Math 347H Lecture on Impulse Functions

March 16, 2020

Homework for the week of March 16 to March 20.

Section 6.4. 1, 3, 7, 9, 11.

Section 6.5. 1, 5, 7, 9, 25*

Section 6.6. 3, 5, 10, 13, 17.

Impulse is a large magnitude of force (voltage) over a very short time. That is, $g(t)$ is big in an interval $[t_0 - \tau, t_0 + \tau]$, and is zero other places.

How to measure the strength of a pulse? It is quite natural that we define $I(\tau) = \int g(t)dt$ is the total pulse of the force g.

Example $g = d_\tau(t - t_0) = \frac{1}{2\tau}\begin{cases} 1, & t - t_0 \in [-\tau, \tau], \\ 0, & \text{otherwise,} \end{cases}$ then $I(\tau) = 1$. When $\tau \to 0$, $g(t) \to 0$ for any $t \neq t_0$. The limit is called the delta function at t_0, and is denoted as $\delta(t - t_0)$.

What is δ? It is the limit of a sequence functions $f_n \geq 0$ supported in $(-1/n.1/n)$ such that $\int f_n(t)dt = 1$.

To be precise, the delta function is not a function, it is just a notation, but is very useful in many applications, and we can compute its Laplace transform.

$$\int_0^\infty \delta(t - t_0)e^{-st}dt = \lim_{\tau\to 0} \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}dt = \lim_{\tau\to 0} \frac{1}{2\tau}\frac{1}{-s}[e^{-s(t_0+\tau)} - e^{-s(t_0-\tau)}]$$

Using L'Hopital's rule, we have

$$\int_0^\infty \delta(t-t_0)e^{-st}dt = e^{-st_0}.$$

Actually for any sequence function $f_n \to \delta_(t-t_0)$, we have

$$\begin{aligned} & \int_0^\infty \delta(t-t_0)e^{-st}dt = \lim_{n\to\infty}\int f_n e^{-st}dt \\ = & \lim_{n\to\infty}\int f_n(t)[e^{-st}-e^{-st_0}] + e^{-st_0}\int f_n(t)dt = e^{-st_0}. \end{aligned}$$

Furthermore, for any continuous function $g(t)$, we have

$$\int \delta(t-t_0)g(t)dt = \int \delta(t-t_0)[g(t)-g(t_0)] + g(t_0)\int \delta(t-t_0)dt = g(t_0).$$

In more advanced mathematics, it is called a distribution, a continuous functional for differentiable functions. it has "derivatives"!

Example 1. Find the solution of

$$\begin{cases} y'' + 4y = 2\delta(t-4\pi), \\ y(0) = 1/2, \ y'(0) = 0. \end{cases}$$

The system is acted by the force near $t_0 = 4\pi$ for a very short time and total force is 2.

Using Laplace transform to find

$$Y(s)(s^2+4) = s/2 + 2e^{-4\pi s}, \ \ Y(s) = \frac{2e^{-4\pi s}}{s^2+2^2} + \frac{s}{2(s^2+4)},$$

$$\begin{aligned} y(t) &= \tfrac{1}{2}\cos 2t + \sin 2(t-4\pi)u_{4\pi}(t) = \tfrac{1}{2}\cos 2t + \sin 2t \ u_{4\pi}(t) \\ &= \begin{cases} \tfrac{1}{2}\cos 2t, & t \in [0, 4\pi), \\ \tfrac{1}{2}\cos 2t + \sin 2t, & t \geq 4\pi. \end{cases} \end{aligned}$$

Check that $y(t)$ is continuous everywhere, but there is a jump of $y'(t)$ at $t = 4\pi$.

Physical Explanation?

Example 2. Find the solution of

$$\begin{cases} y'' + 2y' + 5y = \delta(t-5) \\ y(0) = 0 = y'(0) \end{cases}$$

The system is acted by the force near $t_0 = 5$ for a very short time and total force is 1.

Using Laplace transform to find

$$Y(s)(s^2 + 2s + 5) = e^{-5s}, \qquad Y(s) = \frac{e^{-5s}}{(s+1)^2 + 2^2},$$

From $L^{-1}\{\frac{2}{s^2+2^2}\} = \sin(2t)$, we have $L^{-1}\{\frac{2}{(s+1)^2+2^2}\} = e^{-t}\ \sin(2t)$, and

$$y(t) = \frac{1}{2}\, e^{-t+5}\ u_5(t)\ \sin 2(t-5).$$

Is $y(t)$ continuous? Is $y'(t)$ continuous?

Work out the following examples during the class if time permits.

Example 3. Consider the equation $y'' + 9y = -\delta(t-\pi) + \delta(t-2\pi), \quad y(0) = 0, \ \ y'(0) = 0.$.

Find the following values. $y(1), \ \ y'(1), \ \ y(\pi), \ \ y(\frac{5\pi}{2}), \ \ y'(\frac{5\pi}{2})$.

Example. 4*. Consider the problem $y'' + 9y = -\delta(t-\pi) + k\delta(t-t_0), \quad y(0) = 0, \ \ y'(0) = 0.$.

The system is excited when $t = \pi$. Can you find a constant k, and time $t_0 > \pi$ such that it bring the system to rest again after one cycle ($t \geq \pi + 2\pi/3$) ?

Can you bring the system to the rest with a damping term? if yes, size of k?