Topology

from the context of the course
MTH 461: Metric and Topological Spaces

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Contents

Chapter 1

Introduction

1.1 Functions

Definition 1.1.1. A **function** $f : A \to B$ is a subset of $X \times Y$ such that $\forall x \in X$, $\exists$ exactly one element $y \in B$, $(x, y) \in f$.

Definition 1.1.2. The **domain** of a function $f : A \to B$ is $\{a \in A : \exists b \in B$ such that $(a, b) \in f\}$.

Definition 1.1.3. The **range** of a function $f : A \to B$ is $\{b \in B : \exists a \in A$ such that $(a, b) \in f\}$.

Definition 1.1.4. A function is a **injective** denoted $f : A \hookrightarrow B$ iff $f(x) = f(u) \Rightarrow x = y$.

Definition 1.1.5. A function is a **surjection** denoted $f : A \twoheadrightarrow B$ iff the range of f equals B.

Definition 1.1.6. A function is a **bijection** denoted $f : A \hookrightarrow\!\!\!\!\rightarrow B$ iff it is both an injection and a surjection.

Definition 1.1.7. An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

1.2 Relations

Definition 1.2.1. A **relation** on a set A is a subset of $A \times A$. Conventionally written xRy rather than $(x, y) \in R$.

Definition 1.2.2. For a relation R on a set A, R is

- **Reflexive** iff xAx for all $x \in A$
- **Antireflexive** iff $\nexists x \in A$ such that xAx
- **Transitive** iff xRy and $yRz \Rightarrow xRz$, for any $x, y, z \in A$.
- **Symmetric** iff $xRy \Leftrightarrow yRx$, for any $x, y \in A$.
- **Antisymmetric** iff xRy and $yRx \Rightarrow x = y$, for any $x, y \in A$.
- **Connex** iff for every $x, y \in R$ at least on of xRy, yRx, or $x = y$ hold.

Definition 1.2.3. The **equivalence class** of $a \in A$ for a relation $\sim$ is $[x] := \{b \in A | a \sim b\}$.

Definition 1.2.4. A **partition** of a set A is a set of subsets X such that $\bigcup X = A$ and $\forall B, C \in X, A \neq B \Rightarrow A \cap B = \emptyset$.

Lemma 1.2.1. Let $x, y \in A$ and $\sim$ be an equivalence class on A, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

Corollary 1.2.1.1. Any partition defines and equivalence relation and vice versa.

1.3 Order

Definition 1.3.1. An **order** on a set A is a relation that is antireflexive, transitive, and connex.

Definition 1.3.2. A **partial order** on a set A is a relation that is reflexive, antisymmetric, and transitive.

Definition 1.3.3. Two ordered sets have the same **order type** if there exists a bijection that preserves order.

Definition 1.3.4. Let $(X, \leq)$ be an ordered set, and let $A \subseteq X$.

- The **maximum** of A is an element $a_{max} \in A$ such that $\forall a \in A, a \leq a_{max}$.
- The **minimum** of A is an element $a_{min} \in A$ such that $\forall a \in A, a \geq a_{min}$.
- An **upper bound** of A is an element $x \in X$ such that $\forall a \in A, a \leq x$.
- An **lower bound** of A is an element $x \in X$ such that $\forall a \in A, a \geq x$.
- The **supremum** of A is the least upper bound of A.
- The **infimum** of A is the greatest lower bound of A.

Definition 1.3.5. An **interval** on an ordered set $(X, <)$ is

- $(a, b) = \{x \in X : a < x < b\}$ for some $a, b \in X$
- $[a, b) = \{x \in X : a \leq x < b\}$ for some $a, b \in X$
- $(a, b] = \{x \in X : a < x \leq b\}$ for some $a, b \in X$
- $[a, b] = \{x \in X : a \leq x \leq b\}$ for some $a, b \in X$

1.4 Cardinality

Definition 1.4.1. A set A is **finite** if there exists a bijection $f : A \hookrightarrow\!\!\!\!\rightarrow \{1, 2, 3, ..., n\}$ for some $n \in \mathbb{N}$.

Definition 1.4.2. The **cardinality** of a finite set A is $n \in \mathbb{N}$ such that $f : A \hookrightarrow\!\!\!\!\rightarrow \{1, 2, 3, ..., n\}$ is a bijection.

Theorem 1.4.1. Let A be a finite set with cardinality $n \in \mathbb{N}$ and $B \subsetneq A$ be a proper nonempty subset, then

$$\nexists \text{ a bijection } B \hookrightarrow\!\!\!\!\rightarrow \{1, ..., n\}$$

$$\exists \text{ a bijection } B \hookrightarrow\!\!\!\!\rightarrow \{1, ..., m\} \text{ for some } m \in \mathbb{N}$$

Corollary 1.4.1.1. For finite sets A there is no bijection between A and any proper nonempty subset $B \subsetneq A$.

Definition 1.4.3. A set A is **countable** iff $\exists A \hookrightarrow\!\!\!\!\rightarrow \mathbb{N}$ or A is finite.

Theorem 1.4.2. Let A be a nonempty set, then the following are equivalent.

- A is countable
- There exists a surjection $g : \mathbb{N} \twoheadrightarrow A$.
- There exists an injection $f : A \hookrightarrow \mathbb{N}$.

Corollary 1.4.2.1. Every subset $A \subset \mathbb{N}$ is countable.

Corollary 1.4.2.2. A countable union of countable sets is countable.

Definition 1.4.4. The **power set** of a set A denoted $P(A)$ is the set of all subsets of A.

Theorem 1.4.3. The Cantor Theorem states that for a nonempty set A there is no injection $f : P(A) \hookrightarrow A$ and no surjection $g : A \twoheadrightarrow P(A)$.

1.5 Topologies

Definition 1.5.1. A **topology** on a set A is a set of subsets $J \subset P(A)$ with the following properties

1. $\emptyset, A \in J$.
2. Any union of elements in J is also in J.
3. Any finite intersection of elements in J is also in J.

Definition 1.5.2. A **topological space** is a pair $(X, \mathcal{T})$ of a set X and a topology $\mathcal{T}$ on X.

Definition 1.5.3. A subset $A \subset X$ is **open** iff $A \in \mathcal{T}$ where $(X, \mathcal{T})$ is a topological space.

Definition 1.5.4. A **basis** is a collection $\mathcal{B}$ of subsets of a set X such that

1. $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in B$.
2. $\forall B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_1$.

Proposition 1.5.1. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C} \subset P(X)$. If $\forall U \in \mathcal{T}$, $\forall x \in U$, $\exists D \in \mathcal{C}$ such that $x \in D \subseteq U$, then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Definition 1.5.5. The **topology generated by** a basis $\mathcal{B}$ on a set X is

$$\mathcal{T} = \{U \in P(X) : U = \bigcap_{B_i \in \mathcal{C}} B_i, \mathcal{C} \subset \mathcal{B}\}$$

Theorem 1.5.1. Let $\mathcal{B}, \mathcal{B}' \subset P(X)$ be bases of the topological spaces $(X, \mathcal{T}), (X, \mathcal{T}')$. The following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. $\forall x \in X$ and any basis element $B \in \mathcal{B}$ such that $x \in B$ there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

Definition 1.5.6. A **homeomorphism** is a bijection $f : \mathcal{X} \hookrightarrow\!\!\!\!\rightarrow \mathcal{Y}$ between topologies $\mathcal{X}$ and $\mathcal{Y}$.

Definition 1.5.7. A topology is **second countable** if it has a countable basis.

1.5.1 Examples of Topologies

Definition 1.5.8. The **discrete topology** on a set X is $\mathcal{T} = P(X)$.

Definition 1.5.9. The **indiscrete topology** on a set X is $\mathcal{T} = \{\emptyset, X\}$.

Definition 1.5.10. The **finite compliment topology** on a set X is $\mathcal{T} = \{U \subset X : X - U \text{ is finite}\}$.

Definition 1.5.11. The **standard topology** on $\mathbb{R}$ is the topology generated by the basis

$$\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.12. The **lower limit topology** on $\mathbb{R}$ denoted $\mathbb{R}_\ell$ is the topology generated by the basis

$$\mathcal{B} = \{[a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.13. The **upper limit topology** on $\mathbb{R}$ is the topology generated by the basis

$$\mathcal{B} = \{(a, b] \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.14. The **K-topology** on $\mathbb{R}$ denoted $\mathbb{R}_K$ is the topology generated by the basis

$$\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\} \cap \{(a, b) - K \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

$$K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

Definition 1.5.15. The **order topology** on a ordered set S with more than 1 element is the topology generated by the a basis

$$\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\} \cap \{(a, b_0] \subset \mathbb{R} : a \in \mathbb{R}\} \cap \{[a_0, b) \subset \mathbb{R} : b \in \mathbb{R}\}$$

where a_0 is the smallest element and b_0 is the largest element.

1.6 Well Ordered Sets

Definition 1.6.1. A **well ordered set** X is an ordered set such that any subset $S \subseteq X$ has a smallest element $s_0 \in S$ such that $s_0 \leq s, \forall s \in S$.

Corollary 1.6.0.1. Any finite ordered set is well ordered.

Definition 1.6.2. The **section** of a well ordered set X by $a \in X$ denoted S_a is

$$S_a = \{x \in X : x < a\}$$

Theorem 1.6.1. Any set A admits a well ordering.

Corollary 1.6.1.1. There exists an uncountable well ordered set.

Theorem 1.6.2. There exists a well ordered set S such that any section is countable S_Ω where Ω is the largest element.

Definition 1.6.3. The **minimal uncountable well-ordered set** denoted S_Ω is the uncountable well-ordered set such that any section is countable.

Theorem 1.6.3. If $A \subset S_\Omega$ is a countable subset of S_Ω then A has an upper bound in S_Ω.

"At least if you believe in calculus."

"I hope that homework didn't kill anyone too much."
"I'm actually currently a zombie now."
"Yeah, that's the thing about homework."

"It's like set theory but actually interesting!"