PHY 410 - Reference Sheet

Stirling's approximation - for very large N the factorial can be very accurately approximated with the following

$$\log N! \approx N \log N - N$$

$$N! \approx \sqrt{2\pi N} N^N e^{-N}$$

Fractional uncertainty of $\mathbb{X}$ is uncertainty of expected value per particle.

$$\frac{\Delta \mathbb{X}}{N} = \frac{\sqrt{\langle \mathbb{X}^2 \rangle - \langle \mathbb{X} \rangle^2}}{N}$$

Boltzmann's constant
$k_B = 1.380649 \times 10^{-23} \mathrm{m}^2\, \mathrm{s}^{-2}\, \mathrm{K}^{-1}$
Entropy $S = k_B \sigma, \quad \sigma_{TOT} = \sigma_1 + \sigma_2$
Temperature $T = \tau / k_B$

Microcanonical Ensemble

Multiplicity function

$$g = \#\text{ of microstates}, \quad \mathcal{P}(n) = \frac{1}{g}$$

Expected value of $\mathbb{X}$ is the average across all microstates.

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n) \mathcal{P}(n) = \frac{1}{g} \sum_n \mathbb{X}(n)$$

Entropy can be written in terms of the multiplicity function.

$$\sigma(N, T, U, V, P) \equiv \log[g(N, T, U, V, P)]$$

Binary System

A **binary system** is a system of N particles where each particles has two possible states. Let $N_\uparrow$ is the number of particle in the up state and $N_\downarrow$ be the number of particles in the down state.

$$g(N, N_\uparrow) = \frac{N!}{N_\uparrow!(N - N_\uparrow)!}, \quad \sum_{N_\uparrow = 0}^{N} g(N, N_\uparrow) = 2^N$$

The binary system can be rewritten in terms of the differnce between up states and down states this is the **spin excess**.

$$2S = N_\uparrow - N_\downarrow$$

$$g(N, S) = \frac{N!}{(\frac{N}{2} + S)!(\frac{N}{2} - S)!}$$

$$\sum_{S = -\frac{N}{2}}^{S = \frac{N}{2}} g(N, N_\uparrow) = 2^N$$

Applying Stirling's approximation to the binary model, for large N the multiplicity function and fractional uncertainty are

$$g(N, S) \approx g(N, 0) e^{-2s^2/N}$$

$$g(N, S) \approx \sqrt{\frac{2}{\pi N}} 2^N e^{-2s^2/N}$$

$$\frac{\Delta S}{N} \approx \frac{1}{\sqrt{N}}$$

An example of a binary system is N spin 1/2 particles in an external **magnetic field** B. The total energy U and magnetization M of the system are

$$U = \sum_{i=1}^{N} -\vec{m_i} \cdot \vec{B} = -(N_\uparrow - N_\downarrow) m B = -2SmB$$

$$M = 2Sm = -U/B$$

$$g(N, U) = \frac{N!}{(\frac{N}{2} - \frac{U}{2mB})!(\frac{N}{2} + \frac{U}{2mB})!}$$

Einstein Solid

An **einstein solid** is a system of N atoms where each atom is modeled as a harmonic oscillator the total energy of the system is determined by the number of atoms n oscillating at frequency ω.

$$U = n\hbar\omega$$

$$g(N, n) = \frac{(n + N - 1)!}{n!(N - 1)!}$$

Thermal Equilibrium

Temperature

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_{N, V}$$

Thermal Equilibrium

$$\left(\frac{\partial \sigma_1}{\partial U_1}\right)_{N_1, V_1} = \left(\frac{\partial \sigma_2}{\partial U_2}\right)_{N_2, V_2}$$

$$\frac{1}{\tau_1} = \frac{1}{\tau_2}$$

2nd law of thermo - Change in entropy ≥ 0.
Sharpness of Equilibrium For a two binary systems, the number of states in a configuration of deviation δ from equilibirum is

$$g_1 g_2 = (g_1 g_2)_{max} e^{\left(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}\right)}$$

Canonical Ensemble

Partition Function - partition by energy levels for a fixed temperature

$$z = \sum_n e^{-\varepsilon_n/\tau}, \quad \mathcal{P}(n) = \frac{1}{z} e^{-\varepsilon_n/\tau}$$

$$z = \sum_\alpha g(\varepsilon_\alpha) e^{-\varepsilon_\alpha/\tau}, \quad \text{for degeneracy } g(\varepsilon_\alpha)$$

Expected Value of $\mathbb{X}$ is the average across all energies (Thermal Average).

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n) \mathcal{P}(n) = \frac{1}{z} \sum_n \mathbb{X}(n) e^{-\varepsilon_n/\tau}$$

Expected Energy in the canonical ensemble is

$$U = \langle \varepsilon \rangle = \frac{1}{z} \sum_n \varepsilon_n e^{-\varepsilon_n/\tau}$$

$$U = \langle \varepsilon \rangle = \tau^2 \frac{1}{z} \frac{\partial z}{\partial \tau} = \tau^2 \frac{\partial}{\partial \tau} \log z$$

The total partition function and expected energy for N non-interacting particles is simply

$$z_N = z_1^N$$

$$U_N = \langle \varepsilon \rangle_N = N U_1 = N \langle \varepsilon \rangle_1$$

(this also applies for expected value of any $\mathbb{X}$)

Theromodynamic Relations

1st Law of Thermo

$$dU = dQ + dW = \tau d\sigma - P dV$$

$$d\sigma = \frac{1}{\tau} dU + \frac{P}{\tau} dV$$

Temperature $\tau = \left(\frac{\partial U}{\partial \sigma}\right)_V$
Quasi-static Compression Equilibirum the equilibrium condition for quasi-static compression is

$$\left(\frac{\partial U_1}{\partial V_1}\right)_{\sigma_1} = \left(\frac{\partial U_2}{\partial V_2}\right)_{\sigma_2}$$

Helmholtz Free Energy

$$F = U - \tau\sigma = U - ST = -\tau \log z$$

$$dF = -\sigma d\tau - P dV$$

Entropy $\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V, \quad S = k_B \sigma$
Pressure

$$P = -\left(\frac{\partial U}{\partial V}\right)_\sigma = \tau \left(\frac{\partial \sigma}{\partial V}\right)_U = -\left(\frac{\partial F}{\partial V}\right)_\tau$$

Energy

$$U = -\tau^2 \frac{\partial}{\partial \tau} \left(\frac{F}{\tau}\right)$$

Ideal Gas

DeBroglie Thermal Wavelength is the wavelength of the wave functions of matter at a given temperature.

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{m\tau}}$$

Concentration of a system is the inverse of the volume

$$n = \frac{1}{V}$$

Quantum Concentration is the density of quantum state per particle. It is used to define when a system will behave classically ($n << n_Q$) and when a system will be dominated by quantum effects ($n >> n_Q$).

$$n_Q = \frac{1}{\lambda_T^3}$$

Single Particle Ideal Gas is a system in the canonical ensemble consisting of a signle particle in a box of side lengths L. The energy levels , partition function and average energy are

$$E_{n_x, n_y, n_z} = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$

$$z_1 = \frac{V}{\lambda_T^3}$$

$$U_1 = \frac{3}{2}\tau$$

$$\sigma_1 = \log\left(\frac{V}{\lambda_T^3}\right) + \frac{3}{2}$$

Gibbs Resolution states that for systems in the classical regime the partition function for an ideal gas with N particles is

$$z_N = \frac{1}{N!}(z_1)^N$$

$$U_N = \frac{3}{2} N\tau$$

$$\sigma_N = N\left[\log\left(\frac{V}{N\lambda_T^3} + \frac{5}{2}\right)\right]$$

N-Particle Ideal Gas - by applying Gibbs resolution and properties of expected values we can find the classical ideal gas results

$$PV = N\tau$$

$$U = \frac{3}{2} N\tau$$

$$\sigma = N\left[\log\left(\frac{V}{N\lambda_T^3}\right) + \frac{5}{2}\right]$$

DOG (bork)