CHAPTER

2

First Order Differential Equations

This chapter deals with differential equations of first order

$$\frac{dy}{dt} = f(t, y), \tag{1}$$

where f is a given function of two variables. Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and our object is to determine whether such functions exist and, if so, to develop methods for finding them. Unfortunately, for an arbitrary function f, there is no general method for solving the equation in terms of elementary functions. Instead, we will describe several methods, each of which is applicable to a certain subclass of first order equations. The most important of these are linear equations (Section 2.1), separable equations (Section 2.2), and exact equations (Section 2.6). Other sections of this chapter describe some of the important applications of first order differential equations, introduce the idea of approximating a solution by numerical computation, and discuss some theoretical questions related to the existence and uniqueness of solutions. The final section includes an example of chaotic solutions in the context of first order difference equations, which have some important points of similarity with differential equations and are simpler to investigate.

2.1 Linear Equations; Method of Integrating Factors

If the function f in Eq. (1) depends linearly on the dependent variable y, then Eq. (1) is called a first order linear equation. In Sections 1.1 and 1.2 we discussed a restricted type of first order linear equation in which the coefficients are constants.

A typical example is

$$\frac{dy}{dt} = -ay + b, \tag{2}$$

where a and b are given constants. Recall that an equation of this form describes the motion of an object falling in the atmosphere.

Now we want to consider the most general first order linear equation, which is obtained by replacing the coefficients a and b in Eq. (2) by arbitrary functions of t. We will usually write the general **first order linear equation** in the standard form

$$\frac{dy}{dt} + p(t)y = g(t), \tag{3}$$

where p and g are given functions of the independent variable t. Sometimes it is more convenient to write the equation in the form

$$P(t)\frac{dy}{dt} + Q(t)y = G(t), \tag{4}$$

where P, Q, and G are given. Of course, as long as $P(t) \neq 0$, you can convert Eq. (4) to Eq. (3) by dividing Eq. (4) by $P(t)$.

In some cases it is possible to solve a first order linear equation immediately by integrating the equation, as in the next example.

EXAMPLE 1

Solve the differential equation

$$(4 + t^2)\frac{dy}{dt} + 2ty = 4t. \tag{5}$$

The left side of Eq. (5) is a linear combination of dy/dt and y, a combination that also appears in the rule from calculus for differentiating a product. In fact,

$$(4 + t^2)\frac{dy}{dt} + 2ty = \frac{d}{dt}[(4 + t^2)y];$$

it follows that Eq. (5) can be rewritten as

$$\frac{d}{dt}[(4 + t^2)y] = 4t. \tag{6}$$

Thus, even though y is unknown, we can integrate both sides of Eq. (6) with respect to t, thereby obtaining

$$(4 + t^2)y = 2t^2 + c, \tag{7}$$

where c is an arbitrary constant of integration. By solving for y we find that

$$y = \frac{2t^2}{4 + t^2} + \frac{c}{4 + t^2}. \tag{8}$$

This is the general solution of Eq. (5).

Unfortunately, most first order linear equations cannot be solved as illustrated in Example 1 because their left sides are not the derivative of the product of y and some other function. However, Leibniz discovered that if the differential equation is multiplied by a certain function $\mu(t)$, then the equation is converted into one that is immediately integrable by using the product rule for derivatives, just as in Example 1. The function $\mu(t)$ is called an **integrating factor** and our main task is to determine how to find it for a given equation. We will show how this method works first for an example and then for the general first order linear equation in the standard form (3).

EXAMPLE 2

Find the general solution of the differential equation

$$\frac{dy}{dt} + \tfrac{1}{2}y = \tfrac{1}{2}e^{t/3}. \tag{9}$$

Draw some representative integral curves; that is, plot solutions corresponding to several values of the arbitrary constant c. Also find the particular solution whose graph contains the point $(0, 1)$.

The first step is to multiply Eq. (9) by a function $\mu(t)$, as yet undetermined; thus

$$\mu(t)\frac{dy}{dt} + \tfrac{1}{2}\mu(t)y = \tfrac{1}{2}\mu(t)e^{t/3}. \tag{10}$$

The question now is whether we can choose $\mu(t)$ so that the left side of Eq. (10) is the derivative of the product $\mu(t)y$. For any differentiable function $\mu(t)$ we have

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y, \tag{11}$$

Thus the left side of Eq. (10) and the right side of Eq. (11) are identical, provided that we choose $\mu(t)$ to satisfy

$$\frac{d\mu(t)}{dt} = \tfrac{1}{2}\mu(t). \tag{12}$$

Our search for an integrating factor will be successful if we can find a solution of Eq. (12). Perhaps you can readily identify a function that satisfies Eq. (12): what well-known function from calculus has a derivative that is equal to one-half times the original function? More systematically, rewrite Eq. (12) as

$$\frac{d\mu(t)/dt}{\mu(t)} = \tfrac{1}{2},$$

which is equivalent to

$$\frac{d}{dt}\ln|\mu(t)| = \tfrac{1}{2}. \tag{13}$$

Then it follows that

$$\ln|\mu(t)| = \tfrac{1}{2}t + C,$$

or

$$\mu(t) = ce^{t/2}. \tag{14}$$

The function $\mu(t)$ given by Eq. (14) is an integrating factor for Eq. (9). Since we do not need the most general integrating factor, we will choose c to be 1 in Eq. (14) and use $\mu(t) = e^{t/2}$.

Now we return to Eq. (9), multiply it by the integrating factor $e^{t/2}$, and obtain

$$e^{t/2}\frac{dy}{dt} + \tfrac{1}{2}e^{t/2}y = \tfrac{1}{2}e^{5t/6}. \tag{15}$$

By the choice we have made of the integrating factor, the left side of Eq. (15) is the derivative of $e^{t/2}y$, so that Eq. (15) becomes

$$\frac{d}{dt}(e^{t/2}y) = \tfrac{1}{2}e^{5t/6}. \tag{16}$$

By integrating both sides of Eq. (16), we obtain

$$e^{t/2}y = \tfrac{3}{5}e^{5t/6} + c, \tag{17}$$

where c is an arbitrary constant. Finally, on solving Eq. (17) for y, we have the general solution of Eq. (9), namely,

$$y = \tfrac{3}{5}e^{t/3} + ce^{-t/2}. \tag{18}$$

To find the solution passing through the point $(0, 1)$, we set $t = 0$ and $y = 1$ in Eq. (18), obtaining $1 = (3/5) + c$. Thus $c = 2/5$, and the desired solution is

$$y = \tfrac{3}{5}e^{t/3} + \tfrac{2}{5}e^{-t/2}. \tag{19}$$

Figure 2.1.1 includes the graphs of Eq. (18) for several values of c with a direction field in the background. The solution satisfying $y(0) = 1$ is shown by the black curve.

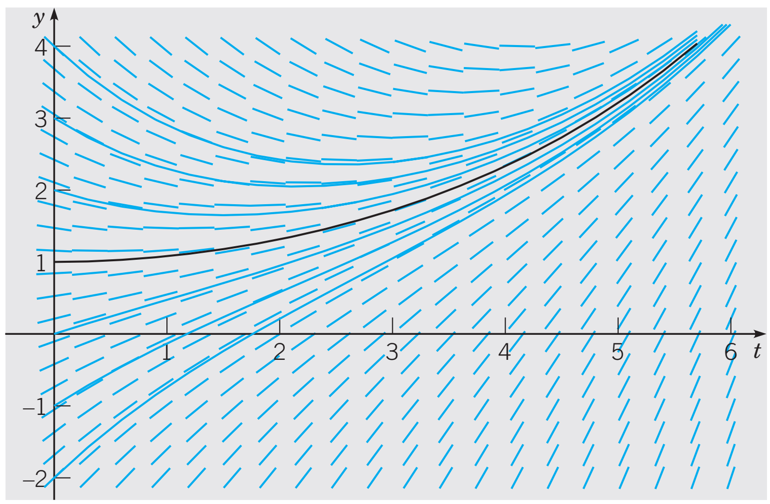

FIGURE 2.1.1 Direction field and integral curves of $y' + \frac{1}{2}y = \frac{1}{2}e^{t/3}$; the black curve passes through the point $(0, 1)$.

Let us now extend the method of integrating factors to equations of the form

$$\frac{dy}{dt} + ay = g(t), \tag{20}$$

where a is a given constant and $g(t)$ is a given function. Proceeding as in Example 2, we find that the integrating factor $\mu(t)$ must satisfy

$$\frac{d\mu}{dt} = a\mu, \tag{21}$$

rather than Eq. (12). Thus the integrating factor is $\mu(t) = e^{at}$. Multiplying Eq. (20) by $\mu(t)$, we obtain

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t),$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t). \tag{22}$$

By integrating both sides of Eq. (22), we find that

$$e^{at}y = \int e^{at}g(t)\,dt + c, \tag{23}$$

where c is an arbitrary constant. For many simple functions $g(t)$, we can evaluate the integral in Eq. (23) and express the solution y in terms of elementary functions, as in Example 2. However, for more complicated functions $g(t)$, it is necessary to leave the solution in integral form. In this case

$$y = e^{-at}\int_{t_0}^{t} e^{as}g(s)\,ds + ce^{-at}. \tag{24}$$

Note that in Eq. (24) we have used s to denote the integration variable to distinguish it from the independent variable t, and we have chosen some convenient value t_0 as the lower limit of integration.

EXAMPLE 3

Find the general solution of the differential equation

$$\frac{dy}{dt} - 2y = 4 - t \tag{25}$$

and plot the graphs of several solutions. Discuss the behavior of solutions as $t \to \infty$.

Equation (25) is of the form (20) with $a = -2$; therefore, the integrating factor is $\mu(t) = e^{-2t}$. Multiplying the differential equation (25) by $\mu(t)$, we obtain

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = 4e^{-2t} - te^{-2t},$$

or

$$\frac{d}{dt}(e^{-2t}y) = 4e^{-2t} - te^{-2t}. \tag{26}$$

Then, by integrating both sides of this equation, we have

$$e^{-2t}y = -2e^{-2t} + \tfrac{1}{2}te^{-2t} + \tfrac{1}{4}e^{-2t} + c,$$

where we have used integration by parts on the last term in Eq. (26). Thus the general solution of Eq. (25) is

$$y = -\tfrac{7}{4} + \tfrac{1}{2}t + ce^{2t}. \tag{27}$$

A direction field and graphs of the solution (27) for several values of c are shown in Figure 2.1.2. The behavior of the solution for large values of t is determined by the term ce^{2t}. If $c \neq 0$, then the solution grows exponentially large in magnitude, with the same sign as c itself. Thus the solutions diverge as t becomes large. The boundary between solutions that ultimately grow positively and those that ultimately grow negatively occurs when $c = 0$. If we substitute $c = 0$ into Eq. (27) and then set $t = 0$, we find that $y = -7/4$ is the separation point on the y-axis. Note that for this initial value, the solution is $y = -\frac{7}{4} + \frac{1}{2}t$; it grows positively, but linearly rather than exponentially.

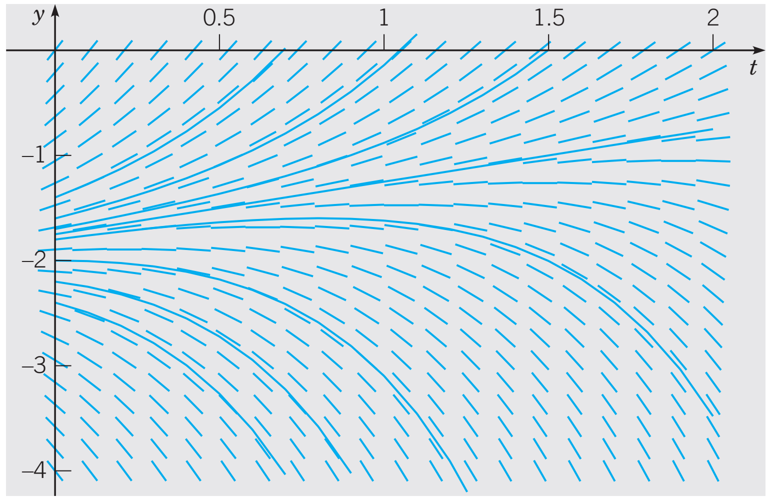

FIGURE 2.1.2 Direction field and integral curves of $y' - 2y = 4 - t$.

Now we return to the general first order linear equation (3)

$$\frac{dy}{dt} + p(t)y = g(t),$$

where p and g are given functions. To determine an appropriate integrating factor, we multiply Eq. (3) by an as yet undetermined function $\mu(t)$, obtaining

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t). \tag{28}$$

Following the same line of development as in Example 2, we see that the left side of Eq. (28) is the derivative of the product $\mu(t)y$, provided that $\mu(t)$ satisfies the equation

$$\frac{d\mu(t)}{dt} = p(t)\mu(t). \tag{29}$$

If we assume temporarily that $\mu(t)$ is positive, then we have

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t),$$

and consequently

$$\ln \mu(t) = \int p(t)\,dt + k.$$

By choosing the arbitrary constant k to be zero, we obtain the simplest possible function for μ, namely,

$$\mu(t) = \exp \int p(t)\,dt. \tag{30}$$

Note that $\mu(t)$ is positive for all t, as we assumed. Returning to Eq. (28), we have

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t). \tag{31}$$

Hence

$$\mu(t)y = \int \mu(t)g(t)\,dt + c, \tag{32}$$

where c is an arbitrary constant. Sometimes the integral in Eq.(32) can be evaluated in terms of elementary functions. However, in general this is not possible, so the general solution of Eq. (3) is

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\,ds + c\right], \tag{33}$$

where again t_0 is some convenient lower limit of integration. Observe that Eq. (33) involves two integrations, one to obtain $\mu(t)$ from Eq. (30) and the other to determine y from Eq. (33).

EXAMPLE 4

Solve the initial value problem

$$ty' + 2y = 4t^2, \tag{34}$$

$$y(1) = 2. \tag{35}$$

In order to determine $p(t)$ and $g(t)$ correctly, we must first rewrite Eq. (34) in the standard form (3). Thus we have

$$y' + (2/t)y = 4t, \tag{36}$$

so $p(t) = 2/t$ and $g(t) = 4t$. To solve Eq. (36), we first compute the integrating factor $\mu(t)$:

$$\mu(t) = \exp \int \frac{2}{t}\,dt = e^{2\ln|t|} = t^2.$$

On multiplying Eq. (36) by $\mu(t) = t^2$, we obtain

$$t^2y' + 2ty = (t^2y)' = 4t^3,$$

and therefore

$$t^2y = t^4 + c,$$

where c is an arbitrary constant. It follows that

$$y = t^2 + \frac{c}{t^2} \tag{37}$$

is the general solution of Eq. (34). Integral curves of Eq. (34) for several values of c are shown in Figure 2.1.3. To satisfy the initial condition (35), it is necessary to choose $c = 1$; thus

$$y = t^2 + \frac{1}{t^2}, \qquad t > 0 \tag{38}$$

is the solution of the initial value problem (34), (35). This solution is shown by the black curve in Figure 2.1.3. Note that it becomes unbounded and is asymptotic to the positive y-axis as $t \to 0$ from the right. This is the effect of the infinite discontinuity in the coefficient $p(t)$ at the origin. The function $y = t^2 + (1/t^2)$ for $t < 0$ is not part of the solution of this initial value problem.

This is the first example in which the solution fails to exist for some values of t. Again, this is due to the infinite discontinuity in $p(t)$ at $t = 0$, which restricts the solution to the interval $0 < t < \infty$.

Looking again at Figure 2.1.3, we see that some solutions (those for which $c > 0$) are asymptotic to the positive y-axis as $t \to 0$ from the right, while other solutions (for which $c < 0$)

are asymptotic to the negative y-axis. The solution for which $c = 0$, namely, $y = t^2$, remains bounded and differentiable even at $t = 0$. If we generalize the initial condition (35) to

$$y(1) = y_0, \tag{39}$$

then $c = y_0 - 1$ and the solution (38) becomes

$$y = t^2 + \frac{y_0 - 1}{t^2}, \qquad t > 0 \text{ if } y_0 \neq 1. \tag{40}$$

As in Example 3, this is another instance where there is a critical initial value, namely, $y_0 = 1$, that separates solutions that behave in one way from others that behave quite differently.

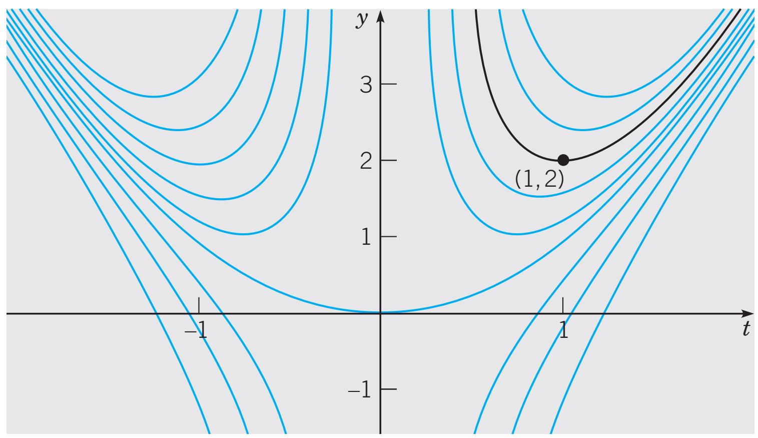

FIGURE 2.1.3 Integral curves of the differential equation $ty' + 2y = 4t^2$; the black curve passes through the point $(1, 2)$.

EXAMPLE 5

Solve the initial value problem

$$2y' + ty = 2, \tag{41}$$
$$y(0) = 1. \tag{42}$$

To convert the differential equation (41) to the standard form (3), we must divide by 2, obtaining

$$y' + (t/2)y = 1. \tag{43}$$

Thus $p(t) = t/2$, and the integrating factor is $\mu(t) = \exp(t^2/4)$. Then multiply Eq. (43) by $\mu(t)$, so that

$$e^{t^2/4}y' + \frac{t}{2}e^{t^2/4}y = e^{t^2/4}. \tag{44}$$

The left side of Eq. (44) is the derivative of $e^{t^2/4}y$, so by integrating both sides of Eq. (44), we obtain

$$e^{t^2/4}y = \int e^{t^2/4}\,dt + c. \tag{45}$$

The integral on the right side of Eq. (45) cannot be evaluated in terms of the usual elementary functions, so we leave the integral unevaluated. However, by choosing the lower limit of integration as the initial point $t = 0$, we can replace Eq. (45) by

$$e^{t^2/4}y = \int_0^t e^{s^2/4}\,ds + c, \tag{46}$$

where c is an arbitrary constant. It then follows that the general solution y of Eq. (41) is given by

$$y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + ce^{-t^2/4}. \tag{47}$$

The initial condition (42) requires that $c = 1$.

The main purpose of this example is to illustrate that sometimes the solution must be left in terms of an integral. This is usually at most a slight inconvenience, rather than a serious obstacle. For a given value of t, the integral in Eq. (47) is a definite integral and can be approximated to any desired degree of accuracy by using readily available numerical integrators. By repeating this process for many values of t and plotting the results, you can obtain a graph of a solution. Alternatively, you can use a numerical approximation method, such as those discussed in Chapter 8, that proceed directly from the differential equation and need no expression for the solution. Software packages such as Maple and Mathematica readily execute such procedures and produce graphs of solutions of differential equations.

Figure 2.1.4 displays graphs of the solution (47) for several values of c. From the figure it may be plausible to conjecture that all solutions approach a limit as $t \to \infty$. The limit can be found analytically (see Problem 32).

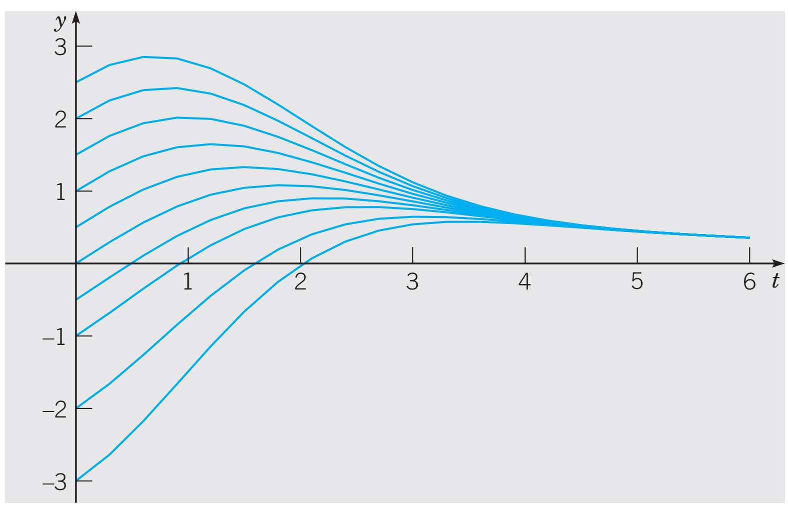

FIGURE 2.1.4 Integral curves of $2y' + ty = 2$.

PROBLEMS

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.

(b) Based on an inspection of the direction field, describe how solutions behave for large t.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

1. $y' + 3y = t + e^{-2t}$
2. $y' - 2y = t^2 e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + (1/t)y = 3\cos 2t, \qquad t > 0$
5. $y' - 2y = 3e^t$
6. $ty' + 2y = \sin t, \qquad t > 0$
7. $y' + 2ty = 2te^{-t^2}$
8. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$
9. $2y' + y = 3t$
10. $ty' - y = t^2 e^{-t}, \qquad t > 0$
11. $y' + y = 5\sin 2t$
12. $2y' + y = 3t^2$

In each of Problems 13 through 20, find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}, \qquad y(0) = 1$
14. $y' + 2y = te^{-2t}, \qquad y(1) = 0$
15. $ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, \quad t > 0$
16. $y' + (2/t)y = (\cos t)/t^2, \qquad y(\pi) = 0, \quad t > 0$
17. $y' - 2y = e^{2t}, \qquad y(0) = 2$
18. $ty' + 2y = \sin t, \qquad y(\pi/2) = 1, \quad t > 0$
19. $t^3 y' + 4t^2 y = e^{-t}, \qquad y(-1) = 0, \quad t < 0$
20. $ty' + (t + 1)y = t, \qquad y(\ln 2) = 1, \quad t > 0$

In each of Problems 21 through 23:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

21. $y' - \frac{1}{2}y = 2\cos t, \qquad y(0) = a$
22. $2y' - y = e^{t/3}, \qquad y(0) = a$
23. $3y' - 2y = e^{-\pi t/2}, \qquad y(0) = a$

In each of Problems 24 through 26:

(a) Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

(b) Solve the initial value problem and find the critical value a_0 exactly.

(c) Describe the behavior of the solution corresponding to the initial value a_0.

24. $ty' + (t + 1)y = 2te^{-t}, \qquad y(1) = a, \quad t > 0$
25. $ty' + 2y = (\sin t)/t, \qquad y(-\pi/2) = a, \quad t < 0$
26. $(\sin t)y' + (\cos t)y = e^t, \qquad y(1) = a, \quad 0 < t < \pi$
27. Consider the initial value problem

$$y' + \tfrac{1}{2}y = 2\cos t, \qquad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

28. Consider the initial value problem

$$y' + \tfrac{2}{3}y = 1 - \tfrac{1}{2}t, \qquad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

29. Consider the initial value problem

$$y' + \tfrac{1}{4}y = 3 + 2\cos 2t, \qquad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behavior for large t.
(b) Determine the value of t for which the solution first intersects the line $y = 12$.

30. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

31. Consider the initial value problem

$$y' - \tfrac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

32. Show that all solutions of $2y' + ty = 2$ [Eq. (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, Eq. (47), and use L'Hôpital's rule on the first term.

33. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 34 through 37, construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

34. All solutions have the limit 3 as $t \to \infty$.
35. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.
36. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.
37. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.
38. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \tag{i}$$

(a) If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left[-\int p(t)\,dt\right], \tag{ii}$$

where A is a constant.
(b) If $g(t)$ is not everywhere zero, assume that the solution of Eq. (i) is of the form

$$y = A(t)\exp\left[-\int p(t)\,dt\right], \tag{iii}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t) \exp\left[\int p(t)\, dt\right]. \tag{iv}$$

(c) Find $A(t)$ from Eq. (iv). Then substitute for $A(t)$ in Eq. (iii) and determine y. Verify that the solution obtained in this manner agrees with that of Eq. (33) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.6 in connection with second order linear equations.

In each of Problems 39 through 42, use the method of Problem 38 to solve the given differential equation.

39. $y' - 2y = t^2 e^{2t}$
40. $y' + (1/t)y = 3\cos 2t, \qquad t > 0$
41. $ty' + 2y = \sin t, \qquad t > 0$
42. $2y' + y = 3t^2$

2.2 Separable Equations

In Section 1.2 we used a process of direct integration to solve first order linear equations of the form

$$\frac{dy}{dt} = ay + b, \tag{1}$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of equations.

We will use x, rather than t, to denote the independent variable in this section for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first order equation is

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Linear equations were considered in the preceding section, but if Eq. (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first order equations that can be solved by direct integration.

To identify this class of equations, we first rewrite Eq. (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{3}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. If it happens that M is a function of x only and N is a function of y only, then Eq. (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4}$$

Such an equation is said to be **separable**, because if it is written in the differential form

$$M(x)\,dx + N(y)\,dy = 0, \tag{5}$$

then, if you wish, terms involving each variable may be placed on opposite sides of the equation. The differential form (5) is also more symmetric and tends to suppress the distinction between independent and dependent variables.

A separable equation can be solved by integrating the functions M and N. We illustrate the process by an example and then discuss it in general for Eq. (4).

EXAMPLE 1

Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \tag{6}$$

is separable, and then find an equation for its integral curves.

If we write Eq. (6) as

$$-x^2 + (1-y^2)\frac{dy}{dx} = 0, \tag{7}$$

then it has the form (4) and is therefore separable. Recall from calculus that if y is a function of x, then by the chain rule,

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\,\frac{dy}{dx} = f'(y)\frac{dy}{dx}.$$

For example, if $f(y) = y - y^3/3$, then

$$\frac{d}{dx}(y - y^3/3) = (1-y^2)\frac{dy}{dx}.$$

Thus the second term in Eq. (7) is the derivative with respect to x of $y - y^3/3$, and the first term is the derivative of $-x^3/3$. Thus Eq. (7) can be written as

$$\frac{d}{dx}\left(-\frac{x^3}{3}\right) + \frac{d}{dx}\left(y - \frac{y^3}{3}\right) = 0,$$

or

$$\frac{d}{dx}\left(-\frac{x^3}{3} + y - \frac{y^3}{3}\right) = 0.$$

Therefore, by integrating, we obtain

$$-x^3 + 3y - y^3 = c, \tag{8}$$

where c is an arbitrary constant. Equation (8) is an equation for the integral curves of Eq. (6). A direction field and several integral curves are shown in Figure 2.2.1. Any differentiable function $y = \phi(x)$ that satisfies Eq. (8) is a solution of Eq. (6). An equation of the integral curve passing through a particular point (x_0, y_0) can be found by substituting x_0 and y_0 for x and y, respectively, in Eq. (8) and determining the corresponding value of c.

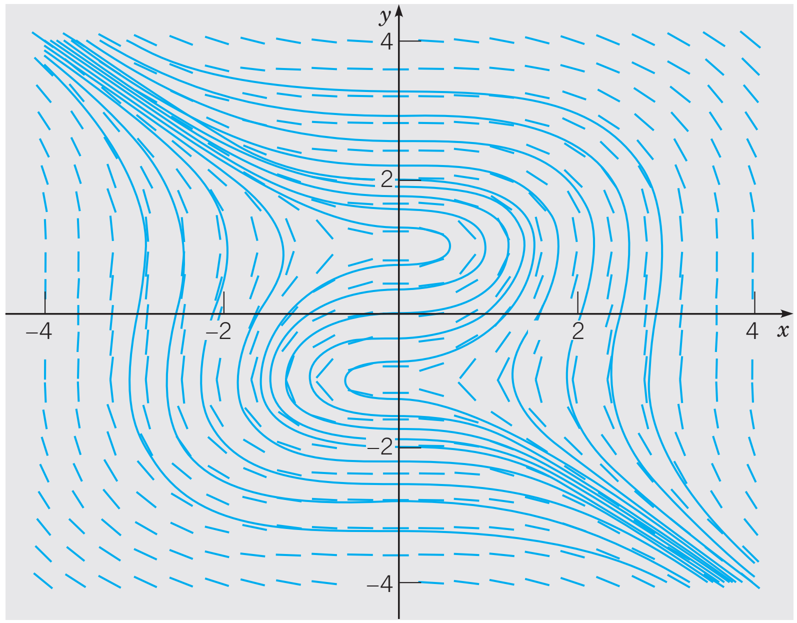

FIGURE 2.2.1 Direction field and integral curves of $y' = x^2/(1 - y^2)$.

Essentially the same procedure can be followed for any separable equation. Returning to Eq. (4), let H_1 and H_2 be any antiderivatives of M and N, respectively. Thus

$$H_1'(x) = M(x), \qquad H_2'(y) = N(y), \tag{9}$$

and Eq. (4) becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0. \tag{10}$$

If y is regarded as a function of x, then according to the chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\,\frac{dy}{dx} = \frac{d}{dx}H_2(y). \tag{11}$$

Consequently, we can write Eq. (10) as

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0. \tag{12}$$

By integrating Eq. (12), we obtain

$$H_1(x) + H_2(y) = c, \tag{13}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies Eq. (13) is a solution of Eq. (4); in other words, Eq. (13) defines the solution implicitly rather than explicitly. In practice, Eq. (13) is usually obtained from Eq. (5) by integrating the first term with respect to x and the second term with respect to y. The justification for this is the argument that we have just given.

The differential equation (4), together with an initial condition

$$y(x_0) = y_0, \tag{14}$$

forms an initial value problem. To solve this initial value problem, we must determine the appropriate value for the constant c in Eq. (13). We do this by setting $x = x_0$ and $y = y_0$ in Eq. (13) with the result that

$$c = H_1(x_0) + H_2(y_0). \tag{15}$$

Substituting this value of c in Eq. (13) and noting that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)\,ds, \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)\,ds,$$

we obtain

$$\int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(s)\,ds = 0. \tag{16}$$

Equation (16) is an implicit representation of the solution of the differential equation (4) that also satisfies the initial condition (14). Bear in mind that to determine an explicit formula for the solution, you need to solve Eq. (16) for y as a function of x. Unfortunately, it is often impossible to do this analytically; in such cases you can resort to numerical methods to find approximate values of y for given values of x.

EXAMPLE 2

Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1, \tag{17}$$

and determine the interval in which the solution exists.

The differential equation can be written as

$$2(y-1)\,dy = (3x^2 + 4x + 2)\,dx.$$

Integrating the left side with respect to y and the right side with respect to x gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \tag{18}$$

where c is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x = 0$ and $y = -1$ in Eq. (18), obtaining $c = 3$. Hence the solution of the initial value problem is given implicitly by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3. \tag{19}$$

To obtain the solution explicitly, we must solve Eq. (19) for y in terms of x. That is a simple matter in this case, since Eq. (19) is quadratic in y, and we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{20}$$

Equation (20) gives two solutions of the differential equation, only one of which, however, satisfies the given initial condition. This is the solution corresponding to the minus sign in Eq. (20), so we finally obtain

$$y = \phi(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{21}$$

as the solution of the initial value problem (17). Note that if we choose the plus sign by mistake in Eq. (20), then we obtain the solution of the same differential equation that satisfies the initial condition $y(0) = 3$. Finally, to determine the interval in which the solution (21) is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$. Some integral curves of the differential equation are shown in Figure 2.2.2. The black curve passes through the point $(0, -1)$ and thus is the solution of the initial value problem (17). Observe that the boundary of the interval of validity of the solution (21) is determined by the point $(-2, 1)$ at which the tangent line is vertical.

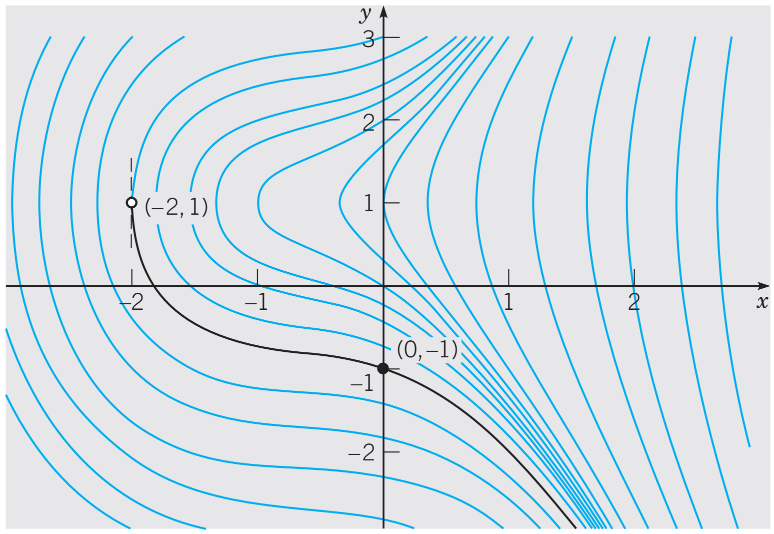

FIGURE 2.2.2 Integral curves of $y' = (3x^2 + 4x + 2)/2(y - 1)$; the solution satisfying $y(0) = -1$ is shown in black and is valid for $x > -2$.

EXAMPLE 3

Solve the equation

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3} \tag{22}$$

and draw graphs of several integral curves. Also find the solution passing through the point $(0, 1)$ and determine its interval of validity.

Rewriting Eq. (22) as

$$(4 + y^3)\,dy = (4x - x^3)\,dx,$$

integrating each side, multiplying by 4, and rearranging the terms, we obtain

$$y^4 + 16y + x^4 - 8x^2 = c, \tag{23}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies Eq. (23) is a solution of the differential equation (22). Graphs of Eq. (23) for several values of c are shown in Figure 2.2.3.

To find the particular solution passing through $(0, 1)$, we set $x = 0$ and $y = 1$ in Eq. (23) with the result that $c = 17$. Thus the solution in question is given implicitly by

$$y^4 + 16y + x^4 - 8x^2 = 17. \tag{24}$$

It is shown by the black curve in Figure 2.2.3. The interval of validity of this solution extends on either side of the initial point as long as the function remains differentiable. From the figure we see that the interval ends when we reach points where the tangent line is vertical. It follows from the differential equation (22) that these are points where $4 + y^3 = 0$, or

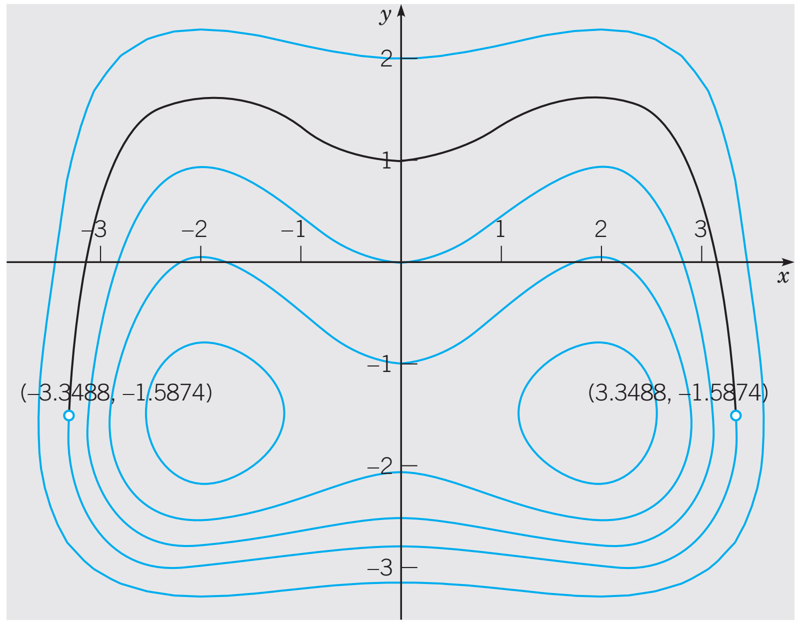

FIGURE 2.2.3 Integral curves of $y' = (4x - x^3)/(4 + y^3)$. The solution passing through $(0, 1)$ is shown by the black curve.

$y = (-4)^{1/3} \cong -1.5874$. From Eq. (24) the corresponding values of x are $x \cong \pm 3.3488$. These points are marked on the graph in Figure 2.2.3.

Note 1: Sometimes an equation of the form (2)

$$\frac{dy}{dx} = f(x, y)$$

has a constant solution $y = y_0$. Such a solution is usually easy to find because if $f(x, y_0) = 0$ for some value y_0 and for all x, then the constant function $y = y_0$ is a solution of the differential equation (2). For example, the equation

$$\frac{dy}{dx} = \frac{(y - 3)\cos x}{1 + 2y^2} \tag{25}$$

has the constant solution $y = 3$. Other solutions of this equation can be found by separating the variables and integrating.

Note 2: The investigation of a first order nonlinear equation can sometimes be facilitated by regarding both x and y as functions of a third variable t. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{26}$$

If the differential equation is

$$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}, \tag{27}$$

then, by comparing numerators and denominators in Eqs. (26) and (27), we obtain the system

$$dx/dt = G(x, y), \qquad dy/dt = F(x, y). \tag{28}$$

At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

Note 3: In Example 2 it was not difficult to solve explicitly for y as a function of x. However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 1 and 3. Thus, in the problems below and in other sections where nonlinear equations appear, the words "solve the following differential equation" mean to find the solution explicitly if it is convenient to do so, but otherwise to find an equation defining the solution implicitly.

PROBLEMS

In each of Problems 1 through 8, solve the given differential equation.

1. $y' = x^2/y$
2. $y' = x^2/y(1 + x^3)$
3. $y' + y^2 \sin x = 0$
4. $y' = (3x^2 - 1)/(3 + 2y)$
5. $y' = (\cos^2 x)(\cos^2 2y)$
6. $xy' = (1 - y^2)^{1/2}$
7. $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$
8. $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$

In each of Problems 9 through 20:

(a) Find the solution of the given initial value problem in explicit form.

(b) Plot the graph of the solution.

(c) Determine (at least approximately) the interval in which the solution is defined.

9. $y' = (1 - 2x)y^2, \quad y(0) = -1/6$
10. $y' = (1 - 2x)/y, \quad y(1) = -2$
11. $x\,dx + ye^{-x}dy = 0, \quad y(0) = 1$
12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$
13. $y' = 2x/(y + x^2y), \quad y(0) = -2$
14. $y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1$
15. $y' = 2x/(1 + 2y), \quad y(2) = 0$
16. $y' = x(x^2 + 1)/4y^3, \quad y(0) = -1/\sqrt{2}$
17. $y' = (3x^2 - e^x)/(2y - 5), \quad y(0) = 1$
18. $y' = (e^{-x} - e^x)/(3 + 4y), \quad y(0) = 1$
19. $\sin 2x\,dx + \cos 3y\,dy = 0, \quad y(\pi/2) = \pi/3$
20. $y^2(1 - x^2)^{1/2}dy = \arcsin x\,dx, \quad y(0) = 1$

Some of the results requested in Problems 21 through 28 can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion about the advantages and disadvantages of each approach.

21. Solve the initial value problem

$$y' = (1 + 3x^2)/(3y^2 - 6y), \qquad y(0) = 1$$

and determine the interval in which the solution is valid.

Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

22. Solve the initial value problem

$$y' = 3x^2/(3y^2 - 4), \qquad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

23. Solve the initial value problem

$$y' = 2y^2 + xy^2, \qquad y(0) = 1$$

and determine where the solution attains its minimum value.

24. Solve the initial value problem

$$y' = (2 - e^x)/(3 + 2y), \qquad y(0) = 0$$

and determine where the solution attains its maximum value.

25. Solve the initial value problem

$$y' = 2\cos 2x/(3 + 2y), \qquad y(0) = -1$$

and determine where the solution attains its maximum value.

26. Solve the initial value problem

$$y' = 2(1 + x)(1 + y^2), \qquad y(0) = 0$$

and determine where the solution attains its minimum value.

27. Consider the initial value problem

$$y' = ty(4 - y)/3, \qquad y(0) = y_0.$$

(a) Determine how the behavior of the solution as t increases depends on the initial value y_0.
(b) Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

28. Consider the initial value problem

$$y' = ty(4 - y)/(1 + t), \qquad y(0) = y_0 > 0.$$

(a) Determine how the solution behaves as $t \to \infty$.
(b) If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.
(c) Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

29. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where a, b, c, and d are constants.

Homogeneous Equations. If the right side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous.[1] Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

30. Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \tag{i}$$

(a) Show that Eq. (i) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \tag{ii}$$

thus Eq. (i) is homogeneous.

(b) Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.

(c) Replace y and dy/dx in Eq. (ii) by the expressions from part (b) that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \tag{iii}$$

Observe that Eq. (iii) is separable.

(d) Solve Eq. (iii), obtaining v implicitly in terms of x.

(e) Find the solution of Eq. (i) by replacing v by y/x in the solution in part (d).

(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (i) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.

(b) Solve the differential equation.

(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

31. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

32. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$

33. $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

34. $\dfrac{dy}{dx} = -\dfrac{4x + 3y}{2x + y}$

[1] The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.

35. $\dfrac{dy}{dx} = \dfrac{x + 3y}{x - y}$

36. $(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$

37. $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$

38. $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$

2.3 Modeling with First Order Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if not impossible, in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and they may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.

Construction of the Model. In this step you translate the physical situation into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton's laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of change (derivative) and consequently, when expressed mathematically, leads to a differential equation. The differential equation is a mathematical model of the process.

It is important to realize that the mathematical equations are almost always only an approximate description of the actual process. For example, bodies moving at speeds comparable to the speed of light are not governed by Newton's laws, insect populations do not grow indefinitely as stated because of eventual lack of food or space, and heat transfer is affected by factors other than the temperature difference. Thus you should always be aware of the limitations of the model so that you will use it only when it is reasonable to believe that it is accurate. Alternatively, you can adopt the point of view that the mathematical equations exactly describe the operation of