Real Analysis

from the context of the course
MTH 429H: Honors Real Analysis

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Chapter 1

Introduction

1.1 General Notation

$\forall$ - For all
$\exists$ - Exists

1.1.1 Common Sets

$\mathbb{C}$ - Set of all Complex Numbers
$\mathbb{R}$ - Set of all Real Numbers
$\mathbb{Q}$ - Set of all Rational Numbers
$\mathbb{Z}$ - Set of all Integers
$\mathbb{N}$ - Set of all Natural Numbers

1.2 Set Notation

$\in$ - "In" := is an element of

Example. $\vec{v} \in \mathbb{R}^3$

$\notin$ - "Not In" := is not an element of

Example. $\vec{v} \notin \mathbb{R}^3$

$\{,\}$ - Set := elements of the set are listed inside the brackets the elements must be unique.
$\{\}$ *or* $\emptyset$ - The Empty Set

Definition 1.2.1. $|\ |$ - **Cardinality** := The size of a set or the number of elements in a set.

Example. $|A| = n$ "set A has a cardinality of n"

Definition 1.2.2. $\cap$ - **Intersection** := The **Intersection** of two sets is the set of all elements that are contained in both sets.

Example. $A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}$

the **Intersection** of many sets can be denoted: $\cap_{i=1}^{k} A_i$*For the set of elements that appear in all of* $A_1 \cdots A_k$

Definition 1.2.3. $\cup$ - **Union** := The **Union** of two sets is the set of all elements that are contained either of the two sets.

Example. $A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}$

the **Union** of many sets can be denoted: $\cup_{i=1}^{k} A_i$ *For the set of elements that appear in any of* $A_1 \cdots A_k$

Definition 1.2.4. $\subseteq$ - **Subset** := Set A is a **Subset** of B if all the elements of A are also elements of B. We denote this by $A \subseteq B$.

Definition 1.2.5. $\subsetneq$ - **Proper Subset** := Set A is a **Proper Subset** of B if $A \subseteq B$ and $A \neq B$

$\vee$ - or

Example. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$ - and

Example. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

1.3 Review of Set Theory

Definition 1.3.1. Two sets are consider to be equal if $A \subseteq B$ and $A \supseteq B$

Definition 1.3.2. **Pairwise Disjoint** := A set of sets $\Im$ is considered to be **Pairwise Disjoint** *if for* $S, T \in \Im$

$$S \neq T \Rightarrow S \cup T = \emptyset$$

There are two way of taking "differences" of sets:

$$X \backslash Y = \{x \in X : x \notin Y\}$$

$$X \Delta Y = (X \cup Y) \backslash (X \cap Y)$$

Definition 1.3.3. Given a set X and a set $\mathscr{S}$ whose elements are sets.

1. We say that $\mathscr{S}$ **covers** X if $X \subseteq \bigcup \mathscr{S}$
2. We say that $\mathscr{S}$ **partitions** X if $X = \bigcup \mathscr{S}$, the elements of $\mathscr{S}$ are non-empty, and $\mathscr{S}$ is pairwise disjoint

Definition 1.3.4. **Ordered Pair (tuple)** := an ordered list of two elements, each of which can be an arbitrary mathematical object and may or may not be the same. Denoted for $n \in \mathbb{N}$, an n-tuple is an ordered list of n elements, written as $(x_1, \ldots, x_n)$

Definition 1.3.5. For two sets X, Y the **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. More generally we can write a **Cartesian product** for n sets denoted by

$$X_1 \times X_2 \times \ldots \times X_n \; or \; \prod_{i=1}^{n} X_i$$

Remark. When taking the **Cartesian product** of the same set we use the shorthand: X^n

Remark. Additionally, the notation 2^X indicates the set of all possible subsets of X

Definition 1.3.6. We say that the **diagonal** of X^n is the subset $\{(x_1, \ldots, x_n) \in X^n : x_1 = x_2 = \ldots = x_n\}$

Definition 1.3.7. Given two sets X, Y we say that f is a **function** with domain X and codomain Y denoted $f : X \to Y$, if f is a subset of $X \times Y$ such that every element of X appears as exactly the first component of exactly one element of f.

Example. We used the notation $f(x)$ to refer to the element y such that $(x, y) \in f$ is the unique ordered pair that refers to the element $x \in X$.

Definition 1.3.8. The **Identity Function** is a function with the same domain and codomain X written $\mathbf{1}_X : X \to X$ corresponding to the diagonal 1.3.6 of X^2

Definition 1.3.9. Given $f : X \to W$ and $g : W \to Z$ with $Y \subseteq W$, the composition $g \circ f : X \to Z$ is the function satisfying $g \circ f(x) = g(f(x))$.

Definition 1.3.10. A function is **Injective** if $f(x) = f(u) \Rightarrow x = y$

Definition 1.3.11. A function $f : X \to Y$ is **Surjective** if the range of f equals Y

Definition 1.3.12. A function is **Bijective** if it is both Injective and Surjective

Theorem 1.3.1. If X is non-empty, $f : X \to Y$ is injective $\Leftrightarrow$ f is left invertible

Theorem 1.3.2. $f : X \to Y$ is surjective $\Leftrightarrow$ f is right invertible

Definition 1.3.13. A **Relation** of a set X is a subset of X^2. Conventionally written xRy rather than $(x,y) \in R$

1.3.14. Properties of Relation

1. **Reflexive** if xRx for all $x \in X$
2. **Transitive** if xRy and $yRz \Rightarrow xRz$
3. **Symmetric** if $xRy \Leftrightarrow yRx$
4. **Antisymmetric** if xRy and $yRx \Rightarrow x = y$
5. **Connex** if for every $x, y \in X$ at least on of xRy, yRx, or $x = y$ hold.

Definition 1.3.15. An **Equivalence Relation** is a relation that is Reflexive, Transitive, and Symmetric

Definition 1.3.16. if $\sim$ is an equivalence relation, the **Equivalence Class** of $x \in X$ is $[x] := \{y \in X : x \sim y\}$. Additionally, the notation $X/\sim$ refers to the set of all equivalence classes $\{[x] : x \in X\}$

1.4 The sets $\mathbb{Z}$ and $\mathbb{Q}$

Theorem 1.4.1. The natural numbers $\mathbb{N}$ with it's standard addition and multiplication is a **commutative semiring** with the following properties:

1. $(\mathbb{N}, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(\mathbb{N}, \cdot)$ Multiplication is a commutative semigroup (ie. multiplication is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition 1.4.1. The integers $\mathbb{Z}$ is defined as a set of equivalence classes 1.3.16 $\mathbb{N}/\sim$ where the equivalence relation $\sim$ is

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

Remark. This is simply a rigorous way of representing integers with two natural numbers where the first natural number is the positive component and the second number is the negative component.

Theorem 1.4.2. The integers $\mathbb{Z}$ with it's standard addition and multiplication is a **commutative ring** with the following properties:

1. $(\mathbb{Z}, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(\mathbb{Z}, \cdot)$ Multiplication is a commutative semigroup (ie. multiplication is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.

Definition 1.4.2. The rational numbers $\mathbb{Q}$ is defined as a set of equivalence classes 1.3.16 $(\mathbb{Z} \times \mathbb{N})/\sim$ where the equivalence relation $\sim$ is

$$(a, n) \sim (b, m) \Leftrightarrow am = bn$$

Remark. This is simply a rigorous way of representing rational numbers as a fraction from integers and natural numbers where the integer is the numerator and the natural number is the denominator.

Theorem 1.4.3. (Rudin 1.12) The rational numbers $\mathbb{Q}$ with it's standard addition and multiplication is a **field** with the following properties:

1. $(\mathbb{Q}, +)$ Addition is a commutative semigroup (ie. addition is commutative and associative)
2. $(\mathbb{Q}, \cdot)$ Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.
7. Multiplicative inverse.

1.4.1 Cardinality of Sets

Definition 1.4.3. The **cardinality** of a set is the number of elements in that set.

- $card(A) = card(B)$ if there exists and bijective function: $A \to B$
- $card(A) \leq card(B)$ if there exists an injective(left invertible) function: $A \to B$
- $card(A) \geq card(B)$ if there exists an surjective(right invertible) function: $A \to B$

Corollary 1.4.3.1. (Pigeonhole Principle). Suppose $n < m$ there does not exist an injective(left invertible) function: $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ and there does not exist a surjective(right invertible) function: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$

Definition 1.4.4. A set X is said to be

- **countable** if $card(X) \leq card(\mathbb{N})$
- **uncountable** if $card(X) > card(\mathbb{N})$
- **finite** if $\exists\ n \in \mathbb{N}$ such that $card(X) \leq card(\{1, \ldots, n\})$
- **countably infinite** if $card(X) = card(\mathbb{N})$
- **infinite** if $card(X) \geq card(\mathbb{N})$

1.5 Ordered Sets

Definition 1.5.1. (Rudin 1.5) An **order** on a set S is a relation denoted by $<$ that is connex and transitive.

Definition 1.5.2. (Rudin 1.6) An **ordered set** is a set for which an order is defined.

Remark. The sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ if $a < b$ is defined to mean that $b - a$ is positive.

Definition 1.5.3. Let $(X, \leq)$ be an order. Define the two functions $\uparrow, \downarrow: X \to 2^X$ by

- $\downarrow(x) : \{y \in X : y \leq x\}$, a subset is a **lower set** or **downward closed** if $s \in S \Rightarrow \downarrow(s) \subseteq S$.
- $\uparrow(x) : \{y \in X : x \leq y\}$, a subset is an **upper set** or **upper closed** if $s \in S \Rightarrow \uparrow(s) \subseteq S$

Definition 1.5.4. (Rudin 1.7) Let $(X, \leq)$ and be an order and let $S \subseteq X$, and $z \in X$

- We say that z is an **Upper bound** of S if $S \subseteq \downarrow(z)$. The set s is said to be **bounded above** if it has an upper bound.
- We say that z is a **Lower bound** of S if $S \subseteq \uparrow(z)$. The set s is said to be **bounded bellow** if it has a lower bound.
- We say that S is bounded, if it is bounded both above and below.

1.5.1 Special Elements

Definition 1.5.5. Let $(X, \leq)$ be an ordered set, and let $S \subseteq X$. We say that an element of $s_0 \in S$ is...

- **the maximum of** S if $S \subseteq \downarrow(s_0)$
- **the minimum of** S if $S \subseteq \uparrow(s_0)$
- **a maximal element of** S if, for $s \in S$, $s_0 \in \downarrow(s) \Rightarrow s_0 = s$
- **a minimal element of** S if, for $s \in S$, $s_0 \in \uparrow(s) \Rightarrow s_0 = s$

Definition 1.5.6. (Rudin 1.8) Let $(X, \leq)$ be an ordered set and $S \subseteq X$. We say that an element of $x \in X$ is...

- **the supremum of** S if $x = \min\{y \in X : S \subseteq \downarrow(y)\}$
- **the infimum of** S if $x = \max\{y \in X : S \subseteq \uparrow(y)\}$

1.5.2 Dedekind completeness

Definition 1.5.7. Let $(X, \leq)$ be an order. We say a pair of subsets (X_-, X_+) form a **cut** of X if:

- $\{X_-, X_+\}$ is a partition of X.
- X_- is a lower set and X_+ is an upper set.

Definition 1.5.8. An ordered set X is said to be **Dedekind complete** if in every cut (X_-, X_+), at least one of X_- or X_+ is principal. That is X_- has a maximum or X_+ has a minimum.

Definition 1.5.9. (Ruden 1.10) An ordered set $(X, \leq)$ is said to posses the **least upper bound property** if every nonempty subset that is bounded above has a supremum. Similarly, an ordered set $(X, \leq)$ is said to posses the **greatest lower bound property** if every nonempty subset that is bounded bellow has an infimum.

Theorem 1.5.1. (Rudin 1.11) For an ordered set $(X, \leq)$, the following statements are equivalent.

- X is a Dedekind complete
- X has the least upper bound property
- X has the greatest lower bound property

1.6 The Set $\mathbb{R}$

Theorem 1.6.1. (Rudin 1.19) There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property and contains $\mathbb{Q}$ as a subfield.

Theorem 1.6.2. (Rudin 1.20) The following properties hold for the real numbers $\mathbb{R}$:

- **(The Archimedean Property)** If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there exists a positive integer n such that $nx > y$.
- **($\mathbb{Q}$ is dense in $\mathbb{R}$)** If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ suhc that $x < p < y$.

Definition 1.6.1. (Rudin 1.23) The **extended real number system** is the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. Preserving the original ordering in $\mathbb{R}$ we define:

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

1.7 The Set $\mathbb{C}$

Definition 1.7.1. (Rudin 1.24) A **complex number** is an ordered pair (a, b) of real number. Let x, y be complex numbers where $x = (a, b)$, and $y = (c, d)$. Define the following properties:

- $x = y \Leftrightarrow a = c$ and $b = d$
- $x + y = (a + c, b + d)$
- $xy = (ac - bd, ad + bc)$

Under this definition $\mathbb{R}$ is a subfield of $\mathbb{C}$.

Definition 1.7.2. (Rudin 1.27) $i = (0, 1)$

Theorem 1.7.1. (Rudin 1.28) $i^2 = -1$

Proof. $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$ □

Definition 1.7.3. (Rudin 1.30) If a, b are real and $z = a + bi$, then the complex number $\bar{z} = a - bi$ is the **complex conjugate** of z. Additionally, a is the **real** part of z and b is the **imaginary** part of z.

$$a = \text{Re}(z), \quad b = \text{Im}(z)$$

Theorem 1.7.2. (Rudin 1.31) For $z, y \in \mathbb{C}$ we have

- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z} \cdot \bar{w}$
- $z + \bar{z} = 2\text{Re}(z)$, $z - \bar{z} = 2i\text{Im}(z)$

Definition 1.7.4. (Rudin 1.32) The **absolute value** of a complex numbe $z \in \mathbb{C}$ is defined as $|z| = (z\bar{z})^{\frac{1}{2}}$.

Chapter 2

Metric Spaces and Sets

2.1 Metric Spaces

Definition 2.1.1. **(Rudin 2.15)** A **Metric** is a function $d : X \times X \to \mathbb{R}$ for a set X, that satisfies the following properties:

- **Positivity** - The distance $d(x_1, x_2) \geq 0$, for all $x_1, x_2 \in X$.
- **Non-degeneracy** - The distance $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$, for all $x_1, x_2 \in X$.
- **Symmetry** - The distances $d(x_1, x_2) = d(x_2, x_1)$, for all $x_1, x_2 \in X$.
- **Triangle inequality** - Given $x_1, x_2, x_3 \in X$ their mutual distances satisfy

$$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$$

Definition 2.1.2. A **Metric Space** is a set X equipped with a metric.

Definition 2.1.3. **(Rudin 2.17)** Given $(X, \preceq)$ an order and elements $a, b \in X$. The **open interval** (a, b) is defined to be $\uparrow(a) \cap \downarrow(b) \backslash \{a, b\}$

Definition 2.1.4. **(Rudin 2.17)** Given $(X, \preceq)$ an order and elements $a, b \in X$. The **closed interval** (a, b) is defined to be $\uparrow(a) \cap \downarrow(b)$

Definition 2.1.5. **(Rudin 2.17)** Given a metric space (X, d), an **open ball**, centered at $x \in X$ with radius r is the set $B(x, r) = \{y \in X : d(x, y) < r\}$. Similarly, a **closed ball**, centered at $x \in X$ with radius r is a the set $B(x, r) = \{y \in X : d(x, y) < r\}$.

Remark. **(Rudin 2.18)** A open ball centered at x with radius r is also called the **neighborhood** of x, denoted $N_r(x)$.

Definition 2.1.6. **(Rudin 2.18)** Let X be a metric space. An element $x \in X$ is consider to be a **limit point** of a subset $S \subseteq X$ if every neighborhood of x intersects $S \backslash \{x\}$. If $s \in S$ is not a limit point then s is an **isolated** point in S.

Definition 2.1.7. **(Rudin 2.18)** For a metric space X and a subset $S \subseteq X$.

- S is **open** if for every $s \in S$ there is a neighborhood of s that is a subset of S.
- S is **closed** if every limit point of S is in S.
- S is **clopen** if it is open and closed. (ex $\emptyset$)
- S is **perfect** if S is closed and for any $s \in S$, s is a limit point of S.
- S is **bounded** if for some real number M and a point $x \in X$, $S \subseteq B(x, M)$.
- S is **dense** in X if any $x \in X$ is a limit point of S.

Theorem 2.1.1. **(Rudin 2.20)** If x is a limit point of S, then every neighborhood of x has an infinite intersection with S.

Theorem 2.1.2. **(Rudin 2.22)** Let $\{S_\alpha\}$ be a collection of sets S_α. Then

$$(\bigcup_\alpha S_\alpha)^c = \bigcap_\alpha (E_\alpha^c)$$

Theorem 2.1.3. **(Rudin 2.23)** A set S is open $\Leftrightarrow$ the compliment of S is closed.

Definition 2.1.8. **(Rudin 2.26)** For a subset $S \subseteq X$ of a metric space X, the **closure** of S is the union of S and the limit points of S, denoted $\bar{S}$.

2.2 Compact Sets

Definition 2.2.1. (Rudin 2.31) An **open cover** of a subset $S \subseteq X$, where X is a metric space, is a collection of open subsets $\{C_\alpha\}$ such that $S \subseteq \bigcup_\alpha \{C_\alpha\}$.

Definition 2.2.2. (Rudin 2.32) A subset $S \subseteq X$ of a metric space X is **compact** if every open cover of K contains a finite subcover.

Theorem 2.2.1. (Rudin 2.33) Let $K \subseteq Y \subseteq X$. Then K is compact relative to $X \Leftrightarrow K$ is compact relative to Y.

Theorem 2.2.2. (Rudin 2.34) Compact subsets of metric spaces are closed.

Theorem 2.2.3. (Rudin 2.35) Closed subsets of compact sets are compact.

Theorem 2.2.4. (Rudin 2.41) For a subsets S of $\mathbb{R}^k$ the following statements are equivalent.

- S is closed and bounded.
- S is compact.
- Every infinite subset of S has a limit point in S.

Theorem 2.2.5. Rudin 2.42 (Weirstrass) Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

2.3 Perfect Sets

Theorem 2.3.1. (Rudin 2.43) Perfect nonempty subsets of $\mathbb{R}^k$ are uncountable.

Theorem 2.3.2. Any closed interval in $\mathbb{R}^k$ is perfect.

2.4 Connected Sets

Definition 2.4.1. (Rudin 2.45) Subsets A and B of a metric space X are **separated** if $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$, where $\bar{A}$ is the closure of A, and $\bar{B}$ is the closure of B.

Definition 2.4.2. (Rudin 2.45) A subsets S of a metric space X is **connected** if it cannot be partitioned by two nonempty separated sets.

Theorem 2.4.1. A set X is connected if and only if X has no clopen subsets other than X and $\emptyset$.

Chapter 3

Sequences and Series

3.1 Sequences

Definition 3.1.1. For a subset $S \subseteq X$ and a sequence $\{a_n\}$ we say that

- $\{a_n\}$ is **eventually** in S if there exists some $N \in \mathbb{N}$ such that for every $n > N$ we have $a_n \in S$.
- $\{a_n\}$ is **frequently** in S if for any $N \in \mathbb{N}$ the intersection $\{a_{\uparrow(N)}\} \cap S \neq \emptyset$.

3.1.1 Limits

Definition 3.1.2. (Rudin 3.1) A sequence $\{a_n\}$ in a metric space X **converges** to a point $x \in X$ if for every $\epsilon > 0$ there is an integer $N \in \mathbb{N}$ such that for every $n \geq N$, $d(a_n, x) < \epsilon$, denoted:

$$\lim_{n\to\infty} a_n = x$$

Corollary 3.1.0.1. A sequence $\{a_n\}$ converges to a point x if and only if $\{a_n\}$ is eventually in every neighborhood of x.

Theorem 3.1.1. (Rudin 3.2) Let $\{a_n\}$ be a sequence in a metric space X.

- $\{a_n\}$ converges to $a \in X \Leftrightarrow$ every neighborhood of a contains a_n for all but finitely many n.
- $\{a_n\}$ converges to a and b then $a = b$.
- $\{a_n\}$ converges $\rightarrow \{a_n\}$ is bounded.
- If $E \subseteq X$ and E has a limit point a, then there is a sequence $\{a_n\}$ in E such that $\lim a_n = a$.

Theorem 3.1.2. (Rudin 3.3) Suppose $\{a_n\}$ and $\{b_n\}$ are complex sequences and $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$

- $\lim_{n\to\infty} a_n + b_n = a + b$.
- $\lim_{n\to\infty} a_n + c = a + c,\ \lim_{n\to\infty} ca_n = ca$, for any number c.
- $\lim_{n\to\infty} a_n b_n = ab$.
- $\lim_{n\to\infty} frac1a_n = \frac{1}{a}$ provided that $0 \notin \{a_n\}$ and $a \neq 0$.

3.1.2 Subsequences

Definition 3.1.3. (Rudin 3.5) Given a sequence $\{a_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_k < n_k + 1$ for all $k \in \mathbb{N}$. The sequence $\{a_{n_i}\}$ is a **subsequence** of $\{a_n\}$. If $\{a_{n_i}\}$ converges, its limit is called an **accumulation** point of $\{a_n\}$.

Corollary 3.1.2.1. A point $x \in X$ is an accumulation point of a sequence $\{a_n\}$ if and only if $\{a_n\}$ is frequently in every neighborhood of x.

Theorem 3.1.3. A sequence $\{a_n\}$ converges to a point a if and only if every subsequence of $\{a_n\}$ converges to a.

Theorem 3.1.4. (Rudin 3.6) (Weierstrass) If $\{a_n\}$ is a sequence in a compact metric space X, then $\{a_n\}$ has an accumulation point in X.

Theorem 3.1.5. (Rudin 3.7) The set of all accumulation points of a sequence in a metric space is a closed set.

3.1.3 Cauchy Sequences

Definition 3.1.4. (Rudin 3.8) A sequence $\{a_n\}$ is a **Cauchy sequence** if for every $\epsilon > 0$ there is an integer N such that $d(a_n, a_m) < \epsilon$ if $n > N$ and $m > N$

Definition 3.1.5. (Rudin 3.9) Let E be a nonempty subset of a metric space. The **diameter** of E is $\sup S$ where S is the set of all distances $d(a, b)$ between two points $a, b \in E$.

Theorem 3.1.6. (Rudin 3.11)

- Every convergent sequence is Cauchy.
- A Cauchy sequence taking points in a compact metric space must converge to a point in that metric space.
- In $\mathbb{R}^n$ every Cauchy sequence converges.

Theorem 3.1.7. (Rudin 3.12) A metric space in which every Cacuhy sequences converges is Dedekind complete.

Definition 3.1.6. (Rudin 3.13) Let $\{a_n\}$ be a sequence of real numbers.

- **monotonically increasing** if $a_n \leq a_{n+1}$.
- **monotonically decreasing** if $a_n \geq a_{n+1}$.
- **monotonic** if $\{a_n\}$ is either monotonically increasing or decreasing.

Theorem 3.1.8. (Rudin 3.14) Let $\{a_n\}$ be a monotonic series. $\{a_n\}$ converges if and only if $\{a_n\}$ is bounded.

3.1.4 Upper and Lower Limits

Definition 3.1.7. (Rudin 3.15) Let $\{a_n\}$ be a sequence and M be any real number.

- $a_n \to +\infty$ if and only if $\exists N$ such that $n > N \Rightarrow a_n > M$.
- $a_n \to -\infty$ if and only if $\exists N$ such that $n > N \Rightarrow a_n < M$.

Definition 3.1.8. (Rudin 3.16) Let $\{s_n\}$ be a sequence of real numbers. Let S be the set of all accumulation points of $\{s_n\}$. Use our definition of supremum and infimum to define **limit superior** and **limit inferior**:

$$\limsup s_n := \sup S$$

$$\liminf s_n := \inf S$$

Theorem 3.1.9. Alternate Definition and proof of equivalence.

3.2 Series

Definition 3.2.1. (Rudin 3.22) $\sum a_n$ converges if and only if for every $\epsilon > 0$ there is an integer N such that $m \geq n \geq N$ implies that

$$\left| \sum_{k=n}^{m} \right|$$

3.2.1 Definition of e

3.2.2 The Root Test

3.2.3 The Ratio Test

3.2.4 Alternating Series

Theorem 3.2.1. (Rudin 3.43) If a series $\sum a_n$ has the following properties

- $|a_1| \geq |a_2| \geq |a_3| \geq \ldots$
- $a_{2m-1} \geq 0$, $a_{2m} \leq 0$ $(m = 1, 2, 3, \ldots)$
- $\lim_{n\to\infty} c_n = 0$

then $\sum a_n$ converges.

3.2.5 Power Series

3.2.6 Absolute Convergence

3.2.7 Enumerations

Chapter 4

Continuity

4.1 Continuous Functions

Definition 4.1.1. (Rudin 4.5) Suppose X and Y are metric spaces, $E \subset X$, a function $f : E \to Y$ is **continuous** at a point $a \in E$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(a), f(b)) < \epsilon$$

for all $b \in E$ such that $d(a, b) < \delta$.

Theorem 4.1.1. (Rudin 4.9) If f and g are continuous functions on a metric space X, then $f+g$, fg, and $\frac{f}{g}$ are continuous on X.

4.1.1 Uniform Continuity

Definition 4.1.2. (Rudin 4.18) Let f be a function $f : X \to Y$ where X and Y are metric spaces. f is **uniformly continuous** if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(a), f(b)) < \epsilon$ for any $a, b \in X$ such that $d_X(a, b) < \delta$.

Theorem 4.1.2. (Rudin 4.19) If f is a continuous function from a compact metric space X to a metric space Y, then f is uniformly continuous.

4.1.2 Continuity and Compactness

Theorem 4.1.3. (Rudin 4.16) Suppose f is a continuous real valued function on a compact metric space X with

$$M = \sup_{p \in X} f(p),\ m = \inf_{p \in X} f(p)$$

There exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

4.1.3 Continuity and Connectedness

4.2 Discontinuities

4.3 Monotonic Functions

Chapter 5

Differentiation

5.1 Mean Value Theorems

Theorem 5.1.1. (Rudin 5.8) Let f be a function $f : [a, b] \to \mathbb{R}$ with a local maximum at a point $x \in (a, b)$. If $f'(x)$ exists, then $f'(x) = 0$.

Theorem 5.1.2. (Rudin 5.10) Let $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b), then there exists a point $x \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(x)$$

Theorem 5.1.3. (Rudin 5.19) Let $f : [a, b] \to \mathbb{R}^n$ be a continuous function that is differentiable on (a, b). There exists a point $x \in (a, b)$ such that

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a)|\mathbf{f}'(x)|$$

Chapter 6

The Riemann-Stieltjes Integral

Chapter 7

Sequences and Series of Functions

Definition 7.0.1. (Rudin 7.1) A sequence of functions $\{f_n\}$ on a set E is **pointwise convergent** if $\{f_n(x)\}$ converges for every $x \in E$. In this case we define the following function:

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Definition 7.0.2. (Rudin 7.1) A series of functions $\sum f_n$ on a set E is **pointwise convergent** if $\sum f_n(x)$ converges for every $x \in E$. In this case we define the following function:

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

7.1 Uniform Convergence

Definition 7.1.1. (Rudin 7.7) A sequence of functions $\{f_n\}$ on a set E **converges uniformly** if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $x \in E$.

Definition 7.1.2. (Rudin 7.7) A series of functions $\sum f_n$ on a set E **converges uniformly** if for every the sequence of partial sums

$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$

converges uniformly on E.

Theorem 7.1.1. (Rudin 7.8) The sequence of functions $\{f_n\}$ converges uniformly on E if and only if for every $\epsilon > 0$ there exists and integer N such that $n \geq N$, $m \geq N$ implies

$$|f_n(x) - f_m(x)| \leq \epsilon$$

for all $x \in E$.

Theorem 7.1.2. (Rudin 7.10) Let $\{f_n\}$ be a sequence of functions on the set E. If there exists a convergent series $\sum M_n$ such that

$$|f_n(x)| \leq M_n$$

for all $x \in E$. Then $\sum f_n$ converges uniformly on E.

7.1.1 Uniform Convergence and Continuity

Theorem 7.1.3. (Rudin 7.11) Let $\{f_n\}$ converge uniformly to f on a set E and x be a limit point of E.

$$\lim_{t\to x} f(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

Theorem 7.1.4. (Rudin 7.12) If a sequence of continuous functions $\{f_n\}$ converges uniformly to f then f is continuous on E.

7.1.2 Uniform Convergence and Integration

7.1.3 Uniform Convergence and Differentiation

7.2 Equicontinuous Functions

Definition 7.2.1. (Rudin 7.19) A sequence of functions $\{f_n\}$ on a set E is **pointwise bounded** if every sequence $\{f_n(x)\}$ is bounded for every $x \in E$. That is there exists a function ϕ on E such that

$$|f_n(x)| < \phi(x)$$

Definition 7.2.2. (Rudin 7.19) A sequence of functions $\{f_n\}$ on a set E is **uniformly bounded** if there exists a number M such that

$$|f_n(x)| < M$$

Definition 7.2.3. (Rudin 7.22) A family (or sequence) of functions $\mathscr{F}$ is said to be **equicontinuous** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta$ for all $f \in \mathscr{F}$.

7.3 The Stone Weierstrass Theorem

Chapter 8

Some Special Functions

Chapter 9

Multivariable Functions

9.1 Vector Spaces

Definition 9.1.1. (Rudin 1.36) Let $\mathbb{R}^n$ be the set of all ordered n-tuples, where $n \in \mathbb{N}$. We define the following for elements in $\mathbb{R}^n$.

- For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
- For $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$
- For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the **inner product** of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

- For $\mathbf{x} \in \mathbb{R}^n$ the **norm** of $\mathbf{x}$ is defined by

$$|x| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}}$$

Theorem 9.1.1. (Rudin 1.37) Let $x, y, z \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the following are always true:

- $|\mathbf{x}| \geq 0$
- $|\mathbf{x}| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
- $|\alpha\mathbf{x}| = |\alpha||\mathbf{x}|$
- $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$
- $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$
- $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$

Definition 9.1.2. A **vector space** is a set V equipped with operations of addition and scalar multiplication with the following properties

1. Commutativity: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$
2. Associativity: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$
3. Zero Vector: $\exists\ \mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$
4. Additive Inverse: For any $\mathbf{x} \in V$ there exists $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$
5. Multiplicative Identity: $(1)\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$
6. $x + y \in V$ and $\alpha\mathbf{x} \in V$ for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$

Theorem 9.1.2. (Rudin 9.1) Any nonempty subset $X \subseteq \mathbb{R}^n$ is a vector space if and only if $x + y \in X$ and $\alpha\mathbf{x} \in X$ for all $\mathbf{x}, \mathbf{y} \in X$ and $\alpha \in \mathbb{R}$.

Definition 9.1.3. (Rudin 9.1) For a set of vectors $\{\mathbf{x_1}, \cdots, \mathbf{x_n}\}$ and scalars $\{\alpha_1, \cdots, \alpha_n\}$ a **linear combination** is defined by

$$\sum_{i=1}^{n} \alpha_i \mathbf{x_i}$$

9.2 Differentiation

Definition 9.2.1. (Rudin 9.11) A function $\mathbf{f} : (a, b) \to \mathbb{R}^m$ is **diffentiable** at $x \in (a, b)$ with derivative $f'(x)$ if

$$\lim_{t \to x} \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = 0$$

Definition 9.2.2. Suppose $E \in \mathbb{R}^n$, $f : E \to \mathbb{R}^m$ $x \in E$ if $\exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{|f(x + h) - f(x) - Ah|}{|h|} = 0$$

then the that the derivative of f at x is

$$f'(x) = A$$

Theorem 9.2.1. (Rudin 9.17) Suppose $\mathbf{f}$ maps an open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ and $\mathbf{f}$ is differentiable at a point $\mathbf{x} \in E$. Then the partial derivatives $(D_j f_i)(\mathbf{x})$ exist, and

$$\mathbf{f}'(\mathbf{x}) = \sum_{i=1}^{m} (D_j f_i)(\mathbf{x})\mathbf{e}_i$$

Theorem 9.2.2. (Chain Rule) Let $E \in \mathbb{R}^n$ be a vector space, $f : E \to \mathbb{R}^m$, and $g : f(E) \to \mathbb{R}^k$. Suppose for $x \in E$ f is differentiable at x and g is differentiable at $f(x)$ are differentiable with derivatives $f'(x)$ and $g'(f(x))$. The derivative of the composition $f \circ g$ is

$$F'(x) = g'(f(x))f'(x)$$

Theorem 9.2.3. (Mean Value Theorem for multivariable functions) Let $f : [a, b] \to \mathbb{R}^m$ be differentiable and continuous on (a, b), Then $\exists$ $x \in (a, b)$ such that

$$|f(b) - f(a)| \leq |f'(x)|(b - a)$$