PROBABILITY THEORY
Session 1

Basic Concepts of Probability Theory

Topics:
- Introduction
- Terminology
- Axioms of probability
- How to compute probability using counting methods
- Conditional probability and Bayes' formula
- Independent events

Deterministic models vs Probabilistic models

- Deterministic model: the conditions under which an experiment is carried out determine the exact outcome of the experiment

- Probabilistic model: the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions

Frequency interpretation of probability

- Example: flipping a coin

$$\frac{N_0(n)}{n}$$

Frequency interpretation of probability

$$p = \lim_{n \to \infty} \frac{N_0(n)}{n}$$

- Not possible to perform an experiment infinite number of times
- Situations when an experiment is not repeatable
- → a mathematical theory of probability

Lecture plan

- Terminology
- Axioms of probability
- How to compute probability using counting methods
- Conditional probability and Bayes' formula
- Independent events

Terminology

Experiment → Outcome → Sample space → Event

- A random experiment is an experiment in which outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.
- Outcome is a result of an experiment that can not be decomposed into other results.
- The sample space S is defined as the set of all possible outcomes.
  - Discrete and continuous sample spaces
- An event is defined as a subset of S
  - Certain event S = all possible outcomes
  - Impossible (null) event = no outcomes

Exercise

Denmark wins

# of goals 31

DK scores less then 20 goals

Final Denmark – France
Observe a number of goals DK scores in a handball match

0, 1, 2, 3, 4, ….

- Our goal is to assign probability to certain **events**

  Outcome: A result of a random experiment.

  Sample Space: The set of all possible outcomes.

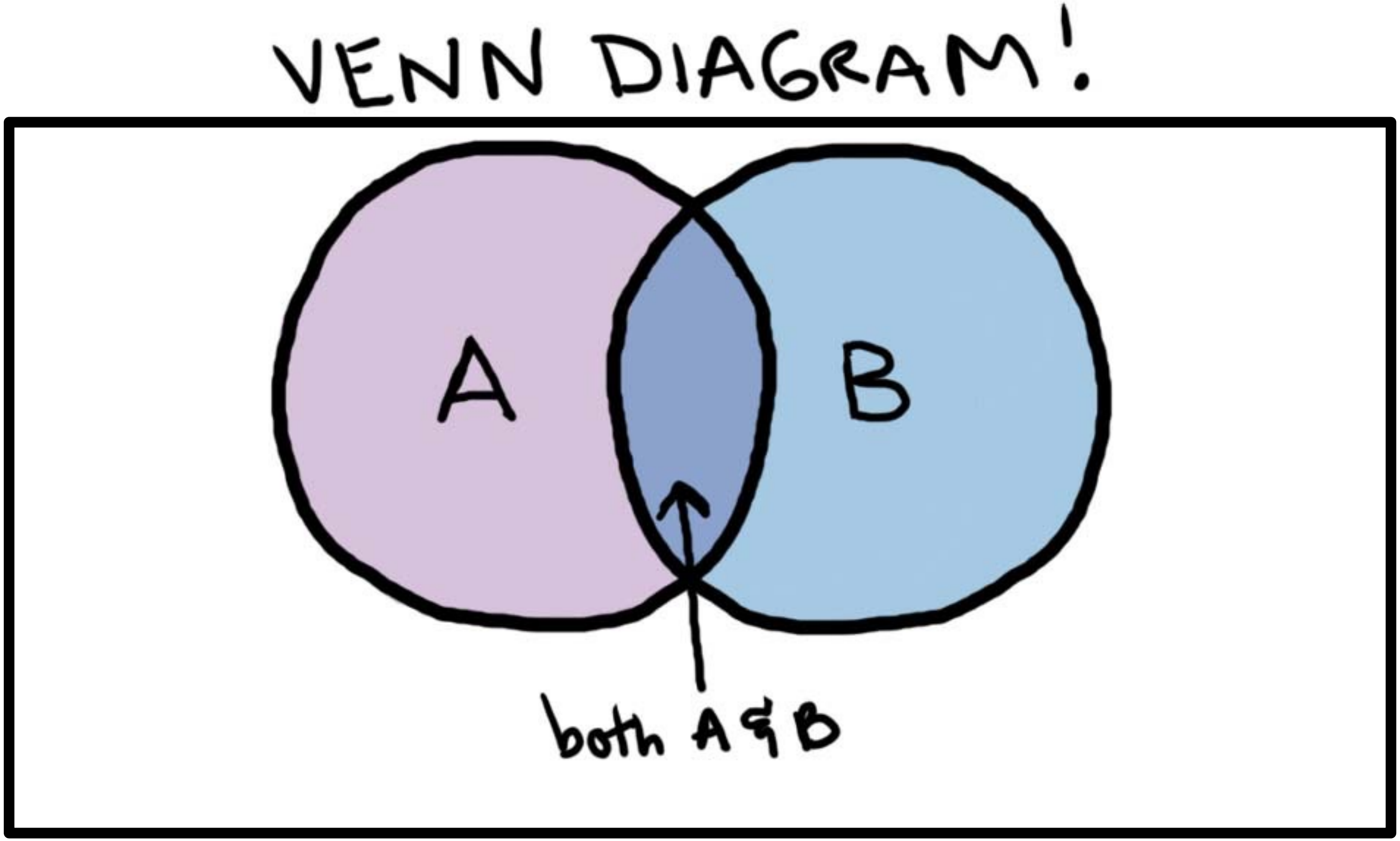
  Event: A subset of the sample space.

Set operations

- Union of A and B $A \bigcup B$ = {all outcomes that are either in A or B}
- Intersection of A and B = $AB = A \bigcap B$ {all outcomes that are both in A and B}
- Two events are mutually exclusive, if $AB = \emptyset$
- The complement of an event A = $A^c = \bar{A}$ {all events that are not in A}
- If all outcomes of B are in A, B is contained in A: $B \subset A$

- The definitions can be generalized for the case of n events

- Graphical representation of events can be made by Venn diagrams


VENN DIAGRAM!
A
B
both A & B

Set operations

Commutative law	$E \bigcup F = F \bigcup E$	$EF = FE$
Associative law	$(E \bigcup F) \bigcup G = E \bigcup (F \bigcup G)$	$(EF)G = E(FG)$
Distributive law	$(E \bigcup F)G = EG \bigcup FG$	$EF \bigcup G = (E \bigcup G)(F \bigcup G)$

DeMorgan's laws:

$$(E \bigcup F)^c = E^c F^c$$

$$(EF)^c = E^c \bigcup F^c$$

- What is the precedence of set operations?
    1. Parentheses
    2. Complement (or negation)
    3. Intersection
    4. Union
    5. Set difference

Exercise

S={1,2,3,4,5,6}, and A={1,2}, B={2,4,5},C={1,5,6}

Find 1) $A \cup B$; 2) $A \cap B$; 3) not A ; 4) $A \cap (B \cup C)$

Axioms of probability

- Let *E* be a random experiment. A probability law for the experiment *E* is a rule that assigns to each event *A* a number *p(A),* called the probability of A, that satisfies the following axioms:

- Axiom 1. $0 \leq P(A) \leq 1$

- Axiom 2. $P(S) = 1$

- Axiom 3. For any sequence of mutually exclusive events

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

Propositions

- Proposition 1.

$$P(A^c) = 1 - P(A)$$

Propositions

Proposition 2.

$$P(A \bigcup B) = P(A) + P(B) - P(AB)$$

Propositions

Proposition 3.

$$\text{If } A \subset B, \text{ then } P(A) \leq P(B)$$

Exercise

- In a presidential election, there are 3 candidates. Call them A, B, C. It is estimated that A has a 20 percent chance of winning the election, while B has a 40percent chance of winning.
- What is the probability that C wins?
- What is the probability that A **or** B win?

Suppose we have the following information:

1. There is a 60 percent chance that it will rain today.
2. There is a 50 percent chance that it will rain tomorrow.
3. There is a 30 percent chance that it does not rain either day.

a. Find the probability that it will rain today or tomorrow.
b. Find the probability that it will rain today and tomorrow.

Computing probabilities

- Sample space having equally likely outcomes
  – If *S* is a finite space, we ennumerate all possible outcomes
    S={1, 2, ..., N}

$$P(A) = \frac{Number\ of\ points\ in\ A}{N}$$

- The calculation of probabilities reduces to counting the number of outcomes in the event.

Multiplication principle

Suppose 2 experiments are to be performed. If there are *m* possible outcomes for experiment 1, and for each possible outcome of an experiment 1, there are *k* possible outcomes for experiment 2, then there are *mk* possible outcomes of the 2 experiments.

- If third experiment is to be performed with *l* possible outcomes → sample space of 3 experiments consists of *mkl* elements.

Example: roads between towns

Sampling with/without Replacement and with/without Ordering

N objects in the basket. We choose k objects.
Number of possible outcomes?

- Sampling with Replacement and with Ordering $\boxed{n^k}$
- Sampling without Replacement and with Ordering

$$\boxed{n(n-1)\cdots(n-k+1)}$$

- Sampling without Replacement and without Ordering

$$\boxed{\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}}$$

- Sampling with Replacement and without Ordering $\boxed{\binom{n-1+k}{k}}$

Sampling with Replacement and with Ordering

- A={1,2,3}; k=2 . Ennumerate all possibilities. How many?

Sampling without Replacement and with Ordering

- A={1,2,3}; k=2 . Ennumerate all possibilities. How many?

Sampling without Replacement and without Ordering

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Sampling with Replacement and without Ordering

- A={1,2,3}; k=2 . Ennumerate all possibilities. How many?

Birthday paradox

- If k people are at a party, what is the probability that at least two of them have the same birthday? Suppose that there are n=365 days in a year and all days are equally likely to be the birthday of a specific person.

Conditional probability

- We are often interested in calculating probabilities when some partial information concerning the results of the experiment is available; or recalculating it in light of new information
- It is often turns out that it is easier to compute the probability of an event if we first ”condition” on the occurence or non-occurence of a secondary event.

- Definition. The conditional probability is defined as

$$P(A|B) = \frac{P(AB)}{P(B)}, \quad P(B) > 0$$

Exercise

- When A and B are disjoint, find $P(A|B)$

- If $B \subset A$, find $P(A|B)$
-

Conditional probability

- It is often turns out that it is easier to compute the probability of an event if we first “condition” on the occurrence or non-occirence of a secondary event

Formula of total probability

- A and B are two events $A = AB \bigcup AB^c$

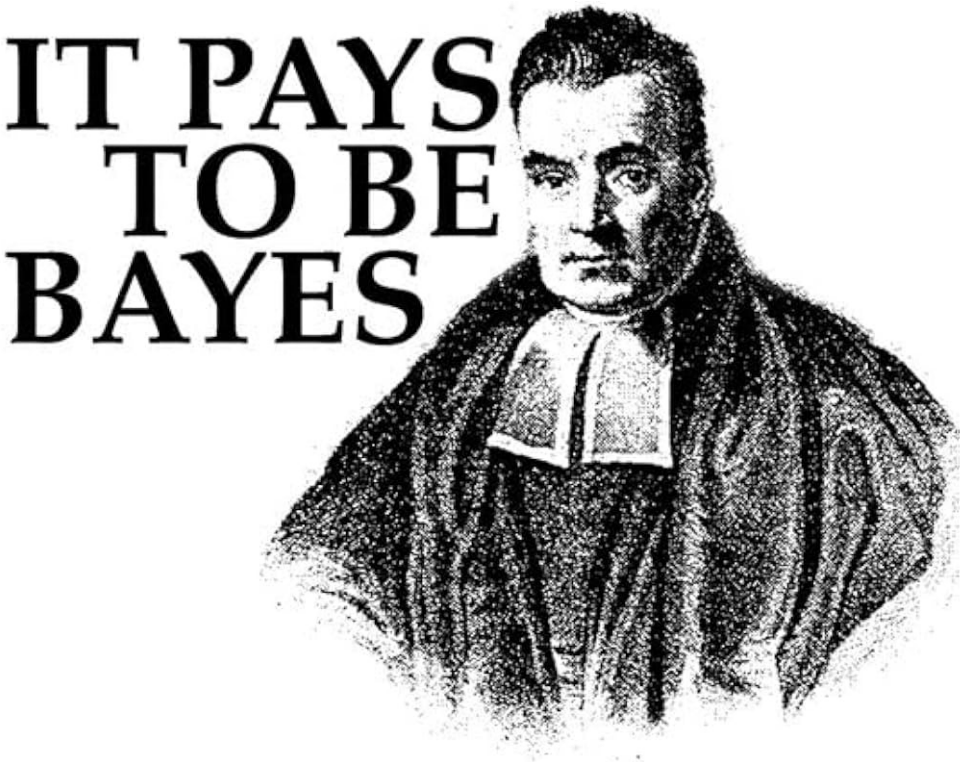
$$\boxed{\begin{gathered} P(A) = P(AB) + P(AB^c) = \\ P(A|B)P(B) + P(A|B^c)P(B^c) \end{gathered}}$$

- The probability of event A is a weighted average of conditional probabilities

Bayes' formula

- Suppose that A has occured and we are interested in determined if B has also occured:

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$


IT PAYS
TO BE
BAYES

Example

Peter and Eric are chefs at Restaurant. Peter works 6 days a week while Erik works one day a week. In 90% of the cases Peter's food is high quality while Eric's food is high quality in 50% of the cases.

One evening Restaurant serves an awful meal.

Whom should we blame?

Is it fair to conclude that Eric prepared the food that evening?

Independence of events

- Generally, knowing that B has occured, changes the chances of A's occurence. If it does not, then $P(A|B)=P(A)$

- Definition. Two events are independent, if $\boxed{P(AB)=P(A)P(B)}$

- Definition. Three events are independent, if

$$\boxed{\begin{gathered} P(ABC)=P(A)P(B)P(C) \\ P(AB)=P(A)P(B) \\ P(BC)=P(B)P(C) \\ P(AC)=P(A)P(C) \end{gathered}}$$

- Definition. N events are independent, if for any subset

$$\boxed{P(A_{r_1} \dots A_{r_k}) = P(A_{r_1}) \dots P(A_{r_k})}$$

Independent?

- You flip a coin and get a head **and** you flip a second coin and get a tail
- There is a sun shine **and** a lecture today is cancelled
- You draw obe card from a deck **and** its black and you draw a second card and it's black
- There is a snow storm **and** there is a traffic chaos

How do we know that there is independence?

Example

- We roll a dice twice. Let us define A as the event that the first outcome is odd. Let E be the event that the second outcome is 6. Let B be the event that both outcomes are the same. Finally, let C be the event that the sum of outcomes is even.
- Are A and E independent?
- Are A and B independent?
- Are A and C independent?