MTH102 Engineering Mathematics II

Lesson 9: Jointly distributed random variables

Term: 2024

Outline

Distribution of multiple random variables

Motivations: In many practical cases, we have to take more than one measurement of a random observation. For example, we need to measure the height H and the weight W during a physical examination. Furthermore, there may be some pattern between height and weight that needs to determine. For example,

$$P(W > 60kg \cap H = 1.60m) \neq P(W > 60kg \cap H = 1.70m).$$

This then leads to the analysis on multiple random variables during a random experiment.

In this chapter, we aim at introducing the distributions of two random variables, called bivariante distributions. All the notions and results on bivariante distributions can be generalized in the case of multiple variables.

Outline

Joint probability mass function

Definition

Let (X, Y) *be a pair of discrete random variables. Let* S *denote the corresponding two-dimensional support of* X *and* Y*. The probability that* $X = x$ *and* $Y = y$ *is denoted by*

$$f(x, y) = P(X = x, Y = y).$$

The function $f(x, y)$ *is called the* **joint probability mass function (joint pmf)** *of* (X, Y) *and has the following properties:*

(a) $0 \leq f(x, y) \leq 1$.

(b) $\sum\sum_{(x,y)\in S} f(x, y) = 1$.

(c) *If* A *is a subset of* S*, then* $P((X, Y) \in A) = \sum\sum_{(x,y)\in A} f(x, y)$.

Example 1

Sample space $S = \{(i,j); i,j \in \{1,\cdots,6\}\}$, $|S| = 6\times 6 = 36$

Roll a pair of fair dice. For each of the 36 sample points with probability $1/36$, let X denote the smaller and Y the larger outcome on the dice. If and only if the outcome is $(2,3)$ or $(3,2)$, the observed values are $X = 2,\ Y = 3$. Then

$$P(X = 2, Y = 3) = \frac{2}{36} = \frac{1}{18}.$$

The event $\{X = 2, Y = 2\}$ can occur only when the outcome is $(2,2)$. Thus

$$P(X = 2, Y = 2) = \frac{1}{36}.$$

In general, the joint pmf is

$$f(x,y) = \begin{cases} \frac{1}{36}, & 1 \le x = y \le 6, \\ \frac{1}{18}, & 1 \le x < y \le 6, \end{cases}$$

$$\Rightarrow \sum_{x=1}^{6}\sum_{y=1}^{6} f(x,y) = \sum_{1\le x=y\le 6}\frac{1}{36} + \sum_{1\le x<y\le 6}\frac{1}{18} = \frac{6}{36} + \frac{1}{18}(5+4+3+2+1) = 1$$

when x and y are integers.

Marginal probability mass function

Definition

Let (X, Y) have the joint pmf $f(x, y)$ with support S; X and Y have supports S_X and S_Y, respectively. The pmf of X alone, which is called the **marginal probability mass function (marginal pmf)** *of X, is defined by*

$$f_X(x) = \sum_{y \in S_Y} f(x, y) = P(X = x), \ x \in S_X.$$

Similarly, the **marginal pmf** *of Y is defined by*

$$f_Y(y) = \sum_{x \in S_X} f(x, y) = P(Y = y), \ y \in S_Y.$$

Example 2

Recall the previous example: roll a pair of fair dice. For each of the 36 sample points with probability $1/36$, let X denote the smaller and Y the larger outcome on the dice. Recall that the joint pmf is

$$f(x,y) = \begin{cases} \frac{1}{36}, & 1 \le x = y \le 6, \\ \frac{1}{18}, & 1 \le x < y \le 6, \end{cases}$$

when x and y are integers. Then

$$f_X(1) = P(X=1) = \sum_{y=1}^{6} f(1,y) = \frac{1}{36} + 5 \cdot \frac{1}{18} = \frac{11}{36},$$

$$f_X(2) = P(X=2) = \sum_{y=2}^{6} f(2,y) = \frac{1}{36} + 4 \cdot \frac{1}{18} = \frac{9}{36},$$

and so on.

Example 2

In general, the pmf and marginal pmf are displayed in the following table.

$X \backslash Y$	1	2	3	4	5	6	$P(X=x)$
1	$1/36$	$2/36$	$2/36$	$2/36$	$2/36$	$2/36$	$11/36$
2	0	$1/36$	$2/36$	$2/36$	$2/36$	$2/36$	$9/36$
3	0	0	$1/36$	$2/36$	$2/36$	$2/36$	$7/36$
4	0	0	0	$1/36$	$2/36$	$2/36$	$5/36$
5	0	0	0	0	$1/36$	$2/36$	$3/36$
6	0	0	0	0	0	$1/36$	$1/36$
$P(Y=y)$	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$	1

Independence of random variables

Definition

The discrete random variables X and Y are called **independent**, *if for every $x \in S_X$ and every $y \in S_Y$,*

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

or, equivalently,

$$f(x, y) = f_X(x)f_Y(y);$$

otherwise, X and Y are said to be **dependent**.

Recall the previous example: roll a pair of fair dice. For each of the 36 sample points with probability $1/36$, let X denote the smaller and Y the larger outcome on the dice. Note that

$$f_X(1)f_Y(1) = \frac{11}{36} \cdot \frac{1}{36} \neq \frac{1}{36} = f(1, 1).$$

Therefore, X and Y are dependent.

Example 3

Let the joint pmf of X and Y be

$$f(x,y) = \frac{xy^2}{30}, \; x = 1,2,3, \; y = 1,2.$$

The marginal pmfs are

$$f_X(x) = \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x}{6}, \; x = 1,2,3,$$

and

$$f_Y(y) = \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{y^2}{5}, \; y = 1,2.$$

Then $f(x,y) = f_X(x)f_Y(y)$ for $x = 1,2,3$ and $y = 1,2$. Thus, X and Y are independent.

Example 3

In general, the pmf and marginal pmf are displayed in the following table.

$X \backslash Y$	1	2	$P(X = x)$
1	$1/30$	$4/30$	$5/30$
2	$2/30$	$8/30$	$10/30$
3	$3/30$	$12/30$	$15/30$
$P(Y = y)$	$6/30$	$24/30$	1

Outline

Joint probability density function

Definition

Let (X, Y) be a pair of continuous random variables. The **joint probability density function (joint pdf)** *of X and Y is an integrable function $f(x, y)$ with the following properties:*

(a) $f(x, y) \geq 0$, *where* $f(x, y) = 0$ *when* (x, y) *is not in the space* S *of* X *and* Y.

(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$.

(c) $P[(X, Y) \in A] = \iint\limits_{A} f(x, y) dxdy$, *where* $\{(X, Y) \in A\}$ *is an event defined in the plane.*

Property (c) implies that $P[(X, Y) \in A]$ is the volume of the solid over the region A in the xy-plane and bounded by the surface $z = f(x, y)$.

Marginal probability density function

The respective **marginal pdfs** of continuous-type random variables X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, \; x \in S_X,$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx, \; y \in S_Y,$$

where S_X and S_Y are respectively the supports of X and Y.

Example 5

Let X and Y have the joint pdf

$$f(x,y) = \left(\frac{4}{3}\right)(1-xy),\; 0 \le x, y \le 1.$$

The marginal pdfs are

$$f_X(x) = \int_0^1 \left(\frac{4}{3}\right)(1-xy)dy = \left(\frac{4}{3}\right)\left(1-\frac{x}{2}\right),\; 0 \le x \le 1,$$

and

$$f_Y(y) = \int_0^1 \left(\frac{4}{3}\right)(1-xy)dx = \left(\frac{4}{3}\right)\left(1-\frac{y}{2}\right),\; 0 \le y \le 1.$$

The following probability is computed by a double integral:

$$P(Y \le X/2) = \int_0^1 \int_0^{x/2} \left(\frac{4}{3}\right)(1-xy)dydx = \frac{7}{24}.$$

Example 5

The mean of X is

$$\mu_X = E(X) = \int_0^1 x f_X(x) dx = \int_0^1 x \left(\frac{4}{3}\right)\left(1 - \frac{x}{2}\right) dx = \frac{4}{9}.$$

The variance of X is

$$Var(X) = E(X^2) - [E(X)]^2 = \int_0^1 x^2 \left(\frac{4}{3}\right)\left(1 - \frac{x}{2}\right) dx - \left(\frac{4}{9}\right)^2 = \frac{13}{162}.$$

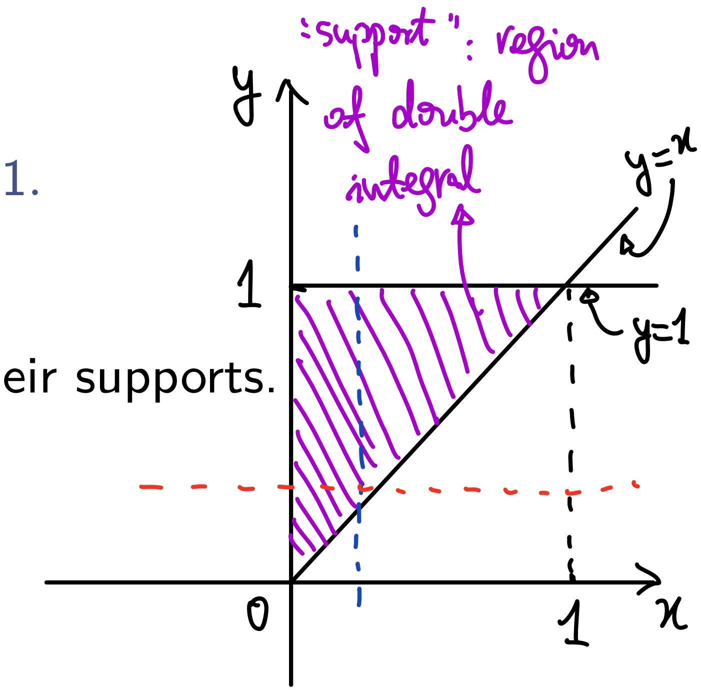
Likewise, the mean and variance of Y are

$$E(Y) = \frac{4}{9} \text{ and } Var(Y) = \frac{13}{162}.$$

Example 6

Let X and Y have the joint pdf

$$f(x,y) = 2,\ 0 \le x \le y, 0 \le y \le 1.$$



(a) Find the marginal pdf of X and Y, and specify their supports.

(b) Find $P(X < 1/4, Y < 1/2)$.

Solution.

(a) $f_X(x) = \int_x^1 f(x,y)dy = 2(1-x),\ 0 \le x \le 1.$
$f_Y(y) = \int_0^y f(x,y)dx = 2y,\ 0 \le y \le 1.$
The supports of X and Y are both $[0,1]$.

(b)

$$P(X < 1/4, Y < 1/2) = \int_0^{1/4} \int_x^{1/2} f(x,y)dydx = \frac{3}{16}.$$

Independence of random variables

Definition

The continuous random variables X and Y are called **independent**, *if for every $x \in S_X$ and every $y \in S_Y$,*

$$f(x,y) = f_X(x) f_Y(y);$$

otherwise, X and Y are said to be **dependent**.

In the previous Example 5 and Example 6, X and Y are dependent.

Outline

Covariance

Definition

Let (X, Y) be a pair of random variables with support S.

(a) *If X and Y are both discrete with joint pmf $p(x, y)$, then their* **covariance** *is defined by*

$$Cov(X, Y) = \sum\sum_{(x,y)\in S}(x - \mu_X)(y - \mu_Y)p(x, y) = \sigma_{XY}.$$

(b) *If X and Y are both continuous with joint pdf $f(x, y)$, then their* **covariance** *is defined by*

$$Cov(X, Y) = \iint_{(x,y)\in S}(x - \mu_X)(y - \mu_Y)\, f(x, y)\, dx\, dy = \sigma_{XY}.$$

Correlation coefficient

Definition

If the standard deviations σ_X *and* σ_Y *are positive, then*

$$\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

is called the **correlation coefficient** *of* X *and* Y.

Properties of covariance

Let X and Y be two random variables. The following holds.

$$Cov(X, Y) = E(XY) - E(X)E(Y).$$

More precisely,

(a) If X and Y are both discrete with joint pmf $p(x, y)$, then

$$Cov(X, Y) = \left[\sum\sum_{(x,y)\in S} xy\, p(x, y) \right] - E(X)E(Y).$$

(b) If X and Y are both continuous with joint pdf $f(x, y)$, then

$$Cov(X, Y) = \left[\iint_{(x,y)\in S} xy\, f(x, y)\, dx\, dy \right] - E(X)E(Y).$$

Example 8

Problem. Let X and Y have the joint pmf

$$f(x,y) = \frac{x+2y}{18}, \ x = 1,2, \ y = 1,2.$$

Compute the correlation coefficient of X and Y.
Solution. The marginal pmfs are

$$f_X(x) = \sum_{y=1}^{2} \frac{x+2y}{18} = \frac{x+3}{9}, \ x = 1,2,$$

$$f_Y(y) = \sum_{x=1}^{2} \frac{x+2y}{18} = \frac{3+4y}{18}, \ y = 1,2.$$

Example 8

The mean and variance are

$$\mu_X = \sum_{x=1}^{2} x \frac{x+3}{9} = \frac{14}{9},$$

$$\mu_Y = \sum_{y=1}^{2} y \frac{3+4y}{18} = \frac{29}{18},$$

$$\sigma_X^2 = \sum_{x=1}^{2} x^2 \frac{x+3}{9} - \left(\frac{14}{9}\right)^2 = \frac{20}{81},$$

$$\sigma_Y^2 = \sum_{y=1}^{2} y^2 \frac{3+4y}{18} - \left(\frac{29}{18}\right)^2 = \frac{77}{324}.$$

Example 8

Then we have

$$E(XY) = \sum_{x=1}^{2}\sum_{y=1}^{2} xy\frac{x+2y}{18} = \frac{45}{18}.$$

Hence,

$$Cov(X,Y) = E(XY) - \mu_X\mu_Y = -\frac{1}{162}.$$

Therefore,

$$\rho = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = -\frac{1}{\sqrt{1540}}.$$

Example 9

Let X and Y have the joint pdf

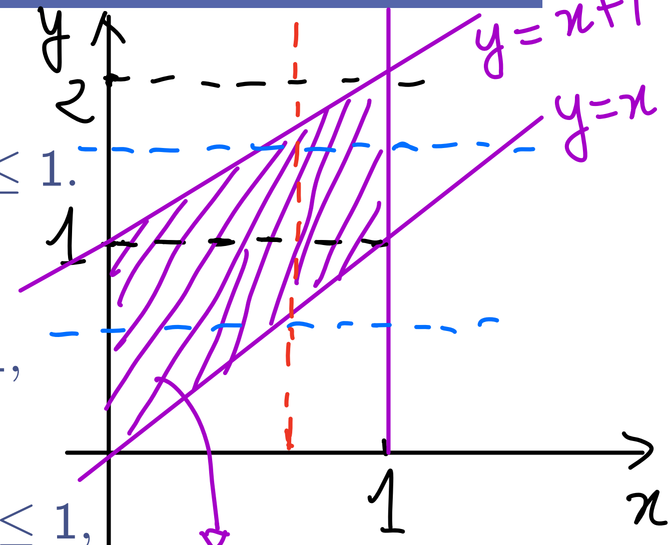
$$f(x,y)=1,\ x\le y\le x+1,\ 0\le x\le 1.$$

Then

$$f_X(x)=\int_x^{x+1}1dy=1,\ 0\le x\le 1,$$

and

$$f_Y(y)=\begin{cases}\int_0^y 1dx=y, & 0\le y\le 1,\\ \int_{y-1}^1 1dx=2-y, & 1\le y\le 2.\end{cases}$$



Support of (X, Y): "Parallelogram"

Also,

$$E(X)=\int_0^1 x\cdot 1dx=\frac{1}{2},\ E(Y)=\int_0^1 y\cdot ydy+\int_1^2 y(2-y)dy=1,$$

$$E(X^2)=\int_0^1 x^2\cdot 1dx=\frac{1}{3},\ E(Y^2)=\int_0^1 y^2\cdot ydy+\int_1^2 y^2(2-y)dy=\frac{7}{6}.$$

Example 9

Therefore,

$$E(XY) = \int_0^1 \int_x^{x+1} xy \cdot 1 dy dx = \frac{7}{12},$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{12},$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{6},$$

$$Cov(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{12},$$

and finally

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\sqrt{2}}{2}.$$

Properties of independence

Theorem

Let X and Y be two independent random variables. The following holds.

(a)
$$E(XY) = E(X)E(Y).$$

(b)
$$Cov(X, Y) = 0, \ \ and \ \rho = 0.$$

(c)
$$Var(X + Y) = Var(X) + Var(Y).$$