CAN207 Continuous and Discrete Time Signals and Systems

Lecture 24

Final Revision_Part 2

Zhao Wang & Qing Liu

Qing.Liu@xjtlu.edu.cn

Room SC340

Content

- Again: Important changes of exam rules
- Office hour arrangement
- Walk through every lecture (L14-L22)

- Please fill Module Questionnaire! Please :)

1. Information of final exam

- Time: 30-Dec-2024, 14:00-17:10
- Exam room: GYM-GMG01-Basketball Court

- Questions
  - 5 questions (20 marks each)
  - 15 subquestions

- Important notice:
  - Calculator: University approved Casio FS82/83 series (991 sereis is not allowed)
  - Reading time:
    - 14:00-14:10: reading
    - 14:10-17:10: answering

2024/25 Semester 1 – **Final** Exam

Bachelor Degree – Year 3

Continuous and Discrete Time Signals and Systems

Writing Time: 180 minutes

Reading Time: 10 minutes (no writing or annotating allowed anywhere)

INSTRUCTIONS TO CANDIDATES

a) This is a **closed-book** examination. **NO** notes or books are permitted.
b) Total marks available are 100. The number on the right indicates the mark for each question.
c) Attempt **ALL** questions. Write all the answers in the answer booklet provided.
d) Only solutions written in **English** will be accepted.
e) Correct answers do not guarantee a full score: mark penalties may be imposed for missing intermediate solution steps or illogical solution processes.
f) **No annotating is allowed in reading time or after the end of writing time.**
g) ALL communications-enabled & network accessible devices **MUST** be **switched OFF** & placed in the storage area.
h) **ALL** materials must be returned to the exam invigilators upon completion of the exam period. Failure to do so will be deemed academic misconduct and will be dealt with accordingly.

AUTHORISED MATERIALS

1. Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, and one scientific calculator.
2. Only the university approved calculator - **Casio FS82ES/83ES** can be used. Calculators are permitted in accordance with the rules of the university. They may be used for the processing of numerical solutions **ONLY**. They must not have been programmed nor should they store additional information.
3. You are **NOT** permitted to have on your desk or on your person any unauthorised materials. This includes but is not limited to laptops, tablets, mobile phones, smart watches and bands, smart glasses, cheat sheet, draft paper, and electronic dictionaries. Unauthorised material will be confiscated.

Equation List on LMO

2. Important rules and tips (from Registry)

1. Bring <u>two documents</u> for admission携带双证参加考试:
   a. XJTLU Student ID Card学生证
   b. Official Identity Verification Document 官方身份证件
      i. Mainland China: Resident Identity Card（居民身份证）;
      ii. Hong Kong, Macau, Taiwan: Mainland Travel Permit （通行证）
      iii. International: Passport (护照)

2. Use the washroom before admission check.

   <u>No washroom breaks are allowed within the first two hours and last 15 minutes of each exam</u>.

   入场前如厕。考试开始后两小时内及考试结束前15分钟不得离场如厕。

2. Important rules and tips (from Registry)

3. (Students) Arrive at least <u>**30 minutes early**</u> for admission and <u>**metal scanner**</u> check.

   至少提前**30**分钟到达考场门口进行入场检查。

4. Any unauthorized materials or misbehaviors are strictly prohibited. Violation of exam rules will result in <u>**disciplinary actions**</u> and <u>**the imposition of demerit points on transcripts**</u>.

   严禁携带任何考试违禁品及任何违纪行为。违反考试规则者将受到纪律处分，并在成绩单上记录违规积分。

3. Office hour arrangement

Week 14							
December	16 Mon.	17 Tue.	18 Wed.	19 Thu.	20 Fri.	21 Sat.	22 Sun.
10:00-12:00	Liu (SC340)	Liu (SC340)	Liu (SC340)	Liu (SC340)			
13:00-15:00		Wang (EE322)			Wang (EE322)		

Week 15							
December	23 Mon.	24 Tue.	25 Wed.	26 Thu.	27 Fri.	28 Sat.	29 Sun.
10:00-12:00	Liu (SC340)						Wang (EE322)
13:00-15:00		Wang (EE322)			Wang (EE322)		Wang (EE322)
15:00-17:00							

4. Walk through every lecture

- The following slides are the “outline” of every lecture;
- We will work through some after-class quiz on LMO for each one.

Lecture 14 DTFS and DTFT_1

- 1. Discrete-Time Fourier Series (for periodic sequences)
  - Review the concepts of eigenfunction for LTI systems
  - Definition of DTFS
  - Examples

- 2. Discrete-Time Fourier Transform (for general sequences)
  - From DTFS to DTFT
  - ☆ Definition of DTFT
  - ☆ Examples (Calculation of DTFT)
  - DTFT of periodic signals (optional)

Synthesis & Analysis Equation for DTFS and DTFT

- DTFS for a periodic discrete-time signal $x[n]$:
  - period: N; fundamental frequency: $\omega_0 = \frac{2\pi}{N}$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$ Synthesis equation of DTFS

$$a_k = \sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}$$ Analysis equation of DTFS

a_k only has *N* distinct values, and is periodic, i.e. $a_k = a_{k+N}$

- DTFT for a general aperiodic discrete-time signal $x[n]$:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$ Synthesis equation of DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ Analysis equation of DTFT

$X\left(e^{j\omega}\right)$, or $X(\omega)$ is continuous and periodic in ω (period = 2π),
i.e. $X(\omega) = X(\omega + 2\pi)$.

Quiz

Q9.

For a continuous-time periodic signal, determine the Fourier series coefficients a_k:

$x(t) = 2 + cos(\frac{2\pi}{3}t) + 4sin(\frac{5\pi}{3}t)$

a_1	Choose...	$=0$
a_{-5}	Choose...	$=2j$
a_{-2}	Choose...	$=0.5$
a_0	Choose...	$=2$
a_5	Choose...	$=-2j$
a_2	Choose...	$=0.5$

DTFS: $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$

$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\omega_0 n}$ $\left(N = \frac{2\pi}{\omega_0}\right)$

$T_0 = LCM(3, \frac{6}{5}) = 6 = N.$

fundamental freq. $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{3}.$

$x(t) = 2 + \cos(\frac{2\pi}{3}t) + 4\sin(\frac{5\pi}{3}t)$

$= 2 + \frac{1}{2}(e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}) + \frac{4}{2j}(e^{j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t})$

$= 2 + \frac{1}{2}e^{j\cdot 2\cdot\frac{\pi}{3}t} + \frac{1}{2}e^{-j2\cdot\frac{\pi}{3}t} - 2je^{j5\cdot\frac{\pi}{3}t} + 2je^{-j5\cdot\frac{\pi}{3}t}.$

$\rightarrow a_0 = 2,\ a_2 = a_{-2} = \frac{1}{2},\ a_5 = -2j,\ a_{-5} = 2j.$

$a_1 = 0.\ (a_{\pm 3}, a_{\pm 4} = 0)$

Quiz

Q8.

For the given Fourier transform, i.e.,

$$X(w) = 4 + e^{-jw} + 2e^{-j2w} + 4e^{-j4w} - e^{jw}$$

Determine x[n].

- a. x[n]={-1,4,1,2,0,4}
- b. x[n]={-4,0,-2,-1,4,1}
- c. x[n]={4,0,2,1,4,-1}
- d. x[n]={1,4,-1,-2,0,-4}

Recall: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$X(w) = \underset{x[0]}{4} \cdot e^{-j\omega \cdot 0} + \underset{x[1]}{1} \cdot e^{-j\omega \cdot 1} + \underset{x[2]}{2} \cdot e^{-j\omega \cdot 2} + \underset{x[4]}{4} e^{-j\omega \cdot 4} \underset{x[-1]}{-1} \cdot e^{-j\omega \cdot (-1)}$$

$x[n] = \{-1, \underline{4}, 1, 2, 0, 4\}$

Lecture 15 DTFT_2

- 1. Definition revisit
  - From CTFT to DTFT
  - Existence of DTFT (convergence)

☆ 2. DTFT properties (for easy calculation of DTFT)
  - Periodicity, linearity, time-reversal, shifting, differencing in TD, differentiation in FD...
  - Convolution property
  - Modulation property
  - ~~Duality~~

DTFT Properties

$$X(\omega) = X(\omega + 2\pi k) \quad \text{for any integer } k$$

$$ax_1[n] + bx_2[n] \xleftrightarrow{\text{DTFT}} aX_1(\omega) + bX_2(\omega)$$

$$x[-n] \xleftrightarrow{\text{DTFT}} X(-\omega)$$

$$x^*[n] \xleftrightarrow{\text{DTFT}} X^*(-\omega)$$

$$x[n-M] \xleftrightarrow{\text{DTFT}} e^{-j\omega M}X(\omega)$$

$$e^{j\omega_0 n}x[n] \xleftrightarrow{\text{DTFT}} X(\omega - \omega_0)$$

$$nx[n] \xleftrightarrow{\text{DTFT}} j\frac{dX(\omega)}{d\omega}$$

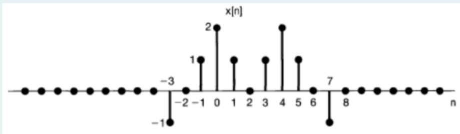
$$y[n] = x[n] * h[n] \xleftrightarrow{\text{DTFT}} X(\omega) \cdot H(\omega) \ = Y(\omega)$$

$$x[n] \cdot h[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\gamma) \cdot H(\omega - \gamma)d\gamma$$

Quiz

Q1.

Let $X(e^{j\omega})$ denote the DTFT of the signal x[n] shown in below figure.
Evaluate values of the following calculations:



① $X(e^{j\pi})$	Choose...	2
② $X(e^{j0})$	Choose...	6
③ $\measuredangle X(e^{j\omega})$	Choose...	-2ω
④ $\int_{-\pi}^{\pi} \left\lvert \frac{dX(e^{j\omega})}{d\omega} \right\rvert^2 d\omega$	Choose...	316π
⑤ $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$	Choose...	4π

DTFT: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) e^{j\omega n} d\omega$

① $X(e^{j\pi}) = \sum_n x[n] e^{-j\pi n} = \sum_n x[n] (-1)^n = 2.$

② $X(e^{j0}) = \sum_n x[n] e^{j \cdot 0} = \sum_n x[n] = 6.$

③ Recall: $x^*[n] \xleftrightarrow{DTFT} X^*(-\omega).$

for this problem. note: $x[n+2]$ is even (& real)

$\rightarrow e^{j2\omega} X(\omega)$ is real (& even) $\rightarrow \measuredangle e^{j2\omega} X(\omega) = 0$

$\rightarrow \measuredangle e^{j2\omega} + \measuredangle X(\omega) = 0 \rightarrow \measuredangle X(\omega) = -\measuredangle e^{j2\omega} = -2\omega.$

④ $n x[n] \longleftrightarrow j \frac{dX(e^{j\omega})}{d\omega}$

parseval's theorem:

$\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_n |x[n]|^2$

$\rightarrow \int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = 2\pi \cdot \sum_n |n|^2 |x[n]|^2$

$= 316\pi.$

⑤ from inverse DTFT:

$\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega \cdot 0} d\omega$

$= 2\pi \cdot x[0] = 4\pi.$

Lecture 16 DTFT_3

☆ 1. Commonly used DTFT pairs

– Use DTFT properties when needed (Equation list provided)

★ 2. Inverse DTFT

– Definition and calculation

★ Partial Fraction Expansion

• 3. DTFT in LTID Systems

★ Relationship between impulse response $h[n]$, LCCDE and frequency response (or transfer function, system function) $H(\omega)$

– Magnitude ~~and phase~~ spectrum

Quiz

Q5.

Determine the numerical value of

$A = \sum_0^\infty n(\frac{1}{2})^n$

Hints: use $X(e^{j0}) = \sum_{-\infty}^{\infty} x[n]$

- a. -1
- b. 1
- c. 0
- d. 2 ✓

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{df(x)}{dx}g(x) - f(x)\frac{dg(x)}{dx}}{g^2(x)}$$

Recall: $a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$, $|a|<1$.

$n x[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$.

$(\frac{1}{2})^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

$n(\frac{1}{2})^n u[n] \longleftrightarrow j\frac{(-1)(-\frac{1}{2})(-j)e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2} = \frac{\frac{1}{2}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2}$

Recall: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$\rightarrow \sum_{-\infty}^{\infty} x[n] = \sum_{-\infty}^{\infty} x[n]e^{-j0\cdot n} = X(e^{j0})$

$A = \left.\frac{\frac{1}{2}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2}\right|_{\omega=0} = \frac{1/2}{1/4} = 2.$

Inverse DTFT by PFE

- Most frequently used pair:

$$a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1 - ae^{-j\omega}}, \qquad |a| < 1$$

- Consider a rational transform in the form

$$X(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{(1 - \alpha_1 e^{-j\omega})(1 - \alpha_2 e^{-j\omega})\ldots(1 - \alpha_N e^{-j\omega})}$$

  – where the poles $\alpha_1, \alpha_2, \ldots, \alpha_N$ are distinct.
  – the order of the numerator polynomial of $e^{-j\omega}$ is less than the order of the denominator polynomial.

- The transform $X(e^{j\omega})$ can be expanded into partial fractions in the form

$$X(e^{j\omega}) = \frac{k_1}{1 - \alpha_1 e^{-j\omega}} + \frac{k_2}{1 - \alpha_2 e^{-j\omega}} + \ldots + \frac{k_N}{1 - \alpha_N e^{-j\omega}}$$

  – the coefficients $k_1, k_2, \ldots, k_N$ can be computed by

$$k_i = (1 - \alpha_i e^{-j\omega})\, X(e^{j\omega})\Big|_{e^{j\omega} = \alpha_i} \qquad i = 1, 2, \ldots, N$$

- $x[n]$ can be obtained:

$$x[n] = k_1 \cdot \alpha_1{}^n u[n] + k_2 \cdot \alpha_2{}^n u[n] + \cdots + k_N \cdot \alpha_N{}^n u[n]$$

Quiz

Q9.

The input-output relationship of an LTID system is given by the following LCCDE:

$$y[n+2] - \frac{3}{4}y[n+1] + \frac{1}{8}y[n] = 2x[n+2]$$

Find the impulse response of the system.

- a. $h[n] = 4(\frac{1}{2})^n u[n] + 2(\frac{1}{4})^n u[-n]$
- ✓ b. $h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$
- c. $h[n] = 4(\frac{1}{2})^n u[n] + 2(\frac{1}{4})^n u[n]$
- d. $h[n] = 4(\frac{-1}{2})^n u[n] - 2(\frac{-1}{4})^n u[-n]$

Apply DTFT:

$$e^{2j\omega}Y(\omega) - \frac{3}{4}e^{j\omega}Y(\omega) + \frac{1}{8}Y(\omega) = 2e^{2j\omega}X(\omega)$$

$$\rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2e^{2j\omega}}{e^{2j\omega} - \frac{3}{4}e^{j\omega} + \frac{1}{8}}$$

$$= \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{k_1}{1-\frac{1}{2}e^{-j\omega}} + \frac{k_2}{1-\frac{1}{4}e^{-j\omega}}$$

Recall PFE: $X(e^{j\omega}) = \frac{k_1}{1-\alpha_1 e^{-j\omega}} + \frac{k_2}{1-\alpha_2 e^{-j\omega}} + \cdots + \frac{k_N}{1-\alpha_N e^{-j\omega}}$

$k_i = (1-\alpha_i e^{-j\omega})X(e^{j\omega})\big|_{e^{j\omega}=\alpha_i}$. $i = 1 \cdots N$.

use PFE: $k_1 = 4$, $k_2 = -2$.

$$h[n] = DTFT^{-1}\{H(\omega)\}$$

$$= 4\cdot(\tfrac{1}{2})^n u[n] - 2(\tfrac{1}{4})^n u[n].$$

Recall: $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$. $|a|<1$

Lecture 17 Filtering

- 1. Continuous-Time (analogue) vs. Discrete-Time (digital) filtering
  ☆ Frequency mapping (analogue frequency vs. digital frequency)

☆ 2. Ideal and Practical filters
  – Four types: lowpass, highpass, bandpass and bandstop

- 3. CT Filters examples
  – Simple electrical circuit filters
  – Butterworth filters
- 4. DT Filters examples
  – Recursive and non-recursive filters
  – FIR vs. IIR filters.

Analogue and Digital Filters

- Continuous-Time (Analogue) filters:

$$x(t) \longrightarrow \boxed{\text{LTIC filter}} \longrightarrow y(t)$$

- Discrete-Time (Digital) filters:

$$x(t) \longrightarrow \boxed{\text{Sampling}} \xrightarrow{x[n]} \boxed{\text{LTID filter}} \xrightarrow{y[n]} \boxed{\text{Recontruction}} \longrightarrow y(t)$$

Note: in discussing discrete-time signals and systems, Ω is used to denote the analog frequency of the Continuous-Time Fourier Transform (CTFT), and ω is used to denote the digital frequency of the Discrete-Time Fourier Transform (DTFT) for clarity.

- Frequency Mapping:

$$F_s = \frac{1}{T_s};\ \Omega_s = \frac{2\pi}{T_s} = 2\pi F_s.$$

$$\frac{\Omega}{\Omega_s} = \frac{\omega}{2\pi},\ \text{or}\ \omega = \Omega T_s;\ \text{or}\ \frac{F}{F_s} = \frac{\omega}{2\pi},\ \text{or}\ \omega = 2\pi F/F_s = 2\pi F \cdot T_s$$

- T_s: sampling interval or sampling period (sec, or sec/sample);
- Ω: analogue angular frequency (rad/s); Ω_s: angular sampling frequency (rad/s);
- F: analogue linear frequency (Hz); F_s: linear sampling frequency (Hz);
- ω: angular digital frequency (rad/sample).

Quiz

Q10.

Consider removing noise from some continuous-time signal, the cuffoff frequency of an analog filter is 500π. Assume we perform filtering using digital filter. Assume the sampling frequency is 1000Hz. Find the corresponding cuffoff frequency of the digital filter.

- a. 500Hz
- b. 0.5π ✓
- c. 5π
- d. π

Recall: $\omega = \Omega T_s$.

$$\Omega_c = 500\pi.$$

$$f_s = 1000HZ \rightarrow T_s = \frac{1}{f_s} = 0.001S.$$

$$\omega_c = \Omega_c \cdot T_s = 0.5\pi.$$

Filter Types

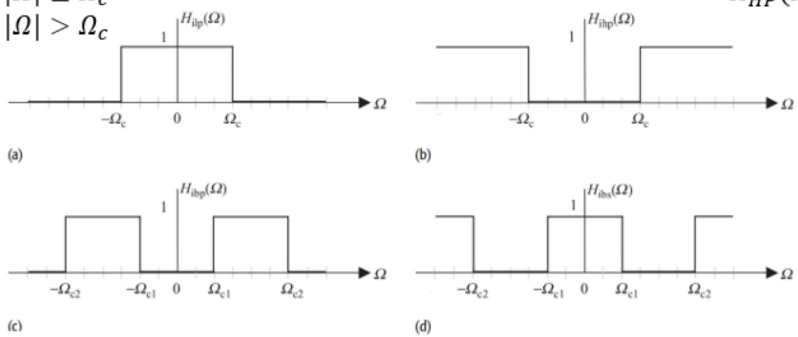
- Four types of CT filters

Low-pass

$$H_{LP}(\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

High-pass

$$H_{HP}(\Omega) = \begin{cases} 1, & |\Omega| \ge \Omega_c \\ 0, & |\Omega| < \Omega_c \end{cases}$$



CT: focus on $[0, \infty)$

Band-pass

$$H_{BP}(\Omega) = \begin{cases} 1, & \Omega_{c1} \le |\Omega| \le \Omega_{c2} \\ 0, & \text{others} \end{cases}$$

Band-stop

$$H_{BS}(\Omega) = \begin{cases} 1, & \text{others} \\ 0, & \Omega_{c1} \le |\Omega| \le \Omega_{c2} \end{cases}$$

- Four types of DT filters

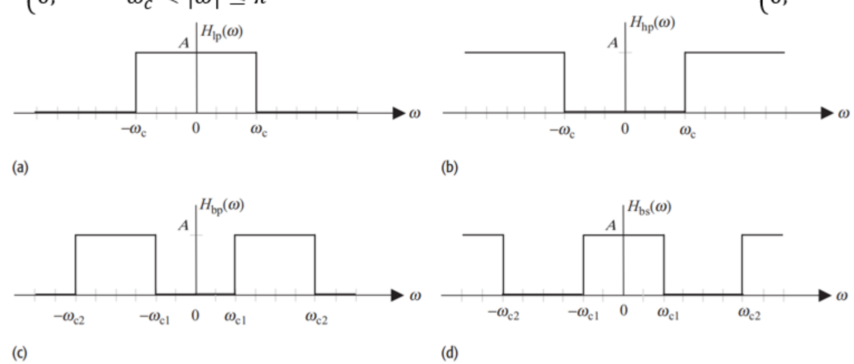
Low-pass

$$H_{LP}(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

High-pass

$$H_{HP}(\omega) = \begin{cases} 1, & \omega_c \le |\omega| < \pi \\ 0, & |\omega| < \omega_c \end{cases}$$

DT: focus on $[0, \pi)$



Band-pass

$$H_{BP}(\omega) = \begin{cases} 1, & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & \text{others} \end{cases}$$

Band-stop

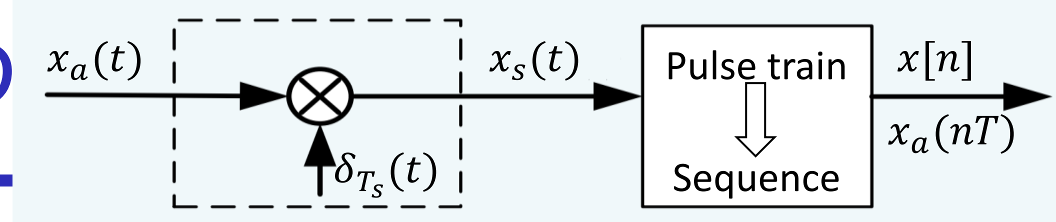
$$H_{BS}(\omega) = \begin{cases} 1, & \text{others} \\ 0, & \omega_{c1} \le |\omega| \le \omega_{c2} \end{cases}$$

Lecture 18 Sampling

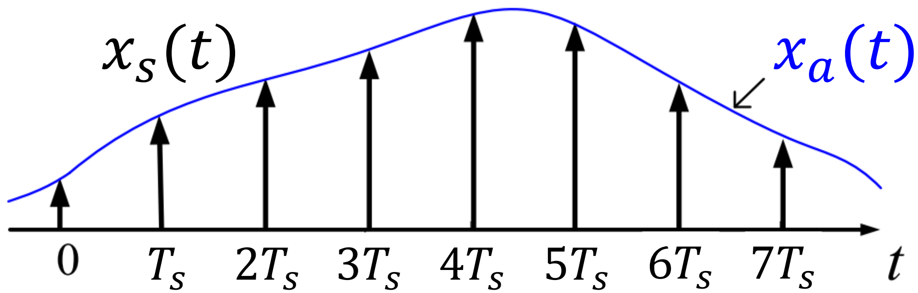
- 1. Sampling
  - 1.0. What and Why?
  - 1.1. Sampling in Time domain (TD)
  - 1.2. Sampling in Frequency domain (FD)
  - 1.3. Nyquist Theorem

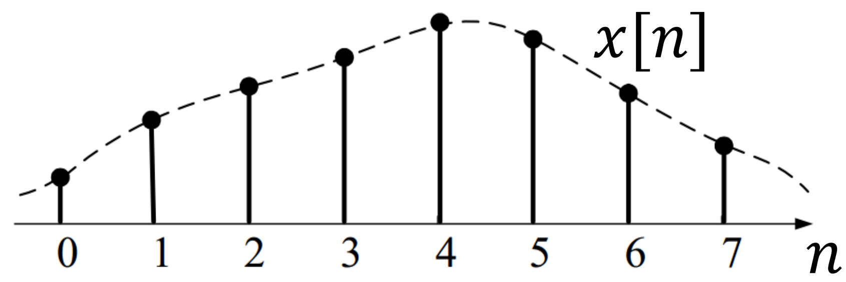
- 2. Reconstruction
  - 2.1 Interpolation
  - 2.2 Reconstruction theory
  - 2.3 Reconstruction in FD – filtering
  - 2.4 Reconstruction in TD – interpolation
  - 2.5 Realisation

Sampling in TD and FD

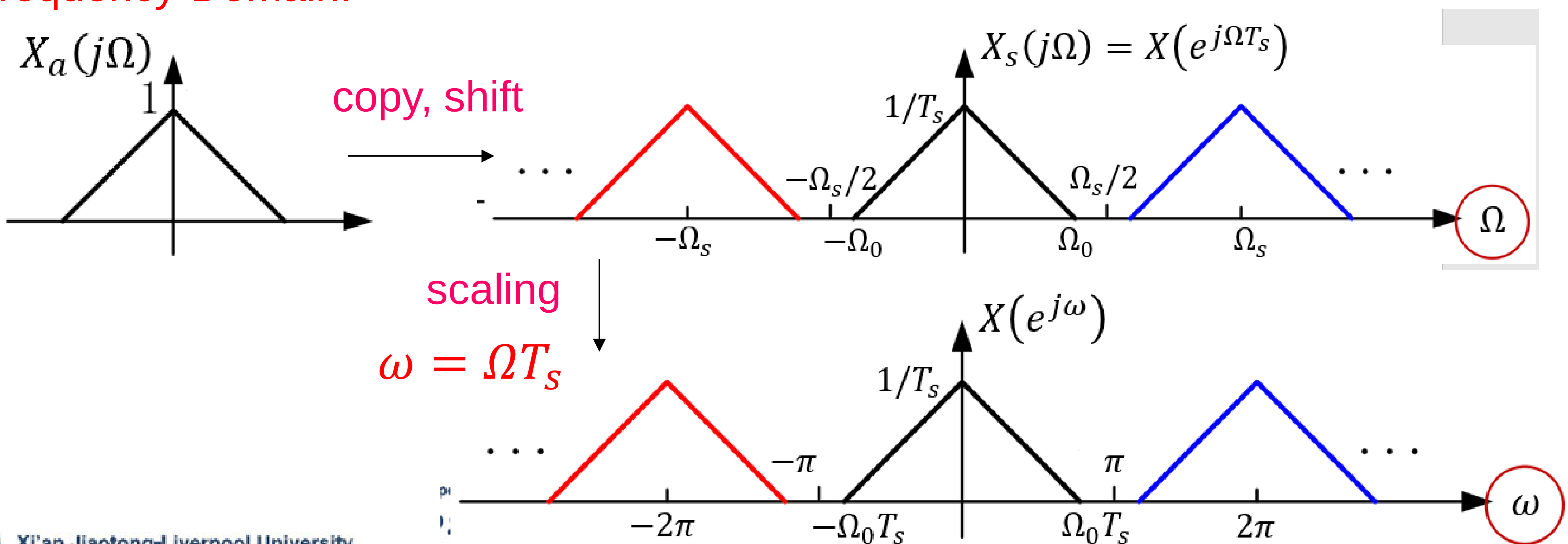


Time-Domain:





Frequency-Domain:



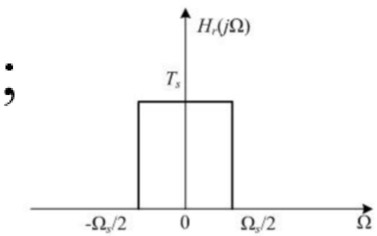
Nyquist-Shannon Theorem and Ideal Reconstruction

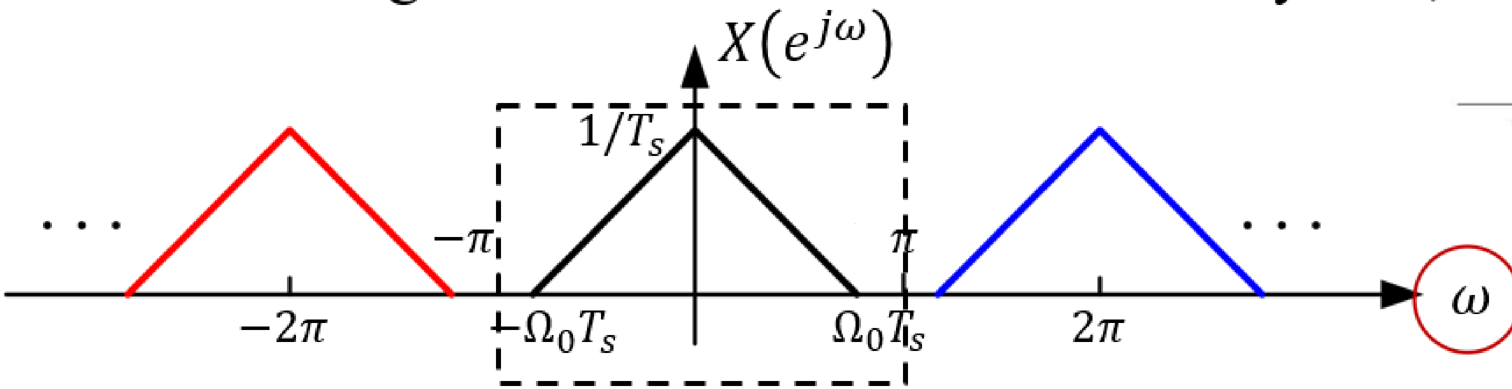
- The Nyquist-Shannon theorem!

$$\Omega_0 T_s \le \pi \iff \Omega_s \ge 2\Omega_0$$

where Ω_0 is the maximum (highest) frequency in analogue signal.

  – So the CT signal can be recovered without any loss;





$$\Omega = \omega / T_s$$

$$X_a(j\Omega) \longrightarrow x_a(t)$$

Quiz

Q2.

Let x(t) be a signal with Nyquist rate ω_0 , determine the Nyquist rate for each of the following signals:

a) $x(t)+x(t-1)$

b) $\frac{dx(t)}{dt}$

c) $x(t)cos(\omega_0 t)$

- a. ω_0 , ω_0 , $3\omega_0$ ✓
- b. ω_0 , $2\omega_0$, $3\omega_0$
- c. ω_0 , 0.5 ω_0 , 2 ω_0
- d. $2\omega_0$, ω_0 , ω_0

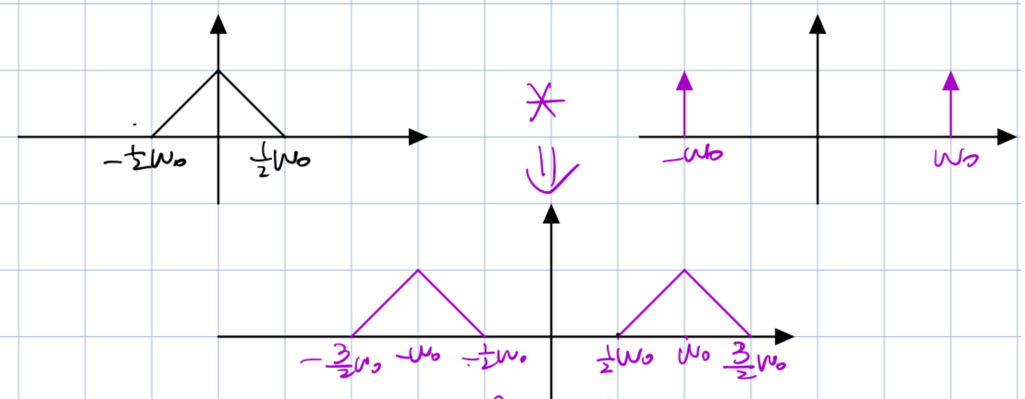
a). $X(\Omega) + e^{j\Omega} X(\Omega) \rightarrow$ no change in bandwidth.

$\rightarrow \omega_0$.

b). $\frac{dx}{dt} \rightarrow j\Omega X(\Omega) \rightarrow$ no change in bandwidth

$\rightarrow \omega_0$

c) $x(t)\cos(\omega_0 t) \rightarrow X(\Omega) * \left(\frac{1}{2} e^{j\omega_0 t} - \frac{1}{2} e^{-j\omega_0 t}\right)$



$\rightarrow f_{max} = \frac{3}{2}\omega_0 \Rightarrow$ N. rate $= 3\omega_0$.

Quiz

Determine the Nyquist rate corresponding to each of the following signals:

a) $x(t) = 1 + cos(2000\pi t) + sin(4000\pi t)$

b) $x(t) = (\frac{sin(4000\pi t)}{\pi t})^2$

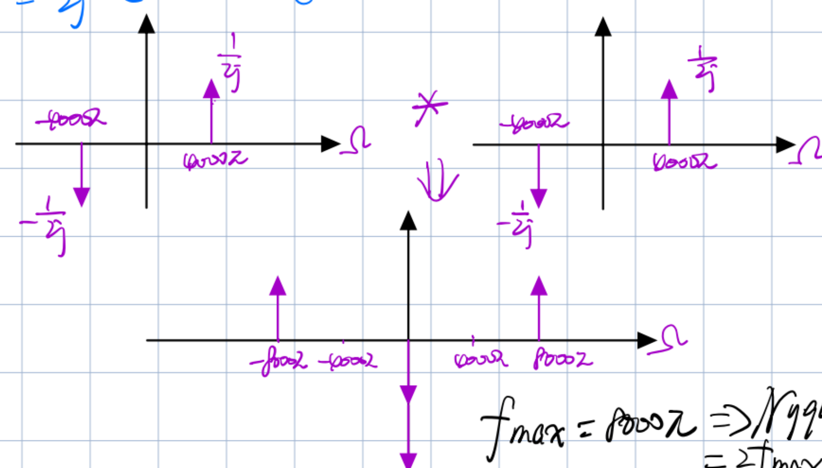
- a. $8000\pi, 4000\pi$
- b. $4000\pi, 16000\pi$
- c. $8000\pi, 16000\pi$ ✓
- d. $8000\pi, 8000\pi$

a) $f_{max} = 4000\pi \rightarrow$ Nyquist rate $= 2f_{max}$
$= 8000\pi$ (rad/s).

b) $x(t) = \underbrace{\left(\frac{1}{\pi t}\right)^2}_{\text{constant}} \sin(4000\pi t)\cdot\sin(4000\pi t)$

multiplication in time-domain
↳ Convolution in frequency-domain

$\sin 4000\pi t = \frac{1}{2j}\left(e^{j4000\pi t} - e^{-j4000\pi t}\right)$



$f_{max} = 8000\pi \Rightarrow$ Nyquist rate
$= 2f_{max} = 16000\pi$.

Quiz

Q5.

Find the Nyquist rate and Nyquist interval for the signal f(t) = -10 sin 40πt cos 300πt.

a. 260Hz, 1/260s

b. 340Hz, 1/340s

c. 260Hz, 1/340s

d. 340Hz, 1/260s

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cos\alpha = \frac{1}{2}(e^{j\alpha} + e^{-j\alpha})$

$\sin\alpha = \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha})$

$f(t) = 10\sin 40\pi t\cos 300\pi t$

$= 10[\sin 340\pi t - \sin 260\pi t)$

$\Omega_{max} = 340\pi \rightarrow \Omega_s = 2\Omega_{max} = 680\pi$

$T_s = \frac{2\pi}{\Omega_s} = \frac{1}{340}s$; $f_s = \frac{1}{T_s} = 340Hz$,

Reconstruction:

Lecture 19 Z-Transform_1

- 1. Definition of z-transform
  - Eigenfunctions
  - ☆ Relationship between DTFT and z-transform
  - Visualisation of DTFT and z-transform
  - s-plane to z-plane
- ☆ 2. Region of Convergence (ROC)
  - Definition and graphical depiction
  - ☆ Zeros and Poles (Zero-pole plot)
  - ☆ ROC properties
- ☆ 3. Commonly use z-transform pairs (Equation list provided)

z-Transform vs. DTFT

- Definition equation of DTFT:

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n] z^{-n}\Bigg|_{z=e^{j\omega}} = H(z)\big|_{z=e^{j\omega}}$$

DTFT is the z-transform of h[n] evaluated on the unit circle

- Definition equation of z-transform ($z = re^{j\omega}$):

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=-\infty}^{\infty} h[n] r^{-n} e^{-j\omega n} = DTFT\{h[n] r^{-n}\}$$

z-transform is the DTFT of r^{-n}-scaled $h[n]$

ROC and Properties

- For a given sequence, the set R of values of z for which its z-transform converges is called the region of convergence (ROC).
  - annular region of the z-plane

    $$R^- < |z| < R^+, \qquad \text{where } 0 \leq R^- < R^+ \leq \infty$$

  - The z-transform must always be specified with its ROC !
- DTFT exists ↔ ROC include |z|=1 ↔ stable
- Right-sided sequence ↔ ROC is outside the outermost pole circle;

  - causal (right sided and equal to 0 for n < 0): ROC =includes infinity $(X(\infty) < \infty) \leftrightarrow$ O(N(z)) ≤ O(D(z));
- Left-sided sequence ↔ ROC is inside the innermost pole circle;

  - anti-causal (left sided and equal to 0 for n > 0): ROC also includes 0 $(X(0) < \infty) \leftrightarrow$ O(N(z)) > O(D(z));
- Double-sided sequence ↔ ROC will be the intersection of the two ROC areas, i.e., a ring shape.

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Lecture 20 Z-Transform_2

☆ 4. Properties of z-transform

- Linearity, time-shifting, time-reversal, time-scaling, z-domain scaling, z-domain differentiation, time-difference, time-accumulation, conjugation, time convolution.
- Comparing with DTFT and Laplace transform

★ 5. Inverse z-Transform

- Table Look-up

☆ Long Division (Power series expansion)

★ Partial Fraction Expansion

  - ROC determination

Z-Transform Properties

$$\alpha x_1[n] + \beta x_2[n] \overset{z}{\longleftrightarrow} \alpha X_1(z) + \beta X_2(z)$$

$$x[n-k] \overset{z}{\longleftrightarrow} z^{-k} X(z)$$

$$x[-n] \overset{z}{\longleftrightarrow} X(z^{-1})$$

$$a^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{a}\right)$$

$$nx[n] \overset{z}{\longleftrightarrow} (-z)\frac{dX(z)}{dz}$$

$$x^*[n] \overset{z}{\longleftrightarrow} X^*(z^*)$$

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z)$$

Always consider ROC!

Inverse z-Transform

- Long-division:
  - When the transform X(z) is expanded as follows:
    $$X(z) = a + bz^{-1} + cz^{-2} + dz^{-3} + \ldots$$
  - The corresponding time-domain sequence should be:
    $$x[n] = a\delta[n] + b\delta[n-1] + c\delta[n-2] + d\delta[n-3] + \ldots$$
- PFE:

Consider a proper fraction transform $X(z)$ given with its denominator factored out as

$$X(z) = \frac{N(z)}{(z-z_1)(z-z_2)\ldots(z-z_N)}$$

Expanding the transform into partial fractions in the form

$$X(z) = \frac{k_1 z}{z-z_1} + \frac{k_2 z}{z-z_2} + \ldots + \frac{k_N z}{z-z_N} \text{ or equivalently, } \frac{X(z)}{z} = \frac{k_1}{z-z_1} + \frac{k_2}{z-z_2} + \ldots + \frac{k_N}{z-z_N}$$

Consider simple poles, the coefficients k_i could be obtained from:

$$k_i = \left[(z-z_i)\frac{X(z)}{z}\right]_{z=z_i}$$

Each one has its inverse transform like:

$$x_i[n] = \mathcal{Z}^{-1}\{X_i(z)\} = \mathcal{Z}^{-1}\left\{\frac{k_i z}{z-z_i}\right\} = \begin{cases} a^n u[n], & ROC: |z| > |a| \\ -a^n u[-n-1], & ROC: |z| < |a| \end{cases}$$

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Quiz

Assume $X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}$, ROC is $\frac{1}{3} < |z| < \frac{1}{2}$. Determine $x[n]$. Does DTFT of $x[n]$ exist?

a. $x[n] = \left(\frac{1}{3}\right)^n u[n] - (2)^n u[-n-1]$; not exist

b. $x[n] = -\left(\frac{1}{3}\right)^n u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$; exist

c. $x[n] = \left(\frac{1}{3}\right)^n u[n] - (2)^n u[-n-1]$; exist

d. $x[n] = -\left(\frac{1}{3}\right)^n u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$; not exist

e. $x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$; exist

f. $x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$; not exist

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a}, \quad \text{ROC: } |z| > |a|$$
$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a}, \quad \text{ROC: } |z| < |a|$$

Solutions:

$p_1 = \frac{1}{3}$, ROC outside corresponding pole circle → right-sided sequence→ $\left(\frac{1}{3}\right)^n u[n]$

$p_2 = \frac{1}{2}$, ROC inside corresponding pole circle → left-sided sequence→ $-\left(\frac{1}{2}\right)^n u[-n-1]$

So $x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$.

ROC doesn't include unit circle → DTFT does not exist.

Quiz

Use partial fraction expansion, determine which of the following time sequence is corresponding to the below Z transform:

$F(z) = \frac{z^2+z}{(z^2-z+1)(z-1)}, |z| > 1$

- a. $f[n] = (2 - 2cos\frac{\pi}{3}n)u[n]$ ✓
- b. $f[n] = (2 + 2cos\frac{\pi}{3}n)u[n]$
- c. $f[n] = (2 + 2sin\frac{\pi}{3}n)u[n]$
- d. $f[n] = (2 - 2sin\frac{\pi}{3}n)u[n]$

$z^2 - z + 1 = 0 \longrightarrow z_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}j$

$F(z) = \frac{z^2+z}{(z-\frac{1}{2}+\frac{\sqrt{3}}{2}j)(z-\frac{1}{2}-\frac{\sqrt{3}}{2}j)(z-1)}$

$= \frac{k_1 z}{z-\frac{1}{2}+\frac{\sqrt{3}}{2}j} + \frac{k_2 z}{z-\frac{1}{2}-\frac{\sqrt{3}}{2}j} + \frac{k_3 z}{z-1}$

use PFE: $k_i = \frac{F(z)}{z}(z-p_i)\Big|_{z=p_i}$

$k_1 = -1, \quad k_2 = -1, \quad k_3 = 2.$

$\frac{1}{2} \pm \frac{\sqrt{3}}{2}j = e^{\pm j\pi/3}$

$\rightarrow f[n] = \{-(e^{j\pi/3})^n - (e^{-j\pi/3})^n + 2\}u[n]$

$e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n} = 2\cos\frac{\pi}{3}n.$

$\rightarrow f[n] = (2 - 2\cos\frac{\pi}{3}n)u[n].$

Lecture 21 Z-Transform_3

- 6. Analysis of LTID systems using z-transform
  - Impulse response $h[n]$, LCCDE and system transfer function $H(z)$
  - Zero-pole plot
  - Causality and stability determination from ROC
  - Geometric Evaluation of DTFT based on zero-pole locations
  - System behavior
- 7. Block diagram representation
  - Direct form I and II (canonic form)
  - Cascade and parallel form
- 8. ~~Unilateral z-transform (optional)~~

LTID System Analysis

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad\qquad X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

- LCCDE (Linear Constant Coefficient Difference Equation):

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Big\updownarrow \ \text{z-transform}$$

- Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Impulse response

$$\text{z-transform} \ \Big\updownarrow$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Quiz

A causal LTI system has transfer function

$F(z) = \frac{2z^2+z}{z^2-1.5z+0.5}$

Determine its impulse response.

- a. $f[n] = -6u[-n-1] - 4(0.5)^n u[n]$
- b. $f[n] = 6u[n] + 4(0.5)^n u[-n-1]$
- c. $f[n] = 6u[n] - 4(0.5)^n u[n]$
- d. $f[n] = 6u[n] + 4(0.5)^n u[n]$

LCCDE?

Use PFE.

$$F(z) = \frac{k_1 z}{z-p_1} + \frac{k_2 z}{z-p_2} + \cdots + \frac{k_N z}{z-p_N}$$

$$k_i = \frac{F(z)}{z}(z-p_i)\Big|_{z=p_i}$$

$$F(z) = \frac{k_1 z}{z-1} + \frac{k_2 z}{z-0.5}$$

$$k_1 = \frac{2z+1}{z-0.5}\Big|_{z=1} = 6,$$

$$k_2 = \frac{2z+1}{z-1}\Big|_{z=0.5} = -4.$$

Recall: $a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} = \frac{z}{z-a}$. $|z|>|a|$ (causal)

$$f[n] = 6u[n] - 4(0.5)^n u[n].$$

anticausal case.

$$-a^n u[-n-1] \longleftrightarrow \frac{1}{1-az^{-1}}. \quad |z|<|a|$$

LTID System Properties

- Causal: $h[n] = 0 \quad \text{for} \quad n < 0$
  - (a) the ROC is the exterior of a circle outside the outermost pole;
  - (b) with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator; or equivalently, ROC includes infinity.

- Stable ↔ impulse response being absolutely summable:
  - ROC include unit circle |z|=1

- Causal & Stable:
  - all of the poles of $H(z)$ lie inside the unit circle, i.e., they must all have magnitude smaller than 1.

Quiz

Suppose the Z transform of x[n] is

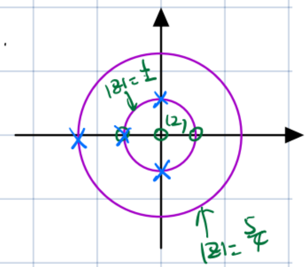
$$X(z) = \frac{1-\frac{1}{4}z^{-2}}{(1+\frac{1}{4}z^{-2})(1+\frac{7}{4}z^{-1}+\frac{5}{8}z^{-2})} = \frac{z^2(z^2-\frac{1}{4})}{(z^2+\frac{1}{4})(z+\frac{1}{2})(z+\frac{5}{4})}.$$

There are possibly three ROCs could be correspond to X(z). Determine causality and stability of x[n] for each case.

b) $\frac{1}{2} < |z| < \frac{5}{4}$ Choose... nC, S

c) $\frac{5}{4} < |z| < \infty$ Choose... C, nS.

a) $0 \le |z| < \frac{1}{2}$ Choose... nC, nS.



Note:
in option c), although infinity is not explicitly included in ROC, for this specific X(z), note the order of numerator and order of denominator are equal (both = 4), so ROC also includes infinity. → causal.

Recall: $H(z) = \frac{N(z)}{D(z)}$ → polynomial of z.

- causal ⟺ ROC { 1. outside of outermost pole circle. 2. includes ∞. (or. order of $N(z) \le$ order of $D(z)$)

- stable ⟺ $\sum_{-\infty}^{\infty} |h[n]| < \infty$ ⟺ DTFT exists ⟺ ROC include $|z|=1$.

- causal & stable ⟺ all poles inside $|z|=1$ ($|P_i| < 1$ ∀ all poles).

Zero-pole plot & Geometric Evaluation of DTFT

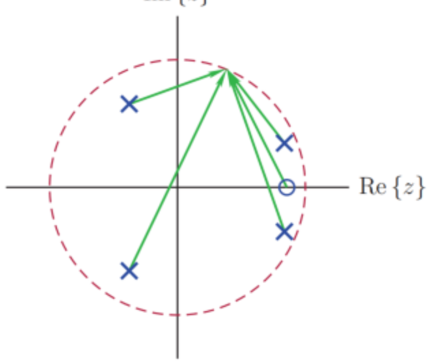
- Consider a more general system function in the form:

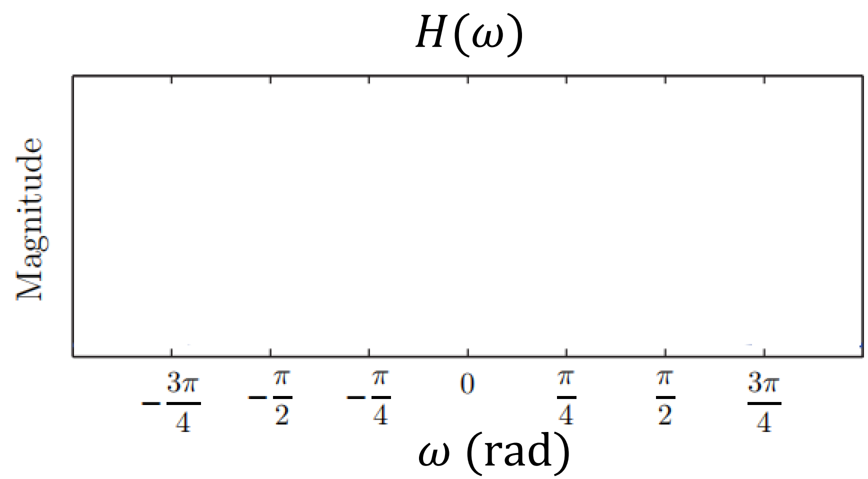
$$H(z) = K\frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$$

  – M zeros and N poles;

  – The magnitude of the system function is:

$$\left|\overrightarrow{H(z_a)}\right| = K\frac{|\overrightarrow{z_a - z_1}||\overrightarrow{z_a - z_2}|\ldots|\overrightarrow{z_a - z_M}|}{|\overrightarrow{z_a - p_1}||\overrightarrow{z_a - p_2}|\ldots|\overrightarrow{z_a - p_N}|}$$



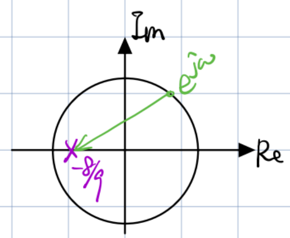


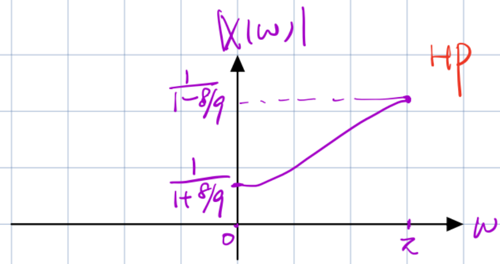
Quiz

By considering the geometric interpretation of the magnitude of the Fourier transform from the pole-zero plot, determine for each of the following z transforms, whether the corresponding signal has an approximately highpass, lowpass, bandpass, or bandstop characteristic?

b) $X(z) = \frac{1+\frac{8}{9}z^{-1}}{1-\frac{16}{9}z^{-1}+\frac{64}{81}z^{-2}}, \|z\| > \frac{8}{9}$	Choose...	lowpass
c) $X(z) = \frac{1}{1+\frac{64}{81}z^{-2}}, \|z\| > \frac{8}{9}$	Choose...	bandpass
d) $X(z) = 1 + \frac{64}{81}z^{-2}, \|z\| > 0$	Choose...	bandstop
a) $X(z) = \frac{z^{-1}}{1+\frac{8}{9}z^{-1}}, \|z\| > \frac{8}{9}$	Choose...	highpass

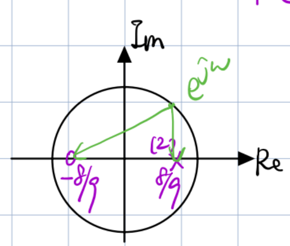
a) $X(z) = \frac{1}{z + 8/9}$. $|X(\omega)| = \frac{1}{|e^{j\omega} + 8/9|}$

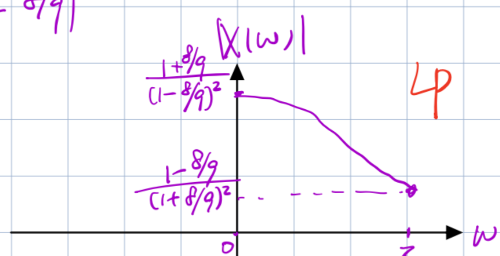




b) $X(z) = \frac{z + 8/9}{z^2 - \frac{16}{9}z + \frac{64}{81}} = \frac{z + 8/9}{(z - 8/9)^2}$

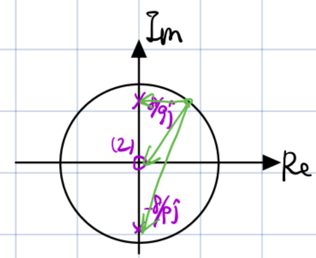
$|X(\omega)| = \frac{|e^{j\omega} + 8/9|}{|e^{j\omega} - 8/9|^2}$

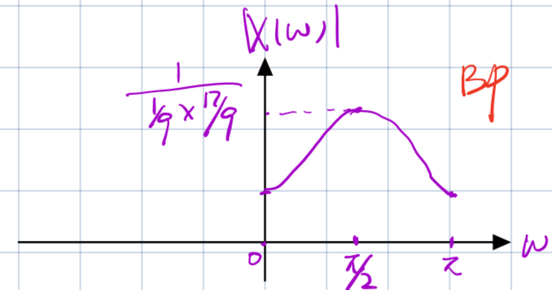




c) $X(z) = \frac{z^2}{z^2 + 64/81} = \frac{z^2}{(z + 8/9j)(z - 8/9j)}$

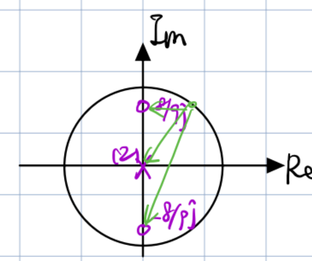
$|X(\omega)| = \frac{|e^{j\omega}|^2 = 1}{|e^{j\omega} + 8/9j||e^{j\omega} - 8/9j|}$

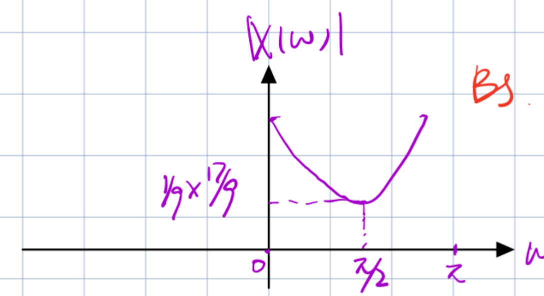




d) $X(z) = \frac{z^2 + 64/81}{z^2} = \frac{(z + 8/9j)(z - 8/9j)}{z^2}$

$|X(\omega)| = \frac{|e^{j\omega} + 8/9j||e^{j\omega} - 8/9j|}{|e^{j\omega}|^2 = 1}$



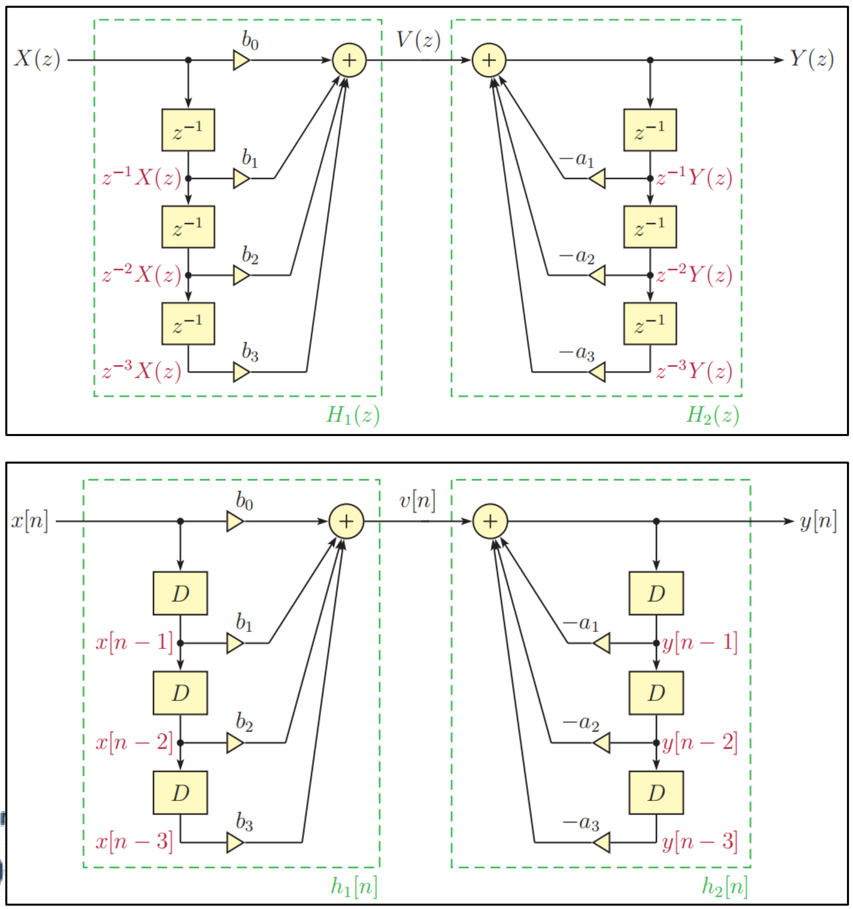


Implementation structures for LTID system

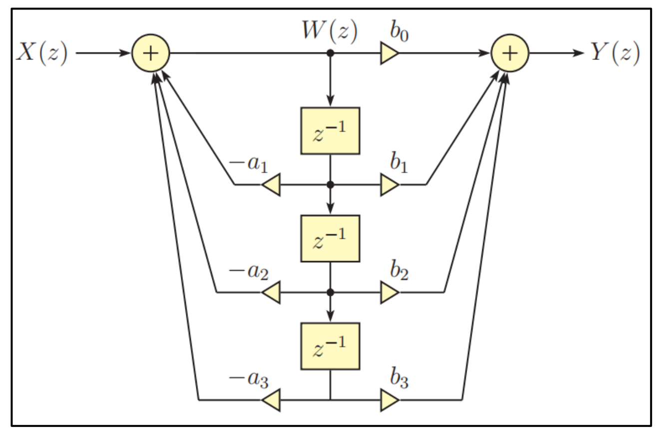
- Take a 3rd-order LTID system as example:

$$H(Z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

Direct Form I

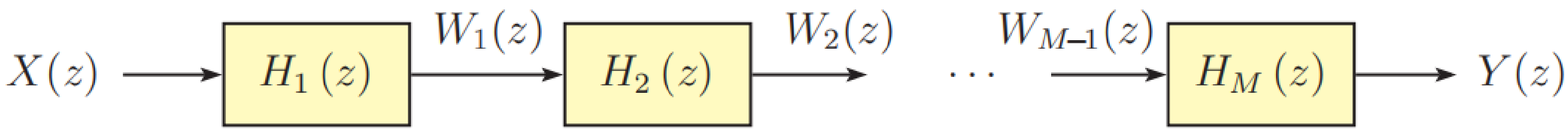


Direct Form II
(canonic form)



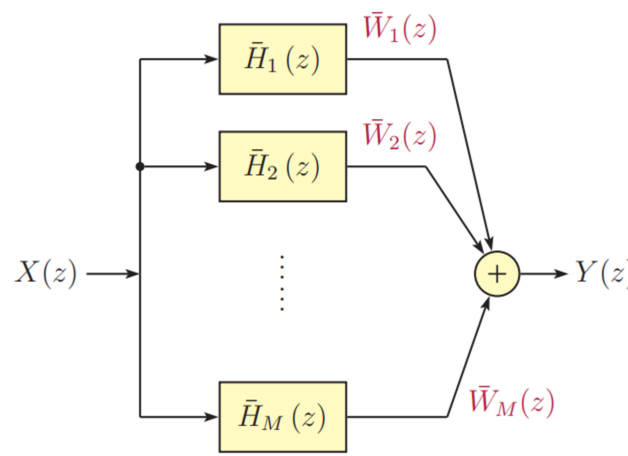
Cascade Form

$$H(Z) = H_1(z)\, H_2(z) \ldots H_M(z)$$



Parallel Form

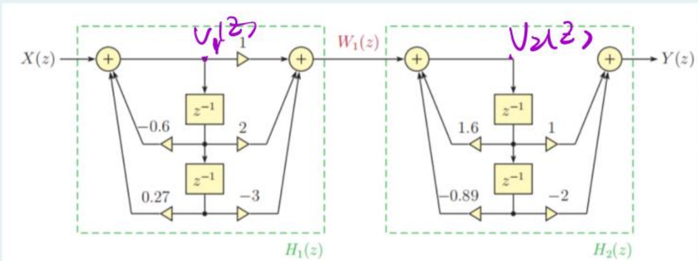
$$H(Z) = \widetilde{H}_1(z) + \widetilde{H}_2(z) + \ldots + \widetilde{H}_M(z)$$



- The sub-systems could be any order, but usually are 2nd order.
  - Especially for conjugate pole and zero pairs, they are often combined to be 2nd order sub-systems.

Quiz

Obtain transform function of the system described by the following cascade form block diagram:



- a. $H(z) = \frac{(z+3)(z-1)(z-2)}{(z-0.9)(z+0.3)(z+0.8-j0.5)(z+0.8+j0.5)}$
- ✓ b. $H(z) = \frac{z^2+2z-3}{z^2+0.6z-0.27} \cdot \frac{z-2}{z^2-1.6z+0.89}$
- ✓ c. $H(z) = \frac{(z+3)(z-1)(z-2)}{(z+0.9)(z-0.3)(z-0.8-j0.5)(z-0.8+j0.5)}$
- d. $H(z) = \frac{z^2+2z-3}{z^2+0.6z-0.27} + \frac{z-2}{z^2-1.6z+0.89}$

$H_1(z)$: $V_1(z) = X(z) - 0.6z^{-1}V_1(z) + 0.27z^{-2}V_1(z)$

$W_1(z) = V_1(z) + 2z^{-1}V_1(z) - 3z^{-2}V_1(z)$

$$\Rightarrow H_1(z) = \frac{W_1(z)}{X(z)} = \frac{W_1(z)}{V_1(z)} \cdot \frac{V_1(z)}{X(z)} = \frac{1+2z^{-1}-3z^{-2}}{1+0.6z^{-1}-0.27z^{-2}}.$$

$H_2(z)$: $V_2(z) = W_1(z) + 1.6z^{-1}V_2(z) - 0.89z^{-2}V_2(z)$

$Y(z) = z^{-1}V_2(z) - 2z^{-2}V_2(z)$

$$\Rightarrow H_2(z) = \frac{Y(z)}{W_1(z)} = \frac{Y(z)}{V_2(z)} \cdot \frac{V_2(z)}{W_1(z)} = \frac{1-2z^{-2}}{1-1.6z^{-1}+0.89z^{-2}}$$

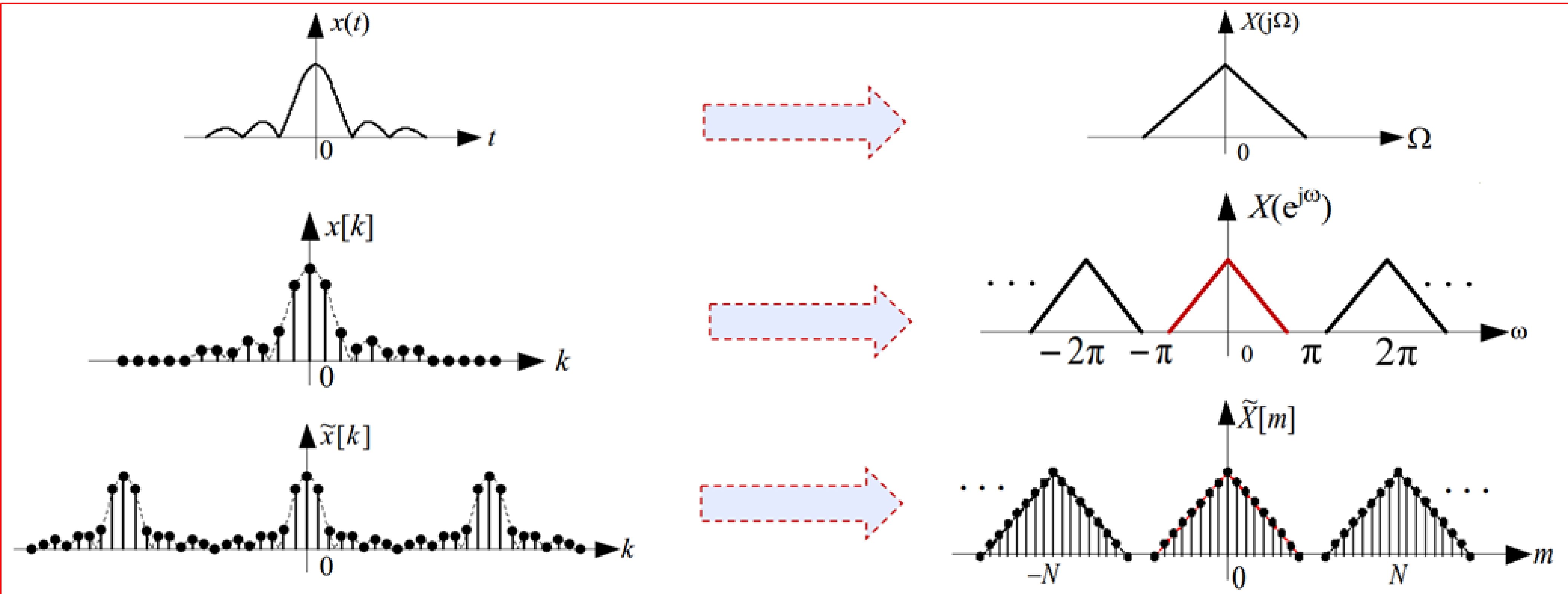
$H(z) = H_1(z) \cdot H_2(z)$ → coverts to function 对 z.

Lecture 22 DFT

- 1. Definition of DFT
  - 1.1 DFT definition
  - 1.2 DFT: synthesis and analysis equations
  - 1.3 Relationships among CTFT, DTFT and DFT
- 2. Computation of DFT
  - 2.1 Computing DFT based on twiddle factor
- 3. DFT Properties
  - 3.1 Periodicity
  - 3.2 Parseval's theorem
- 4. Circular Convolution
  - 4.1 Circular shift
  - 4.2 Circular reversal
  - 4.3 Circular convolution

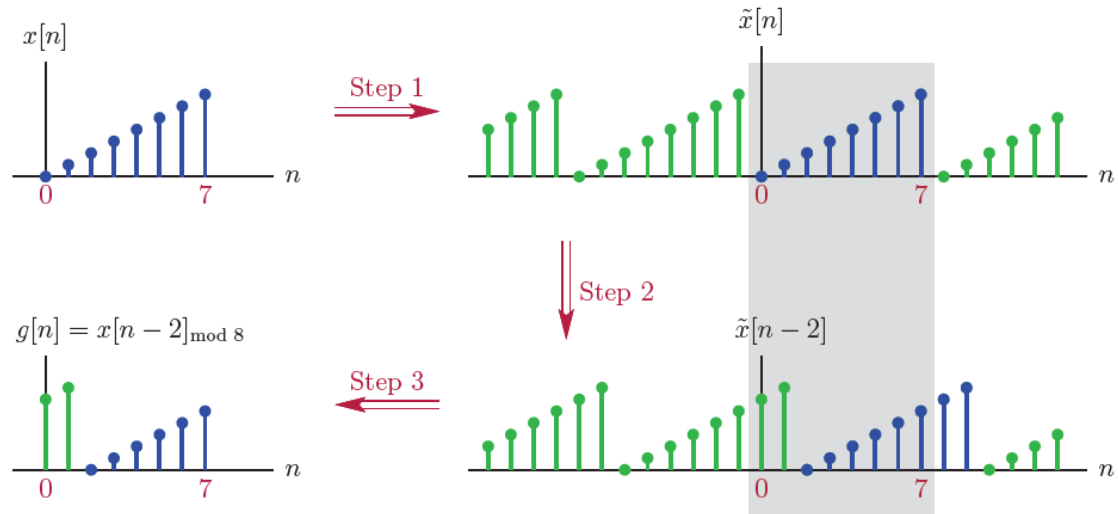
CTFT - DTFT - DFT

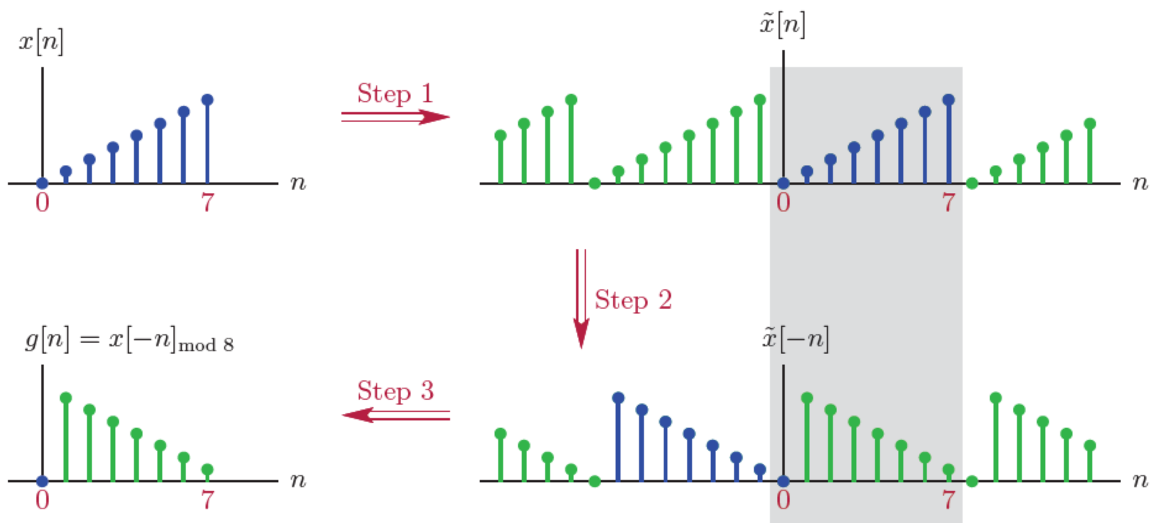
Time Domain	$\boldsymbol{x(t)}$	$\boldsymbol{x[k]}$	$\boldsymbol{\tilde{x}[k]}$
	$\updownarrow$ CTFT	$\updownarrow$ DTFT	$\updownarrow$ DFT
Frequency Domain	$\boldsymbol{X(j\Omega)}$	$\boldsymbol{X(e^{j\omega})}$	$\boldsymbol{X[m]}$



Circular Shift and Reversal

$$g[n] = x[n-m]_{\text{mod } N}$$





$$g[n] = x[-n]_{\text{mod } N}$$

Circular time reversal of a length-8 signal.

Quiz

Q9.

Assume x[n] = {$\underline{1}$, 2, 3, 4}. Then sequence {3, 4, 1, 2} is corresponding to:

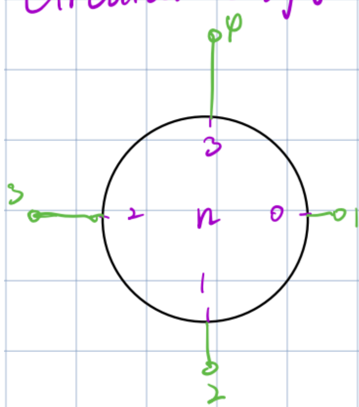
- a. x[n+2]
- ✓ b. $x[\langle n-6\rangle_4]$
- ✓ c. $x[\langle n+2\rangle_4]$
- d. x[n-2]
- ✓ e. $x[\langle n-2\rangle_4]$
- f. $x[\langle n+1\rangle_4]$

Linear shift: $x[n+2] = \{1, 2, \underline{3}, 4\}$

$x[n-2] = \{\underline{0}, 0, 1, 2, 3, 4\}$

Circular shift: $x[\langle n-6\rangle_4] = x[\langle n-2\rangle_4]$

$= \{\underline{3}, 4, 1, 2\}$



$x[\langle n+2\rangle_4] = \{\underline{3}, 4, 1, 2\}$

$x[\langle n+1\rangle_4] = \{\underline{2}, 3, 4, 1\}$

Circularly Conjugate Symmetry

For any signal $x[n]$, it can be decomposed into two components:

$$x_E[n] = \frac{x[n]+x^*[-n]_{\text{mod } N}}{2} \text{ (conjugate symmetric component)}$$

$$x_O[n] = \frac{x[n]-x^*[-n]_{\text{mod } N}}{2} \text{ (conjugate antisymmetric component)}$$

So that

$$x[n] = x_E[n]+x_O[n]$$

In similar manner, its DFT can also be decomposed into

$$X[k] = X_E[k]+X_O[k]$$

$$\boxed{\begin{aligned} x_r[n] + jx_i[n] &\xleftrightarrow{\text{DFT}} X_E[k] + X_o[k] \\ x_E[n] + x_O[n] &\xleftrightarrow{\text{DFT}} X_r[k] + jX_i[k] \end{aligned}}$$

Xi'an Jiaotong-Liverpool University 西交利物浦大學

Circular vs. Linear Convolution

- Circular convolution:

$$y[k] = x_1[k] \circledast_N x_2[k] = \sum_{n=0}^{N-1} x_1[\langle n \rangle_N] x_2[\langle k-n \rangle_N]$$

- Expressed in matrix form, take N=4 as an example:

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} x_2[0] & x_2[3] & x_2[2] & x_2[1] \\ x_2[1] & x_2[0] & x_2[3] & x_2[2] \\ x_2[2] & x_2[1] & x_2[0] & x_2[3] \\ x_2[3] & x_2[2] & x_2[1] & x_2[0] \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ x_1[3] \end{bmatrix}$$

- More convenient method?

Quiz

Determine circular convoluation of {5, 4, 3, 2} and {1, 2, 4}.

- [ ] a. {22, 31, 24, 21}
- [x] b. {22, 31, 24, 21} ✓
- [ ] c. {21, 22, 31, 24}
- [x] d. {21, 22, 31, 24} ✓

Linear conv. $y_L[n]$

		5	4	3	2
			1	2	4
		20	16	12	8
	10	8	6	4	
5	4	3	2		
5	14	31	24	16	8

Recall: $y[n] = x_1[n] * x_2[n]$

$N_1 \le n \le M_1$, $N_2 \le n \le M_2$.

then, range of $y[n]$: $N_1+N_2 \sim M_1+M_2$

So, $y_L[n]$: $-2 \le n \le 3$

$y_L[n] = \{5, 14, \underline{31}, 24, 16, 8\}$

Circular Convolution.

$\text{len}\{y_c[n]\} = \max\{\text{len}\{x_1[n]\}, \text{len}\{x_2[n]\}\}$

So, $y_c[n]$ is of length 4.

$y_c[n] = \{5, 14, \underline{31}, 24, 16, 8\} = \{21, 22, \underline{31}, 24\}$

$y_c[n]$ is periodic (or circular), can also be written as

$\{22, \underline{31}, 24, 21\}$, $\{\underline{31}, 24, 21, 22\}$, $\{24, 21, 22, \underline{31}\}$.

$N = 4$.

... 24 16 8 ← $y_L[n+8]$

5 14 31 24 16 8 ← $y_L[n+4]$

5 14 $\underline{31}$ 24 16 8 ← $y_L[n]$

$y_L[n-4]$ → 5 14 31 24 16 8

$y_L[n-8]$ → 5 14 ...

→ ...

$y_c[n]$: ...	21	22	31	24	21	22	$\underline{31}$	24	21	22	31	24	21	22 ...
n: ...	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7

Thank You ! &
GOOD LUCK☺

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