Topology
from the context of the course
MTH 461: Metric and Topological Spaces

Kaedon Cleland-Host

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Contents

Chapter 1

Fundamentals

1.1 Functions

Definition 1.1.1. A **function** $f : A \to B$ is a subset of $X \times Y$ such that $\forall x \in X$, $\exists$ exactly one element $y \in B$, $(x, y) \in f$.

Definition 1.1.2. The **domain** of a function $f : A \to B$ is $\{a \in A : \exists b \in B$ such that $(a, b) \in f\}$.

Definition 1.1.3. The **range** of a function $f : A \to B$ is $\{b \in B : \exists a \in A$ such that $(a, b) \in f\}$.

Definition 1.1.4. A function is a **injective** denoted $f : A \hookrightarrow B$ iff $f(x) = f(u) \Rightarrow x = y$.

Definition 1.1.5. A function is a **surjection** denoted $f : A \twoheadrightarrow B$ iff the range of f equals B.

Definition 1.1.6. A function is a **bijection** denoted $f : A \hookrightarrow\!\!\!\!\!\rightarrow B$ iff it is both an injection and a surjection.

1.2 Relations

Definition 1.2.1. A **relation** on a set A is a subset of $A \times A$. Conventionally written xRy rather than $(x, y) \in R$.

Definition 1.2.2. For a relation R on a set A, R is

- **Reflexive** iff xAx for all $x \in A$
- **Antireflexive** iff $\nexists x \in A$ such that xAx
- **Transitive** iff xRy and $yRz \Rightarrow xRz$, for any $x, y, z \in A$.
- **Symmetric** iff $xRy \Leftrightarrow yRx$, for any $x, y \in A$.
- **Antisymmetric** iff xRy and $yRx \Rightarrow x = y$, for any $x, y \in A$.
- **Connex** iff for every $x, y \in R$ at least on of xRy, yRx, or $x = y$ hold.

Definition 1.2.3. An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

Definition 1.2.4. The **equivalence class** of $a \in A$ for a relation $\sim$ is $[x] := \{b \in A | a \sim b\}$.

Definition 1.2.5. A **partition** of a set A is a set of subsets X such that $\bigcup X = A$ and $\forall B, C \in X, A \neq B \Rightarrow A \cap B = \emptyset$.

Lemma 1.2.1. Let $x, y \in A$ and $\sim$ be an equivalence class on A, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

Corollary 1.2.1.1. Any partition defines and equivalence relation and vice versa.

1.3 Order

Definition 1.3.1. An **order** on a set A is a relation that is antireflexive, transitive, and connex.

Definition 1.3.2. A **partial order** on a set A is a relation that is reflexive, antisymmetric, and transitive.

Definition 1.3.3. Two ordered sets have the same **order type** if there exists a bijection that preserves order.

Definition 1.3.4. Let $(X, \leq)$ be an ordered set, and let $A \subseteq X$.

- The **maximum** of A is an element $a_{max} \in A$ such that $\forall a \in A, a \leq a_{max}$.
- The **minimum** of A is an element $a_{min} \in A$ such that $\forall a \in A, a \geq a_{min}$.
- An **upper bound** of A is an element $x \in X$ such that $\forall a \in A, a \leq x$.
- An **lower bound** of A is an element $x \in X$ such that $\forall a \in A, a \geq x$.
- The **supremum** of A is the least upper bound of A.
- The **infimum** of A is the greatest lower bound of A.

Definition 1.3.5. An **interval** on an ordered set $(X, <)$ is

- $(a, b) = \{x \in X : a < x < b\}$ for some $a, b \in X$
- $[a, b) = \{x \in X : a \leq x < b\}$ for some $a, b \in X$
- $(a, b] = \{x \in X : a < x \leq b\}$ for some $a, b \in X$
- $[a, b] = \{x \in X : a \leq x \leq b\}$ for some $a, b \in X$

1.4 Cardinality

Definition 1.4.1. A set A is **finite** if there exists a bijection $f : A \hookrightarrow\!\!\!\!\rightarrow \{1, 2, 3, ..., n\}$ for some $n \in \mathbb{N}$.

Definition 1.4.2. The **cardinality** of a finite set A is $n \in \mathbb{N}$ such that $f : A \hookrightarrow\!\!\!\!\rightarrow \{1, 2, 3, ..., n\}$ is a bijection.

Theorem 1.4.1. Let A be a finite set with cardinality $n \in \mathbb{N}$ and $B \subsetneq A$ be a proper nonempty subset, then

$$\nexists \text{ a bijection } B \hookrightarrow\!\!\!\!\rightarrow \{1, ..., n\}$$

$$\exists \text{ a bijection } B \hookrightarrow\!\!\!\!\rightarrow \{1, ..., m\} \text{ for some } m \in \mathbb{N}$$

Corollary 1.4.1.1. For finite sets A there is no bijection between A and any proper nonempty subset $B \subsetneq A$.

Definition 1.4.3. A set A is **countable** iff $\exists A \hookrightarrow\!\!\!\!\rightarrow \mathbb{N}$ or A is finite.

Theorem 1.4.2. Let A be a nonempty set, then the following are equivalent.

- A is countable
- There exists a surjection $g : \mathbb{N} \twoheadrightarrow A$.
- There exists an injection $f : A \hookrightarrow \mathbb{N}$.

Corollary 1.4.2.1. Every subset $A \subset \mathbb{N}$ is countable.

Corollary 1.4.2.2. A countable union of countable sets is countable.

Definition 1.4.4. The **power set** of a set A denoted $P(A)$ is the set of all subsets of A.

Theorem 1.4.3. The Cantor Theorem states that for a nonempty set A there is no injection $f : P(A) \hookrightarrow A$ and no surjection $g : A \twoheadrightarrow P(A)$.

1.5 Topologies

Definition 1.5.1. A **topology** on a set A is a set of subsets $J \subset P(A)$ with the following properties

1. $\emptyset, A \in J$.
2. Any union of elements in J is also in J.
3. Any finite intersection of elements in J is also in J.

Definition 1.5.2. A **topological space** is a pair $(X, \mathcal{T})$ sometimes denoted $\mathcal{T}_X$ of a set X and a topology $\mathcal{T}$ on X.

Definition 1.5.3. A subset $A \subset X$ is **open** iff $A \in \mathcal{T}$ where $(X, \mathcal{T})$ is a topological space.

Definition 1.5.4. A subset $A \subset X$ is **closed** iff $X - A$ is open.

Definition 1.5.5. A **basis** is a collection $\mathcal{B}$ of subsets of a set X such that

1. $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in B$.
2. $\forall B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then $\exists B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_1$.

Proposition 1.5.1. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C} \subset P(X)$. If $\forall U \in \mathcal{T}$, $\forall x \in U$, $\exists D \in \mathcal{C}$ such that $x \in D \subseteq U$, then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Definition 1.5.6. The **topology generated by a basis** $\mathcal{B}$ on a set X is

$$\mathcal{T} = \{U \in P(X) : U = \bigcap_{B_i \in \mathcal{C}} B_i, \mathcal{C} \subset \mathcal{B}\}$$

Definition 1.5.7. A **subbasis** for a topology $\mathcal{T}$ on X is a collection $\mathcal{S}$ of subsets of X such that the collection of all unions and finite intersections of elements in $\mathcal{S}$ is $\mathcal{T}$.

Definition 1.5.8. The **topology generated by a subbasis** $\mathcal{S}$ on a set X is the collection of all unions and finite intersections of elements in $\mathcal{S}$.

Definition 1.5.9. A topology $\mathcal{T}'$ is **finer** than another topology $\mathcal{T}$ iff $\mathcal{T}' \subseteq \mathcal{T}$.

Theorem 1.5.1. Let $\mathcal{B}, \mathcal{B}' \subset P(X)$ be bases of the topological spaces $(X, \mathcal{T}), (X, \mathcal{T}')$. The following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. $\forall x \in X$ and any basis element $B \in \mathcal{B}$ such that $x \in B$ there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$

Definition 1.5.10. A **homeomorphism** is a bijection $f : \mathcal{X} \hookrightarrow\!\!\!\!\rightarrow \mathcal{Y}$ between topologies $\mathcal{X}$ and $\mathcal{Y}$.

Definition 1.5.11. A topological space $(X, \mathcal{T})$ is **first countable** iff $\forall x \in X$, $\exists$ a countable set $\mathcal{B} \subset \mathcal{T}$ so that for every set $U \in \mathcal{T}$ containing x, $V \subseteq U$ for some $V \in \mathcal{B}$.

Definition 1.5.12. A topology is **second countable** iff it has a countable basis.

1.5.1 Examples of Topologies

Definition 1.5.13. The **discrete topology** on a set X is $\mathcal{T} = P(X)$.

Definition 1.5.14. The **indiscrete topology** on a set X is $\mathcal{T} = \{\emptyset, X\}$.

Definition 1.5.15. The **finite compliment topology** on a set X is $\mathcal{T} = \{U \subset X : X - U \text{ is finite}\}$.

Definition 1.5.16. The **standard topology** on $\mathbb{R}$ is the topology generated by the basis

$$\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.17. The **lower limit topology** on $\mathbb{R}$ denoted $\mathbb{R}_\ell$ is the topology generated by the basis

$$\mathcal{B} = \{[a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.18. The **upper limit topology** on $\mathbb{R}$ is the topology generated by the basis

$$\mathcal{B} = \{(a, b] \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

Definition 1.5.19. The **K-topology** on $\mathbb{R}$ denoted $\mathbb{R}_K$ is the topology generated by the basis

$$\mathcal{B} = \{(a,b) \subset \mathbb{R} : a,b \in \mathbb{R}\} \cap \{(a,b) - K \subset \mathbb{R} : a,b \in \mathbb{R}\}$$

$$K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

Definition 1.5.20. The **order topology** on a ordered set S with more than 1 element is the topology generated by the a basis

$$\mathcal{B} = \{(a,b) \subset \mathbb{R} : a,b \in \mathbb{R}\} \cap \{(a,b_0] \subset \mathbb{R} : a \in \mathbb{R}\} \cap \{[a_0,b) \subset \mathbb{R} : b \in \mathbb{R}\}$$

where a_0 is the smallest element and b_0 is the largest element.

1.6 Well Ordered Sets

Definition 1.6.1. A **well ordered set** X is an ordered set such that any subset $S \subseteq X$ has a smallest element $s_0 \in S$ such that $s_0 \leq s$, $\forall s \in S$.

Corollary 1.6.0.1. Any finite ordered set is well ordered.

Definition 1.6.2. The **section** of a well ordered set X by $a \in X$ denoted S_a is

$$S_a = \{x \in X : x < a\}$$

Theorem 1.6.1. Any set A admits a well ordering.

Corollary 1.6.1.1. There exists an uncountable well ordered set.

Theorem 1.6.2. There exists a well ordered set S such that any section is countable S_Ω where Ω is the largest element.

Definition 1.6.3. The **minimal uncountable well ordered set** denoted S_Ω is the uncountable well-ordered set such that any section is countable.

Theorem 1.6.3. If $A \subset S_\Omega$ is a countable subset of S_Ω then A has an upper bound in S_Ω.

1.7 Product Topology

Definition 1.7.1. The **Product Topology** denoted $\mathcal{T} \times \mathcal{T}'$ for two topologies $\mathcal{T}, \mathcal{T}'$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}, V \in \mathcal{T}'\}$$

Theorem 1.7.1. For topologies $\mathcal{T}$ and $\mathcal{T}'$ with bases $\mathcal{B}$ and $\mathcal{B}'$ the product topology is equivalently generated by the basis

$$\mathscr{B} = \{B \times B' : B \in \mathcal{B}, B' \in \mathcal{B}'\}$$

Definition 1.7.2. The function $\pi_n : \prod_{i \in I} X_i \to X_n$ is the function mapping $(\ldots, x_n, \ldots) \mapsto x_n$.

Theorem 1.7.2. The product topology on $X \times Y$ is the weakest topology such that $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are both open $\forall$ open $U \subset X$, $V \subset Y$.

Lemma 1.7.3. Let X, Y be topological spaces the set $\{U \times Y : U \in X\} \cap \{X \times V : V \subset Y\}$ is a subbasis for $X \times Y$.

1.8 Subspace Topology

Definition 1.8.1. The **subspace topology** denoted $\mathcal{T}_Y$ for a subset $Y \subset X$ of a topological space $(X, \mathcal{T})$ is

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

Lemma 1.8.1. If $\mathcal{B}$ is a basis for a topological space X and $Y \subset X$ then $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

Lemma 1.8.2. For $Y \subset X$ with the subspace topology, if $U \subset Y$ is open in Y and $Y \subset X$ is open in X then U is open in X.

Theorem 1.8.3. Let $A \subset X$, $B \subset Y$ be topological spaces with subspace topologies, then for $A \times B \subset X \times Y$ the product topology agrees with the subspace topology.

Definition 1.8.2. A subset $Y \subset X$ is **convex** iff $\forall a, b, c \in Y$, if $a < c < b$ then $c \in Y$.

Theorem 1.8.4. If X be an ordered set with a convex subset $Y \subset X$, then the subspace topology on Y is the order topology on Y.

1.9 Interior and Closure

Definition 1.9.1. For a subset $V \subset X$ of a topological space X the **interior** of V denoted V^o is the largest open set in V or equivalently

$$V^o = \bigcup_i U_i \quad \forall U_i \in V$$

Definition 1.9.2. For a subset $V \subset X$ of a topological space X the **closure** of V denoted $\bar{V}$ is the smallest closed set containing V or equivalently

$$\bar{V} = X - (X - V)^o = \bigcap_j F_j \quad \forall F_j \in \mathcal{T} \text{ such that } V \subset F_j$$

Lemma 1.9.1. Let $A \subset X$ be a subset of a topological space X, then $x \in \bar{A}$ if and only if every open set containing $x \in X$ intersects A.

Lemma 1.9.2. Let $A \subset X$ be a subset of a topological space X and $\mathcal{B}$ be a basis for X, then $x \in \bar{A}$ if and only if every basis element $B \in \mathcal{B}$ containing $x \in X$ intersects A.

Definition 1.9.3. For a subset $A \subset X$ of a space X, a point $x \in X$ is a **limit point** or **cluster point** of A iff for every neighborhood $U \in \mathcal{T}$ of x, $U - \{x\}$ intersects A.

Theorem 1.9.3. Let $A \subset X$ be a subset of a topological space X and A' be the set of limit points of A, then $\bar{A} = A \cup A'$.

1.10 Hausdorff Topologies

Definition 1.10.1. A topological space X is **Hausdorff** iff $\forall x, y \in X$ such that $x \neq y$, there exists $U \in \mathcal{T}$ and $V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 1.10.1. Every finite subset of a Hausdorff space is closed.

Definition 1.10.2. A **sequence** in a space X is a series of points $x_i \in X$ for $i \in \mathbb{N}$.

Definition 1.10.3. A sequence **converges** to a point $x \in X$ iff for all open subsets $U \subset X$ such that $x \in U$ $\exists N$ such that for all $n \geq N$, $x_n \in U$.

Proposition 1.10.1. Let $A \subset X$ be a subset of a topological space X. If a sequence $x_n \in A$ converges to $x \in A$, then $x \in \overline{A}$.

Theorem 1.10.2. If a space X is Hausdorff, then any sequence $x_n \in X$ can only converge to at most one point.

1.11 Continuity

Definition 1.11.1. A function $f : X \to Y$ is **continuous** iff for any open subset $V \subset Y$ in the range of f, $f^{-1}(V)$ is open.

Proposition 1.11.1. A function $f : X \to Y$ is continuous iff for $\mathcal{B}$ basis of Y, $f^{-1}(\mathcal{B})$ is open $\forall B \in \mathcal{B}$.

Theorem 1.11.1. If $f : \mathbb{R} \to \mathbb{R}$ is continuous under the epsilon delta definition from real analysis, then it is continuous.

Definition 1.11.2. A **homomorphism** is a function between spaces that is continuous in both directions.

Theorem 1.11.2. Let $f : X \to Y$ a function between spaces. The following are equivalent"

- f is continuous.
- $\forall A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
- $\forall B \subset Y$ closed, $f^{-1}(B)$ is closed
- $\forall x \in X$, if $f(x) \in B$ is a neighborhood of $f(x)$ then $f^{-1}(B)$ contains a neighborhood of x.

Theorem 1.11.3. Let X, Y, Z be spaces.

- $f : X \to Y$ defined by $x \mapsto y$ for some $y \in Y$ and $\forall x \in X$, is continuous.
- For $A \subset X$ with the subspace topology, $j : A \to X$ defined by $x \mapsto x$ is continuous.
- If $f : X \to Y$ and $g : Y \to Z$ are both continuous then $g \circ f = g(f(x))$ is continuous.
- If $f : X \to Y$ is continuous. For $A \subset X$, the restriction $f_A : A \to Y$ is continuous.
- For $Y \subset Z$ with the subspace topology, if $f : X \to Y$ is continuous then $f : X \to Z$ is also continuous.
- The map $f : X \to Y$, where $X = \bigcup_\alpha U_\alpha$ for open subsets U_α is continuous if and only if $f_{\bigcup_{\alpha \in A} U_\alpha}$ is open for all A.

Theorem 1.11.4. The Pasting Theorem states that for $X = A \cup B$ where $A, B \subset X$ are closed subsets. If $f : A \to Y$ is continuous, $g : B \to Y$ is continuous and $f = g$ on $A \cap B$, then $h : X \to Y$ is continuous where

$$h(x) = \left\{ \begin{array}{ll} f(x) & x \in A \\ g(x) & x \in B \end{array} \right\}$$

Theorem 1.11.5. Let $f : X \to Y \times Z$ for spaces X, Y, Z where $f(x) = (f_1(x), f_2(x))$. f is continuous iff f_1, f_2 are continuous.

Proposition 1.11.2. The functions $+, \times, /$ are continuous, defined by

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{defined by } (a, b) \mapsto a + b$$

$$\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{defined by } (a, b) \mapsto ab$$

$$/ : \mathbb{R} \times \mathbb{R} - \{0\} \to \mathbb{R} \quad \text{defined by } (a, b) \mapsto a/b$$

1.12 Metric Spaces

Definition 1.12.1. A **metric space** X is a set with a function $d : X \to \mathbb{R}$ such that $\forall x, y, z \in X$,

- $d(x, y) \geq 0$
- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) + d(y, z) \leq d(x, z)$

Definition 1.12.2. The **metric ball** denoted $B(x, \varepsilon)$ for a point $x \in X$ in a metric space (X, d) and a real number $\varepsilon > 0$ is the set

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

Definition 1.12.3. The **topology on a metric space** is the topology generated by the basis

$$\mathcal{B} = \{B(x, \varepsilon) : x \in X, \varepsilon > 0)\}$$

Definition 1.12.4. A topological space X is **metrizable** iff $\exists$ a metric on X whose metric topology is that of X.

Theorem 1.12.1. For two metrics (X, d) and (X, d'), the metric topology generated by d' is finer than d if and only if $d(x, y) \leq d'(x, y)$

Corollary 1.12.1.1. For two metrics (X, d) and (X, d') with metric topologies $\mathcal{T}$ and $\mathcal{T}'$. $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $x \in X$ and all $\varepsilon > 0 \in \mathbb{R}$, $\exists\ \delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$

Definition 1.12.5. A subset set $A \subseteq X$ of a metric space X is bounded iff $\exists k > 0 \in \mathbb{R}$ so that $d(x, y) < k$ for all $x, y \in A$.

Definition 1.12.6. The diameter of a metric space is $\sup_{x,y \in A} d(x, y)$.

Theorem 1.12.2. Let (X, d) be a metric space, then $\overline{d} : X \times X \to \mathbb{R}$ defined by $\overline{d}(x, y) = \min\{d(x, y), 1\}$ is a metric that induces the same topology.

Proposition 1.12.1. Let $A \subset X$, where X is a metric $\forall x \in \overline{A}$, $\exists$ a sequence $x_n \in A$ for $n \in \mathbb{N}$ that converges to X.

Theorem 1.12.3. Let X be a metric space and $f : X \to Y$ a function. f is continuous if and only if $\lim f(x_n) = f(x)$ for all sequences $x_n \in X$ that converge to $x \in X$.

Proposition 1.12.2. Every metrizable space is first countable.

Theorem 1.12.4. Let X, Y be spaces and $f : X \to Y$ a function.
If f is continuous, then for every sequences $x_n \in X$ converging to $x \in X$, $f(x_n)$ converges to $f(x)$.
If X is first countable and for every sequence $x_n \in X$ converging to $x \in X$, $f(x_n)$ converges to $f(x)$, then f is continuous.

Chapter 2

Product Topology

"At least if you believe in calculus."

"I hope that homework didn't kill anyone too much."
"I'm actually currently a zombie now."
"Yeah, that's the thing about homework."

"It's like set theory but actually interesting!"

"If you want to make class more interesting just replace 'bases' with Al-Qaeda."

"If you're trapped in the U-ball, you're screwed!"

"It's in all these weird languages that nobody should be speaking with too many consonants."

"When you see a circle and you see a line; You don't see the same thing!"

"Hmm, maybe"

"The inverse function theorem is basically the reason the world works."