



PHY 410 - Reference Sheet

Stirling’s approximation - for very large N :

$$\log N! \approx N \log N - N$$

$$N! \approx \sqrt{2\pi N} N^N e^{-N}$$

Fractional uncertainty of \mathbb{X} is uncertainty of expected value per particle.

$$\frac{\Delta \mathbb{X}}{N} = \frac{\sqrt{\langle \mathbb{X}^2 \rangle - \langle \mathbb{X} \rangle^2}}{N}$$

Boltzmann’s constant

$$k_B = 1.380649 \times 10^{-23} \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}$$

Entropy $S = k_B \sigma, \quad \sigma_{TOT} = \sigma_1 + \sigma_2$

Temperature $T = \tau / k_B$

Microcanonical Ensemble

Multiplicity function

$$g = \# \text{ of microstates}, \quad \mathcal{P}(n) = \frac{1}{g}$$

Expected value of \mathbb{X} is the average across all microstates.

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n) \mathcal{P}(n) = \frac{1}{g} \sum_n \mathbb{X}(n)$$

Entropy can be written in terms of the multiplicity function.

$$\sigma(N, T, U, V, P) \equiv \log[g(N, T, U, V, P)]$$

Binary System

A **binary system** is a system of N particles where each particles has two possible states. Let N_{\uparrow} is the number of particle in the up state and N_{\downarrow} be the number of particles in the down state.

$$g(N, N_{\uparrow}) = \frac{N!}{N_{\uparrow}!(N - N_{\uparrow})!}, \quad \sum_{N_{\uparrow}=0}^N g(N, N_{\uparrow}) = 2^N$$

The binary system can be rewritten in terms of the difference between up states and down states this is the **spin excess**.

$$2S = N_{\uparrow} - N_{\downarrow}$$

$$g(N, S) = \frac{N!}{(\frac{N}{2} + S)!(\frac{N}{2} - S)!}$$

$$\sum_{S=-\frac{N}{2}}^{\frac{N}{2}} g(N, N_{\uparrow}) = 2^N$$

Applying Stirling’s approximation to the binary model, for large N the multiplicity function and fractional uncertainty are

$$g(N, S) \approx g(N, 0) e^{-2s^2/N}$$

$$g(N, S) \approx \sqrt{\frac{2}{\pi N}} 2^N e^{-2s^2/N}$$

$$\frac{\Delta S}{N} \approx \frac{1}{\sqrt{N}}$$

An example of a binary system is N spin 1/2 particles in an external **magnetic field** B . The total energy U and magnetization M of the system are

$$U = \sum_{i=1}^N -m \vec{i} \cdot \vec{B} = -(N_{\uparrow} - N_{\downarrow}) m B = -2S m B$$

$$M = 2S m = -U/B$$

$$g(N, U) = \frac{N!}{(\frac{N}{2} - \frac{U}{2mB})!(\frac{N}{2} + \frac{U}{2mB})!}$$

$$\sigma(N, S) \approx -\left(\frac{N}{2} + S\right) \log\left(\frac{1}{2} + \frac{S}{N}\right) - \left(\frac{N}{2} - S\right) \log\left(\frac{1}{2} - \frac{S}{N}\right)$$

$$M = N m \tanh(mB/\tau)$$

Einstein Solid

An **einstein solid** is a system of N atoms where each atom is modeled as a harmonic oscillator the energy of the system is determined by the number of atoms n oscillating at frequency ω .

$$U = n \hbar \omega$$

$$g(N, n) = \frac{(n + N - 1)!}{n!(N - 1)!}$$

$$g(N, n) \approx \frac{\left(\frac{n+N}{n}\right)^n \left(\frac{n+N}{n}\right)^N}{\sqrt{2\pi n(n+N)/N}}$$

Thermal Equilibrium Temperature

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_{N, V}$$

Thermal Equilibrium

$$\left(\frac{\partial \sigma_1}{\partial U_1}\right)_{N_1, V_1} = \left(\frac{\partial \sigma_2}{\partial U_2}\right)_{N_2, V_2}$$

$$\frac{1}{\tau_1} = \frac{1}{\tau_2}$$

2nd law of thermo - Change in entropy ≥ 0 . **Sharpness of Equilibrium** For a two binary systems, the number of states in a configuration of deviation δ from equilibrium is

$$g_1 g_2 = (g_1 g_2)_{max} e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

Canonical Ensemble

Partition Function - partition by energy levels for a fixed temperature

$$z = \sum_n e^{-\varepsilon_n/\tau}, \quad \mathcal{P}(n) = \frac{1}{z} e^{-\varepsilon_n/\tau}$$

$$z = \sum_{\alpha} g(\varepsilon_{\alpha}) e^{-\varepsilon_{\alpha}/\tau}, \quad \text{for degeneracy } g(\varepsilon_{\alpha})$$

Expected Value of \mathbb{X} is the average across all energies (Thermal Average).

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n) \mathcal{P}(n) = \frac{1}{z} \sum_n \mathbb{X}(n) e^{-\varepsilon_n/\tau}$$

Expected Energy in the canonical ensemble is

$$U = \langle \varepsilon \rangle = \frac{1}{z} \sum_n \varepsilon_n e^{-\varepsilon_n/\tau}$$

$$U = \langle \varepsilon \rangle = \tau^2 \frac{1}{z} \frac{\partial z}{\partial \tau} = \tau^2 \frac{\partial}{\partial \tau} \log z$$

The total partition function and expected value for N non-interacting particles is simply

$$z_N = z_1^N$$

$$\langle \mathbb{X} \rangle_N = N \langle \mathbb{X} \rangle_1 \Rightarrow U_N = N U_1$$

Helmholtz Free Energy

$$F = U - \tau \sigma = U - S T = -\tau \log z$$

$$dF = -\sigma d\tau - P dV$$

Entropy $\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V, \quad S = k_B \sigma$

Temperature $\tau = \left(\frac{\partial U}{\partial \sigma}\right)_V$

Pressure

$$P = -\left(\frac{\partial U}{\partial V}\right)_{\sigma} = \tau \left(\frac{\partial \sigma}{\partial V}\right)_U = -\left(\frac{\partial F}{\partial V}\right)_{\tau}$$

Energy $U = -\tau^2 \frac{\partial}{\partial \tau} \left(\frac{F}{\tau}\right)$

Ideal Gas

DeBroglie Thermal Wavelength is the wavelength of the wave functions of matter at a given temperature.

$$\lambda_T = \sqrt{\frac{2\pi \hbar^2}{m\tau}}$$

Concentration of a system is the inverse of the volume

$$n = \frac{1}{V}$$

Quantum Concentration is the density of quantum state per particle. It is used to define when a system will behave classically ($n \ll n_Q$) and when a system will be dominated by quantum effects ($n \gg n_Q$).

$$n_Q = \frac{1}{\lambda_T^3}$$

Single Particle Ideal Gas is a system in the canonical ensemble consisting of a single particle in a box of side lengths L . The energy levels, partition function and average energy are

$$E_{n_x, n_y, n_z} = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$

$$z_1 = \frac{V}{\lambda_T^3}$$

$$U_1 = \frac{3}{2} \tau$$

$$\sigma_1 = \log\left(\frac{V}{\lambda_T^3}\right) + \frac{3}{2}$$

Gibbs Resolution states that for systems in the classical regime the partition function for an ideal gas with N particles is

$$z_N = \frac{1}{N!} (z_1)^N$$

$$U_N = \frac{3}{2} N \tau$$

$$\sigma_N = N \left[\log\left(\frac{V}{N \lambda_T^3} + \frac{5}{2}\right) \right]$$

N-Particle Ideal Gas - by applying Gibbs resolution and properties of expected values we can find the classical ideal gas results

$$PV = N \tau$$

$$U = \frac{3}{2} N \tau$$

$$\sigma = N \left[\log\left(\frac{V}{N \lambda_T^3} + \frac{5}{2}\right) \right]$$

Thermal Radiation

Single Frequency Photon Gas is a system in the canonical ensemble that considers photons of a specific frequency ω .

$$\varepsilon_s = s \hbar \omega, \quad s = 0, 1, 2, 3, \dots$$

$$z = \sum_{s=0}^{\infty} e^{-s \hbar \omega / \tau} = \frac{1}{1 - e^{-\hbar \omega / \tau}}$$

$$\mathcal{P}(s) = \frac{e^{-s \hbar \omega / \tau}}{z}$$

$$\langle s \rangle = \frac{1}{z} \sum_{s=0}^{\infty} s e^{-s \hbar \omega / \tau} = \frac{1}{e^{\hbar \omega / \tau} - 1}$$

Photon Gas is an expansion of the single frequency photon gas that considers all the possible cavity modes. The modes are 2 fold degenerate for the 2 independent polarizations.

$$\omega_n = \frac{c\pi}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{c\pi n}{L}$$

$$U = \langle \epsilon \rangle = 2 \sum_n \frac{\hbar \omega_n}{e^{\hbar \omega_n / \tau} - 1} = \frac{\pi^2 V}{15(\hbar c)^3} \tau^4$$

Stefan-Boltzmann Law

U/V = (h / (pi^2 c^3)) * integral from 0 to infinity of (omega^3 / (e^{h*omega/tau} - 1)) d*omega = (pi^2 / (15(hc)^3)) * tau^4

Spectral Density Function

(d/d*omega) (U/V) = (h / (pi^2 c^3)) * (omega^3 / (e^{h*omega/tau} - 1))

Flux Density (sigma_B = Stefan-Boltzmann constant)

J_mu = (1/4) * (cU/V) = sigma_B * tau^4 = (pi^2 / (60(hc)^3)) * tau^4

Phonons in a Solid (Debye Model)

Phonons in a solid is a system in the canonical ensemble that is very similar to thermal radiation except there is 3 fold degeneracy from 3 polarizations of phonons and an upper cutoff frequency omega_D due to the separation distance between atoms.

omega_n = (pi*c_s / L) * sqrt(n_x^2 + n_y^2 + n_z^2) = (pi*c_s / L) * n

Debye cutoff frequency

omega_D = c_s * (6*pi^2*N / V)^(1/3), omega_D = (pi*c_s / L) * n_D

Grand Canonical Ensemble
Chemical Potential

mu = (dF/dN)_{tau,V}

mu = tau * log((N*lambda_T^3) / V) = tau * log(n / n_Q)

mu = (dU/dN)_{sigma,V} = -tau * (d sigma / dN)_{U,V}

Grand Partition Function - partition by energy levels for a fixed temperature and all possible values of N

z_lambda = sum_N sum_{n(N)} e^{-(epsilon_n^N - mu*N)/tau}

P(N, epsilon_n) = (1/z_lambda) * e^{-(epsilon_n^N - mu*N)/tau}

Fugacity

lambda = e^{mu/tau}

z_lambda = sum_N lambda^N sum_{s(N)} e^{-epsilon_s^N/tau} = sum_N lambda^N z_N

Expected Value of X is the average across all energies (Diffusive Average).

<X> = (1/z_lambda) * sum_N sum_s X(N,s) e^{-(epsilon_s^N - mu*N)/tau}

Expected Number of Particles in the grand canonical ensemble is

N = <N> = tau * (d/d*mu) log z_lambda = lambda * (d/d*lambda) log z_lambda

Expected Energy in the grand canonical ensemble is

U = <epsilon> = (1/z_lambda) * sum_N sum_{n(N)} epsilon_n^N e^{-(epsilon_n^N - mu*N)/tau}

U = <epsilon> = tau^2 * (d/d*tau) log z_lambda)_lambda

Concentration and DeBroglie Wavelength

n = N/V, n_Q = 1/lambda_T^3, lambda_T = sqrt(2*pi*h^2 / (m*tau))

Grand Potential

Omega = U - sigma*tau - mu*N

Omega = -tau * log z_lambda

sigma = (-d*Omega/d*tau)_{V,mu}, P = (-d*Omega/d*V)_{tau,mu}, N = (-d*Omega/d*mu)_{tau,V}

System of Non-interacting Particles

The grand partition function for a system with M energy states where n_alpha is the number of particles occupying a state is

z_lambda = product_{alpha=1}^M z_alpha, z_alpha = sum_{n_alpha} e^{-n_alpha*(epsilon_alpha - mu)/tau}

U = sum_{alpha=1}^M epsilon_alpha * f(epsilon_alpha), N = sum_{alpha=1}^M f(epsilon_alpha)

Fermions

n_alpha = 0, 1

z_alpha = 1 + e^{-(epsilon_alpha - mu)/tau} = 1 + lambda * e^{-epsilon_alpha/tau}

Fermi-Dirac Distribution is the expected number of a particles in a particular energy epsilon_alpha.

<n_alpha> = f(epsilon_alpha) = 1 / (e^{(epsilon_alpha - mu)/tau} + 1) = 1 / (lambda^{-1} * e^{epsilon_alpha/tau} + 1)

For tau -> 0: f(epsilon_alpha) = theta(epsilon_alpha - mu)

Bosons (Bonsons)

n_alpha = 0, 1, 2, 3, ...

z_alpha = 1 / (1 - e^{-(epsilon_alpha - mu)/tau}) = 1 / (1 - lambda * e^{-epsilon_alpha/tau})

Boson Distribution is the expected number of a particles in a particular energy epsilon_alpha.

<n_alpha> = f(epsilon_alpha) = 1 / (e^{(epsilon_alpha - mu)/tau} - 1) = 1 / (lambda^{-1} * e^{epsilon_alpha/tau} - 1)

Ideal Gas

Both fermions and bosons behave identically at the classical limit epsilon_alpha - mu >> tau.

epsilon_n = (h^2 * pi^2 / (2 * m * L^2)) * n^2 = (h^2 * pi^2 / (2 * m * L^2)) * (n_x^2 + n_y^2 + n_z^2)

epsilon_n = p * c = (pi * h * c / L) * n = (pi * h * c / L) * sqrt(n_x^2 + n_y^2 + n_z^2)

<n_alpha> = f(epsilon_alpha) = e^{-(epsilon_alpha - mu)/tau}

z_lambda = sum_N lambda^N z_N = sum_N lambda^N (1/N!) * z_1^N = e^{lambda * z_1}

lambda = n / n_Q, P * V = N * tau, U = (3/2) * N * tau, mu = tau * log(n / n_Q)

sigma = N * [log(n_Q / n) + 5/2], F = N * tau * [log(n / n_Q) - 1]

Heat Capacity measures the change in heat energy per unit temperature

C_P > C_V, C_V = (dU/dT)_V = tau * (d sigma / dT)_V

C_P = (dU/dT)_P + P * (dV/dT)_P = tau * (d sigma / dT)_P

Monoatmc gas C_V = (3/2) * N * k_B, C_P = (5/2) * N * k_B

Isothermal Expansion sigma_f - sigma_i = N * log(V_f / V_i)

Q = N * tau * log(V_f / V_i)

Isoentropic Expansion tau_f / tau_i = (V_i / V_f)^(2/3)

Internal Excitations

Expansion of the ideal gas to take into account the additional energy states from internal excitations.

z_int = sum_alpha e^{-epsilon_alpha/tau}, z_lambda = 1 + lambda * z_int * e^{-epsilon_n/tau}

Internal Excitation Corrections

lambda = n / (n_Q * z_int), mu = tau * (log(n / n_Q) - log z_int)

F = N * tau * [log(n / n_Q) - 1] - N * tau * log z_int

sigma = N * [log(n / n_Q) + 5/2] - (dF_int/d*tau)_V

Fermi Gas

DOG (bork)



Kaedon.net/reference