Linear Algebra

from the context of the course

MTH 317H: Honors Linear Algebra

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Contents

Chapter 1

Vectors and Linear Transformations

1.1 Vectors and Bases

Definition 1.1.1. **Vector Space** is a set equipped with operations of addition and scalar multiplication such that the following holds:

- Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \ \forall \ \mathbf{v}, \mathbf{w} \in V$
- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{w} + \mathbf{v}) \ \forall \ \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$
- Zero Vector: $\exists$ a vector $\mathbf{0}$ such that for any vector $\mathbf{v} \in V,\ \mathbf{v} + \mathbf{0} = \mathbf{v}$
- Additive Inverse: for any vector $\mathbf{v} \in V$ there exists a vector $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$
- Multiplicative Identity: for any vector $\mathbf{v} \in V, \mathbf{v} \cdot 1 = \mathbf{v}$
- Additive Conservation: for any two vectors $\mathbf{v}, \mathbf{w} \in V,\ \mathbf{v} + \mathbf{w} \in V$
- Multiplicative Conservation: for any vector $\mathbf{v} \in V$ and any scalar a, we have $a\mathbf{v} \in V$

Remark. If the scalars of a vectors space are elements in $\mathbb{R}$ then it is a real vector space. Likewise if the scalars of a vectors space are elements in $\mathbb{C}$ then it is a complex vector space.

Example. $\mathbb{R}^n$ defines a real vector space with columns of size n,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \text{ for any } \mathbf{v} \in \mathbb{R}^n$$

with real number entries. To make this a vector space define addition and multiplication as

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}, \alpha \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix}$$

Definition 1.1.2. A **linear combination** of a collection of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n} \in V$ is the sum

$$\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i}$$

for scalars $\alpha_1, \alpha_2, \dots, \alpha_n$.

Definition 1.1.3. A system of vectors is **generating** or **spanning** in V if any vector $\mathbf{v} \in V$ can be represented as the linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \cdots + \alpha_n \mathbf{v_n} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i}$$

Definition 1.1.4. A linear combination is **trivial** if $\alpha_1, \alpha_2, \dots, \alpha_n = 0$.

Definition 1.1.5. A system of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n} \in V$ is **linearly independent** if only the trivial linear combination equal $\mathbf{0}$.

Definition 1.1.6. A system of vectors is **linearly dependent** if $\mathbf{0}$ can be represented by a linear combination that is not trivial.

Proposition 1.1.1. A system of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n} \in V$ is linearly dependent if and only if one of the vectors $\mathbf{v_k}$ can be represented as a linear combination of the other vectors,

$$\mathbf{v_k} = \sum_{i=1, i\neq k}^{n} \beta_i \mathbf{v_i}$$

Definition 1.1.7. A system of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ is a **basis** if every $\mathbf{v} \in V$ can be represented as a unique linear combination.

Proposition 1.1.2. A system of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n} \in V$ is a basis if and only if it is linearly independent and generating.

Proposition 1.1.3. Any (finite) generating system contains a basis.

Definition 1.1.8. The **dimension** of a vectors space V is the number of vectors in a basis denoted $\dim(V)$.

Proposition 1.1.4. Every basis in a vector space has the same number of vectors.

Proposition 1.1.5. For a system of vectors $\mathbf{v_1}, \dots, \mathbf{v_n} \in V$ if $n < \dim(V)$ then the system is not generating.

Proposition 1.1.6. For a system of vectors $\mathbf{v_1}, \dots, \mathbf{v_n} \in V$ if $n > \dim(V)$ then the system is not linearly independent.

Proposition 1.1.7. For a generating system of vectors $\mathbf{v_1}, \dots, \mathbf{v_n} \in V$ if $n = \dim(V)$ then it is a basis in V.

Proposition 1.1.8. For a linearly independent system of vectors $\mathbf{v_1}, \dots, \mathbf{v_n} \in V$ if $n = \dim(V)$ then it is a basis in V.

Proposition 1.1.9. A subset of a linearly independent system of vectors is also linearly independent.

1.2 Linear Transformations

Definition 1.2.1. A function $T : V \to W$ is a **linear transformation** if the following hold for all $\mathbf{v}, \mathbf{w} \in V$ and any scalar α.

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$
2. $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$

Definition 1.2.2. A **matrix** is a rectangular array with rows and columns. The elements of the array are called the entries of the matrix. An $m \times n$ matrix A is denoted as

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}$$

Proposition 1.2.1. Any linear transformation can be represented by where it takes basis vectors of its input space.

Definition 1.2.3. Define **matrix-vector multiplication** for a matrix A and a vector $\mathbf{v} \in V$ as

$$A\mathbf{v} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = v_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{pmatrix} + v_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \end{pmatrix} + \cdots + v_m \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,m} \end{pmatrix}$$

Remark. Under this definition it is clear that for any linear transformation T the matrix A with columns of its values on the standard basis represents the transformation such that

$$T(\mathbf{v}) = A\mathbf{v}$$

Definition 1.2.4. Define **matrix multiplication** by applying matrix-vector multiplication to the columns of the second matrix. That is for an $n \times m$ matrix A and a $m \times k$ matrix B:

$$AB = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,k} \\ b_{2,1} & b_{2,2} & \dots & b_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \dots & b_{m,k} \end{pmatrix} = \left(A \begin{pmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{m,1} \end{pmatrix} \quad A \begin{pmatrix} b_{1,2} \\ b_{2,2} \\ \vdots \\ b_{m,2} \end{pmatrix} \quad \dots \quad A \begin{pmatrix} b_{1,k} \\ b_{2,k} \\ \vdots \\ b_{m,k} \end{pmatrix} \right)$$

Remark. Under this definition it is clear that multiplication of matrices represents composition of linear transformations.

$$T_A(T_B(\mathbf{v})) = AB\mathbf{v}$$

Proposition 1.2.2. The following properties of matrix multiplication hold for any matrices A, B, C and scalar α.

1. **Associativity**: $A(BC) = (AB)C$
2. **Distributivity**: $A(B + C) = AB + AC, (A + B)C = AC + BC$
3. **Scalar Distributivity**: $A(\alpha B) = (\alpha A)B = \alpha(AB) = \alpha AB$

Definition 1.2.5. Define **matrix addition** as element-wise addition. That is for two $n \times m$ matrices A, B:

$$A + B = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix} + \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m} \\ b_{2,1} & b_{2,2} & \dots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,m} \end{pmatrix} = \begin{pmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \dots & a_{1,m} + b_{1,m} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \dots & a_{2,m} + b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} + b_{n,1} & a_{n,2} + b_{n,2} & \dots & a_{n,m} + b_{n,m} \end{pmatrix}$$

Definition 1.2.6. Define **scalar-matrix multiplication** as element-wise multiplication. That is for a scalar α and a matrix A as

$$\alpha A = \alpha \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix} = \begin{pmatrix} \alpha a_{1,1} & \alpha a_{1,2} & \dots & \alpha a_{1,m} \\ \alpha a_{2,1} & \alpha a_{2,2} & \dots & \alpha a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n,1} & \alpha a_{n,2} & \dots & \alpha a_{n,m} \end{pmatrix}$$

Remark. With these definitions the set of matrices and hence the set of linear transformations is itself a vector space.

Definition 1.2.7. The **transpose** of an $n \times m$ matrix is the corresponding $m \times n$ matrix with the row and columns swapped. Denoted A^T.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}, A^T = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{n,1} \\ a_{1,2} & a_{2,2} & \dots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,m} & a_{2,m} & \dots & a_{n,m} \end{pmatrix}$$

Proposition 1.2.3. For any two matrices A, B such that AB is defined, $(AB)^T = A^T B^T$

Definition 1.2.8. The **trace** of a square matrix A (denoted trace$A$) is the sum of the diagonal entries.

$$\text{trace}A = \sum_{k=1}^{n} a_{k,k}$$

Theorem 1.2.1. Let A and B be matrices of size $m \times n$ and $n \times m$ respectively (hence both products AB and BA are well defined). Then

$$\text{trace}(AB) = \text{trace}(BA)$$

1.3 Invertible Transformations

Definition 1.3.1. The **identity** matrix (denoted I) or identity transformation is the transformation such that $I\mathbf{v} = \mathbf{v}, \forall \mathbf{v}$.

Definition 1.3.2. A linear transformation A is **left invertible** if there exists a linear transformation A^{-1} such that

$$A^{-1}A = I$$

Definition 1.3.3. A linear transformation A is **right invertible** if there exists a linear transformation A^{-1} such that

$$AA^{-1} = I$$

Definition 1.3.4. A linear transformation is **invertible** if it is both left and right invertible.

Theorem 1.3.1. A linear transformation is invertible if and only if it's left and right inverses exist and are equal.

Proposition 1.3.1. An invertible matrix must be square.

Theorem 1.3.2. If a square matrix has either a left or right inverse, then it is invertible.

Theorem 1.3.3. (Inverse of the product) If linear transformations A and B are invertible then the product AB (if it exists) is also invertible.

Theorem 1.3.4. (Inverse of Transpose) If a matrix is invertible then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof. From 1.2.3 we see that

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

and similarly

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

□

Chapter 2

Linear Systems

Chapter 3

Determinant

Chapter 4

Spectral Theory

Chapter 5

Inner product spaces