PHY 321 HW6 Exercise 1

Maya Joyce, Kaedon Cleland-Host, John Graham II

3/1/2021

1 Newton's Laws

1.1 Newton's 2nd Law

The force on an object is related to its mass and its acceleration.

$$\mathbf{F} = m \cdot \mathbf{a}$$

Note that acceleration has the same direction as force.
inertial mass: mass of an object determined by measuring acceleration for a given applied force.

1.2 Newton's 1st Law

An object in a state of motion tends to remain in motion unless an external force changes its state of motion. (if the acceleration is 0, velocity is constant)

$$\mathbf{F} = 0 = m \cdot \mathbf{a}$$

Definition of Acceleration:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 0$$

Inertial System: An inertial system is a system that is not accelerated (reference system must not be accelerated linearly or rotation)

1.3 Newton's 3rd Law

For every action there is an equal and opposite reaction. Thus, for any force from object A on object B, there is an equivalent force from object B on object A.

$$\mathbf{F_{1,2}} = \mathbf{F_{2,1}}$$

1.4 Kinematics

1.4.1 Discretized Equations for Motion

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}, \frac{d\mathbf{r}}{dt} = \mathbf{v}$$

1.4.2 Euler's Method

Euler's method is a numerical method for approximating a solution to the discretized equations of motion shown above where Δt is a chosen time step:

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \Delta t \cdot \mathbf{v}_i$$

$$\mathbf{v}_{i+1} = \mathbf{v}_i + \Delta t \cdot \mathbf{a}_i$$

Note that this method does not conserve energy. The Euler Cromer method and Velocity Verlet methods are other approximations that do conserve energy (see section 2.3 and 2.4 respectively).

2 Conservative Forces and Energy Conservation

2.1 Work-Energy Theorem

Kinetic energy K with a given velocity is defined as follows

$$K = \frac{1}{2}mv^2$$

Now, taking the derivative we can derive the Work-Energy Theorem

$$\frac{dK}{dt} = m\frac{d\mathbf{v}}{dt}\mathbf{v} = \mathbf{F}\frac{d\mathbf{r}}{dt}$$

$$\int \frac{dK}{dt}dt = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int \mathbf{F}d\mathbf{r}$$

2.2 Conservative Forces

A conservative force is a force has a path independent integral. Under this definition two immediate condition appear for a force to be conservative. First, the force must depend only on the spatial degrees of freedom and secondly, the the curl of the force must be zero, that is

$$\nabla \times \mathbf{F} = 0$$

Under a conservative force any circular integral is guaranteed to be zero by Stoke's theorem so from the Work-Energy theorem energy is conserved. In this case we can define a potential energy $V(\mathbf{r})$ that depends only on position.

$$V(\mathbf{r}) = -\nabla\mathbf{V}(\mathbf{r})$$

2.3 Total Energy

The total energy E of a system is the sum of the kinetic and potential energies

$$E = K + V(\mathbf{r})$$

With the potential energy defined as follows

$$V(\mathbf{r}) = -W(\mathbf{r_0} \to \mathbf{r_f}) = -\int_{\mathbf{r_0}}^{\mathbf{r_f}} \mathbf{F}(\mathbf{r})d\mathbf{r}$$

2.4 Euler-Cromer Method

Recall that Euler's method we have the Taylor expansion

$$y_{i+1} = y(t_i) + \Delta t f(t_i, y_i) + O(\Delta t^2)$$

This method always has an error of $NO(\Delta t^2)$. We can reduce this error by decreasing Δt buck this increases the computation necessary. A better improvement to Euler's method can be made to reduce the time asymmetry of Euler's method. Instead of using v_n to determine the y_{n+1} position we can use the v_{n+1} velocity, that is

$$y_{n+1} = y_n + v_{n+1} + O(\Delta t^2)$$

and

$$v_{n+1} = v_n + a_n + O(\Delta t^2)$$

2.5 Velocity Verlet Method

Another way we can improve upon Euler's method is by including higher order terms of the Taylor expansion. This reduces the global error from $NO(\Delta t^2)$ to $NO(\Delta t^3)$. The method is described as follows:

$$x_{i+1} = x_i + (\Delta t)v_i + \frac{(\Delta t)^2}{2}a_i + O((\Delta t)^3)$$

$$v_{i+1} = v_i + \frac{\Delta t}{2}(a_{i+1} + a_i) + O((\Delta t)^3)$$

3 Momentum and Angular Momentum

3.1 Linear Momentum

Linear momentum is a vector quantity with dimension (mass $\times$ length)/time, and for a single object, with mass m and velocity $\mathbf{v}$, is given by

$$\mathbf{p} = m\mathbf{v}.$$

For a collection of many (say N) particles, the total linear momentum is just given by the sum over all the individual linear momenta:

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = \sum_{i=1}^{N} m_i \mathbf{v}_i$$

Taking the derivative with respect to time, we see

$$\frac{d}{dt}\mathbf{P} = \sum_{i=1}^{N} \frac{d}{dt}\mathbf{p}_i = \sum_{i=1}^{N} m_i \frac{d\mathbf{v}_i}{dt} = \sum_{i=1}^{N} \mathbf{F}_i,$$

where $\mathbf{F}_i$ is the net force on the i^{th} object. However, we can split each $\mathbf{F}_i$ into internal and external forces, so

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij}.$$

Now, if we assume we have an isolated system, that is there are no external forces, we may derive conservation of total momentum:

$$\frac{d}{dt}\mathbf{P} = \sum_{i=1}^{N} \mathbf{F}_i = \sum_{ij, j \neq i}^{N} \mathbf{F}_{ij} = \sum_{i}^{N} \sum_{j>i} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) = 0$$

3.2 Angular Momentum

Angular momentum of a single particle with position $\mathbf{r}$ and momemtum $\mathbf{p}$ is defined as

$$\mathbf{L} := \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}$$

Taking the derivative with respect to time gives

$$\frac{d\mathbf{L}}{dt} = m\mathbf{v} \times \mathbf{v} + m\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{r} + \mathbf{F}$$

This defines the **torque**: $\boldsymbol{\tau} := \mathbf{r} \times \mathbf{F}$. Note that if $\mathbf{F}$ is parallel to $\mathbf{r}$, then $\boldsymbol{\tau} = 0$ and angular momentum is conserved.