



PHY 410 - Reference Sheet

Stirling's approximation - for very large N :

$$\log N! \approx N \log N - N$$

$$N! \approx \sqrt{2\pi N} N^N e^{-N}$$

Fractional uncertainty of \mathbb{X} is uncertainty of expected value per particle.

$$\frac{\Delta \mathbb{X}}{N} = \frac{\sqrt{\langle \mathbb{X}^2 \rangle - \langle \mathbb{X} \rangle^2}}{N}$$

Boltzmann's constant

$$k_B = 1.380649 \times 10^{-23} \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}$$

Entropy $S = k_B \sigma$, $\sigma_{TOT} = \sigma_1 + \sigma_2$

Temperature $T = \tau / k_B$

Microcanonical Ensemble

Multiplicity function

$$g = \# \text{ of microstates, } \mathcal{P}(n) = \frac{1}{g}$$

Expected value of \mathbb{X} is the average across all microstates.

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n) \mathcal{P}(n) = \frac{1}{g} \sum_n \mathbb{X}(n)$$

Entropy can be written in terms of the multiplicity function.

$$\sigma(N, T, U, V, P) \equiv \log[g(N, T, U, V, P)]$$

Binary System

A **binary system** is a system of N particles where each particle has two possible states. Let N_\uparrow is the number of particle in the up state and N_\downarrow be the number of particles in the down state.

$$g(N, N_\uparrow) = \frac{N!}{N_\uparrow!(N - N_\uparrow)!}, \quad \sum_{N_\uparrow=0}^N g(N, N_\uparrow) = 2^N$$

The binary system can be rewritten in terms of the difference between up states and down states this is the **spin excess**.

$$2S = N_\uparrow - N_\downarrow$$

$$g(N, S) = \frac{N!}{(\frac{N}{2} + S)!(\frac{N}{2} - S)!}$$

$$\sum_{S=-\frac{N}{2}}^{\frac{N}{2}} g(N, N_\uparrow) = 2^N$$

Applying Stirling's approximation to the binary model, for large N the multiplicity function and fractional uncertainty are

$$g(N, S) \approx g(N, 0) e^{-2s^2/N}$$

$$g(N, S) \approx \sqrt{\frac{2}{\pi N}} 2^N e^{-2s^2/N}$$

$$\frac{\Delta S}{N} \approx \frac{1}{\sqrt{N}}$$

An example of a binary system is N spin 1/2 particles in an external **magnetic field** B . The total energy U and magnetization M of the system are

$$U = \sum_{i=1}^N -m_i \cdot \vec{B} = -(N_\uparrow - N_\downarrow) m B = -2SmB$$

$$M = 2Sm = -U/B$$

$$g(N, U) = \frac{N!}{(\frac{N}{2} - \frac{U}{2mB})!(\frac{N}{2} + \frac{U}{2mB})!}$$

$$\sigma(N, S) \approx -\left(\frac{N}{2} + S\right) \log\left(\frac{1}{2} + \frac{S}{N}\right) - \left(\frac{N}{2} - S\right) \log\left(\frac{1}{2} - \frac{S}{N}\right)$$

$$M = Nm \tanh(mB/\tau)$$

Einstein Solid

An **einstein solid** is a system of N atoms where each atom is modeled as a harmonic oscillator the energy of the system is determined by the number of atoms n oscillating at frequency ω .

$$U = n\hbar\omega$$

$$g(N, n) = \frac{(n + N - 1)!}{n!(N - 1)!}$$

$$g(N, n) \approx \left(\frac{n+N}{n}\right)^n \left(\frac{n+N}{N}\right)^N \frac{1}{\sqrt{2\pi n(n+N)/N}}$$

Thermal Equilibrium

Temperature

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_{N, V}$$

Thermal Equilibrium

$$\left(\frac{\partial \sigma_1}{\partial U_1}\right)_{N_1, V_1} = \left(\frac{\partial \sigma_2}{\partial U_2}\right)_{N_2, V_2}$$

$$\frac{1}{\tau_1} = \frac{1}{\tau_2}$$

2nd law of thermo - Change in entropy ≥ 0 .

Sharpness of Equilibrium For a two binary systems, the number of states in a configuration of deviation δ from equilibrium is

$$g_1 g_2 = (g_1 g_2)_{max} e^{-\left(\frac{2\delta^2}{N_1} + \frac{2\delta^2}{N_2}\right)}$$

Canonical Ensemble

Partition Function - partition by energy levels for a fixed temperature

$$z = \sum_n e^{-\varepsilon_n/\tau}, \quad \mathcal{P}(n) = \frac{1}{z} e^{-\varepsilon_n/\tau}$$

$$z = \sum_\alpha g(\varepsilon_\alpha) e^{-\varepsilon_\alpha/\tau}, \quad \text{for degeneracy } g(\varepsilon_\alpha)$$

Expected Value of \mathbb{X} is the average across all energies (Thermal Average).

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n) \mathcal{P}(n) = \frac{1}{z} \sum_n \mathbb{X}(n) e^{-\varepsilon_n/\tau}$$

Expected Energy in the canonical ensemble is

$$U = \langle \varepsilon \rangle = \frac{1}{z} \sum_n \varepsilon_n e^{-\varepsilon_n/\tau}$$

$$U = \langle \varepsilon \rangle = \tau^2 \frac{1}{z} \frac{\partial z}{\partial \tau} = \tau^2 \frac{\partial}{\partial \tau} \log z$$

The total partition function and expected value for N non-interacting particles is simply

$$z_N = z_1^N$$

$$\langle \mathbb{X} \rangle_N = N \langle \mathbb{X} \rangle_1 \Rightarrow U_N = N U_1$$

Helmholtz Free Energy

$$F = U - \tau \sigma = U - ST = -\tau \log z$$

$\Delta F \leq 0$ - helmholtz free energy decreases

$dF = 0$ - helmholtz free energy minimized

Entropy $\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V$, $S = k_B \sigma$

Temperature $\tau = \left(\frac{\partial U}{\partial \sigma}\right)_V$

Pressure

$$P = -\left(\frac{\partial U}{\partial V}\right)_\sigma = \tau \left(\frac{\partial \sigma}{\partial V}\right)_U = -\left(\frac{\partial F}{\partial V}\right)_\tau$$

Energy $U = -\tau^2 \frac{\partial}{\partial \tau} \left(\frac{F}{\tau}\right)$

Concentration and DeBroglie Wavelength

$$n = \frac{N}{V}, \quad n_Q = \frac{1}{\lambda_T^3}, \quad \lambda_T = \sqrt{\frac{2\pi\hbar^2}{m\tau}}$$

Single Particle Ideal Gas

A system in the canonical ensemble consisting of a single particle in a box of side lengths L . The energy levels, partition function and average energy are

$$\varepsilon_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2 = \frac{\hbar^2 \pi^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2)$$

and for the ultra-relativistic case:

$$\varepsilon_n = pc = \frac{\pi \hbar c}{L} n = \frac{\pi \hbar c}{L} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

$$z_1 = \frac{V}{\lambda_T^3}, \quad U_1 = \frac{3}{2} \tau$$

$$\sigma_1 = \log\left(\frac{V}{\lambda_T^3}\right) + \frac{3}{2}, \quad F_1 = -\tau \log \frac{V}{\lambda_T^3}$$

N-Particle Ideal Gas

Gibbs Resolution for systems of N identical particles the partition function is

$$z_N = \frac{1}{N!} (z_1)^N$$

$$PV = N\tau, \quad U = \frac{3}{2} N\tau$$

$$\sigma = N \left[\log\left(\frac{V}{N\lambda_T^3}\right) + \frac{5}{2} \right]$$

$$F = N\tau \left[\log \frac{n}{n_Q} - 1 \right]$$

Thermal Radiation

Single Frequency Photon Gas is a system in the canonical ensemble that considers photons of a specific frequency ω .

$$\varepsilon_s = \hbar\omega, \quad s = 0, 1, 2, 3, \dots$$

$$z = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}}$$

$$\mathcal{P}(s) = \frac{e^{-s\hbar\omega/\tau}}{z}$$

$$\langle s \rangle = \frac{1}{z} \sum_{s=0}^{\infty} s e^{-s\hbar\omega/\tau} = \frac{1}{e^{\hbar\omega/\tau} - 1}$$

Photon Gas is an expansion of the single frequency photon gas that considers all the possible cavity modes. The modes are 2 fold degenerate for the 2 independent polarizations.

$$\omega_n = \frac{c\pi}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{c\pi n}{L}$$

$$U = \langle \varepsilon \rangle = 2 \sum_n \frac{\hbar\omega_n}{e^{\hbar\omega_n/\tau} - 1} = \frac{\pi^2 V}{15(\hbar c)^3} \tau^4$$

Stefan-Boltzmann Law

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3} \int_0^{\infty} \frac{\omega^3}{e^{\hbar\omega/\tau} - 1} d\omega = \frac{\pi^2}{15(\hbar c)^3} \tau^4$$

Spectral Density Function

$$\frac{\partial U}{\partial \omega} \frac{1}{V} = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/\tau} - 1}$$

Flux Density (σ_B =Stefan-Boltzmann constant)

$$J_\mu = \frac{1}{4} \frac{cU}{V} = \sigma_B \tau^4 = \frac{\pi^2}{60(\hbar c)^3} \tau^4$$

Phonons in a Solid (Debye Model)

Phonons in a solid is a system in the canonical ensemble that is very similar to thermal radiation except there is 3 fold degeneracy from 3 polarizations of phonons and an upper cutoff frequency ω_D due to the separation distance between atoms.

$$\omega_n = \frac{\pi c_S}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\pi c_S}{L} n$$

Debye cutoff frequency

$$\omega_D = c_S \left(\frac{6\pi^2 N}{V}\right)^{1/3}, \quad \omega_D = \frac{\pi c_S}{L} n_D$$

Grand Canonical Ensemble

Chemical Potential

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{\tau,V}$$
$$\mu = \tau \log \left(\frac{N \lambda_T^3}{V}\right) = \tau \log \left(\frac{n}{n_Q}\right)$$
$$\mu = \left(\frac{\partial U}{\partial N}\right)_{\sigma,V} = -\tau \left(\frac{\partial \sigma}{\partial N}\right)_{U,V}$$

Grand Partition Function - partition by energy levels for a fixed temperature and all possible values of N

$$z_\epsilon = \sum_N \sum_{n(N)} e^{-(\epsilon_n^N - \mu N)/\tau}$$
$$\mathcal{P}(N, \epsilon_n) = \frac{1}{z_\epsilon} e^{-(\epsilon_n^N - \mu N)/\tau}$$

Fugacity

$$z_\epsilon = \sum_N \lambda^N \sum_{s(N)} e^{-\epsilon_s^N/\tau} = \sum_N \lambda^N z_N$$

Expected Value of \mathbb{X} is the average across all energies (Diffusive Average).

$$\langle \mathbb{X} \rangle = \frac{1}{z_\epsilon} \sum_N \sum_s \mathbb{X}(N, s) e^{-(\epsilon_s^N - \mu N)/\tau}$$

Expected Number of Particles in the grand canonical ensemble is

$$N = \langle N \rangle = \tau \frac{\partial}{\partial \mu} \log z_\epsilon = \lambda \frac{\partial}{\partial \lambda} \log z_\epsilon$$

Expected Energy in the grand canonical ensemble is

$$U = \langle \epsilon \rangle = \frac{1}{z_\epsilon} \sum_N \sum_{n(N)} \epsilon_n^N e^{-(\epsilon_n^N - \mu N)/\tau}$$
$$U = \langle \epsilon \rangle = \tau^2 \left(\frac{\partial}{\partial \tau} \log z_\epsilon\right)_\lambda$$

Concentration and DeBroglie Wavelength

$$n = \frac{N}{V}, \quad n_Q = \frac{1}{\lambda_T^3}, \quad \lambda_T = \sqrt{\frac{2\pi\hbar^2}{m\tau}}$$

Grand Potential

$$\Omega = U - \sigma\tau - \mu N$$
$$\Omega = -\tau \log z_\epsilon$$
$$\sigma = \left(\frac{-\partial \Omega}{\partial \tau}\right)_{V,\mu} \quad P = \left(\frac{-\partial \Omega}{\partial V}\right)_{\tau,\mu} \quad N = \left(\frac{-\partial \Omega}{\partial \mu}\right)_{\tau,V}$$

System of Non-interacting Particles

The grand partition function for a system with M energy states where n_α is the number of particles occupying a state is

$$z_\epsilon = \prod_{\alpha=1}^M z_{\alpha}, \quad z_{\alpha} = \sum_{n_{\alpha}} e^{-n_{\alpha}(\epsilon_{\alpha}-\mu)/\tau}$$
$$U = \sum_{\alpha=1}^M \epsilon_{\alpha} f(\epsilon_{\alpha}), \quad N = \sum_{\alpha=1}^M f(\epsilon_{\alpha})$$

Fermions

$$n_{\alpha} = 0, 1$$
$$z_{\alpha} = 1 + e^{-(\epsilon_{\alpha}-\mu)/\tau} = 1 + \lambda e^{-\epsilon_{\alpha}/\tau}$$

Fermi-Dirac Distribution is the expected number of a particles in a particular energy ϵ_{α} .

$$\langle n_{\alpha} \rangle = f(\epsilon_{\alpha}) = \frac{1}{e^{(\epsilon_{\alpha}-\mu)/\tau} + 1} = \frac{1}{\lambda^{-1} e^{\epsilon_{\alpha}/\tau} + 1}$$

For $\tau \rightarrow 0$: $f(\epsilon_{\alpha}) = \theta(\epsilon_{\alpha} - \mu)$

Bosons (Bosons)

$$n_{\alpha} = 0, 1, 2, 3, \dots$$
$$z_{\alpha} = \frac{1}{1 - e^{-(\epsilon_{\alpha}-\mu)/\tau}} = \frac{1}{1 - \lambda e^{-\epsilon_{\alpha}/\tau}}$$

Boson Distribution is the expected number of a particles in a particular energy ϵ_{α} .

$$\langle n_{\alpha} \rangle = f(\epsilon_{\alpha}) = \frac{1}{e^{(\epsilon_{\alpha}-\mu)/\tau} - 1} = \frac{1}{\lambda^{-1} e^{\epsilon_{\alpha}/\tau} - 1}$$

Ideal Gas

Both fermions and bosons behave identically at the classical limit $\epsilon_{\alpha} - \mu \gg \tau$.

$$\langle n_{\alpha} \rangle = f(\epsilon_{\alpha}) = e^{-(\epsilon_{\alpha}-\mu)/\tau}$$
$$z_{\epsilon} = \sum_N \lambda^N z_N = \sum_N \lambda^N \frac{1}{N!} z_1^N = e^{\lambda z_1}$$
$$\lambda = \frac{n}{n_Q}, \quad PV = N\tau, \quad U = \frac{3}{2}N\tau, \quad \mu = \tau \log \frac{n}{n_Q}$$
$$\sigma = N \left[\log \frac{n_Q}{n} + \frac{5}{2} \right], \quad F = N\tau \left[\log \frac{n}{n_Q} - 1 \right]$$

Heat Capacity measures the change in heat energy per unit temperature

$$C_P > C_V, \quad C_V = \left(\frac{\partial U}{\partial T}\right)_V = \tau \left(\frac{\partial \sigma}{\partial T}\right)_V$$
$$C_P = \left(\frac{\partial U}{\partial T}\right)_P + P \left(\frac{\partial V}{\partial T}\right)_P = \tau \left(\frac{\partial \sigma}{\partial T}\right)_P$$

Monoatmc gas $C_V = \frac{3}{2}Nk_B, \quad C_P = \frac{5}{2}Nk_B$

Isothermal Expansion $\sigma_f - \sigma_i = N \log \frac{V_f}{V_i}$
 $Q = N\tau \log \frac{V_f}{V_i}$

Isoentropic Expansion $\frac{\tau_f}{\tau_i} = \left(\frac{V_i}{V_f}\right)^{2/3}$

Internal Excitations

Expansion of the ideal gas to take into account the additional energy states from internal excitations.

$$z_{int} = \sum_{\alpha} e^{-\epsilon_{\alpha}/\tau}, z_{\epsilon} = 1 + \lambda z_{int} e^{-\epsilon_n/\tau}$$

Internal Excitation Corrections

$$\lambda = \frac{n}{n_Q z_{int}}, \mu = \tau \left(\log \frac{n}{n_Q} - \log z_{int} \right)$$
$$F = N\tau \left[\log \frac{n}{n_Q} - 1 \right] - N\tau \log z_{int}$$
$$\sigma = N \left[\log \frac{n}{n_Q} + \frac{5}{2} \right] - \left(\frac{\partial F_{int}}{\partial \tau} \right)_V$$

Fermi Gas

DOG (bork)



Kaedon.net/reference