Problem 4.1

Given $P(a\ transmitted\ digit\ is\ received\ incorrectly) = 0.2$.

Let X be a r.v. representing a number of incorrectly received digits out of 5. X is a binomial r.v. with parameters (5, 0.2) $X \sim b(n = 5, p = 0.2)$.

The message is decoded correctly if at least 3 out of 5 digits are received correctly, or in other words, the number of errors is less or equal to 2.

$$P(X \leq 2) = \binom{5}{0} \cdot 0.2^0 \cdot 0.8^5 + \binom{5}{1} \cdot 0.2^1 \cdot 0.8^4 + \binom{5}{2} \cdot 0.2^2 \cdot 0.8^3 =$$

$$0.3277 + 0.4096 + 0.2048 = 0.9421$$

Thus, probability that the message is decoded incorrectly can be found as

$$P(incorrectly) = 1 - P(X \leq 2) = 0.0579$$

The above derivations are done under assumption that errors in transmitted digits are independent.

Problem 4.2

It is given that the parents are of hybrid type, that is they have genes r_1d_1 and r_2d_2. Since children receive 1 gene from each parent, they can have the following genes:

$$r_1r_2 \; or \; r_1d_2 \; or \; d_1r_2 \; or \; d_1d_2$$

All these 4 outcomes are equiprobable. A child will have the appearance of the dominant gene in the last 3 cases (r_1d_2 *or* d_1r_2 *or* d_1d_2). Thus, probability that a child has the appearance of the dominant gene is

$$p = \frac{3}{4}$$

Let X be a r.v. representing a number of children that have the appearance of the dominant gene. $X \sim b(n = 4, p = 0.75)$

Probability that X is equal to exactly 3 can be found as

$$P(X = 3) = \binom{4}{3} \cdot \frac{3}{4}^3 \cdot \left(1 - \frac{3}{4}\right)^{4-3} = 0.4219$$

Problem 4.3

Given $n = 50$ and $p = \frac{1}{100}$. Then $\lambda = n \cdot p = 50\frac{1}{100} = 0.5$

Let r.v. X represents the number of times you win a prize. If n is large and p is small, we can assume that X is approximately Poisson distributed.

a)

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda} = 1 - e^{-0.5} = 0.3935$$

b)

$$P(X = 1) = e^{-\lambda}\frac{\lambda^1}{1!} = e^{-0.5}\frac{1}{2} = 0.3033$$

c)

$$P(X \geq 2) = 1 - P(X = 2) = 1 - P(X = 0) - P(X = 1) =$$

$$1 - e^{-0.5} - 0.3033 = 0.0902$$

Problem 4.4

First, we find how $P(X = i+1)$ and $P(X = i)$ are connected. Then, observing the coefficient (if it is greater or smaller than 1), we can determine if the function $P(X = i)$ is increasing or decreasing when i increases.

$$P(X = i+1) = e^{-\lambda}\frac{\lambda^{i+1}}{(i+1)!} = \frac{\lambda}{i+1}e^{-\lambda}\frac{\lambda^{i}}{(i)!} = \frac{\lambda}{i+1}P(X = i)$$

$P(X = i)$ is increasing when

$$\frac{\lambda}{i+1} > 1 \;\Leftrightarrow\; i < \lambda - 1$$

$P(X = i)$ is decreasing if $i > \lambda - 1$.

Problem 4.5

(b)

$P(X < 6) = P(X = 2)+P(X = 4)+P(X = 5) = 0.1+0.2+0.2 = 0.5$

(c)

$P(2.5 < X < 7.5) = P(X = 4) + P(X = 5) = 0.2 + 0.2 = 0.4$

(d) We apply definition for condition probability:

$P(X = 2|X < 6) = \frac{P(X=2\ \ and\ \ X<6)}{P(X<6)} = \frac{P(X=2)}{P(X<6)} = \frac{0.1}{0.1+0.2+0.2} = 0.2$

Problem 4.6

(a) X and Y have the same distribution:
$P(X = i) = 1/6$ for $i = 1, \ldots, 6$

(b) X and Y are independent. Their joint pmf is
$P(X = i, Y = j) = 1/36$ for all $i = 1, \ldots, 6$ and $j = 1, \ldots, 6$.

(c) Since X and Y are independent,
$P(X > 3|Y = 2) = P(X > 3) = 1/6 + 1/6 + 1/6 = 1/2$

(d) $E[Z] = E[X+Y] = E[X]+E[Y] = 2\frac{1}{6}(1+2+3+4+5+6) = 7$

(e) $P(X = 1|Z = 3) = \frac{P(X=1,Z=3)}{P(Z=3)} = \frac{P(X=1,Y=2)}{P(Z=3)} = \frac{1/6 \cdot 1/6}{2 \cdot 1/36} = \frac{1}{2}$

(f) since X and Y are independent, we can write

$Var(Z) = Var(X + Y) = Var(X) + Var(Y) = 2Var(X) = 29.2$
$Var(X) = E[X^2] - E[X]^2 = \frac{91}{6} - \frac{49}{4} = 14.6$

(g) Since X and Y are independent, $Cov(X, Y) = 0$.

Problem 4.7

Let X be a random variable indicating an outcome of a trial. For each trial probability to observe a number greater than 4 is $p = P(X > 4) = P(X = 5) + P(X = 6) = 2/6 = 1/3$.

N is a geometric random variable with parameter $p = 1/3$. Therefore, $P(N = k) = p(1 - p)^{k-1}$ where $p = 1/3$.

On average we will need $E[N] = 1/p = 3$ trials.

Problem 4.8

Since the time interval we are working with is 1.5 hours, the number of customers in this interval is Poisson distributed with parameter $\lambda = 1.5 \cdot 10 = 15$.

$$P(10 < X \leq 15) = \sum_{i=11}^{15} \frac{e^{-15}15^k}{k!} = 0.4496$$