Math 347H Lecture on Laplace Transform of Step Functions

March 13, 2020

Homework for the week of March 9 to March 13.

Section 6.1. 5, 7, 9, 15, 23

Section 6.2. 3, 5, 13, 14, 23

Section 6.3. 1, 5, 7, 11, 15, 19, 21, 23, 25

Goal of this lecture is to compute the Laplace transform of the simplest discontinuous functions quickly. We can use it to solve the differential equation with discontinuous external forces, which is very useful in many practical applications.

Recall: For any function defined on $[0, \infty)$, we defined its Laplace transform

$$F(s) = L\{f\}(s) = \int_0^\infty f(t)e^{-st}dt.$$

Notation. We will use $u_c(t)$ to denote the following function

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

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What is the the Laplace transform of this step function?

$$L\{u_c\} = \int_c^\infty e^{-st}dt = \frac{1}{s}e^{-sc}, \;\; c \geq 0.$$

What is $u_0(t)$? $u_\infty(t)$?

We can express other step functions using this simple step function.

How to rewrite the function $f_2(t) = \begin{cases} 1, & 0 \leq t < c, \\ 0, & t \geq c \end{cases}$ using that simple step function?

The answer is $f_2(t) = 1 - u_c = u_0(t) - u_c(t)$. $F_2(s) = \frac{1}{s} - \frac{e^{-sc}}{s}$.

Find the Laplace transform of $f_3(t) = \begin{cases} 1, & 2 \le t < 4, \\ 0, & \text{otherwise.} \end{cases}$.

It is called a rectangular pulse, and $f_3(t) = u_2(t) - u_4(t)$. We just need to worry about the nonzero part! $F_3(s) = \frac{1}{s}[e^{-2s} - e^{-4s}]$

Example. Sketch the graph of $y = h(t)$ and express $h(t)$ in terms of u_c, where

$$h(t) = \begin{cases} 2, & t \in [0, 4), \\ 3, & t \in [4, 6), \\ -2, & t \in [6, 7), \\ 3, & t \ge 7. \end{cases}$$

$$h(t) = 2(1 - u_4) + 3[u_4 - u_6] - 2[u_6 - u_7] + 3u_7(t) = 2 + u_4 - 5u_6 + 5u_7.$$

$$H(s) = \frac{1}{s}[2 + e^{-4s} - 5e^{-5s} + 5e^{-7s}.$$

We can also use the simple step function u_c to express more complicated piece-wise continuous functions. For example.

$$f(t) = \begin{cases} \sin t, & t \in [0, \pi/2), \\ 2\cos t, & t \ge \pi/2. \end{cases}$$

We can rewrite as $f(t) = [u_0 - u_{\pi/2}](t) \sin t + 2u_{\pi/2} \cos t$.

How do we find its Laplace transform quickly?

In general, for $g(t) = u_c(t) f(t - c)$ shifting $f(t)$ by a distance c in the positive t direction, we have, by the change of variable $z = t - c$,

$$L\{g\}(s) = \int_0^\infty g(t) e^{-st} dt = \int_c^\infty f(t - c) e^{-st} dt = \int_0^\infty e^{-cs} e^{-sz} f(z) dz = e^{-cs} F(s).$$

When $f(t) \equiv 1$, $g(t) = u_c(t)$. We get the special case $L\{u_c(t)\}(s) = e^{-sc}\frac{1}{s}$.

Let us come back to our previous example $f(t) = [u_0 - u_{\pi/2}](t)\sin t + 2u_{\pi/2}\cos t = \sin t - u_{\pi/2}(t)\sin t + 2u_{\pi/2}(t)\cos t$.

We have to be careful to rewrite $u_{\pi/2}\sin t = u_{\pi/2}\sin(t + \pi/2 - \pi/2)$, we have to start from $\sin(t + \pi/2) = \cos t$ Not original $\sin t$!

Rewrite $u_{\pi/2}(t)\cos t = u_{\pi/2}(t)\cos(t+\pi/2-\pi/2)$, we have to start from $\cos(t+\pi/2) = -\sin t$.

Therefore

$$L\{u_{\pi/2}\sin t\}(s) = e^{-\pi s/2}L\{\cos t\}(s) = \frac{s}{s^2+1}e^{-\pi s/2}$$

$$L\{u_{\pi/2}\cos t\}(s) = -e^{-\pi s/2}L\{\sin t\}(s) = -\frac{1}{s^2+1}e^{-\pi s/2}$$

$$F(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1}e^{-\pi s/2} - \frac{2}{s^2+1}e^{-\pi s/2}.$$

Two important shifting formulas of Laplace transform:

$$L\{u_c f(t-c)\}(s) = e^{-c\,s}F(s); \qquad L\{e^{ct}f(t)\}(s) = F(s-c).$$

One is shifting the original function $f(t)$, another is shifting the Laplace transform. Be careful with the signs before c!

Example. Find the inverse Laplace transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

Solution. The important thing is to find the inverse of $1/s^2$. The rest can be obtained from it. Since $L^{-1}\{\frac{1}{s^2}\} = t$, we have

$$f(t) = t - u_2(t)(t-2) = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \ge 2. \end{cases}$$

Example. Find the inverse Laplace transform of $G(s) = \frac{1}{s^2-4s+13}$.

Solution. Note that $G(s) = \frac{1}{(s-2)^2+3^2}$, which is a translation of $\frac{1}{s^2+3^2}$. Note that $L^{-1}\{\frac{1}{s^2+3^2}\} = \frac{1}{3}L^{-1}\{\frac{3}{s^2+3^2}\} = \frac{1}{3}\sin 3t$. Hence $g(t) = \frac{1}{3}\sin(3t)e^{2t}$

Work out more examples together if we have more time.

1. Sketch the graph of the function $f(t) = (t-3)u_2(5) - (t-2)u_3(t)$

2. Sketch the graph of $g(t) = \begin{cases} t^2, & t \in [0, 2), \\ 1, & t \geq 2, \end{cases}$ express it in terms of unit step function u_c, and find its Laplace transform.

3. Find the inverse Laplace transform of $F(s) = \frac{e^{-2s}}{s^2+s-2}$.