Math Reference

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Contents

Chapter 1

Introduction

This chapter will offer reference and information that applies to the entire book.

1.1 Structure of This Book

1.1.1 Categories

Each section of this book will focus on one of these general categories

- **Notation -** The way that we choose to represent mathematics as is written down, each topic will have a notation page with symbol definitions and other important information
- **Number Systems -** Representations of a numbers and fundamental operations that we can run on these numbers (i.e. numbers, vectors, counting, complex numbers)
- **Structures -** Ways to organize numbers operations and units to represent something or to indicate something (i.e. equations, logical statements, foundation of proofs)
- **Methods -** Strategies for going between structures and representations of real things (i.e. integrals, derivatives, trigonometry, rref)

Chapter 2

Linear Algebra

2.1 Notation

General
$\forall$ - For all
$\exists$ - Exists

Common Sets
$\mathbb{C}$ - Set of all Complex Numbers
$\mathbb{R}$ - Set of all Real Numbers
$\mathbb{Q}$ - Set of all Rational Numbers
$\mathbb{Z}$ - Set of all Integers
$\mathbb{N}$ - Set of all Natural Numbers

Set Notation
$\in$ - "In" := is an element of

Example. $\vec{v} \in \mathbb{R}^3$

$\notin$ - "Not In" := is not an element of

Example. $\vec{v} \notin \mathbb{R}^3$

$\{,\}$ - Set := elements of the set are listed inside the brackets
Example: $A = \{1, 2, 3\}$ "A is a set containing the elements 1, 2, and 3"
Note: elements in a set must be unique
$\{\}$ *or* $\emptyset$ - The Empty Set

Definition 2.1.1. $|\,|$ **- Cardinality** := The size of a set or the number of elements in a set.

Example. $|A| = n$ "set A has a cardinality of n"

Definition 2.1.2. $\cap$ **- Intersection** := The **Intersection** of two sets in the set of all elements that are contained in both sets.

Example. $A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}$

Definition 2.1.3. $\cup$ **- Union** := The **Union** of two sets in the set of all elements that are contained either of the two sets.

Example. $A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}$

$\vee$ - or

Example. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$ - and

Example. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

2.2 Vectors and Bases

Definition 2.2.1. Vector Space := a collection of vectors equiped with operations of addition and scalar multiplication such that the following axioms are true:

- Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v} \; \forall \; \vec{v}, \vec{w} \in V$
- Associativity: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{w} + \vec{v}) \; \forall \; \vec{u}, \vec{v}, \vec{w} \in V$
- Zero Vector: $\exists$ a vector $\vec{0}$ such that for any vector $\vec{v} \in V, \; \vec{v} + \vec{0} = \vec{v}$
- Additive Inverse: for any vector $\vec{v} \in V$ there exists a vector $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$
- Multiplicative Identity: for any vector $\vec{v} \in V, \; (1)\vec{v} = \vec{v}$
-

Chapter 3

Real Analysis

3.1 Notation

General
$\forall$ - For all
$\exists$ - Exists

Common Sets
$\mathbb{C}$ - Set of all Complex Numbers
$\mathbb{R}$ - Set of all Real Numbers
$\mathbb{Q}$ - Set of all Rational Numbers
$\mathbb{Z}$ - Set of all Integers
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Definition 3.1.1. $|\,|$ - **Cardinality** := The size of a set or the number of elements in a set.

Example. $|A| = n$ "set A has a cardinality of n"

Definition 3.1.2. $\cap$ - **Intersection** := The **Intersection** of two sets is the set of all elements that are contained in both sets.

Example. $A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}$

the **Intersection** of many sets can be denoted: $\cap_{i=1}^{k} A_i$ *For the set of elements that appear in all of* $A_1 \cdots A_k$

Definition 3.1.3. $\cup$ - **Union** := The **Union** of two sets is the set of all elements that are contained either of the two sets.

Example. $A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}$

the **Union** of many sets can be denoted: $\cup_{i=1}^{k} A_i$ *For the set of elements that appear in any of* $A_1 \cdots A_k$

Definition 3.1.4. $\subseteq$ - **Subset** := Set A is a **Subset** of B if all the elements of A are also elements of B. We denote this by $A \subseteq B$.

Definition 3.1.5. $\subsetneq$ - **Proper Subset** := Set A is a **Proper Subset** of B if $A \subseteq B$ and $A \neq B$

$\vee$ - or

Example. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$ - and

Example. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

3.2 Review of Set Theory

Definition 3.2.1. Two sets are consider to be equal if $A \subseteq B$ and $A \supseteq B$

Definition 3.2.2. Pairwise Disjoint := A set of sets $\Im$ is considered to be **Pairwise Disjoint** *if for* $S, T \in \Im$

$$S \neq T \Rightarrow S \cup T = \emptyset$$

There are two way of taking "differences" of sets:

$$X \backslash Y = \{x \in X : x \notin Y\}$$

$$X \Delta Y = (X \cup Y) \backslash (X \cap Y)$$

Proof. For any three finite sets X, Y, Z:
From the definition of Δ we find that:

$$(X \Delta Y)\Delta Z = \{x \in (X \cup Y) : x \notin (X \cap Y)\}\Delta Z = \{x \in ((X \cup Y) \cup Z) : x \notin ((X \cap Y) \cap Z)\}$$

Now, since $\cup$ and $\cap$ are associative we have:

$$\{x \in ((X \cup Y) \cup Z) : x \notin ((X \cap Y) \cap Z)\} = \{x \in (X \cup (Y \cup Z)) : x \notin (X \cap (Y \cap Z))\}$$

Now, from the definition of Δ we find that:

$$\{x \in (X \cup (Y \cup Z)) : x \notin (X \cap (Y \cap Z))\} = X\Delta\{x \in (Y \cup Z) : x \notin (Y \cap Z)\} = X\Delta(Y\Delta Z)$$

Therefore:

$$(X \Delta Y)\Delta Z = X\Delta(Y\Delta Z)$$

$\square$

Definition 3.2.3. Given a set X and a set $\mathscr{S}$ whose elements are sets.

1. We say that $\mathscr{S}$ **covers** X if $X \subseteq \bigcup \mathscr{S}$
2. We say that $\mathscr{S}$ **partitions** X if $X = \bigcup \mathscr{S}$, the elements of $\mathscr{S}$ are non-empty, and $\mathscr{S}$ is pairwise disjoint

Definition 3.2.4. Ordered Pair (tuple) := an ordered list of two elements, each of which can be an arbitrary mathematical object and may or may not be the same. Denoted for $n \in \mathbb{N}$, an n-tuple is an ordered list of n elements, written as $(x_1, \ldots, x_n)$

Definition 3.2.5. For two sets X, Y the **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. More generally we can write a **Cartesian product** for n sets denoted by

$$X_1 \times X_2 \times \ldots \times X_n \textit{ or } \prod_{i=1}^{n} X_i$$

Remark. When taking the **Cartesian product** of the same set we use the shorthand: X^n

Remark. Additionally, the notation 2^X indicates the set of all possible subsets of X

Definition 3.2.6. We say that the **diagonal** of X^n is the subset $\{(x_1, \ldots, x_n) \in X^n : x_1 = x_2 = \ldots = x_n\}$

Definition 3.2.7. Given two sets X, Y we say that f is a **function** with domain X and codomain Y denoted $f : X \to Y$, if f is a subset of $X \times Y$ such that every element of X appears as exactly the first component of exactly one element of f.

Example. We used the notation $f(x)$ to refer to the element y such that $(x, y) \in f$ is the unique ordered pair that refers to the element $x \in X$.

Definition 3.2.8. The **Identity Function** is a function with the same domain and codomain X written $\mathbf{1}_X : X \to X$ corresponding to the diagonal 3.2.6 of X^2

Definition 3.2.9. Given $f : X \to W$ and $g : W \to Z$ with $Y \subseteq W$, the composition $g \circ f : X \to Z$ is the function satisfying $g \circ f(x) = g(f(x))$.

Definition 3.2.10. A function is **Injective** if $f(x) = f(u) \Rightarrow x = y$

Definition 3.2.11. A function $f : X \to Y$ is **Surjective** if the range of f equals Y

Definition 3.2.12. A function is **Bijective** if it is both Injective and Surjective

Theorem 3.2.1. If X is non-empty, $f : X \to Y$ is injective $\Leftrightarrow$ f is left invertible

Theorem 3.2.2. $f : X \to Y$ is surjective $\Leftrightarrow$ f is right invertible

Definition 3.2.13. A **Relation** of a set X is a subset of X^2. Conventionally written xRy rather than $(x, y) \in R$

3.2.14. Properties of Relation

1. **Reflexive** if xRx for all $x \in X$
2. **Transitive** if xRy and $yRz \Rightarrow xRz$
3. **Symmetric** if $xRy \Leftrightarrow yRx$
4. **Antisymmetric** if xRy and $yRx \Rightarrow x = y$
5. **Connex** if for every $x, y \in X$ at least on of xRy or yRx hold.

Definition 3.2.15. An **Equivalence Relation** is a relation that is Reflexive, Transitive, and Symmetric

Definition 3.2.16. if $\sim$ is an equivalence relation, the **Equivalence Class** of $x \in X$ is $[x] := \{y \in X : x \sim y\}$. Additionally, the notation $X/\sim$ refers to the set of all equivalence classes $\{[x] : x \in X\}$

3.2.1 The sets $\mathbb{Z}$ and $\mathbb{Q}$

Definition 3.2.17. The natural numbers $\mathbb{N}$ with it's addition and multiplication forms a commuative semiring.

Definition 3.2.18. A **commutative semi-ring** is set R equipped with binary operations addition and multiplication such that

1. $(R, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(R, \cdot)$ Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition 3.2.19. A **commutative ring** is set R equipped with binary operations addition and multiplication such that

1. $(R, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(R, \cdot)$ Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.

Definition 3.2.20. The integers $\mathbb{Z}$ is defined as a set of equivalence classes 3.2.16 $\mathbb{N}/\sim$ where the equivalence relation $\sim$ is

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

Remark. This is simply a rigorous way of representing integers with two natural numbers where the first natural number is the positive component and the second number is the negative component.

Definition 3.2.21. A **field** is set R equipped with binary operations addition and multiplication such that

1. $(R, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(R, \cdot)$ Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.
7. Multiplicative inverse.

Definition 3.2.22. The rational numbers $\mathbb{Q}$ is defined as a set of equivalence classes 3.2.16 $(\mathbb{Z} \times \mathbb{N})/\sim$ where the equivalence relation $\sim$ is

$$(a, n) \sim (b, m) \Leftrightarrow am = bn$$

Remark. This is simply a rigorous way of representing rational numbers as a fraction from integers and natural numbers where the integer is the numerator and the natural number is the denominator.

3.2.2 Cardinality of Sets

Definition 3.2.23. The **cardinality** of a set is the number of elements in that set.

- $card(A) = card(B)$ if there exists and bijective function: $A \to B$
- $card(A) \leq card(B)$ if there exists an injective(left invertible) function: $A \to B$
- $card(A) \geq card(B)$ if there exists an surjective(right invertible) function: $A \to B$

Proposition 3.2.1. (Pigeonhole Principle). Suppose $n < m$ there does not exist an injective(left invertible) function: $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ and there does not exist a surjective(right invertible) function: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$

Definition 3.2.24. A set X is said to be

- **countable** if $card(X) \leq card(\mathbb{N})$
- **uncountable** if $card(X) > card(\mathbb{N})$
- **finite** if $\exists\ n \in \mathbb{N}$ such that $card(X) \leq card(\{1, \ldots, n\})$
- **countably infinite** if $card(X) = card(\mathbb{N})$
- **infinite** if $card(X) \geq card(\mathbb{N})$

3.3 Partial Orders

Definition 3.3.1. **A Partial Order** is a relation $\preceq$ that is transitive, reflexive, and antisymmetric

Definition 3.3.2. **Poset** is a set that is equipped with a partial order.

Definition 3.3.3. Let $(X, \preceq)$ and $(U, \trianglelefteq)$ be posets we say a function $f : X \to Y$ is...

- **increasing** if $x_1 \preceq x_2 \Rightarrow f(x_1) \trianglelefteq f(x_2)$
- **decreasing** if $x_1 \preceq x_2 \Rightarrow f(x_2) \trianglelefteq f(x_1)$
- **monotone** if it is either increasing or decreasing (*note: the constant function is both increasing and decreasing*)
- **strictly increasing/decreasing/monotone** if it is increasing/decreasing/monotone and injective.
- **an order isomorphism** if it is invertible and both f and f^{-1} are increasing.

Definition 3.3.4. Let $(X, \preceq)$ and be a poset. Define the two functions $\uparrow, \downarrow: X \to 2^X$ by

- $\downarrow(x) : \{y \in X : y \preceq x\}$, a subset is a **lower set** or **downward closed** if $s \in S \Rightarrow \downarrow(s) \subseteq S$.
- $\uparrow(x) : \{y \in X : x \preceq y\}$, a subset is an **upper set** or **upper closed** if $s \in S \Rightarrow \uparrow(s) \subseteq S$

Definition 3.3.5. a lower(upper) set S is said to be **principal** if there exists $x \in X$ such that $\downarrow(x) = S(\uparrow(x) = S)$

Definition 3.3.6. Let $(X, \preceq)$ and be a poset and let $S \subseteq X$, and $z \in X$

- We say that z is an **Upper bound** of S if $S \subseteq \downarrow(z)$. The set s is said to be **bounded above** if it has an upper bound.
- We say that z is a **Lower bound** of S if $S \subseteq \uparrow(z)$. The set s is said to be **bounded bellow** if it has a lower bound.
- We say that S is order bounded (*or just bounded*), if it is bounded both above and below.

Definition 3.3.7. Let $(X, \preceq)$ and be a poset a subset $S \subseteq X$ is said to be...

- **downward directed** if every finite subset has a lower bound $z \in S$
- **upward directed** if every finite subset has a upper bound $z \in S$

3.3.1 Special Elements

Definition 3.3.8. Let $(X, \preceq)$ be a poset, and let $S \subseteq X$. We say that an element of $s_0 \in S$ is...

- **the maximum of** S if $S \subseteq \downarrow (s_0)$
- **the minimum of** S if $S \subseteq \uparrow (s_0)$
- **a maximal element of** S if, for $s \in S$, $s_0 \in \downarrow (s) \Rightarrow s_0 = s$
- **a minimal element of** S if, for $s \in S$, $s_0 \in \uparrow (s) \Rightarrow s_0 = s$

Definition 3.3.9. Let $(X, \preceq)$ be a poset and $S \subseteq X$. We say that an element of $x \in X$ is...

- **the supremum of** S if $x = \min\{y \in X : S \subseteq \downarrow (y)\}$
- **the infimum of** S if $x = \max\{y \in X : S \subseteq \uparrow (y)\}$

3.4 Total Orders

Definition 3.4.1. **A Total Order** is a relation $\preceq$ that is transitive, reflexive, antisymmetric, and connex.

Definition 3.4.2. **A Well Ordered Set** is totally ordered set where every non-empty subset has a minimum.

Theorem 3.4.1. Totally ordered sets cannot contain imaginary numbers.

Definition 3.4.3. A **totally ordered field** is a field F equipped with a total order $\preceq$ such that

- $\preceq$ respects addition: $a \preceq b \Rightarrow a + c \preceq b + c$
- $\preceq$ respects positive multiplication: $0 \preceq a \Rightarrow a + c \preceq b + c$

Definition 3.4.4. In a totally ordered field, the set of **positive** elements is $\uparrow (0) \backslash 0$. The set of **negative** elements is $\downarrow (0) \backslash 0$.

Definition 3.4.5. Given a totally ordered field $(F, \preceq)$. The absolute value function $F \Rightarrow F$, denoted by $x \to |x|$, is

$$|x| = \begin{cases} x & 0 \preceq x \\ -x & x \preceq 0 \end{cases}$$

Proposition 3.4.1. Let $(X, \preceq)$ be a total order. Let A, B be both upper(both lower) sets. Either $A \subseteq B$ or $B \subseteq A$

Definition 3.4.6. Let $\{X, \preceq\}$ be a total order. We say a pair of subsets (X_-, X_+) form a **cut** of X if:

- $\{X_-, X_+\}$ is a partition of X.
- X_- is a lower set and X_+ is an upper set.

Definition 3.4.7. A totally ordered set X is said to be **Dedekind complete** if in every cut (X_-, X_+), at least one of X_- or X_+ is principal. That is $\exists x \in X_-$ such that $\downarrow (x) = X_-$ or $\exists x \in X_+$ such that $\uparrow (x) = X_+$

Proposition 3.4.2. Let $(X, \preceq)$ be a Dedekind total order. The total order restricted to $\uparrow (a)$ and $\downarrow (a)$ for any $a \in X$ is also Dedekind complete.

Definition 3.4.8. A poset $(X, \preceq)$ is said to posses the **least upper bound property** if every nonempty subset that is bounded above has a supremum. Similarly, a poset $(X, \preceq)$ is said to posses the **greatest lower bound property** if every nonempty subset that is bounded bellow has an infimum.

Theorem 3.4.2. For a totally ordered set $(X, \preceq)$

- X is a Dedekind complete
- X has the least upper bound property
- X has the greatest lower bound property

Remark. This theorem also holds for posets if you remove the Dedekind complete line since the definition of Dedekind complete relies a total order. In general we can also define Dedekind complete as a poset that has the least upper bound property and greatest lower bound property.

Definition 3.4.9. Given $(X, \preceq)$ a partial order and elements $a, b \in X$. The **open interval** (a, b) is defined to be $\uparrow(a) \cap \downarrow(b) \backslash \{a, b\}$

Definition 3.4.10. Given $(X, \preceq)$ a partial order and elements $a, b \in X$. The **closed interval** (a, b) is defined to be $\uparrow(a) \cap \downarrow(b)$

Definition 3.4.11. Given $(X, \preceq)$ as total order with no max and no min, an **entourage mapping** is a function $f : X \to 2^X$ such that $f(x)$ is an open interval that contains x.

Definition 3.4.12. Given $(X, \preceq)$ as total order with no max and no min, we say that it possesses the **Heine-Borel property** if, for every closed interval $[a, b]$ and every entourage mapping f, there exists a finite subset $S \subseteq [a, b]$ such that $f(S)$ covers $[a, b]$.

Theorem 3.4.3. Suppose $(X, \preceq)$ as total order with no max and no min. Then it is Dedekind complete if and only if it possesses the Heine-Borel property.

3.4.1 The Reals

Definition 3.4.13. The set $\mathbb{R}$ is defined to be the set of all cuts (X_-, X_+) of $\mathbb{Q}$ such that X_- has no maximum.

Definition 3.4.14. We equip $\mathbb{R}$ with the relation $\leq$ defined as $(X_-, X_+) \leq (Y_-, Y_+)$ if $X_- \subseteq Y_-$.

Theorem 3.4.4. Archimedean Property of Reals If x, y are positive real numbers then $\exists n \in \mathbb{N}$ such that $n \cdot x > y$.

Definition 3.4.15. We say a subset $U \subseteq \mathbb{R}$ is open if if for every $x \in U$, there is an open interval $(a_X, b_X) \subseteq U$ with $x \in (a_X, b_X)$

3.5 Nets and Limits

Definition 3.5.1. A **directed set** is a pair $(X, \preceq)$ where X is a set equipped with a relation $\preceq$ such that

- $\preceq$ is reflexive.
- $\preceq$ is transitive.
- $\preceq$ is upward directed; for any $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.

Lemma 3.5.1. Let $(X, \preceq)$ be a directed set. Let $x \in X$ the set $\uparrow(x)$ equipped with the restricted order, is a directed set. Additionally any total order is a directed set.

3.5.1 Nets

Definition 3.5.2. A **net** is a function $f : A \to B$ from a directed set $(A, \preceq)$.

Remark. Notationally instead of $f(a)$ we usually write f_a.

Definition 3.5.3. Given a net $f : A \to B$, we denote the **tail sets** of f by

$$f_{\uparrow(\alpha_0)} = \{f_\alpha \in B : \alpha \preceq \alpha_0\}$$

Definition 3.5.4. For a net $f : A \to B$ and a subset $S \subseteq B$ we say that:

- f is **eventually** in S if there exists some $a \in A$ such that $f_{\uparrow(a)} \subseteq S$.
- f is **frequently** in S if for every $a \in A$, the intersection $f_{\uparrow(a)} \cap S \neq \emptyset$
- f is **infrequently** in S if there exists some $a \in A$ such that $f_{\uparrow(a)}$ is disjoint from S.

3.5.2 Limits

Definition 3.5.5. A real valued net f is said to **Converge** to z if for every open interval I containing z, the net f is eventually in I. When f converges to z we sat that z is the **limit** of f and write $z = \lim f$.

Lemma 3.5.2. Let B be a set and $f : A \to B$ be a non-empty net. Fix a subset $S \subseteq B$.

- A net f is either frequently in S or infrequently in S.
- A net f is eventually in S if and only if f is infrequently in $X \backslash S$.

- If a net f is eventually in S, then f is frequently in S.

Definition 3.5.6. A real valued net f is said to **accumulate** (or **cluster**) at the real number z if for every open interval I containing z, the net x is frequently in I.

Proposition 3.5.1. If a real valued net f covers z, then z is its unique accumulations point.

Theorem 3.5.3. Arithmetic of Limits states that if x, y are real valued nets with the same domain then

$$\lim(x + y) = \lim(x) + \lim(y)$$

$$\lim(x \cdot y) = \lim(x) \cdot \lim(y)$$

Theorem 3.5.4. Limit Characterization of Open Sets states that a set $S \subseteq \mathbb{R}$ is open if an only if every real valued net f with an accumulation point in S is frequently in S.

Corollary 3.5.4.1. A set $S \subset \mathbb{R}$ is closed if and only if for every real valued net $f : A \to S$ has all of it's accumulation points in S.

Theorem 3.5.5. Monotone Convergence states that if f is a non-empty real valued net:

- If f is increasing and bounded above, then f converges to the supremum of its range.
- If f is decreasing and bounded below, then f converges to the infimum of its range.

Definition 3.5.7. Given a bounded, non-empty, interval $I \subseteq \mathbb{R}$, its width, which we denote by $|I|$, is a real number $|I| := \sup I - \inf I$. Necessarily $|I| \geq 0$

Definition 3.5.8. A real valued net f is a **Cauchy net** if for any positive real number ω, there exists an open interval I with a width $0 < |I| \leq \omega$ such that f is eventually in I.

Theorem 3.5.6. Cauchy's Criterion states that a real-valued net f is convergent if and only if f is Cauchy.

3.5.3 Limit superior and limit inferior

Definition 3.5.9. A real-valued net f is said to be eventually bounded (above/below) if there exists a_0 such that $f_{\uparrow(a_0)}$ is bounded (above/below)

Definition 3.5.10. Let f be a non-empty real-valued net.

- If f is eventually bounded above then its limit superior is defined as

  $$\limsup f := \lim U$$

  where $U :\uparrow (a_0) \to \mathbb{R}$ is defined as $U_a = \sup f_{\uparrow(a)}$.

- If f is eventually bounded below then its limit inferior is defined as

  $$\liminf f := \lim L$$

  where $L :\downarrow (a_0) \to \mathbb{R}$ is defined as $L_a = \inf f_{\downarrow(a)}$.

Proposition 3.5.2. Given a real-valued net f, it limit superior and limit inferior, when they exist, are the largest and smallest (respectively) accumulation points of x.

Theorem 3.5.7. Bolzano-Weierstrass Every non-empty bounded real-valued net f has an accumulation point.

Theorem 3.5.8. A real-valued net f converges if and only if it is eventually bounded and $\limsup f = \liminf f$.

Theorem 3.5.9. The Squeeze Theorem states that if x, y, z are real-valued nets with the same index set, such that both $y - x$ and $z - y$ are evntually non-negative. Then if z is eventually bounded above, and x is eventually bounded below, and $\limsup z = \liminf x = r$, then all three sequences converge to r.

3.6 Metric Spaces

Definition 3.6.1. A **Metric** is a function $f : X \times X \to \mathbb{R}$ for a set X, that satisfies the following properties:

- **Positivity** - The distance $d(x_1, x_2) \geq 0$, for all $x_1, x_2 \in X$.
- **Non-degeneracy** - The distance $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$, for all $x_1, x_2 \in X$.
- **Symmetry** - The distances $d(x_1, x_2) = d(x_2, x_1)$, for all $x_1, x_2 \in X$.
- **Triangle inequality** - Given $x_1, x_2, x_3 \in X$ their mutual distances satisfy

$$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$$

Definition 3.6.2. A **Metric Space** is a set X equipped with a metric.

Proposition 3.6.1. Given a metric space (X, d) and $Y \subseteq X$, then (Y, d) is a metric space.

Definition 3.6.3. Given a metric space (X, d), an **open ball**, centered at $x \in X$ with radius r is the set $B(x, r) = \{y \in X : d(x, y) < r\}$

Definition 3.6.4. Given a metrics space (X, d), a subset $S \subseteq X$ is said to be

- **Open** if for every $x \in S$, there exists $r > 0$ such that $B(x, r) \subseteq S$.
- **Closed** if $X \backslash S$ is open

3.6.1 Nets in Metric spaces

Definition 3.6.5. Let (X, d) be a metric space, and let $f : A \to X$ be a net.

- We say the net f converges to $x \in X$ if f is eventually in every open ball centered as x.
- We say the net f accumulates at $x \in X$ if f is frequently in every ball centered at x.

Lemma 3.6.1. If $f : A \to X$ is a convergent net, then $\lim f$ is a unique accumulation point of X.

Theorem 3.6.2. Let (X, d) be a metric space.

1. a set $S \subseteq X$ is open $\Leftrightarrow$ every net f in X with an accumulation point in S is frequently in S.
2. a set $S \subseteq X$ is open $\Leftrightarrow$ event net f in X taking values in S has all of its accumulation points in S.

3.6.2 Completeness and Compactness

Definition 3.6.6. A f taking values in a metric space (X, d) is a **Cauchy** net if, for every real number $r > 0$, there exists a point $x \in X$ such that f is eventually in $B(x, r)$.

Lemma 3.6.3. If f is a convergent net in a metric space, then f is **Cauchy**

Definition 3.6.7. A metric space (X, d) is **Cauchy Complete** if every **Cauchy** net is X converges.

Definition 3.6.8. Let (X, d) be a metric space, A subset S is **totally bounded** if, for every $r > 0$, there exists a finite subset $E \subseteq X$ such that $\{B(x, r) : x \in E\}$ covers S.

Theorem 3.6.4. Let (X, d) be a complete metric space. If S is a totally bounded subset, and f is a net that takes values in S, then f has at least one accumulation point.

Corollary 3.6.4.1. If the S from the previous theorem is also closed, then f has at least one accumulation point in S.

Definition 3.6.9. Let (X, d) be a complete metric space. A subset $K \subseteq X$ is said to be compact if every net taking values in K has an accumulation point in K

Theorem 3.6.5. Let (X, d) be a complete metric space. The following statements about a subset K are equivalent.

1. K is compact.
2. K is closed and totally bounded.
3. $\mathscr{S}$ is a collection of open subsets of X and $\mathscr{S}$ covers K, then there exists a finite subset $\mathscr{D} \subseteq \mathscr{S}$ that also covers K.
4. If $f : K \to \mathbb{R}$ is a positive function, then there exists a finite subset $S \subseteq K$ such that $\{B(x, f(x)) : x \in S\}$ covers K.

3.7 Subnets

Definition 3.7.1. Given a net $x : A \to X$, a net $y : B \to X$ is said to be a subnet of x, if there exists a function $\varphi : B \to A$ such that

- $y = x \circ \varphi$
- φ is increasing: $b_1 \leq b_2 \Rightarrow \varphi(b_1) \leq \varphi(b_2)$
- For every $a \in A$, there exists $b \in B$ with $\varphi(b) \geq a$.

Proposition 3.7.1. If y is a subnet of x, and z is a subnet of y, then z is a subnet of x.

Proposition 3.7.2. Let x be a net in X. Suppose $S \subseteq X$ is such that x is frequently in S. Then there exists a subnet y of x that is eventually in S.

Theorem 3.7.1. If x is a net in X, and y is a subnet.

- If x is eventually in S, then y is eventually in S.
- If y is frequently in S, then x is frequently in S.

Corollary 3.7.1.1. Let x be a net and y be a subnet. Then if x converges to a point z, then so does y. If y accumulates at a point z' then so does x.

Theorem 3.7.2. Let X be the real line or a metric space. Let $x : A \to B$ be a net, and z an accumulation points of x. Then there exists a subnet y of x that converges to z.

Corollary 3.7.2.1. Let x be a net, then z is an accumulation points of x if and only if there exists a subnet y that converges to z

Theorem 3.7.3. Let x be a net, then the set of all of its accumulation points is a closed set.

3.8 Infinite sums

Let $\mathcal{I}$ be an arbitrary set. Consider a function $\tau : \mathcal{I} \to \mathbb{R}$. Let A be the set of all finite subsets of $\mathcal{I}$ ordered by inclusion. A is a subset of the poset $2^{\mathcal{I}}$ and hence is a poset, it is also directed since if α_1, α_2 are two finite subsets of $\mathcal{I}$, so is the set $\alpha_1 \cup \alpha_2$ which succeeds both α_1 and α_2. Now we can construct a net $x : A \to \mathbb{R}$ where

$$x_a = \sum_{i \in \alpha} \tau(i)$$

This is well defined since α is finite and real numbers are closed under addition. We can interpret the limits of the net x, if it exists as the infinite sum of τ.

Proposition 3.8.1. If $\tau(u) \neq 0$ for uncountably many, then the net x cannot converge.

Assumption 3.8.1. For this section we will assume we are adding a countable infinite list of non-zero numbers. We will fix the following notations.

- We let $\mathcal{I}$ be a countable infinite set, and $\tau : \mathcal{I} \to \mathbb{R} \backslash \{0\}$ be the list on non-vanishing terms.
- We will denote by $A \subseteq 2^{\mathcal{I}}$ the set of all finite subsets of natural numbers.
- We have the net $x : A \to \mathbb{R}$, where $x_\alpha = \sum_{i \in \alpha} \tau(i)$ is the infinite sum.

Definition 3.8.1. Given a particular enumeration $\eta : \mathbb{N} \to \mathcal{I}$, the **associated series** is the sequence σ where $\sigma_n = \sum_{j=1}^{n} \tau(\eta(j))$. The series is said to converge if the sequence σ converges, in which case we write $\sum_\eta \tau = \lim \sigma$.

Lemma 3.8.1. σ is a subnet of x.

Corollary 3.8.1.1. If $\lim x$ exists, then for every enumeration η of $\mathcal{I}$, the corresponding series converges $\sum_\eta \tau = \lim x$.

3.8.1 Absolute convergence

Definition 3.8.2. We say that the infinite sum of τ converges absolutely if the net x converges. In this case we write $\sum_{abs} \tau = \lim x$.

Lemma 3.8.2. If there exists an enumeration n such that the corresponding $\sum_n \tau$ converges, then for every $r > 0$, the set $\{i \in \mathcal{I} : |\tau(i)| \geq r\}$ is finite.

Theorem 3.8.3. Suppose there exists and enumeration η for which $\sum_\eta \tau$ converges, and suppose that exactly one of $\mathcal{I}^{+-}$ is finite, then the infinite sum of τ converges absolutely and $\sum_{abs} \tau = \sum_\eta \tau$.

Theorem 3.8.4. Let τ take only positive values, and suppose $\sum_{abs} \tau$ converges absolutely. If $\mu : \mathcal{I} \to \mathbb{R}$ is any function such that for every i, $|\mu(i)| \leq \tau(i)$, then the infinite sum of μ also converges absolutely.

Theorem 3.8.5. Denote by τ^+ the restriction of τ to $\mathcal{I}^+$ and τ^- the restriction of τ to $\mathcal{I}^-$. Suppose the sums $\sum_{abs} \tau^+$ and $\sum_{abs} \tau^-$ both converge absolutely, then the infinite sum for τ converges absolutely and $\sum_{abs} \tau = \sum_{abs} \tau^+ + \sum_{abs} \tau^-$.

3.8.2 Conditional convergence

Definition 3.8.3. We say the infinite sum of τ converges conditionally if the net x does not converge and for some enumeration η the sum $\sum_\eta \tau$ converges.

Lemma 3.8.6. Suppose x is divergent but there exists an enumeration η such that the series σ converges. Then there exists enumerations of v_{+-} of $\mathcal{I}^{+-}$ respectively, such that $\tau \circ v_+$ is a decreasing function, and $\tau \circ v_-$ is an increasing function.

Theorem 3.8.7. Riemann rearrangement theorem states that if x is divergence and there exists an enumeration η_0 such that the corresponding series converges. Let $x \in \mathbb{R}$. Then there exists a possibly different enumeration of η of $\mathcal{I}$ such that the corresponding series converges, with $\sum_\eta \tau = z$.

Theorem 3.8.8. If there exists an enumeration η whose series converges, then the net x corresponding to the infinite sum of τ satisfies exatctly on of the following:

- x converges; or
- the set of all accumulation points of x is $\mathbb{R}$.

3.8.3 Convergence tests

Define,

$$\mathcal{J}_0 := \{i \in \mathcal{I} : |\tau(i)| \geq 1\}$$

and for $k \in \mathbb{N}$,

$$\mathcal{J}_0 := \{i \in \mathcal{I} : |\tau(i)| \in [2^{-k}, 2^{1-k})\}$$

We shall assume that each of the these sets are finite since if they were infinite the infinite sum would have no convergent series

Proposition 3.8.2. If the series $\sum_{k=1}^{\infty} 2^{-k} \rho_k$ converges then the infinite sum $\sum_{abs} \tau$ converges absolutely.

Proposition 3.8.3. If there exists $C > 0$ and $s < 1$ such that $\rho_k \leq C 2^{sk}$ for all $k \in \mathbb{N}$, then $\sum_{abs} \tau$ converges absolutely.

3.9 Continuity

3.9.1 Other modes of continuity

3.10 Interpolation and Extrapolation

3.10.1 Intermediate Value Theorem

3.10.2 Extensions from dense subsets

3.11 Continuous functions and compact sets