PDE and Numerical Methods
Project - Mean Field Games with Congestion Effects

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Abstract

In this project, we look for a numerical approximation over a domain $\Omega =]0,1[$ of the probability density $\hat{m}(t,\cdot)$ in a mean field model with congestion effects. More precisely, $\hat{m}(t,\cdot)$ is the density of X_t^ν for $\nu = \hat{\nu}$ with the following controlled dynamics:

$$dX_t^\nu = \nu_t dt + \sigma dW_t,$$

with the density of X_0^ν being $m_0(\cdot)$. Moreover, $\hat{\nu}$ is the optimal feedback control that minimizes the following criteria:

$$J_{\hat{m}}(\nu) = \mathbb{E}\left[\int_0^T f(X_t^\nu, \hat{m}(t, X_t^\nu), \nu(t, X_t^\nu))dt + \phi(X_T^\nu)\right],$$

where $f(x,\mu,\gamma) = \frac{\beta}{\beta-1}(c_0 + c_1\mu)^{\frac{\alpha}{\beta-1}}|\gamma|^{\frac{\beta}{\beta-1}} + \tilde{f}_0(\mu) + g(x)$. Unless stated otherwise, we use the following functions and parameters throughout this project:

$$m_0(x) = \sqrt{\frac{300}{\pi}}e^{-300(x-0.2)^2},\ \phi(x) = -e^{-40(x-0.7)^2},\ \tilde{f}_0(\mu) = \frac{\mu}{10},\ g(x) = 0.$$

The spatial mesh step is $h = \frac{1}{N_h - 1}$ and the time step is $\Delta t = \frac{T}{N_T}$, with $T = 1$, $N_h = 201$, $N_T = 100$. We denote $t_n = n\Delta t$ and $x_i = ih$.

Finally, the numerical results (code, graphics, videos) are available on our Github repository[1].

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[1] github.com/Dracdarc/EDP_MeanFieldGames

Contents

1 Mean Field Game

1.1 From continuous representation to finite difference schemes

We consider a Nash equilibrium, with the number of players tending to infinity. Each individual player is selfish and tries to minimize its own individual cost via its feedback strategy given the other players' strategies. It can be proven that the value function $u(t,x)$ of the optimal control problem for a player and the density $\hat{m}(t,x)$ satisfy the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{cases} \partial_t u(t,x) - \nu \partial_x^2 u(t,x) + H_0(\partial_x u(t,x), m(t,x)) = g(x) + \tilde{f}_0(m(t,x)) & \forall (t,x) \in [0,T[\, \times \Omega, \\ \partial_x u(t,0) = \partial_x u(t,1) = 0 & \forall t \in \,]0,T[\,, \\ u(T,x) = \phi(x) & \forall x \in \Omega, \end{cases} \tag{1}$$

where $H_0(p,\mu) = \frac{1}{\beta}\frac{|p|^\beta}{(c_0+c_1\mu)^\alpha}$ is related to the Hamiltonian of the control problem, and $\nu = \frac{\sigma^2}{2}$. They also satisfy the Kolmogorov-Fokker-Planck (KFP) equation:

$$\begin{cases} \partial_t m(t,x) - \nu \partial_x^2 m(t,x) - \partial_x \left(\frac{m(t,\cdot)}{(c_0+c_1 m(t,\cdot))^\alpha} |\partial_x u(t,\cdot)|^{\beta-2} \partial_x u(t,\cdot) \right)(x) = 0 & \forall (t,x) \in [0,T[\, \times \Omega, \\ \partial_x m(t,0) = \partial_x m(t,1) = 0 & \forall t \in \,]0,T[\,, \\ m(0,x) = m_0(x) & \forall x \in \Omega. \end{cases} \tag{2}$$

These two equations can be discretized using finite difference schemes where we approximate u and m such that $u(t_n,x_i) \approx U_i^n$ and $m(t_n,x_i) \approx M_i^n$. Let us define the following discrete operators $\forall W \in \mathbb{R}^{N_T+1}$:

$$\begin{aligned} (D_t W)^n &= \tfrac{1}{\Delta t}(W^{n+1} - W^n) \\ (DW)_i &= \tfrac{1}{h}(W_{i+1} - W_i) \\ (\Delta_h W)_i &= -\tfrac{1}{h^2}(2W_i - W_{i+1} - W_{i-1}) \\ [\nabla_h W]_i &= ((DW)_i,\, (DW)_{i-1})^T \end{aligned}$$

It yields the following discrete HJB equation:

$$\begin{cases} -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) = g(x_i) + \tilde{f}_0(M_i^{n+1}) & 0 \le i < N_h,\ 0 \le n < N_T, \\ U_{-1}^n = U_0^n & 0 \le n < N_T, \\ U_{N_h}^n = U_{N_h-1}^n & 0 \le n < N_T, \\ U_i^{N_T} = \phi(x_i) & 0 \le i < N_h, \end{cases} \tag{3}$$

where $\tilde{H}(p_1,p_2,\mu) = \frac{1}{\beta}\frac{\left((p_1)_-^2 + (p_2)_+^2\right)^{\frac{\beta}{2}}}{(c_0+c_1\mu)^\alpha}$ is the discrete Hamiltonian[2], ie the discrete version of

[2] We denote $(x)_{+/-}^2 = (x_{+/-})^2$.

$H_0(p,\mu)$. Similarly, the discrete KFP equation is:

$$\begin{cases} (D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}, \tilde{M}^{n+1}) = 0 & 0 \le i < N_h,\ 0 \le n < N_T, \\ M_{-1}^n = M_0^n & 0 \le n < N_T, \\ M_{N_h}^n = M_{N_h-1}^n & 0 \le n < N_T, \\ M_i^0 = \bar{m}_0(x_i) & 0 \le i < N_h, \end{cases} \tag{4}$$

where $\bar{m}_0(x) = m_0(x)$ and $\tilde{M}^{n+1}$ is fixed as described in Section 1.2.2. Moreover, $\mathcal{T}_i$ is the discrete transport operator such that:

$$\begin{aligned} \mathcal{T}_i(U^n, M, \tilde{M}) = \frac{1}{h}\Big(& M_i \tilde{H}_{p_1}([\nabla_h U^n]_i, \tilde{M}_i) - M_{i-1}\tilde{H}_{p_1}([\nabla_h U^n]_{i-1}, \tilde{M}_{i-1}) \\ & + M_{i+1}\tilde{H}_{p_2}([\nabla_h U^n]_{i+1}, \tilde{M}_{i+1}) - M_i \tilde{H}_{p_2}([\nabla_h U^n]_i, \tilde{M}_i)\Big), \end{aligned}$$

where $\tilde{H}_{p_1}$ and $\tilde{H}_{p_2}$ are the derivatives of $\tilde{H}$ wrt p_1 and p_2. More precisely,

$$\tilde{H}_{p_1}(p_1, p_2, \mu) = \frac{-(p_1)_-}{(c_0 + c_1\mu)^\alpha}\left((p_1)_-^2 + (p_2)_+^2\right)^{\frac{\beta-2}{2}}, \tag{5}$$

$$\tilde{H}_{p_2}(p_1, p_2, \mu) = \frac{(p_2)_+}{(c_0 + c_1\mu)^\alpha}\left((p_1)_-^2 + (p_2)_+^2\right)^{\frac{\beta-2}{2}}. \tag{6}$$

Notice that we can restate the discrete KFP equation (4) in a vector form:

$$\frac{1}{\Delta t}(M^{n+1} - M^n) + (\Lambda - A(U^n, \tilde{M}^{n+1}))M^{n+1} = 0, \tag{7}$$

where $\Lambda \in \mathbb{R}^{N_h \times N_h}$ is a tridiagonal matrix such that:

$$\begin{cases} \Lambda_{i,i} & = \frac{2\nu}{h^2}, \\ \Lambda_{i,i+1} = \Lambda_{i+1,i} & = -\frac{\nu}{h^2}. \end{cases}$$

The matrix $A(U^n, \tilde{M}^{n+1}) \in \mathbb{R}^{N_h \times N_h}$ is the matrix of the discrete transport operator linear application $M \mapsto \mathcal{T}(U, M, \tilde{M}) = \left(\mathcal{T}_i(U, M, \tilde{M})\right)_{0 \le i < N_h}$ valued at $U = U^n$ and $\tilde{M} = \tilde{M}^{n+1}$:

$$A(U, \tilde{M}) = \begin{pmatrix} \gamma_0 - \varepsilon_0 & \varepsilon_1 & 0 & 0 \\ -\gamma_0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \varepsilon_{N_h-1} \\ 0 & 0 & -\gamma_{N_h-2} & \gamma_{N_h-1} - \varepsilon_{N_h-1} \end{pmatrix}, \tag{8}$$

where $\gamma_i = \frac{1}{h}\tilde{H}_{p_1}\left([\nabla_h U]_i, \tilde{M}_i\right)$, and $\varepsilon_i = \frac{1}{h}\tilde{H}_{p_2}\left([\nabla_h U]_i, \tilde{M}_i\right)$. Notice that we use $U_{-1}^n = U_0^n$ for γ_0 and ε_0, and $U_{N_h}^n = U_{N_h-1}^n$ for γ_{N_h-1} and ε_{N_h-1}.

1.2 Solving the discrete equations

In order to solve the whole forward-backward system of equations (3)-(4), we will use fixed points iterations. It enables to separate the resolution of the discrete HJB equation (3) from the resolution of the discrete KFP equation (4). Let us define $\mathcal{U}^{(k)} = \left(U^{n,(k)}\right)_{0\leq n\leq N_T} \in \mathbb{R}^{(N_T+1)\times N_h}$ and $\mathcal{M}^{(k)} = \left(M^{n,(k)}\right)_{0\leq n\leq N_T} \in \mathbb{R}^{(N_T+1)\times N_h}$ the solution vectors for the k-th fixed point iteration, and $x = (x_0, \cdots, x_{N_h-1})^T$. The algorithm structure is then the following:

- Initialize $\mathcal{U}^{(0)} = (\phi(x))_{0\leq n\leq N_T}$ and $\mathcal{M}^{(0)} = (\bar{m}_0(x))_{0\leq n\leq N_T}$.
- While $\|(\mathcal{U}^{(k+1)}, \mathcal{M}^{(k+1)}) - (\mathcal{U}^{(k)}, \mathcal{M}^{(k)})\|$ is above a threshold[3] of $2\cdot 10^{-5}$:
  - $\hat{\mathcal{U}}^{(k+1)} = \text{HJB solver}\left(\mathcal{M}^{(k)}\right)$, and then $\hat{\mathcal{M}}^{(k+1)} = \text{KFP solver}\left(\hat{\mathcal{U}}^{(k+1)}, \mathcal{M}^{(k)}\right)$.
  - $(\mathcal{U}^{(k+1)}, \mathcal{M}^{(k+1)}) = \theta(\hat{\mathcal{U}}^{(k+1)}, \hat{\mathcal{M}}^{(k+1)}) + (1-\theta)(\mathcal{U}^{(k)}, \mathcal{M}^{(k)})$, where θ is a relaxation parameter set to $\theta = 0.02$.

1.2.1 HJB solver

Now, we need to solve the nonlinear parabolic backward equation (3) with unknown $(U^n)_n$, given $(M^n)_n$. At time iteration n, this system with unknown U^n can be written as $\mathcal{F}(U^n, U^{n+1}, M^{n+1}) = 0 \in \mathbb{R}^{N_h}$, where:

$$\mathcal{F}_i(U^n, U^{n+1}, M^{n+1}) = \frac{U_i^n - U_i^{n+1}}{\Delta t} - \nu(\Delta_h U^n)_i + \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) - g(x_i) - \tilde{f}_0(M_i^{n+1}).$$

Notice that U^{n+1} and M^{n+1} are given. We will solve this equation with a Newton-Raphson method. The Jacobian matrix associated to the map $U \mapsto \mathcal{F}(U, U^{n+1}, M^{n+1})$ is the tridiagonal matrix:

$$\mathcal{J}(U, M^{n+1}) = \frac{1}{\Delta t} I_{N_h} + \Lambda - A(U, M^{n+1})^T,$$

where $A(\cdot,\cdot)$ is the matrix of the discrete transport operator defined in (8). Thus, the Newton method is such that we initialize $U^{n,0} = U^{n+1}$ and we compute iteratively until convergence:

$$U^{n,j+1} = U^{n,j} - \mathcal{J}^{-1}(U^{n,j}, M^{n+1})\mathcal{F}(U^{n,j}, U^{n+1}, M^{n+1}).$$

The HJB solver structure is then the following:

- Initialize $\hat{U}^{N_T,(k+1)} = \phi(x)$.
- For time iteration n decreasing:
  - $\hat{U}^{n,(k+1)} = \text{Newton}\left(U^{n+1} = \hat{U}^{n+1,(k+1)}, M^{n+1} = M^{n+1,(k)}\right)$.

1.2.2 KFP solver

Finally, we need to solve the linear parabolic forward equation (4) with unknown $(M^n)_n$. In theory, $\tilde{M}^{n+1} = M^{n+1}$, but we will set $\tilde{M}^{n+1}$ to M^{n+1} from the previous fixed point iteration, so

[3] We define $\|(\mathcal{X},\mathcal{Y})\|$ as $\frac{\|z\|_2}{(N_T+1)N_h}$ such that the norm of full ones matrices $\mathcal{X}$ and $\mathcal{Y}$ is equal to 1, and where z is the vector flattened from $(\mathcal{X},\mathcal{Y})$.

that we have a classic implicit scheme for $(M^n)_n$, as presented in (7). The KFP solver structure is then the following:

- Initialize $\hat{M}^{0,(k+1)} = \bar{m}_0(x)$.
- For time iteration n increasing:
  - $\hat{M}^{n+1,(k+1)} = B^{-1}\hat{M}^{n,(k+1)}$, where $B = I_{N_h} + \Delta t \left[\Lambda - A\left(\hat{U}^{n,(k+1)}, M^{n+1,(k)}\right)\right]$.

1.3 Results

We consider the following sets of parameters:

$$\begin{cases} (a) & \beta = 2 & c_0 = 0.1 & c_1 = 1 & \alpha = 0.5 & \sigma = 0.02, \\ (b) & \beta = 2 & c_0 = 0.1 & c_1 = 5 & \alpha = 1 & \sigma = 0.02, \\ (c) & \beta = 2 & c_0 = 0.01 & c_1 = 2 & \alpha = 1.2 & \sigma = 0.1, \\ (d) & \beta = 2 & c_0 = 0.01 & c_1 = 2 & \alpha = 1.5 & \sigma = 0.2, \\ (e) & \beta = 2 & c_0 = 1 & c_1 = 3 & \alpha = 2 & \sigma = 0.002. \end{cases}$$

We obtain the following results:

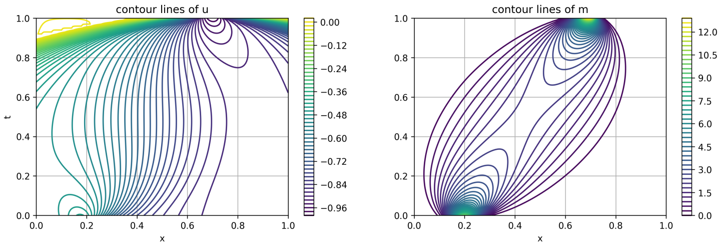

Figure 1: Set (a). Contour lines of u and m.

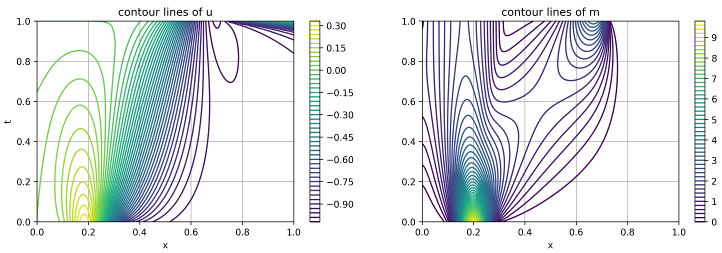

Figure 2: Set (b). Contour lines of u and m.

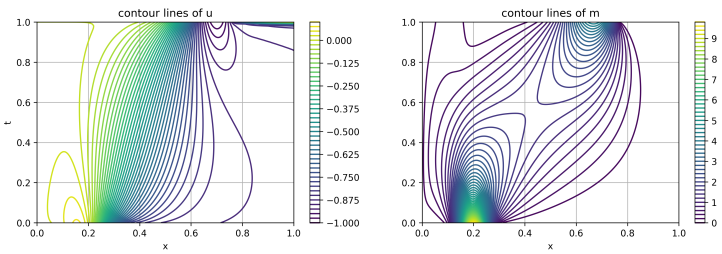

Figure 3: Set (c). Contour lines of u and m.

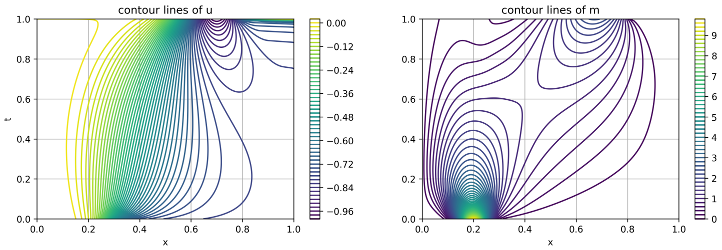

Figure 4: Set (d). Contour lines of u and m.

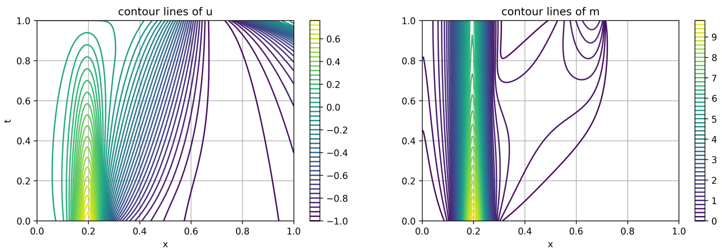

Figure 5: Set (e). Contour lines of u and m.

1.4 Stability and uniqueness of discrete HJB and KFP equations

1.4.1 Discrete HJB equation: uniqueness

Let $(M^n)_n$ be fixed, we show in this section that the solution of the discrete HJB equation will be unique. To do so, we take $(U^n)_n$ and $(V^n)_n$ two solutions of the discrete HJB equation (3) when coupled to $(M^n)_n$. By defining $\delta = U - V$, we get:

$$\begin{cases} (D_t\delta_i)^n + \nu(\Delta_h\delta^n)_i & 0 \le i < N_h \\ \quad = \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) - \tilde{H}([\nabla_h V^n]_i, M_i^{n+1}) & 0 \le n < N_T, \\ \delta_{-1}^n = \delta_0^n & 0 \le n < N_T, \\ \delta_{N_h}^n = \delta_{N_h-1}^n & 0 \le n < N_T, \\ \delta_i^{N_T} = 0 & 0 \le i < N_h. \end{cases}$$

Let $(n_0, i_0) \in \arg\max_{n,i}(\delta_i^n)$, then, in particular $(D_t\delta_{i_0})^{n_0}, (\Delta_h\delta^{n_0})_{i_0} \le 0$. We deduce that:

$$\tilde{H}([\nabla_h U^{n_0}]_{i_0}, M_{i_0}^{n_0+1}) \le \tilde{H}([\nabla_h V^{n_0}]_{i_0}, M_{i_0}^{n_0+1})$$

From this inequality and using the monotonicity of $\tilde{H}$ (non-increasing in its first variable, and non-decreasing in its second variable), we deduce that $(DV^{n_0})_{i_0} \le (DU^{n_0})_{i_0}$ and $(DV^{n_0})_{i_0-1} \ge (DU^{n_0})_{i_0-1}$. Since $(D\delta^{n_0})_{i_0} \le 0$ and $(D\delta^{n_0})_{i_0-1} \ge 0$, it yields $(D\delta^{n_0})_{i_0} = (D\delta^{n_0})_{i_0-1} = 0$. Thus, we have shown that $\delta_{i_0+1}^{n_0} = \delta_{i_0-1}^{n_0} = \max_{n,i}(\delta_i^n)$. By induction in both directions, we can retrieve $\{(n_0, i)\ ;\ i\} \subset \arg\max_{n,i}(\delta_i^n)$. Plugging this result into our difference of discrete HJB equations gives: $-(D_t\delta_i)^{n_0} = 0$. So $\delta_i^n = \max\delta$ for any i and $n \ge n_0$. In particular, $\max\delta = \delta_0^{N_T} = 0$. As the problem is symmetric, we can consider $\delta' = V - U = -\delta$ and get $\min\delta = 0$, leading to $\delta = 0$. In conclusion, we have shown $U = V$ which demonstrates uniqueness in the discrete HJB equation.

1.4.2 Discrete KFP equation: stability and uniqueness

First, we show that the mass of M^n does not depend on n. Let us begin by denoting $\Sigma M^n = \sum_{i=0}^{N_h-1} M_i^n$ and sum the discrete KFP equation (4) over $0 \leq i < N_h$:

$$(D_t \Sigma M)^n - \nu \sum_{i=0}^{N_h-1} (\Delta_h M^{n+1})_i - \sum_{i=0}^{N_h-1} \mathcal{T}_i(U^n, M^{n+1}, \tilde{M}^{n+1}) = 0.$$

However, $\sum_{i=0}^{N_h-1} (\Delta_h M^{n+1})_i = -\frac{1}{h^2}\left[M_{N_h-1}^n - M_{N_h}^n - M_{-1}^n + M_0^n\right] = 0$, and:

$$\begin{aligned}\sum_{i=0}^{N_h-1} \mathcal{T}_i(U^n, &M^{n+1}, \tilde{M}^{n+1}) \\ &= \frac{1}{h}\Big(M_{N_h-1}\tilde{H}_{p_1}([\nabla_h U^n]_{N_h-1}, \tilde{M}_{N_h-1}) - M_{-1}\tilde{H}_{p_1}([\nabla_h U^n]_{-1}, \tilde{M}_{-1}) \\ &\qquad + M_{N_h-1}\tilde{H}_{p_2}([\nabla_h U^n]_{N_h}, \tilde{M}_{N_h}) - M_0\tilde{H}_{p_2}([\nabla_h U^n]_0, \tilde{M}_0)\Big) \\ &= 0.\end{aligned}$$

Thus, we get the equation $(D_t \Sigma M)^n = 0$, that is, $\Sigma M^{n+1} = \Sigma M^n$, which means that the mass of M^n does not depend on n. This result is intuitive because it is natural to conserve the total mass and stay positive if the initial position is positive, since the KFP equation is a differential equation on probability distribution.

Then, to show the uniqueness of the discrete KFP equation, we need a lemma that guarantees the positivity of the mass over time. To prove it, let us proceed by induction and assume $M^n \geq 0$. We will show that $M^{n+1} \geq 0$. From the vector form of the discrete KFP equation (7), we have $B^n M^{n+1} = M^n$ with:

$$B^n = I_n + \Delta t(\Lambda - A(U^n, \tilde{M}^{n+1}))$$

Let us verify that $(B^n)^T$ satisfies the M-property:

- $\forall i$, $B_{i,i}^n = 1 + \Delta t(\frac{2\nu}{h^2} - \gamma_i + \varepsilon_i) > 0$ because $\gamma_i \leq 0$ and $\epsilon_i \geq 0$ from equations (5)-(6).
- To show that $\forall i \neq j$, $B_{i,j}^n \leq 0$, we have to make the distinction between two cases:
  - $\forall i < j$, $B_{i,j}^n = \Delta t(-\frac{\nu}{h^2} - \varepsilon_j \mathbf{1}_{j=i+1})$. However, $\varepsilon_j \geq 0$ so $B_{i,j}^n \leq 0$.
  - $\forall i > j$, $B_{i,j}^n = \Delta t(-\frac{\nu}{h^2} + \gamma_j \mathbf{1}_{j=i-1})$. However, $\gamma_j \leq 0$ so $B_{i,j}^n \leq 0$.
- $\sum_{i=0}^{N_h-1} B_{i,j}^n = 1 + \Delta t\left[\left(\frac{\nu}{h^2} + \varepsilon_0\right)\mathbf{1}_{j=0} + \left(\frac{\nu}{h^2} - \gamma_{N_h-1}\right)\mathbf{1}_{j=N_h-1}\right] > 0$ as $\gamma_{N_h-1} < 0 < \epsilon_0$.

Thus, $(B^n)^T$ satisfies the M-property. It follows that $(B^n)^T$ is invertible with $\left((B^n)^T\right)^{-1}$ having non-negative elements. It follows that B^n is also invertible with $(B^n)^{-1}$ having non-negative elements. As $M^{n+1} = (B^n)^{-1} M^n$, we deduce that $M^{n+1} \geq 0$. Finally, we have proven by induction that $(M^n)_n \geq 0$.

Thus we can conclude on the uniqueness of the discrete KFP equation's solution. Let us start by fixing $(U^n)_n$ and $(\tilde{M}^n)_n$, and be $(M^n)_n$, $(\hat{M}^n)_n$ two solutions of the discrete KFP equation (4) when coupled to $\Big((U^n)_n, (\tilde{M}^n)_n\Big)$. Defining $\delta^n = M^n - \hat{M}^n$, we get a new KFP equation:

$$\begin{cases} (D_t\delta_i)^n - \nu(\Delta_h\delta^{n+1})_i - \mathcal{T}_i(U^n, \delta^{n+1}, \tilde{M}^{n+1}) = 0 & 0 \leq i < N_h,\ 0 \leq n < N_T, \\ \delta^n_{-1} = \delta^n_0 & 0 \leq n < N_T, \\ \delta^n_{N_h} = \delta^n_{N_h-1} & 0 \leq n < N_T, \\ \delta^0_i = 0 & 0 \leq i < N_h, \end{cases}$$

However, from the stability statement, we know that the mass of δ^n does not depend on n. Thus, $\sum_{i=0}^{N_h-1} \delta^n_i = \sum_{i=0}^{N_h-1} \delta^0_i = 0$. Since $\delta^0 \geq 0$, we get from the previous lemma that $\delta^n \geq 0$, and so $\delta^n = 0$. Thus, $(M^n)_n = (\hat{M}^n)_n$, i.e. the solution is unique.

2 Mean Field Control

We now focus on another kind of asymptotic regime different from the Nash equilibrium studied in Section 1. Here, we assume that all agents use the same distributed feedback strategy and the common feedback is optimized when the number of players tends to infinity. It amounts to solve a control problem driven by the McKean-Vlasov dynamics, which is called a mean-filed control. Using the same congestion model, the system of PDEs arising is:

$$\begin{cases} \partial_t u(t,x) - \nu \partial_x^2 u(t,x) + H_0(\partial_x u(t,x), m(t,x)) + G_0(\partial_x u(t,x), m(t,x)) & \\ \quad = g(x) + \tilde{f}_0(m(t,x)) + m(t,x)\tilde{f}_0'(m(t,x)) & \forall (t,x) \in [0,T[\times \Omega, \\ \partial_x u(t,0) = \partial_x u(t,1) = 0 & \forall t \in]0,T[, \\ u(T,x) = \phi(x) & \forall x \in \Omega, \end{cases}$$

and we have the same KFP equation as in (2). Moreover, $G_0(p,\mu) = -\frac{c_1 \alpha \mu}{c_0 + c_1 \mu} H_0(p,\mu)$. The only difference with the previous equations is the addition of the term $G_0(\partial_x u(t,x), m(t,x))$ on the left-hand side and $m(t,x)\tilde{f}_0'(m(t,x))$ on the right-hand side, compared to the HJB equation in (1).

Thus, we can use exactly the same idea as in Section 1, modifying the discrete HJB equation (3) into:

$$\begin{cases} -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \left(1 - \frac{c_1 \alpha M_i^{n+1}}{c_0 + c_1 M_i^{n+1}}\right) \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) & \\ \quad = g(x_i) + \tilde{f}_0(M_i^{n+1}) + M_i^{n+1} \tilde{f}_0'(M_i^{n+1}) & 0 \le i < N_h,\ 0 \le n < N_T, \\ U_{-1}^n = U_0^n & 0 \le n < N_T, \\ U_{N_h}^n = U_{N_h-1}^n & 0 \le n < N_T, \\ U_i^{N_T} = \phi(x_i) & 0 \le i < N_h, \end{cases}$$

Then, we only need to change the functions in the Newton method for the resolution of the discrete HJB equation, such that:

$$\begin{aligned} \mathcal{F}_i(U^n, U^{n+1}, M^{n+1}) = & \frac{U_i^n - U_i^{n+1}}{\Delta t} - \nu(\Delta_h U^n)_i + \left(1 - \frac{c_1 \alpha M_i^{n+1}}{c_0 + c_1 M_i^{n+1}}\right) \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) \\ & - g(x_i) - \tilde{f}_0(M_i^{n+1}) - M_i^{n+1} \tilde{f}_0'(M_i^{n+1}), \end{aligned}$$

and the Jacobian matrix is now given by:

$$\left[\mathcal{J}(U, M^{n+1})\right]_{i,j} = \frac{1}{\Delta t} + \Lambda_{i,j} - \left(1 - \frac{c_1 \alpha M_i^{n+1}}{c_0 + c_1 M_i^{n+1}}\right) \left[A(U, M^{n+1})^T\right]_{i,j}.$$

2.1 Results

In this section, we consider $\tilde{f}_0 = 0$, and the following set of parameters:

$$\beta = 2 \quad c_0 = 0.1 \quad c_1 = 1 \quad \alpha = 0.5 \quad \sigma = 0.02.$$

Due to convergence issues in the fixed point method, we had to set a larger spatial mesh step h such that $N_h = 101$ instead of $N_h = 201$ for the mean-field *control* problem. In Figures 6 and 7 , we compare the solutions obtained for the mean-field *game* (as in Section 1) and the mean-field *control* problems, with the set of parameters above.

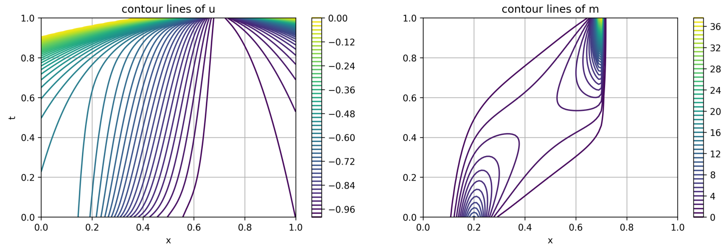

Figure 6: Mean field control problem. Contour lines of u and m.

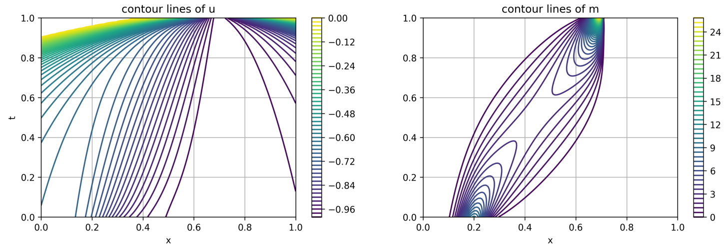

Figure 7: Mean field game problem. Contour lines of u and m.

In Figure 8, we compare the fixed point iterations convergence of the mean field control and the mean field game problems. The number of iterations necessary to reach the target of fixed point increment norm is slightly higher for the mean-field control problem.

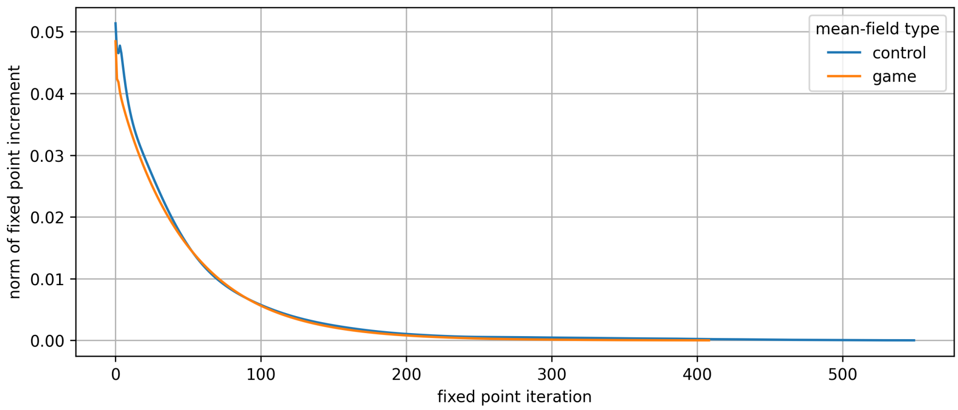

Figure 8: Mean field control vs game problem. Fixed point iterations convergence.

3 Conclusion

In this project, we have seen how to approximate the probability density $m(\cdot, \cdot)$ in a mean-field game problem. First, we discretized the Hamilton-Jacobi-Bellman and Kolmogorov-Fokker-Planck equations by introducing a discrete Hamiltonian operator and a discrete transport operator. They satisfy special properties leading to uniqueness in these two discrete equations. We solved this system using fixed point iterations, which enabled us to separate the resolution of the discrete HJB and KFP equations. For the first one, we used a Newton method while the second one amounts to solve a linear system, at each time step. Finally, we have extended this approach to a mean-field control problem, which features a more complex HJB equation. We managed to keep good convergence properties at the cost of a larger spatial step mesh.