Math Reference

Kaedon Cleland-Host

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Contents

Chapter 1

Introduction

This chapter will offer reference and information that applies to the entire book.

1.1 Structure of This Book

1.1.1 Categories

Each section of this book will focus on one of these general categories

- **Notation -** The way that we choose to represent mathematics as is written down, each topic will have a notation page with symbol definitions and other important information
- **Number Systems -** Representations of a numbers and fundamental operations that we can run on these numbers (i.e. numbers, vectors, counting, complex numbers)
- **Structures -** Ways to organize numbers operations and units to represent something or to indicate something (i.e. equations, logical statements, foundation of proofs)
- **Methods -** Strategies for going between structures and representations of real things (i.e. integrals, derivatives, trigonometry, rref)

Chapter 2

Algebra

Chapter 3

Statistics

Chapter 4

Linear Algebra

4.1 Notation

General
$\forall$ - For all
$\exists$ - Exists

Common Sets
$\mathbb{C}$ - Set of all Complex Numbers
$\mathbb{R}$ - Set of all Real Numbers
$\mathbb{Q}$ - Set of all Rational Numbers
$\mathbb{Z}$ - Set of all Integers
$\mathbb{N}$ - Set of all Natural Numbers

Set Notation
$\in$ - "In" := is an element of

Example. $\vec{v} \in \mathbb{R}^3$

$\notin$ - "Not In" := is not an element of

Example. $\vec{v} \notin \mathbb{R}^3$

$\{,\}$ - Set := elements of the set are listed inside the brackets
Example: $A = \{1, 2, 3\}$ "A is a set containing the elements 1, 2, and 3"
Note: elements in a set must be unique
$\{\}$ *or* $\emptyset$ - The Empty Set

Definition 4.1.1. $|\,|$ - **Cardinality** := The size of a set or the number of elements in a set.

Example. $|A| = n$ "set A has a cardinality of n"

Definition 4.1.2. $\cap$ - **Intersection** := The **Intersection** of two sets in the set of all elements that are contained in both sets.

Example. $A \cap B = \{x : (x \in A) \textit{ and } (x \in B)\}$

Definition 4.1.3. $\cup$ - **Union** := The **Union** of two sets in the set of all elements that are contained either of the two sets.

Example. $A \cup B = \{x : (x \in A) \textit{ or } (x \in B)\}$

$\vee$ - or

Example. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$ - and

Example. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

4.2 Vectors and Bases

Definition 4.2.1. Vector Space := a collection of vectors equiped with operations of addition and scalar multiplication such that the following axioms are true:

- Commutativity: $\vec{v} + \vec{w} = \vec{w} + \vec{v} \; \forall \; \vec{v}, \vec{w} \in V$
- Associativity: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{w} + \vec{v}) \; \forall \; \vec{u}, \vec{v}, \vec{w} \in V$
- Zero Vector: $\exists$ a vector $\vec{0}$ such that for any vector $\vec{v} \in V, \; \vec{v} + \vec{0} = \vec{v}$
- Additive Inverse: for any vector $\vec{v} \in V$ there exists a vector $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$
- Multiplicative Identity: for any vector $\vec{v} \in V, \; (1)\vec{v} = \vec{v}$
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Chapter 5

Real Analysis

5.1 Notation

General
$\forall$ - For all
$\exists$ - Exists

Common Sets
$\mathbb{C}$ - Set of all Complex Numbers
$\mathbb{R}$ - Set of all Real Numbers
$\mathbb{Q}$ - Set of all Rational Numbers
$\mathbb{Z}$ - Set of all Integers
$\mathbb{N}$ - Set of all Natural Numbers

Set Notation
$\in$ - "In" := is an element of

Example. $\vec{v} \in \mathbb{R}^3$

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$\{,\}$ - Set := elements of the set are listed inside the brackets
Example: $A = \{1, 2, 3\}$ "A is a set containing the elements 1, 2, and 3"
Note: elements in a set must be unique
$\{\}$ *or* $\emptyset$ - The Empty Set

Definition 5.1.1. $|\,|$ - **Cardinality** := The size of a set or the number of elements in a set.

Example. $|A| = n$ "set A has a cardinality of n"

Definition 5.1.2. $\cap$ - **Intersection** := The **Intersection** of two sets is the set of all elements that are contained in both sets.

Example. $A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}$

the **Intersection** of many sets can be denoted: $\cap_{i=1}^{k} A_i$*For the set of elements that appear in all of* $A_1 \cdots A_k$

Definition 5.1.3. $\cup$ - **Union** := The **Union** of two sets is the set of all elements that are contained either of the two sets.

Example. $A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}$

the **Union** of many sets can be denoted: $\cup_{i=1}^{k} A_i$ *For the set of elements that appear in any of* $A_1 \cdots A_k$

Definition 5.1.4. $\subseteq$ - **Subset** := Set A is a **Subset** of B if all the elements of A are also elements of B. We denote this by $A \subseteq B$.

Definition 5.1.5. $\subsetneq$ - **Proper Subset** := Set A is a **Proper Subset** of B if $A \subseteq B$ and $A \neq B$

$\vee$ - or

Example. $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$ - and

Example. $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

5.2 Week 1

5.2.1 Review of Set Theory

Definition 5.2.1. Two sets are consider to be equal if $A \subseteq B$ and $A \supseteq B$

Definition 5.2.2. **Pairwise Disjoint** := A set of sets $\Im$ is considered to be **Pairwise Disjoint** *if for* $S, T \in \Im$

$$S \neq T \Rightarrow S \cup T = \emptyset$$

There are two way of taking "differences" of sets:

$$X \backslash Y = \{x \in X : x \notin Y\}$$

$$X \Delta Y = (X \cup Y) \backslash (X \cap Y)$$

Proof. For any three finite sets X, Y, Z:
From the definition of Δ we find that:

$$(X \Delta Y)\Delta Z = \{x \in (X \cup Y) : x \notin (X \cap Y)\}\Delta Z = \{x \in ((X \cup Y) \cup Z) : x \notin ((X \cap Y) \cap Z)\}$$

Now, since $\cup$ and $\cap$ are associative we have:

$$\{x \in ((X \cup Y) \cup Z) : x \notin ((X \cap Y) \cap Z)\} = \{x \in (X \cup (Y \cup Z)) : x \notin (X \cap (Y \cap Z))\}$$

Now, from the definition of Δ we find that:

$$\{x \in (X \cup (Y \cup Z)) : x \notin (X \cap (Y \cap Z))\} = X \Delta \{x \in (Y \cup Z) : x \notin (Y \cap Z)\} = X \Delta (Y \Delta Z)$$

Therefore:

$$(X \Delta Y)\Delta Z = X \Delta (Y \Delta Z)$$

$\square$

Definition 5.2.3. Given a set X and a set $\mathscr{S}$ whose elements are sets.

1. We say that $\mathscr{S}$ **covers** X if $X \subseteq \bigcup \mathscr{S}$
2. We say that $\mathscr{S}$ **partitions** X if $X = \bigcup \mathscr{S}$, the elements of $\mathscr{S}$ are non-empty, and $\mathscr{S}$ is pairwise disjoint

Definition 5.2.4. **Ordered Pair (tuple)** := an ordered list of two elements, each of which can be an arbitrary mathematical object and may or may not be the same. Denoted for $n \in \mathbb{N}$, an n-tuple is an ordered list of n elements, written as $(x_1, \ldots, x_n)$

Definition 5.2.5. For two sets X, Y the **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. More generally we can write a **Cartesian product** for n sets denoted by

$$X_1 \times X_2 \times \ldots \times X_n \text{ or } \prod_{i=1}^{n} X_i$$

Remark. When taking the **Cartesian product** of the same set we use the shorthand: X^n

Remark. Additionally, the notation 2^X indicates the set of all possible subsets of X

Definition 5.2.6. We say that the **diagonal** of X^n is the subset $\{(x_1, \ldots, x_n) \in X^n : x_1 = x_2 = \ldots = x_n\}$

Definition 5.2.7. Given two sets X, Y we say that f is a **function** with domain X and codomain Y denoted $f : X \to Y$, if f is a subset of $X \times Y$ such that every element of X appears as exactly the first component of exactly one element of f.

Example. We used the notation $f(x)$ to refer to the element y such that $(x, y) \in f$ is the unique ordered pair that refers to the element $x \in X$.

Definition 5.2.8. The **Identity Function** is a function with the same domain and codomain X written $\mathbf{1}_X : X \to X$ corresponding to the diagonal /refdef:functionaldiagonal of X^2

Definition 5.2.9. Given $f : X \to W$ and $g : W \to Z$ with $Y \subseteq W$, the composition $g \circ f : X \to Z$ is the function satisfying $g \circ f(x) = g(f(x))$.

Definition 5.2.10. A function is **Injective** if $f(x) = f(u) \Rightarrow x = y$

Definition 5.2.11. A function $f : X \to Y$ is **Surjective** if the range of f equals Y

Definition 5.2.12. A function is **Bijective** if it is both Injective and Surjective

Theorem 5.2.1. If X is non-empty, $f : X \to Y$ is injective $\Leftrightarrow$ f is left invertible

Theorem 5.2.2. $f : X \to Y$ is surjective $\Leftrightarrow$ f is right invertible

Definition 5.2.13. A **Relation** of a set X is a subset of X^2. Conventionally written xRy rather than $(x, y) \in R$

5.2.14. Properties of Relation

1. **Transitive** if xRy and $yRz \Rightarrow xRz$
2. **Symmetric** if $xRy \Leftrightarrow yRx$
3. **Antisymmetric** if xRy and $yRx \Leftarrow x = y$
4. **Connex** if for every $x, y \in X$ at least on of xRy or yRx hold.
5. **Reflexive** if xRx for all $x \in X$

Definition 5.2.15. An **Equivalence Relation** is a relation that is Reflexive, Transitive, and Symmetric

Chapter 6

Calculus

Chapter 7

Multivariable Calculus

Chapter 8

Ordinary Differential Equations

Chapter 9

Partial Differential Equations

Chapter 10

Analysis