Lecture on Kepler's Laws – Math 347H

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Johannes Kepler (1571-1630)

Kepler was born poor and sickly in what is now Germany. His father left home when he was five and never returned. It is believed he was killed in a war. While he was pursuing higher education his mother was tried as a witch. He hired a legal team which was able to obtain her release, mostly on legal technicalities. Although he had an eventful life, Kepler is most remembered for "cracking the code" that describes the orbits of the planets. Prior to Kepler's discoveries, the predominate theory of the solar system was an Earth-centered geometry. A Sun-centered theory had been proposed by Copernicus, but its predictions were plagued with inaccuracies.

Newton's Law of Gravity

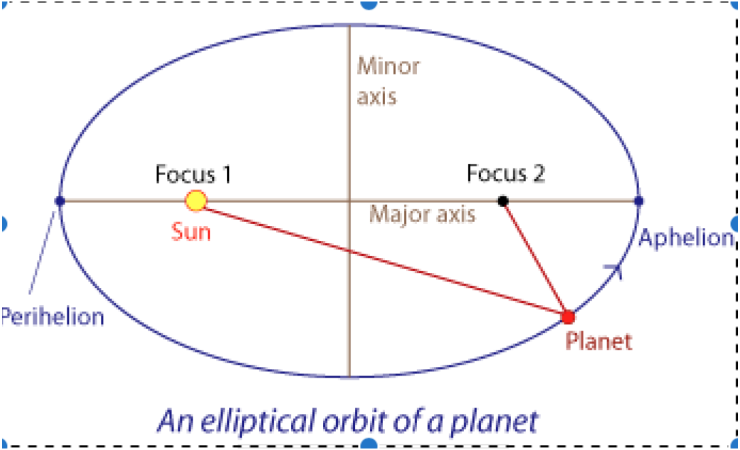
Working in Prague at the Royal Observatory of Denmark, Kepler succeeded by using the notes of his predecessor, Tycho Brahe, which recorded the precise position of Mars relative to the Sun and Earth.

Kepler developed his laws empirically from observation, as opposed to deriving them from some fundamental theoretical principles.

About 30 years after Kepler died, Isaac Newton (1642-1727) was able to derive Kepler's Laws from basic laws of gravity, which astounded the world of science of his day, and was one of the most remarkable achievements in the history of science. This course is a suitable place to study this beautiful proof.

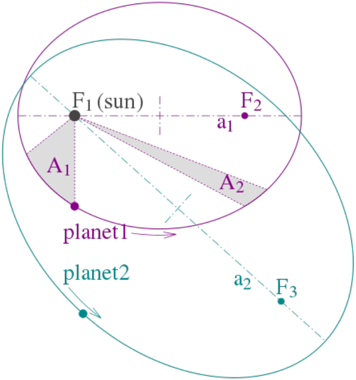
Kepler's First Law

The orbit of any planet is an ellipse with the sun at one focus.



Kepler's Second Law

The position vector from the sun to the planet sweeps out equal area in equal time.



Kepler's Third Law: Law of Period

$T =$ Period of the orbit

$a =$ Semi major axis of the orbit

$T^2 = \frac{4\pi^2}{MG} a^3$.

If we use the following units:

T: Years

a: AU = Astronomical units. $a = 1$ AU for earth

M: Solar masses (1 solar mass is the mass of the sun)

$\frac{4\pi^2}{MG} = 1$. Use this unit, $T^2 = a^3$.

Table: Planet Data

Planet	Semi-major axis [AU]	Period [days]	$\frac{a^3}{T^2} * 10^{-6}[\frac{\mathrm{AU}^3}{\mathrm{day}^2}]$
Mercury	0.38710	87.9693	7.496
Venus	0.72333	224.7008	7.496
Earth	1	365.2564	7.496
Mars	1.52366	686.9796	7.495
Jupiter	5.20336	4332.8201	7.504
Saturn	9.53707	10775.599	7.498
Uranus	19.1913	30687.153	7.506
Neptune	30.0690	60190.03	7.504

Gravitational Force

Let M be the mass of the sun, and m the mass of a planet. Let us fix the center of the mass of the sun at the origin $(0, 0, 0)$. The gravitational force on the planet is given by

$$\vec{F} = -\frac{GmM}{r^3}\vec{r},$$

where $\vec{r} = (x, y, z)$ is the position vector, G is the gravitational constant ($G = 6.67 \times 10^{-11}$ newton meter^2/Kilogram). The force $\vec{F}$ is also called central force since it directed toward the origin. By Newton's law, we have

$$m\vec{r}'' = F = -\frac{GmM}{r^3}\vec{r}, \quad \vec{r}'' = -\frac{GM}{r^3}\vec{r}.$$

First we want to make sure that the planet is moving on a plane.

Review inner and cross products during the lecture

Moving on a plane

Theorem 1. If a particle is moving under a central force $\vec{F} = f(\vec{r})\vec{r}$, then the particle is moving on a plane.
Proof. Consider $\vec{H} = \vec{r} \times \vec{v}$, where $\vec{v} = \vec{r}\,'$. Then

$$\vec{H}'(t) = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{0}$$

since $\vec{a} = \frac{\vec{F}}{m}$ is parallel to $\vec{r}$. Hence $\vec{H}(t) \equiv \vec{C}$.
If $\vec{C} = 0$, then $\vec{v}(t) = g(t)\vec{r}(t)$, which implies that

$$\vec{r}(t) = e^{\int_0^t g(s)ds}\vec{r}(0),$$

and the particle moves along a line.
If $\vec{C} \neq \vec{0}$, then from the definition of cross product, $\vec{C} \cdot \vec{r} = 0$, which implies that the particle moves on the plane through the origin and perpendicular to $\vec{C}$.

Polar Coordinates

Now we can restrict our attention to a central force $\vec{F}$ on xy-plane. Introduce the polar coordinates as usual. Any vector $\vec{r} = x\vec{i} + y\vec{j}$ can be represented as $\vec{r}(t) = r(t)\hat{r}$, where $r = \|\vec{r}\|$, $\hat{r}$ is the unit vector in the direction of $\vec{r}$. The unit vector perpendicular to $\hat{r}$ such that $(\hat{r}, \hat{\theta}, \hat{k})$ forms a right-handed system is denoted by $\hat{\theta}$.

$$\hat{r} = \cos\theta\vec{i} + \sin\theta\vec{j}, \hat{\theta} = -\sin\theta\vec{i} + \cos\theta\vec{j}.$$

By the chain rule,

$$\frac{d}{dt}\hat{r}(t) = \theta'(t)[-\sin\theta\vec{i} + \cos\theta\vec{j}] = \theta'(t)\hat{\theta}(t).$$

Conservation of Angular Momentum

Using the position function $\vec{r}(t) = r(t)\hat{r}(t)$, its derivative is give by

$$\vec{r}\,'(t) = r\,'(t)\hat{r} + r(t)\theta\,'(t)\hat{\theta}. \tag{0.1}$$

From $\vec{r} \times \vec{r}\,'$ is a constant vector, we know that

$$r\hat{r} \times [r\,'\hat{r} + r\theta\,'\hat{\theta}] = r^2\theta\,'\vec{k} \equiv \vec{C},$$

which implies the following conservation of Angular momentum.
Theorem 2. For a particle moving under the central force, we have

$$m(r(t))^2\theta'(t) \equiv h. \tag{0.2}$$

If $h = 0$, then $\theta'(t) = 0$, the particle moves on a straight line. This situation is not very interesting. If $h \neq 0$, the sign of $\theta'(t)$ is fixed along the curve, i.e., the θ is always increasing or alway decreasing with time. Hence r is also a function of θ along the curve.

Kepler's Second Law

It is enough to prove the Kepler's second law now: The line segment joining a planet to the sun sweeps out equal areas in equal times.

Proof. Let $A(t)$ be the area swept out by the vector $\vec{r}$ from time 0 to time t. We want to show that $A'(t) \equiv c$. Note that

$$\frac{A(t+\triangle t)-A(t)}{\triangle t} \sim \frac{1}{2} r^2(t) \frac{\theta(t+\triangle t)-\theta(t)}{\triangle t}.$$

$$A'(t) = \frac{1}{2} r^2(t) \theta'(t) \equiv \frac{h}{2m}.$$

We now turn our attention to Kepler's first law by using ellipse equation in polar coordinates. .

Ellipse Equation in Polar Coordinates

The idea is to show that $r(1+\epsilon\cos\theta) \equiv B\epsilon$.
In fact, if we can show this, then

$$r^2 = (B\epsilon - \epsilon r\cos\theta)^2,$$

$$x^2 + y^2 = \epsilon^2[B^2 - 2Bx + x^2],$$

$$(1-\epsilon^2)(x + \frac{B\epsilon^2}{1-\epsilon^2})^2 + y^2 = \epsilon^2 B^2[1 + \frac{\epsilon^2}{1-\epsilon^2}] = \frac{B^2\epsilon^2}{1-\epsilon^2},$$

$$\frac{(x+x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x_0 = \frac{B\epsilon^2}{1-\epsilon^2}, \qquad a^2 = \frac{B^2\epsilon^2}{(1-\epsilon^2)^2}, \qquad b^2 = \frac{B^2\epsilon^2}{1-\epsilon^2}, \tag{0.3}$$

which is the equation of an ellipse with $(0,0)$ a focus. $\epsilon \in [0,1)$ is the eccentricity.

More Remarks on Ellipse during the class

What happens if $\epsilon = 1$, $\epsilon > 1$?

Total Energy

In order to prove Kepler's first law, we first prove the following conservation law of the energy.

Theorem 3. The total energy of the planet is conserved. i.e.,

$$\frac{1}{2}m\|\vec{r}\,'\|^2 - \frac{mMG}{\|\vec{r}\|} \equiv E.$$

Proof. The first term in the expression of E is usually called the kinetic energy, and the second term is called the potential energy. The proof is very easy. A direct computation shows that

$$\frac{dE}{dt} = m\vec{r}\,' \cdot \vec{r}\,'' + \frac{mMG}{r^3}\vec{r} \cdot \vec{r}\,' = 0$$

from Newton's law.

In terms of polar coordinates, we have

$$E = \frac{1}{2}m[(r'(t))^2 + (r(t)\theta'(t))^2 - \frac{mMG}{r}. \qquad (0.4)$$

Differential Equation

Since the equation for $r = r(\theta)$ is relatively complicated. Instead we consider $u = \frac{1}{r}$, using the conservation of angular momentum $mr^2\theta' = h$,

$$\begin{aligned} \frac{du}{d\theta} &= -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} \\ &= -\frac{1}{r^2}\frac{dr}{dt}\frac{1}{\theta'} = -\frac{mr'(t)}{h}. \end{aligned} \tag{0.5}$$

The conservation of energy implies that

$$\frac{1}{2}m(\frac{h}{m}\frac{du}{d\theta})^2 + \frac{h^2u^2}{2m} = E + mMGu,$$

$$(\frac{du}{d\theta})^2 + u^2 = \frac{2m}{h^2}(E + mMGu).\text{Nonlinear Equation!} \tag{0.6}$$

Differentiating with respect to θ on the both sides

$$\frac{d^2u}{d\theta^2} + u = \frac{m^2GM}{h^2}. \tag{0.7}$$

Solution of u Using Undetermined Coefficients

We know that the solution of (0.7) has the form

$$u = \frac{m^2 GM}{h^2} + A\ \cos(\theta - \theta_0), \tag{0.8}$$

where $A \geq 0$ is a constant, $\theta_0 \in [0, 2\pi)$. To simplify the final polar equation of the orbit further, we choose the direction of the polar axis so that r is minimum when $\theta = 0$, this requires that $\theta_0 = 0$ gives the maximum of u. Hence $\theta_0 = 0$. i.e.,

$$u = \frac{m^2 GM}{h^2} + A\ \cos(\theta).$$

Kepler's First Law

$$r = \frac{1}{u} = \frac{B\epsilon}{1 + \epsilon \cos\theta}, \tag{0.9}$$

where

$$B\epsilon = \frac{h^2}{m^2 MG}, \quad \epsilon = \frac{A\ h^2}{m^2 MG}.$$

We can determine A from (0.6). Using $\frac{du}{d\theta} = -A\ \sin\theta$, we have

$$A^2 \sin^2\theta + A^2\cos^2\theta + \frac{1}{B^2\epsilon^2} + \frac{2\ A}{B\epsilon}\cos\theta = \frac{2mE}{h^2} + \frac{2}{B\epsilon}[\frac{1}{B\epsilon} + A\cos\theta].$$

$$A^2 = \frac{2mE}{h^2} + \frac{1}{B^2\epsilon^2} = \frac{2mEh^2 + m^4G^2M^2}{h^4},$$

$$A = \frac{1}{h^2}[2mEh^2 + m^4G^2M^2]^{1/2}, \quad B = 1/A,$$

$$\epsilon = \frac{[2mEh^2 + m^4G^2M^2]^{1/2}}{m^2MG} = [1 + \frac{2Eh^2}{m^3M^2G^2}]^{1/2},$$

Kepler's Third Law

Recall that the area of the ellipse is given by πab, where $a = \frac{B\epsilon}{1-\epsilon^2}$, major axis, and $b = \frac{B\epsilon}{\sqrt{1-\epsilon^2}}$, minor axis. i.e.,

$$\text{Area of ellipse} = \frac{\pi B^2 \epsilon^2}{(1-\epsilon^2)^{3/2}}.$$

On the other hand, we can use a different method to compute the area of the ellipse from Kepler's second law. If T is the period, then the area of the ellipse is also given by

$$TA'(t) = T(\frac{1}{2}r^2\theta') = \frac{T}{2m}mr^2\theta' = \frac{Th}{2m}.$$

$$\frac{Th}{2m} = \frac{\pi B^2 \epsilon^2}{(1-\epsilon^2)^{3/2}},$$

$$T^2 = \frac{4m^2}{h^2}\frac{\pi^2 B^4 \epsilon^4}{(1-\epsilon^2)^3} = \frac{4m^2\pi^2 B\epsilon}{h^2}a^3 = \frac{4\pi^2}{GM}a^3.$$

Questions

What is a geostationary satellite?

What is a geosynchronous satellite?

How high above the earth is any geostationary or geosynchronous satellite? Can you compute this number?

Why GPS satellites are not on the geostationary orbit?

How high above the earth is the international space station (ISS)? What is its period?