Math 347H Summary of Laplace Transform,

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The Laplace transform is motivated by solve any constant second order linear differential equation

$$y'' + ay' + by = f(t), \;\; y(0) = c, \;\; y'(o) = d$$

with any force term $f(t)$, especially for discontinuous and even δ function in many practical problems. In this case, the solution $y(t)$ may not have continuous second order derivative.

1. Definition of the Laplace transform

$$L\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt, \;\; t > a.$$

2. When we try to solve this initial value problem by using the method of Laplace transform, we end up with

$$(s^2 + as + b)Y(s) - \; [y'(0) + sy(0)] - ay(0) = F(s).$$

$$(s^2 + as + b)Y(s) = y'(0) + sy(0)] + ay(0) + F(s) = F(s) + cs + d + a\; c.$$

$$Y(s) = [F(s) + cs + d + a\; c]\frac{1}{s^2 + as + b}.$$

We have to know (a) how to compute $F()s)$ quickly; and (b) how to find $y(t)$ from the expression of $Y(s)$.

3. To finish step 2, we have the following table of Elementary Laplace transforms, which is Table 6.2.1 from Page 321 in the text book

The most important are items 13, 14, 16, 17, and 18. We proved them in the class.

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^nf(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29

3. Connection between the method of Variation of Parameters and Laplace transform

For $c = d = 0$, we know the solution for the differential equation is $y = Ay_1 + By_2 + y_1u_1 + y_2u_2$ with

$$u_1 = -\int \frac{y_2(t)f(t)}{W(t)}dt, \quad u_2 = \int \frac{y_1(t)f(t)}{W(t)}dt, \quad W(t) = \det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}.$$

Especially, we can choose a particular u_1 and u_2 as the following definite integrals

$$u_1 = -\int_0^t \frac{y_2(\tau)f(\tau)}{W(\tau)}d\tau, \qquad u_2 = \int_0^t \frac{y_1(\tau)f(\tau)}{W(\tau)}d\tau.$$

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = \int_0^t [y_2(t)y_1(\tau) - y_1(t)y_2(\tau)]\frac{f(\tau)}{W(\tau)}d\tau$$

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = \int_0^t \det\begin{pmatrix} y_1(\tau) & y_2(\tau) \\ y_1(t) & y_2(t) \end{pmatrix}\frac{f(\tau)}{W(\tau)}d\tau$$

Set

$$H(t,\tau) = \det\begin{pmatrix} y_1(\tau) & y_2(\tau) \\ y_1(t) & y_2(t) \end{pmatrix}/W(\tau) = \det\begin{pmatrix} y_1(\tau) & y_2(\tau) \\ y_1(t) & y_2(t) \end{pmatrix}/\det\begin{pmatrix} y_1(\tau) & y_2(\tau) \\ y_1'(\tau) & y_2'(\tau) \end{pmatrix}.$$

It is easy to check that $y_p(0) = \int_0^0 H(0,\tau)f(\tau)d\tau = 0$.

$$y_p'(t) = H(t,t)f(t) + \int_0^t \frac{d}{dt}H(t,\tau)f(\tau)d\tau, \quad y_p'(0) = H(0,0)f(0) = 0!$$

We got the solution of the equation for $y(0) = c = 0 = d = y'(0)$.

If we used the Laplace transform, we should get $Y(s) = F(s)\frac{1}{s^2+as+b} = F(s)G(s)$. From the convolution formula, we should have

$$y(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Compare two formulas, we should have $H(t,\tau) = g(t-\tau)$ with $G = \dfrac{1}{s^2+as+b}$.

No matter how you choose linearly independent y_1 and y_2, $H(t,\tau)$ is the same thing! Especially, $H(t,t) = g(0)$ should be independent of t. This is obviously truce since $H(t,t) \equiv 0$! Why 0?

Example 1. $y'' + 16y = f(t)$. In this case, we can choose $y_(t) = \cos 4t$, $y_2(t) = \sin 4t$. $W(t) = W(0) = \det \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = 4$ from Abel's theorem.

$$H(t,\tau) = \frac{1}{4}\det \begin{pmatrix} \cos 4\tau & \sin 4\tau \\ \cos 4t & \sin 4t \end{pmatrix} = \frac{1}{4}[\cos 4\tau \sin 4t - \sin 4\tau \cos 4t = \frac{\sin 4(t-\tau)}{4}.$$

In this case,

$$g(t) = \frac{\sin 4t}{4}, \qquad G(s) = \frac{1}{4}\frac{4}{s^2+4^2} = \frac{1}{s^2+16}.$$

Example 2. $y'' - 3y' + 2y = f(t)$. In this case, we can choose $y_(t) = e^t$ $y_2(t) = e^{2t}$. $W(t) = \det \begin{pmatrix} e^t. & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix} = e^{3t}$.

$$H(t,\tau) = \frac{1}{e^{3\tau}} \det \begin{pmatrix} e^{\tau} & e^{2\tau} \\ e^{t} & e^{2t} \end{pmatrix} = \frac{1}{e^{3\tau}}[e^{\tau+2t} - e^{2\tau+t}] = e^{2(t-\tau)} - e^{t-\tau}.$$

In this case,

$$g(t) = e^{2t} - e^{t}, \quad G(s) = \frac{1}{s-2} - \frac{1}{s-1} = \frac{1}{(s-1)(s-2)} = \frac{1}{s^2 - 3s + 2}.$$

3. Connection between Laplace transform and Fourier transform

We already know the definition of Laplace transform of $f(t)$. You can think that as

$$F(s) = \int_{-\infty}^{\infty} f_e(t)\, e^{-s\,t} dt, \quad s \geq a, \quad f_e(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0. \end{cases}$$

The Fourier transform of a function f defined on $\mathbb{R}$ is

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-is\,t} dt.$$

The advantage of the Fourier transform is that we have an easy inverse Laplace transform formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)\, e^{is\,t} ds.$$

It is almost the Fourier transform of $\hat{f}$, the difference is the sign of exponential function.

Look at carefully, we see that $F(a + i\ s) = \widehat{f_e(t) e^{-a\,t}}(s)$. We should have

$$f_e(t) e^{-a\,t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(a + i\ s)\, e^{is\,t} ds.$$

For $t > 0$, and when $F(a+i\ s)\ e^{is\,t}$ is analytic function if $Im(s) \leq 0$, and it has finite many singular points $a_1, ..., a_k$. ($Im(a_k) > 0$), then the Residue theorem says that

$$f(t)e^{-a\,t} = i\sum_{j=1}^{k} Res(F(a+i\ s)\ e^{is\,t}, a_j).$$

How to compute the residue at a pole $z_0 \in \mathbb{C}$ of $H(z)$? You expand $H(z) = \sum_{l=-m}^{\infty} c_l(z-z_0)^l$, the coefficient c_{-1} is the residue at that point.

Example 3. Find the inverse Laplace inverse transform of $F(s) = \frac{1}{s+1}$.

The function F is analytic for $Re(s) \geq 0$. (choose $a = 0$). then $F(i\ s) = \frac{1}{1+i\ s}$ has a singularity at $s = i$, note that

$$i\ F(i\ s)e^{i\ s\ t} = \frac{1}{s-i}[e^{it(s-i)}e^{-t}] = e^{-t}\frac{1}{s-i}[1+O(s-i)].$$

The coefficient before $(s-i)^{-1}$ is e^{-t}. Therefore

$$f(t) = iRes(F(i\ s)e^{i\ s\ t},\ s = i) = e^{-t}.$$

Example 4. Find the inverse Laplace inverse transform of $F(s) = \frac{1}{(s+2)^2}$.

The function F is analytic for $Re(s) \geq 0$. (choose $a = 0$). then $F(i\ s) = \frac{1}{(2+i\ s)^2}$ has a singularity at $s = 2i$, note that

$$i\ F(i\ s)e^{i\ s\ t} = \frac{-i}{(s-2i)^2}[e^{-2t} + ite^{-2t}(s-2i) + ...].$$

The coefficient before $(s-2i)^{-1}$ is te^{-2t}. Therefore

$$f(t) = iRes(F(i\ s)e^{i\ s\ t},\ s = 2i) = t\ e^{-2t}.$$