Modeling with First Order Equations

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Methods Studied

- For the linear Ordinary Differential Equation

$$y' + p(x)y = q(x)$$

Step 1. Find the integrating factor $\mu(x) = e^{\int p(x)dx}$.
Step 2. The general solution is

$$y(x) = \frac{\int \mu(x)q(x)dx + C}{\mu(x)}.$$

- For Seperable equations $M(x)dx + N(y)dy = 0$. The general solution is

$$\int M(x)dx + \int N(y)dy = C.$$

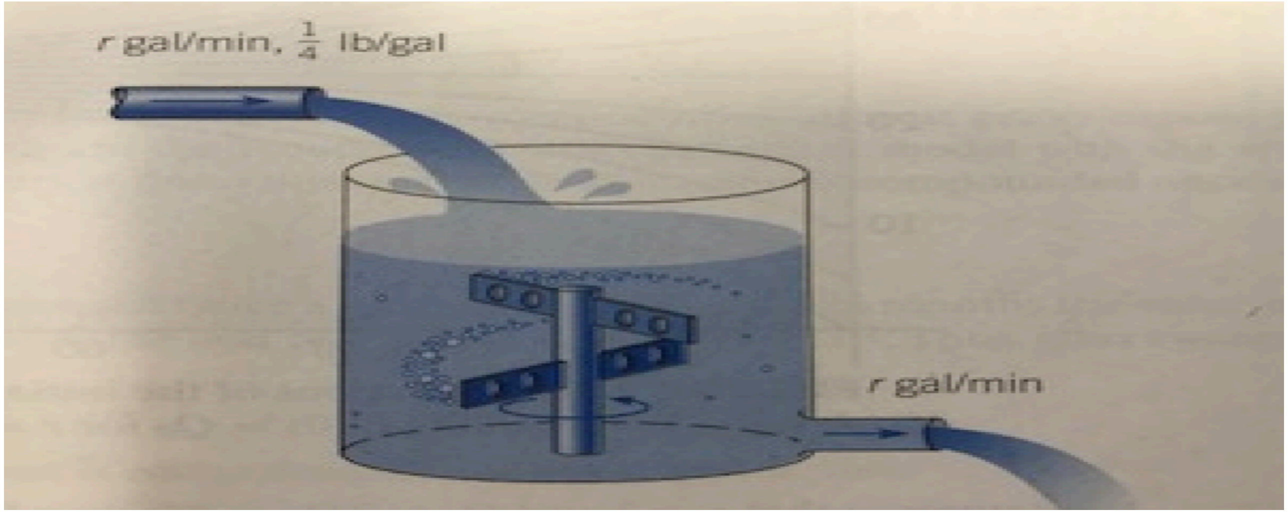
Steps of Modeling

- Step 1. Construction of model: Translate physical or real world problem into mathematical terms, using known physical principle or other knowledge. Establish a DE.
- Step 2. Mathematical analysis of model: Solve the DE and study the behavior of solutions, simplification may be necessary.
- Step 3. Interpret the mathematical results, what does it mean in practice?
- Step 4. Compare with experimental results or observation. If there is a big discrepancy, go back to find a better model.

Mixing Problem

At time t=0, a tank containing Q_0 Lb of salt dissolved in V_0 gallons of water. Assume that water containing ¼ Lb of salt per gallon is entering the tank at the rate of 3 gallon/minute, and well mixed solution is drained from the tank at the same rate. Find the amount of salt, Q(t), in the tank at any time t, and also find the limiting amount Q_L that is present after a long time.

$M = S$



Modeling Mixing Problem

- Let $Q(t)$ be the amount of the salt in the tank at time t. We want to find an equation for $Q(t)$.
- Think the change of salt from time t to time $t + \Delta t$.

$Q(t+\Delta t) - Q(t)$ = salt in – salt out during this period

= water in x concentration of incoming fluid

- water out x concentration of outgoing fluid

$= \Delta t\ 3 \times ¼ - \Delta t\ 3 \times Q(t)/V_{0.}$ Let Δt tend to 0,

$Q'(t) = 3/4 - 3\ Q/V_{0,}$ $Q(0) = Q_{0.}$

Solution from standard form $Q'(t) + 3\ Q/V_0 = 3/4_,$

$Q(t) = V_0/4 + [Q_0 - V_0/4]\ \exp(-3\ t/V_0)$.

As t tends to infinity, $Q(t)$ tends to $V_0/4$.

Very reasonable, why?

Compound Interest

- Suppose that John deposited $\$S_0$ in a bank at age of 25 that pays interest at an annual rate r. The value S(t) of the investment at any time t depends on the frequency with which the interest is compounded (yearly, quartly, monthly and daily). If it compounds continuously, we see
- $S(t+\Delta t) - S(t)$ = interest earned = $S(t)\, r\, \Delta t$
- Let Δt tends to zero, we have
- $S'(t) = r\, S(t)$, $S(0) = S_{0.}$ (t =0 corresponding to 25 years old).
- The solution is $S(t) = S_0 \exp(r\, t)$.

Interest Problem Continued

- Now let us assume that John can add \$k/year to the account, then
- $S(t+\Delta t) - S(t)$ = interest earned + deposited

  $= S(t)\, r\, \Delta t + k\, \Delta t$

  $S'(t) = r\, S(t) + k, \;\; S(0) = S_0$.
- $S(t) = S_0 \exp(rt) + k/r\, [\exp(rt) - 1]$.
- The second term is the contribution from k.

Let us plug in a few numbers to get some ideas of John's investment after 40 years (65 years old, about retirement time).

With No Further Addition (k=0)

- Assume r = 8%. S_0 = \$1.00. Comparison of Different Compound Methods.

Years	m=4 (quarterly)	m=12 (monthly)	m=365 (daily)	m = ∞(Continuously)
1	1.08243216	1.082999507	1.083277572	1.083287068
2	1.171659381	1.172887932	1.173490298	1.173510871
5	1.485947396	1.489845708	1.491759314	1.491824698
10	2.208039664	2.219640235	2.22534585	2.225540928
20	4.875439156	4.926802771	4.95216415	4.953032424
30	10.76516303	10.93572966	11.02027794	11.02317638
40	23.76990696	24.27338554	24.52392977	24.5325302

With Annual Addition

- Assume $S_0 = 5000$, $K = 5000$
- $S(t) = S_0 \exp(rt) + k/r\,[\exp(rt) - 1]$.

Years	r = 7%	r = 8 %	r = 9 %	r = 10 %
1	10541.69671	10621.87707	10702.77607	10784.40049
2	16485.21177	16711.98379	16942.60699	17177.1517
5	37028.73407	38198.16709	39414.46012	40679.66989
10	82479.67121	87724.01267	93387.07729	99505.50057
20	238504.5689	271829.6886	310784.2076	356398.0854
30	552700.129	681564.4057	845494.8656	1054704.531
40	1185412.289	1593445.788	2160670.864	2952898.252

Mortgage or Car Loan Payments

- The same idea can be used to compute the loan payments.
- Let S_0 be the initial loan amount. The bank charge annual interest r. If you want to pay off the loan in T years, what is the monthly payment M?
- Let S(t) be the loan balance at time t, then

$S(t+ \Delta t) - S(t)$ = interest charge – payment during this Δt time

$= S(t)\, r\, \Delta t - 12M\, \Delta t$, Let Δt tends to zero,

$S'(t) = r\, S(t) - 12\, M$, $S(0) = S_0$ is the initial value problem.

The solution is $S(t) = S_0 \exp(rt) - 12\, M/r\, [\exp(rt) - 1]$.

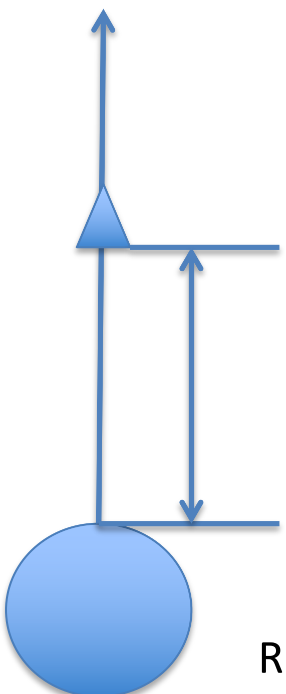
The Goal is $S(T) = 0$, we should have

$0 = S(T) = S_0 \exp(rT) - 12\, M/r\, [\exp(rT) - 1]$.

$M = S_0\, r/[12\, (1 - \exp(-rT))]$

Example. S_0 = 200, 000, r = 4%, T = 30 years. M = $ 954.

Escape Velocity

- An object of mass m is launched from the earth in the direction perpendicular to the earth's surface.
- 

Height = $y(t)$

Newton's law: $F = m\,a = m\,y''(t)$

F = gravitational force $= -\,m\,M\,G/(R + y)^2$

$= -\,m\,g\,R^2/(R + y)^2$

When $y = 0$, the force should be $-\,m\,g$

$g = 9.8\ \mathrm{m/S^2}$

Radius = R = 6400 km

Mass = M

Escape Velocity Continued

- From $y'' = - g R^2/(R + y)^2$. This is a second order equation, we want to change to first order equation. Note
- $Y''(t) = dv/dt = (dv/dy)(dy/dt) = v\, dv/dy$

  $v\, dv/dy = - g R^2/(R + y)^2$ Separable equation

  Integrating to get

  $v^2/2 = g R^2/(R + y) + C$,

  At $t=0$, $y(0) =0$, $v = v_0$, we can determine C

$$v = \pm\sqrt{v_0^2 - 2gR + 2gR^2/(R+y)}$$

Case 1. $v_0^2 > 2gR$

- In this case, v keeps the positive sign, the object will continue to fly away from the earth, will never return to the earth.
- This number = square root of 2gR

= 11,200 m/S is called the escape velocity

Extra Problem: Find the escape velocity of the moon. If you fire a rifle on the moon, can the bullet fly away from the moon?

Case 2. $v_0^2 < 2gR$

- In this case, the object will reach a maximum height, and return to the earth.
- What is this maximum height? (Hint. At this maximum height, the velocity must be zero!)

  It turns out $H_{max} = v_0^2 R/(2 g R - v_0^2)$

Extra Problem: How much time does it take to reach the mars if the initial velocity is the escape velocity (ignore the gravitational force of the mars)?

Homework: Section 2.3. #1, 3, 7, 8, 9, 29.