



PHY 410 - Reference Sheet

Stirling’s approximation - for very large N:

log N! ≈ N log N - N
N! ≈ sqrt(2*pi*N) * N^N * e^-N

Fractional uncertainty of X is uncertainty of expected value per particle.

Delta X / N = (sqrt(X^2) - (X)^2) / N

Boltzmann’s constant
k_B = 1.380649 x 10^-23 m^2 s^-2 K^-1
Entropy S = k_B sigma, sigma_TOT = sigma_1 + sigma_2
Temperature T = tau / k_B

Microcanonical Ensemble
Multiplicity function

g = # of microstates, P(n) = 1/g

Expected value of X is the average across all microstates.

X = sum(n) X(n)P(n) = 1/g sum(n) X(n)

Entropy can be written in terms of the multiplicity function.

sigma(N, T, U, V, P) = log[g(N, T, U, V, P)]

Binary System

A binary system is a system of N particles where each particles has two possible states. Let N_up is the number of particle in the up state and N_down be the number of particles in the down state.

g(N, N_up) = N! / (N_up!(N - N_up)!), sum(N_up=0 to N) g(N, N_up) = 2^N

The binary system can be rewritten in terms of the difference between up states and down states this is the spin excess.

2S = N_up - N_down

g(N, S) = N! / ((N/2 + S)!(N/2 - S)!)
sum(S=-N/2 to N/2) g(N, N_up) = 2^N

Applying Stirling’s approximation to the binary model, for large N the multiplicity function and fractional uncertainty are

g(N, S) ≈ g(N, 0)e^-2s^2/N
g(N, S) ≈ sqrt(2/(pi*N)) * 2^N * e^-2s^2/N
Delta S / N ≈ 1/sqrt(N)

An example of a binary system is N spin 1/2 particles in an external magnetic field B. The total energy U and magnetization M of the system are

U = sum(i=1 to N) -m_i . B = -(N_up - N_down)mB = -2SmB
M = 2Sm = -U/B
g(N, U) = N! / ((N/2 - U/(2mB))!(N/2 + U/(2mB))!)
sigma(N, S) ≈ -(N/2 + S) log(N/2 + S) - (N/2 - S) log(N/2 - S)
M = Nm tanh(mB/tau)

Einstein Solid

An einstein solid is a system of N atoms where each atom is modeled as a harmonic oscillator the energy of the system is determined by the number of atoms n oscillating at frequency omega.

U = nh*omega
g(N, n) = (n + N - 1)! / (n!(N - 1)!)
g(N, n) ≈ ((n+N)/n)^n * ((n+N)/n)^N / sqrt(2*pi*n*(n+N)/N)

Thermal Equilibrium
Temperature

1/tau = (d sigma / d U)_{N, V}

Thermal Equilibrium

(d sigma_1 / d U_1)_{N_1, V_1} = (d sigma_2 / d U_2)_{N_2, V_2}
1/tau_1 = 1/tau_2

2nd law of thermo - Change in entropy >= 0.
Sharpness of Equilibrium For a two binary systems, the number of states in a configuration of deviation delta from equilbirum is

g1g2 = (g1g2)max * e^(-(2*delta^2/N1 - 2*delta^2/N2))

Canonical Ensemble

Partition Function - partition by energy levels for a fixed temperature

z = sum(n) e^-epsilon_n/tau, P(n) = 1/z * e^-epsilon_n/tau
z = sum_alpha g(epsilon_alpha) e^-epsilon_alpha/tau, for degeneracy g(epsilon_alpha)

Expected Value of X is the average across all energies (Thermal Average).

X = sum(n) X(n)P(n) = 1/z sum(n) X(n)e^-epsilon_n/tau

Expected Energy in the canonical ensemble is

U = <epsilon> = 1/z sum(n) epsilon_n e^-epsilon_n/tau
U = <epsilon> = tau^2 * 1/z * d z / d tau = tau^2 * d / d tau log z

The total partition function and expected value for N non-interacting particles is simply

z_N = z_1^N
X_N = N(X)_1 => U_N = NU_1

Helmholtz Free Energy

F = U - tau sigma = U - ST = -tau log z
Delta F <= 0 - helmholtz free energy decreases
dF = 0 - helmholtz free energy minimized
Entropy sigma = -(dF/dtau)_V, S = k_B sigma
Temperature tau = (dU/dsigma)_V
Pressure

P = -(dU/dV)_sigma = tau (d sigma / d V)_U = -(dF/dV)_tau

Energy U = -tau^2 * d / d tau (F/tau)
Concentration and DeBroglie Wavelength

n = N/V, n_Q = 1/lambda_T^3, lambda_T = sqrt(2*pi*h^2/m*tau)

Single Particle Ideal Gas

A system in the canonical ensemble consisting of a signle particle in a box of side lengths L. The energy levels , partition function and average energy are

epsilon_n = h^2*pi^2/(2m*L^2) * n^2 = h^2*pi^2/(2m*L^2) * (n_x^2 + n_y^2 + n_z^2)
epsilon_n = pc = pi*h*c/L * n = pi*h*c/L * sqrt(n_x^2 + n_y^2 + n_z^2)
z_1 = V/lambda_T^3, U_1 = 3/2 * tau

and for the ultra-reletavistic case:

sigma_1 = log(V/lambda_T^3) + 3/2, F_1 = -tau log V/lambda_T^3

N-Particle Ideal Gas

Gibbs Resolution for systems of N identical particles the partition function is

z_N = 1/N! * (z_1)^N
PV = N*tau, U = 3/2 * N*tau
sigma = N * [log(V/(N*lambda_T^3)) + 5/2]
F = N*tau * [log(n/n_Q) - 1]

Thermal Radiation

Single Frequency Photon Gas is a system in the canonical ensemble that considers photons of a specific frequency omega.

epsilon_s = sh*omega, s = 0, 1, 2, 3, ...
z = sum(s=0 to inf) e^-sh*omega/tau = 1/(1 - e^-h*omega/tau)
P(s) = e^-sh*omega/tau / z
<s> = 1/z * sum(s=0 to inf) s * e^-sh*omega/tau = 1/(e^h*omega/tau - 1)

Photon Gas is an expansion of the single frequency photon gas that considers all the possible cavity modes. The modes are 2 fold degenerate for the 2 independent polarizations.

omega_n = c*pi/L * sqrt(n_x^2 + n_y^2 + n_z^2) = c*pi*n/L

U = <epsilon> = 2 * sum(n) h*omega_n / (e^h*omega_n/tau - 1) = pi^2*V / (15*(h*c)^3) * tau^4

Stefan-Boltzmann Law

U/V = h / (pi^2*c^3) * integral from 0 to inf of omega^3 / (e^h*omega/tau - 1) * d omega = pi^2 / (15*(h*c)^3) * tau^4

Spectral Density Function

d U / d omega * V = h / (pi^2*c^3) * omega^3 / (e^h*omega/tau - 1)

Flux Density(sigma_B=Stefan-Boltzmann constant)

J_mu = 1/4 * cU/V = sigma_B*tau^4 = pi^2 / (60*(h*c)^3) * tau^4

Phonons in a Solid (Debye Model)

Phonons in a solid is a system in the canonical ensemble that is very similar to thermal radiation except there is 3 fold degeneracy from 3 polarizations of phonons and an upper cutoff frequency omega_D due to the separation distance between atoms.

omega_n = pi*c_S/L * sqrt(n_x^2 + n_y^2 + n_z^2) = pi*c_S/L * n

Debye cutoff frequency

omega_D = c_S * (6*pi^2*N/V)^(1/3), omega_D = pi*c_S/L * n_D

Grand Canonical Ensemble

Chemical Potential

μ = (∂F / ∂N)_{τ,V}

μ = τ log ((N λ_T^3) / V) = τ log (n / n_Q)

μ = (∂U / ∂N)_{σ,V} = -τ (∂σ / ∂N)_{U,V}

Grand Partition Function - partition by energy levels for a fixed temperature and all possible values of N

z_ε = ∑_N ∑_{n(N)} e^{-(ε_N^N - μN) / τ}

P(N, ε_n) = 1 / z_ε e^{-(ε_n^N - μN) / τ}

Fugacity

z_ε = ∑_N λ^N ∑_{s(N)} e^{-ε_s^N / τ} = ∑_N λ^N z_N

Expected Value of X is the average across all energies (Diffusive Average).

⟨X⟩ = 1 / z_ε ∑_N ∑_s X(N, s) e^{-(ε_s^N - μN) / τ}

Expected Number of Particles in the grand canonical ensemble is

N = ⟨N⟩ = 1 / z_ε ∑_N ∑_s N e^{-(ε_s^N - μN) / τ}

N = ⟨N⟩ = τ ∂ / ∂μ log z_ε = λ ∂ / ∂λ log z_ε

Expected Energy in the grand canonical ensemble is

U = ⟨ε⟩ = 1 / z_ε ∑_N ∑_{n(N)} ε_n^N e^{-(ε_n^N - μN) / τ}

U = ⟨ε⟩ = τ^2 (∂ / ∂τ log z_ε)_λ

Grand Potential

Ω = U - στ - μN

Ω = -τ log z_ε

σ = (-∂Ω / ∂τ)_{V,μ} P = (-∂Ω / ∂V)_{τ,μ} N = (-∂Ω / ∂μ)_{τ,V}

System of Non-interacting Particles

The grand partition function for a system with M energy states where n_α is the number of particles occupying a state is

z_ε = ∏_{α=1}^M z_{αε}, z_{αε} = ∑_{n_α} e^{-n_α(ε_α - μ) / τ}

U = ∑_{α=1}^M ε_α f(ε_α), N = ∑_{α=1}^M f(ε_α)

Fermions

n_α = 0, 1

z_{αε} = 1 + e^{-(ε_α - μ) / τ} = 1 + λ e^{-ε_α / τ}

Fermi-Dirac Distribution is the expected number of a particles in a particular energy ε_α.

⟨n_α⟩ = f(ε_α) = 1 / (e^{(ε_α - μ) / τ} + 1) = 1 / (λ^{-1} e^{ε_α / τ} + 1)

For τ → 0: f(ε_α) = θ(ε_α - μ)

Bosons

n_α = 0, 1, 2, 3, ...

z_{αε} = 1 / (1 - e^{-(ε_α - μ) / τ}) = 1 / (1 - λ e^{-ε_α / τ})

Boson Distribution is the expected number of a particles in a particular energy ε_α.

⟨n_α⟩ = f(ε_α) = 1 / (e^{(ε_α - μ) / τ} - 1) = 1 / (λ^{-1} e^{ε_α / τ} - 1)

Ideal Gas

Both fermions and bosons behave identically at the classical limit ε_α - μ >> τ.

⟨n_α⟩ = f(ε_α) = e^{-(ε_α - μ) / τ}

z_ε = ∑_N λ^N z_N = ∑_N λ^N 1 / N! z_1^N = e^{λ z_1}

λ = n / n_Q, PV = Nτ, U = 3 / 2 Nτ, μ = τ log n / n_Q

σ = N [log (n_Q / n) + 5 / 2], F = Nτ [log (n / n_Q) - 1]

Heat Capacity measures the change in heat energy per unit temperature

C_P > C_V, C_V = (∂U / ∂T)_V = τ (∂σ / ∂T)_V

C_P = (∂U / ∂T)_P + P (∂V / ∂T)_P = τ (∂σ / ∂T)_P

Monoatomic gas C_V = 3 / 2 Nk_B, C_P = 5 / 2 Nk_B

Isothermal Expansion σ_f - σ_i = N log (V_f / V_i)

Q = Nτ log (V_f / V_i)

Isoentropic Expansion τ_f / τ_i = (V_i / V_f)^{2 / 3}

Internal Excitations

Expansion of the ideal gas to take into account the additional energy states from internal excitations.

z_{int} = ∑_α e^{-ε_α / τ}, z_ε = 1 + λ z_{int} e^{-ε_n / τ}

Internal Excitation Corrections

λ = n / (n_Q z_{int}), μ = τ (log (n / n_Q) - log z_{int})

F = Nτ [log (n / n_Q) - 1] - Nτ log z_{int}

σ = N [log (n / n_Q) + 5 / 2] - (∂F_{int} / ∂τ)_V

Fermi Gas

Fermions behave differently at quantum concentrations.

Density of States

⟨X⟩ = ∑_n f(ε_n) X_n = ∫_0^∞ D(ε) f(ε) X(ε) dε

Finding Density of States

Σ(ε) = g_S ∑_n θ(ε - ε_n)

D(ε) = dΣ(ε) / dε

Expected Energy and Expected Number of Particles written in terms of the density of states:

U = ∫_0^∞ ε D(ε) f(ε) dε

N = ∫_0^∞ D(ε) f(ε) dε

At τ << ε_F, the integrals can be reduced

U(τ = 0) = ∫_0^{ε_F} ε D(ε) dε

N(τ = 0) = ∫_0^{ε_F} D(ε) dε

Degenerate Fermi Gas

Fermi Energy - ε_F = μ(τ = 0)

Ground State Energy - U_0 = U(τ = 0)

Finding Fermi Energy

N = (1 / (2^3 * 3) n_F) n_F^2 = π n_F^3 / 3 ⇒ n_f = (3N / π)^{1 / 3}

ε_F = (ħ^2 π^2 / 2mL^2) n_F^2 = (ħ^2 / 2m) (3π^2 N / V)^{2 / 3} = τ_F

N = ∫_0^{ε_F} D(ε) dε

Sommerfeld Expansion

for finite τ << ε_F:

μ(τ << ε_F) ≈ ε_F + (τ / ε_F)^2 ε_F

U(τ << ε_F) ≈ U_0 + (τ / ε_F)^2 U_0

Bose-Einstein Condensate

Thermodynamics

First Law - dU = dQ + dW

Reversible process: dU = τ dσ + dW

Heat Engines

Refrigerators

Gibbs Free Energy

Enthalpy

Chemical Reactions

DOG (bork)



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