Introduction to Probability and Statistics

Session 3 Exercises

Israel Leyva-Mayorga

Exercise 1: Adapted from Chapter 8, problem 25

It is claimed that a certain type of bipolar transistor has a mean value of current gain that is at least 210. A sample of these transistors is tested. If the sample mean value of the current gain is 200 and it is known that the standard deviation is 35, would the claim be rejected at the 5 percent level of significance if

a) The sample size is 25?
b) The sample size is 64?

Tips: First, formulate the null hypothesis H_0. Then, identify the type of test:

- Is it one-side or two-side test?
- What is the test statistic T that you should use?
- Should you perform a z-test or a t-test?
- Will you compare T to the threshold c or the p-value to α?

Try to interpret the results.

Solution:

a) The sample size is 25?

   The null hypothesis is $\mu \geq 210$ and we have that the sample mean is $\overline{X}_n = \hat{\mu}_n = 200$ and we know that the standard deviation is $S_n = 35$.

   Since we know the standard deviation $\sigma = 35$, we perform a one-sided test using the normal distribution.

Several of the exercises were taken from the book "Introduction to Probability and Statistics for Engineers and Scientists" by Sheldon M. Ross, 3rd Ed.

The test statistic for this type of one-sided test is $T = (\hat{\mu}_n - \mu_0)\sqrt{n}/S_n$ and the rejection region is $R = \{x : T(x) < c\}$. We also need to calculate the value of $c = -Z_\alpha$.

Option 1: We get that $T = -1.4285$ and $c = -1.6449$. Since $T > c$, we **cannot reject** H_0.

Option 2: We calculate the p-value as $v = \Phi(T) = 0.0766$. Since $v > \alpha$, we **cannot reject** H_0.

b) The sample size is 64?

Now we have that $n = 64$ and re-do the calculations.

Option 1: We get that $T = -2.2857$ and $c = -1.6449$. Since $T < c$, we **reject** H_0.

Option 2: We calculate the p-value as $v = \Phi(T) = 0111$. Since $v < \alpha$, we **reject** H_0.

Interpretation

In this case, we cannot reject H_0 when the sample size is $n = 25$. This is because, even though the sample mean is lower than the value of $\mu_0 > \hat{\mu}_n$, this outcome is not so extreme with μ_0 and a relatively small sample size of 25 as indicated by the p-value of 0.0766.

As the sample size increases, it would be expected that extreme outcomes in the sample mean happen more rarely because the standard deviation of the sample mean $\sigma/\sqrt{n}$ decreases as n increases. However, this is not the case and the outcome for a sample mean of 200 is considered an extreme case as indicated by the p-value of 0.0111. Therefore, H_0 is rejected for $n = 64$ but not for $n = 25$.

Exercise 2: Chapter 8, problem 56

According to the U.S. Bureau of Census, 25.5 percent of the population of those age 18 or over smoked in 1990. A scientist has recently claimed that this percentage has since increased, and to prove her claim she randomly sampled 500 individuals from this population. If 138 of them were smokers, is her claim proved? Use the 5 percent level of significance.

Tips: Which type of distribution do the RVs have when there are only two options for the outcomes? What is the type of distribution that results from the sum of several of these RVs?

Solution:

The point here is that the scientist wants to find evidence to reject the hypothesis that the ratio of smokers in the U.S. is still 25.5 percent or lesser. So, we make the hypothesis $H_0 : p \leq p_0$, where $p_0 = 0.255$. We find that the p-value for this hypothesis is

$$v = \Pr(X \geq 138; p_0) = \sum_{k=138}^{500} \binom{500}{k} p_0^k (1-p_0)^{500-k} = 0.1525. \tag{1}$$

Since $v = 0.1525 > \alpha = 0.05$, the hypothesis H_0 cannot be rejected and we can assume that there has not been an increase in the percentage of smokers.

Another option is using the normal approximation use $\hat{p}_n = 0.276$ and performing a t-test with test statistic $T = (\hat{p}_n - p_0)\sqrt{n}/\sqrt{p_0(1-p_0)}$. The rejection region is $T > T_{n-1,\alpha}$. The calculated test statistic from the evidence is $T = 1.0773$ and $T_{n-1,\alpha} = 1.6479$, so the null hypothesis cannot be rejected.

Exercise 3: The students were asked to choose between the following two options for the rest of the course using an online poll.

1) Israel gives the rest of the theory this Friday morning. Students do the hypothesis testing exercises on their own. We do the workshop exercises on April 12.
2) Israel gives half of the theory this Friday and the rest on April 12. We do the exercises for hypothesis testing and the workshop mixed with the lectures.

Naturally, not all the students responded to the poll. As Fig. 1 with a total of $n = 5$ responses, 3 of them selecting option 2).

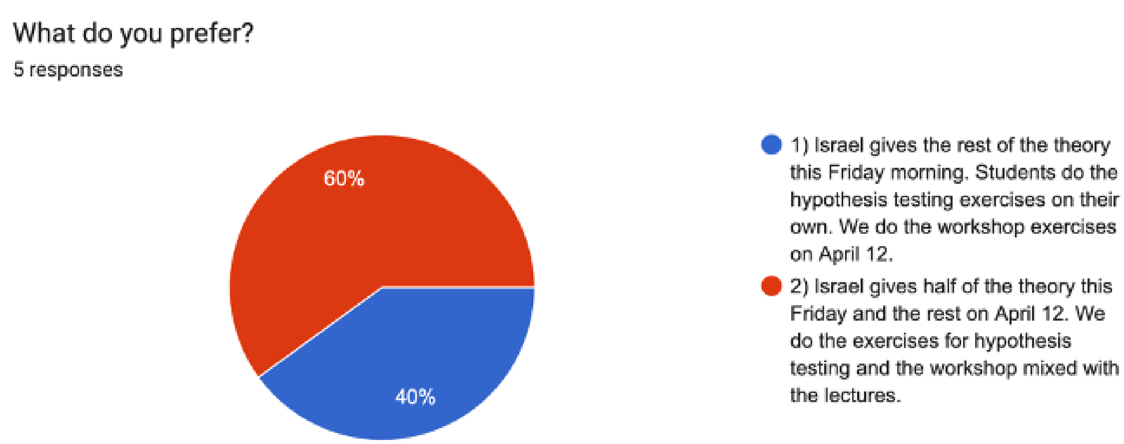

Fig. 1. Results of the voting for the next sessions of statistics.

To make a decision that is fair, Israel decided to conduct a test. First, he defined $X_i \sim \text{Bernoulli}(p)$, where $X_i = 0$ if the selected option was 1) and $X_i = 1$ if the selected option was 2). With this, he set $p_0 = 0.5$, defined the null hypothesis H_0: "the students prefer option 2)", which can be formulated as $p \geq p_0$. Then, he estimated

$$\hat{p}_5 = \frac{1}{5}\sum_{i=1}^{n} X_i = 0.6$$

a) Perform a one-sided test with $\alpha = 0.05$ using the null hypothesis H_0 and calculating the p-value for the Binomial distribution $v = P(X \leq k; n, p_0)$, where $k = 3$ is the number of students that chose option 2). Can you reject Israel's hypothesis?

b) Change the null hypothesis to be H_0': "the students prefer option 1)". What is the mathematical formulation of this? Can you reject this hypothesis with the collected evidence?

c) Assume that the result of the poll is now $k = 1$, meaning that only one student preferred option 2). Can the original null hypothesis H_0 be rejected?

Solution:

a) We calculate the p-value as

$$v = \Pr(X \leq k; p_0) = \sum_{j=0}^{k} \binom{n}{j} p_0^j (1-p_0)^{n-j} = 0.8125. \tag{2}$$

Since $v = 0.8125 > \alpha = 0.05$, Israel's hypothesis H_0 cannot be rejected.

b) The new null hypothesis to H_0': "the students prefer option 1)", is defined as $p \leq p_0$. Now we calculate the p-value as

$$v = \Pr(X \geq k; p_0) = \sum_{j=k}^{n} \binom{n}{j} p_0^j (1-p_0)^{n-j} = 0.5. \tag{3}$$

Since $v = 0.5 > \alpha = 0.05$, we cannot reject hypothesis H_0' either. This means that the evidence is not enough to give a strong statement for rejecting any of the two hypotheses.

c) We have the same problem for the hypothetical case of $k = 1$. By calculating the p-value $v = \Pr(X \leq k; p_0) = 0.1875$, we observe that is still higher than α. This means that, because the sample size is so small, the only case where the hypothesis H_0 can be rejected is if $k = 0$.

Exercise 4: Repeat **Exercise 1** but now assuming that the real standard deviation is not known and the estimated one from the sample is 35.

Solution:

a) The sample size is 25?

The null hypothesis is $\mu \geq 210$ and we have that the sample mean is $\overline{X}_n = \hat{\mu}_n = 200$ and the sample standard deviation is $S_n = 35$.

Since we have only the estimated value of the standard deviation $\hat{\sigma}_n = S_n$, we need to perform a one-sided test using the t-distribution.

The test statistic for this type of one-sided test is $T = (\hat{\mu}_n - \mu_0)\sqrt{n}/S_n$ and the rejection region is $R = \{x : T(x) < c\}$. We also need to calculate the value of $c = -T_{n-1,\alpha}$.

Option 1: We get that $T = -1.4285$ and $c = -1.7108$. Since $T > c$, we **cannot reject** H_0.

Option 2: We calculate the p-value as $v = P(T_{25-1} \leq T) = 0.083$. Since $v > \alpha$, we **cannot reject** H_0.

b) The sample size is 64?

Now we have that $n = 64$ and re-do the calculations.

Option 1: We get that $T = -2.2857$ and $c = -1.6694$. Since $T < c$, we **reject** H_0.

Option 2: We calculate the p-value as $v = P(T_{64-1} \leq T) = 0.0128$. Since $v < \alpha$, we **reject** H_0.

Exercise 5: Chapter 8, problem 43

A question of medical importance is whether jogging leads to a reduction in one's heart rate. To test this hypothesis, 8 nonjogging volunteers agreed to begin a 1-month jogging program. After the month, their pulse rates were determined and compared with their earlier values. If the data are as follows, can we conclude that jogging has an effect on the pulse rates with a 5 percent level of significance?

Subject	1	2	3	4	5	6	7	8
Pulse rate before	74	86	98	102	78	84	79	70
Pulse rate after	70	85	90	110	71	80	69	74

Tips: Which type of test should you use? The paired t-test. How should you formulate the null-hypothesis so that it represents that jogging has no effect on pulse rates?

Solution:

The test to use is a paired two-sided t-test where the null hypothesis is $H_0 : \mu_A = \mu_B$.

We calculate the mean of the pulse rate before $\overline{X}_B = 81.125$ and after $\overline{X}_A = 83.875$ jogging. Then, we calculate the variance before $S_B^2 = \frac{1}{8-1}\sum_{i=1}^{8}(X_i - \overline{X})^2 = 193.2678$ and after jogging $S_A^2 = 125.8392$.

Next, for the difference of the RVs, we calculate the mean $\hat{\mu} = -2.75$ and the standard deviation for the sample mean is $S_\mu = \sqrt{(S_B^2 + S_A^2)/n} = 6.3157$.

With these values, we calculate the test statistic as $T = |\mu|/S = 0.4354$ and the threshold is $c = T_{2n-2,\alpha/2} = 2.1447$. since $T < c$, we accept the null hypothesis H_0 that jogging has no effect on the pulse rate. Alternatively, we could calculate the p-value $v = \Phi(T) - \Phi(-T) = 0.3301$, which is greater than α.

Exercise 6:

Two different types of cable insulation have been tested. We need to determine the voltage level at which failures occur. The tests revealed that the individual cables failed at the following voltage levels.

TABLE I

VOLTAGE LEVEL AT WHICH FAILURES OCCURRED

Type A cables		Type B cables	
36	54	52	60
44	52	64	44
41	37	38	48
53	51	68	46
38	44	66	70
36	35	52	62
34	44		

We know that the voltage level that cables of type A can withstand is normally distributed with unknown mean μ_A and known variance $\sigma_A^2 = 40$. We also know that the voltage level that cables of type B can withstand is normally distributed with unknown mean μ_B and known variance $\sigma_B^2 = 100$.

With these data, can you reject the hypothesis $\mu_A = \mu_B$ with

a) 5 percent level of significance?

b) 10 percent level of significance?

c) 1 percent level of significance?

Tips: Which type of test should you use? What happens to the result of a test when we increase or decrease the level of significance?

Solution:

Again, our null hypothesis is $H_0 : \mu_A = \mu_B$.

Besides, now we are given the variances $\sigma_A^2 = 40$ and $\sigma_B^2 = 100$ and we know that the values are normally distributed. Therefore, we can solve this problem with a z-test.

We know that the distribution of the difference of means with normal RVs and known variance is $N\left(\mu_A - \mu_B, \sigma_A^2/n_A + \sigma_B^2/n_B\right)$. Therefore, the difference of means has a distribution $N(-13.0476, 11.1904)$.

The test statistic is $T = |\mu_A - \mu_B| / \sqrt{\sigma_A^2/n_A + \sigma_B^2/n_B} = 3.9004$.

a) Since $T = 3.9004 > Z_{\alpha/2}$ for $\alpha = 0.05$, the null hypothesis is rejected.

b) Given that we rejected the null hypothesis with $\alpha = 0.05$, we can immediately conclude that we will reject the null hypothesis for all $\alpha' > \alpha = 0.05$, including the case where $\alpha' = 0.1$. You can make the calculations to confirm this.

c) Even though the significance level is dropped to $\alpha = 0.01$, the threshold $c = Z_{0.005} = 2.575$ is still below the test statistic $T = 3.9004$. So, we also reject H_0 in this case.

Exercise 7: Chapter 9, problem 12

The following data set presents the heights of 12 male law school classmates whose law school examination scores were roughly equal. It also gives their annual salaries 5 years after graduation. Each of them went into corporate law. The height is in inches and the salary in units of $1,000.

Height	Salary
64	91
65	94
66	88
67	103
69	77
70	96
72	105
72	88
74	122
74	102
75	90
76	114

a) Do the above data establish the hypothesis that a lawyer's salary is related to his height? Use the 5 percent level of significance.

b) What was the null hypothesis in part a)

Solution:

a) By performing linear regression on the data, we obtain the regression coefficients $\hat{\beta}_0 = -4.9857$ and $\hat{\beta}_1 = 1.4571$ and a coefficient of determination $R^2 = 0.2341$. Therefore, the data exhibits a linear relationship with positive slope. That is, a linear and positive correlation.

b) To make this test, we start with the null hypothesis being H_0: $\beta_1 = 0$. If H_0 is rejected, we can confidently say that the data does not support the claim of a linear relationship. To proceed with the test, we find our test statistic to be $T = \left|\hat{\beta}_1 - 0\right| / \hat{\text{se}} = 1.7483$ and the t-value $T_{n-2,\alpha/2}$. By setting $\alpha = 0.05$, we obtain $T_{n-2,\alpha/2} = 2.2281$. In a two-sided t-test, the rejection region is $T > T_{n-2,\alpha/2}$. Since this is not the case, we the evidence is not sufficient to reject the fact that the trend can be observed even when there is no real linear relationship between the height and the salary.