Math 347H Lecture: Review of Matrixes Continued

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Homework: Section 7.3. 9, 17, 20, 23, 31.

6. For any matrix $A(t) = (a_{ij}(t))$. we have $A'(t) = (a'_{ij}(t))$.

$[A(t)B(t)]' = A'(t)B(t) + A(t)B'(t)$.

What is $[\det(A(t))]'$?

7. How to solve the system $Ax = b$ with $A_{m\times n}$?

Practically, using the row operations to change the form $(A|b)$ to $(U|b*)$ with U an upper triangular matrix.

Possibilities: Unique solution; many solutions; no solutions.

Theoretically. A is invertible, $rankA = n = rank(A|b) \Longrightarrow$ Unique solution.

A is not invertible, $rankA = rank(A|b) < n \Longrightarrow$ Infinitely many solutions

$rankA < rank(A|b) \Longrightarrow$ No solution!

rank A = the number of linearly independent column vectors in A.

If A is non singular, $Ax = b$ has the unique solution, $x = A^{-1}b$. $x_j = \frac{\det A_j}{\det A}$ with A_j the same as A except the jth column of A is replaced by b.

Example 4. Solve the equation $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix}$.

The solution is $(2, -1, 1)^T$. Using Gauss elimination and formula during the class.

8. Subspace of $\mathbb{R}^n$ == Set of vectors S in $\mathbb{R}^n$ such that $\alpha x+\beta y \in S$ whenenver $x, y \in S$ for any $\alpha, \beta \in \mathbb{R}^n$

9. How to measure the size of any subspace S? = Dimension of a subspace = # of linearly independent vectors in S.

Definition of linearly independence: A set $\{x_1, x_2, ..., x_m\}$ is called a linearly independent set if the following is true:

For any linearly combination $c_1x_1+...+c_mx_m = 0$, then $c_1 = c_2 = ... = c_m = 0$.

Or equivalently, any x_j can not be written as a linear combination of other elements.

Give some examples in the class to clarify the concepts.

10. Eigenvalue and Eigenvectors (one of the most important parts of linear algebra, many applications in data science or data analysis). $A\eta = \lambda\eta$ with $\eta \neq 0$. iff $(A - \lambda I)\eta = 0$ iff $A - \lambda I$ is singular iff $\det(A - \lambda I) = 0$. This is called the characteristic polynomial. The eigenvector is not unique since 2 times the eigenvector is still an eigenvector.

Example 5. Find eigenvalues and associated eigenvectors of $A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$.

$\lambda_1 = 2, \lambda_2 = -1,\ \eta_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \eta_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Work out details during the class.

Example 6. $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The eigenvalues are 1 and 1, a repeated eigenvalue, we only have one linearly independent eigenvector $\eta = \begin{pmatrix} 1. \\ 0 \end{pmatrix}$.

When can we have n linearly independent vectors?

Theorem. If $A = A^T$, $a_{ij} \in \mathbb{R}$, then all eigenvalues are real, and it has full set of linearly independent eigenvectors, regardless of multiplicities of the eigenvalues. Actually the eigenvectors can be chosen orthogonal to each other.

Theorem.. If $A_{n\times n}$ has n distinct eigenvalues $\lambda_1, ..., \lambda_n$, and $\eta_1, ..., \eta_n$ are their corresponding eigenvectors, then $\eta_1, ..., \eta_n$ are linearly independent.