

Real Analysis  
from the context of the course  
MTH 429H: Honors Real Analysis

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# Chapter 1

## Introduction

### 1.1 General Notation

$\forall$  - For all

$\exists$  - Exists

#### 1.1.1 Common Sets

$\mathbb{C}$  - Set of all Complex Numbers

$\mathbb{R}$  - Set of all Real Numbers

$\mathbb{Q}$  - Set of all Rational Numbers

$\mathbb{Z}$  - Set of all Integers

$\mathbb{N}$  - Set of all Natural Numbers

### 1.2 Set Notation

$\in$  - "In" := is an element of

*Example.*  $\vec{v} \in \mathbb{R}^3$

$\notin$  - "Not In" := is not an element of

*Example.*  $\vec{v} \notin \mathbb{R}^3$

$\{, \}$  - Set := elements of the set are listed inside the brackets

Example:  $A = \{1, 2, 3\}$  "A is a set containing the elements 1, 2, and 3"

Note: elements in a set must be unique

$\{\}$  or  $\emptyset$  - The Empty Set

**Definition 1.2.1.**  $||$  - **Cardinality** := The size of a set or the number of elements in a set.

*Example.*  $|A| = n$  "set A has a cardinality of n"

**Definition 1.2.2.**  $\cap$  - **Intersection** := The **Intersection** of two sets is the set of all elements that are contained in both sets.

*Example.*  $A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}$

the **Intersection** of many sets can be denoted:  $\cap_{i=1}^k A_i$  For the set of elements that appear in all of  $A_1 \cdots A_k$

**Definition 1.2.3.**  $\cup$  - **Union** := The **Union** of two sets is the set of all elements that are contained either of the two sets.

*Example.*  $A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}$

the **Union** of many sets can be denoted:  $\cup_{i=1}^k A_i$  For the set of elements that appear in any of  $A_1 \cdots A_k$

**Definition 1.2.4.**  $\subseteq$  - **Subset** := Set A is a **Subset** of B if all the elements of A are also elements of B. We denote this by  $A \subseteq B$ .

**Definition 1.2.5.**  $\subsetneq$  - **Proper Subset** := Set A is a **Proper Subset** of B if  $A \subseteq B$  and  $A \neq B$

$\vee$  - or

*Example.*  $A \cup B = \{x : (x \in A) \vee (x \in B)\}$

$\wedge$  - and

*Example.*  $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$

### 1.3 Review of Set Theory

**Definition 1.3.1.** Two sets are considered to be equal if  $A \subseteq B$  and  $A \supseteq B$

**Definition 1.3.2. Pairwise Disjoint** := A set of sets  $\mathfrak{S}$  is considered to be **Pairwise Disjoint** if for  $S, T \in \mathfrak{S}$

$$S \neq T \Rightarrow S \cap T = \emptyset$$

There are two ways of taking "differences" of sets:

$$X \setminus Y = \{x \in X : x \notin Y\}$$

$$X \Delta Y = (X \cup Y) \setminus (X \cap Y)$$

**Definition 1.3.3.** Given a set  $X$  and a set  $\mathcal{S}$  whose elements are sets.

1. We say that  $\mathcal{S}$  **covers**  $X$  if  $X \subseteq \bigcup \mathcal{S}$
2. We say that  $\mathcal{S}$  **partitions**  $X$  if  $X = \bigcup \mathcal{S}$ , the elements of  $\mathcal{S}$  are non-empty, and  $\mathcal{S}$  is pairwise disjoint

**Definition 1.3.4. Ordered Pair (tuple)** := an ordered list of two elements, each of which can be an arbitrary mathematical object and may or may not be the same. Denoted for  $n \in \mathbb{N}$ , an  $n$ -tuple is an ordered list of  $n$  elements, written as  $(x_1, \dots, x_n)$

**Definition 1.3.5.** For two sets  $X, Y$  the **Cartesian product**  $X \times Y$  is the set of all ordered pairs  $(x, y)$  with  $x \in X$  and  $y \in Y$ . More generally we can write a **Cartesian product** for  $n$  sets denoted by

$$X_1 \times X_2 \times \dots \times X_n \text{ or } \prod_{i=1}^n X_i$$

*Remark.* When taking the **Cartesian product** of the same set we use the shorthand:  $X^n$

*Remark.* Additionally, the notation  $2^X$  indicates the set of all possible subsets of  $X$

**Definition 1.3.6.** We say that the **diagonal** of  $X^n$  is the subset  $\{(x_1, \dots, x_n) \in X^n : x_1 = x_2 = \dots = x_n\}$

**Definition 1.3.7.** Given two sets  $X, Y$  we say that  $f$  is a **function** with domain  $X$  and codomain  $Y$  denoted  $f : X \rightarrow Y$ , if  $f$  is a subset of  $X \times Y$  such that every element of  $X$  appears as exactly the first component of exactly one element of  $f$ .

*Example.* We used the notation  $f(x)$  to refer to the element  $y$  such that  $(x, y) \in f$  is the unique ordered pair that refers to the element  $x \in X$ .

**Definition 1.3.8.** The **Identity Function** is a function with the same domain and codomain  $X$  written  $1_X : X \rightarrow X$  corresponding to the diagonal 1.3.6 of  $X^2$

**Definition 1.3.9.** Given  $f : X \rightarrow W$  and  $g : W \rightarrow Z$  with  $Y \subseteq W$ , the composition  $g \circ f : X \rightarrow Z$  is the function satisfying  $g \circ f(x) = g(f(x))$ .

**Definition 1.3.10.** A function is **Injective** if  $f(x) = f(u) \Rightarrow x = u$

**Definition 1.3.11.** A function  $f : X \rightarrow Y$  is **Surjective** if the range of  $f$  equals  $Y$

**Definition 1.3.12.** A function is **Bijective** if it is both Injective and Surjective

**Theorem 1.3.1.** If  $X$  is non-empty,  $f : X \rightarrow Y$  is injective  $\Leftrightarrow f$  is left invertible

**Theorem 1.3.2.**  $f : X \rightarrow Y$  is surjective  $\Leftrightarrow f$  is right invertible

**Definition 1.3.13.** A **Relation** of a set  $X$  is a subset of  $X^2$ . Conventionally written  $xRy$  rather than  $(x, y) \in R$

### 1.3.14. Properties of Relation

1. **Reflexive** if  $xRx$  for all  $x \in X$
2. **Transitive** if  $xRy$  and  $yRz \Rightarrow xRz$
3. **Symmetric** if  $xRy \Leftrightarrow yRx$
4. **Antisymmetric** if  $xRy$  and  $yRx \Rightarrow x = y$
5. **Connex** if for every  $x, y \in X$  at least one of  $xRy$ ,  $yRx$ , or  $x = y$  hold.

**Definition 1.3.15.** An **Equivalence Relation** is a relation that is Reflexive, Transitive, and Symmetric

**Definition 1.3.16.** if  $\sim$  is an equivalence relation, the **Equivalence Class** of  $x \in X$  is  $[x] := \{y \in X : x \sim y\}$ . Additionally, the notation  $X/\sim$  refers to the set of all equivalence classes  $\{[x] : x \in X\}$

## 1.4 The sets $\mathbb{Z}$ and $\mathbb{Q}$

**Theorem 1.4.1.** The natural numbers  $\mathbb{N}$  with its standard addition and multiplication is a **commutative semiring** with the following properties:

1.  $(\mathbb{N}, +)$  Addition is a commutative semigroup (ie. addition is commutative and associative)
2.  $(\mathbb{N}, \cdot)$  Multiplication is a commutative semigroup (ie. multiplication is commutative and associative)
3. Multiplication distributes over addition  $a \cdot (b + c) = a \cdot b + a \cdot c$

**Definition 1.4.1.** The integers  $\mathbb{Z}$  is defined as a set of equivalence classes 1.3.16  $\mathbb{N}/\sim$  where the equivalence relation  $\sim$  is

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

*Remark.* This is simply a rigorous way of representing integers with two natural numbers where the first natural number is the positive component and the second number is the negative component.

**Theorem 1.4.2.** The integers  $\mathbb{Z}$  with its standard addition and multiplication is a **commutative ring** with the following properties:

1.  $(\mathbb{Z}, +)$  Addition is a commutative semigroup (ie. addition is commutative and associative)
2.  $(\mathbb{Z}, \cdot)$  Multiplication is a commutative semigroup (ie. multiplication is commutative and associative)
3. Multiplication distributes over addition  $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.

**Definition 1.4.2.** The rational numbers  $\mathbb{Q}$  is defined as a set of equivalence classes 1.3.16  $(\mathbb{Z} \times \mathbb{N})/\sim$  where the equivalence relation  $\sim$  is

$$(a, n) \sim (b, m) \Leftrightarrow am = bn$$

*Remark.* This is simply a rigorous way of representing rational numbers as a fraction from integers and natural numbers where the integer is the numerator and the natural number is the denominator.

**Theorem 1.4.3. (Rudin 1.12)** The rational numbers  $\mathbb{Q}$  with its standard addition and multiplication is a **field** with the following properties:

1.  $(\mathbb{Q}, +)$  Addition is a commutative semigroup (ie. addition is commutative and associative)
2.  $(\mathbb{Q}, \cdot)$  Multiplication is a commutative semigroup (ie. addition is commutative and associative)
3. Multiplication distributes over addition  $a \cdot (b + c) = a \cdot b + a \cdot c$
4. Additive identity.
5. Additive inverse.
6. Multiplicative identity.
7. Multiplicative inverse.

### 1.4.1 Cardinality of Sets

**Definition 1.4.3.** The **cardinality** of a set is the number of elements in that set.

- $\text{card}(A) = \text{card}(B)$  if there exists a bijective function:  $A \rightarrow B$
- $\text{card}(A) \leq \text{card}(B)$  if there exists an injective(left invertible) function:  $A \rightarrow B$
- $\text{card}(A) \geq \text{card}(B)$  if there exists a surjective(right invertible) function:  $A \rightarrow B$

**Corollary 1.4.3.1.** (Pigeonhole Principle). Suppose  $n < m$  there does not exist an injective(left invertible) function:  $\{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$  and there does not exist a surjective(right invertible) function:  $\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, m\}$

**Definition 1.4.4.** A set  $X$  is said to be

- **countable** if  $\text{card}(X) \leq \text{card}(\mathbb{N})$
- **uncountable** if  $\text{card}(X) > \text{card}(\mathbb{N})$
- **finite** if  $\exists n \in \mathbb{N}$  such that  $\text{card}(X) \leq \text{card}(\{1, \dots, n\})$
- **countably infinite** if  $\text{card}(X) = \text{card}(\mathbb{N})$
- **infinite** if  $\text{card}(X) \geq \text{card}(\mathbb{N})$

## 1.5 Ordered Sets

**Definition 1.5.1. (Rudin 1.5)** An **order** on a set  $S$  is a relation denoted by  $<$  that is connex and transitive.

**Definition 1.5.2. (Rudin 1.6)** An **ordered set** is a set for which an order is defined.

*Remark.* The sets  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{Q}$  if  $a < b$  is defined to mean that  $b - a$  is positive.

**Definition 1.5.3.** Let  $(X, \leq)$  be an order. Define the two functions  $\uparrow, \downarrow: X \rightarrow 2^X$  by

- $\downarrow(x) : \{y \in X : y \leq x\}$ , a subset is a **lower set** or **downward closed** if  $s \in S \Rightarrow \downarrow(s) \subseteq S$ .
- $\uparrow(x) : \{y \in X : x \leq y\}$ , a subset is an **upper set** or **upper closed** if  $s \in S \Rightarrow \uparrow(s) \subseteq S$

**Definition 1.5.4. (Rudin 1.7)** Let  $(X, \leq)$  be an order and let  $S \subseteq X$ , and  $z \in X$

- We say that  $z$  is an **Upper bound** of  $S$  if  $S \subseteq \downarrow(z)$ . The set  $s$  is said to be **bounded above** if it has an upper bound.
- We say that  $z$  is a **Lower bound** of  $S$  if  $S \subseteq \uparrow(z)$ . The set  $s$  is said to be **bounded below** if it has a lower bound.
- We say that  $S$  is bounded, if it is bounded both above and below.

### 1.5.1 Special Elements

**Definition 1.5.5.** Let  $(X, \leq)$  be an ordered set, and let  $S \subseteq X$ . We say that an element of  $s_0 \in S$  is...

- **the maximum of  $S$**  if  $S \subseteq \downarrow(s_0)$
- **the minimum of  $S$**  if  $S \subseteq \uparrow(s_0)$
- **a maximal element of  $S$**  if, for  $s \in S$ ,  $s_0 \in \downarrow(s) \Rightarrow s_0 = s$
- **a minimal element of  $S$**  if, for  $s \in S$ ,  $s_0 \in \uparrow(s) \Rightarrow s_0 = s$

**Definition 1.5.6. (Rudin 1.8)** Let  $(X, \leq)$  be an ordered set and  $S \subseteq X$ . We say that an element of  $x \in X$  is...

- **the supremum of  $S$**  if  $x = \min\{y \in X : S \subseteq \downarrow(y)\}$
- **the infimum of  $S$**  if  $x = \max\{y \in X : S \subseteq \uparrow(y)\}$

## 1.5.2 Dedekind completeness

**Definition 1.5.7.** Let  $(X, \leq)$  be an order. We say a pair of subsets  $(X_-, X_+)$  form a **cut** of  $X$  if:

- $\{X_-, X_+\}$  is a partition of  $X$ .
- $X_-$  is a lower set and  $X_+$  is an upper set.

**Definition 1.5.8.** An ordered set  $X$  is said to be **Dedekind complete** if in every cut  $(X_-, X_+)$ , at least one of  $X_-$  or  $X_+$  is principal. That is  $X_-$  has a maximum or  $X_+$  has a minimum.

**Definition 1.5.9. (Rudin 1.10)** An ordered set  $(X, \leq)$  is said to possess the **least upper bound property** if every nonempty subset that is bounded above has a supremum. Similarly, an ordered set  $(X, \leq)$  is said to possess the **greatest lower bound property** if every nonempty subset that is bounded below has an infimum.

**Theorem 1.5.1. (Rudin 1.11)** For an ordered set  $(X, \leq)$ , the following statements are equivalent.

- $X$  is a Dedekind complete
- $X$  has the least upper bound property
- $X$  has the greatest lower bound property

## 1.6 The Set $\mathbb{R}$

**Theorem 1.6.1. (Rudin 1.19)** There exists an ordered field  $\mathbb{R}$  which has the least-upper-bound property and contains  $\mathbb{Q}$  as a subfield.

**Theorem 1.6.2. (Rudin 1.20)** The following properties hold for the real numbers  $\mathbb{R}$ :

- **(The Archimedean Property)** If  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ , and  $x > 0$ , then there exists a positive integer  $n$  such that  $nx > y$ .
- **( $\mathbb{Q}$  is dense in  $\mathbb{R}$ )** If  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ , and  $x < y$ , then there exists a  $p \in \mathbb{Q}$  such that  $x < p < y$ .

**Definition 1.6.1. (Rudin 1.23)** The **extended real number system** is the real field  $\mathbb{R}$  and two symbols,  $+\infty$  and  $-\infty$ . Preserving the original ordering in  $\mathbb{R}$  we define:

$$-\infty < x < +\infty$$

for every  $x \in \mathbb{R}$ .

## 1.7 The Set $\mathbb{C}$

**Definition 1.7.1. (Rudin 1.24)** A **complex number** is an ordered pair  $(a, b)$  of real number. Let  $x, y$  be complex numbers where  $x = (a, b)$ , and  $y = (c, d)$ . Define the following properties:

- $x = y \Leftrightarrow a = c$  and  $b = d$
- $x + y = (a + c, b + d)$
- $xy = (ac - bd, ad + bc)$

Under this definition  $\mathbb{R}$  is a subfield of  $\mathbb{C}$ .

**Definition 1.7.2. (Rudin 1.27)**  $i = (0, 1)$

**Theorem 1.7.1. (Rudin 1.28)**  $i^2 = -1$

*Proof.*  $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$  □

**Definition 1.7.3. (Rudin 1.30)** If  $a, b$  are real and  $z = a + bi$ , then the complex number  $\bar{z} = a - bi$  is the **complex conjugate** of  $z$ . Additionally,  $a$  is the **real part** of  $z$  and  $b$  is the **imaginary part** of  $z$ .

$$a = \operatorname{Re}(z), \quad b = \operatorname{Im}(z)$$

**Theorem 1.7.2. (Rudin 1.31)** For  $z, y \in \mathbb{C}$  we have

- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z} \cdot \bar{w}$
- $z + \bar{z} = 2\operatorname{Re}(z)$ ,  $z - \bar{z} = 2i\operatorname{Im}(z)$

**Definition 1.7.4. (Rudin 1.32)** The **absolute value** of a complex number  $z \in \mathbb{C}$  is defined as  $|z| = (z\bar{z})^{\frac{1}{2}}$ .