Basic Theory of System of First Order Liner Equations

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Homework. Section 7.4. # 2, 3, 4, 6,

Part 1. Consider the system $X' = A(t)X(t) + g(t)$. This system is called linear. When we use the matrix notation, we can not only save a lot of space and make the computation more compact, but also see the similarity between the system of equation and a single (scalar) equation.

We will assume that $A(t)$ and $g(t)$ are continuous on an interval $\alpha < t < \beta$.

Theorem 1 (Existence theorem).. For any $t_0 \in (\alpha, \beta)$, the initial value problem $X' = A(t)X(t) + g(t), \ \ X(t_0) = X_0$ has one and only one solution $X(t)$ for $t \in (\alpha, \beta)$.

Part 2. (Principle of superposition) Consider the associated homogenous system $X' = A(t)X(t)$. If two vector functions $X^{(1)}, X^{(2)}$ are two solutions of the system, then any linear combination $c_1X^{(1)} + c_2X^{(2)}$is also a solution for any constants c_1 and c_2. That is easy to check.

$$\tfrac{d}{dt}[c_1X^{(1)} + c_2X^{(2)}] = c_1\tfrac{d}{dt}X^{(1)} + c_2\tfrac{d}{dt}X^{(2)}$$

$$= c_1A(t)X^{(1)}. + c_2A(t)X^{(2)} = A(t)[c_1X^{(1)} + c_2X^{(2)}]$$

By repeated use this principle, we can see that if $X^{(1)}..., X^{(k)}$ are k solutions, then $X = c_1X^{(1)} + ...c_kX^{(k)}$ is also a solution.

Part 3. Question: How many linearly independent solutions for homogeneous system?

We can obviously find n linearly independent solutions. For example, $X^{(j)}(t_0) = e_j$, where $e_j \in \mathbb{R}^n$ is the vector such that jth element of e_j is 1, the rest are zero.

Can we find more than n linearly independent solutions ?

Question: If it is linearly independent at t_0, does it linearly independent at any other point $t \in (\alpha, \beta)$?

We know that it was true for second order equation from the Wronskian. Similarly we have the following

Definition: Let $X^{(1)}, ..., X^{(n)}$ be n solutions of the system $X' = A(t)X$, the determinant

$$\det \begin{pmatrix} X^{(1)}(t) & ... & X^{(n)}(t) \end{pmatrix} = W[X^{(1)}, ..., X^{(n)}](t)$$

is called the Wronskian of $X^{(1)}, ..., X^{(n)}$.

Abel's Theorem $W'(t) = Trace(A(t))\, W(t)$. That is,

$$W(t) = C \exp\{\int [a_{11}(t) + ...a_{nn}(t)]dt\},$$

where C is a constant.

Abel's Theorem If the vector functions $X^{(1)}, ..., X^{(n)}$ are n linearly independent solutions of the homogenous system at any point $t_0 \in (\alpha, \beta)$, then every solution of the system $\phi(t)$ is a linear combination of $X^{(1)}, ..., X^{(n)}$. That is,

$$\phi(t) = c_1 X^{(1)} + ... + c_n X^{(n)}.$$

That means that the system has at most n linearly independent solutions. The set $X^{(1)}, ..., X^{(n)}$ is called a fundamental set of solutions. The matrix. $X = \begin{pmatrix} X^{(1)} & ... & X^{(n)} \end{pmatrix}$ is called a fundamental matrix. We note that $X' = AX$.

Prove these theorem if time permits during the lecture. The ideas are the same as before, using Gronwall's Inequality.

Part 4. The general solution for the non-homogeneous system is given by $X = X_h + X_p$, X_h is homogeneous solution, and X_p is ANY particular solution. The proof is exactly the same as non-homogeneous scalar equation.

Part 5. Particular solution can be obtained using the method of variation of parameters. Try $X_p = XC(t)$ with X a fundamental matrix. Constant vector C is changed to a vector function $C(t0$. Then $X'(t)C + X(t)C'(t) = AX(t)C(t) + g(t)$, we need $X(t)C'(t) = g(t), C'(t) = X^{-1}g(t)$. Hence

$$C(t) = \int X^{-1} g(t) dt, \quad X_p = X \int X^{-1} g(t) dt.$$

Proof of the Uniqueness theorem in Class.