Math 347H Laplace Transform of Convolution Integral

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When we try to solve initial value problem $y'' + ay' + by = f(t), \quad y(0) = c, \quad y'(o) = d$ by using the method of Laplace transform, we end up with

$$(s^2 + as + b)Y(s) - \ [y'(0) + sy(0)] - ay(0) = F(s).$$

$$Y(s) = (F(s) + cs + d + ac)\frac{1}{s^2 + as + b}.$$

In many cases, we want to know the Laplace inverse transform of $H(s) = F(s)G(s)$.

Do we have a simple formula for $h(t) = L^{-1}\{H\}(t)$ in terms of $f(t)$ and $g(t)$?

Warning: In general, it is not $f(t)\ g(t)$!

Can anybody give a simple counter example?

We do have the following (Theorem 6.6.1 in the book)

Theorem. If $L\{f(t)\} = F(s),\ L\{g(t)\} = G(s)$ for $s > a$, then

$$H(s) = F(s)\ G(s) = L\{h(t)\}, \ \ s > a,$$

where

$$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau = f * g(t).$$

h is called the the convolution of f and g, the integral is called the convolution integral.

From the definition of convolution integral, we can see that

$$\begin{aligned} f * g &= g * f, && \text{Commutative property} \\ f * (g_1 + g_2) &= f * g_1 + f * g_2, && \text{Distributive property} \\ (f * g) * h &= f * (g * h), && \text{Associative property} \end{aligned}$$

The proof of these properties are left as exercises.

Is $f * 0 = 0$?

Is $f * 1 = f$? . Look at $f(t) = \cos t$.

Let us turn to the proof of theorem. Note that

$$F(s) = \int_0^\infty e^{-t\tau} f(\tau) d\tau, \;\; G(s) = \int_0^\infty e^{-t\eta} f(\eta) d\eta, \;\; h(t) = \int_0^t f(\tau) g(t-\tau) d\tau.$$

From the definition of Laplace transform, we have

$$H(s) = \int_0^\infty h(t) e^{-st} dt = \int_0^\infty \int_0^t f(\tau) g(t-\tau) e^{-st} d\tau dt.$$

Let us change the order of integration,

$$\begin{aligned} H(s) &= \textstyle\int_0^\infty f(\tau) \int_\tau^\infty g(t-\tau) e^{-st} dt d\tau \\ &= \textstyle\int_0^\infty f(\tau) e^{-s\tau} \int_\tau^\infty g(t-\tau) e^{-s(t-\tau)} dt d\tau \\ &= \textstyle\int_0^\infty f(\tau) e^{-s\tau} [\int_0^\infty g(\eta) e^{-s\eta} d\eta] \, d\tau \;\; \eta = t - \tau \\ &= F(s) \, G(s). \end{aligned}$$

Next, we will work out a few examples.

Example 1. $f(t) = \cos t, \quad g(t) = 1$. We already saw that $f * g(t) = \sin t = h(t)$. $H(s) = \frac{1}{s^2+1} = F(s) G(s) = \frac{s}{s^2+1} \, \frac{1}{s}$.

Example 2. Find the Laplace transform of $h(t) = \int_0^t (t-\tau)^2 \sin 3\tau d\tau$.

We have $f(t) = t^2, \; g(t) = \sin 3t, \; h(t) = f * g, \quad H(s) = \frac{2}{s^3} \frac{3}{s^2+9} = \frac{6}{s^3(s^2+9)}$.

Don't try to compute the convolution integral !

Example 3. Find the inverse Laplace transform of $H(s) = \frac{s}{(s+1)(s^2+4)}$ by using two methods: (a) Convolution formula; (b) Partial fraction.

(a) Note that $H(s) = \frac{1}{s+1}\ \frac{s}{s^2+4} = F(s)\ G(s)$ with $F(s) = \frac{1}{s+1}$, $G(s) = \frac{s}{s^2+4}$. $f(t) = e^{-t}$, $g(t) = \cos 2t$. Hence

$$\begin{aligned} h(t) &= \int_0^t \cos 2\tau\ e^{-(t-\tau)} d\tau = e^{-t} \int_0^t \cos 2\tau\ e^{\tau} d\tau \quad \text{Tricky! which one } \tau,\ t-\tau \\ &= e^{-t} Re \int_0^t e^{\tau + 2i\tau} d\tau = e^{-t} Re \tfrac{1}{1+2i} e^{(1+2i)\tau}|_0^t \\ &= e^{-t} Re \tfrac{1}{1+2i} [e^{(1+2i)t} - 1] = \tfrac{1}{5} e^{-t} Re (1-2i)\{e^t[\cos 2t + i \sin 2t] - 1\} \\ &= \tfrac{1}{5} e^{-t} [e^t \cos 2t - 1 + 2e^t \sin 2t] = \tfrac{1}{5}(\cos 2t + 2 \sin 2t - e^{-t}) \end{aligned}$$

(2) Use the partial fraction,

$$H(s) = \frac{s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+c}{s^2+4}.$$

From $A(s^2+4) + (Bs+c)(s+1) = s$. From $s = -1$, we have $5A = -1, A = -1/5$. Then $(Bs+c)(s+1) = s + \frac{s^2+4}{5}$. From $s = 0$, $c = 4/5$. From $s = 1$, $B + c = 2/2 = 1$, $B = 1/5$. $H(s) = \frac{1}{5}[-\frac{1}{s+1} + \frac{s+4}{s^2+4}]$.

$$h(t) = \frac{1}{5}\ [-e^{-t} + \cos 2t + 2 \sin 2t].$$

Example 4. Let $g(t)$ be any function on $[0, \infty)$. Prove that the solution to $y'' + 4y = g(t),\ y(0) = 3,\ y'(0) = -2$ is given by

$$y(t) = 3 \cos 2t - \sin 2t + \frac{1}{2} \int_0^t g(\tau) \sin 2(t-\tau) d\tau.$$

The first two terms is a solution to homogeneous equation, the third term is a particular solution.

Solution. Take Laplace transform on both sides of the equation,

$$(s^2+4)Y(s) + 2 - 3s = G(s), \quad Y(s) = \frac{3s}{s^2+4} - \frac{2}{s^2+4} + \frac{G(s)}{s^2+4}.$$

Note that $L^{-1}\{\frac{1}{s^2+4}\} = \frac{1}{2}\sin 2t$. Hence, using the convolution integral,

$$y(t) = 3\cos 2t - \sin 2t + \frac{1}{2}\int_0^t g(\tau)\sin 2(t-\tau)d\tau.$$

The last term corresponding to a particular solution $y_p(0) = 0,\ y_p'(0) = 0$.

Can you get the same formula with the method of variation of parameters?

The Laplace transform can be used to solve higher order equation and linear systems.

If the time permits, find the solution of

$$y'' + 4y' + 4 = -2\delta(t-2),\ y(0) = 2,\ y'(0) = 1.$$

Take Laplace transform, we have

$$(s^2+4s+4)Y(s) - (1+2s) - 8 = -2e^{-2s}.\ Y(s) = \frac{2s+9}{(s+2)^2} - \frac{2e^{-2s}}{(s+2)^2}.$$

Rewrite $2s+9 = 2(s+2)+5$, we have

$$y(t) = 2e^{-2t} + 5\ t\ e^{-2t} - 2\ (t-2)\ e^{-2(t-2)}\ u_2(t).$$

What is Laplace transform of $\int_0^t \sin\tau\cos(t-\tau)d\tau$?

Find the inverse Laplace transform of $\frac{1}{s^3(s^2+1)}$? (You can use convolution integral).