Xi'an Jiaotong-Liverpool University
西交利物浦大学

MEC208 Instrumentation and Control System

2024-25 Semester 2

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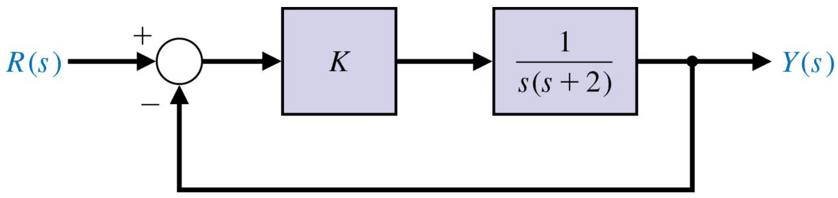
Lecture 17

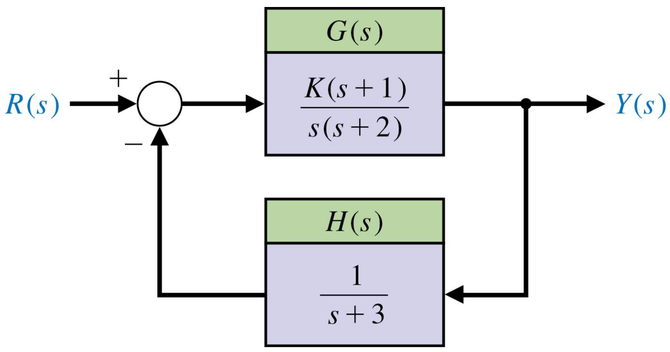
Outline

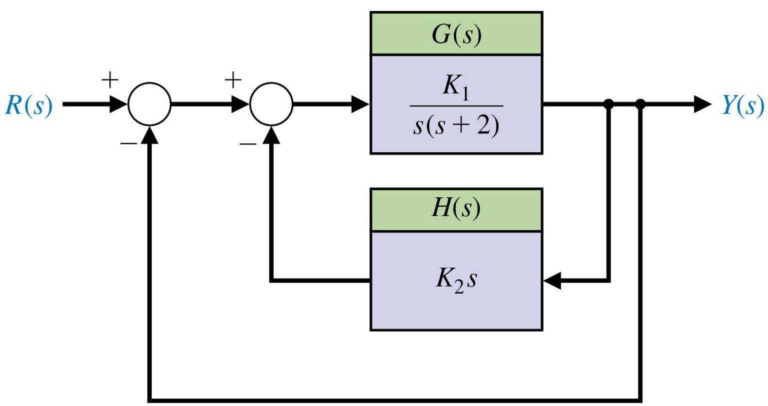
The Root Locus Method

- The Root Locus Concept
- Root Locus Plotting Procedure
- Root Locus Using Matlab
- Parameter Design using the Root Locus Method
- PID Controllers
  - Concept
  - PID Tuning
- Design Examples

Feedback control systems: how to design the control/feedback parameters?







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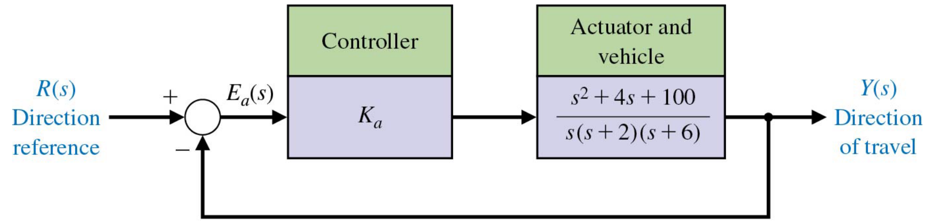

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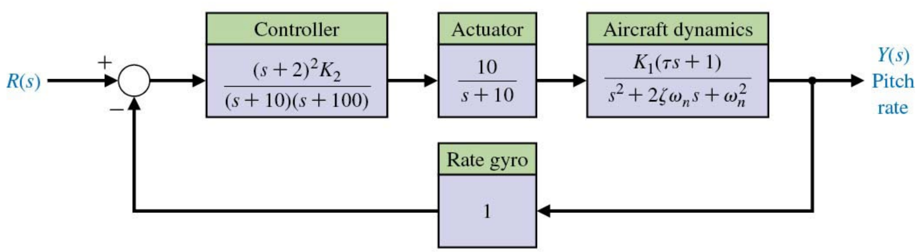
Feedback control systems: how to design the control/feedback parameters?

(a)



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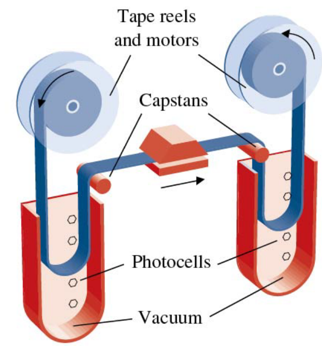
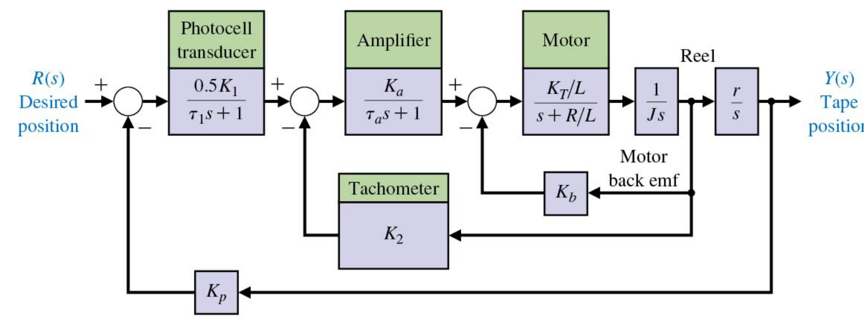


(a)

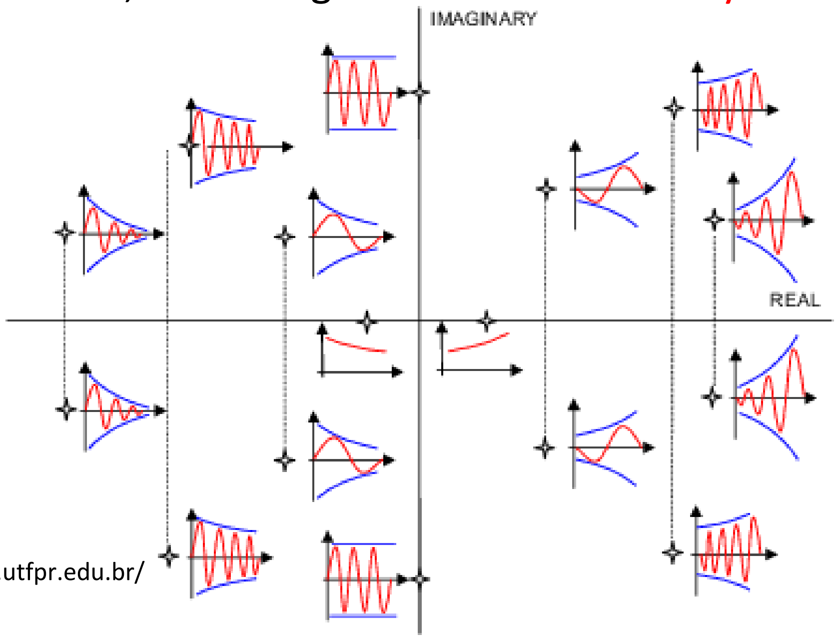


(b)

We can design these parameters using the methods established from classical control theory: <u>Root locus method</u>, Bode plot, Nyquist plot, etc.

Introduction - Root Locus Method

- Why "roots"?
  - The roots of the characteristic equation indicate the features of the homogenous response of the transfer function of concern, reflecting the relative stability and transient response.

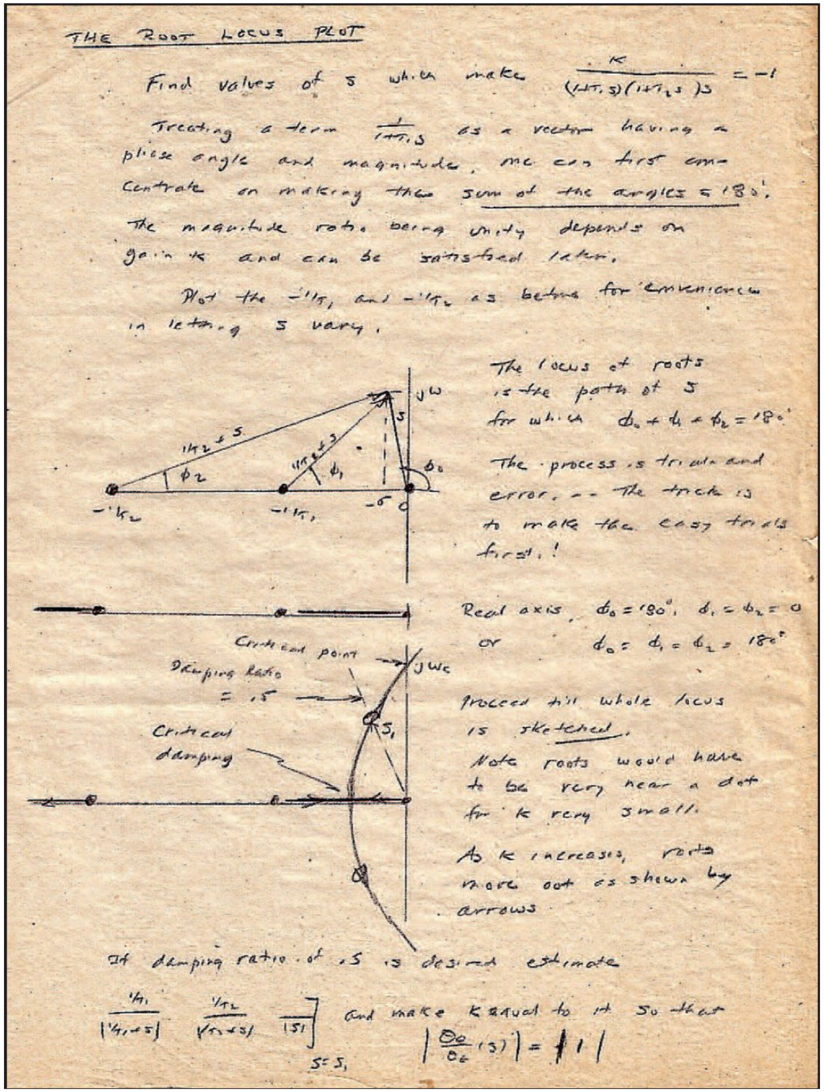


http://nupet.daelt.ct.utfpr.edu.br/

- What is root locus?
  - The root locus is the path of the roots of the characteristic equation of a CL system traced out in the s-plane as a single gain parameter varies from zero to infinity.

- What is RLM?
  - A method to plot the loci of closed-loop poles with respect to the change of a scalar parameter by only referring to the open-loop transfer function information.

Origin of RLM



THE ROOT LOCUS PLOT

Find values of s which make $\frac{K}{(1+T_1 s)(1+T_2 s)s} = -1$

Treating a term $\frac{1}{1+T_1 s}$ as a vector having a phase angle and magnitude, we can first concentrate on making the sum of the angles = 180°.

The magnitude ratio being unity depends on gain K and can be satisfied later.

Plot the $-1/T_1$ and $-1/T_2$ as before for convenience in letting s vary.

The locus of roots is the path of s for which $\phi_0 + \phi_1 + \phi_2 = 180°$

The process is trial and error. -- The trick is to make the easy trials first.!

$j\omega$, s, $1+T_2 s$, $1+T_1 s$, ϕ_2, ϕ_1, ϕ_0, $-1/T_2$, $-1/T_1$, $-\sigma$, 0

Real axis, $\phi_0 = 180°$, $\phi_1 = \phi_2 = 0$ or $\phi_0 = \phi_1 = \phi_2 = 180°$

Critical point

Damping Ratio = .5

$j\omega_c$

S_1

Critical damping

Proceed till whole locus is sketched.

Note roots would have to be very near a dot for K very small.

As K increases, roots move out as shown by arrows.

If damping ratio of .5 is desired estimate

$\frac{1/T_1}{|1/T_1+s|}$ $\frac{1/T_2}{|1/T_2+s|}$ $\frac{1}{|s|}\Big]_{s=s_1}$ and make K equal to it so that $\left|\frac{\theta_o}{\theta_\varepsilon}(s)\right| = |1|$

- Walters Richard Evans came up with the “root-locus” idea as evidenced by a letter written by him in June 1949.
- The last page of the letter is shown. He showed how the characteristic equation could be solved by plotting the locus of points “*s*” that have a simple relationship with other known points, that is, angles that sum to 180◦.
- Evans developed a simple, sequential process, which engineers used to generate sketches in a short time, and a specialized protractor, which supported high accuracy sketches.
  - Of course, almost the whole plotting process has now been substituted by computers, but the process of analysis and using the tool to solve control problems remain the same!
- RLM is first used by North American Aviation designers and taught at Uni. California LA, but the application and instruction of Evans’ new method spread rapidly to other universities (and industries).

Basic Concept

- As the system performance is closely related with the location of its poles, in practice, it is frequently necessary to adjust one or more system parameters in order to obtain suitable root locations.
- Therefore, it is worthwhile to determine how the roots of the characteristic equation of a given system migrate about the *s*-plane as the parameters are varied.

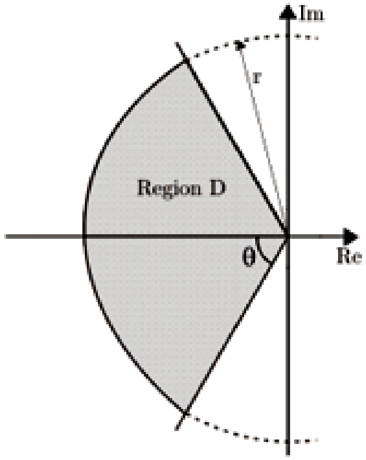
The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter varies from zero to infinity.

- Root locus method is a graphical method for sketching the locus of roots in the s-plane as a parameter is varied.
  - Being an approximate sketch can be used to obtain ***qualitative*** information concerning the stability and characteristic performance of the CL system.
  - If the root locations are not satisfactory, the necessary parameter adjustments often can be ascertained from the root locus.

Relevance of RLM

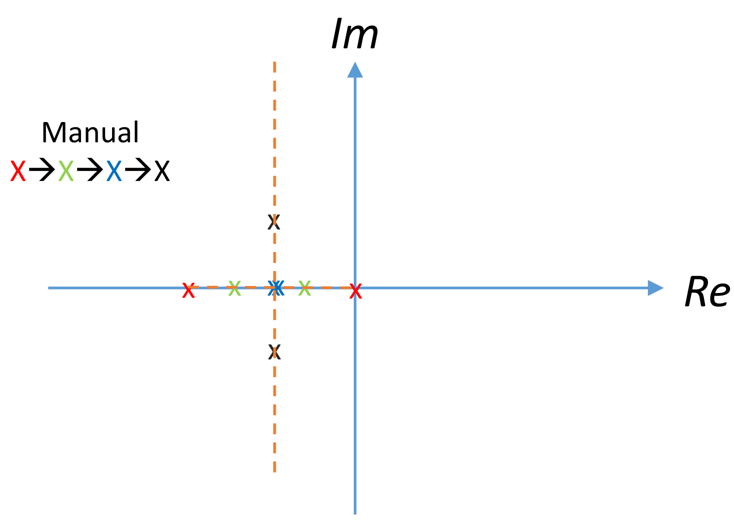
- The strength of RLM is the simplicity of designing a CL control system by "operating" on the loop TF (known in some books as open-loop TF).
  - The loci of the CL poles are sketched by analyzing the OL poles and zeroes.

- Broadly speaking, RLM can be used:
  - To inform the range/value of controller gain for which the closed loop (CL) system is stable/unstable.
  - To determine the effect of different controller gain/parameter on the CL pole positions, allowing further tuning to the desired CL performance (e.g., Region D).



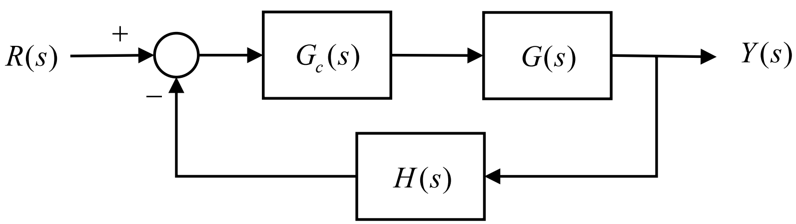
Locus/Loci of the roots

- E.g., $K(s) = k$, $G(s) = \frac{1}{s(s+2)}$
- Closed-loop transfer function: $T(s) = \frac{k}{s(s+2)+k}$
- Let's trace the position of the CL poles as *k* changes from 0 to ∞.
  - Note that when the CL poles are changing from real to complex conjugate pair.



- *G*(*s*) has rank 2, which indicates that there are two zeroes at *inf* (as shown).

Fundamentals of Root Locus Method/Analysis



Closed-loop TF: $T_{CL}(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)H(s)}$

Loop TF: $T_L(s) = G_c(s)G(s)H(s)$

Characteristic function $\Delta(s) = 1 + G_c(s)G(s)H(s)$

Assume $G_c(s)G(s)H(s) = \frac{kb(s)}{a(s)} = kL(s)$, where $a(s) = \prod_{j=1}^{n}(s + p_j)\,, b(s) = \prod_{i=1}^{m}(s + z_i)\,, and\ 0 < k < \infty$:

$$T_{CL}(s) = \frac{G_c(s)G(s)}{1 + \frac{kb(s)}{a(s)}}$$

$$T_L(s) = \frac{kb(s)}{a(s)}$$

CL system poles: $a(s) + kb(s) = 0$

$a(s) = 0$ → OL poles

$b(s) = 0$ → OL zeros

- By comparing the poles-zeroes of the two systems, we can establish that:
  - when k is zero, the CL poles coincide with the LTF poles (or simply, OL poles).
  - When k is ∞, the CL poles coincide with the LTF zeros (or simply, OL zeros) .
  - So, it is logical to expect that, for $0 < k < \infty$, as k increases from 0 to *inf*, the CL poles are moving from the OL poles towards the OL zeroes. The traces of these CL poles form the loci of the CL characteristic equation's roots, a.k.a. "root locus".

Fundamentals of Root Locus Method/Analysis

- Characteristic equation: $a(s) + kb(s) = 0$

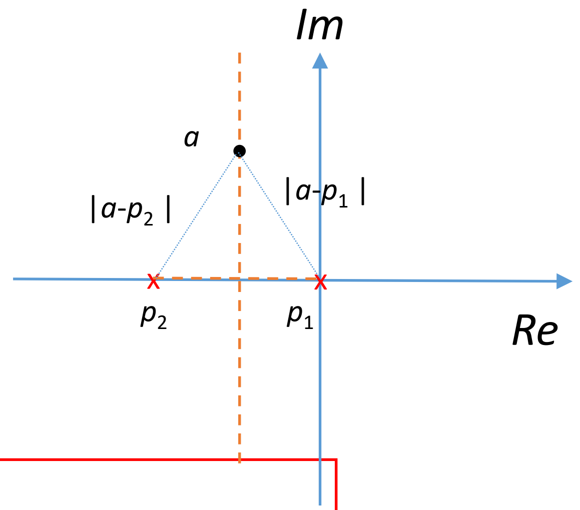
$$\prod_{j=1}^{n}(s + p_j) + k\prod_{i=1}^{m}(s + z_i) = 0$$

- Further insights based on inspection:
  - When k is near zero, the **closed-loop (CL) poles** are at the same location as the **OL poles**.
  - As k increases, two scenarios may result, depending on the rank (=*n*-*m*) of the LTF:
    - Rank 0: CL poles move towards the **OL zeroes**.
    - Rank 1,2,3,…: CL poles of the rank number move **towards *inf*** and the remaining move towards the **OL zeroes**.

Arbitrary points on the locus/loci

Recall $L(s) = \frac{kb(s)}{a(s)} = \frac{k\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$

$$T_{CL} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)}$$



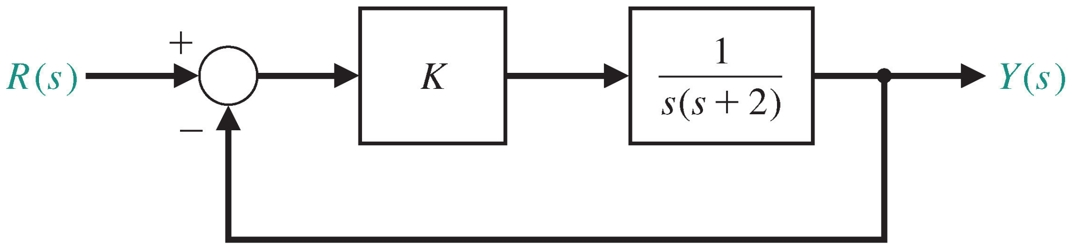
- Let $G_c(s)G(s)H(s) = kL(s)$, where $0 \le k < \infty$.
- Characteristic equation: $$1 + kL(s) = 0$$ $$kL(s) = -1$$
- Therefore,

$$\boxed{\begin{array}{c} |kL(s)|=1 \\ \angle kL(s) = 180^o + m360^o, m = 0, \pm1, \pm2, \ldots\ (e.g. \pm180^o, \pm540^o, \ldots) \end{array}}$$

- Two **very important** insights can be gathered from these two equations:
  - Since $k = \frac{\left|\prod_{all\ j}(s+p_j)\right|}{\left|\prod_{all\ i}(s+z_i)\right|}$, we can determine k for any arbitrary point '*a*' on the root locus. Draw vectors from every OL pole (p_j) and OL zero (z_i) to the point '*a*' and insert these vectors' magnitudes into this k expression.
  - Sum of the angles of the point '*a*' to all the poles minus that to all the zeroes must be equal to 180^o.

Example 17.1: Manual Sketch of the Locus

Consider the second-order system shown in the following figure



- The characteristic equation is

$$\Delta(s) = 1 + KG(s) = 1 + \frac{K}{s(s+2)} = 0$$

$$\Delta(s) = s^2 + 2s + K = 0$$

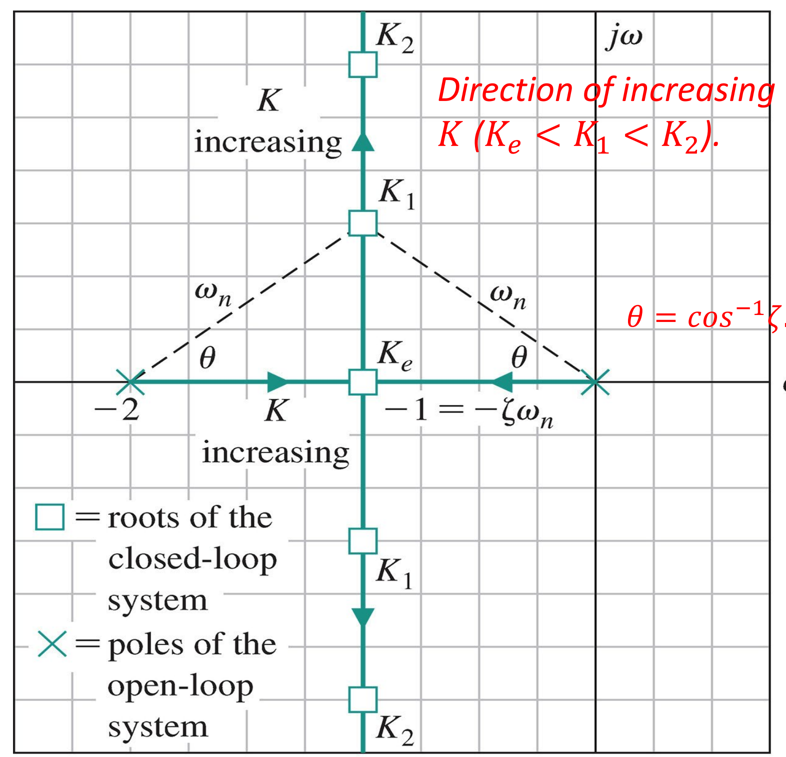
Poles of the closed-loop system are,

for $K \leq 1$:

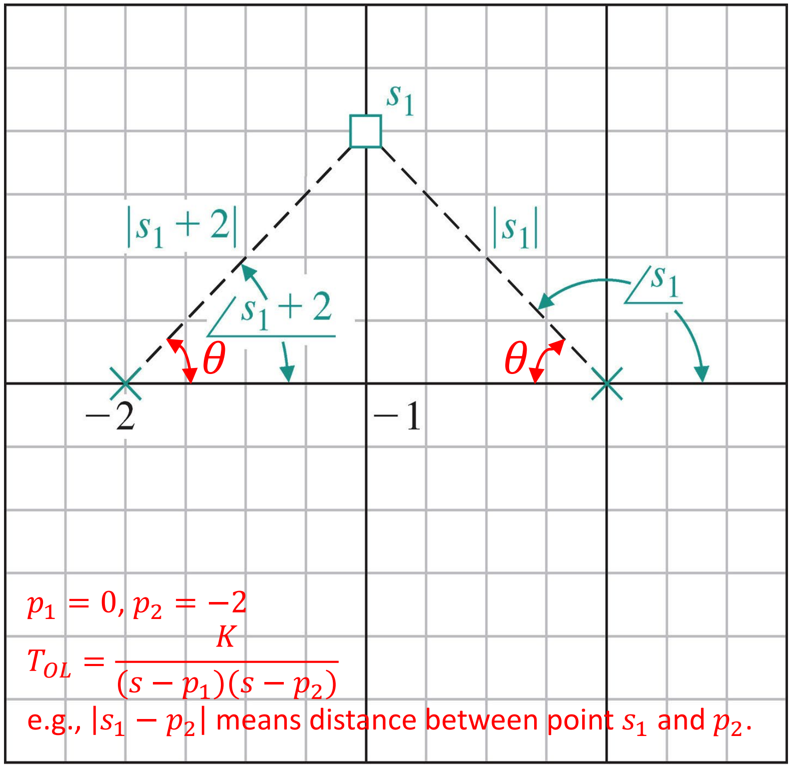
$$s_1, s_2 = -1 \pm \sqrt{1-K}$$

for $K > 1$:

$$s_1, s_2 = -1 \pm j\sqrt{K-1}$$



Example 17.1: Angle and Gain at an Arbitrary Point/Root on the locus



***Note**: s_1 is the vector from origin to s_1; $s_1 + 2$ is the vector from -2 to s_1. $|s_1|$ and $|s_1 + 2|$ are the magnitude of the vectors.*

- **Angle requirement**

  For example, at a root s_1, the angles are

$$\angle \frac{K}{s(s+2)}|_{s=s_1} = -\angle s_1 - \angle(s_1+2)$$
$$= -(180^o - \theta) - \theta = -180^o$$

  Therefore, the angle requirement is satisfied at any point on the root locus.

- **Gain requirement**

  The gain K at root s_1 can be found using

$$\left|\frac{K}{s(s+2)}\right|_{s=s_1} = \frac{K}{|s_1||s_1+2|} = 1$$

  Thus

$$K = |s_1||s_1+2|$$

RLM Plotting Rules 1-4 (out of 7 or 8)

Demo:

$$KGH(s) = \frac{k(s+10)}{s(s^2+8s+41)}$$

Zeros: -10
Poles: $0, -4 \pm j5$

Rule 1: Rule of Symmetry

- The root locus is always symmetrical about the real axis.

Rule 2: Rule of Loci Number

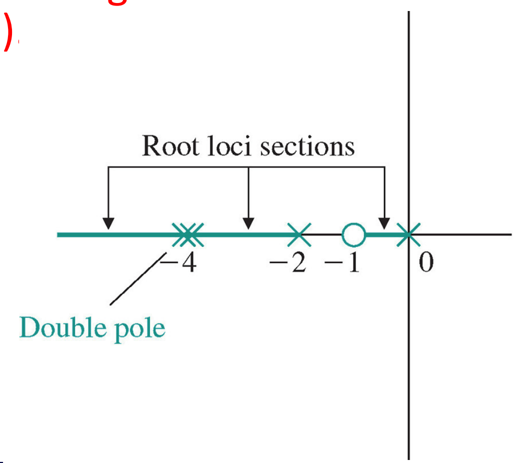
- The number of loci is equal to the number of OL poles.
- The loci start at each of the OL poles at $k = 0$ (or 0^+) and travel towards OL zeroes.

Rule 3: Rule of Rank Number

- The number of zeroes at *inf* is equal to the rank of the LTF (i.e., # of poles minus # of zeroes).

Rule 4: Rule for Loci on the Real Axis

- On the real axis, loci only exists at the region with its RHS having odd number of poles&zeros or "poles-zeros" (note: this does not mean *minus*!).

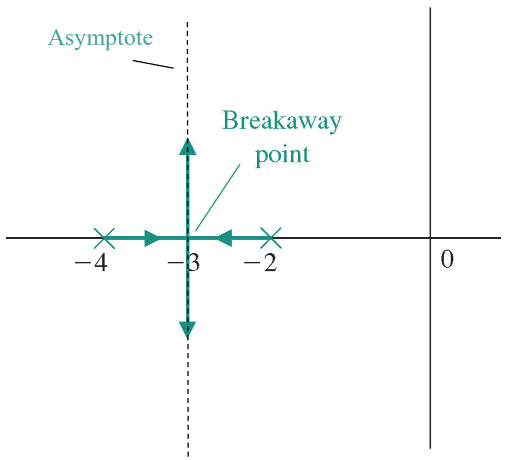


Plotting Rule 5: Rule of Asymptotes

- The zeroes at *inf* have asymptotes inclined at an angle to the real axis. All asymptotes meet the real axis at a common point.
- Angle for asymptotes$= \frac{2q+1}{Rank} \cdot 180^o$, $q = 0,1,2, \ldots (Rank - 1)$
- Point of intersection at the real axis = $\frac{Re(\sum p - \sum z)}{Rank}$

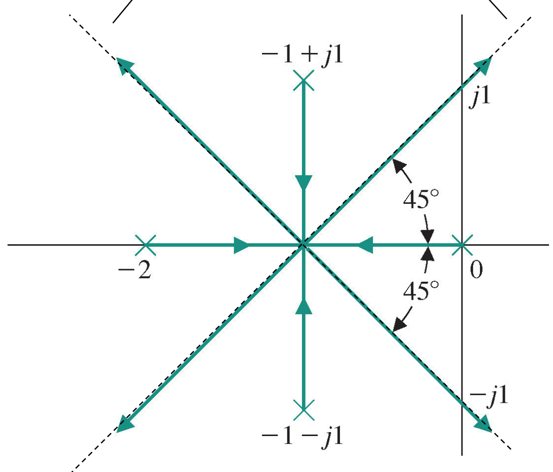
Note: for these two examples, it just happened that the POI@real-axis is the same as the BIBO point, but this is not always the case! POI@real-axis (Rule 5) not BIBO point (Rule 6).

- For two loci at the breakaway point are spaced 180° apart.



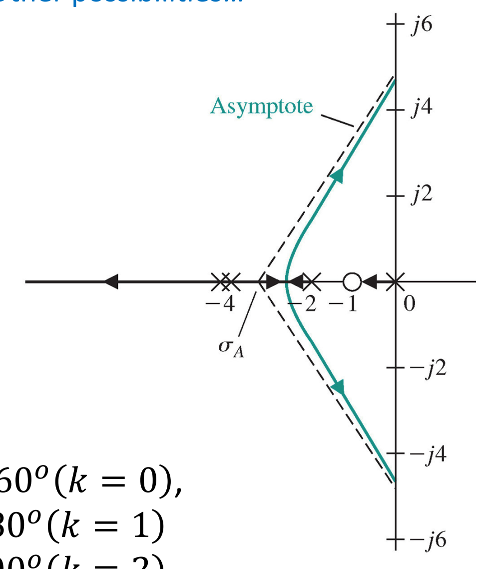
(a)

- For four loci at the breakaway point are spaced 90° apart.



(b)

- Other possibilities...



$$\emptyset_A = +60^o (k = 0),$$
$$\emptyset_A = 180^o (k = 1)$$
$$\emptyset_A = 300^o (k = 2)$$
$$\sigma_A = \frac{(-2) + (-4) + (-4) - (-1)}{4 - 1} = -3$$

Plotting Rule 6: Break-in Break-out Points

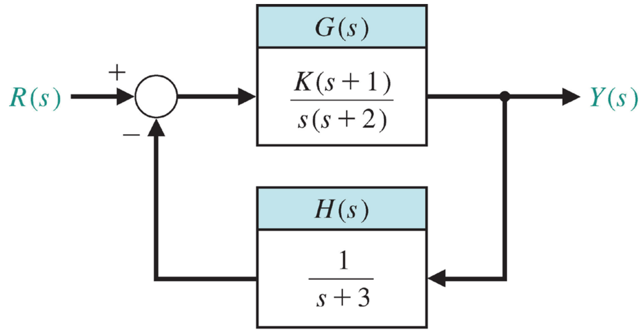
- When the loci move **from** the real axis **into** complex space, there exists a point on the real axis, known as the "**break-out**" or "break-away" point. It is usually located in between two OL poles on the real axis.

- When the loci move **from** the complex space **onto** the real axis, there exists a point on the real axis, known as the "**break-in**" point. It is usually located in between even-multiple (i.e., 2, 4, 6,…) of OL zeroes on the real axis.
  - Technically, break-out/break-in point(s): "The loci leave/enter the real axis where is a multiplicity of roots".
  - There could be more than one break-in-break-out points for a root locus plot.

- If the LTF is $\frac{kb(s)}{a(s)}$, then the break-in-break-out points are the solutions of $\frac{d}{ds}\left[\frac{-a(s)}{b(s)}\right] = 0$. These solutions must be valid for them to be considered further.

Proof (not required in exam): $1 + (k + \Delta k)\left(\frac{b}{a}\right) = 0, 1 + \frac{\Delta kb}{a+kb} = 0$, introduce multiplicity (m) property $\frac{b}{a+kb} = \frac{c}{(s-s_i)^m} = \frac{c}{\Delta s^m}$, and show that break-in/break-out occurs at $\frac{\Delta k}{\Delta s} = \frac{-(\Delta s)^{m-1}}{c} \rightarrow \frac{dk}{ds} = 0$

Plotting Rule 6: Example 18.2



$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0$$

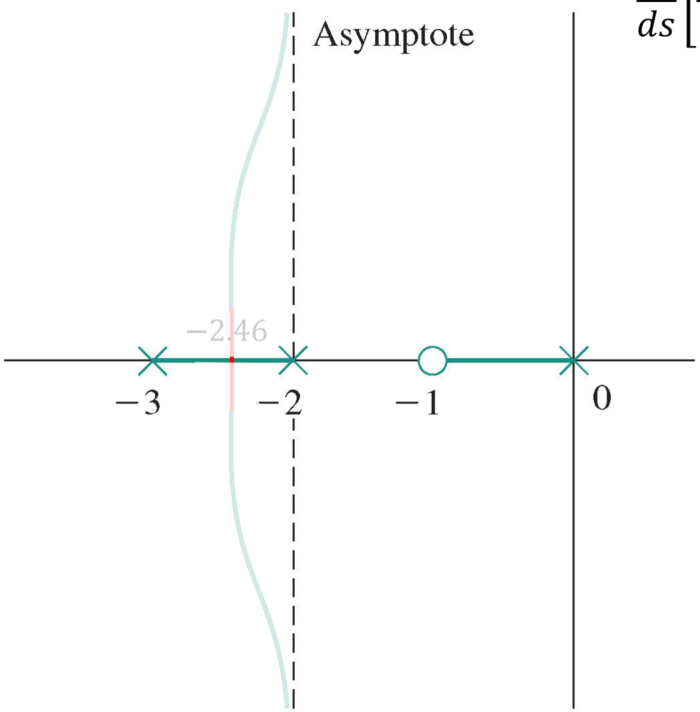
Then we set its first derivative to be zero

$$\frac{d}{ds}\left[\frac{-s(s+2)(s+3)}{(s+1)}\right] = \frac{(s^3+5s^2+6s)-(s+1)(3s^2+10s+6)}{(s+1)^2} = 0$$

$$2s^3 + 8s^2 + 10s + 6 = 0$$

There are three roots for the above equation, but only $s = -2.46$ is on the real axis and between -2 and -3.

The breakaway point is $s = -2.46$.

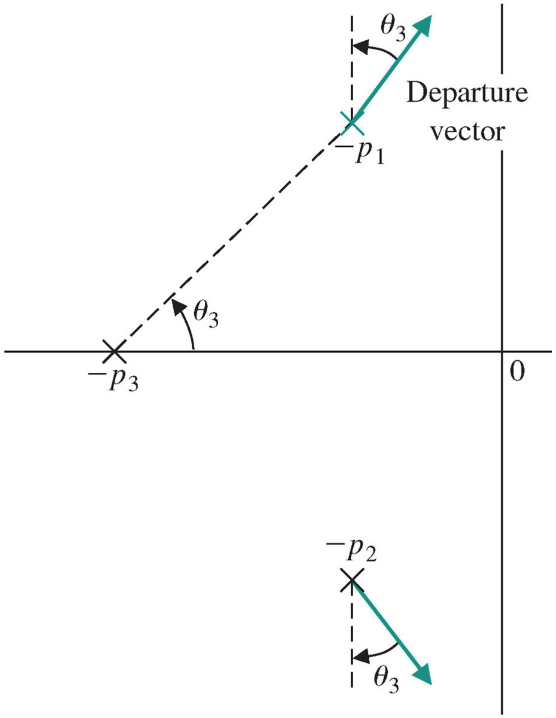


Plotting Rule 7: Angles of Departure

- To accurately plot the root loci, it is useful to know at what angle the loci leave the complex pairs of OL poles. This rule is relevant only when there are complex OL poles!
- **Recall the angle condition**: The sum of the angles of vectors between the arbitrary point and the OL poles and zeroes must be an odd multiple of 180º.
  - For brevity, this can be stated mathematically as

$$\angle target_{pole} + sum(\angle p) - sum(\angle z) = 180^o$$

  - $\angle p$ or $\angle z$ is the angle of the target pole, referencing to other poles p or zeroes z, measured in the counter-clockwise direction
- There exists also "angles of arrival", meant for the OL zeroes. Almost similar principle applies.



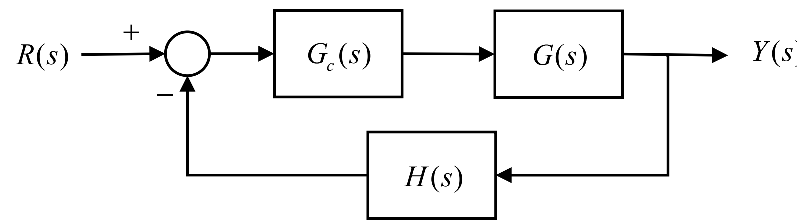
Additional - **Plotting Rule 8***: Imaginary-axis Crossing

- At times it is of our interest to find the intersection of the loci at the imaginary axis (e.g. when one is interested in knowing the min/max gain before the CL system would become stable/unstable).
- Two ways:
  - These intersecting point(s) s can be found by substituting $s=j\omega$ to the CL characteristic equation, then solve for the gain k and angular frequency ω.
  - Alternatively, Routh-Hurwitz stability criterion can be used.

Summary: 7+1 rules to remember! (NOTE: Rules here are slightly different from that in the reference book, but they will produce the same exact RL plot!)

Prepare the OLTF of the CL system, compute the poles and zeroes, then proceed with the following rules:

with a variable of interest, e.g., "*k*"



- Symmetry (about the real axis)
- Number of loci = number of OL poles
- Rank = number of OL zeros at infinity (i.e., no. of finite *p* – no. of finite *z*)
- Odd number of poles-zeros on the real-axis's RHS = loci exists on the real axis
- Asymptotes angle = $\frac{2q+1}{Rank} \cdot 180^o$, *q*=0,1,2…(*Rank*-1);
  - The only POI of the asymptotes at the real axis = $\frac{Re(\sum p - \sum z)}{Rank}$
- Break-in break-out (a.k.a. break-away break-in) points, $\frac{d}{ds}\left[\frac{-a(s)}{b(s)}\right] = 0$ (provided the transfer function is coprime)
- Angle of departure from a complex OL pole
  - $\angle target_{pole} + sum(\angle p) - sum(\angle z) = 180^o$
  - $\angle p$ or $\angle z$ is the angle of the target pole, referencing to other poles p or zeroes z, measured in the counter-clockwise direction
- Lastly, if the loci do cross the imaginary axis, the crossing points and gain can be found through the CL characteristic equation (or RHC!).

Positive feedback (mainly FYI, not for exam)

$$CLTF = \frac{G_c(s)G(s)}{1 - G_c(s)G(s)H(s)}$$

- Characteristic equation: $1 - kL(s) = 0$

$$kL(s) = 1$$

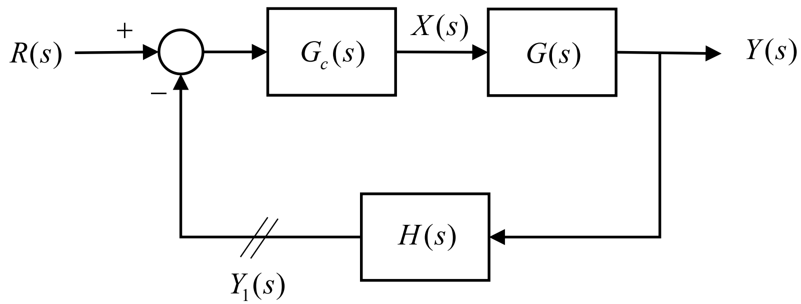
- Therefore,

$$|kL(s)|=1$$

$$\angle kL(s) = m360^o, \mathrm{m} = 0, \pm 1, \pm 2, \ldots$$

- **Revision on the "S N R O A B A" rules**:
- "Odd rule" – loci exist at the real-axis's RHS when there are EVEN number of poles-zeros
- Asymptotes angle = $\frac{2q}{Rank} \cdot 180^o$, q=0,1,2…($Rank$-1);
- Angle of departure: $\angle target_{pole} + sum(\angle p) - sum(\angle z) = 0^o \ (\pm \mathrm{m}360^o)$

Example 17.3 (in-class)



$$G_c(s) = k$$
$$G(s) = \frac{s+1}{s^2}$$
$$H(s) = 1$$

Question: Find the range of k to ensure CL stability, crossing points on the imaginary axis and the corresponding gain, and plot the root locus.

Thought process: (1) Form loop TF; (2) Apply RLM rule ("S N R O A B A"); (3) Determine if there is any crossing of the imaginary axis (the 8th rule).

Answer: $T_L = \frac{k(s+1)}{s^2}$. OL zeroes: -1; OL poles: 0, 0.

S – symmetrical, N = 2, R = 2-1 = 1

O – sketch and recognize this feature in the plot

A – P.O.I. @real axis = 1; Angles of asymptotes: 180°

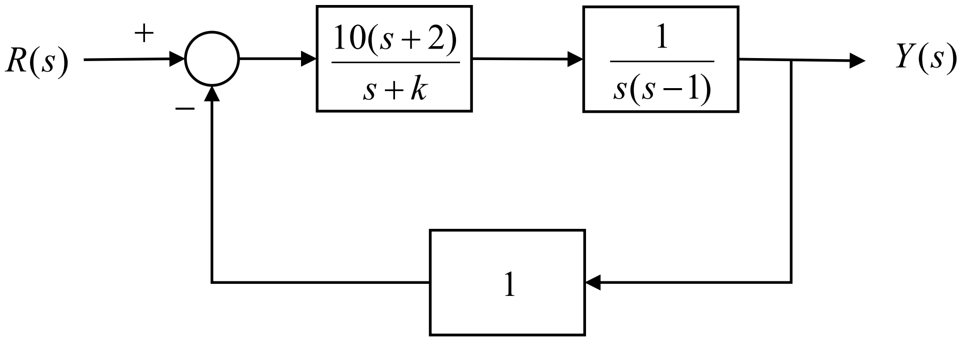
B – Apply the rule, B. I. B. O. points obtained are 0, -2.

A – not relevant here

+1 rule – Apply. Conclude that crossing occurs only at k = 0.

To tune other controller parameters (e.g. time constant of the first-order pole)

- If the parameter under consideration is not the scalar gain but being the pole/zero location, some manipulation can be done to fit the problem to the form required by RLM.



To find the CL poles, we equate the CL characteristic function to zero:

$$0 = \Delta(s)$$

$$0 = 1 + \frac{10(s+2)}{s+k}\frac{1}{s(s-1)}$$

$$0 = s(s+k)(s-1) + 10(s+2)$$

$$0 = s^2(s-1) + 10(s+2) + ks(s-1)$$

$$0 = 1 + \frac{ks(s-1)}{s^2(s-1)+10(s+2)}$$

This changes the location of controller pole *k* to a new LTF where RLM will be relevant:

$$LTF\,(or\;T_L) = \frac{ks(s-1)}{s^2(s-1)+10(s+2)}$$

Next Lecture

- **In our next lecture:** we will continue to develop understanding on Root Locus Method, with an emphasis placed on parameter design, numerical tool, and how to use it in common controllers (e.g., PID).

- **What you can do from now till the next lecture:** revise the material, further self-reading, and group study.

- **How to get in touch**: through LMO Module's "General question and answer forum" section or during my weekly consultation hour(s).

Reference book's RLM (optional, FYR)

Note: The sequence of the following plotting rules (details see Ref. book, Chap. 7) are slightly different from what were presented. Ultimately, they will produce the same Root Locus plot.

Table 7.2 Seven Steps for Sketching a Root Locus

Step	Related Equation or Rule
1. Prepare the root locus sketch.	
(a) Write the characteristic equation so that the parameter of interest, K, appears as a multiplier.	$1 + KP(s) = 0.$
(b) Factor $P(s)$ in terms of n poles and M zeros.	$1 + K\dfrac{\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 0.$
(c) Locate the open-loop poles and zeros of $P(s)$ in the s-plane with selected symbols.	$\times$ = poles, $\bigcirc$ = zeros Locus begins at a pole and ends at a zero.
(d) Determine the number of separate loci, SL.	$SL = n$ when $n \geq M$; n = number of finite poles, M = number of finite zeros.
(e) The root loci are symmetrical with respect to the horizontal real axis.	
2. Locate the segments of the real axis that are root loci.	Locus lies to the left of an odd number of poles and zeros.

Table 7.2 Seven Steps for Sketching a Root Locus	
Step	**Related Equation or Rule**
3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A.	$\sigma_A = \dfrac{\sum(-p_j) - \sum(-z_i)}{n - M}$. $\phi_A = \dfrac{2k+1}{n-M}180°, k = 0, 1, 2, \ldots, (n - M - 1)$.
4. Determine the points at which the locus crosses the imaginary axis (if it does so).	Use Routh–Hurwitz criterion.
5. Determine the breakaway point on the real axis (if any).	a) Set $K = p(s)$. b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s)$.
6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion.	$\angle P(s) = 180° + k360°$ at $s = -p_j$ or $-z_i$.
7. Complete the root locus sketch.	