PROBABILITY THEORY

mm 1

Problem 1.1 (problem 3.3 from Sheldon, 3rd ed.)

$$\begin{aligned}
&S = \{1, 2, 3, 4, 5, 6, 7\}\\
&E = \{1, 3, 5, 7\}\\
&F = \{7, 4, 6\}\\
&G = \{1, 4\}
\end{aligned}$$

(a) $EF = E \bigcap F = \{7\}$

(b) $E \bigcup FG = E \bigcup (F \bigcap G) = \{1, 3, 5, 7\} \bigcup \{4\} = \{1, 3, 4, 5, 7\}$

(c) $EG^c = \{1, 3, 5, 7\} \bigcap \{2, 3, 5, 6, 7\} = \{3, 5, 7\}$

(d) $EF^c \bigcup G = (\{1, 3, 5, 7\} \bigcap \{1, 2, 3, 5\}) \bigcup \{1, 4\} = \{1, 3, 5\} \bigcup \{1, 4\} = \{1, 3, 4, 5\}$

(e) $E^c(F \bigcup G) = \{2, 4, 6\} \bigcap \{1, 4, 6, 7\} = \{4, 6\}$

(f) $EG \bigcup FG = (E \bigcap G) \bigcup (F \bigcap G) = \{1\} \bigcup \{4\} = \{1, 4\}$

Problem 1.2

(a) The first book can be chosen in 10 different ways; the second in 9 ways, the third in 8 ways etc. Thus the answer is $10 \cdot 9 \cdot \ldots \cdot 1 = 10!$.

(b) We can treat this case as placement of 9 books on a shelf (since two books are "glued" together) and we should take into account that there are 2 possibilities to place those two books. Thus, the answer is $9! \cdot 2$.

Problem 1.3 (problem 3.14 from Sheldon, 3rd ed.)

Show that $P(exactly\ one\ of\ the\ events\ E\ or\ F\ occurs) = P(E) + P(F) - 2P(EF)$.

Proof.

$$P(exactly\ one\ of\ the\ events\ E\ or\ F\ occurs) =$$
$$P\left((E\bigcup F)\backslash(E\bigcap F)\right) = \left(P(E) + P(F) - P(E\bigcap F)\right) - P(E\bigcap F) =$$
$$P(E) + P(F) - 2P(E\bigcap F)$$

Problem 1.4

We introduce the following notation: A_t is an event that message A is transmitted; B_t is an event that message B is transmitted; A_r is an event that message A is received; B_r is an event that message B is received.

Probability that message A is received can be found as

$$P(A_r) = P(A_r|A_t) \cdot P(A_t) + P(A_r|B_t) \cdot P(B_t) = \frac{5}{6} \cdot 0.84 + \frac{1}{8} \cdot 0.16 = 0.72$$

In the same way we can find the probability that message B is received:

$$P(B_r) = P(B_r|A_t) \cdot P(A_t) + P(B_r|B_t) \cdot P(B_t) = \frac{1}{6} \cdot 0.84 + \frac{7}{8} \cdot 0.16 = 0.28$$

Probability that A is received given that A is sent can be found by using Bayes formula:

$$P(A_t|A_r) = \frac{P(A_r|A_t) \cdot P(A_t)}{P(A_r)} = \frac{5/6 \cdot 0.84}{0.72} = 0.972$$

Problem 1.5 (problem 3.16 from Sheldon, 3rd ed.)

Proof.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} =$$
$$\frac{n!}{(n-(n-r))!(n-r)!} = \binom{n}{n-r}$$

Every time you are choosing r objects among n objects, there are $n-r$ objects left. It means that choosing n objects is equivalent to choosing $n-r$ objects. This explains the above identity.

Problem 1.6

We can work with the reduced sample space. The problem reduces to computing the probability that a transistor chosen at random from 25 working and 10 defective transistors, is a working transistor: $P = 25/(25 + 10) = 5/7$

See also example 3.6 in the book.

Problem 1.7

(a) $E \bigcup E^c = S$

(b) $E \bigcap E^c =$

(c) $(E \bigcup F) \bigcap (E \bigcup F^c) = E \bigcap E \bigcup E \bigcap F^c \bigcup F \bigcap E \bigcup F \bigcap F^c = E \bigcap (F \bigcup F^c) = E$

(d) $(E \bigcup F)(E^c \bigcup F)(E \bigcup F^c) = F \bigcap (E \bigcup F^c) = (F \bigcap E) \bigcup (F \bigcap F^c) = E \bigcap F$

(e) $(E \bigcup F)(F \bigcup G) = EF \bigcup EG \bigcup FF \bigcup FG = EF \bigcup EG \bigcup F \bigcup FG = F \bigcup EG$

Problem 1.8 (problem 3.10 from Sheldon, 3rd ed.)

Proof.

We can represent event F as a union of two mutually exclusive events:

$$F = E \bigcup (F \setminus E), where E \bigcap (F \setminus E) = \emptyset$$

Then

$$P(F) = P(E) + P(F \backslash E) \geq P(E)$$

Problem 1.9

Let A be an event that the selected integer can not be divided by 2. Let B be an event that the selected integer can not be divided by 3.

$$P(A) = \frac{1}{2}; \quad P(B) = \frac{2}{3}$$

(a) The probability that this number can not be divided by 2 and it can not be divided by 3 is

$$P(AB) = P(A)P(B) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

(b)The probability that this number can not be divided by 2 or it can not be divided by 3 is

$$P(A \bigcup B) = P(A) + P(B) - P(AB) = \frac{1}{2} + \frac{2}{3} - \frac{1}{3} = \frac{5}{6}$$

Problem 1.10

Let A be an event that the tests has showed positive reaction twice. Conditional probability of occurence of event A in case the patetient has disease H_1 is

$$P(A|H_1) = 0.1 \cdot 0.1 = 0.01$$

Similarly,

$$P(A|H_2) = 0.5 \cdot 0.5 = 0.25$$

Using Bayes formula, we obtain

$$P(H_1|A) = \frac{P(H_1) \cdot P(A|H_1)}{P(A)} = \frac{0.6 \cdot 0.01}{0.6 \cdot 0.01 + 0.4 \cdot 0.25} \approx 0.057$$

and

$$P(H_2|A) = \frac{P(H_2) \cdot P(A|H_2)}{P(A)} = \frac{0.4 \cdot 0.25}{0.6 \cdot 0.01 + 0.4 \cdot 0.25} \approx 0.943$$

Problem 1.11

Let A be an event that the student knows the answer and C be the event that the answer is correct.

Using the formula of total probability we get

$$P(C) = P(C|A)P(A) + P(C|A^c)P(A^c) =$$

$$= 1 \cdot p + (1/n) \cdot (1 - p)$$

Using Bayes formula, we obtain

$$P(A|C) = \frac{1 - p}{np + (1 - p)}$$

Calculating for $n = 5, p = 1/2$ we get $P(C) = 3/5$, $P(A|C) = 1/6$. Calculating for $n = 4, p = 1/4$ we get $P(C) = 7/16$, $P(A|C) = 3/7$.