PROBABILITY THEORY WORKSHOP

Problem 1: Parallel system
A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions. Independence of components is usually a key moment in designing a parallel system, e.g. to achieve independency of the components different operating systems or different types of processors can be chosen for different components.

Suppose that we have a parallel system with n independent components, each functions with probability p_i, $i = 1, \ldots, n$. What is the probability that the system work?

Problem 2: series connection of systems
Consider the case in which a system is composed of n components connected in series. Let us denote p_i probability that ith component fails, and let us assume that the components fail independently of each other. Find probability that the system fails. Find this probability if $n = 3$ and $p_i = 0.1$ for all i.

Problem 3: Binary Communication System
An input to a communication system is either 0 or 1. Lets assume that a receiver makes a random decision error with probability p. Given that the receiver has the output 1, find which input is more probable.

Problem 4
Suppose there are 5 jobs in a computer queue: 2 from application no. 1 and 3 from application no. 2. Jobs are selected at random to be performed. What are the probability that the 2 first selected jobs are for application no. 1?

Problem 5: Error correction coding
A communication system transmits binary information over unreliable channel that introduces random bit errors with probability $p = 10^{-3}$. The transmitter sends each bit 3 times; a decoder takes a majority vote of the received bits to decide on the output. Find probability that the receiver will make an incorrect decision.

Problem 6: Quality control
A manufacturing process produces a mix of "good" memory chips and "bad" memory chips. The lifetime of good chips follows the exponential law with a rate of failure α ($P(chip\ functions\ after\ t\ sec) = e^{-\alpha t}$). The lifetime of bad chips also follows the exponential law, but with the rate of failure is 1000α. Suppose that the fraction of good chips is $1 - p$ and of bad chips p.

- (a) Find the probability that a randomly selected chip is still functioning after t sec.
- (b) Suppose that every chip is tested for t seconds prior to leaving the factory. The chips that fail are discarded and the remaining chips are sent out to customers. Find the value of t for which 99% of the chips sent out to customers are good.

  Make the calculations assuming that $\alpha = 1/20000$ and $p = 0.1$.

Problem 7

A communication system consists of n components. Each of components independently functions with probability p.The system is able to operate if at least half of its components functions. As an engineer, we should choose if we should construct a 5-component system or a 3-component system. In order words, for which values of p is a 5-component system more likely to operate effectively than a 3-component system?

Problem 8

The lifetime if a computer is random variable with distribution

$$P(X > t) = 1/(1 + t), t > 0$$

Suppose that three new computers are installed at time $t = 0$. Their lifetimes are independent random variables that follow the equation above. At time $t = 1$ all three computers are still working. Find the probability that at least one computer is still working at time $t = 9$.

Problem 9

Suppose we send three packets of data through a switch with two packet-switched channels. Each packet is routed via one of the channels via a random assignment. Let the event that the packet is routed via channel 1 be A and the event that it is routed via channel 2 be B (Note $B = A^c$). The event that a packet is lost we denote as L. The loss probability for channel 1 is 0.01 and for channel 2 is 0.00. Suppose that we know that 30% of the packets are routed via channel 1.

- (a) if one packet is sent, what is the probability of its being lost?
- (b) if a packet is lost, what i the probability that it was routed via channel 1?
- (c) if 3 packets are sent, what is the probability of all three being lost?

Problem 10: Error control by retransmissions
A sends a message to B. At B an error detection mechanism is applied and upon detection an error, retransmission is required. Let $p = 0.9$ is probability to receive message correctly.

- (a) What is the probability that the message will be transmitted more than twice?
- (b) How many times on average the message will be transmitted?

Problem 11: speech waveforms
The pdf of the samples of the amplitude of speech waveforms is found to decay exponentially at a rate α:

$$f(x) = ce^{-\alpha|x|}, \quad x \in (-\infty, \infty)$$

- (a) Find constant c.
- (b) Find probability $P(|X| < v)$ where v is a certain level of an amplitude.

Problem 12: Transmission errors
A binary communication channel introduces a bit error in a transmission with probability p- Let X be the number of errors in n independent transmissions. Find pmf of X. Find the probability of one or fewer errors.

Problem 13
A communication system accepts a positive voltage V as input and output a voltage $Y = \alpha V + N$, $\alpha = 10^{-2}$, N is noise. N is Gaussian random variable with $\mu = 0$ and $\sigma = 2$. Find value of V that gives $P(Y < 0) = 10^{-3}$.

Problem 14
When we receive a signal, it is usually corrupted by noise, which we assume to be Gaussian with $\mu = 0$ and $\sigma = 0.1$. We are trying to distinguish between the case in which the signal is present at the receiver (in this case the signal amplitude is assumed to be Gaussian with $\mu = 0.5$ and the same $\sigma = 0.1$) and the case in which noise only is present. We do so by comparing the received signal to 0.25 and if the received signal is larger than 0.25, we claim that it is indeed present. Find the probability of the following two types of errors:

- (a) when we say the signal is absent but it is present;
- (b) when we say that the signal is present but it is not.

Problem 15
The lifetime of an electronic component (in years) is an exponential random variable with mean 10. If the component has functioned already for 10 years, what is the probability that it will last for another 10 years?

Problem 16
A packet switch has two input ports and two output ports. At a given time slot a packet arrives at each port with probability 1/2, and it is equally likely to be destined to output port 1 or 2. Let X and Y be the number of packets destined for output ports 1 and 2, respectively.

- (a) find the joint pmf and write it in a tabular form;
- (b) find marginal pmf for X and Y;
- (c) are X and Y independent?

Problem 17
In an Ethernet network messages are divided into packets and transmitted through the network. If the network is congested, packets may be lost. Suppose that in a very congested network the probability of a packet loss is 0.8. A packet is retransmitted until it is received at the destination.

- (a) find the probability that a packet is has to be sent at least three times
- (b) find the probability that a packet need to be sent at most 5 times for successful reception.

Problem 18
Let X be a noise voltage that is uniformly distributed in $S = \{-3, -1, +1, +3\}$ with $P(X = k) = 1/4$.

- (a) Find $E[Z]$ where $Z = X^2$.
- (b) The noise voltage X is amplified and shifted to obtain $Y = 2X + 10$, and then squared to produce $Z = Y^2$. Find $E[Z]$.

Problem 19
Random variables X and Y are independent. It is known that $E[X] = 2$, $E[Y] = -3$, $Var(X) = 1$ and $Var(Y) = 2$. Fund expectation of a random variable $Z = 4X^2Y + 2Y^2 + 1$.

Problem 20
The joint cdf for two random variables X and Y is given by

$$F_{X,Y}(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta x})$$

if $x \geq 0$ and $y \geq 0$ and it is zero otherwise. Find marginal cdfs.

Problem 21: Communication channel with discrete input and continuous output
The input X to a communication channel is $+1$ volt or -1 volt with equal probability. The output Y of the channel is the input plus noise voltage N that is uniformy distributed in he interval from -2 volts to $+2$ volts. Find $P(X = +1, Y \leq 0)$.

Problem 22: Minimum of n random variables
Web page requests arrive at a server from n independent sources. Source j generates packets with exponentially distributed inter-arrival times with rate λ_j. Find the distribution of the interarrival times between consecutive requests at the server.

Problem 23: Maximum of n random variables
A computing cluster has n independent redundant subsystems. each subsystem has an exponentially distributed lifetime with parameter λ. The cluster will operate as long as at least one subsystem is functioning. Find the cdf of the time until the system fails.

Problem 24: sum of iid random variables
Find mean and variance of the sum of n independent, identically distributed (iid) random variables, each with mean μ and variance σ^2.

Problem 25: call arrival

- (a) If calls arrive at a switch according to a Poisson process at a rate of five per second, what is the probability that no call arrives in a half-second interval?
- (b) what is the probability of one of more arrivals in the same interval?
- (c) what is the probability of one or less arrivals in the same interval?