Math Reference

Kaedon Cleland-Host

October 3, 2020

Contents

Chapter 1

Introduction

This chapter will offer reference and information that applies to the entire book.

1.1 Structure of This Book

1.1.1 Categories

Each section of this book will focus on one of these general categories

- **Notation -** The way that we choose to represent mathematics as is written down, each topic will have a notation page with symbol definitions and other important information
- **Number Systems -** Representations of a numbers and fundamental operations that we can run on these numbers (i.e. numbers, vectors, counting, complex numbers)
- **Structures -** Ways to organize numbers operations and units to represent something or to indicate something (i.e. equations, logical statements, foundation of proofs)
- **Methods -** Strategies for going between structures and representations of real things (i.e. integrals, derivatives, trigonometry, rref)

Chapter 2

Linear Algebra

2.1 Notation

General
$\forall$ - For all
$\exists$ - Exists

Common Sets
$\mathbb{C}$ - Set of all Complex Numbers
$\mathbb{R}$ - Set of all Real Numbers
$\mathbb{Q}$ - Set of all Rational Numbers
$\mathbb{Z}$ - Set of all Integers
$\mathbb{N}$ - Set of all Natural Numbers

Set Notation
$\in$ - "In" := is an element of

Example. $\vec{v} \in \mathbb{R}^3$

$\notin$ - "Not In" := is not an element of

Example. $\vec{v} \notin \mathbb{R}^3$

$\{,\}$ - Set := elements of the set are listed inside the brackets
Example: $A = \{1, 2, 3\}$ "A is a set containing the elements 1, 2, and 3"
Note: elements in a set must be unique
$\{\}$ *or* $\emptyset$ - The Empty Set

Definition 2.1.1. $|\,|$ - **Cardinality** := The size of a set or the number of elements in a set.

Example. $|A| = n$ "set A has a cardinality of n"

Definition 2.1.2. $\cap$ - **Intersection** := The **Intersection** of two sets in the set of all elements that are contained in both sets.

Example. $A \cap B = \{x : (x \in A)\ and\ (x \in B)\}$

Definition 2.1.3. $\cup$ - **Union** := The **Union** of two sets in the set of all elements that are contained either of the two sets.

Example. $A \cup B = \{x : (x \in A)\ or\ (x \in B)\}$

$\vee$ - or

Example. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$ - and

Example. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

2.2 Vectors and Bases

Definition 2.2.1. **Vector Space** := a collection of vectors equiped with operations of addition and scalar multiplication such that the following axioms are true:

- Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v} \; \forall \; \vec{v}, \vec{w} \in V$
- Associativity: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{w} + \vec{v}) \; \forall \; \vec{u}, \vec{v}, \vec{w} \in V$
- Zero Vector: $\exists$ a vector $\vec{0}$ such that for any vector $\vec{v} \in V, \; \vec{v} + \vec{0} = \vec{v}$
- Additive Inverse: for any vector $\vec{v} \in V$ there exists a vector $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$
- Multiplicative Identity: for any vector $\vec{v} \in V, \; (1)\vec{v} = \vec{v}$
-

Chapter 3

Real Analysis

3.1 Notation

General
$\forall$ - For all
$\exists$ - Exists

Common Sets
$\mathbb{C}$ - Set of all Complex Numbers
$\mathbb{R}$ - Set of all Real Numbers
$\mathbb{Q}$ - Set of all Rational Numbers
$\mathbb{Z}$ - Set of all Integers
$\mathbb{N}$ - Set of all Natural Numbers

Set Notation
$\in$ - "In" := is an element of

Example. $\vec{v} \in \mathbb{R}^3$

$\notin$ - "Not In" := is not an element of

Example. $\vec{v} \notin \mathbb{R}^3$

$\{,\}$ - Set := elements of the set are listed inside the brackets
Example: $A = \{1, 2, 3\}$ "A is a set containing the elements 1, 2, and 3"
Note: elements in a set must be unique
$\{\}$ *or* $\emptyset$ - The Empty Set

Definition 3.1.1. $|\ |$ - **Cardinality** := The size of a set or the number of elements in a set.

Example. $|A| = n$ "set A has a cardinality of n"

Definition 3.1.2. $\cap$ - **Intersection** := The **Intersection** of two sets is the set of all elements that are contained in both sets.

Example. $A \cap B = \{x : (x \in A)\ \mathit{and}\ (x \in B)\}$

the **Intersection** of many sets can be denoted: $\cap_{i=1}^{k} A_i$*For the set of elements that appear in all of* $A_1 \cdots A_k$

Definition 3.1.3. $\cup$ - **Union** := The **Union** of two sets is the set of all elements that are contained either of the two sets.

Example. $A \cup B = \{x : (x \in A)\ \mathit{or}\ (x \in B)\}$

the **Union** of many sets can be denoted: $\cup_{i=1}^{k} A_i$ *For the set of elements that appear in any of* $A_1 \cdots A_k$

Definition 3.1.4. $\subseteq$ - **Subset** := Set A is a **Subset** of B if all the elements of A are also elements of B. We denote this by $A \subseteq B$.

Definition 3.1.5. $\subsetneq$ - **Proper Subset** := Set A is a **Proper Subset** of B if $A \subseteq B$ and $A \neq B$

$\vee$ - or

Example. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$ - and

Example. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

3.2 Review of Set Theory

Definition 3.2.1. Two sets are consider to be equal if $A \subseteq B$ and $A \supseteq B$

Definition 3.2.2. **Pairwise Disjoint** := A set of sets $\Im$ is considered to be **Pairwise Disjoint** *if for* $S, T \in \Im$

$$S \neq T \Rightarrow S \cup T = \emptyset$$

There are two way of taking "differences" of sets:

$$X \backslash Y = \{x \in X : x \notin Y\}$$

$$X \Delta Y = (X \cup Y) \backslash (X \cap Y)$$

Proof. For any three finite sets X, Y, Z:
From the definition of Δ we find that:

$$(X \Delta Y) \Delta Z = \{x \in (X \cup Y) : x \notin (X \cap Y)\} \Delta Z = \{x \in ((X \cup Y) \cup Z) : x \notin ((X \cap Y) \cap Z)\}$$

Now, since $\cup$ and $\cap$ are associative we have:

$$\{x \in ((X \cup Y) \cup Z) : x \notin ((X \cap Y) \cap Z)\} = \{x \in (X \cup (Y \cup Z)) : x \notin (X \cap (Y \cap Z))\}$$

Now, from the definition of Δ we find that:

$$\{x \in (X \cup (Y \cup Z)) : x \notin (X \cap (Y \cap Z))\} = X \Delta \{x \in (Y \cup Z) : x \notin (Y \cap Z)\} = X \Delta (Y \Delta Z)$$

Therefore:

$$(X \Delta Y) \Delta Z = X \Delta (Y \Delta Z)$$

□

Definition 3.2.3. Given a set X and a set $\mathscr{S}$ whose elements are sets.

1. We say that $\mathscr{S}$ **covers** X if $X \subseteq \bigcup \mathscr{S}$
2. We say that $\mathscr{S}$ **partitions** X if $X = \bigcup \mathscr{S}$, the elements of $\mathscr{S}$ are non-empty, and $\mathscr{S}$ is pairwise disjoint

Definition 3.2.4. **Ordered Pair (tuple)** := an ordered list of two elements, each of which can be an arbitrary mathematical object and may or may not be the same. Denoted for $n \in \mathbb{N}$, an n-tuple is an ordered list of n elements, written as $(x_1, \ldots, x_n)$

Definition 3.2.5. For two sets X, Y the **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. More generally we can write a **Cartesian product** for n sets denoted by

$$X_1 \times X_2 \times \ldots \times X_n \textit{ or } \prod_{i=1}^{n} X_i$$

Remark. When taking the **Cartesian product** of the same set we use the shorthand: X^n

Remark. Additionally, the notation 2^X indicates the set of all possible subsets of X

Definition 3.2.6. We say that the **diagonal** of X^n is the subset $\{(x_1, \ldots, x_n) \in X^n : x_1 = x_2 = \ldots = x_n\}$

Definition 3.2.7. Given two sets X, Y we say that f is a **function** with domain X and codomain Y denoted $f : X \to Y$, if f is a subset of $X \times Y$ such that every element of X appears as exactly the first component of exactly one element of f.

Example. We used the notation $f(x)$ to refer to the element y such that $(x, y) \in f$ is the unique ordered pair that refers to the element $x \in X$.

Definition 3.2.8. The **Identity Function** is a function with the same domain and codomain X written $\mathbf{1}_X : X \to X$ corresponding to the diagonal 3.2.6 of X^2

Definition 3.2.9. Given $f : X \to W$ and $g : W \to Z$ with $Y \subseteq W$, the composition $g \circ f : X \to Z$ is the function satisfying $g \circ f(x) = g(f(x))$.

Definition 3.2.10. A function is **Injective** if $f(x) = f(u) \Rightarrow x = y$

Definition 3.2.11. A function $f : X \to Y$ is **Surjective** if the range of f equals Y

Definition 3.2.12. A function is **Bijective** if it is both Injective and Surjective

Theorem 3.2.1. If X is non-empty, $f : X \to Y$ is injective $\Leftrightarrow$ f is left invertible

Theorem 3.2.2. $f : X \to Y$ is surjective $\Leftrightarrow$ f is right invertible

Definition 3.2.13. A **Relation** of a set X is a subset of X^2. Conventionally written xRy rather than $(x, y) \in R$

3.2.14. Properties of Relation

1. **Reflexive** if xRx for all $x \in X$
2. **Transitive** if xRy and $yRz \Rightarrow xRz$
3. **Symmetric** if $xRy \Leftrightarrow yRx$
4. **Antisymmetric** if xRy and $yRx \Rightarrow x = y$
5. **Connex** if for every $x, y \in X$ at least on of xRy or yRx hold.

Definition 3.2.15. An **Equivalence Relation** is a relation that is Reflexive, Transitive, and Symmetric

Definition 3.2.16. if is an equivalence relation, the **Equivalence Class** of $x \in X$ is $[x] := \{y \in X : x \ \ y\}$. Additionally, the notation $X/$ refers to the set of all equivalence classes $\{[x] : x \in X\}$

3.2.1 The sets $\mathbb{Z}$ and $\mathbb{Q}$

Definition 3.2.17. The natural numbers $\mathbb{N}$ with it's addition and multiplication forms a commuative semiring.

Definition 3.2.18. A **commutative semi-ring** is set R equipped with binary operations addition and multiplication such that

1. $(R, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(R, \cdot)$ Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition 3.2.19. A **commutative ring** is set R equipped with binary operations addition and multiplication such that

1. $(R, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(R, \cdot)$ Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.

Definition 3.2.20. The integers $\mathbb{Z}$ is defined as a set of equivalence classes 3.2.16 $\mathbb{N}/\sim$ where the equivalence relation $\sim$ is

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

Remark. This is simply a rigorous way of representing integers with two natural numbers where the first natural number is the positive component and the second number is the negative component.

Definition 3.2.21. A **field** is set R equipped with binary operations addition and multiplication such that

1. $(R, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(R, \cdot)$ Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.
7. Multiplicative inverse.

Definition 3.2.22. The rational numbers $\mathbb{Q}$ is defined as a set of equivalence classes 3.2.16 $(\mathbb{Z} \times \mathbb{N})/\sim$ where the equivalence relation $\sim$ is

$$(a, n) \sim (b, m) \Leftrightarrow am = bn$$

Remark. This is simply a rigorous way of representing rational numbers as a fraction from integers and natural numbers where the integer is the numerator and the natural number is the denominator.

3.2.2 Cardinality of Sets

Definition 3.2.23. The **cardinality** of a set is the number of elements in that set.

- $card(A) = card(B)$ if there exists and bijective function: $A \to B$
- $card(A) \leq card(B)$ if there exists an injective(left invertible) function: $A \to B$
- $card(A) \geq card(B)$ if there exists an surjective(right invertible) function: $A \to B$

Proposition 3.2.1. (Pigeonhole Principle). Suppose $n < m$ there does not exist an injective(left invertible) function: $\{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ and there does not exist a surjective(right invertible) function: $\{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$

Definition 3.2.24. A set X is said to be

- **countable** if $card(X) \leq card(\mathbb{N})$
- **uncountable** if $card(X) > card(\mathbb{N})$
- **finite** if $\exists\ n \in \mathbb{N}$ such that $card(X) \leq card(\{1, \dots, n\})$
- **countably infinite** if $card(X) = card(\mathbb{N})$
- **infinite** if $card(X) \geq card(\mathbb{N})$

3.3 Partial Orders

Definition 3.3.1. A Partial Order is a relation $\preceq$ that is transitive, reflexive, and antisymmetric

Definition 3.3.2. Poset is a set that is equipped with a partial order.

Definition 3.3.3. Let $(X, \preceq)$ and $(U, \trianglelefteq)$ be posets we say a function $f : X \to Y$ is...

- **increasing** if $x_1 \preceq x_2 \Rightarrow f(x_1) \trianglelefteq f(x_2)$
- **decreasing** if $x_1 \preceq x_2 \Rightarrow f(x_2) \trianglelefteq f(x_1)$
- **monotone** if it is either increasing or decreasing (*note: the constant function is both increasing and decreasing*)
- **strictly increasing/decreasing/monotone** if it is increasing/decreasing/monotone and injective.
- **an order isomorphism** if it is invertible and both f and f^{-1} are increasing.

Definition 3.3.4. Let $(X, \preceq)$ and be a poset. Define the two functions $\uparrow, \downarrow : X \to 2^X$ by

- $\downarrow(x) : \{y \in X : y \preceq x\}$, a subset is a **lower set** or **downward closed** if $s \in S \Rightarrow \downarrow(s) \subseteq S$.
- $\uparrow(x) : \{y \in X : x \preceq y\}$, a subset is an **upper set** or **upper closed** if $s \in S \Rightarrow \uparrow(s) \in subseteqS$

Definition 3.3.5. a lower(upper) set S is said to be **principal** if there exists $x \in X$ such that $\downarrow(x) = S(\uparrow(x) = S)$

Definition 3.3.6. Let $(X, \preceq)$ and be a poset and let $S \subseteq X$, and $z \in X$

- We say that z is an **Upper bound** of S if $S \subseteq \downarrow(z)$. The set s is said to be **bounded above** if it has an upper bound.
- We say that z is a **Lower bound** of S if $S \subseteq \uparrow(z)$. The set s is said to be **bounded bellow** if it has a lower bound.
- We say that S is order bounded (*or just bounded*), if it is bounded both above and below.

Definition 3.3.7. Let $(X, \preceq)$ and be a poset a subset $S \subseteq X$ is said to be...

- **downward directed** if every finite subset has a lower bound $z \in S$
- **upward directed** if every finite subset has a upper bound $z \in S$

3.3.1 Special Elements

Definition 3.3.8. Let $(X, \preceq)$ be a poset, and let $S \subseteq X$. We say that an element of $s_0 \in S$ is...

- **the maximum of** S if $S \subseteq \downarrow (s_0)$
- **the minimum of** S if $S \subseteq \uparrow (s_0)$
- **a maximal element of** S if, for $s \in S$, $s_0 \in \downarrow (s) \Rightarrow s_0 = s$
- **a minimal element of** S if, for $s \in S$, $s_0 \in \uparrow (s) \Rightarrow s_0 = s$

Definition 3.3.9. Let $(X, \preceq)$ be a poset and $S \subseteq X$. We say that an element of $x \in X$ is...

- **the supremum of** S if $x = \min\{y \in X : S \subseteq \downarrow (y)\}$
- **the infimum of** S if $x = \max\{y \in X : S \subseteq \uparrow (y)\}$

3.4 Total Orders

Definition 3.4.1. A Total Order is a relation $\preceq$ that is transitive, reflexive, antisymmetric, and connex.

Definition 3.4.2. A Well Ordered Set is totally ordered set where every non-empty subset has a minimum.

Theorem 3.4.1. Totally ordered sets cannot contain imaginary numbers.

Definition 3.4.3. A **totally ordered field** is a field F equipped with a total order $\preceq$ such that

- $\preceq$ respects addition: $a \preceq b \Rightarrow a + c \preceq b + c$
- $\preceq$ respects positive multiplication: $0 \preceq a \Rightarrow a + c \preceq b + c$

Definition 3.4.4. In a totally ordered field, the set of **positive** elements is $\uparrow (0)\backslash 0$. The set of **negative** elements is $\downarrow (0)\backslash 0$.

Definition 3.4.5. Given a totally ordered field $(F, \preceq)$. The absolute value function $F \Rightarrow F$, denoted by $x \to |x|$, is

$$|x| = \begin{cases} x & 0 \preceq x \\ -x & x \preceq 0 \end{cases}$$

Proposition 3.4.1. Let $(X, \preceq)$ be a total order. Let A, B be both upper(both lower) sets. Either $A \subseteq B$ or $B \subseteq A$

Definition 3.4.6. Let $\{X, \preceq\}$ be a total order. We say a pair of subsets (X_-, X_+) form a **cut** of X if:

- $\{X_-, X_+\}$ is a partition of X.
- X_- is a lower set and X_+ is an upper set.

Definition 3.4.7. A totally ordered set X is said to be **Dedekind complete** if in every cut (X_-, X_+), at least one of X_- or X_+ is principal. That is $\exists x \in X_-$ such that $\downarrow (x) = X_-$ or $\exists x \in X_+$ such that $\uparrow (x) = X_+$

Proposition 3.4.2. Let $(X, \preceq)$ be a Dedekind total order. The total order restricted to $\uparrow (a)$ and $\downarrow (a)$ for any $a \in X$ is also Dedekind complete.

Definition 3.4.8. A poset $(X, \preceq)$ is said to posses the **least upper bound property** if every nonempty subset that is bounded above has a supremum. Similarly, a poset $(X, \preceq)$ is said to posses the **greatest lower bound property** if every nonempty subset that is bounded bellow has an infimum.

Theorem 3.4.2. For a totally ordered set $(X, \preceq)$

- X is a Dedekind complete
- X has the least upper bound property
- X has the greatest lower bound property

Remark. This theorem also holds for posets if you remove the Dedekind complete line since the definition of Dedekind complete relies a total order. In general we can also define Dedekind complete as a poset that has the least upper bound property and greatest lower bound property.

Definition 3.4.9. Given $(X, \preceq)$ a partial order and elements $a, b \in X$. The **open interval** (a, b) is defined to be $\uparrow(a) \cap \downarrow(b) \backslash \{a, b\}$

Definition 3.4.10. Given $(X, \preceq)$ a partial order and elements $a, b \in X$. The **closed interval** (a, b) is defined to be $\uparrow(a) \cap \downarrow(b)$

Definition 3.4.11. Given $(X, \preceq)$ as total order with no max and no min, an **entourage mapping** is a function $f : X \to 2^X$ such that $f(x)$ is an open interval that contains x.

Definition 3.4.12. Given $(X, \preceq)$ as total order with no max and no min, we say that it possesses the **Heine-Borel property** if, for every closed interval $[a, b]$ and every entourage mapping f, there exists a finite subset $S \subseteq [a, b]$ such that $f(S)$ covers $[a, b]$.

Theorem 3.4.3. Suppose $(X, \preceq)$ as total order with no max and no min. Then it is Dedekind complete if and only if it possesses the Heine-Borel property.

3.4.1 The Reals

Definition 3.4.13. The set $\mathbb{R}$ is defined to be the set of all cuts (X_-, X_+) of $\mathbb{Q}$ such that X_- has no maximum.

Definition 3.4.14. We equip $\mathbb{R}$ with the relation $\leq$ defined as $(X_-, X_+) \leq (Y_-, Y_+)$ if $X_- \subseteq Y_-$.

Theorem 3.4.4. Archimedean Property of Reals If x, y are positive real numbers then $\exists n \in \mathbb{N}$ such that $n \cdot x > y$.

Definition 3.4.15. We say a subset $U \subseteq \mathbb{R}$ is open if if for every $x \in U$, there is an open interval $(a_X, b_X) \subseteq U$ with $x \in (a_X, b_X)$

3.5 Nets and Limits

Definition 3.5.1. A **directed set** is a pair $(X, \preceq)$ where X is a set equipped with a relation $\preceq$ such that

- $\preceq$ is reflexive.
- $\preceq$ is transitive.
- $\preceq$ is upward directed; for any $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.

Lemma 3.5.1. Let $(X, \preceq)$ be a directed set. Let $x \in X$ the set $\uparrow(x)$ equipped with the restricted order, is a directed set. Additionally any total order is a directed set.

3.5.1 Nets

Definition 3.5.2. A **net** is a function $f : A \to B$ from a directed set $(A, \preceq)$.

Remark. Notationally instead of $f(a)$ we usually write f_a.

Definition 3.5.3. Given a net $f : A \to B$, we denote the **tail sets** of f by

$$f_{\uparrow(\alpha_0)} = \{f_\alpha \in B : \alpha \preceq \alpha_0\}$$

Definition 3.5.4. For a net $f : A \to B$ and a subset $S \subseteq B$ we say that:

- f is **eventually** in S if there exists some $a \in A$ such that $f_{\uparrow(a)} \subseteq S$.
- f is **frequently** in S if for every $a \in A$, the intersection $f_{\uparrow(a)} \cap S \neq \emptyset$
- f is **infrequently** in S if there exists some $a \in A$ such that $f_{\uparrow(a)}$ is disjoint from S.

3.5.2 Limits

Definition 3.5.5. A real valued net f is said to **Converge** to z if for every open interval I containing z, the net f is eventually in I. When f converges to z we sat that z is the **limit** of f and write $z = \lim f$.

Lemma 3.5.2. Let B be a set and $f : A \to B$ be a non-empty net. Fix a subset $S \subseteq B$.

- A net f is either frequently in S or infrequently in S.
- A net f is eventually in S if and only if f is infrequently in $X \backslash S$.

- If a net f is eventually in S, then f is frequently in S.

Definition 3.5.6. A real valued net f is said to **accumulate** (or **cluster**) at the real number z if for every one interval I containing z, the net x is frequently in I.

Proposition 3.5.1. If a real valued net f covers z, then z is its unique accumulations point.

Theorem 3.5.3. **Arithmetic of Limits** states that if x, y are real valued nets with the same domain then

$$\lim(x+y) = \lim(x) + \lim(y)$$

$$\lim(x \cdot y) = \lim(x) \cdot \lim(y)$$