PHY 410 - Reference Sheet

Stirling's approximation - for very large N:

$$\log N! \approx N\log N - N$$
$$N! \approx \sqrt{2\pi N}N^N e^{-N}$$

Fractional uncertainty of $\mathbb{X}$ is uncertainty of expected value per particle.

$$\frac{\Delta\mathbb{X}}{N} = \frac{\sqrt{\langle\mathbb{X}^2\rangle - \langle\mathbb{X}\rangle^2}}{N}$$

Boltzmann's constant
$k_B = 1.380649\times10^{-23}\mathrm{m}^2\,\mathrm{s}^{-2}\,\mathrm{K}^{-1}$
Entropy $S = k_B\sigma,\quad \sigma_{TOT}=\sigma_1+\sigma_2$
Temperature $T=\tau/k_B$

Microcanonical Ensemble

Multiplicity function

$$g = \#\text{ of microstates},\quad \mathcal{P}(n)=\frac{1}{g}$$

Expected value of $\mathbb{X}$ is the average across all microstates.

$$\langle\mathbb{X}\rangle = \sum_n \mathbb{X}(n)\mathcal{P}(n) = \frac{1}{g}\sum_n\mathbb{X}(n)$$

Entropy can be written in terms of the multiplicity function.

$$\sigma(N,T,U,V,P)\equiv\log[g(N,T,U,V,P)]$$

Binary System

A **binary system** is a system of N particles where each particles has two possible states with number of particles $N_\uparrow$ and $N_\downarrow$.

$$g(N,N_\uparrow)=\frac{N!}{N_\uparrow!(N-N_\uparrow)!},\quad \sum_{N_\uparrow=0}^{N} g(N,N_\uparrow)=2^N$$

Binary system written with the **spin excess**.

$$2S = N_\uparrow - N_\downarrow$$
$$g(N,S)=\frac{N!}{(\frac{N}{2}+S)!(\frac{N}{2} - S)!}$$
$$\sum_{S=-\frac{N}{2}}^{S=\frac{N}{2}} g(N,N_\uparrow)=2^N$$

Applying Stirling's approximation to the binary model, for large N the multiplicity function and fractional uncertainty are

$$g(N,S)\approx g(N,0)e^{-2s^2/N}$$
$$g(N,S)\approx\sqrt{\frac{2}{\pi N}}2^N e^{-2s^2/N}$$
$$\frac{\Delta S}{N}\approx\frac{1}{\sqrt{N}}$$

An example of a binary system is N spin 1/2 particles in an external **magnetic field** B. The total energy U and magnetization M of the system are

$$U=\sum_{i=1}^N -\vec{m}_i\cdot\vec{B} = -(N_\uparrow-N_\downarrow)mB = -2SmB$$
$$M=2Sm=-U/B$$
$$g(N,U)=\frac{N!}{(\frac{N}{2}-\frac{U}{2mB})!(\frac{N}{2}+\frac{U}{2mB})!}$$
$$\sigma(N,S)\approx -\left(\frac{N}{2}+S\right)\log\left(\frac{1}{2}+\frac{S}{N}\right) - \left(\frac{N}{2} - S\right)\log\left(\frac{1}{2}-\frac{S}{N}\right)$$
$$M=Nm\tanh(mB/\tau)$$

Einstein Solid

An **einstein solid** is a system of N atoms where each atom is modeled as a harmonic oscillator the energy of the system is determined by the number of atoms n oscillating at frequency ω.

$$U=n\hbar\omega$$
$$g(N,n)=\frac{(n+N-1)!}{n!(N-1)!}$$
$$g(N,n)\approx\frac{\left(\frac{n+N}{n}\right)^n\left(\frac{n+N}{n}\right)^N}{\sqrt{2\pi n(n+N)/N}}$$

Thermal Equilibrium

Temperature

$$\frac{1}{\tau}=\left(\frac{\partial\sigma}{\partial U}\right)_{N,V}$$

Thermal Equilibrium

$$\left(\frac{\partial\sigma_1}{\partial U_1}\right)_{N_1,V_1}=\left(\frac{\partial\sigma_2}{\partial U_2}\right)_{N_2,V_2}$$
$$\frac{1}{\tau_1}=\frac{1}{\tau_2}$$

2nd law of thermo - Change in entropy ≥ 0.
Sharpness of Equilibrium For a two binary systems, the number of states in a configuration of deviation δ from equilibirum is

$$g_1g_2=(g_1g_2)_{max}e^{\left(-\frac{2\delta^2}{N_1}-\frac{2\delta^2}{N_2}\right)}$$

Grand Canonical Ensemble

Chemical Potential

$$\mu=\left(\frac{\partial F}{\partial N}\right)_{\tau,V}$$
$$\mu=\tau\log\left(\frac{N\lambda_T^3}{V}\right)=\tau\log\left(\frac{n}{n_Q}\right)$$
$$\mu=\left(\frac{\partial U}{\partial N}\right)_{\sigma,V}=-\tau\left(\frac{\partial\sigma}{\partial N}\right)_{U,V}$$

Grand Partition Function - partition by energy levels for a fixed temperature and all possible values of N

$$\mathfrak{z}=\sum_N\sum_{n(N)} e^{-(\varepsilon_n^N-\mu N)/\tau}$$
$$\mathcal{P}(N,\varepsilon_n)=\frac{1}{\mathfrak{z}}e^{-(\varepsilon_n^N-\mu N)/\tau}$$

Fugacity

$$\lambda=e^{\mu/\tau}$$
$$\mathfrak{z}=\sum_N\lambda^N\sum_{s(N)}e^{-\varepsilon_s^N/\tau}=\sum_N\lambda^N z_N$$

Expected Value of $\mathbb{X}$ is the average across all energies (Diffusive Average).

$$\langle\mathbb{X}\rangle=\frac{1}{\mathfrak{z}}\sum_N\sum_s\mathbb{X}(N,s)e^{-(\varepsilon_s^N-\mu N)/\tau}$$

Expected Number of Particles in the grand canonical ensemble is

$$N=\langle N\rangle=\tau\frac{\partial}{\partial\mu}\log\mathfrak{z}=\lambda\frac{\partial}{\partial\lambda}\log\mathfrak{z}$$

Expected Energy in the grand canonical ensemble is

$$U=\langle\varepsilon\rangle=\frac{1}{\mathfrak{z}}\sum_N\sum_{n(N)}\varepsilon_n^N e^{-(\varepsilon_n^N-\mu N)/\tau}$$
$$U=\langle\varepsilon\rangle=\tau^2\left(\frac{\partial}{\partial\tau}\log\mathfrak{z}\right)_\lambda$$

Concentration and DeBroglie Wavelength

$$n=\frac{N}{V},\quad n_Q=\frac{1}{\lambda_T^3},\quad\lambda_T=\sqrt{\frac{2\pi\hbar^2}{m\tau}}$$

Grand Potential

$$\Omega=U-\sigma\tau-\mu N$$
$$\Omega=-\tau\log\mathfrak{z}$$
$$\sigma=\left(\frac{-\partial\Omega}{\partial\tau}\right)_{V,\mu}P=\left(\frac{-\partial\Omega}{\partial V}\right)_{\tau,\mu}N=\left(\frac{-\partial\Omega}{\partial\mu}\right)_{\tau,V}$$

System of Non-interacting Particles

The grand partition function for a system with M energy states where n_α is the number of particles occupying a state is

$$\mathfrak{z}=\prod_{\alpha=1}^M\mathfrak{z}_\alpha,\quad\mathfrak{z}_\alpha=\sum_{n_\alpha}e^{-n_\alpha(\varepsilon_\alpha-\mu)/\tau}$$
$$U=\sum_{\alpha=1}^M\varepsilon_\alpha f(\varepsilon_\alpha),\quad N=\sum_{\alpha=1}^M f(\varepsilon_\alpha)$$

Fermions

$$n_\alpha=0,1$$
$$\mathfrak{z}_\alpha=1+e^{-(\varepsilon_\alpha-\mu)/\tau}=1+\lambda e^{-\varepsilon_\alpha/\tau}$$

Fermi-Dirac Distribution is the expected number of a particles in a particular energy ε_α.

$$\langle n_\alpha\rangle=f(\varepsilon_\alpha)=\frac{1}{e^{(\varepsilon_\alpha-\mu)/\tau}+1}=\frac{1}{\lambda^{-1}e^{\varepsilon_\alpha/\tau}+1}$$

For $\tau\to0$: $f(\varepsilon_\alpha)=\theta(\varepsilon_\alpha-\mu)$

Bosons (Bonsons)

$$n_\alpha=0,1,2,3,\dots$$
$$\mathfrak{z}_\alpha=\frac{1}{1-e^{-(\varepsilon_\alpha-\mu)/\tau}}=\frac{1}{1-\lambda e^{-\varepsilon_\alpha/\tau}}$$

Boson Distribution is the expected number of a particles in a particular energy ε_α.

$$\langle n_\alpha\rangle=f(\varepsilon_\alpha)=\frac{1}{e^{(\varepsilon_\alpha-\mu)/\tau}-1}=\frac{1}{\lambda^{-1}e^{\varepsilon_\alpha/\tau}-1}$$

Ideal Gas

Both fermions and bosons behave identically at the classical limit $\varepsilon_\alpha-\mu>>\tau$.

$$\langle n_\alpha\rangle=f(\varepsilon_\alpha)=e^{-(\varepsilon_\alpha-\mu)/\tau}$$
$$\mathfrak{z}=\sum_N\lambda^Nz_N=\sum_N\lambda^N\frac{1}{N!}z_1^N=e^{\lambda z_1}$$
$$\lambda=\frac{n}{n_Q},\quad PV=N\tau,\quad U=\frac{3}{2}N\tau,\quad\mu=\tau\log\frac{n}{n_Q}$$
$$\sigma=N\left[\log\frac{n_Q}{n}+\frac{5}{2}\right],\quad F=N\tau\left[\log\frac{n}{n_Q}-1\right]$$

Heat Capacity measures the change in heat energy per unit temperature

$$C_P>C_V,\quad C_V=\left(\frac{\partial U}{\partial T}\right)_V=\tau\left(\frac{\partial\sigma}{\partial T}\right)_V$$
$$C_P=\left(\frac{\partial U}{\partial T}\right)_P+P\left(\frac{\partial V}{\partial T}\right)_P=\tau\left(\frac{\partial\sigma}{\partial T}\right)_P$$

Monoatmc gas $C_V=\frac{3}{2}Nk_B,\quad C_P=\frac{5}{2}Nk_B$
Isothermal Expansion $\sigma_f-\sigma_i=N\log\frac{V_f}{V_i}$
$Q=N\tau\log\frac{V_f}{V_i}$
Isoentropic Expansion $\frac{\tau_f}{\tau_i}=\left(\frac{V_i}{V_f}\right)^{2/3}$

Internal Excitations

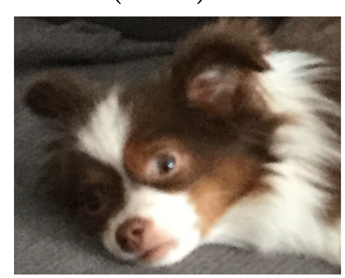
Expansion of the ideal gas to take into account the additional energy states from internal excitations.

$$z_{int}=\sum_\alpha e^{-\varepsilon_\alpha/\tau},\mathfrak{z}=1+\lambda z_{int}e^{-\varepsilon_n/\tau}$$

Internal Excitation Corrections

$$\lambda=\frac{n}{n_Qz_{int}},\mu=\tau\left(\log\frac{n}{n_Q}-\log z_{int}\right)$$
$$F=N\tau\left[\log\frac{n}{n_Q}-1\right]-N\tau\log z_{int}$$
$$\sigma=N\left[\log\frac{n}{n_Q}+\frac{5}{2}\right]-\left(\frac{\partial F_{int}}{\partial\tau}\right)_V$$

DOG (bork)



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