

# Mean Field Games with congestion effects

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## 1 A short introduction to mean field games

The theory of mean field games (*MFGs* in short), has been introduced in the pioneering works of J-M. Lasry and P-L. Lions [2, 3, 4], and aims at studying deterministic or stochastic differential games (Nash equilibria) as the number of players tends to infinity. It supposes that the rational players are indistinguishable and individually have a negligible influence on the game, and that each individual strategy is influenced by some averages of quantities depending on the states (or the controls) of the other players. The applications of MFGs are numerous, from economics and finance to the study of crowd motion. On the other hand, very few MFG problems have explicit or semi-explicit solutions. Therefore, numerical simulations of MFGs play a crucial role in obtaining quantitative information from this class of models.

Let us start by supposing for simplicity that the state space is  $\mathbb{R}^d$  so that no boundary conditions are needed. Boundary conditions will be discussed later. We fix a finite time horizon  $T > 0$ . Starting from Section 2 below, we will focus on the simpler situation in which  $d = 1$ .

**Definition : differential operators in  $\mathbb{R}^d$**  Only for this introductory part, it is useful to define a few differential operators in  $\mathbb{R}^d$ . For a smooth function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ , let  $\nabla\psi$  (the gradient of  $\psi$ ) be the function  $\nabla\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  defined by  $\nabla\psi(x) = \left( \frac{\partial\psi}{\partial x_1}(x), \dots, \frac{\partial\psi}{\partial x_d}(x) \right)^T$  and let  $\Delta\psi$  (the laplacian of  $\psi$ ) be the function  $\Delta\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  defined by  $\Delta\psi(x) = \sum_{i=1}^d \frac{\partial^2\psi}{\partial x_i^2}(x)$ .

For a smooth function  $V : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , let  $\operatorname{div}(V)$  (the divergence of  $V$ ) be the function  $\operatorname{div}(V) : \mathbb{R}^d \rightarrow \mathbb{R}$  defined by  $\operatorname{div}(V)(x) = \sum_{i=1}^d \frac{\partial V_i}{\partial x_i}(x)$ .

Let  $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $(x, m, \gamma) \mapsto f(x, m, \gamma)$  and  $\phi : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $(x, m) \mapsto \phi(x, m)$  be respectively a running cost and a terminal cost, on which assumptions will be made later on.

We consider the following MFG: find a flow of probability densities  $\hat{m} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  and a feedback control  $\hat{v} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfying the following two conditions:

1.  $\hat{v}$  minimizes

$$J_{\hat{m}} : v \mapsto J_{\hat{m}}(v) = \mathbb{E} \left[ \int_0^T f(X_t^v, \hat{m}(t, X_t^v), v(t, X_t^v)) dt + \phi(X_T^v, \hat{m}(T, X_T^v)) \right]$$

subject to the constraint that the process  $X^v = (X_t^v)_{t \geq 0}$  solves the stochastic differential equation (SDE)

$$dX_t^v = b(X_t^v, \hat{m}(t, X_t^v), v(t, X_t^v)) dt + \sigma dW_t, \quad t \geq 0, \quad (1)$$

where  $\sigma$  is the volatility,  $b$  is a given function from  $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$  with values in  $\mathbb{R}^d$ , and  $X_0^v$  is an independent random variable in  $\mathbb{R}^d$ , distributed according to the law  $m_0$ ;

2. For all  $t \in [0, T]$ ,  $\hat{m}(t, \cdot)$  is the law of  $X_t^{\hat{v}}$ .

It is useful to note that for a given feedback control  $v$ , the density  $m_t^v$  of the law of  $X_t^v$  following (1) solves the Kolmogorov-Fokker-Planck (KFP) equation:

$$\begin{cases} \frac{\partial m^v}{\partial t}(t, x) - v \Delta m^v(t, x) + \operatorname{div} \left( m^v(t, \cdot) b(\cdot, \hat{m}(t, \cdot), v(t, \cdot)) \right) (x) = 0, & \text{in } (0, T] \times \mathbb{R}^d, \\ m^v(0, x) = m_0(x), & \text{in } \mathbb{R}^d, \end{cases} \quad (2)$$

where  $v = \sigma^2/2$ .

Let  $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \ni (x, m, p) \mapsto H(x, m, p) \in \mathbb{R}$  be the Hamiltonian of the control problem faced by an infinitesimal player. It is defined by

$$H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \ni (x, m, p) \mapsto H(x, m, p) = \max_{\gamma \in \mathbb{R}^d} \left( -f(x, m, \gamma) - \langle b(x, m, \gamma), p \rangle \right) \in \mathbb{R}.$$

In the sequel, we will assume that the running cost  $f$  and the drift  $b$  are such that  $H$  is well-defined,  $C^1$  with respect to  $(x, p)$ , and strictly convex with respect to  $p$ .

From standard optimal control theory, one can characterize the best strategy through the value function  $u$  of the above optimal control problem for a typical player, which satisfies a Hamilton-Jacobi-Bellman (HJB) equation. Together with the equilibrium condition on the distribution, we obtain that the equilibrium best response  $\hat{v}$  is characterized by

$$\hat{v}(t, x) = \arg \max_{a \in \mathbb{R}^d} \left\{ -f(x, m(t, x), a) - \langle b(x, m(t, x), a), \nabla u(t, x) \rangle \right\},$$

and, denoting  $H_p$  the gradient of  $H$  with respect to  $p$ , that the drift at equilibrium is

$$b(x, m(t, x), \hat{v}(t, x)) = -H_p(x, m(t, x), \nabla u(t, x)),$$

where  $(u, m)$  solves the following forward-backward PDE system:

$$\begin{cases} -\frac{\partial u}{\partial t}(t, x) - v \Delta u(t, x) + H(x, m(t, x), \nabla u(t, x)) = 0, & \text{in } [0, T] \times \mathbb{R}^d, & (3a) \\ \frac{\partial m}{\partial t}(t, x) - v \Delta m(t, x) - \operatorname{div} \left( m(t, \cdot) H_p(\cdot, m(t, \cdot), \nabla u(t, \cdot)) \right) (x) = 0, & \text{in } (0, T] \times \mathbb{R}^d, & (3b) \\ u(T, x) = \phi(x, m(T, x)), \quad m(0, x) = m_0(x), & \text{in } \mathbb{R}^d. & (3c) \end{cases}$$

**Remark 1.1** Note that the Hamilton-Jacobi equation (3a) is a backward parabolic nonlinear equation and is supplemented with a terminal condition, the left part of (3c). The Komogorov-Fokker-Planck equation (3b) is a forward parabolic linear equation and is supplemented with an initial condition, the right part of (3c).

**Example 1.1** ( $f$  depends separately on  $\gamma$  and  $m$ ) Consider the case where the drift is the control, i.e.  $b(x, m, \gamma) = \gamma$ , and the running cost is of the form  $f(x, m, \gamma) = L_0(x, \gamma) + f_0(x, m)$  where  $L_0(x, \cdot) : \mathbb{R}^d \ni \gamma \mapsto L_0(x, \gamma) \in \mathbb{R}$  is strictly convex and such that  $\lim_{|\gamma| \rightarrow \infty} \min_{x \in \mathbb{R}^d} \frac{L_0(x, \gamma)}{|\gamma|} = +\infty$ . We set  $H_0(x, p) = \max_{\gamma \in \mathbb{R}^d} \langle -p, \gamma \rangle - L_0(x, \gamma)$ , which is convex with respect to  $p$ . Then

$$H(x, m, p) = \max_{\gamma \in \mathbb{R}^d} \{ -L_0(x, \gamma) - \langle \gamma, p \rangle \} - f_0(x, m) = H_0(x, p) - f_0(x, m).$$

In particular, for  $1 < \beta' < \infty$  and  $\beta = \beta' / (\beta' - 1)$ ,  $L_0(x, \gamma) = \frac{1}{\beta'} |\gamma|^{\beta'}$  then  $H_0(x, p) = \frac{1}{\beta} |p|^\beta$  and the maximizer in the above expression is  $\hat{\gamma}(p) = -|p|^{\beta-2} p^\beta$ , the Hamiltonian reads  $H(x, m, p) = \frac{1}{\beta} |p|^\beta - f_0(x, m)$  and the equilibrium best response is  $\hat{v}(t, x) = -|\nabla u(t, x)|^{\beta-2} \nabla u(t, x)$  where  $(u, m)$  solves the PDE system

$$\begin{cases} -\frac{\partial u}{\partial t}(t, x) - v \Delta u(t, x) + \frac{1}{\beta} |\nabla u(t, x)|^\beta = f_0(x, m(t, x)), & \text{in } [0, T] \times \mathbb{R}^d, \\ \frac{\partial m}{\partial t}(t, x) - v \Delta m(t, x) - \operatorname{div} \left( m(t, \cdot) |\nabla u(t, \cdot)|^{\beta-2} \nabla u(t, \cdot) \right) (x) = 0, & \text{in } (0, T] \times \mathbb{R}^d, \\ u(T, x) = \phi(x, m(T, x)), \quad m(0, x) = m_0(x), & \text{in } \mathbb{R}^d. \end{cases}$$

**Example 1.2 (A model for congestion)** Consider the case where the drift is the control, i.e.  $b(x, m, \gamma) = \gamma$ , and the running cost is of the form  $f(x, m, \gamma) = L_0(\gamma, m(x)) + f_0(x, m(x))$  where  $L_0 : \mathbb{R}^d \times \mathbb{R}_+ \ni (\gamma, \mu) \mapsto L_0(\gamma, \mu) \in \mathbb{R}$  is given by

$$L_0(\gamma, \mu) = \frac{\beta - 1}{\beta} (c_0 + c_1 \mu)^{\frac{\alpha}{\beta-1}} |\gamma|^{\frac{\beta}{\beta-1}}, \quad (5)$$

where  $\beta > 1$ ,  $0 \leq \alpha \leq \frac{4(\beta-1)}{\beta}$ ,  $c_0 \geq 0$ , and  $c_1 > 0$ . The term  $L_0(\gamma, m(x))$  models congestion effects, i.e. the fact that the cost of motion grows with the density of the population (the denser the population, the higher is the cost of motion). Then

$$H(x, m, p) = \max_{\gamma \in \mathbb{R}^d} \{-L_0(\gamma, m(x)) - \langle \gamma, p \rangle\} - f_0(x, m(x)) = H_0(p, m(x)) - f_0(x, m(x)), \quad (6)$$

where

$$H_0(p, \mu) = \frac{1}{\beta} \frac{|p|^\beta}{(c_0 + c_1 \mu)^\alpha} \quad (7)$$

In this situation, the best response at equilibrium is

$$\hat{v}(t, x, m) = -\frac{1}{(c_0 + c_1 m(t, x))^\alpha} |\nabla u(t, x)|^{\beta-2} \nabla u(t, x) \quad (8)$$

where  $(u, m)$  solves the PDE system

$$\begin{cases} -\frac{\partial u}{\partial t}(t, x) - v \Delta u(t, x) + \frac{1}{\beta} \frac{|\nabla u(t, x)|^\beta}{(c_0 + c_1 m(t, x))^\alpha} = f_0(x, m(t, x)), & \text{in } [0, T] \times \mathbb{R}^d, \\ \frac{\partial m}{\partial t}(t, x) - v \Delta m(t, x) - \operatorname{div} \left( \frac{m(t, \cdot)}{(c_0 + c_1 m(t, \cdot))^\alpha} |\nabla u(t, \cdot)|^{\beta-2} \nabla u(t, \cdot) \right) (x) = 0, & \text{in } (0, T] \times \mathbb{R}^d, \\ u(T, x) = \phi(x, m(T, x)), \quad m(0, x) = m_0(x), & \text{in } \mathbb{R}^d. \end{cases}$$

**Remark 1.2** Deterministic mean field games (i.e. for  $v = 0$ ) are also quite meaningful. The numerical schemes discussed below can also be applied to these situations, even if the convergence results in the available literature are obtained under the hypothesis that  $v$  is bounded from below by a positive constant.

## 2 Finite difference schemes

We present a finite-difference scheme first introduced in [1]. We consider the special case described in Example 1.2, where  $H(x, m, p) = H_0(p, m(x)) - f_0(x, m(x))$  and  $H_0$  is given by (7). We also suppose that  $f_0(x, m(x)) = \tilde{f}_0(m(x)) + g(x)$ .

For simplicity, we focus on the one-dimensional setting, i.e.  $d = 1$ . We also suppose that the state space is the domain  $\Omega = ]0, 1[$ , i.e. the stochastic process involved in the dynamics of the players is reflected at  $\partial\Omega$ . The boundary value problem becomes

$$-\frac{\partial u}{\partial t}(t, x) - v \frac{\partial^2 u}{\partial x^2}(t, x) + \frac{1}{\beta} \frac{|\frac{\partial u}{\partial x}(t, \cdot)|^\beta}{(c_0 + c_1 m(t, x))^\alpha} = g(x) + \tilde{f}_0(m(t, x)), \quad \text{in } [0, T] \times \Omega, \quad (10)$$

$$\frac{\partial m}{\partial t}(t, x) - v \frac{\partial^2 m}{\partial x^2}(t, x) - \frac{\partial}{\partial x} \left( \frac{m(t, \cdot)}{(c_0 + c_1 m(t, \cdot))^\alpha} \left| \frac{\partial u}{\partial x}(t, \cdot) \right|^{\beta-2} \frac{\partial}{\partial x} u(t, \cdot) \right) (x) = 0, \quad \text{in } (0, T] \times \Omega, \quad (11)$$

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, 1) = 0, \quad \text{on } (0, T), \quad (12)$$

$$\frac{\partial m}{\partial x}(t, 0) = \frac{\partial m}{\partial x}(t, 1) = 0, \quad \text{on } (0, T), \quad (13)$$

$$u(T, x) = \phi(x, m(T, x)), \quad m(0, x) = m_0(x), \quad \text{in } \Omega. \quad (14)$$

**Discretization.** Let  $N_T$  and  $N_h$  be two positive integers. We consider  $N_T + 1$  and  $N_h$  points in time and space respectively. Set  $\Delta t = T/N_T$ ,  $h = 1/(N_h - 1)$ , and  $t_n = n \times \Delta t$ ,  $x_i = i \times h$  for  $(n, i) \in \{0, \dots, N_T\} \times \{0, \dots, N_h - 1\}$ .

We approximate  $u$  and  $m$  respectively by vectors  $U$  and  $M \in \mathbb{R}^{(N_T+1) \times N_h}$ , that is,  $u(t_n, x_i) \approx U_i^n$  and  $m(t_n, x_i) \approx M_i^n$  for each  $(n, i) \in \{0, \dots, N_T\} \times \{0, \dots, N_h - 1\}$ . We use a superscript and a subscript respectively for the time and space indices.

To take into account Neumann boundary conditions, we introduce ghost nodes  $x_{-1} = -h$ ,  $x_{N_h} = 1 + h$ , and set

$$U_{-1}^n = U_0^n, \quad U_{N_h}^n = U_{N_h-1}^n, \quad M_{-1}^n = M_0^n, \quad M_{N_h}^n = M_{N_h-1}^n. \quad (15)$$

We introduce the finite difference operators

$$\begin{aligned} (D_t W)^n &= \frac{1}{\Delta t} (W^{n+1} - W^n), & n \in \{0, \dots, N_T - 1\}, & \quad W \in \mathbb{R}^{N_T+1}, \\ (DW)_i &= \frac{1}{h} (W_{i+1} - W_i), & i \in \{0, \dots, N_h - 1\}, & \quad W \in \mathbb{R}^{N_h}, \\ (\Delta_h W)_i &= -\frac{1}{h^2} (2W_i - W_{i+1} - W_{i-1}), & i \in \{0, \dots, N_h - 1\}, & \quad W \in \mathbb{R}^{N_h}, \\ [\nabla_h W]_i &= ((DW)_i, (DW)_{i-1}) \in \mathbb{R}^2, & i \in \{0, \dots, N_h - 1\}, & \quad W \in \mathbb{R}^{N_h}, \end{aligned}$$

in which the special cases  $i = 0$  and  $i = N_h - 1$  can be written thanks to the above mentioned discrete version of the Neumann boundary conditions, see (15).

**Discrete Hamiltonian.** Let  $\tilde{H} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ ,  $(p_1, p_2, \mu) \mapsto \tilde{H}(p_1, p_2, \mu)$  be a discrete Hamiltonian, assumed to satisfy the following properties: for every  $\mu > 0$ ,

1. ( $\tilde{\mathbf{H}}_1$ ) Monotonicity:  $(p_1, p_2) \mapsto \tilde{H}(p_1, p_2, \mu)$  is nonincreasing in  $p_1$  and nondecreasing in  $p_2$ .
2. ( $\tilde{\mathbf{H}}_2$ ) Consistency:  $\tilde{H}(p, p, \mu) = H_0(p, \mu)$ .
3. ( $\tilde{\mathbf{H}}_3$ ) Differentiability:  $(p_1, p_2) \mapsto \tilde{H}(p_1, p_2, \mu)$  is of class  $C^1$ .
4. ( $\tilde{\mathbf{H}}_4$ ) Convexity:  $(p_1, p_2) \mapsto \tilde{H}(p_1, p_2, \mu)$  is convex.

**Example 2.1** Since  $H_0(p, \mu) = \frac{1}{\beta} \frac{|p|^\beta}{(c_0 + c_1 \mu)^\alpha}$ , a sensible choice for the discrete Hamiltonian is

$$\tilde{H}(p_1, p_2, \mu) = \frac{1}{\beta} \frac{((p_1)_-^2 + (p_2)_+^2)^{\frac{\beta}{2}}}{(c_0 + c_1 \mu)^\alpha}, \quad (16)$$

where  $X_+$ , resp.  $X_-$  stand for the positive (resp. negative) part of  $X$ :  $X = X_+ - X_-$  and  $|X| = X_+ + X_-$ , and where we set  $X_+^2 = (X_+)^2$  and  $X_-^2 = (X_-)^2$ .

The monotonicity of the discrete Hamiltonian guarantees uniqueness in the discrete HJB equations and discrete KFP equations below. It also guarantees that the solution of the discrete KFP equation below is non negative if its initial condition is non negative.

The consistency of the discrete Hamiltonian is a key ingredient for convergence of the discrete schemes.

The differentiability of the discrete Hamiltonian makes it possible to use Newton method for solving the discrete HJB equation below.

**Discrete HJB equation.** We consider the following discrete version of the HJB equation (10), supplemented with the Neumann conditions (12) and the terminal condition in (14):

$$\begin{cases} - (D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) = g(x_i) + \tilde{f}_0(M_i^{n+1}), & 0 \leq i < N_h, & (17a) \\ U_{-1}^n = U_0^n, & 0 \leq n < N_T, & (17b) \\ U_{N_h}^n = U_{N_h-1}^n, & 0 \leq n < N_T, & (17c) \\ U_i^{N_T} = \phi(M_i^{N_T}), & 0 \leq i < N_h. & (17d) \end{cases}$$

Note that this is an implicit Euler scheme since the equation is backward in time.

**Discrete KFP equation.** To define an appropriate discretization of the KFP equation (11), together with (13) and the initial condition in (14), we consider the weak form. For a smooth test function  $w \in C^\infty([0, T] \times \Omega)$ , it involves, among other terms, the expression

$$- \int_{\Omega} \partial_x (H_p(\partial_x u(t, x), m(t, x)) m(t, x)) w(t, x) dx = \int_{\Omega} m(t, x) H_p(\partial_x u(t, x), m(t, x)) \partial_x w(t, x) dx, \quad (18)$$

where we used an integration by parts and the Neumann boundary conditions assuming that  $H_p(x, 0, m) = 0$ . In view of what precedes, it is quite natural to propose the following discrete version of the right hand side of (18):

$$h \sum_{i=0}^{N_h-1} M_i^{n+1} \left( \tilde{H}_{p_1}([\nabla_h U^n]_i, M_i^{n+1}) \frac{W_{i+1}^n - W_i^n}{h} + \tilde{H}_{p_2}([\nabla_h U^n]_i, M_i^{n+1}) \frac{W_i^n - W_{i-1}^n}{h} \right),$$

and performing a discrete integration by parts, we obtain the discrete counterpart of the left hand side of

(18) as follows:  $-h \sum_{i=0}^{N_h-1} \mathcal{T}_i(U^n, M^{n+1}, M^{n+1}) W_i^n$ , where  $\mathcal{T}_i$  is the following discrete transport operator:

$$\begin{aligned} \mathcal{T}_i(U, M, \tilde{M}) = & \frac{1}{h} \left( M_i \tilde{H}_{p_1}([\nabla_h U]_i, \tilde{M}_i) - M_{i-1} \tilde{H}_{p_1}([\nabla_h U]_{i-1}, \tilde{M}_i) \right. \\ & \left. + M_{i+1} \tilde{H}_{p_2}([\nabla_h U]_{i+1}, \tilde{M}_{i+1}) - M_i \tilde{H}_{p_2}([\nabla_h U]_i, \tilde{M}_i) \right). \end{aligned}$$

Then, for the discrete version of (11), together with (13) and the initial condition in (14), we consider

$$\begin{cases} (D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}, \tilde{M}^{n+1}) = 0, & 0 \leq i < N_h, 0 \leq n < N_T, & (19a) \\ M_{-1}^n = M_0^n, & 0 < n \leq N_T, & (19b) \\ M_{N_h}^n = M_{N_h-1}^n, & 0 < n \leq N_T, & (19c) \\ M_i^0 = \bar{m}_0(x_i), & 0 \leq i < N_h, & (19d) \end{cases}$$

where  $U$  and  $\tilde{M}$  are fixed, and for example,

$$\bar{m}_0(x_i) = \frac{1}{h} \int_{|x-x_i| \leq h/2} m_0(x) dx,$$

or simply

$$\bar{m}_0(x_i) = m_0(x_i). \quad (20)$$

Here again, the scheme is implicit since the equation is forward in time.

**Remark 2.1** *For the implementation, it is important to realize that the matrix of the linear operator  $M \mapsto -\mathcal{T}(U^n, M, \tilde{M})$  is the conjugate of the Jacobian matrix of the map from  $\mathbb{R}^{N_h}$  to  $\mathbb{R}^{N_h}$ ,  $\mathbb{R}^{N_h} \ni U \mapsto (\tilde{H}([\nabla_h U]_i, \tilde{M}_i))_{0 \leq i < N_h}$ , computed at  $U = U^n$  and  $\tilde{M}^{n+1}$ .*

### 3 Indications for writing the code

**Solving the discrete HJB equation.** Finding  $(U^n)_n$  given  $(M^n)_n$  amounts to solving a discrete version of a nonlinear parabolic equation posed backward in time with Neumann boundary conditions. This is much simpler than solving the complete forward-backward system for  $(U, M)$ , because a backward time-marching procedure can be used. Since, as it was already observed above, the scheme is implicit, each time step consists of solving the discrete version of a nonlinear elliptic partial differential equation. Starting from the terminal time step  $N_T$  for which (17d) gives an explicit formula for  $U^{N_T}$ , the backward loop consists of computing  $U^n$  by solving (17a), (17b) (17c) given  $U^{n+1}, M^{n+1}$ . Let us rewrite this nonlinear system in the compact form  $\mathcal{F}(U^n, U^{n+1}, M^{n+1}) = 0$ . It is solved by means of Newton-Raphson iterations. The  $k$ -th Newton iteration reads

$$U^{n,k+1} = U^{n,k} - \mathcal{J}^{-1}(U^{n,k}, U^{n+1}, M^{n+1}) \mathcal{F}(U^{n,k}, U^{n+1}, M^{n+1}),$$

where  $U^{n,k}$  is the approximation of  $U^n$  at the  $k$ -th Newton iteration and where  $\mathcal{J}(V, U^{n+1}, M^{n+1}) \in \mathcal{M}_{N_h}(\mathbb{R})$  is the Jacobian matrix of the map  $V \mapsto \mathcal{F}(V, U^{n+1}, M^{n+1})$ .

**Remark 3.1** *The drift contribution in the assembly of the Jacobian matrix will also be useful for computing the matrix in the linear systems arising in the discrete Fokker-Planck equation, see Remark 2.1.*

Note that one may choose the initial guess  $U^{n,0} = U^{n+1}$ . The Newton iterations are stopped when the residual  $\|\mathcal{F}(U^{n,k}, U^{n+1}, M^{n+1})\|$  is smaller than a given threshold, say  $10^{-12}$ . It is well known that the Newton algorithm for (17a) is equivalent to an optimal policy iteration algorithm that it is convergent for any initial guess, and that the convergence is quadratic.

**Solving the discrete Fokker-Planck equation.** Finding  $(M^n)_n$  given  $(U^n)_n$  and  $(\tilde{M}^n)_n$  amounts to solving the discrete version of a linear parabolic equation posed forward in time with Neumann boundary conditions. Since the scheme is implicit, each time step consists of solving the discrete version of a linear elliptic partial differential equation. Starting from the initial time step 0 for which (19d) gives an explicit formula for  $M^0$ , the  $n$ -th step of the forward loop consists of computing  $M^{n+1}$  by solving (19a), (19b), (19c) given  $M^n$ . The matrix of the latter system of linear equations must be computed using Remark 2.1, see also the the previous paragraph.

**Fixed point iterations for the whole forward-backward system** Let  $\mathcal{M}$  stand for the collection  $(M^n)_{0,\dots,N_T}$  and  $\mathcal{U}$  stand for the collection  $(U^n)_{0,\dots,N_T}$ .

The program consists of approximating  $(\mathcal{M}, \mathcal{U})$  by fixed point iterations involving a relaxation parameter  $\theta$ .

Let  $\theta$  be a parameter,  $0 < \theta < 1$ ,  $\theta = 0.2$  or  $\theta = 0.02$  are often sensible choices. Let  $(\mathcal{M}^{(k)}, \mathcal{U}^{(k)})$  be the running approximation of  $(\mathcal{M}, \mathcal{U})$ . The next approximation  $(\mathcal{M}^{(k+1)}, \mathcal{U}^{(k+1)})$  is computed as follows:

1. Solve the discrete HJB equation given  $\mathcal{M}^{(k)}$ . The solution is named  $\widehat{\mathcal{U}}^{(k+1)}$ .
2. Solve the discrete FP equation given  $\widehat{\mathcal{U}}^{(k+1)}$  and  $\mathcal{M}^{(k)}$ . The solution is named  $\widehat{\mathcal{M}}^{(k+1)}$ .
3. Set  $(\mathcal{M}^{(k+1)}, \mathcal{U}^{(k+1)}) = (1 - \theta)(\mathcal{M}^{(k)}, \mathcal{U}^{(k)}) + \theta(\widehat{\mathcal{M}}^{(k+1)}, \widehat{\mathcal{U}}^{(k+1)})$ .

These iterations are stopped when the norm of the increment  $(\mathcal{M}^{(k+1)}, \mathcal{U}^{(k+1)}) - (\mathcal{M}^{(k)}, \mathcal{U}^{(k)})$  becomes smaller than a given threshold, say  $10^{-6}$ .

## 4 Work program

1. Simulate the MFG corresponding to the following data:

- $\Omega = ]0, 1[, T = 1$
- $g(x) = 0$
- $H_0$  is given by (7) and the discrete Hamiltonian is given by (16). Try the following sets of parameters:
  - (a)  $\beta = 2, c_0 = 0.1, c_1 = 1, \alpha = 0.5, \sigma = 0.02$
  - (b)  $\beta = 2, c_0 = 0.1, c_1 = 5, \alpha = 1, \sigma = 0.02$
  - (c)  $\beta = 2, c_0 = 0.01, c_1 = 2, \alpha = 1.2, \sigma = 0.1$
  - (d)  $\beta = 2, c_0 = 0.01, c_1 = 2, \alpha = 1.5, \sigma = 0.2$
  - (e)  $\beta = 2, c_0 = 1, c_1 = 3, \alpha = 2, \sigma = 0.002$
- $\tilde{f}(m(x)) = m(x)/10$
- $\phi(x, m) = -\exp(-40(x - 0.7)^2)$
- $m_0(x) = \sqrt{300/\pi} \exp(-300(x - 0.2)^2)$ .
- Choose for example  $N_h = 201, N_T = 100$ .
- The parameter  $\theta$  should be chosen in such a way that the fixed point method converges. Depending on the case of interest, the good values of  $\theta$  may lie in the interval  $[0.001, 0.2]$ . If the fixed point iterations fail to converge, then try to decrease  $\theta$ .

- Stopping criteria in the Newton method:  $10^{-12}$
- Stopping criteria in the fixed point method: not larger than  $2 \cdot 10^{-5}$  (with norms normalized so that  $\|(1, \dots, 1)\| = 1$ )

The outputs will be

- (a) The contour lines of  $u$  and  $m$  in the plane  $(x, t)$
  - (b) Animations (in the mp4 format) showing the evolution of  $u$  and  $m$  with respect to time.
2. Give an interpretation of the results in the different cases.

### How to check that the code is correct

- The Newton method in the HJB solver should converge fast (the number of iteration is of the order of 10) If it does not converge, it usually means that there are errors in the computation of the Jacobian matrix.
- $M$  should be positive if not there is an error in the code.
- Check that the total mass of  $m$  is conserved, i.e.  $\sum_{i=0}^{N_h-1} M_i^n$  should not depend on  $n$
- There may be a large number of fixed point iterations before convergence. If the fixed point method does not converge, decrease  $\theta$ .

### Theoretical questions

1. Understand why the mass of  $M^n$ , i.e.  $\sum_{i=0}^{N_h-1} M_i^n$ , does not depend on  $n$
2. Try to prove uniqueness in the discrete HJB equation, i.e. that given  $(M^n)_n$ ,  $(U^n)_n$  is unique. Hint: take two solutions  $(U^n)_n$  and  $(V^n)_n$ , and consider  $n_0, i_0$  such that  $\max_{(n,i)} (U_i^n - V_i^n)$  is achieved at  $(n_0, i_0)$ , and use the monotonicity of the discrete Hamiltonian.
3. Try to prove uniqueness in the discrete KFP equation, i.e. that given  $(U^n)_n$  and  $(\tilde{M}^n)_n$ ,  $(M^n)_n$  is unique. Hint: prove that the matrices of the linear systems arising in the discrete KFP are the conjugate of M-matrices.
4. Try to prove that if  $M^0$  is positive, then  $M^n$  is positive for all  $n$ .

## 5 Mean field control

As mentioned in the introduction, the theory of mean field games allows one to study Nash equilibria in games with a number of players tending to infinity. In such models, the players are selfish and try to minimize their own individual cost. Another kind of asymptotic regime is obtained by assuming that all the agents use the same distributed feedback strategy and by passing to the limit as  $N \rightarrow \infty$  before optimizing the common feedback. This is the situation when a planner controls a crowd of robots which interact via aggregate global or local quantities, in order to optimize a global criteria. For a fixed common feedback strategy, the asymptotic behavior is given by the McKean-Vlasov theory: the dynamics of a representative agent is found by solving a stochastic differential equation with coefficients depending on a mean field, namely



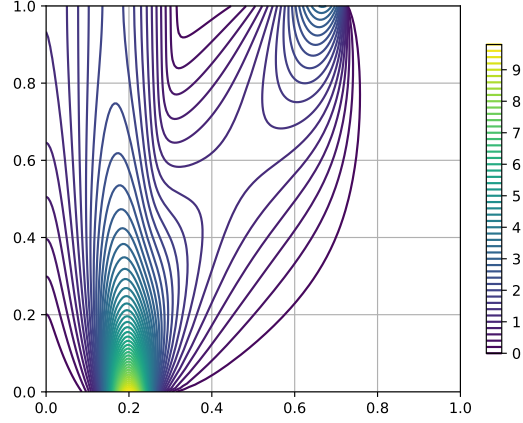


Figure 1: Contour lines of  $m$  in the second example (b) described above ( $x$  in abscissa,  $t$  in ordinate)

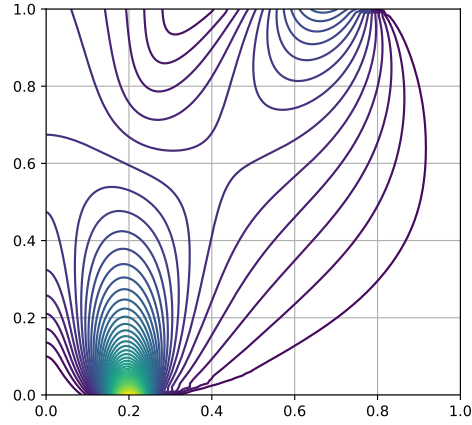


Figure 2: Contour lines of  $m$  in the fourth example (d) described above ( $x$  in abscissa,  $t$  in ordinate)

the statistical distribution of the states, which may also affect the objective function. Since the feedback strategy is common to all agents, perturbations of the latter affect the mean field (whereas in a mean field game, the other players' strategies are fixed when a given player chooses her strategy). Then, having each agent optimize her objective function amounts to solving a control problem driven by the McKean-Vlasov dynamics. The latter is named control of McKean-Vlasov dynamics or mean field control. Besides the aforementioned interpretation as a social optima in collaborative games with a number of agents growing to infinity, mean field control problems have also found applications in finance and risk management for problems in which the distribution of states is naturally involved in the dynamics or the cost function. Mean field control problems lead to a system of forward-backward PDEs which has some features similar to the MFG system, but which can always be seen as the optimality conditions of a minimization problem.

Using the same model of congestion as above, the system of PDEs that arises is the following:

$$-\frac{\partial u}{\partial t}(t, x) - \nu \frac{\partial^2 u}{\partial x^2}(t, x) + \frac{1}{\beta} \frac{|\frac{\partial u}{\partial x}(t, x)|^\beta}{(c_0 + c_1 m(t, x))^\alpha} - \frac{c_1 \alpha}{\beta} \frac{m(t, x) |\frac{\partial u}{\partial x}(t, x)|^\beta}{(c_0 + c_1 m(t, x))^{\alpha+1}} \quad (21)$$

$$= g(x) + \tilde{f}_0(m(t, x)) + m(t, x) \tilde{f}_0'(m(t, x)) \quad \text{in } [0, T] \times \Omega,$$

$$\frac{\partial m}{\partial t}(t, x) - \nu \frac{\partial^2 m}{\partial x^2}(t, x) - \frac{\partial}{\partial x} \left( \frac{m(t, \cdot)}{(c_0 + c_1 m(t, \cdot))^\alpha} \left| \frac{\partial u}{\partial x}(t, \cdot) \right|^{\beta-2} \frac{\partial}{\partial x} u(t, \cdot) \right) (x) = 0, \quad \text{in } (0, T] \times \Omega, \quad (22)$$

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, 1) = 0, \quad \text{on } (0, T), \quad (23)$$

$$\frac{\partial m}{\partial x}(t, 0) = \frac{\partial m}{\partial x}(t, 1) = 0, \quad \text{on } (0, T), \quad (24)$$

$$u(T, x) = \phi(x), \quad m(0, x) = m_0(x), \quad \text{in } \Omega. \quad (25)$$

It can be proved that if  $\beta > 1$ ,  $0 \leq \alpha < 1$  and  $\tilde{f}_0$  is convex, then there is a unique solution.

1. Modify the program in order to approximate numerically the PDE system (21),(22),(23), (24), (25)
2. Test for  $\beta = 2$ ,  $c_0 = 0.1$ ,  $c_1 = 1$ ,  $\alpha = 0.5$ ,  $\tilde{f}_0 = 0$ ,  $g = 0$ ,  $\sigma = 0.02$ . Compare with the solution of the mean field game with the same parameters.

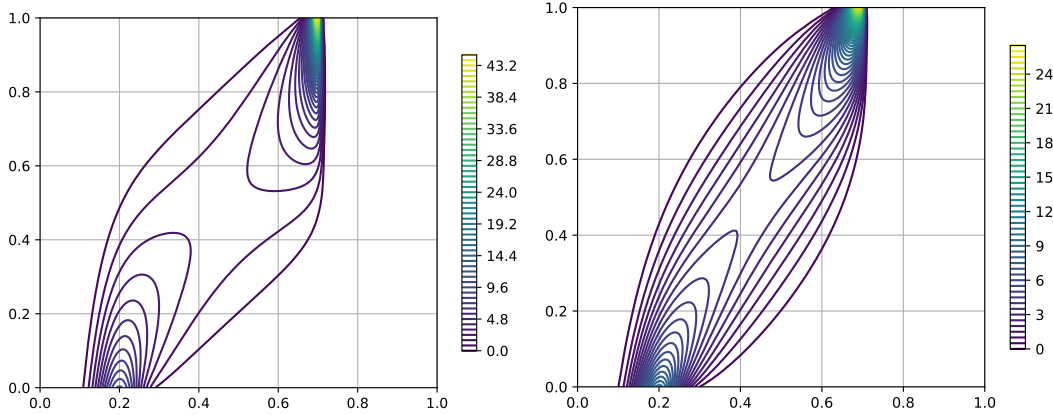


Figure 3: Left: Contour lines of  $m$  in the mean field control problem described above. Right: Contour lines of  $m$  in the mean field game with the same parameters ( $x$  in abscissa,  $t$  in ordinate)

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