PHY 410 - Reference Sheet

Boltzmann's constant
$k_B = 1.380649 \times 10^{-23} \mathrm{m^2\,s^{-2}\,K^{-1}}$
Entropy $S = k_B \sigma$
Temperature $T = \tau / k_B$

Canonical Ensemble

Partition Function - partition by energy levels for a fixed temperature

$$z = \sum_n e^{-\varepsilon_n/\tau}, \quad \mathcal{P}(n) = \frac{1}{z} e^{-\varepsilon_n/\tau}$$

Expected Value of $\mathbb{X}$ is the average across all energies (Thermal Average).

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n)\mathcal{P}(n) = \frac{1}{z}\sum_n \mathbb{X}(n) e^{-\varepsilon_n/\tau}$$

Expected Energy in the canonical ensemble is

$$U = \langle \varepsilon \rangle = \frac{1}{z}\sum_n \varepsilon_n e^{-\varepsilon_n/\tau}$$

$$U = \langle \varepsilon \rangle = \tau^2 \frac{1}{z}\frac{\partial z}{\partial \tau} = \tau^2 \frac{\partial}{\partial \tau}\log z$$

The total partition function and expected value for N non-interacting particles is simply

$$z_N = z_1^N$$
$$\langle \mathbb{X} \rangle_N = N \langle \mathbb{X} \rangle_1 \quad \Rightarrow \quad U_N = N U_1$$

Helmholtz Free Energy

$$F = U - \tau\sigma = U - ST = -\tau \log z$$
$$dF = -\sigma d\tau - P dV$$

Entropy $\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V, \quad S = k_B \sigma$

Pressure

$$P = -\left(\frac{\partial U}{\partial V}\right)_\sigma = \tau \left(\frac{\partial \sigma}{\partial V}\right)_U = -\left(\frac{\partial F}{\partial V}\right)_\tau$$

Energy $U = -\tau^2 \frac{\partial}{\partial \tau}\left(\frac{F}{\tau}\right)$

Thermal Radiation

Single Frequency Photon Gas is a system in the canonical ensemble that considers photons of a specific frequency ω.

$$\varepsilon = s\hbar\omega, \quad s = 0, 1, 2, 3, \ldots$$

$$z = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}}$$

$$\mathcal{P}(s) = \frac{e^{-s\hbar\omega/\tau}}{z}$$

$$\langle s \rangle = \frac{1}{z}\sum_{s=0}^{\infty} s e^{-s\hbar\omega/\tau} = \frac{1}{e^{\hbar\omega/\tau} - 1}$$

Photon Gas is an expansion of the single frequency photon gas that considers all the possible cavity modes. The modes are 2 fold degenerate for the 2 independent polarizations.

$$\omega_n = \frac{c\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{c\pi n}{L}$$

$$U = \langle \epsilon \rangle = 2\sum_n \frac{\hbar\omega_n}{e^{\hbar\omega_n/\tau} - 1} = \frac{\pi^2 V}{15(\hbar c)^3}\tau^4$$

Stefan-Boltzmann Law

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{e^{\hbar\omega/\tau} - 1} d\omega = \frac{\pi^2}{15(\hbar c)^3}\tau^4$$

Spectral Density Function

$$\frac{\partial}{\partial \omega}\frac{U}{V} = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/\tau} - 1}$$

Flux Density(σ_B=Stefan-Boltzmann constant)

$$J_\mu = \frac{1}{4}\frac{cU}{V} = \sigma_B \tau^4 = \frac{\pi^2}{60(\hbar c)^3}\tau^4$$

Phonons in a Solid (Debye Model)

Phonons in a solid is a system in the canonical ensemble that is very similar to thermal radiation except there is 3 fold degeneracy from 3 polarizations of phonons and an upper cutoff frequency ω_D due to the separation distance between atoms.

$$\omega_n = \frac{\pi c_S}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\pi c_s}{L} n$$

Debye cutoff frequency

$$\omega_D = c_S\left(\frac{6\pi^2 N}{V}\right)^{1/3}, \quad \omega_D = \frac{\pi c_S}{L} n_D$$

Grand Canonical Ensemble

Chemical Potential

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{\tau, V}$$

$$\mu = \tau \log\left(\frac{N\lambda_T^3}{V}\right) = \tau \log\left(\frac{n}{n_Q}\right)$$

$$\mu = \left(\frac{\partial U}{\partial N}\right)_{\sigma, V} = -\tau\left(\frac{\partial \sigma}{\partial N}\right)_{U, V}$$

Grand Partition Function - partition by energy levels for a fixed temperature and all possible values of N

$$\mathcal{Z} = \sum_N \sum_{n(N)} e^{(\varepsilon_n - \mu N)/\tau}$$

$$\mathcal{P}(N, \varepsilon_n) = \frac{1}{\mathcal{Z}} e^{-(\varepsilon_n^N - \mu N)/\tau}$$

Fugacity

$$\lambda = e^{\mu/\tau}$$

$$\mathcal{Z} = \sum_N \lambda^N \sum_{s(N)} e^{-\varepsilon_s/\tau} = \sum_N \lambda^N z_N$$

Expected Value of $\mathbb{X}$ is the average across all energies (Diffusive Average).

$$\langle \mathbb{X} \rangle = \frac{1}{\mathcal{Z}}\sum_N \sum_s \mathbb{X}(N, s) e^{(-\varepsilon_s - \mu N)/\tau}$$

Expected Number of Particles in the grand canonical ensemble is

$$N = \langle N \rangle = \tau \frac{\partial}{\partial \mu}\log \mathcal{Z}$$

$$N = \langle N \rangle = \lambda \frac{\partial}{\partial \lambda}\log \mathcal{Z}$$

Expected Energy in the grand canonical ensemble is

$$U = \langle \varepsilon \rangle = \frac{1}{\mathcal{Z}}\sum_N \sum_{n(N)} \varepsilon_n e^{-(\varepsilon_n - \mu N)/\tau}$$

$$U = \langle \varepsilon \rangle = \tau^2 \left(\frac{\partial}{\partial \tau}\log \mathcal{Z}\right)_\lambda$$

Grand Potential

$$\Omega = U - \sigma\tau - \mu N$$
$$\Omega = -\tau \log \mathcal{Z}$$
$$\sigma = -\left(\frac{\partial \Omega}{\partial \tau}\right)_{V, \mu}$$
$$P = -\left(\frac{\partial \Omega}{\partial V}\right)_{\tau, \mu}$$
$$N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{\tau, V}$$

System of Non-interacting particles

The grand partition function for a system with M energy states where n_α is the number of particles occupying a state is

$$\mathcal{Z} = \prod_{\alpha=1}^{M} \mathcal{Z}_\alpha, \quad \mathcal{Z}_\alpha = \sum_{n_\alpha} e^{-n_\alpha(\varepsilon_\alpha - \mu)/\tau}$$

$$U = \sum_{\alpha=1}^{M} \varepsilon_\alpha f(\varepsilon_\alpha), \quad N = \sum_{\alpha=1}^{M} f(\varepsilon_\alpha)$$

Fermions

$$n_\alpha = 0, 1$$
$$\mathcal{Z}_\alpha = 1 + e^{-(\varepsilon_\alpha - \mu)/\tau} = 1 + \lambda e^{-\varepsilon_\alpha/\tau}$$

Fermi-Dirac Distribution is the expected number of a particles in a particular energy ε_α.

$$\langle n_\alpha \rangle = f(\varepsilon_\alpha) = \frac{1}{e^{-(\varepsilon_\alpha - \mu)/\tau} + 1} = \frac{1}{\lambda e^{-\varepsilon_\alpha/\tau} + 1}$$

For $\tau \to 0$: $f(\varepsilon_\alpha) = \theta(\varepsilon_\alpha - \mu)$

Bosons (Bonsons)

$$n_\alpha = 0, 1, 2, 3, \ldots$$
$$\mathcal{Z}_\alpha = \frac{1}{1 - e^{-(\varepsilon_\alpha - \mu)/\tau}} = \frac{1}{1 - \lambda e^{-\varepsilon_\alpha/\tau}}$$

Boson Distribution is the expected number of a particles in a particular energy ε_α.

$$\langle n_\alpha \rangle = f(\varepsilon_\alpha) = \frac{1}{e^{(\varepsilon_\alpha - \mu)/\tau} - 1} = \frac{1}{\lambda^{-1} e^{\varepsilon_\alpha/\tau} - 1}$$

Ideal Gas

Both fermions and bosons behave identically at the classical limit $\varepsilon_\alpha - \mu >> \tau$.

$$\langle n_\alpha \rangle = e^{-(\varepsilon_\alpha - \mu)/\tau}$$

$$\mathcal{Z} = \sum_N \lambda^N z_N = \sum_N \lambda^N \frac{1}{N!} z_1^N$$

$$\mathcal{Z} = e^{\lambda z_1}, \quad \log \mathcal{Z} = \lambda z_1$$

$$PV = N\tau, \quad U = \frac{3}{2}N\tau$$

$$\sigma = N\left[\log\frac{n_Q}{n} + \frac{5}{2}\right], \quad \mu = \tau \log\frac{n}{n_Q}$$

Internal Excitations

Expansion of the ideal gas to take into account the additional energy states from internal excitations.

$$z_{int} = \sum_\alpha e^{-\varepsilon_\alpha/\tau}$$

$$\mathcal{Z} = 1 + \lambda z_{int} e^{-\varepsilon_n/\tau}$$

Internal Excitation Corrections

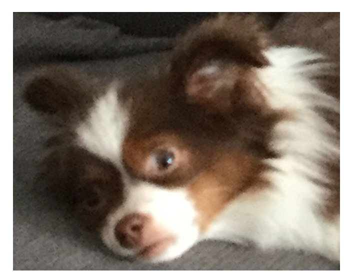
$$\lambda = \frac{n}{n_Q z_{int}}$$

$$\mu = \tau\left(\log\frac{n}{n_Q} - \log z_i nt\right)$$

$$F = N\tau\left[\log\frac{n}{n_Q} - 1\right] - N\tau \log z_{int}$$

$$\sigma = N\left[\log\frac{n}{n_Q} + \frac{5}{2}\right] - \left(\frac{\partial F_{int}}{\partial \tau}\right)_V$$

DOG (bork)



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