Review for Exam 1 – Math 347H

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Review for Examination 1.

Remarks: Time: Friday, February 14 during lecture time, in the regular class room.
Index Card: You can bring a 3×5 index card and the sheet with simple and standard integrals I distributed at the beginning of the semester.
Part I. First Order Equations
Part II. Second Order Linear Equations

First Order Equations

a) $y' = g(t)$, $y(t) = \int g(t)dt + C$.
b) The general linear equation $y' + p(t)y = g(t)$, change to the first case by multiplying the integrating factor $\mu(t) = e^{\int p(t)dt}$.

$$[\mu(t)y]' = \mu(t)g(t), \;\; \mu(t)y(t) = \int \mu(t)g(t)dt + C.$$

c). Separable equations. $M(t)dt + N(y)dy = 0$, the general solution is $\int M(t)dt + \int N(y)dy = c$.
d). Special case: $y' = F(y)$. Equilibrium solutions, their stability from the phase line.
e). Homogeneous equation $y' = F(y/t)$ can be changed to the separable equation by introducing the new variable $v = y/t$.
The equation of the form $y' = F(at + by + c)$ can be changed to the separable equation by introducing the new variable $v = at + by + c$.

First Order equation

f). Exact Equations $M(t, y)dt + N(t, y)dy = 0$ if $M_y = N_t$. The solution is given by $\phi = C$ with $\phi_t x = M, \phi_y = N$.
In practice, $\phi(t, y) = \int M(t, y)dt + h(y)$, then find $h(y)$ by $\phi_y = N(t, y)$.
g). If the equation is not exact, we can try to find an integrating factor $\mu(t, y)$ such that $\mu M dt + \mu N dy = 0$ becomes exact. Especially if we can find $\mu = \mu(t)$ or $\mu = \mu(y)$ in many cases.
The separable equation is an exact equation. The linear equation is changed to an exact equation before solving. Everything is changed to an exact equation!
h). Applications: Mixing, interest rate, heat transfer.
I). Existence and Uniqueness Theorem: For the general first order equation $y' = f(t, y), y(t_0) = y_0$. If $f(t, y)$ is locally Lipschitz with respect to y, or f_y is continuous, then the solution is unique locally. For any continuous function $f(t, y)$ near (t_0, y_0), a solution exists near t_0.

Second Order equations $ay'' + by' + cy = 0$.

Start from the characteristic equation: $ar^2 + br + c = 0$. Three cases gives three different general solutions. Remember the important Euler's formula: $e^{it} = \cos t + i \sin t$.

i) If there are two distinct real roots $r_1 \neq r_2$, the general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

ii) If the roots are complex $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$, the general solution is $y = e^{\alpha t}[c_1 \cos \beta t + c_2 \sin \beta t]$

iii) If roots are repeated $r_1 = r_2$, the general solution is $y = e^{r_1 t}[c_1 + c_2 t]$.

General linear second order homogeneous equations $y'' + p(t)y' + q(t)y = 0$

i) Wronskian of any two solutions y_1, y_2:
$W(y_1, y_2)(t) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = Ce^{-\int p(t)dt}$.

ii) Two solutions are linearly independent iff $W(y_1, y_2) \neq 0$ iff they form a fundamental set of solutions.

iii) The general solution is given by $c_1 y_1 + c_2 y_2$ if they are linearly independent.

iv) If y_1 is known, then you can find the second linearly independent solution $y_2 = y_1 \int \frac{e^{-\int p(t)dt}}{y_1^2} dt$. Reduction of Order or Wronskian.

Nonhomogeneous Equations $y'' + p(t)y' + q(t)y = g(t)$

• The general solution is $y = y_h + y_p$, y_h is the associated homogeneous solution, y_p is any particular solution.
• Special cases: $p(t) = b, q(t) = c$ are constants, $g(t) = g_1(t) + ... + g_k(t)$, $y_p = Y_1 + ... + Y_k$, each g_j is in the following form:

$g_j(t)$	Y_j
$P_n(t) = a_n t^n + ... + a_0$	$t^s Q_n(t)$
$P_n(t)e^{\lambda t}$	$t^s Q_n(t)e^{\lambda t}$
$P_n(t)e^{\lambda t} \cos \beta t$ (*or* $\sin \beta t$)	$t^s e^{\lambda t}[Q_n^1(t) \cos \beta t + Q_n^2(t) \sin \beta t]$

The s is the smallest nonnegative integer ($s = 0, 1, 2$) that will ensure that no term in y_p is a solution of the corresponding homogeneous solution.
For example, if λ is the repeated root of the characteristic equation $r^2 + br + c = 0$, then s must be 2 in the second line of the table.

Variation of parameters

• If y_1 and y_2 are two linearly independent solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

Then the general solution for non-homogeneous equation $y'' + p(t)y' + q(t)y = g(t)$ is

$$y = y_h + y_p, \quad y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = -\int \frac{y_2 g(t)}{W(y_1, y_2)(t)} dt, \, u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)(t)} dt.$$

Application to Mechanic Systems

- For the single spring system $my'' + \gamma y' + ky = 0$
- $k > 0$ is the spring stiffness, $\gamma \geq 0$, the damping coefficient.
- When $\gamma = 0$, the solution $y = A\cos(\omega_0 t - \delta)$.

$A \geq 0$ is the amplitude, $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency, the period is $\frac{2\pi}{\omega_0}$. $\delta \in [0, 2\pi)$ is called the phase shift.

- When $\gamma > 0$, we have three different cases (over damped, critically damped and under damped), and corresponding solutions from the second order linear equations with constant coefficients. In this case, $y(t) \to 0$ as $t \to \infty$.

Application to Mechanic Systems with external force

For the single spring system $my'' + ky = F_0 \cos \omega t$

- When $\omega \neq \omega_0$, then $y = A\cos(\omega_0 t - \delta) + \frac{F_0}{k - m\omega^2} \cos \omega t$.
- When $\omega = \omega_0$, in this case, a particular solution should be $y_p = t[A \sin \omega_0 t + B \cos \omega_0 t]$.

After some computation, we have $y_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$, which is unbounded ! When $\omega = \omega_0$, we have resonance.

For the system with damping $\gamma > 0$, $my'' + \gamma y' + ky = F_0 \cos \omega t$, $y = y_h + y_p$ where $y_h \to 0$ is called the transient solution, $y_p = A\cos(\omega t - \delta)$ is called the steady state, with amplitude $A = F_0[\gamma^2\omega^2 + (k - m\omega^2)^2]^{-1/2}$. The maximum amplitude is $\frac{F_0}{\gamma\omega_0\sqrt{1 - \frac{\gamma^2}{4mk}}}$ when $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}}$.