Quantum Mechanics II

from the context of the courses
PHY 851-852: Quantum Mechanics

Kaedon Cleland-Host

September 26, 2022

Contents

0.1 The SI System

In physics it's often important to have precisely defined units for the purposes of making very accurate measurements or simply having a coherent unit system. It's possible to derive all necessary units from five measurements of **length, mass, time, current, and temperature**. The standard SI units for these properties are listed bellow:

Type	Unit	Definition
Length	Meter(m)	Length of distance light in a vacuum travels in $\frac{1}{299792458}$ seconds
Mass	Kilogram(kg)	Defined by fixing the Planck's constant $h = 6.62607015 \times 10^{-34} kg\ m^2 s^{-1}$
Time	Second(s)	Defined by fixing the ground-state hyperfine transition frequency of the caesium-133 atom, to be $9192631770 s^{-1}$
Current	Ampere(A)	Defined by fixing the charge of an electron as $1.602176634 \times 10^{-19} A \cdot s$
Temperature	Kelvin(K)	Defined by fixing the value of the Boltzmann constant k to $1.380649 \times 10^{-23} kg \cdot m^2 s^{-2} K^{-1}$

Common prefixes are listed bellow:

Prefix	Symbol	Definition
mega	M	10^6
kilo	k	10^3
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}

Additionally, the following are defined constants:

Symbol	Definition
$\hbar$	$\hbar = \frac{h}{2\pi} \approx 1.0546 \times 10^{-34} \mathrm{kg\,m^2\,s^{-1}}$
e	Charge of an electron $e = 1.602176634 \times 10^{-19} \mathrm{C}$

0.2 Why make a second book?

I've found that in the course of study it is incredibly difficult to make a universal reference book across classes. Maybe that is something I will make in the future. Until I know the exact topics that are covered in this course and the depth at which they are covered, I will keep the two books separate to make studying for the current course simpler.

Another major difference between these two books is the notation. The first book used Schrödinger notation with a few examples of Heisenberg notation. With the hope of standardizing notation and consistently representing quantum systems, this book will strictly use Heisenberg notation.

0.3 Stern-Gerlach Experiments

The Stern-Gerlach experiments are a great example of a system that cannot be accurately described by classical mechanics.

Definition 0.3.1. Recall from classical mechanics that **Classical Magnetic Moment** is defined using the following formula

$$\mu = \frac{q}{2m}\mathbf{L} \tag{0.3.1}$$

$$\mathbf{L} = rmv \tag{0.3.1}$$

r is radius, m is mass, v is tangential velocity, q is charge, $\mathbf{L}$ is angular momentum, and μ is magnetic moment.

Definition 0.3.2. Electron, Protons, and Neutrons all have an **intrinsic angular momentum** called **spin** denoted $\mathbf{S}$.

Definition 0.3.3. Electrons, Protons, and Neutrons also have an **intrinsic magnetic moment** defined by

$$\mu = g\frac{q}{2m}\mathbf{S} \tag{0.3.3}$$

g is the dimensionless gyroscopic ratio or g-factor which can be derived using quantum mechanics.

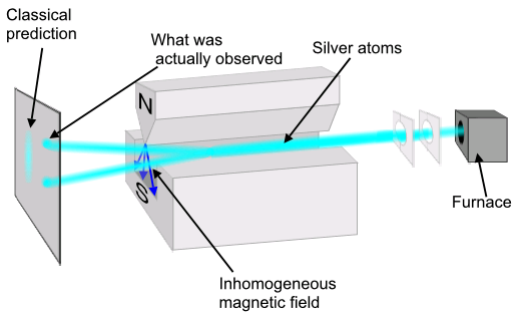


Figure 1: Diagram of the Stern-Gerlach experiment

The first Stern-Gerlach experiment seeks to measure the magnetic moment of the valence electron. A silver atom has 47 electrons and 47 protons. The magnetic moments depends on the inverse of mass, so we can neglect heavy protons and neutrons. Silver has an electron configuration of $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^{10} 4p^6 4d^{10} 5s^1$, so the only electron that contributes to the magnetic moment is the valence electron $5s^1$. Knowing this we expect the magnetic moment of the silver atom to be

$$\mu = -g_e\frac{e}{2m_e}\mathbf{S} \tag{0.3.4}$$

Following the laws of electromagnetism the force in the z direction is

$$F_z = -g_e\frac{e}{2m_e}S_z\frac{\partial B_z}{\partial z} \tag{0.3.5}$$

The deflection of the beam is therefore a measurement of the spin of the valence electron of the silver atoms in the z-direction. Classically, we would expect the magnetic moment to be aligned in random directive and to observe a continuous range of deflection. Instead we observe two distinct magnetic moments. The magnitudes of these deflections are consistent with the spins of

$$S_z = \pm\frac{\hbar}{2} \tag{0.3.6}$$

This is called **quantization** of the electron's spin angular momentum component. The factor $\frac{1}{2}$ in the equation is why we refer to electrons as having **spin-1/2**.

Chapter 1

Quantum Systems

This chapter will outline our system of notation and the fundamental concepts of quantum mechanics.

1.1 Axioms of Quantum Mechanics

Axiom 1. The state of any quantum system can be represented as a ket in a complex ket space.

Axiom 2. Any measurement on a quantum system can be represented as Hermitian operator X where the eigenvalues of X represent the possible outcomes.

1.2 Ket Space

Definition 1.2.1. A **ket** denoted $|\alpha\rangle$ is an element of a ket space.

Definition 1.2.2. A **ket space** K is a set of kets equipped with addition $+$, and complex scalar multiplication such that the following properties hold for any three kets $|\alpha\rangle, |\beta\rangle, |\gamma\rangle \in K$ and complex scalar $z, y \in \mathbb{C}$.

- **Associativity** $|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$.
- **Commutativity** $|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$
- **Identity** $\exists\, |\emptyset\rangle \in K$ such that $|\alpha\rangle + |\emptyset\rangle = |\alpha\rangle$.
- **Inverse** $|\alpha\rangle - |\alpha\rangle = |\emptyset\rangle$.
- **Compatibility** $z(y\,|\alpha\rangle) = (zy)\,|\alpha\rangle$.
- **Identity** $1\,|\alpha\rangle = |\alpha\rangle$.
- **Distributivity** $z(|\alpha\rangle + |\beta\rangle) = z\,|\alpha\rangle + z\,|\beta\rangle$ and $(z+y)\,|\alpha\rangle = z\,|\alpha\rangle + y\,|\beta\rangle$.

Definition 1.2.3.

Definition 1.2.4. The **inner product** of two kets denoted $\langle a|b\rangle$ is a complex number defined as the sum of the element-wise product of $|a\rangle$ and $|b\rangle$ with the following properties:

- $\langle a|b\rangle = \langle b|a\rangle^*$
- $\langle a|a\rangle \geq 0$

Definition 1.2.5. Two kets $|a\rangle$ and $|b\rangle$ are **orthogonal** iff $\langle a|b\rangle = 0$

Definition 1.2.6. The **norm** of a ket $|a\rangle$ is $\sqrt{\langle a|a\rangle}$.

Definition 1.2.7. A ket $|a\rangle$ is **normalized** iff $\langle a|a\rangle = 1$.

Corollary 1.2.8. For any ket $|a\rangle$ the ket $\frac{1}{\sqrt{\langle a|a\rangle}}\,|a\rangle$ is normalized.

1.3 Operators

Definition 1.3.1. An **operator** X acts on kets to produce a new ket with the following properties for any two kets $|a\rangle, |b\rangle$ and any scalar $z \in \mathbb{C}$.

- $X(|a\rangle + |b\rangle) = X|a\rangle + X|b\rangle$
- $X(z|a\rangle) = zX|a\rangle$

Corollary 1.3.2. Addition of operators is associative and commutative. For any linear operators A, B, C, $A + B = B + A$ and $A + (B + C) = (A + B) + C$.

Corollary 1.3.3. Multiplication of operators is associative but not commutative. For any linear operators A, B, C, $A(BC) = (AB)C$ and in general $AB \neq BA$.

Definition 1.3.4. An operator X is **unitary** iff $XX^T = I$.

Definition 1.3.5. The **adjoint** of a matrix X denoted X^T is defined by

$$\langle\alpha| X |\beta\rangle = \langle\beta| X |\alpha\rangle \tag{1.3.5}$$

Definition 1.3.6. The **Hermitian adjoint** of a matrix X denoted with dagger $X^\dagger$ is defined by

$$\langle\alpha| X |\beta\rangle = (\langle\beta| X |\alpha\rangle)^* \tag{1.3.6}$$

Definition 1.3.7. An operator X is **Hermitian** iff $X = X^\dagger$.

Corollary 1.3.8. If X is a Hermitian operator, then X has real eigenvalues and eigenvectors with different eigenvalues are orthogonal.

Definition 1.3.9. The **projection operators** denoted $A_{|\alpha\rangle}$ is an operator defined for any ket $|\alpha\rangle$

$$A_{|\alpha\rangle} = |\alpha\rangle\langle\alpha| \tag{1.3.9}$$

1.4 Bases and Matrix Representation

Definition 1.4.1. The **Kronecker delta** denoted δ_{ij} for any $i, j \in \mathbb{N}$ is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.4.1}$$

Definition 1.4.2. The **Levi-Civita symbol** denoted ε_{ijk} for any $i, j, k \in \mathbb{N}$ is defined as

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k \text{ is } (1,2,3), \text{ or } (2,3,1), \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), \text{ or } (1,3,2), \text{ or } (2,1,3) \\ 0 & \text{if } i = j, \text{ or } i = k, \text{ or } j = k \end{cases} \tag{1.4.2}$$

Definition 1.4.3. A **basis** is a set of kets $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ such that any ket in the ket space can be uniquely represented as a linear combination of basis kets.

Definition 1.4.4. An **orthonormal basis** is a basis $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ such that $\langle e_i|e_j\rangle = \delta_{ij}$.

Corollary 1.4.5. For an orthonormal basis $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ the completeness relation is the identity:

$$\sum_{i=1}^{n} |e_i\rangle\langle e_i| = I \tag{1.4.5}$$

Proposition 1.4.6. For any ket $|\alpha\rangle$ with a basis $\{|e_1\rangle, \ldots, |e_n\rangle\}$, there exists unique complex numbers $\alpha_i \in \mathbb{C}$ such that

$$|\alpha\rangle = \sum_{i=0}^{n} \alpha_i |\alpha\rangle, \qquad \langle e_i|\alpha\rangle = \alpha_i \tag{1.4.6}$$

Definition 1.4.7. We represent bras and kets as **row and column vectors** with respect to a basis $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$. For any ket $|\alpha\rangle = \sum_{i=0}^{n} \alpha_i |\alpha\rangle$.

$$\text{if } |\alpha\rangle = \sum_{i=0}^{n} \alpha_i |\alpha\rangle, \quad \langle\alpha| = \left(\alpha_1^* \alpha_2^* \ldots \alpha_n^*\right), \quad |\alpha\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \tag{1.4.7}$$

Definition 1.4.8. We represent operators as a **matrix** with respect to a basis $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$.

$$X = \begin{pmatrix} \langle e_1| X |e_1\rangle & \langle e_1| X |e_2\rangle & \langle e_1| X |e_3\rangle & \ldots & \langle e_1| X |e_n\rangle \\ \langle e_2| X |e_1\rangle & \langle e_2| X |e_2\rangle & \langle e_2| X |e_3\rangle & \ldots & \langle e_2| X |e_n\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_n| X |e_1\rangle & \langle e_n| X |e_2\rangle & \langle e_n| X |e_3\rangle & \ldots & \langle e_n| X |e_n\rangle \end{pmatrix} \tag{1.4.8}$$

Corollary 1.4.9. Under these definitions standard vector and matrix multiplication are consistent.

Theorem 1.4.10. Given two basis $\{|a_1\rangle, \ldots, |a_n\rangle\}$ and $\{|b_1\rangle, \ldots, |a_n\rangle\}$ there exsits a unitary operator such that for any operator X_a written in the basis $\{|a_1\rangle, \ldots, |a_n\rangle\}$ and the same operator X_b written in the basis $\{|b_1\rangle, \ldots, |b_n\rangle\}$,

$$U = \sum_{i=1}^{n} \langle b_i | a_i \rangle, \quad |b_i\rangle = U |a\rangle, X_b = U^T X_a \tag{1.4.10}$$

1.5 Eigenvalues and Eigenvectors

Definition 1.5.1. For an operator X an **eigenvector** is a ket $|\alpha\rangle$ such that $X |\alpha\rangle = \lambda |\alpha\rangle$ for some **eigenvalue** $\lambda \in \mathbb{C}$.

Definition 1.5.2. An **eigenspace** for an eigenvalue λ of an operator X is the set of eigenvectors with λ as an eigenvalue.

Definition 1.5.3. An eigenvalue is **degenerate** iff the corresponding eigenspace has more than one linearly independent eigenvector.

Definition 1.5.4. An eigenvalue is **non-degenerate** iff it is not degenerate.

1.6 Uncertainty Principle

Definition 1.6.1. The **commutator** of two operators A and B denoted $[A, B]$ is defined as

$$[A, B] = AB - BA \tag{1.6.1}$$

Definition 1.6.2. The **anticommutator** of two operators A and B denoted $\{A, B\}$ is defined as

$$\{A, B\} = AB + BA \tag{1.6.2}$$

Definition 1.6.3. Two operators A and B are **compatible operators** iff $[A, B] = 0$.

Definition 1.6.4. Two operators A and B are **incompatible operators** iff $[A, B] \neq 0$.

Corollary 1.6.5. Compatible operators have the same eigenvalues and eigenvectors.

Definition 1.6.6. The **expectation value** of a Hermitian operator X for a state $|\psi\rangle$ denoted $\langle X\rangle$ is defined as

$$\langle X\rangle = \langle\psi| X |\psi\rangle = \sum_{i=1}^{n} P(\alpha_i)\alpha_i \tag{1.6.6}$$

Definition 1.6.7. The **variance** of a Hermitian operator X for a state $|\psi\rangle$ denoted $\langle(\Delta X)^2\rangle$ is defined as

$$\langle(\Delta X)^2\rangle = \langle X^2\rangle - \langle X\rangle^2 = \langle(X - \langle X\rangle)^2\rangle \tag{1.6.7}$$

Definition 1.6.8. The **standard deviation** of a Hermitian operator X for a state $|\psi\rangle$ denoted σ_X is defined as

$$\sigma_X = \sqrt{\langle(\Delta X)^2\rangle} \tag{1.6.8}$$

Theorem 1.6.9. The Uncertainty Principle states that for any two observables A and B, the following relation holds

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{3} |\langle[A, B]\rangle|^2 \tag{1.6.9}$$

1.7 Spin-1/2 System

The spin 1/2 system is the simplist finite system in quantum mechanics it consists of single particles such as an electron with two possible quantum eigenvalues.

Definition 1.7.1. The spin-1/2 system is a 2 dimensional complex ket space where the system will collapse into either spin up or spin down in the direction of measurement. The standard basis is $\{|+\rangle, |-\rangle\}$ where $|+\rangle$ is spin up along the z-axis and $|-\rangle$ is spin down along the z-axis.

$$\begin{aligned} |+\rangle_X &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, |-\rangle_X = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \\ |+\rangle_Y &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, |-\rangle_Y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \\ |+\rangle &= \begin{pmatrix}1\\0\end{pmatrix}, |-\rangle = \begin{pmatrix}0\\1\end{pmatrix} \end{aligned} \tag{1.7.1}$$

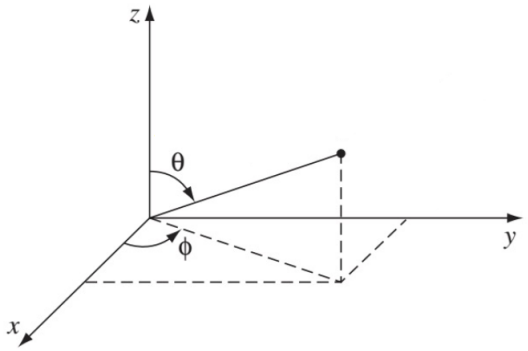
Definition 1.7.2. The S_z, S_y, S_z operators measure the spin along the x, y or z axis. The possible eigenvalues are $\pm\frac{\hbar}{2}$.

$$\begin{aligned} S_x &= \frac{\hbar}{2}\left[(|+\rangle\langle -|) + (|-\rangle\langle +|)\right] = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \\ S_y &= \frac{\hbar}{2}\left[i\,(|-\rangle\langle +|) - i\,(|+\rangle\langle -|)\right] = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \\ S_z &= \frac{\hbar}{2}\left[(|+\rangle\langle +|) - (|-\rangle\langle -|)\right] = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \end{aligned} \tag{1.7.2}$$

Definition 1.7.3. The **general spin operator** $S_{\hat{\mathbf{n}}}$ is defined by

$$S_{\hat{n}} = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z\cos\theta = \begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix} \tag{1.7.3}$$

for a unit vector $\hat{\mathbf{n}}$ defined by $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$



Corollary 1.7.4. The eigenvalues and eigenvectors for the $S_{\hat{\mathbf{n}}}$ operator are

$$\begin{aligned} |+\rangle_n &= \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle \\ |-\rangle_n &= \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}e^{i\phi}|-\rangle \end{aligned} \tag{1.7.4}$$

Chapter 2

Continuous Ket Space