

Condensed Matter Physics  
from the context of the courses  
PHY 491: Solid State Physics

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## 0.1 The SI System

In physics it's often important to have precisely defined units for the purposes of making very accurate measurements or simply having a coherent unit system. It's possible to derive all necessary units from five measurements of **length, mass, time, current, and temperature**. The standard SI units for these properties are listed below:

| Type        | Unit             | Definition   |
|-------------|------------------|--|
| Length      | Meter( $m$ )     | Length of distance light in a vacuum travels in $\frac{1}{299792458}$ seconds  |
| Mass        | Kilogram( $kg$ ) | Defined by fixing the Planck's constant $h = 6.62607015 \times 10^{-34} kg \cdot m^2 s^{-1}$                         |
| Time        | Second( $s$ )    | Defined by fixing the ground-state hyperfine transition frequency of the caesium-133 atom, to be $9192631770 s^{-1}$ |
| Current     | Ampere( $A$ )    | Defined by fixing the charge of an electron as $1.602176634 \times 10^{-19} A \cdot s$                               |
| Temperature | Kelvin( $K$ )    | Defined by fixing the value of the Boltzmann constant $k$ to $1.380649 \times 10^{-23} kg \cdot m^2 s^{-2} K^{-1}$   |

Common prefixes are listed below:

| Prefix | Symbol | Definition |
|--------|--------|------------|
| mega   | M      | $10^6$     |
| kilo   | k      | $10^3$     |
| milli  | m      | $10^{-3}$  |
| micro  | $\mu$  | $10^{-6}$  |
| nano   | $n$    | $10^{-9}$  |
| pico   | $p$    | $10^{-12}$ |
| femto  | $f$    | $10^{-15}$ |

Additionally, the following are defined constants:

| Symbol  | Definition  |
|---------|---|
| $\hbar$ | $\hbar = \frac{h}{2\pi} \approx 1.0546 \times 10^{-34} kg \cdot m^2 s^{-1}$ |

## 0.2 Why Study Condensed Matter

Condensed matter physics is the most broad field of physics that has a large overlap with other fields such as atomic physics, quantum physics, chemistry, and biology. Condensed matter physics describes anything that is "condensed" this includes any solid material, such as metals, glass, wood and electronics. As such it is very important in our world. Condensed matter is simultaneously very fundamental describing microscopic sources for material properties such as superconductivity, charge density waves, topological insulators. These phenomenon can't be described by reductionism and many properties only emerge when you consider larger systems that cannot be solved using fundamental laws alone.

# Chapter 1

## Hydrogen Like Systems

Many systems in condensed matter are similar to the hydrogen atom system and so it is often useful to use the solutions to the basic hydrogen atom Hamiltonian to model these systems.

### 1.1 Simple Hydrogen Atom

**Definition 1.1.1.** The **simple hydrogen atom Hamiltonian** is given by

$$\mathcal{H} = \underbrace{\frac{P_n^2}{2m_n}}_{\text{Nucleus Kinetic Energy}} + \underbrace{\frac{P_e^2}{2m_e}}_{\text{Electron Kinetic Energy}} - \underbrace{\frac{Ze^2}{4\pi\epsilon_0|\vec{x}_e - \vec{x}_n|}}_{\text{Electrostatic Potential}} \quad (1.1.1)$$

This simplified Hamiltonian ignores relativity, spin, and magnetism.

**Definition 1.1.2.** **Center of mass coordinates** for the hydrogen atom are defined by

$$\begin{aligned} \vec{X} &= \frac{m_e \vec{x}_e + m_n \vec{x}_n}{m_e + m_n} && \text{center of mass} \\ \vec{r} &= \vec{x}_e - \vec{x}_n && \text{relative coordinate} \end{aligned} \quad (1.1.2)$$

**Corollary 1.1.3.** The simple hydrogen atom Hamiltonian in center of mass coordinates is

$$\mathcal{H} = \underbrace{-\frac{\hbar^2}{2(m_e + m_n)}\nabla_X^2}_{\mathcal{H}_k} - \underbrace{\frac{\hbar^2}{2\mu}\nabla_r^2 - \frac{Ze^2}{4\pi\epsilon_0|\vec{r}|}}_{\mathcal{H}_r} \quad (1.1.3)$$

**Proposition 1.1.4.** The Hamiltonians  $\mathcal{H}_k$  and  $\mathcal{H}_r$  are compatible observables.

$$[\mathcal{H}_k, \mathcal{H}_r] = 0, \quad [\mathcal{H}_k, \mathcal{H}] = 0, \quad [\mathcal{H}_r, \mathcal{H}] = 0 \quad (1.1.4)$$

**Proposition 1.1.5.** The Hamiltonian  $\mathcal{H}_r$  can be further separated in terms of radial and angular dependence.

$$\mathcal{H}_r = \underbrace{\frac{\hbar^2 L^2}{2\mu|\vec{r}|^2}}_{\text{Tangential Kinetic Energy}} - \underbrace{\frac{\hbar}{2\mu} \frac{P_r^2}{\hbar^2}}_{\text{Radial Kinetic Energy}} - \underbrace{\frac{Ze^2}{4\pi\epsilon_0|\vec{r}|}}_{\text{Electrostatic Potential}} \quad (1.1.5)$$

The solution to the Hamiltonian  $\mathcal{H}_r$  can be written in terms of independent radial and angular components.

$$\begin{aligned} n &= 1, 2, 3, 4, 5, \dots \\ \ell &= 0, 1, 2, 3, \dots, n-1 \\ m_\ell &= -\ell, -\ell+1, \dots, -1, 0, 1, \dots, \ell-1, \ell \end{aligned} \quad (1.1.6)$$

$$\psi_{n\ell m}(r, \theta, \phi) = U_n^\ell(r) Y_\ell^m(\theta, \phi) \quad (1.1.7)$$

**Result 1.1.8.** For the solution to the hydrogen atom  $\psi_{n\ell m}(r, \theta, \phi)$ , the following eigenvalue equations hold:

$$\mathcal{H}\psi_{n\ell m}(r, \theta, \phi) = -\frac{ZRyd}{n^2}\psi_{n\ell m}(r, \theta, \phi) \quad (1.1.8)$$

$$L^2\psi_{n\ell m}(r, \theta, \phi) = \ell(\ell+1)\hbar^2\psi_{n\ell m}(r, \theta, \phi) \quad (1.1.9)$$

$$L_z\psi_{n\ell m}(r, \theta, \phi) = m\hbar\psi_{n\ell m}(r, \theta, \phi) \quad (1.1.10)$$

**Definition 1.1.11.** The **Bohr radius** is the average radius for the first energy level of the hydrogen atom given by

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu Ze^2} \quad (1.1.11)$$

**Definition 1.1.12.** The **Rydberg constant** is the first energy level of hydrogen given by

$$Ryd = \frac{\hbar^2}{2a_0^2\mu} = \frac{e^2}{8\pi\epsilon_0 a_0} \quad (1.1.12)$$

**Result 1.1.13.** The **energy eigenvalues of hydrogen** are

$$E_n = -\frac{Z^2\hbar^2}{2a_0^2\mu} \frac{1}{n^2} = -\frac{Z^2 Ryd}{n^2} \quad (1.1.13)$$

## 1.2 Radial Component of Hydrogen

**Definition 1.2.1.** The **Laguerre polynomials**  $L_q(z)$  are polynomials of degree  $q$  defined by

$$L_q(z) = e^z \frac{d^q}{dx^q} (z^q e^{-z}) \quad (1.2.1)$$

**Definition 1.2.2.** The **associated Laguerre polynomials** are polynomials of degree  $q - p$  defined by

$$L_q^p(z) = \frac{d^p}{dx^p} L_q(x) \quad (1.2.2)$$

**Result 1.2.3.** The **radial component**  $Y_\ell^m(\theta, \phi)$  has eigenvalues with eigenstates given by

$$U_n^\ell(r) = -\sqrt{\left(\frac{2Z}{na_0} \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right)^3} e^{-Zr/na_0} \left(\frac{2Zr}{na_0}\right)^\ell L_{n+\ell}^{2\ell+1}(2Zr/na_0) \quad (1.2.3)$$

## 1.3 Angular Component of Hydrogen

**Definition 1.3.1.** The **Legendre polynomials** denoted  $P_\ell(z)$  are polynomials of degree  $\ell$  that appear in spherically symmetric systems. They are defined with Rodrigues' formula given by

$$P_\ell(z) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dz^\ell} (z^2 - 1)^\ell \quad (1.3.1)$$

**Definition 1.3.2.** The **associated Legendre polynomials** denoted  $P_\ell^m(z)$  are polynomials calculated from the Legendre polynomials. They are equivalently defined by the following formulas

$$\begin{aligned} P_\ell^m(z) &= (-1)^m (1-z^2)^{m/2} \frac{d^m}{dz^m} P_\ell(z) \\ &= \frac{(-1)^m}{2^\ell \ell!} (1-z^2)^{m/2} \frac{d^{\ell+m}}{dz^{\ell+m}} (z^2 - 1)^\ell \end{aligned} \quad (1.3.2)$$

**Result 1.3.3.** The **angular solution**  $Y_\ell^m(\theta, \phi)$  has eigenvalues with eigenstates given by

$$Y_\ell^m(\theta, \phi) = (-1)^m \sqrt{\frac{2(\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi} \quad (1.3.3)$$