Math 347H Lecture: Review of Matrixes

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Homework: Section 7.2. 11, 13, 17, 23, 25.

In this lecture, we quickly review some basic facts on Linear Algebra. I only chose those parts closely related to this course.

1. For any matrix $A = A_{m\times n} = (a_{ij})_{m\times n})$, we can define

$\overline{A} = (\bar{a}_{ij})_{m\times n}$, $A^T = (b_{ij})_{n\times m}$, where $b_{ij} = a_{ji}$. $A^* = (c_{ij})_{n\times m}$, where $c_{ij} = \bar{a}_{ji}$, the complex conjugae of a_{ji}. That is $A^* = \overline{A}^T$. $0_{m\times n}$ matrix means that every entry is 0. $I_{n\times n}$, the $n\times n$ identity matrix, is the matrix with diagonal $a_{ii} = 1$, any other element 0. Why?

Example 1. Let $A = \begin{pmatrix} 2 & 2-i & 3+2i \\ 4+3i & -5+2i & 6-i \end{pmatrix}$. Then

$$\overline{A} = \begin{pmatrix} 2. & 2+i & 3-2i \\ 4-3i & -5-2i & 6+i \end{pmatrix}, \quad A^T = \begin{pmatrix} 2 & 4+3i \\ 2-i & -5+2i \\ 3+2i & 6-i \end{pmatrix},$$

$$A^* = \begin{pmatrix} 2 & 4-3i \\ 2+i & -5-2i \\ 3-2i & 6+i \end{pmatrix}.$$

2. Operations of matrices: $A_{m\times n} \pm B_{m\times n}$, $\lambda A_{m\times n}$, and $A_{m\times n}B_{n\times p} = C_{m\times p}$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}.$$

Notice the size of A and B for these operations.

We have usual associative, commutative, distributive properties except the following

Warning $AB \neq BA$ in general.

Example. 2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

$$AB = 0_{2\times 2}, \quad BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$IA = AI = A$ for any matrix. That is why we choose the identity matrix as defined before, it serves the same purpose as the number 1.

Also 0 matrix serves the same purpose as the number 0. We have $0+A = A+0$ and $0A = A0 = 0$.

3. Any matrix $A_{m\times n}$ is associated with a linear operator from $\mathbb{R}^m \to \mathbb{R}^n$. The multiplication is the composition of the linear maps. $(AB)C = A(BC)$.

4. Inner product $(X, Y) = X^T\bar{Y} = (x_1\bar{y}_1 + ... + x_n\bar{y}_n$ for any $X \in \mathbb{C}^n, Y \in \mathbb{C}^n$. $\sqrt{(X, X)}$ is called the norm of X. When $X, Y \in \mathbb{R}^n$,

$$(X, Y) = X^T Y = x_1y_1 + ... + x_ny_n.$$

Two vectors X and Y are said to be orthogonal if $(X, Y) = 0$.

5. A is called invertible if there is a B such that $AB = BA = I$. A must be a square matrix. $B = A^{-1}$ is called the inverse of A.

Why multiply B on both sides in this definition?

Can you find two different inverses?

When is A invertible? iff $\det(A) \neq 0$. How to find $\det(A)$? Using row or column expansion. How to find A^{-1} if exists?

Practical method. Using elementary row operations to change $(A|I) \to (I|B)$. The resulting B is the inverse of A.

Three elementary row operaions: (a) Exchange rows; (2) λRow for some $\lambda \neq 0$; (3) λ times ith row is added to jth row.

Theoretical method. For each (i, j), let A_{ij} be the submatrix of A with ith row and jth column removed. $M_{ij} = \det A_{ij}$, $C_{ij} = (-1)^{i+j} M_{ij}$, cofactors. Then $A^{-1} = \frac{1}{\det A}(B_{ij})$ with $B_{ij} = C_{ji}$.

This formula is only useful for theoretical purpose and for small matrices.

Especially if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. $\det A = a_{11}a_{22} - a_{21}a_{12}$, $A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

Example 3. Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$.

$$.\text{Answer: } A^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}.$$

Work out the details in the class.