Math 347H Lecture: System of First Order Equations

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Homework: Section 7.1. 3, 5, 7, 11, 23 .

Many practical problems involves on several dependent variables.

Example 1. Two-mass, three spring system in Figure 7.1.1. Masses have m_1 and m_2. Spring constants are k_1, k_2 and k_3 from left to right.

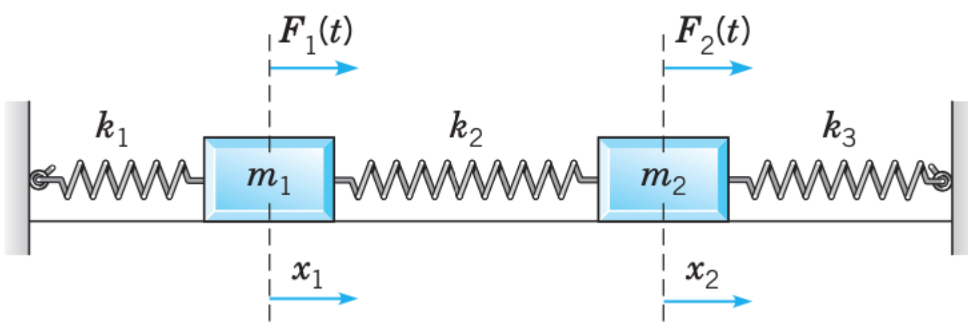

FIGURE 7.1.1 A two-mass, three-spring system.

$$
\begin{aligned}
m_1 x_1'' &= k_2(x_2 - x_1) - k_1 x_1 + F_1(t) \\
m_2 x_2'' &= -k_2(x_2 - x_1) - k_3 x_2 + F_2(t)
\end{aligned}
$$

How does the system move?

Any periodic solutions? What are periods?

Any synchronized movements?

Example 2. Connected tanks Problem.

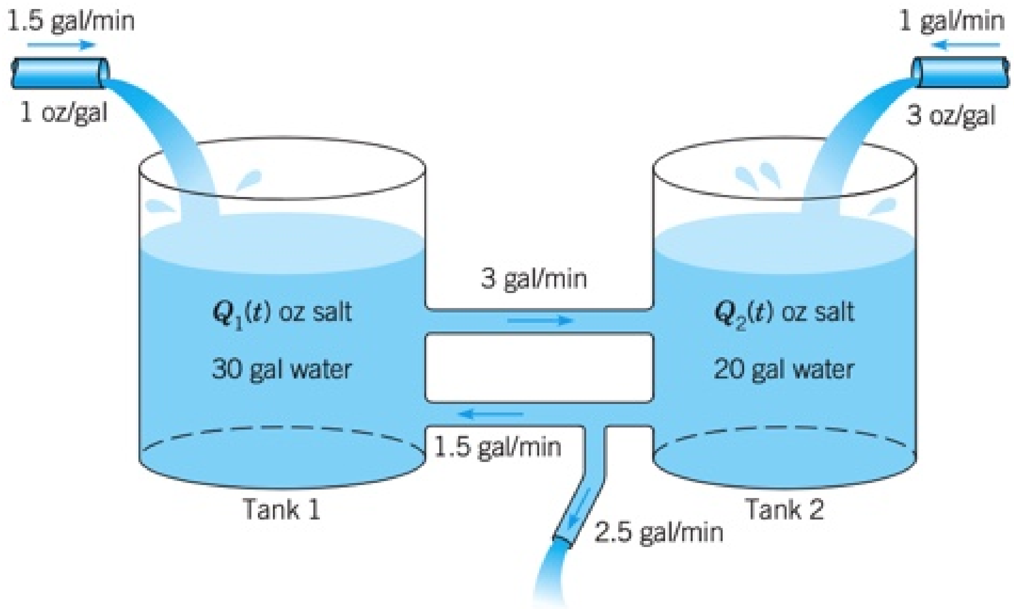

Figure 7.1.2: Two interconnectd tanks

At $t = 0$, Tank 1 contains 30 gal of water and 25 oz of salt, and Tank 2 contains 20 gal of water and 15 oz of salt.

(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equation and initial conditions that model the flow process.

$$\begin{cases} Q_1' &= 1.5 - \frac{1}{10}Q_1 + \frac{3}{40}Q_2, & Q_1(0) = 25, \\ Q_2' &= 3 + \frac{1}{10}Q_1 - \frac{1}{5}Q_2, & Q_2(0) = 15. \end{cases}.$$

(b) Find the value of $Q_1 = Q_1^E$ and $Q_2 = Q_2^E$ for which the system is in equilibrium. That is, they don't depend on time. We expect or hope that the solutions will tend to these equilibrium state. Can you guess on those values?

This can be done by solving

$$1.5 - \frac{1}{10}Q_1 + \frac{3}{40}Q_2. = 0, \quad 3 + \frac{1}{10}Q_1 - \frac{1}{5}Q_2 = 0.$$

The answer is $Q_1^E = 42, \;\; Q_2^E = 36.$

Will the solution tend to these equilibrium state? If yes, how fast? We will study these questions later.

Any second order equation can be written as an equivalent system of two equations.

Any higher order equation can be written as an equivalent first order system.

The system of first order equations is called linear if $X' = A(t)X + F(t)$.

Example 3. The motion of a spring mass system is described by the second order differential equation

$$u'' + 0.2u' + 2u = 0, \;\; u(0) = 1, \;\; u'(0) = 2.$$

Rewrite this equation as an equivalent system of first order equations.

Let $x_1 = u, x_2 = u'$. Then it follows that $x_1' = u' = x_2$. $x_2' = u'' = -0.2u' - 2u = -0.2x_2 - 2x_1$.

Then we have the following equivalent system

$$\begin{cases} x_1' = x_2, & x_1(0) = 1, \\ x_2' = -2x_1 - 0.2x_2, & x_2(0) = 2. \end{cases}$$

If we use the matrix notation, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -2 & -0.2 \end{pmatrix}$, we have

$$X' = AX, \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

In general, we can transform an arbitrary nth order differential equation

$$y^{(n)} = F(t, y, y', ..., y^{(n-1)})$$

into a system of n first order equations, we extend the method of Example 3 by introducing the new variables

$$x_1 = y,\ x_2 = y',\ x_3 = y'',\ ...,\ x_n = y^{(n-1)}.$$

It has the following equivalent system of n equations

$$\begin{cases} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= F(t, x_1, ..., x_{n-1}). \end{cases}$$

Note that any system of differential equations can be rewritten as an equivalent system of first order equations $X' = F(t, X)$ with $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. $F(t, X) = \begin{pmatrix} f_1(t, X) \\ \vdots \\ f_n(t, X) \end{pmatrix}$

Example 4. Some systems can be solved from higher order equation.

$$x_1' = 3x_1 - 2x_2,\ x_2' = 2x_1 - 2x_2,\ x_1(0) = 3, x_2(0) = 3.$$

Solve the first equation for $x_2 = (3x_1 - x_1')/2$. Plug into the second equation to get

$$\frac{1}{2}[(3x_1' - x_1''] = 2x_1 - (3x_1 - x_1').$$

$$x_1'' - x_1' - 2x_1 = 0.$$

The associated characteristic equation is $r^2 - r - 2 = (r + 1)(r - 2) = 0$, $r_1 = -1$, $r_2 = 2$. We know that

$$x_1 = c_1 e^{-t} + c_2 e^{2t},$$

$$x_2 = \frac{1}{2}(3x_1 - x_1') = \frac{1}{2}[3x_1 - x_1'] = \frac{1}{2}[3c_1e^{-t} + 3c_2e^{2t} - (-c_1e^{-t} + 2c_2e^{2t})] = \frac{1}{2}[4c_1e^{-t} + c_2e^{2t}].$$

Using the initial data. $x_1(0) = 3$, and $x_2(0) = 3$, we can determine the coefficients c_1 and c_2.

$$c_1 + c_2 = 3,\ \frac{1}{2}[4c_1 + c_2] = 3, \quad c_1 = 1, \quad c_2 = 2.$$

We have

$$x_1 = e^{-t} + 2e^{2t}, \quad x_2(t) = 2e^{-t} + e^{2t}.$$

It is a bit complicated. We hope to find a better way to solve the linear system, which needs Linear Algebra.