PHY 410 - Reference Sheet

Stirling's approximation - for very large N:

$$\log N! \approx N \log N - N$$

$$N! \approx \sqrt{2\pi N} N^N e^{-N}$$

Fractional uncertainty of $\mathbb{X}$ is uncertainty of expected value per particle.

$$\frac{\Delta \mathbb{X}}{N} = \frac{\sqrt{\langle \mathbb{X}^2 \rangle - \langle \mathbb{X} \rangle^2}}{N}$$

Boltzmann's constant
$k_B = 1.380649 \times 10^{-23} \mathrm{m}^2\,\mathrm{s}^{-2}\,\mathrm{K}^{-1}$
Entropy $S = k_B \sigma, \quad \sigma_{TOT} = \sigma_1 + \sigma_2$
Temperature $T = \tau / k_B$

Microcanonical Ensemble

Multiplicity function

$$g = \#\text{ of microstates}, \quad \mathcal{P}(n) = \frac{1}{g}$$

Expected value of $\mathbb{X}$ is the average across all microstates.

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n)\mathcal{P}(n) = \frac{1}{g}\sum_n \mathbb{X}(n)$$

Entropy can be written in terms of the multiplicity function.

$$\sigma(N,T,U,V,P) \equiv \log[g(N,T,U,V,P)]$$

Binary System

A **binary system** is a system of N particles where each particles has two possible states. Let $N_\uparrow$ is the number of particle in the up state and $N_\downarrow$ be the number of particles in the down state.

$$g(N, N_\uparrow) = \frac{N!}{N_\uparrow!(N - N_\uparrow)!}, \quad \sum_{N_\uparrow = 0}^{N} g(N, N_\uparrow) = 2^N$$

The binary system can be rewritten in terms of the differnce between up states and down states this is the **spin excess**.

$$2S = N_\uparrow - N_\downarrow$$

$$g(N,S) = \frac{N!}{(\frac{N}{2}+S)!(\frac{N}{2}-S)!}$$

$$\sum_{S=-\frac{N}{2}}^{S=\frac{N}{2}} g(N, N_\uparrow) = 2^N$$

Applying Stirling's approximation to the binary model, for large N the multiplicity function and fractional uncertainty are

$$g(N,S) \approx g(N,0)e^{-2s^2/N}$$

$$g(N,S) \approx \sqrt{\frac{2}{\pi N}}2^N e^{-2s^2/N}$$

$$\frac{\Delta S}{N} \approx \frac{1}{\sqrt{N}}$$

An example of a binary system is N spin 1/2 particles in an external **magnetic field** B. The total energy U and magnetization M of the system are

$$U = \sum_{i=1}^{N} -\vec{m}_i \cdot \vec{B} = -(N_\uparrow - N_\downarrow)mB = -2SmB$$

$$M = 2Sm = -U/B$$

$$g(N,U) = \frac{N!}{(\frac{N}{2} - \frac{U}{2mB})!(\frac{N}{2} + \frac{U}{2mB})!}$$

$$\sigma(N,S) \approx -\left(\frac{N}{2}+S\right)\log\left(\frac{1}{2}+\frac{S}{N}\right) - \left(\frac{N}{2}-S\right)\log\left(\frac{1}{2}-\frac{S}{N}\right)$$

$$M = Nm\tanh(mB/\tau)$$

Einstein Solid

An **einstein solid** is a system of N atoms where each atom is modeled as a harmonic oscillator the energy of the system is determined by the number of atoms n oscillating at frequency ω.

$$U = n\hbar\omega$$

$$g(N,n) = \frac{(n+N-1)!}{n!(N-1)!}$$

$$g(N,n) \approx \frac{\left(\frac{n+N}{n}\right)^n \left(\frac{n+N}{n}\right)^N}{\sqrt{2\pi n(n+N)/N}}$$

Thermal Equilibrium

Temperature

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_{N,V}$$

Thermal Equilibrium

$$\left(\frac{\partial \sigma_1}{\partial U_1}\right)_{N_1,V_1} = \left(\frac{\partial \sigma_2}{\partial U_2}\right)_{N_2,V_2}$$

$$\frac{1}{\tau_1} = \frac{1}{\tau_2}$$

2nd law of thermo - Change in entropy ≥ 0.
Sharpness of Equilibrium For a two binary systems, the number of states in a configuration of deviation δ from equilibirum is

$$g_1 g_2 = (g_1 g_2)_{max} e^{\left(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}\right)}$$

Canonical Ensemble

Partition Function - partition by energy levels for a fixed temperature

$$z = \sum_n e^{-\varepsilon_n/\tau}, \quad \mathcal{P}(n) = \frac{1}{z}e^{-\varepsilon_n/\tau}$$

$$z = \sum_\alpha g(\varepsilon_\alpha)e^{-\varepsilon_\alpha/\tau}, \quad \text{for degeneracy } g(\varepsilon_\alpha)$$

Expected Value of $\mathbb{X}$ is the average across all energies (Thermal Average).

$$\langle \mathbb{X} \rangle = \sum_n \mathbb{X}(n)\mathcal{P}(n) = \frac{1}{z}\sum_n \mathbb{X}(n)e^{-\varepsilon_n/\tau}$$

Expected Energy in the canonical ensemble is

$$U = \langle \varepsilon \rangle = \frac{1}{z}\sum_n \varepsilon_n e^{-\varepsilon_n/\tau}$$

$$U = \langle \varepsilon \rangle = \tau^2 \frac{1}{z}\frac{\partial z}{\partial \tau} = \tau^2 \frac{\partial}{\partial \tau}\log z$$

The total partition function and expected value for N non-interacting particles is simply

$$z_N = z_1^N$$

$$\langle \mathbb{X} \rangle_N = N\langle \mathbb{X} \rangle_1 \quad \Rightarrow \quad U_N = NU_1$$

Helmholtz Free Energy

$$F = U - \tau\sigma = U - ST = -\tau \log z$$

$\Delta F \leq 0$ - helmholtz free energy decreases
$dF = 0$ - helmholtz free energy minimized
Entropy $\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V, \quad S = k_B\sigma$
Temperature $\tau = \left(\frac{\partial U}{\partial \sigma}\right)_V$
Pressure

$$P = -\left(\frac{\partial U}{\partial V}\right)_\sigma = \tau\left(\frac{\partial \sigma}{\partial V}\right)_U = -\left(\frac{\partial F}{\partial V}\right)_\tau$$

Energy $U = -\tau^2 \frac{\partial}{\partial \tau}\left(\frac{F}{\tau}\right)$
Concentration and DeBroglie Wavelength

$$n = \frac{N}{V}, \quad n_Q = \frac{1}{\lambda_T^3}, \quad \lambda_T = \sqrt{\frac{2\pi\hbar^2}{m\tau}}$$

Single Particle Ideal Gas

A system in the canonical ensemble consisting of a signle particle in a box of side lengths L. The energy levels , partition function and average energy are

$$\varepsilon_n = \frac{\hbar^2\pi^2}{2mL^2}n^2 = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$$

and for the ultra-reletavistic case:

$$\varepsilon_n = pc = \frac{\pi\hbar c}{L}n = \frac{\pi\hbar c}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

$$z_1 = \frac{V}{\lambda_T^3}, \quad U_1 = \frac{3}{2}\tau$$

$$\sigma_1 = \log\left(\frac{V}{\lambda_T^3}\right) + \frac{3}{2}, \quad F_1 = -\tau\log\frac{V}{\lambda_T^3}$$

N-Particle Ideal Gas

Gibbs Resolution for systems of N identical particles the partition function is

$$z_N = \frac{1}{N!}(z_1)^N$$

$$PV = N\tau, \quad U = \frac{3}{2}N\tau$$

$$\sigma = N\left[\log\left(\frac{V}{N\lambda_T^3}\right) + \frac{5}{2}\right]$$

$$F = N\tau\left[\log\frac{n}{n_Q} - 1\right]$$

Thermal Radiation

Single Frequency Photon Gas is a system in the canonical ensemble that considers photons of a specific frequency ω.

$$\varepsilon_s = s\hbar\omega, \quad s = 0, 1, 2, 3, \ldots$$

$$z = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}}$$

$$\mathcal{P}(s) = \frac{e^{-s\hbar\omega/\tau}}{z}$$

$$\langle s \rangle = \frac{1}{z}\sum_{s=0}^{\infty} s e^{-s\hbar\omega/\tau} = \frac{1}{e^{\hbar\omega/\tau} - 1}$$

Photon Gas is an expansion of the single frequency photon gas that considers all the possible cavity modes. The modes are 2 fold degenerate for the 2 independent polarizations.

$$\omega_n = \frac{c\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{c\pi n}{L}$$

$$U = \langle \epsilon \rangle = 2\sum_n \frac{\hbar\omega_n}{e^{\hbar\omega_n/\tau} - 1} = \frac{\pi^2 V}{15(\hbar c)^3}\tau^4$$

Stefan-Boltzmann Law

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{e^{\hbar\omega/\tau} - 1}d\omega = \frac{\pi^2}{15(\hbar c)^3}\tau^4$$

Spectral Density Function

$$\frac{\partial}{\partial \omega}\frac{U}{V} = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/\tau} - 1}$$

Flux Density(σ_B=Stefan-Boltzmann constant)

$$J_\mu = \frac{1}{4}\frac{cU}{V} = \sigma_B\tau^4 = \frac{\pi^2}{60(\hbar c)^3}\tau^4$$

Phonons in a Solid (Debye Model)

Phonons in a solid is a system in the canonical ensemble that is very similar to thermal radiation except there is 3 fold degeneracy from 3 polarizations of phonons and an upper cutoff frequency ω_D due to the separation distance between atoms.

$$\omega_n = \frac{\pi c_S}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\pi c_s}{L}n$$

Debye cutoff frequency

$$\omega_D = c_S\left(\frac{6\pi^2 N}{V}\right)^{1/3}, \quad \omega_D = \frac{\pi c_S}{L}n_D$$

Grand Canonical Ensemble

Chemical Potential

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{\tau,V}$$

$$\mu = \tau \log\left(\frac{N\lambda_T^3}{V}\right) = \tau\log\left(\frac{n}{n_Q}\right)$$

$$\mu = \left(\frac{\partial U}{\partial N}\right)_{\sigma,V} = -\tau\left(\frac{\partial \sigma}{\partial N}\right)_{U,V}$$

Grand Partition Function - partition by energy levels for a fixed temperature and all possible values of N

$$\mathfrak{z} = \sum_N \sum_{n(N)} e^{-(\varepsilon_n^N - \mu N)/\tau}$$

$$\mathcal{P}(N,\varepsilon_n) = \frac{1}{\mathfrak{z}} e^{-(\varepsilon_n^N - \mu N)/\tau}$$

Fugacity

$$\lambda = e^{\mu/\tau}$$

$$\mathfrak{z} = \sum_N \lambda^N \sum_{s(N)} e^{-\varepsilon_s^N/\tau} = \sum_N \lambda^N z_N$$

Expected Value of $\mathbb{X}$ is the average across all energies (Diffusive Average).

$$\langle\mathbb{X}\rangle = \frac{1}{\mathfrak{z}}\sum_N\sum_s \mathbb{X}(N,s)e^{-(\varepsilon_s^N-\mu N)/\tau}$$

Expected Number of Particles in the grand canonical ensemble is

$$N = \langle N\rangle = \frac{1}{\mathfrak{z}}\sum_N\sum_s N e^{-(\varepsilon_s^N-\mu N)/\tau}$$

$$N = \langle N\rangle = \tau\frac{\partial}{\partial\mu}\log\mathfrak{z} = \lambda\frac{\partial}{\partial\lambda}\log\mathfrak{z}$$

Expected Energy in the grand canonical ensemble is

$$U = \langle\varepsilon\rangle = \frac{1}{\mathfrak{z}}\sum_N\sum_{n(N)}\varepsilon_n^N e^{-(\varepsilon_n^N-\mu N)/\tau}$$

$$U = \langle\varepsilon\rangle = \tau^2\left(\frac{\partial}{\partial\tau}\log\mathfrak{z}\right)_\lambda$$

Grand Potential

$$\Omega = U - \sigma\tau - \mu N$$

$$\Omega = -\tau\log\mathfrak{z}$$

$$\sigma = \left(\frac{-\partial\Omega}{\partial\tau}\right)_{V,\mu} \quad P = \left(\frac{-\partial\Omega}{\partial V}\right)_{\tau,\mu} \quad N = \left(\frac{-\partial\Omega}{\partial\mu}\right)_{\tau,V}$$

System of Non-interacting Particles

The grand partition function for a system with M energy states where n_α is the number of particles occupying a state is

$$\mathfrak{z} = \prod_{\alpha=1}^M \mathfrak{z}_\alpha, \quad \mathfrak{z}_\alpha = \sum_{n_\alpha} e^{-n_\alpha(\varepsilon_\alpha-\mu)/\tau}$$

$$U = \sum_{\alpha=1}^M \varepsilon_\alpha f(\varepsilon_\alpha), \quad N = \sum_{\alpha=1}^M f(\varepsilon_\alpha)$$

Fermions

$$n_\alpha = 0, 1$$

$$\mathfrak{z}_\alpha = 1 + e^{-(\varepsilon_\alpha-\mu)/\tau} = 1 + \lambda e^{-\varepsilon_\alpha/\tau}$$

Fermi-Dirac Distribution is the expected number of a particles in a particular energy ε_α.

$$\langle n_\alpha\rangle = f(\varepsilon_\alpha) = \frac{1}{e^{(\varepsilon_\alpha-\mu)/\tau}+1} = \frac{1}{\lambda^{-1}e^{\varepsilon_\alpha/\tau}+1}$$

For $\tau\to 0$: $f(\varepsilon_\alpha) = \theta(\varepsilon_\alpha - \mu)$

Bosons

$$n_\alpha = 0, 1, 2, 3, \ldots$$

$$\mathfrak{z}_\alpha = \frac{1}{1-e^{-(\varepsilon_\alpha-\mu)/\tau}} = \frac{1}{1-\lambda e^{-\varepsilon_\alpha/\tau}}$$

Boson Distribution is the expected number of a particles in a particular energy ε_α.

$$\langle n_\alpha\rangle = f(\varepsilon_\alpha) = \frac{1}{e^{(\varepsilon_\alpha-\mu)/\tau}-1} = \frac{1}{\lambda^{-1}e^{\varepsilon_\alpha/\tau}-1}$$

Ideal Gas

Both fermions and bosons behave identically at the classical limit $\varepsilon_\alpha - \mu >> \tau$.

$$\langle n_\alpha\rangle = f(\varepsilon_\alpha) = e^{-(\varepsilon_\alpha-\mu)/\tau}$$

$$\mathfrak{z} = \sum_N \lambda^N z_N = \sum_N \lambda^N \frac{1}{N!}z_1^N = e^{\lambda z_1}$$

$$\lambda = \frac{n}{n_Q}, \quad PV = N\tau, \quad U = \frac{3}{2}N\tau, \quad \mu = \tau\log\frac{n}{n_Q}$$

$$\sigma = N\left[\log\frac{n_Q}{n} + \frac{5}{2}\right], \quad F = N\tau\left[\log\frac{n}{n_Q} - 1\right]$$

Heat Capacity measures the change in heat energy per unit temperature

$$C_P > C_V, \quad C_V = \left(\frac{\partial U}{\partial T}\right)_V = \tau\left(\frac{\partial\sigma}{\partial T}\right)_V$$

$$C_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P = \tau\left(\frac{\partial\sigma}{\partial T}\right)_P$$

Monoatmc gas $C_V = \frac{3}{2}Nk_B$, $\quad C_P = \frac{5}{2}Nk_B$

Isothermal Expansion $\sigma_f - \sigma_i = N\log\frac{V_f}{V_i}$

$Q = N\tau\log\frac{V_f}{V_i}$

Isoentropic Expansion $\frac{\tau_f}{\tau_i} = \left(\frac{V_i}{V_f}\right)^{2/3}$

Internal Excitations

Expansion of the ideal gas to take into account the additional energy states from internal excitations.

$$z_{int} = \sum_\alpha e^{-\varepsilon_\alpha/\tau}, \mathfrak{z} = 1 + \lambda z_{int}e^{-\varepsilon_n/\tau}$$

Internal Excitation Corrections

$$\lambda = \frac{n}{n_Q z_{int}}, \mu = \tau\left(\log\frac{n}{n_Q} - \log z_{int}\right)$$

$$F = N\tau\left[\log\frac{n}{n_Q} - 1\right] - N\tau\log z_{int}$$

$$\sigma = N\left[\log\frac{n}{n_Q} + \frac{5}{2}\right] - \left(\frac{\partial F_{int}}{\partial\tau}\right)_V$$

Density of States

$$\sum_n f(\varepsilon_n) \approx \int_0^\infty D(\varepsilon)f(\varepsilon)d\varepsilon$$

$$\langle\mathbb{X}\rangle = \sum_{\mathbf{n}} f(\varepsilon_{\mathbf{n}})\mathbb{X}_{\mathbf{n}} = \int_0^\infty D(\varepsilon)f(\varepsilon)\mathbb{X}(\varepsilon)d\varepsilon$$

Finding Density of States

$$\Sigma(\varepsilon) = g_S\sum_n \theta(\varepsilon - \varepsilon_n)$$

$$D(\varepsilon) = \frac{d\Sigma(\varepsilon)}{d\varepsilon}$$

Expected Energy and **Expected Number of Particles** written in terms of the density of states:

$$U = \int_0^\infty \varepsilon D(\varepsilon)f(\varepsilon)d\varepsilon$$

$$N = \int_0^\infty D(\varepsilon)f(\varepsilon)d\varepsilon$$

At $\tau << \varepsilon_F$, the integrals can be reduced

$$U(\tau=0) = \int_0^{\varepsilon_F}\varepsilon D(\varepsilon)d\varepsilon$$

$$N(\tau=0) = \int_0^{\varepsilon_F} D(\varepsilon)d\varepsilon$$

Degenerate Fermi Gas

Fermions behave differently at quantum concentrations.

Fermi Energy - $\varepsilon_F = \mu(\tau=0)$

Groud State Energy - $U_0 = U(\tau=0)$

Finding Fermi Energy

$$N = \left(\frac{1}{2^3}\frac{4\pi}{3}n_F\right)n_F^2 = \pi\frac{n_F^3}{3} \Rightarrow n_f = \left(\frac{3N}{\pi}\right)^{1/3}$$

$$\varepsilon_F = \frac{\hbar^2\pi^2}{2mL^2}n_F^2 = \frac{\hbar^2}{2m}\left(\frac{3\pi^2N}{V}\right)^{2/3} = \tau_F$$

$$N = \int_0^{\varepsilon_F} D(\varepsilon)d\varepsilon$$

Sommerfeld Expansion

for finite $\tau << \varepsilon_F$:

$$\mu(\tau << \varepsilon_F) \approx \varepsilon_F + \left(\frac{\tau}{\varepsilon_F}\right)^2\varepsilon_F$$

$$U(\tau << \varepsilon_F) \approx U_0 + \left(\frac{\tau}{\varepsilon_F}\right)^2 U_0$$

Bose-Einstain Condensate

Bosons behave differently at quantum concentrations.

$N_0(\tau)$ is the number of ground state particles.

$N_e(\tau)$ is the number of excited state particles.

$$N_0(\tau) = \langle n_{\varepsilon_0}\rangle = f(\varepsilon_0,\tau) = \frac{1}{e^{(\varepsilon_0-\mu)/\tau}-1}$$

$$N_e(\tau) = \int_0^\infty f(\varepsilon)D(\varepsilon)d\varepsilon$$

Limits at $(\tau\approx 0)$:

$$N_0(\tau) \approx \frac{\tau}{\varepsilon_0 - \mu}$$

$$\mu \approx \varepsilon_0 - \frac{\tau}{N}$$

BEC Possible? N_e converges $\Rightarrow$ BEC

N_e diverges $\Rightarrow$ NO BEC

Critical Temperature The maximum temperature τ_E where BEC is possible.

$$N = N_e(\tau)|_{\mu=0}$$

Critical Concentration The smallest concentration n_E where BEC is possible. For $\tau < \tau_E$ the normal phase and condensate are approximately

$$N_e(\tau) = N\left(\frac{\tau}{\tau_E}\right)^{3/2}$$

$$N_0(\tau) = N\left(1 - \left(\frac{\tau}{\tau_E}\right)^{3/2}\right)$$

Boson Ideal Gas

$$\varepsilon_n = \frac{\hbar^2\pi^2}{2mL^2}\vec{n}^2, \quad \vec{n} = 1,2,3,4,\ldots$$

$$N_0 = \frac{1}{e^{-\mu/\tau}-1} = \frac{1}{\lambda^{-1}-1}$$

$$N_e = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{\lambda^{-1}e^{\varepsilon/\tau}-1}$$

$$(N_e)_{max} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\tau^{3/2}\int_0^\infty\frac{x^{1/2}dx}{e^x-1}$$

$$(N_e)_{max} \approx 2.612\left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2} = 2.612 n_Q V$$

$$\frac{N_e(\tau)}{N} = 2.612\frac{n_Q}{n}$$

$$n_E = \frac{(N_e)_{max}}{V} = 2.612 n_Q$$

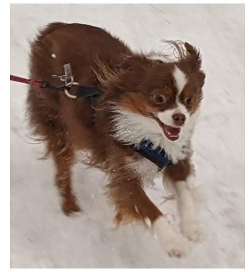
$$\tau_E = \frac{2\pi\hbar^2}{m}\left(\frac{n}{2.612}\right)^{3/2}$$

Thermodynamics

First Law - $dU = \text{đ}Q + \text{đ}W$
Reversible process: $dU = \tau d\sigma + \text{đ}W$

Heat Engines
Refrigerators
Gibbs Free Energy
Enthalpy
Chemical Reactions

DOG (bork)



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