bmatrix} 4e^{-\tau} - 2e^{-2\tau} \\ 2e^{-\tau} - 2e^{-2\tau} \end{bmatrix} d\tau = \begin{bmatrix} -4e^{-\tau} + e^{-2\tau} \\ -2e^{-\tau} + e^{-2\tau} \end{bmatrix}\Bigg|_0^t$$

$$= \begin{bmatrix} -4e^{-t} + e^{-2t} + 3 \\ -2e^{-t} + e^{-2t} + 1 \end{bmatrix}$$

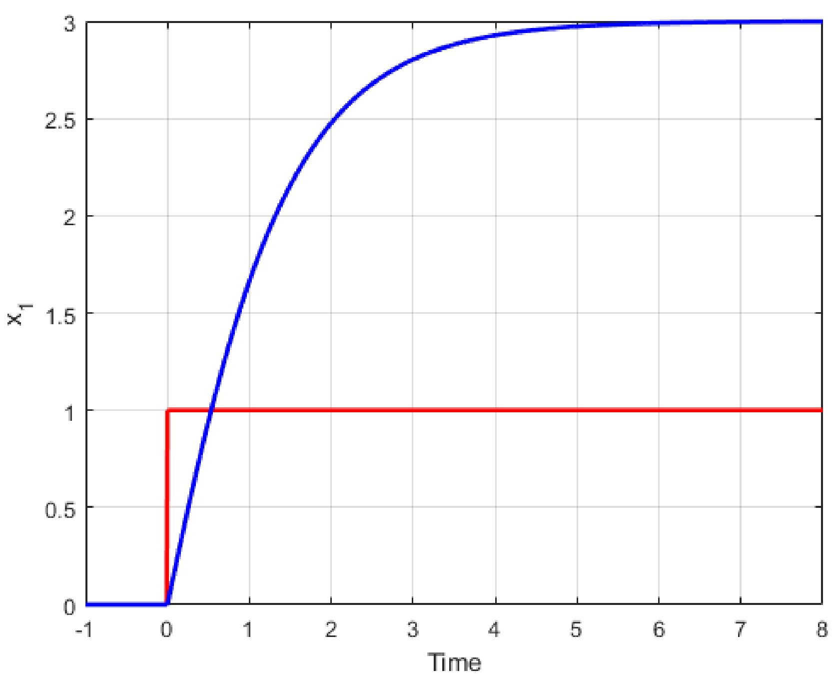
So the time response is

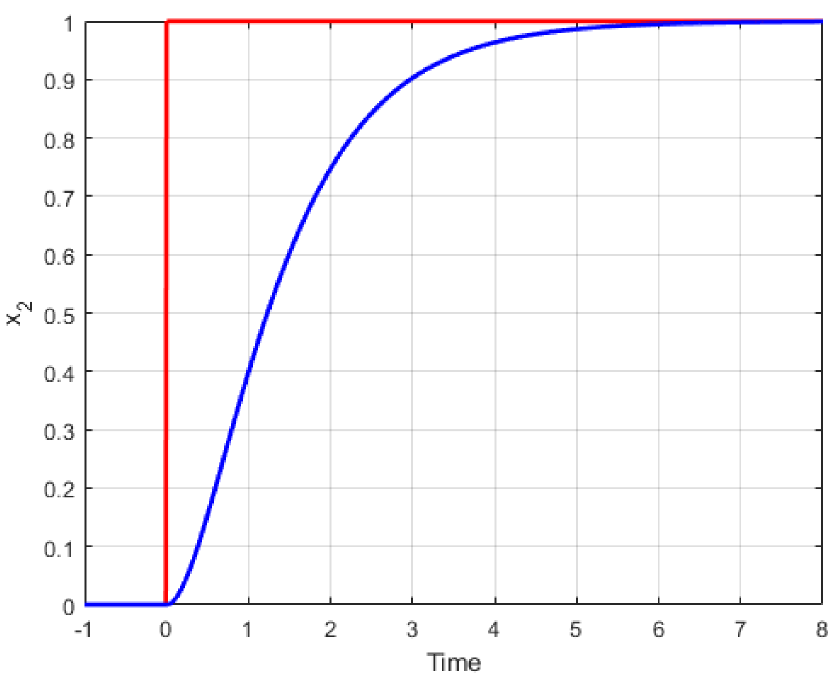
$$\mathbf{x}(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} -4e^{-t} + e^{-2t} + 3 \\ -2e^{-t} + e^{-2t} + 1 \end{bmatrix}$$

Example 9.6: Time Response of Forced Systems

If $x(0) = 0$, then

$$\mathbf{x}(t) = \begin{bmatrix} -4e^{-t} + e^{-2t} + 3 \\ -2e^{-t} + e^{-2t} + 1 \end{bmatrix}$$





Quiz 9.2

The state-space model of a system is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

(1) Find the state-transition matrix;

(2) If initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the time response of the state variables when $u(t)$=0.

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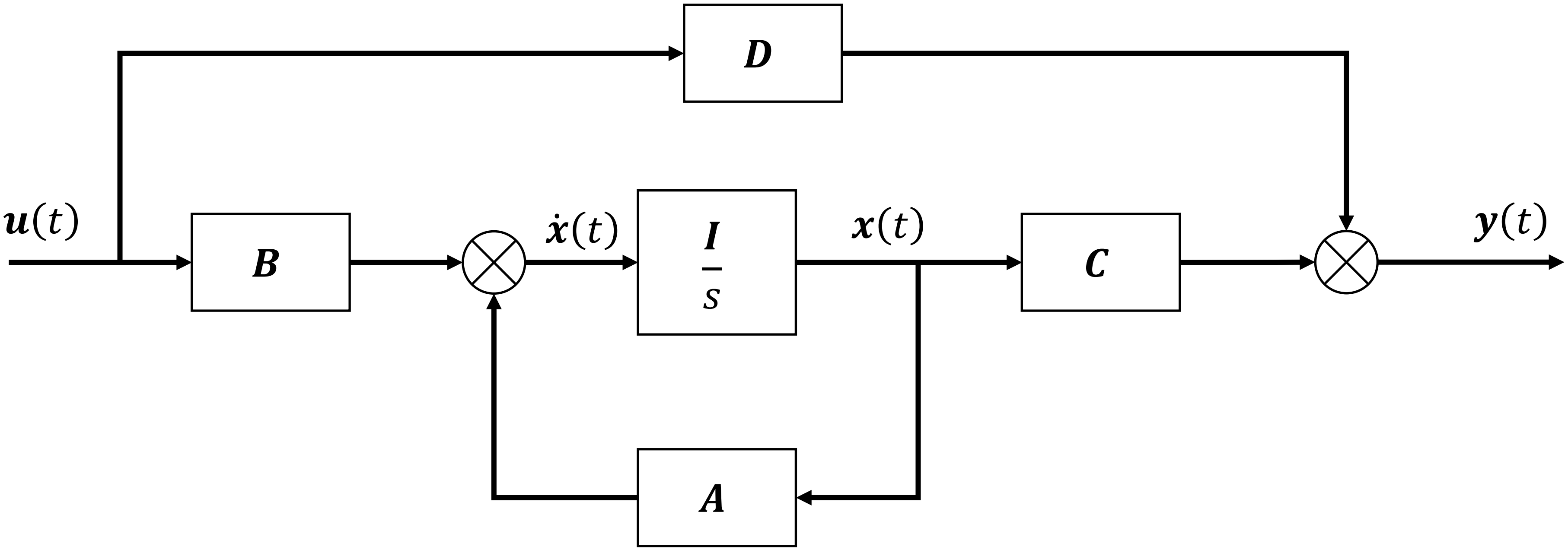
- ❑ **Analysis of the State-space Models using Matlab**

Covert State-space Model to Transfer Function

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$$s\boldsymbol{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\boldsymbol{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$



Covert State-space Model to Transfer Function

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

since $$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

Remember definition of transfer function requires that the initial conditions be set to zero, $\mathbf{x}(0) = 0$, thus:

$$\mathbf{Y}(s) = (\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D})\mathbf{U}(s)$$

Then transfer function between y(t) and u(t) is:

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

Covert State-space Model to Transfer Function

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

In general, if a linear system has q inputs and p outputs, then:

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

$$\mathbf{G}(s) = \begin{bmatrix} G_{11} & \dots & G_{1q} \\ G_{12} & \dots & G_{2q} \\ \vdots & \ddots & \vdots \\ G_{p1} & \dots & G_{pq} \end{bmatrix}$$

The transfer function between jth input and ith output is:

$$G_{ij}(s) = \frac{Y_i(s)}{U_j(s)}$$

Characteristic Equation from State Equations

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

Note: for a 2×2 matrix M=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, assume its inverse matrix is M^{-1}, (i.e., $\mathrm{M}^{-1}\mathrm{M} = \mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Its adjugate is adj(M)=$\begin{bmatrix} \mathrm{d} & -b \\ -c & \mathrm{a} \end{bmatrix}$, its determinant is det(M)=$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathrm{ad} - bc$.

$$\mathrm{M}^{-1} = \frac{adj(\mathrm{M})}{\det(\mathrm{M})} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

$$\mathbf{G}(s) = \mathbf{C}\frac{adj(s\mathbf{I} - \mathbf{A})}{det(s\mathbf{I} - \mathbf{A})}\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}[adj(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|}$$

Characteristic Equation from State Equations

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

$$\mathbf{G}(s) = \mathbf{C}\frac{adj(s\mathbf{I} - \mathbf{A})}{det(s\mathbf{I} - \mathbf{A})}\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}[adj(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|}$$

Setting the denominator of the transfer function matrix **G**(s) to be zero, we get the **characteristic equation:**

$$|s\mathbf{I} - \mathbf{A}| = \mathbf{0}$$

- Hence, it is clear that stability is decided by the pole location in the complex plane.
- Performance is also decided by the pole location.
- Specifically, we call $\mathbf{A}$ system matrix, $\mathbf{B}$ input matrix, $\mathbf{C}$ output matrix, $\mathbf{D}$ feed-forward / feed-through matrix

Example 10.1

Obtain Transfer function for the system:

$$\frac{dx_1}{dt} = -2x_2(t) + 3u(t) \rightarrow \dot{x_1} = -2x_2 + 3u$$

$$\frac{dx_2}{dt} = 3x_1(t) - 5x_2(t) \rightarrow \dot{x_2} = 3x_1 - 5x_2$$

$$y(t) = v_o(t) = 2x_2(t) \rightarrow y = 2x_2$$

$$A = \begin{bmatrix} 0 & -2 \\ 3 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{G}(s) = \frac{\mathbf{C}[adj(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|}$$

$$\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \frac{\begin{bmatrix} 0 & 2 \end{bmatrix}\left(\mathrm{adj}\begin{bmatrix} s & 2 \\ -3 & s+5 \end{bmatrix}\right)\begin{bmatrix} 3 \\ 0 \end{bmatrix}}{\det\begin{vmatrix} s & 2 \\ -3 & s+5 \end{vmatrix}} = \frac{\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} s+5 & -2 \\ 3 & s \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}}{s^2 + 5s + 6} = \frac{18}{s^2 + 5s + 6}$$

Covert Transfer Function to State-space Model

How to obtain the state space model from the transfer function directly without a clear knowledge of the physical system?

Method 1: to develop graphic model of the system and use this model to determine state variables.

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \quad n \geq m}$$

Recall **Mason's Signal-flow Gain Formula**:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_k P_k \Delta_k}{\Delta}.$$

Divided by s^n

$$\begin{aligned}\Delta = 1 &- (\text{sum of all different loop gains}) \\ &+ (\text{sum of the gain products of all combinations of two nontouching loops}) \\ &- (\text{sum of the gain products of all combinations of three nontouching loops}) \\ &+ \cdots.\end{aligned}$$

$$G(s) = \frac{b_m s^{-(n-m)} + b_{m-1} s^{-(n-m+1)} + \cdots + b_1 s^{-(n-1)} + b_0 s^{-n}}{1 + a_{n-1} s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n}}.$$

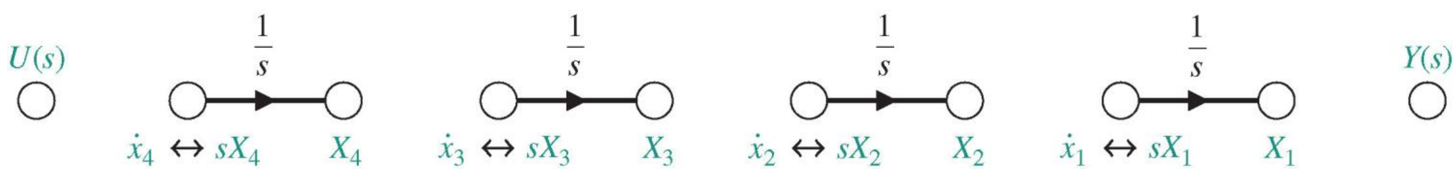
Simple Case

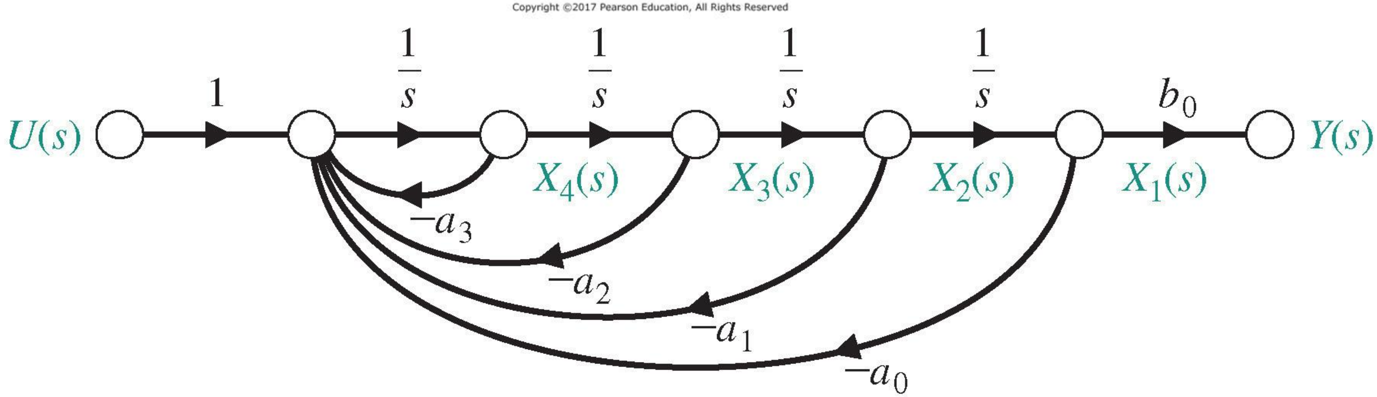
To illustrate the derivation of signal-flow graph from transfer function, let's consider a simple case, when $n = 4$, and $b_m \ldots b_2, b_1 =, 0$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$

The system is fourth order, hence we need to identify four state variables: $x_1(t), x_2(t), x_3(t), x_4(t)$

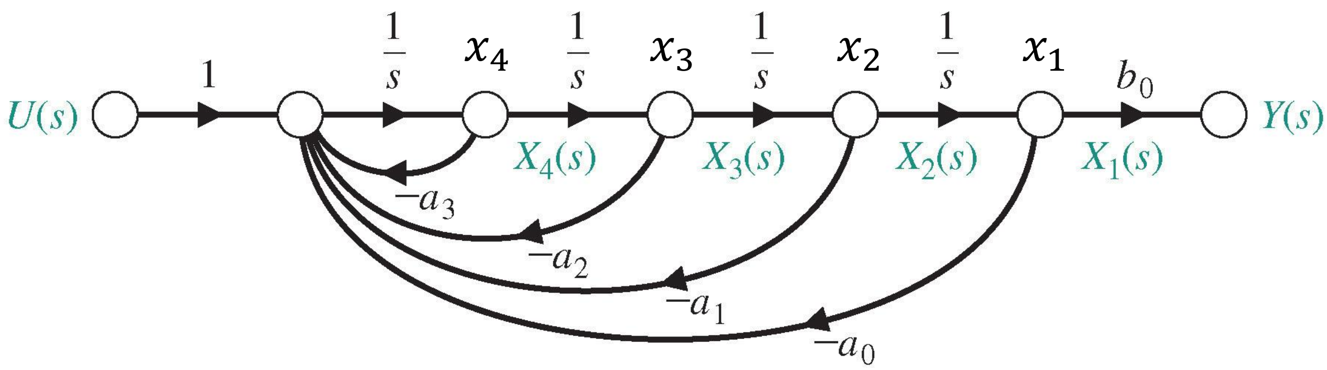




Simple Case

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u$$
$$y = b_0 x_1$$

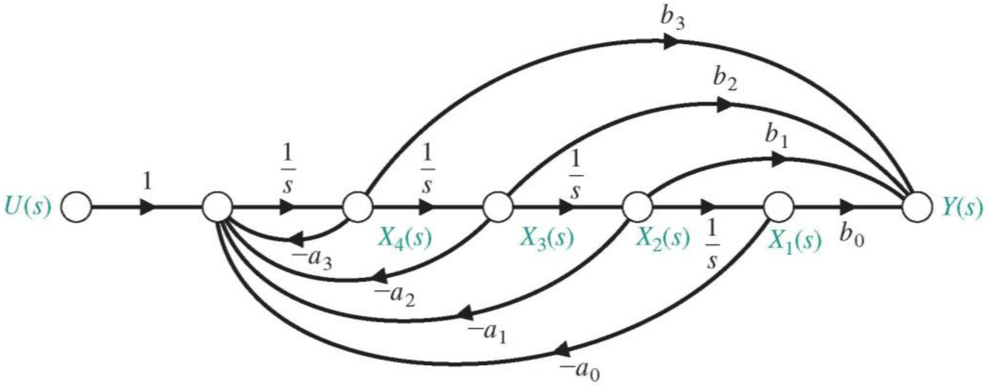
Simple Case

Now consider the numerator is a polynomial in s:

$$G(s) = \frac{\sum_k P_k}{1 - \sum_{q=1}^{N} L_q}$$

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4,$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u.$$

In this equation, $x_1, x_2, \ldots x_n$ are the n **phase variables**.

$$y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3 + b_3 x_4$$

Simple Case

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4,$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u.$$

$$y = b_0 x_1 + b_1 x_2 + b_2 x_3 + b_3 x_4$$

A, B, C, D?

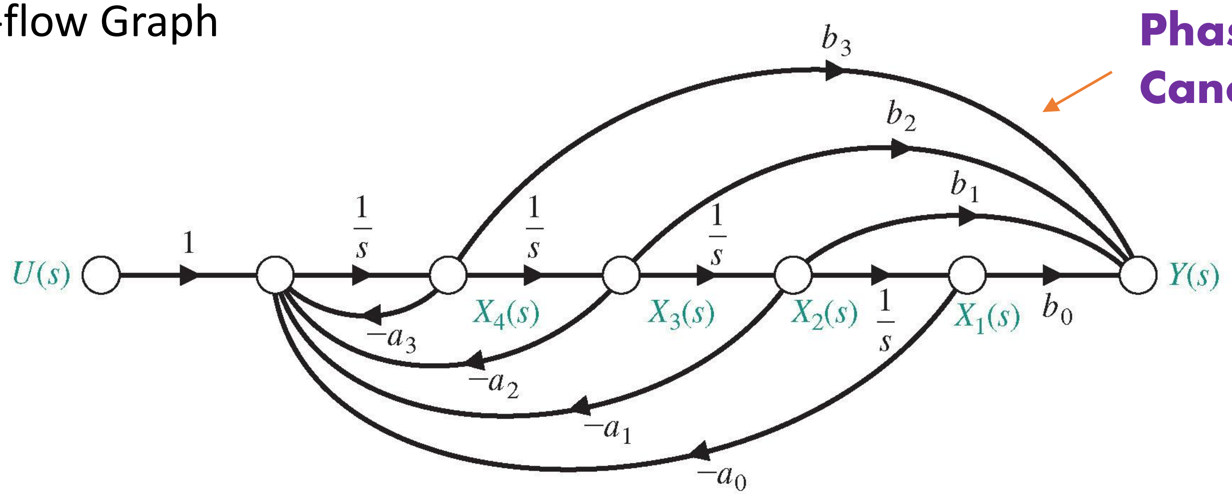
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t). \qquad y(t) = \mathbf{C}\mathbf{x} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

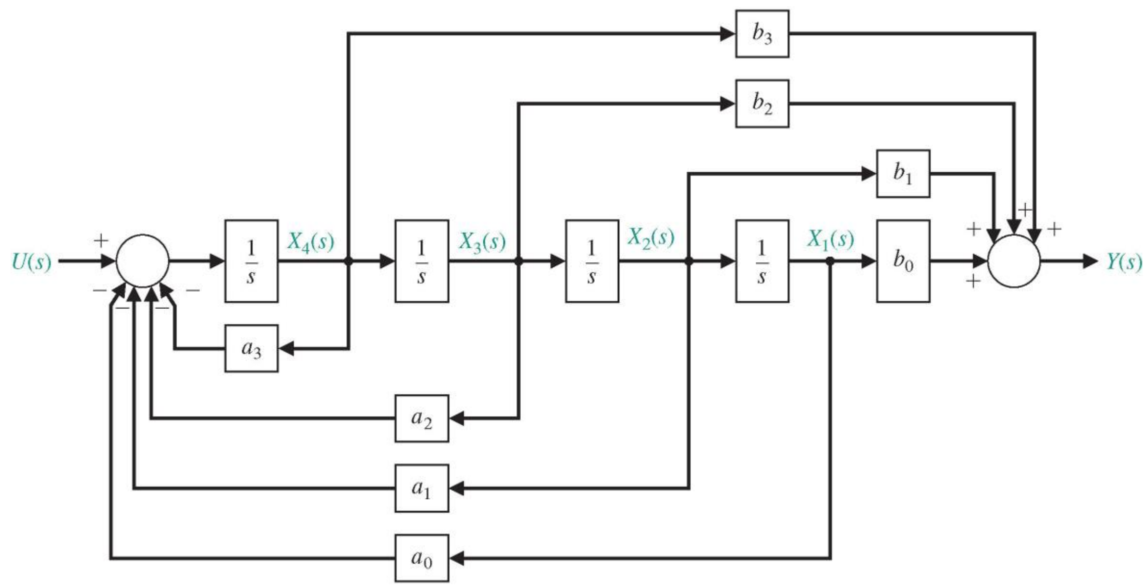
General form of State-space Model

- Signal-flow Graph

Phase Variable Canonical Form

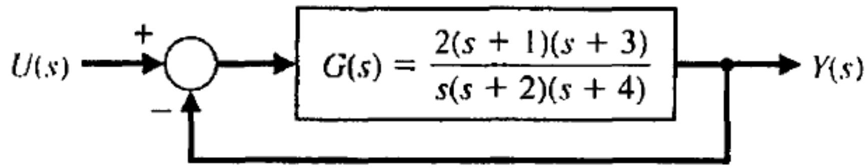


- Equivalent Block Diagram



Example 10.2

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$



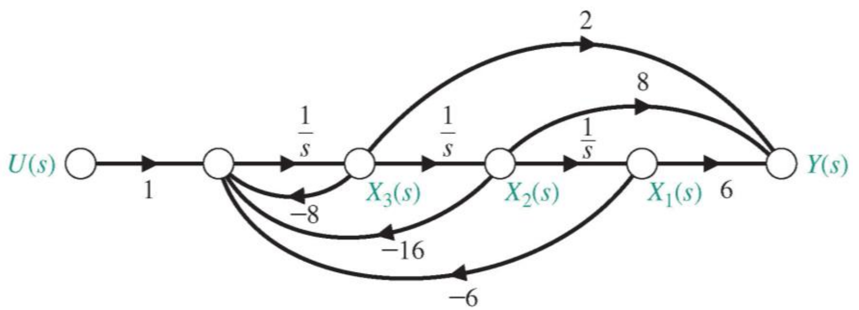
Applying the Phase variable state model:

Multiplying the numerator and denominator by s^{-3}, we have

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [6 \quad 8 \quad 2]\mathbf{x}(t) + [0]u(t)$$



(a)

Covert Transfer Function to State-space Model

Method 2: State-space Model can be also obtained by introducing an intermediate variable $Z(s)$.

For simplicity, assume $n = 4$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \frac{Z(s)}{Z(s)}$$

$$Y(s) = (b_3 s^3 + b_2 s^2 + b_1 s + b_0) Z(s)$$

$$U(s) = (s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0) Z(s)$$

Then taking inverse Laplace transform of both equations:

$$y = b_3 \frac{d^3 z}{dt^3} + b_2 \frac{d^2 z}{dt^2} + b_1 \frac{dz}{dt} + b_0 z$$

$$u = \frac{d^4 z}{dt^4} + a_3 \frac{d^3 z}{dt^3} + a_2 \frac{d^2 z}{dt^2} + a_1 \frac{dz}{dt} + a_0 z.$$

Covert Transfer Function to State-space Model

$$y = b_3\frac{d^3z}{dt^3} + b_2\frac{d^2z}{dt^2} + b_1\frac{dz}{dt} + b_0z \qquad u = \frac{d^4z}{dt^4} + a_3\frac{d^3z}{dt^3} + a_2\frac{d^2z}{dt^2} + a_1\frac{dz}{dt} + a_0z.$$

Define the four state variables as follows:

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{x}_1 = \dot{z} \\ x_3 &= \dot{x}_2 = \ddot{z} \\ x_4 &= \dot{x}_3 = \dddot{z}. \end{aligned}$$

Then the differential equation can be written equivalently as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4, \end{aligned}$$

and

$$\dot{x}_4 = -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + u,$$

and the corresponding output equation is

$$y = b_0x_1 + b_1x_2 + b_2x_3 + b_3x_4.$$

Covert Transfer Function to State-space Model

Method 3: Select state variable with physical meanings.

- Often the control system designer studies an actual control system block diagram that represents physical devices and variables.
- In practice, we wish to select the physical variables as the state variables.
  - While the state-space model could be derived in many forms in math, in practice, we always select physical variables as states such that the **model will be useful**.

Example

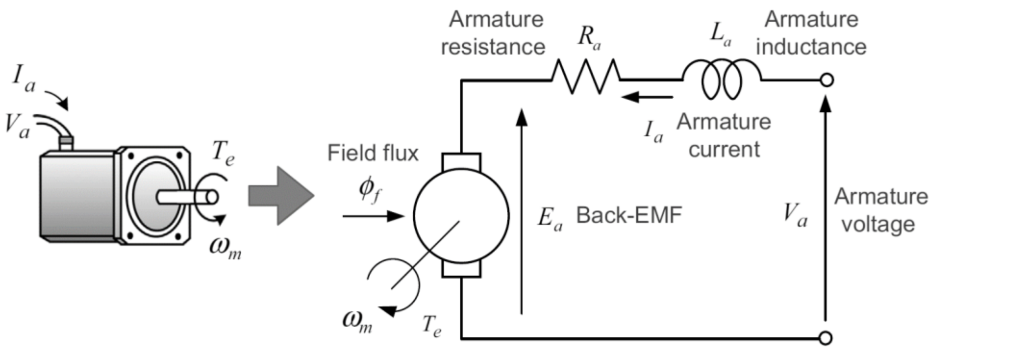
- Recall the model of a DC motor

Voltage equation:

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + k_e \phi_f \omega_m$$

Motion equation

$$k_T \phi_f i_a = J \frac{d\omega_m}{dt} + B\omega_m + T_L$$



Assume $k_e\phi_f = K_T, \phi_f = k_c, B = 0, T_L = 0$

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + k_c \omega_m$$

$$k_c i_a = J \frac{d\omega_m}{dt}$$

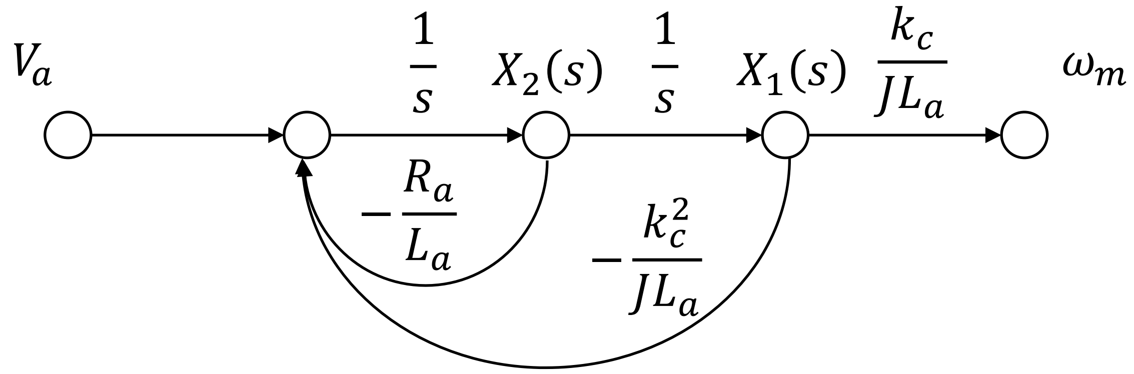
Example

If use method 1 to build state space model:

- The transfer function is

$$\frac{\omega_m}{V_a} = \frac{k_c}{JL_a s^2 + JR_a s + k_c^2} = \frac{\frac{k_c}{JL_a} s^{-2}}{1 + \frac{R_a}{L_a} s^{-1} + \frac{k_c^2}{JL_a} s^{-2}}$$

- Signal flow graph:



Example

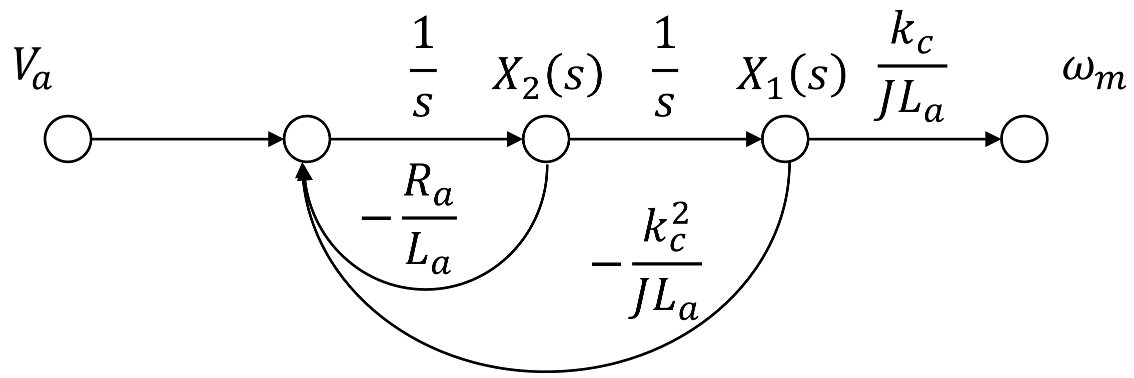
- State space model:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k_c^2}{JL_a}x_1 - \frac{R_a}{L_a}x_2 + V_a$$

$$y = \omega_m = \frac{k_c}{JL_a}x_1$$

Here the state space variables do not have specific physical meaning



Example

If select physical variable as states:

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + k_c \omega_m$$

$$k_c i_a = J \frac{d\omega_m}{dt}$$

The state variables are useful as they could be measured.

- Select state as $x_1 = \omega_m, x_2 = I_a, y = \omega_m, u = V_a$, then

$$\frac{d\omega_m}{dt} = \dot{x}_1 = \frac{k_c}{J} i_a = \frac{k_c}{J} x_2$$

$$\frac{di_a}{dt} = \dot{x}_2 = -\frac{R_a}{L_a} i_a - \frac{k_c}{L_a} \omega_m + \frac{V_a}{L_a} = -\frac{R_a}{L_a} x_2 - \frac{k_c}{L_a} x_1 + \frac{1}{L_a} u$$

$$y = x_1$$

Quiz 10.1

Obtain a state-space model & block diagram for the system with the following transfer equation:

$$\mathbf{G}(s) = \frac{s+2}{s^2+7s+12}$$

Quiz 10.2

Consider following state space model

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

What is the characteristic equation of the system and what are the poles of the system?

Outline

State Variable Models

- ❑ **Introduction**
- ❑ **State Variables**
- ❑ **State-space Modeling**
- ❑ **State Space Representation in Matrix Form**
- ❑ **Time-domain response (Solution of State-space Models)**
- ❑ **Conversion between State-space Model and Transfer Function**
- ❑ ***<u>Analysis of the State-space Models using Matlab</u>***

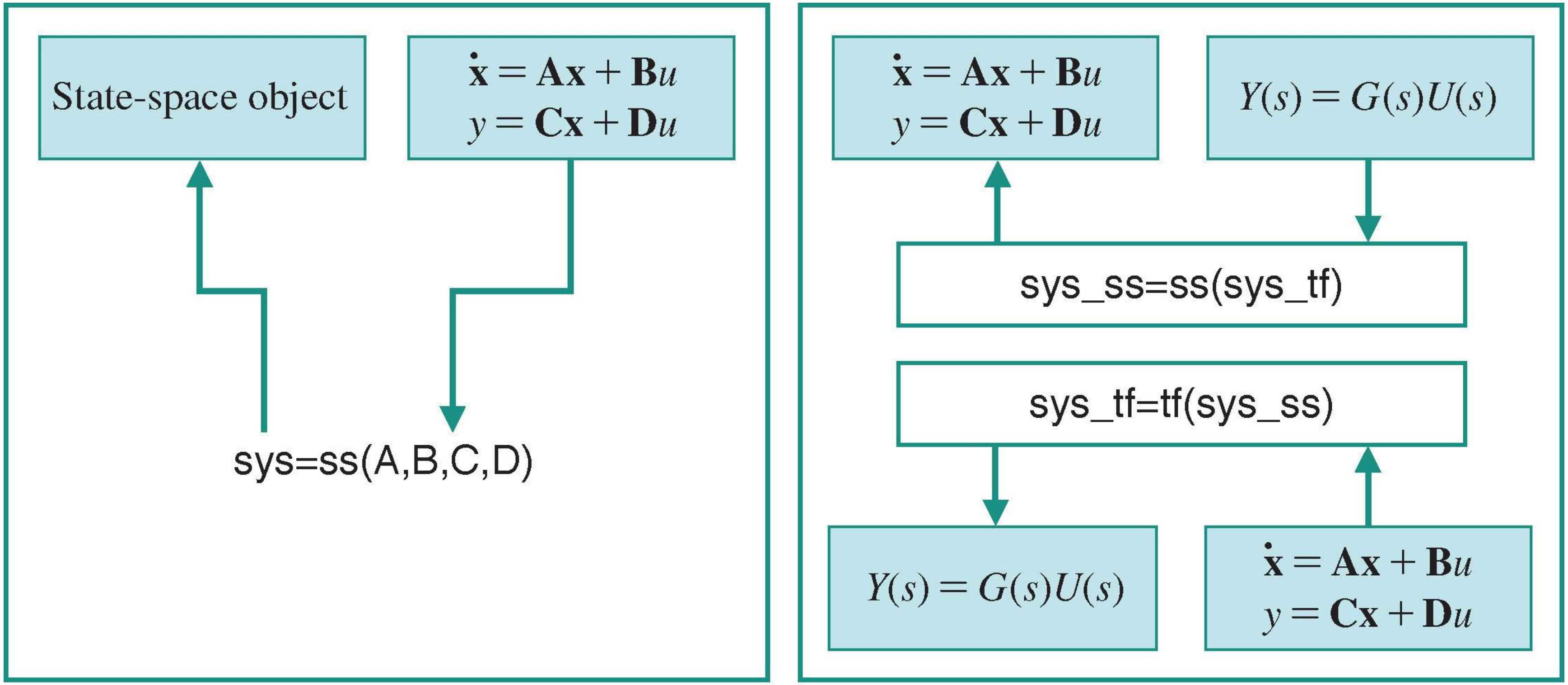
Transfer Function to State Space

- Build state-space model with function *ss*

```
>> help ss
 ss  State-space models.

   Construction:
     SYS = ss(A,B,C,D) creates an object SYS representing the continuous-
     time state-space model
          dx/dt = Ax(t) + Bu(t)
           y(t) = Cx(t) + Du(t)
     You can set D=0 to mean the zero matrix of appropriate size. SYS is
     of type ss when A,B,C,D are dense numeric arrays, of type GENSS when
     A,B,C,D depend on tunable parameters (see REALP and GENMAT), and
     of type USS when A,B,C,D are uncertain matrices (requires Robust
     Control Toolbox). Use SPARSS when A,B,C,D are sparse matrices.

     SYS = ss(A,B,C,D,Ts) creates a discrete-time state-space model with
     sample time Ts (set Ts=-1 if the sample time is undetermined).
```

Transfer Function to State Space

- Covert between state space model and transfer function (ss, tf)



(a) (b)

Example

- Consider the following third order system

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$

Compute the state space model using Matlab.

Example

```
>> num = [2 8 6]; den = [1 8 16 6]; sys_tf = tf(num, den)

sys_tf =

     2 s^2 + 8 s + 6
  ----------------------
  s^3 + 8 s^2 + 16 s + 6

Continuous-time transfer function.

>>
```

```
>> sys_ss = ss(sys_tf)

sys_ss =

  A =
          x1     x2     x3
   x1     -8     -4   -1.5
   x2      4      0      0
   x3      0      1      0

  B =
        u1
   x1    2
   x2    0
   x3    0

  C =
          x1     x2     x3
   y1      1      1   0.75

  D =
        u1
   y1    0

Continuous-time state-space model.
```

Please note: a transfer function can be converted to various state space models by choosing different sets of state variables; therefore, it is possible that when using the ss function, the state space model generated will be different, depending on the specific software and version.

Matrix Exponential Function

- Recall solution of state-space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\Big\downarrow \; s \text{ domain solution}$$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\Big\downarrow \; t \text{ domain solution}$$

$$\mathbf{x}(t) = \boxed{\mathbf{\Phi}(t)}\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$$ State transition matrix

Matrix Exponential Function

- Compute state transition matrix using function *expm*

```
>> help expm
 expm  Matrix exponential.
    expm(A) is the matrix exponential of A and is computed using
    a scaling and squaring algorithm with a Pade approximation.

    Although it is not computed this way, if A has a full set
    of eigenvectors V with corresponding eigenvalues D then
    [V,D] = EIG(A) and expm(A) = V*diag(exp(diag(D)))/V.
```

Example

- Consider the RLC network with following state-space model

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

Set $u(t) = 0$, $x_1(0) = x_2(0) = 1$, compute its time response when $t = 0.2s$.

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$
$$y(t) = C\mathbf{x}(t)$$

Compute Time Response

- Method 1: Calculate the output and state response with expm

Compute the state trajectory first and then compute the output.

- The function expm could compute matrix in time sequence.
  - Input time series, e.g., *t=0:0.01:10*

Example

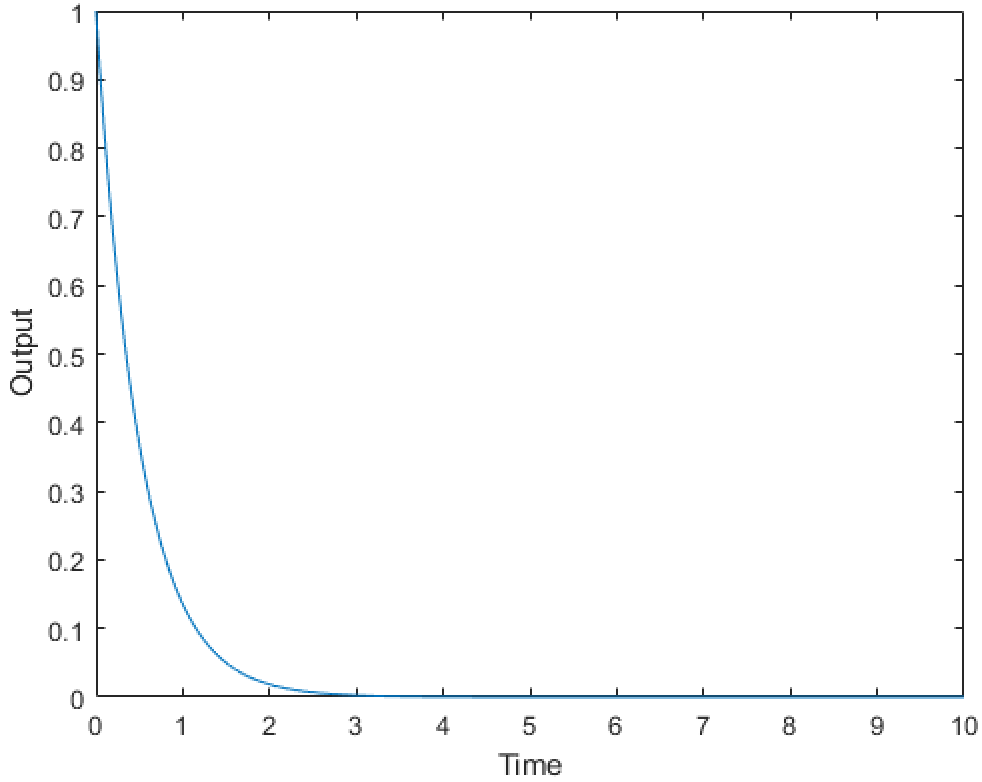
- Consider the RLC network with following state-space model

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

Set $u(t) = 0$, $x_1(0) = x_2(0) = 1$, compute its time response when $t \in [0,10]s$.

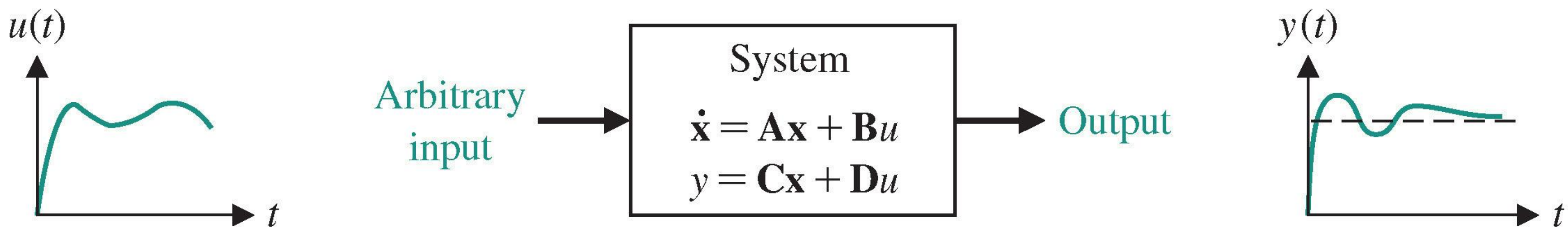
$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$
$$y(t) = C\mathbf{x}(t)$$

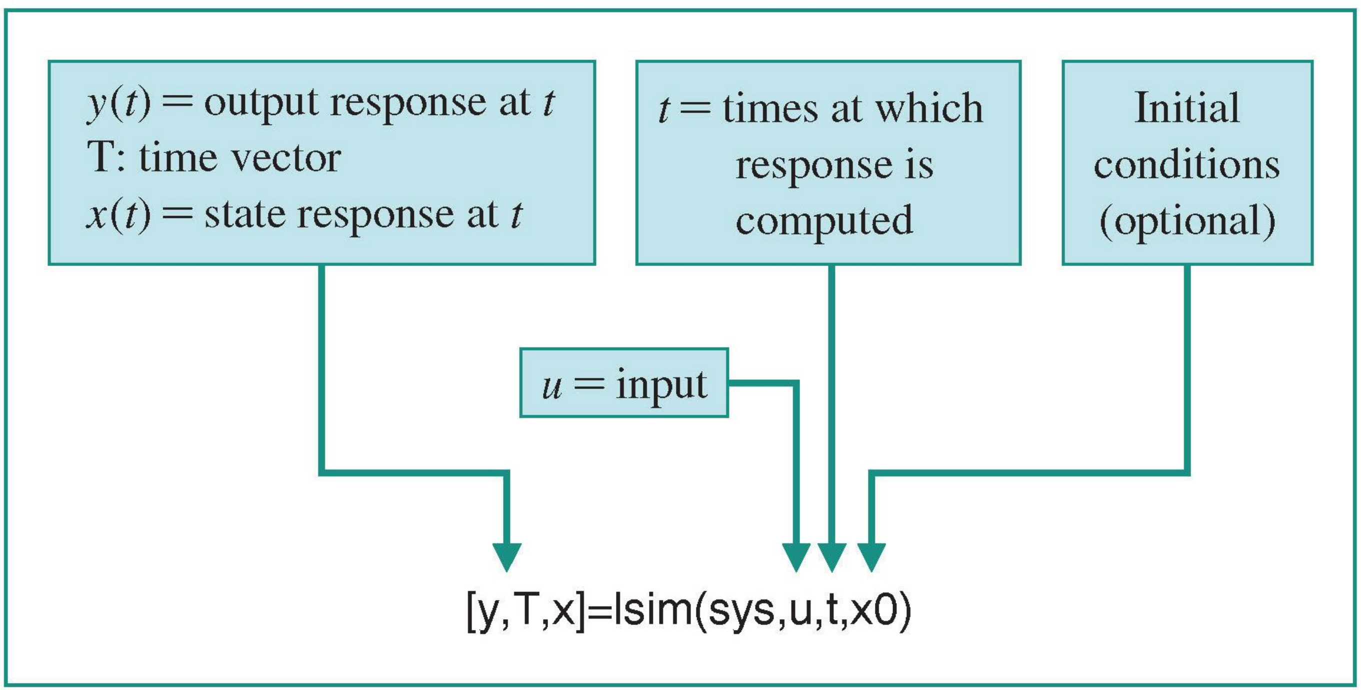
Example



Compute Time Response

Method 2: Calculate the output and state response (lsim)





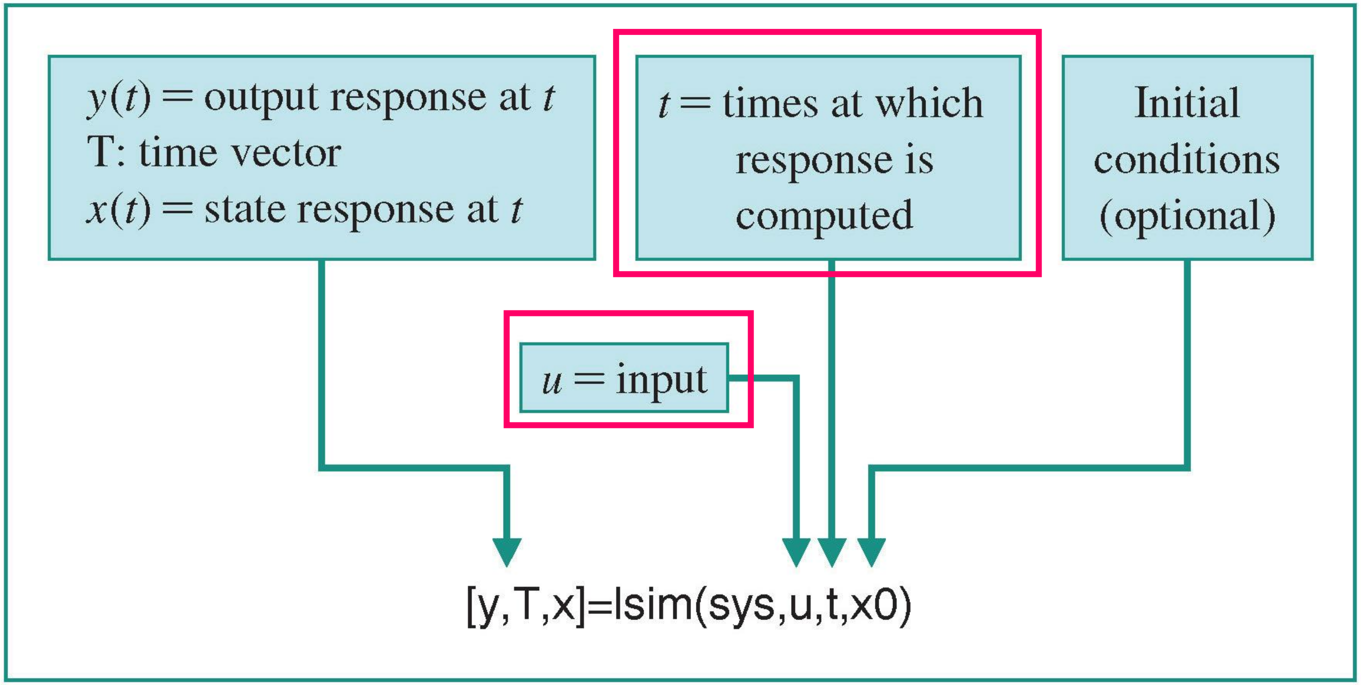
Compute Time Response

Method 2: Calculate the output and state response (lsim)

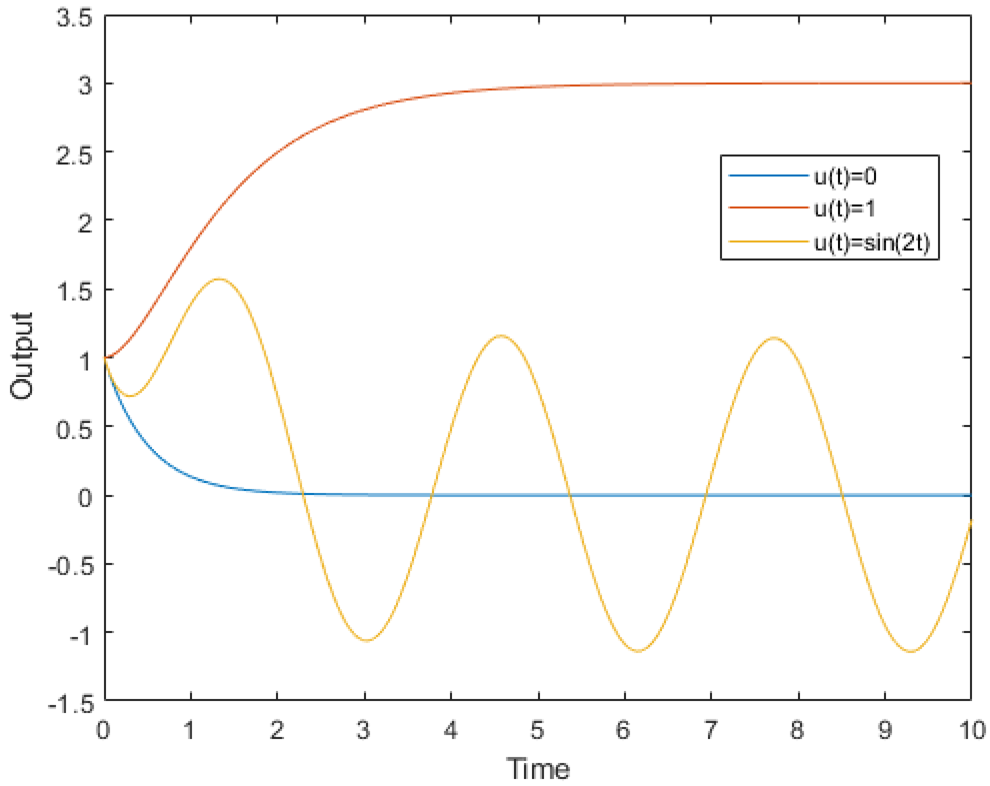
```
>> help lsim
 lsim  Simulate time response of dynamic systems to arbitrary inputs.

    lsim(SYS,U,T) plots the time response of the dynamic system SYS to the
    input signal described by U and T. The time vector T is expressed in the
    time units of SYS and consists of regularly spaced time samples. The
    matrix U has as many columns as inputs in SYS and its i-th row specifies
    the input value at time T(i). For example,
            t = 0:0.01:5;   u = sin(t);   lsim(sys,u,t)
    simulates the response of a single-input model SYS to the input
    u(t)=sin(t) during 5 time units.

    For discrete-time models, U should be sampled at the same rate as SYS
    (T is then redundant and can be omitted or set to the empty matrix).
    For continuous-time models, choose the sampling period T(2)-T(1) small
    enough to accurately describe the input U.  lsim issues a warning when
    U is undersampled and hidden oscillations may occur.

    lsim(SYS,U,T,X0) specifies the initial state vector X0 at time T(1)
    (for state-space models only). X0 is set to zero when omitted.

    lsim(SYS1,SYS2,...,U,T,X0) simulates the response of several systems
    SYS1,SYS2,... on a single plot. The initial condition X0 is optional.
    You can also specify a color, line style, and marker for each system,
    for example
       lsim(sys1,'r',sys2,'y--',sys3,'gx',u,t).
```

Example

- Consider the same problem in previous example. Compute the time response when control is defined as (1) $u(t) = 0$, (2) $u(t) = 1\ (t)$, and (3) $u(t) = \sin 2t$. The time is set within $t \in [0,10]s$.



Example



3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
Output
0
1
2
3
4
5
6
7
8
9
10
Time
u(t)=0
u(t)=1
u(t)=sin(2t)

Thank You !