CAN209 Advanced Electrical Circuits and Electromagnetics

Lecture 2 Static Fields I

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OUTLINE

- ➢ Review of Coulomb’s Law
- ➢ Visualisation of Electric Field
- ➢ Maxwell’s Equations for Static Fields
  - ✓ Gauss’s Law for Electric field
  - ✓ Gauss’s Law for Magnetic field
  - ✓ Electric Field Loop Theorem
  - ✓ Magnetic Field Loop Theorem (Ampere’s Law)

MAXWELL’S EQUATIONS

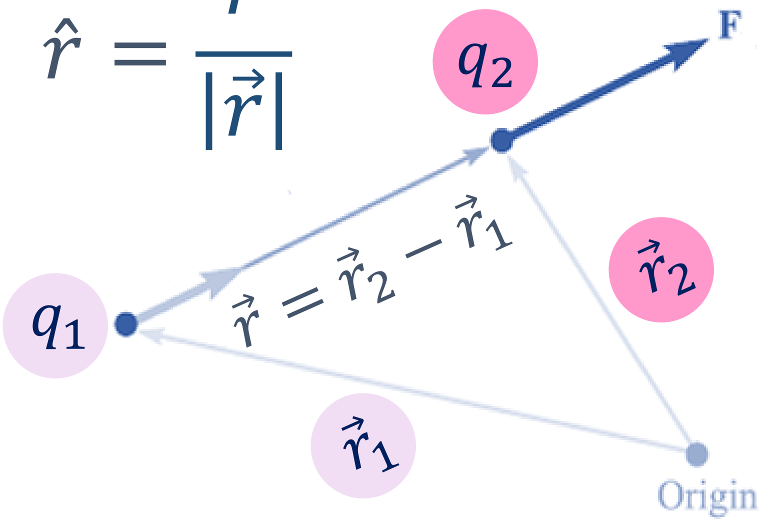
Maxwell’s equations are a set of four equations that describe properties of the electric and magnetic fields and relate them to their sources.

Law	Integral	Differential	Physical meaning
Gauss's law for **E-field**	$\oiint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = \frac{Q}{\varepsilon}$	$\nabla \cdot \vec{\boldsymbol{E}} = \frac{\rho}{\varepsilon}$	Electric flux through a closed surface is proportional to the charges enclosed
E-field Loop Theorem	$\oint_C \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = 0$	$\nabla \times \vec{\boldsymbol{E}} = 0$	Work done by moving a charge in the E-field along a closed loop is 0
Gauss's law for **H-field**	$\oiint_S \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{s}} = 0$	$\nabla \cdot \vec{\boldsymbol{B}} = 0$	The total magnetic flux through a closed surface is 0
H-field Loop Theorem	$\oint_C \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{l}} = I$	$\boldsymbol{\nabla} \times \vec{\boldsymbol{H}} = \vec{\boldsymbol{J}}$	The H-field produced by an electric current is proportional to the current

1. COULOMB'S LAW

The vector form of Coulomb's law is

$$\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{|q_1 q_2|}{r^2}\hat{r}} \longrightarrow \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$



where

Coulomb constant: $k = \frac{1}{4\pi\varepsilon_0} = 8.9875 \times 10^9 N \cdot m^2/C^2$

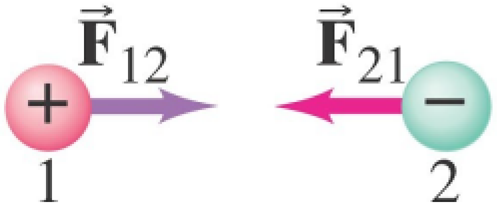
Free space permittivity: $\varepsilon_0 = 8.85 \times 10^{-12} C^2/N \cdot m^2 \approx \frac{1}{36\pi} \times 10^{-9} C^2/N \cdot m^2$

Unit vector $\hat{r}$ is from q_1 to q_2.

1. COULOMB'S LAW

The force is along the line connecting the charges, and is attractive if the charges are opposite, and repulsive if they are the same.

Attractive example:



Repulsive example:

$\vec{F}_{12}$: force on charge 1 due to charge 2

$\vec{F}_{21}$: force on charge 2 due to charge 1

QUIZ 1.1

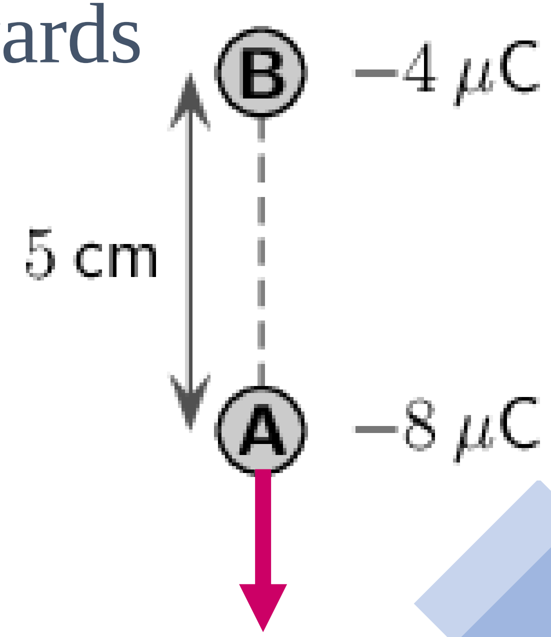
Two charged spheres A and B are placed in a vacuum at a distance of 5 cm apart. What is the electrostatic force (magnitude and direction) experienced by sphere A due to the charge on Sphere B?

A) 115N; downwards

B) 115×10^6 N; downwards

C) 1.15N; downwards

D) 1.15N; upwards



QUIZ 1.2

Two small positively charged spheres have a combined charge of 5.0×10^{-5} C. If each sphere is repelled from the other by an electrostatic force of 1 N when the spheres are 2 m apart, what is the charge on each sphere?

2.1 ELECTRIC FIELD INTENSITY

- $\vec{E}$: Electric field intensity or E-field
- A brief definition: the electric field at any location is the number of newtons of electrical force exerted on each coulomb of charge at that location.

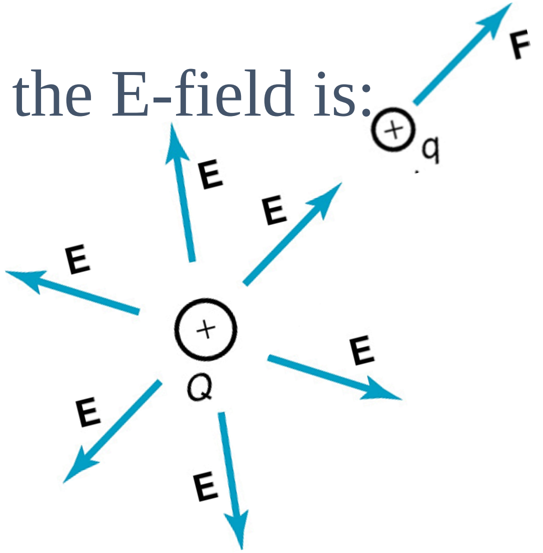
$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{r}$$

$$\vec{E} = \lim_{q_0\to 0}\frac{\vec{F}}{q_0} = \frac{\vec{F}}{q_0}$$

SI Unit: N/C = $Jm^{-1}C^{-1}$ =V/m

For a point charge source Q, the E-field is:

$$\vec{E} = \frac{\vec{F}}{q} = \frac{Q}{4\pi\varepsilon r^2}\hat{r}$$



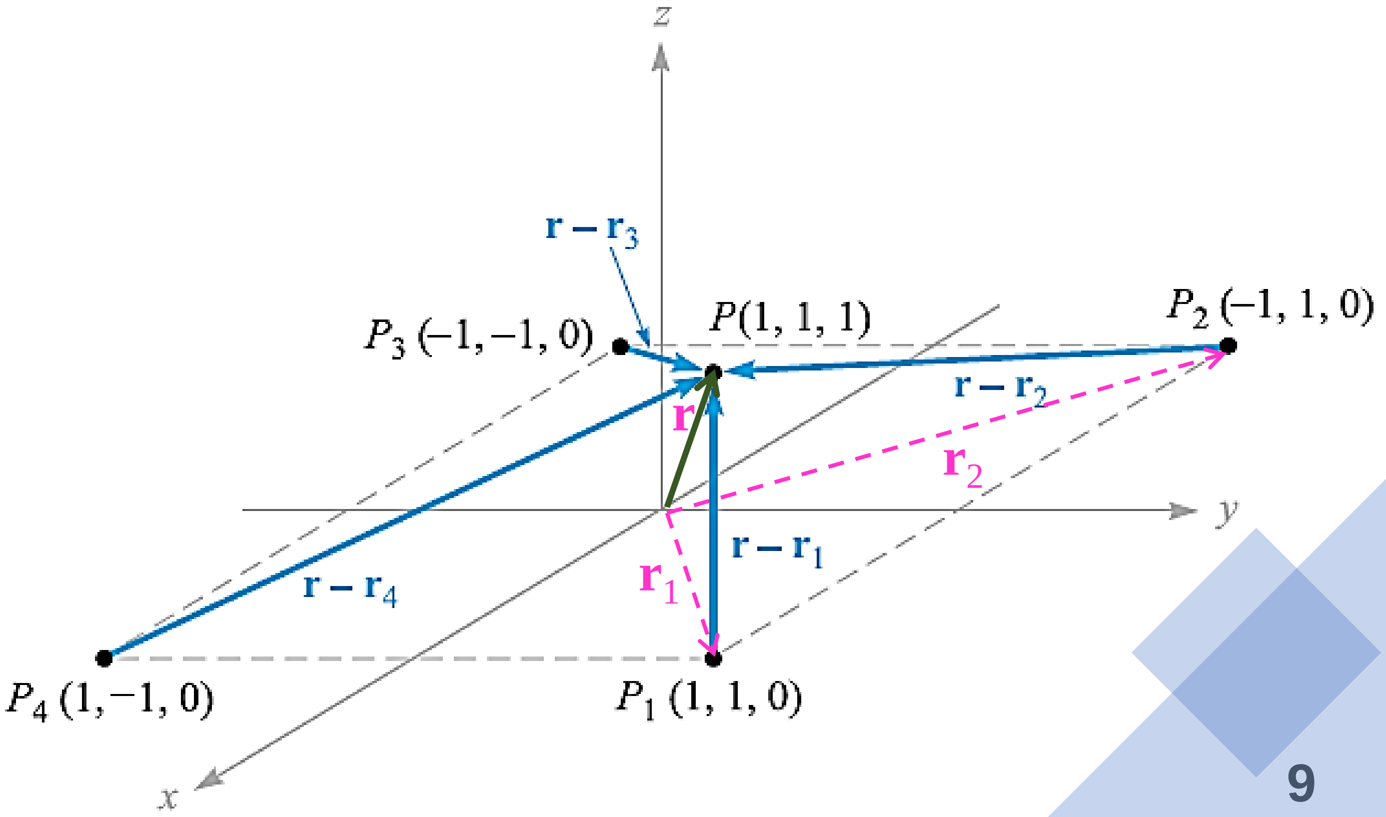
Superposition Principle:

The total electric field due to a group of charges is equal to the ***vector sum*** of the electric fields of individual charges.

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N = \sum_{i=1}^{N}\vec{E}_i$$

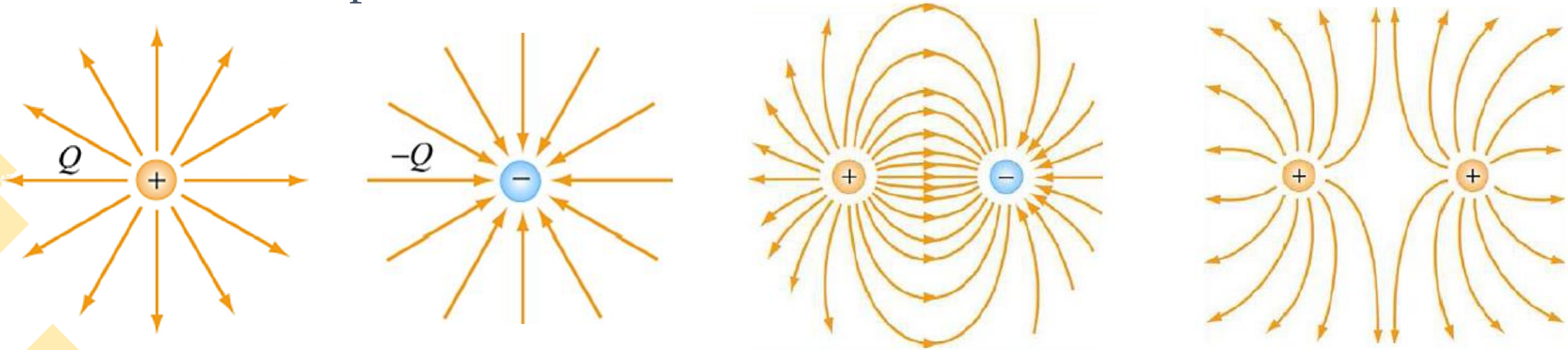
QUIZ 2.1

The point charges locate at P_1 (1, 1, 0) m, P_2 (-1, 1, 0) m, P_3 (-1, -1, 0) m, and P_4 (1, -1, 0) m in free space are shown below. Determine the total Electric field intensity at point P (1, 1, 1) m in the rectangular coordinate system caused by four identical point charges with 3 nC.



2.2. VISUALISATION OF E-FIELD

Electric field lines provide a convenient graphical representation of the electric field in space.

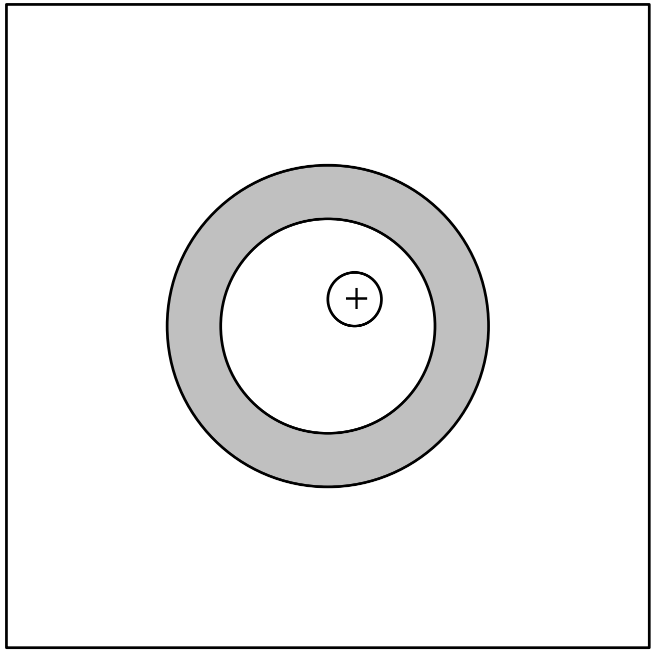


Properties:

1. The field lines **must** begin on positive charges (or at infinity) and then terminate on negative charges (or at infinity).
2. The **direction of the E-field** at any point is **tangent** to the field lines at that point.
3. The number of lines per unit area through a surface **perpendicular** to the line is devised to be proportional to the magnitude of the E-field in a given region.
4. **No** two field lines can cross each other.

QUIZ 2.2

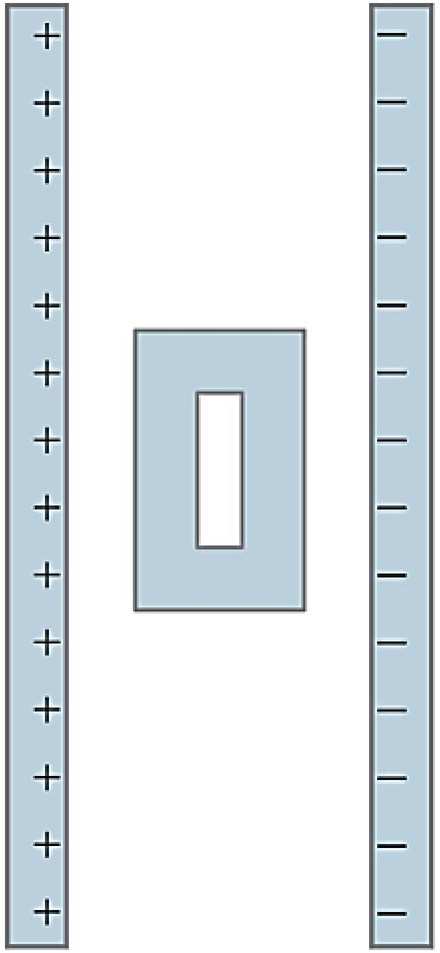
A single positive point charge Q is positioned arbitrarily inside an uncharged spherical conducting shell. Plot the electric field lines inside the boxed region.



Draw, sketch, plot, illustrate, …

QUIZ 2.3

A neutral hollow metal box is placed between two parallel charged plates. Draw the field lines between the plates?



OUTLINE

- ➢ Review of Coulomb's Law
- ➢ Visualisation of Electric Field
- ➢ Maxwell's Equations for Static Fields
  - Gauss's Law for **Electric** field
  - Gauss's Law for **Magnetic** field
  - Electric Field Loop Theorem
  - Magnetic Field Loop Theorem (Ampere's Law)

3.1 GAUSS'S LAW – E FIELD

1. The Integral Form

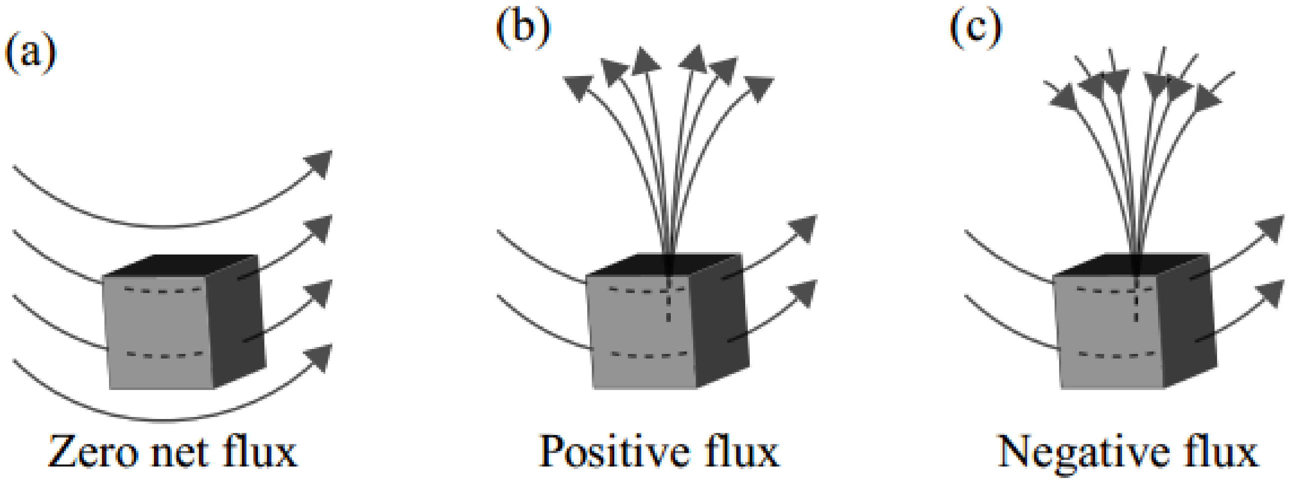
High-symmetric structure

$$\oiint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon}$$

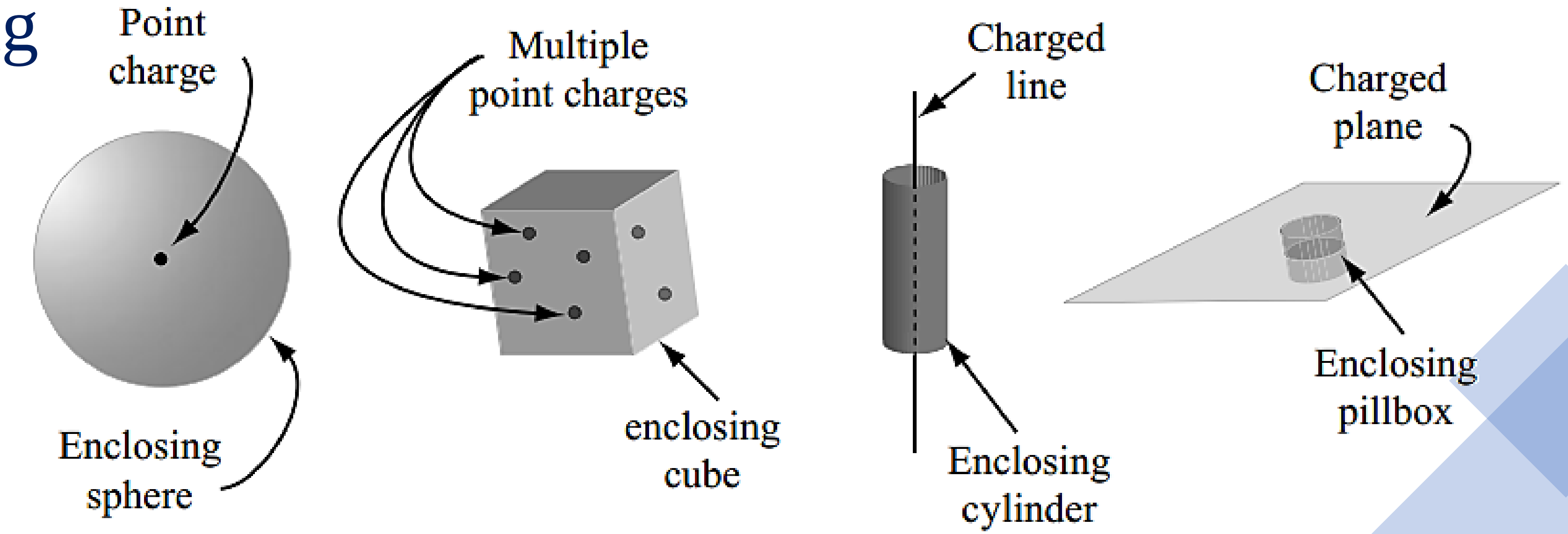
Electric charges produce an E-field, and the flux of that field passing through any closed surface is proportional to the total charge contained within that surface.

3.1 FLUX & ENCLOSED Q

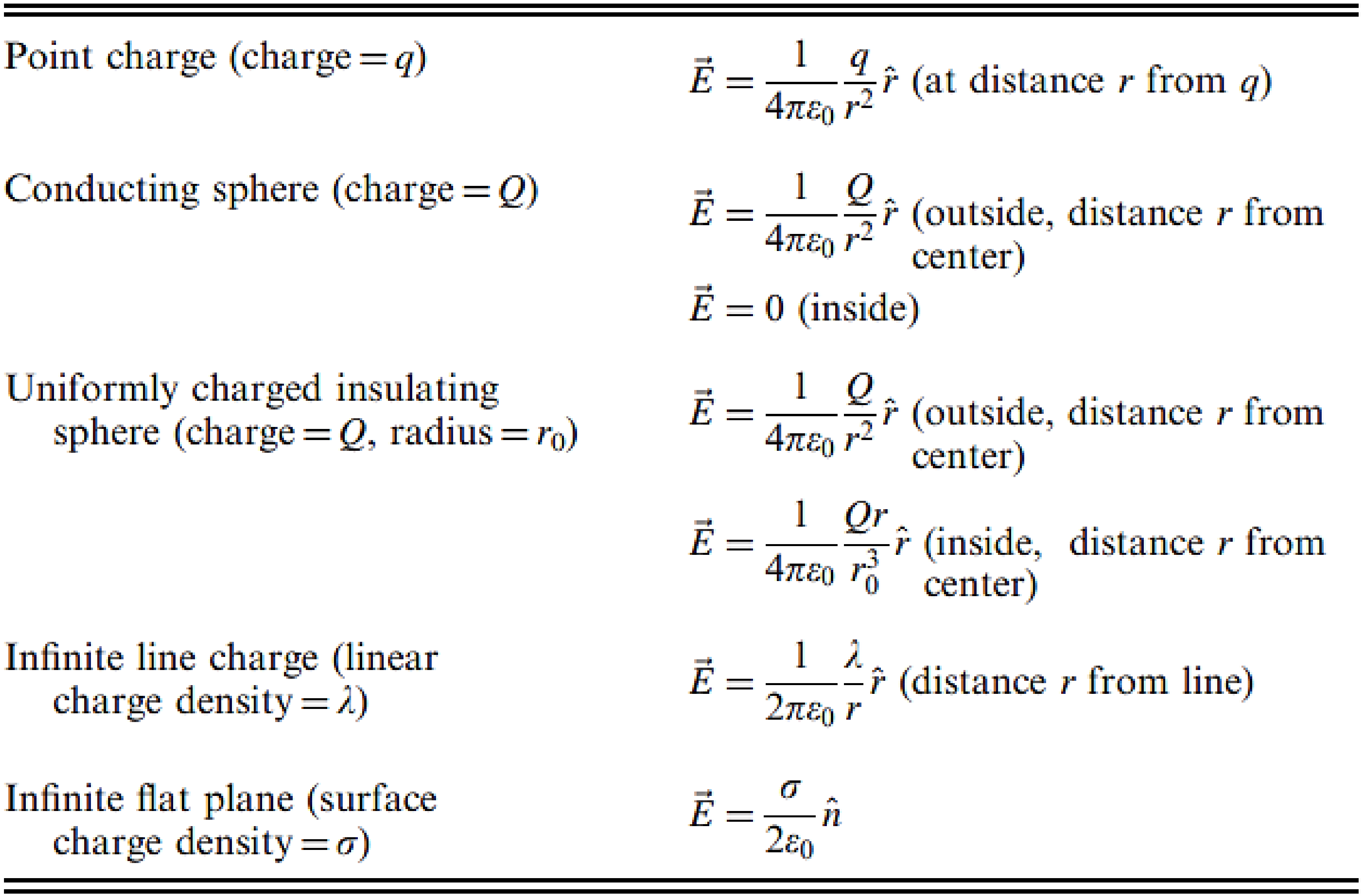
- Flux lines penetrating closed surfaces:



- Surfaces enclosing known charges:



3.1 GAUSS’S LAW – TYPICAL EXAMPLES

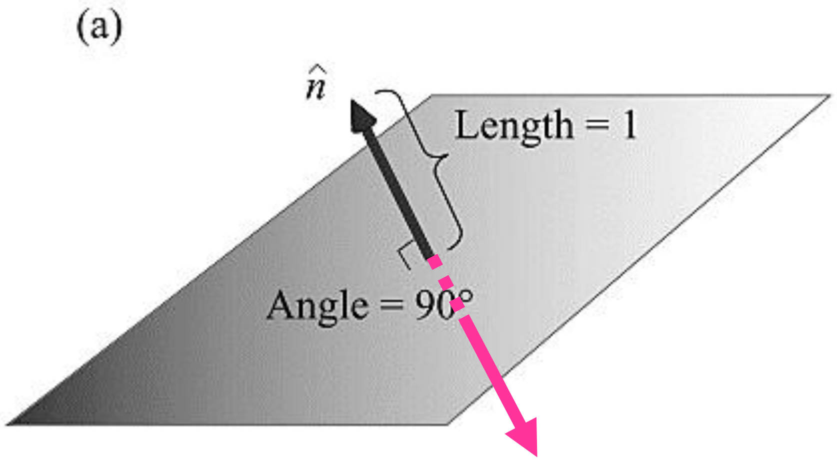


Point charge (charge $= q$)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ (at distance r from q)
Conducting sphere (charge $= Q$)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside, distance r from center) $\vec{E} = 0$ (inside)
Uniformly charged insulating sphere (charge $= Q$, radius $= r_0$)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside, distance r from center) $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{r_0^3}\hat{r}$ (inside, distance r from center)
Infinite line charge (linear charge density $= \lambda$)	$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{r}$ (distance r from line)
Infinite flat plane (surface charge density $= \sigma$)	$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

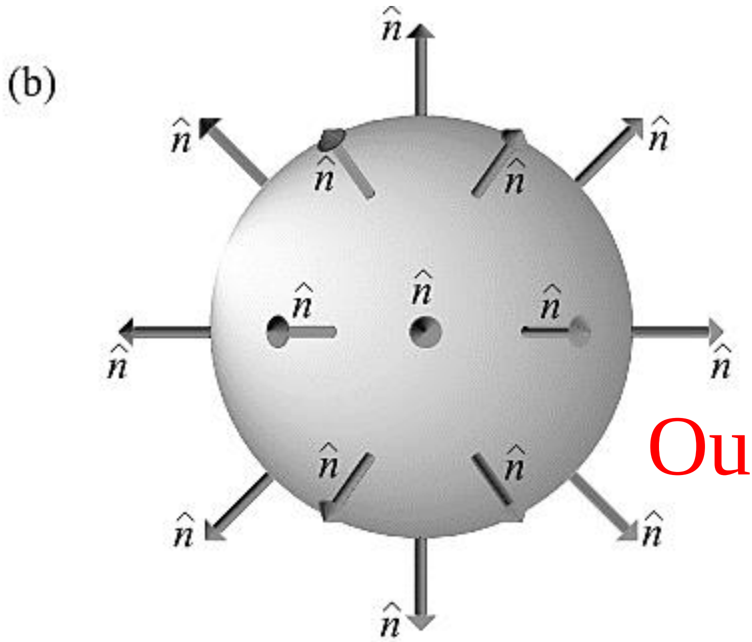
3.1 SURFACE AREA ELEMENT

The surface area element (segment) $d\vec{s}$ contains two parts:

$$d\vec{s} = \overbrace{\hat{n}}^{\text{Unit normal vector}}\underbrace{ds}_{\text{Area element}}$$



Planar surface



Outwards

Spherical surface

3.1 FLUX THROUGH A SURFACE

For a large, curved surface, we can divide it into small vector areas, each one has an area of ds_i and direction $\mathbf{n_i}$;

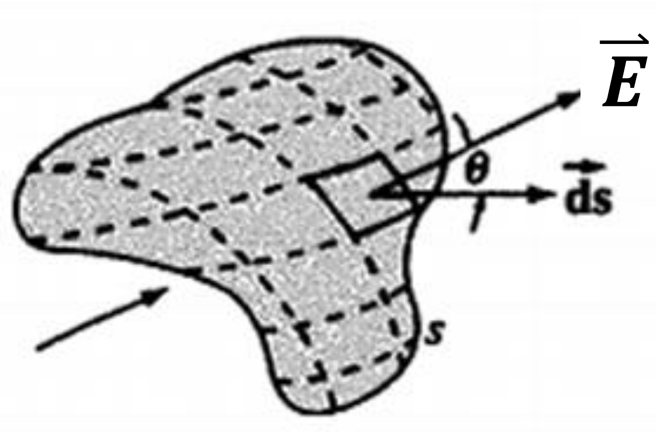
A vector field flows through this surface having different values and different directions at any point on the surface, illustrated as $\boldsymbol{E_i}$;

The flux flows through one area element is:

$$\Delta\Phi_i = \vec{\boldsymbol{E}}_i \cdot \Delta\vec{\boldsymbol{s}}_i$$

The total flux flows through the whole surface is:

$$\Phi = \sum \vec{\boldsymbol{E}}_i \cdot \Delta\vec{\boldsymbol{s}}_i$$

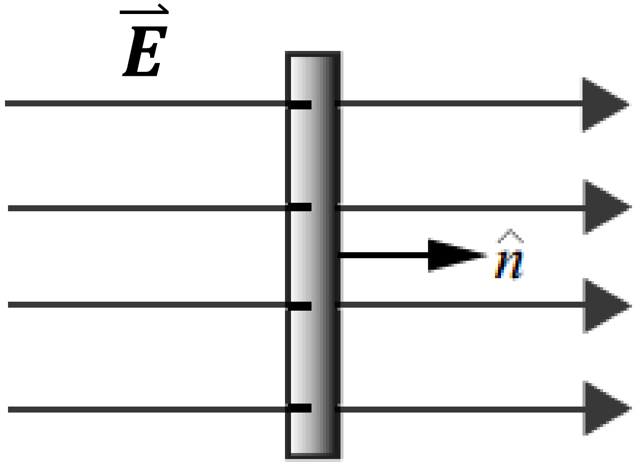


If the area elements are small enough, the summation becomes integration:

$$\Phi = \lim_{\Delta s_i \to 0} \sum \vec{E}_i \cdot \Delta\vec{s}_i = \iint \vec{E} \cdot d\vec{s}$$

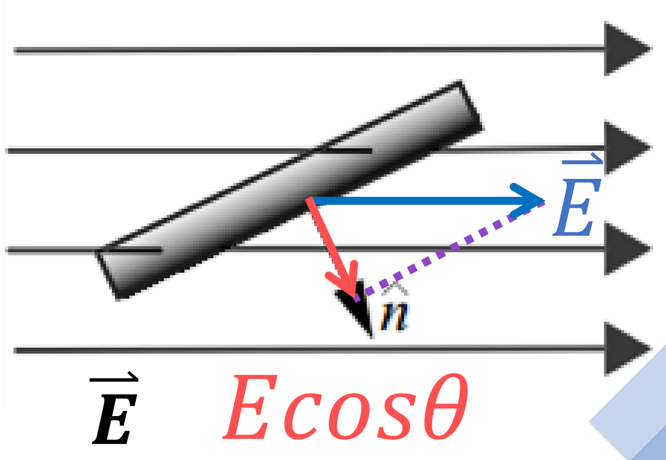
3.1 ELECTRICAL FLUX

In the simplest case of a uniform vector field $\vec{E}$ and a surface $\vec{S}$ perpendicular to the direction of the field, the electrical flux Φ is defined as the product of the field magnitude and the area of the surface:



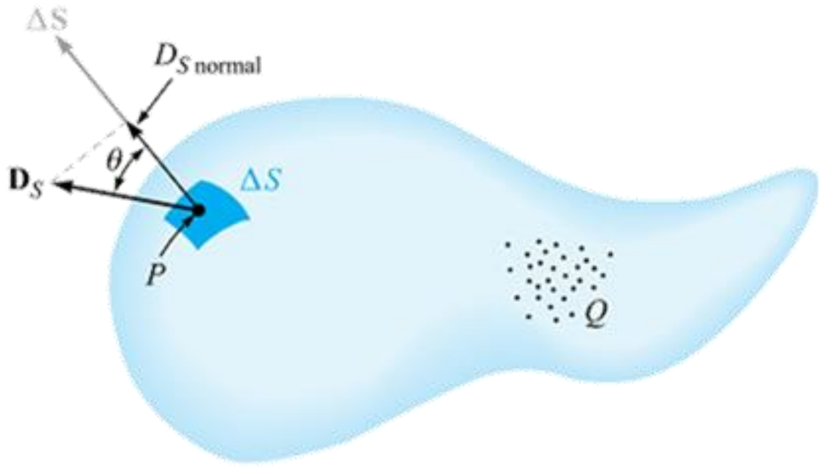
$$\Phi = \vec{E} \cdot \vec{S} = E\hat{n} \cdot S\hat{n} = ES(\hat{n} \cdot \hat{n}) = ES$$

If the vector field is uniform but **not** vertical to the surface, Φ can be determined by finding the component of $\vec{E}$ perpendicular to the surface and then multiplying that value by the surface area:



$$\Phi = \int Ecos\theta \, ds = \int E_{\perp} \, ds = \int \vec{E} \cdot d\vec{s}$$

3.1 ELECTRIC FLUX DENSITY



More general Gauss's law states that the **net** outward flux passing through a closed surface equals the total charge enclosed by that surface (where *Q* is the charge of a point source, which equals the number of field lines):

Electric flux density

$$\vec{D} = \varepsilon\vec{E} \qquad \therefore \oiint \vec{D}\cdot d\vec{s} = Q$$

$$\oiint \vec{D}\cdot d\vec{s} = \iiint_V \rho dv$$

Left side of the equation can be extended to a volume of discrete or continuously distributed charges with the charge density *ρ*:

$$Q = \iiint_V \rho dv$$

The total electric flux emanating from a closed surface numerically equals the net positive charge inside that surface.

In **free space**, Gauss's law becomes:

$$\oiint \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0} = \Phi_E \quad \text{or} \quad \oiint \vec{D}\cdot d\vec{s} = Q = \varepsilon_0\Phi_E$$

3.1 WHEN APPLY GAUSS'S LAW

Gauss's law is valid for any distribution of charges and for any **closed** surface.

Gauss's law can be used in two ways:

1. If the charge distribution is known and it has **enough symmetry** **(cylindrical/planar/spherical)** to evaluate the integral, we can find the field.
2. If the field is known, we can find the charge distribution.

$$\oiint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon}$$

Steps:

1. Identify regions where to calculate E-field
2. Select Gaussian surface: Symmetry (associated with the charge distribution)
3. Calculate the enclosed charge
4. Apply Gauss's Law

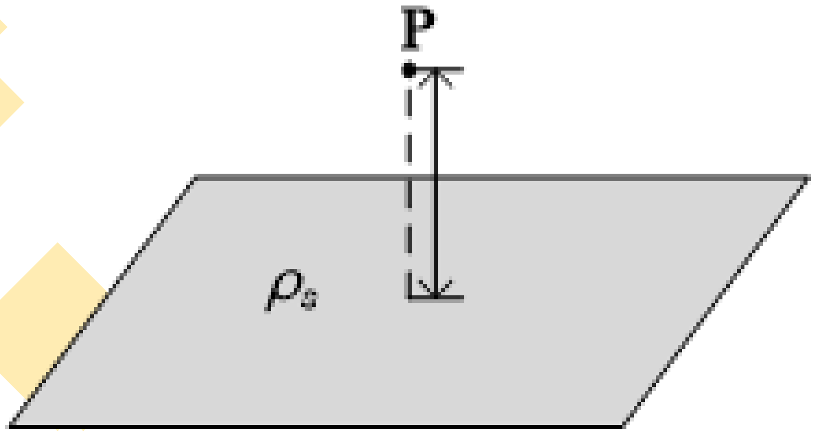
3.1 GAUSSIAN SURFACE

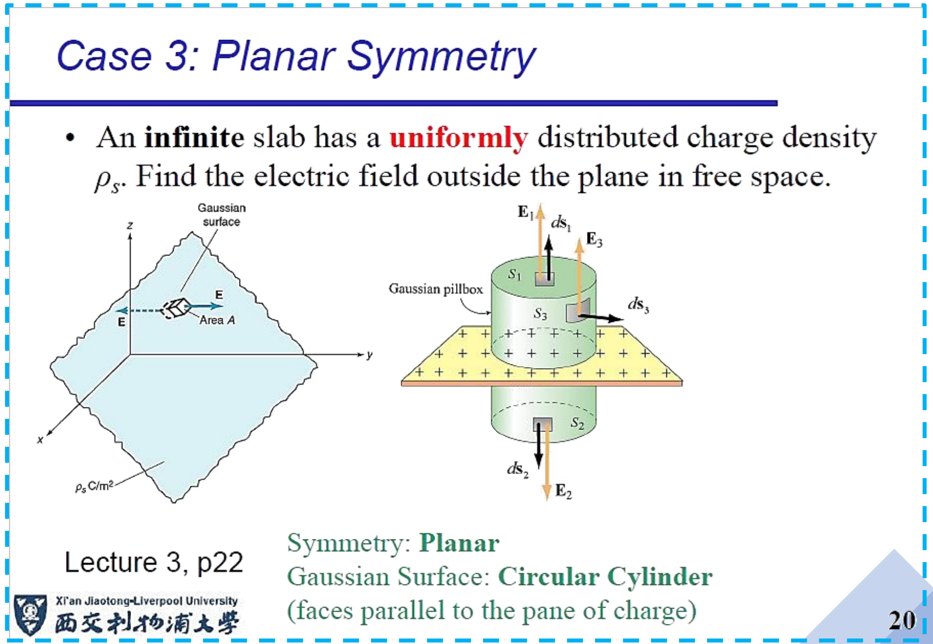
1. If you want to find the field at a particular point, then that point should lie on your Gaussian surface.
2. The Gaussian surface does not have to be a real physical surface. It is an imaginary geometric surface, such as: empty surface, embedded in a solid body, or both.
3. If the charge distribution has cylindrical or spherical symmetry, choose the Gaussian surface to be a coaxial cylinder or a concentric sphere, respectively.

Symmetry	**Gaussian Surface**
Spherical	Concentric Sphere
Cylindrical	Coaxial Cylinder
Planar	Gaussian “Pillbox”

QUIZ 3.1

Find the magnitude of the electric field intensity at a point P above the ***infinitely*** large conducting sheet **in free space** as shown below. The surface charge density of the conducting sheet is $+\rho_s$.





Case 3: Planar Symmetry

- An **infinite** slab has a **uniformly** distributed charge density ρ_s. Find the electric field outside the plane in free space.

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Symmetry: **Planar**
Gaussian Surface: **Circular Cylinder**
(faces parallel to the pane of charge)

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$$\oiint \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

Recall CAN102 Lecture 4

3.1 GAUSS'S LAW – E FIELD

2. The Differential Form

$$\nabla \cdot \overrightarrow{D} = \rho$$

or

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

The electric field produced by electric charge diverges from positive charge and converges upon negative charge.

3.1 DIVERGENCE THEOREM

How to link the integral and differential forms together?

Recall the "Divergence Theorem": $\iiint_V \nabla \cdot \vec{A} dv = \oiint_S \vec{A} \cdot d\vec{s}$

Therefore,

$$\oiint_S \vec{D} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{D} dv = \iiint_V \rho \cdot dv$$

This is true for any volume *v* bounded by a surface *s*.
So, the two integrands must be equal.

Thus, at any point in space, we have

$$\nabla \cdot \vec{D} = \rho \quad \text{or} \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

QUIZ 3.2

Given that a potential field $V = x^2y - 5z$ in free space, determine the electric field intensity, the electric flux density, and the volume charge density which produces this field.

$$\nabla V = \left(\frac{\partial V}{\partial x}\hat{x}, \frac{\partial V}{\partial y}\hat{y}, \frac{\partial V}{\partial z}\hat{z}\right)$$

if $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

OUTLINE

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3.2 SOURCES OF MAGNETIC FIELDS

B – “Magnetic flux density” or “Magnetic induction”

What is a magnetic field?

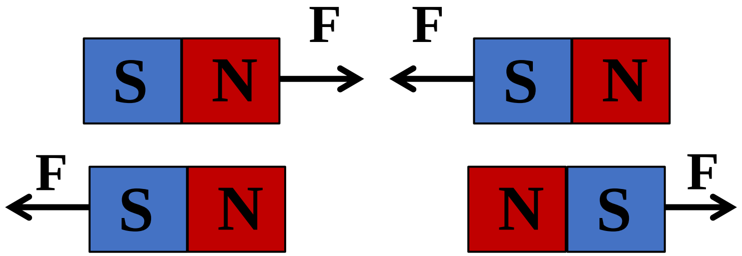
Vector field

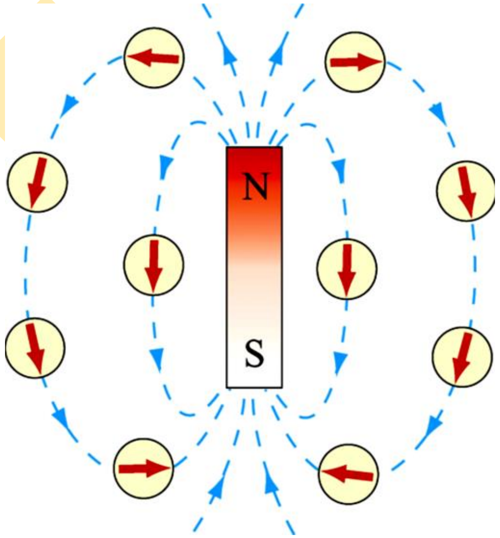
A field of force;

More precisely – a space in which the magnetic force is experienced by a moving charged particle.

Source of a magnetic field:

1. Permanent magnets



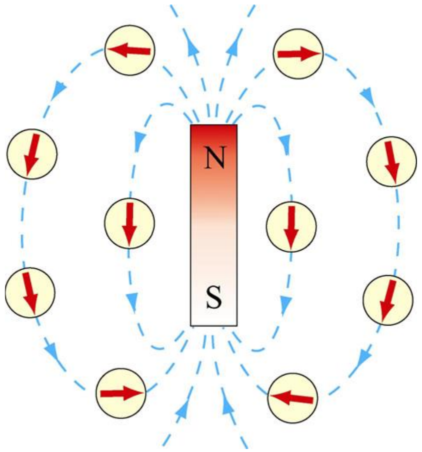


- A bar magnet is a source of a magnetic field.
- The bar magnet consists of two poles: the north (N) & the south (S).
- Magnetic fields are strongest at the poles.
- The magnetic field lines leave from N & enter S.
- The like poles repel each other while the opposite poles attract.

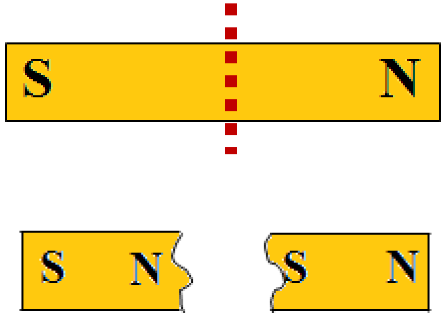
3.2 SOURCES OF MAGNETIC FIELDS

Source of a magnetic field:

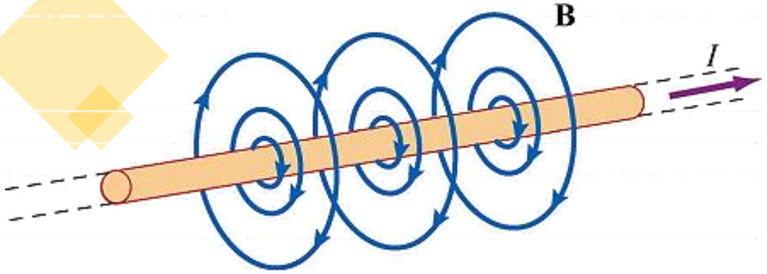
1. Permanent magnets



Bar magnets are dipoles!



2. Electric current



Oersted's observation:
An electric current produces a magnetic field as it flows through a wire.

Right-hand rule for the magnetic field due to current *I*:

- ✓ Point the thumb of the right hand in the direction of the *I*.
- ✓ The four fingers curl around the current element in the direction of the magnetic field lines.

3.2 REVIEW: BIOT-SAVART LAW

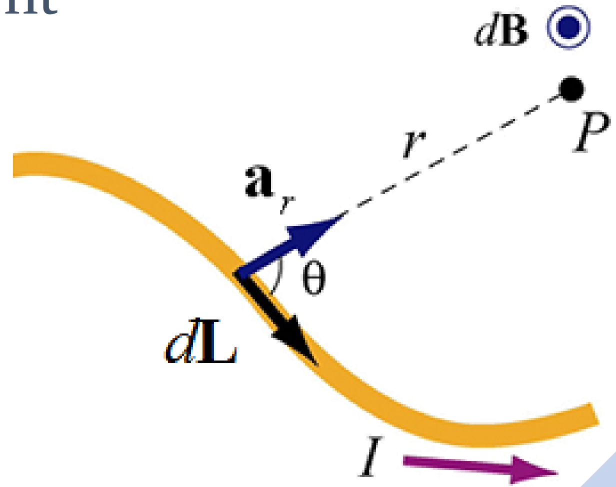
French scientists Jean Biot and Felix Savart arrived at an expression that results the magnetic flux density $\vec{B}$ at a point in space to the current I that generates $\vec{B}$, known as the Biot-Savart Law.

$$\vec{B} \longleftrightarrow I$$

It states that the differential magnetic flux density $d\vec{B}$ generated by a steady current I flowing through a different length *dl* is given by:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\boxed{Id\vec{l}\times\hat{r}}}{\boxed{r^2}} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\vec{r}}{r^3}$$

$$= \frac{\vec{r}}{r}$$



Adding up these contributions to find the magnetic field:

$$\therefore \vec{B} = \int_L d\vec{B}$$

SI Unit: N/(Cm/s) = N/(Am) = Vs/m^2 = T (Tesla)

3.2 GAUSS'S LAW – MAGNETIC FIELD

1. The Integral Form

$$\oiint_S \vec{B} \cdot d\vec{s} = 0$$

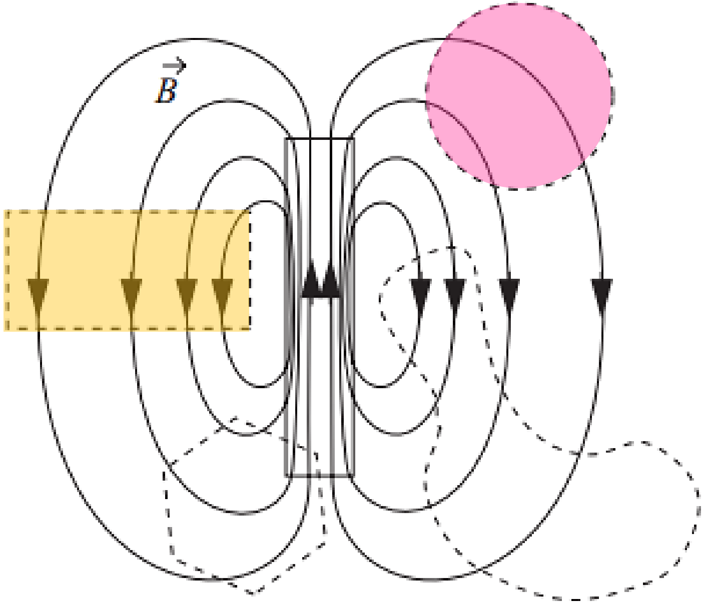
The total magnetic flux passing through any closed surface is zero.

Gauss's law for magnetic fields arises directly from **the lack of magnetic monopole** in nature.

- To date, all efforts to detect magnetic monopoles have failed, and every magnetic north pole is accompanied by a magnetic south pole, no matter how small they are.
- In other words, if you have a real or imaginary closed surface of any size or shape, the total magnetic flux through that surface must be zero.

3.2 MAGNETIC FLUX

Like the electric flux Φ_E, the magnetic flux Φ_B through a surface may be thought of as the "amount" of magnetic field "flowing" through the surface.



- When you think about the number of magnetic field lines through a surface, don't forget that magnetic fields are continuous in space, and that "number of field lines" only has meaning once you've established a relationship between the number of lines you draw and the strength of the field.
- No matter what shape of surface you choose, and no matter where in the magnetic field you place that surface, you'll find that the number of field lines entering the volume enclosed by the surface is exactly equal to the number of field lines leaving that volume.

The physical meaning behind Gauss's law should now be clear:

The net magnetic flux passing through any closed surface must be zero because magnetic field lines always form complete loops.

3.2 GAUSS'S LAW – H FIELD

2. The Differential Form

Recall the 'Divergence Theorem', we have $\iiint_V \nabla \cdot \vec{B} dv = \oiint_S \vec{B} \cdot d\vec{s} = 0$

This must be true for any volume *v* bounded by a surface *s*, so the two integrands must be equal. Therefore, at any point in space, we have

$$\nabla \cdot \vec{B} = 0$$

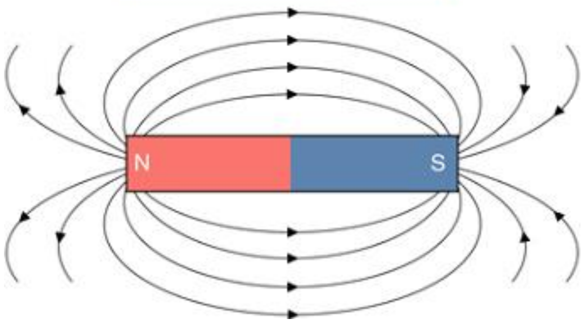
Differential form of Gauss's Law for magnetism*: the divergence of the magnetic field – the tendency of the magnetic field to either "flow" away or towards a point, is* ***zero****.*

Vector fields with zero divergence are called "solenoidal" fields.
All static magnetic fields are solenoidal.

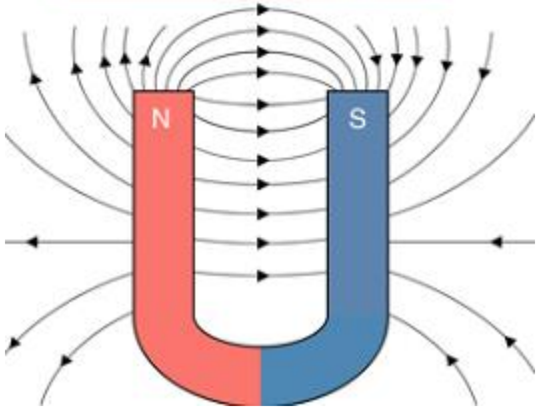
3.2 VISUALISATION

It is helpful to visualise the magnetic field in the vicinity of a current source:

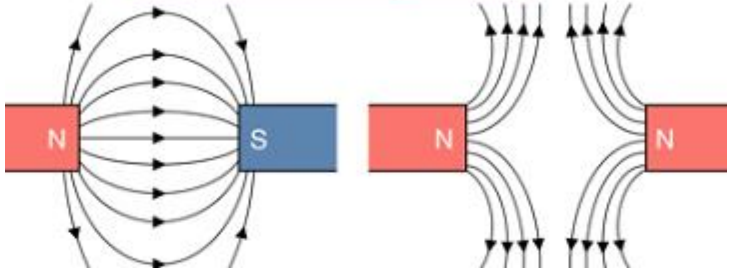
- At each point, the field line is tangent to the $\vec{B}$.
- The more densely the field lines are packed, the stronger the field is at that point.
- At each point, the field lines point in the same direction a compass would, so magnetic field lines point away from N poles and toward S poles.
- Field lines never intersect since the direction of $\vec{B}$ at each point is unique.



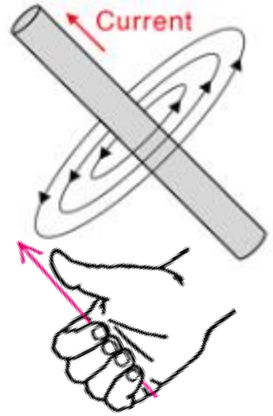
1. Bar Magnet



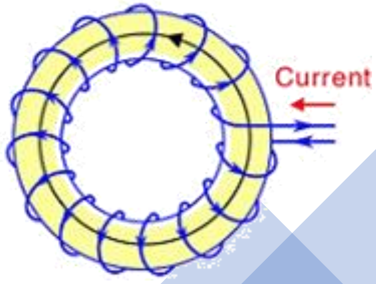
2. Horseshoe Magnet



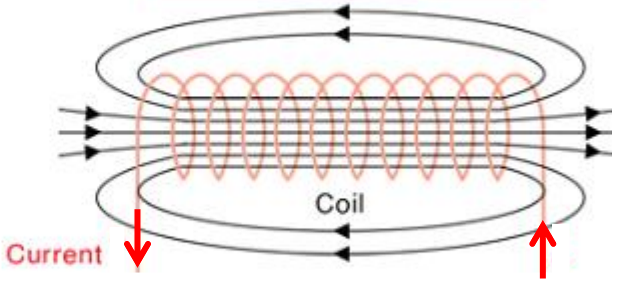
3. Two Bar Magnets



4. Current-carrying wire

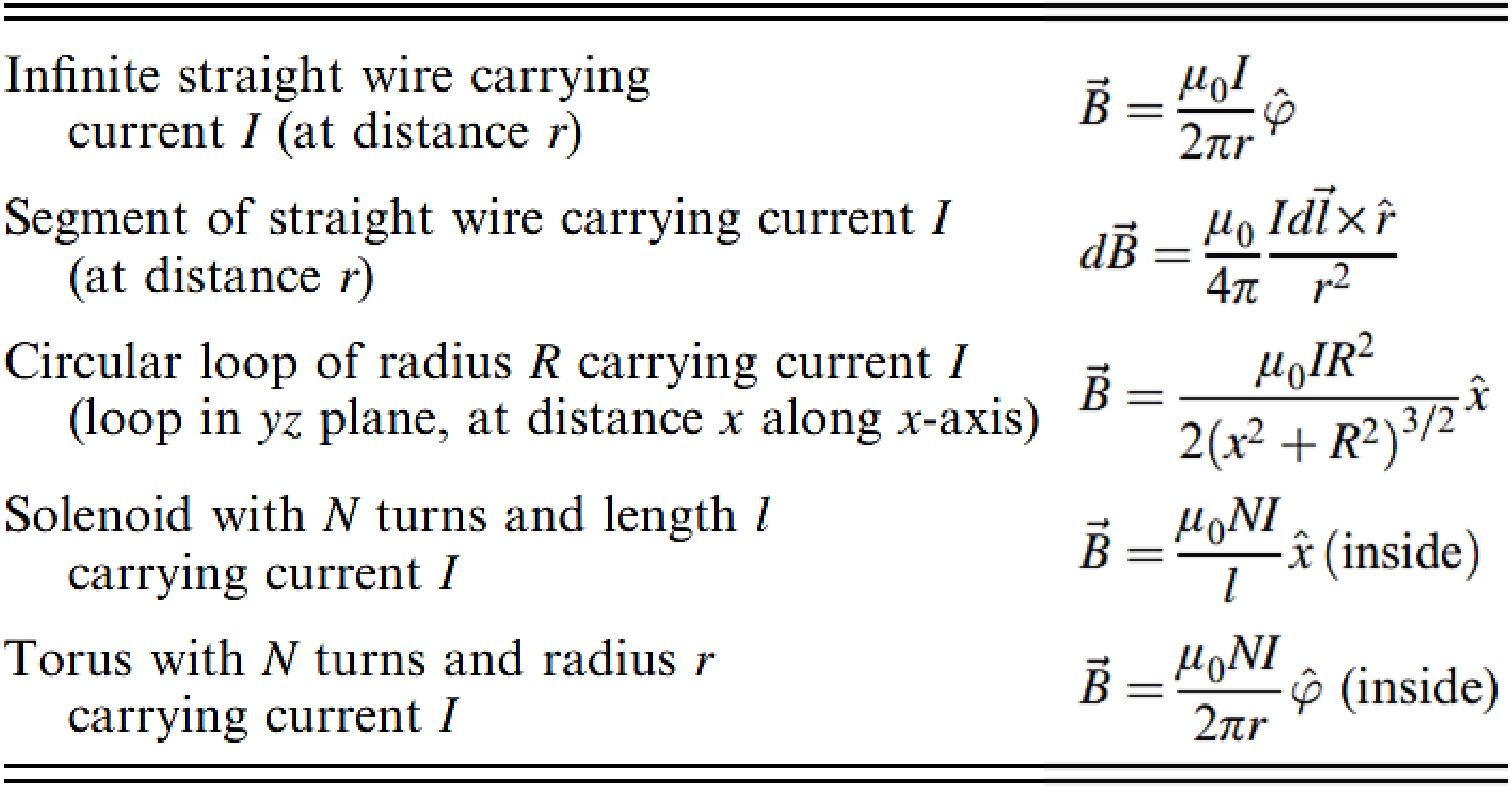


5. Toroid



6. Solenoid

3.2 TYPICAL EXAMPLES



Infinite straight wire carrying current *I* (at distance *r*)	$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$
Segment of straight wire carrying current *I* (at distance *r*)	$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l}\times\hat{r}}{r^2}$
Circular loop of radius *R* carrying current *I* (loop in *yz* plane, at distance *x* along *x*-axis)	$\vec{B} = \frac{\mu_0 I R^2}{2(x^2+R^2)^{3/2}}\hat{x}$
Solenoid with *N* turns and length *l* carrying current *I*	$\vec{B} = \frac{\mu_0 N I}{l}\hat{x}$ (inside)
Torus with *N* turns and radius *r* carrying current *I*	$\vec{B} = \frac{\mu_0 N I}{2\pi r}\hat{\varphi}$ (inside)

QUIZ 3.3

Two same bar magnets are placed as follows. Plot the magnetic field lines produced by **the whole structure**.

NEXT….

- Electric Field Loop Theorem
- Magnetic Field Loop Theorem

  (Ampere’s Law)