

1.2.2 Set Operations

The union of two sets is a set containing all elements that are in A or in B (possibly both). For example, $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$. Thus, we can write $x \in (A \cup B)$ if and only if $(x \in A)$ or $(x \in B)$. Note that $A \cup B = B \cup A$. In Figure 1.4, the union of sets A and B is shown by the shaded area in the Venn diagram.

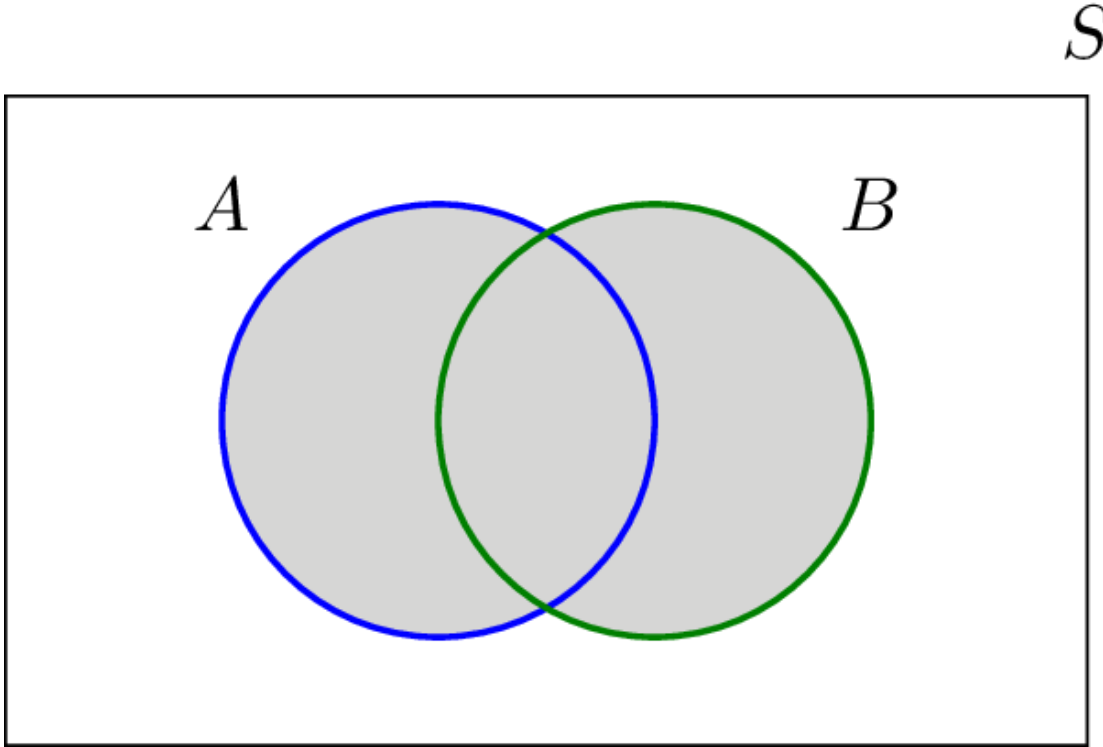


Fig.1.4 - The shaded area shows the set $B \cup A$.

Similarly we can define the union of three or more sets. In particular, if $A_1, A_2, A_3, \dots, A_n$ are n sets, their union $A_1 \cup A_2 \cup A_3 \dots \cup A_n$ is a set containing all elements that are in at least one of the sets. We can write this union more compactly by

$$\bigcup_{i=1}^n A_i.$$

For example, if $A_1 = \{a, b, c\}$, $A_2 = \{c, h\}$, $A_3 = \{a, d\}$, then $\bigcup_i A_i = A_1 \cup A_2 \cup A_3 = \{a, b, c, h, d\}$. We can similarly define the union of infinitely many sets $A_1 \cup A_2 \cup A_3 \cup \dots$.

The intersection of two sets A and B , denoted by $A \cap B$, consists of all elements that are both in A and B . For example, $\{1, 2\} \cap \{2, 3\} = \{2\}$. In Figure 1.5, the intersection of sets A and B is shown by the shaded area using a Venn diagram.

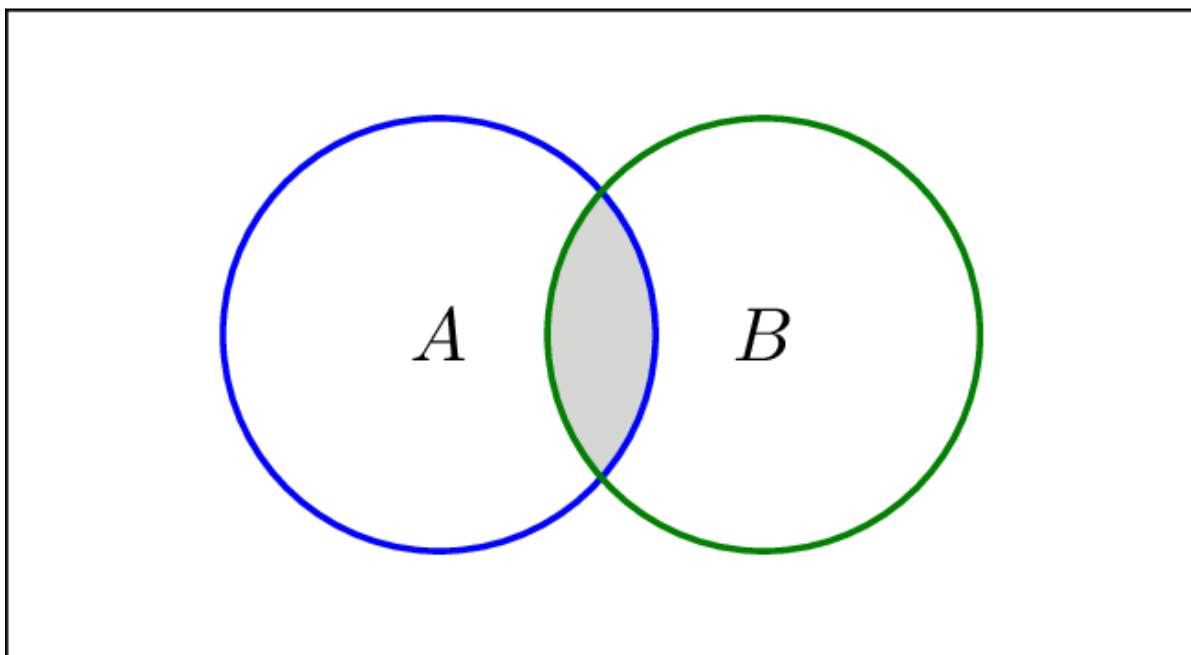
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Fig.1.5 - The shaded area shows the set $B \cap A$.

More generally, for sets A_1, A_2, A_3, \dots , their intersection $\bigcap_i A_i$ is defined as the set consisting of the elements that are in all A_i 's. Figure 1.6 shows the intersection of three sets.

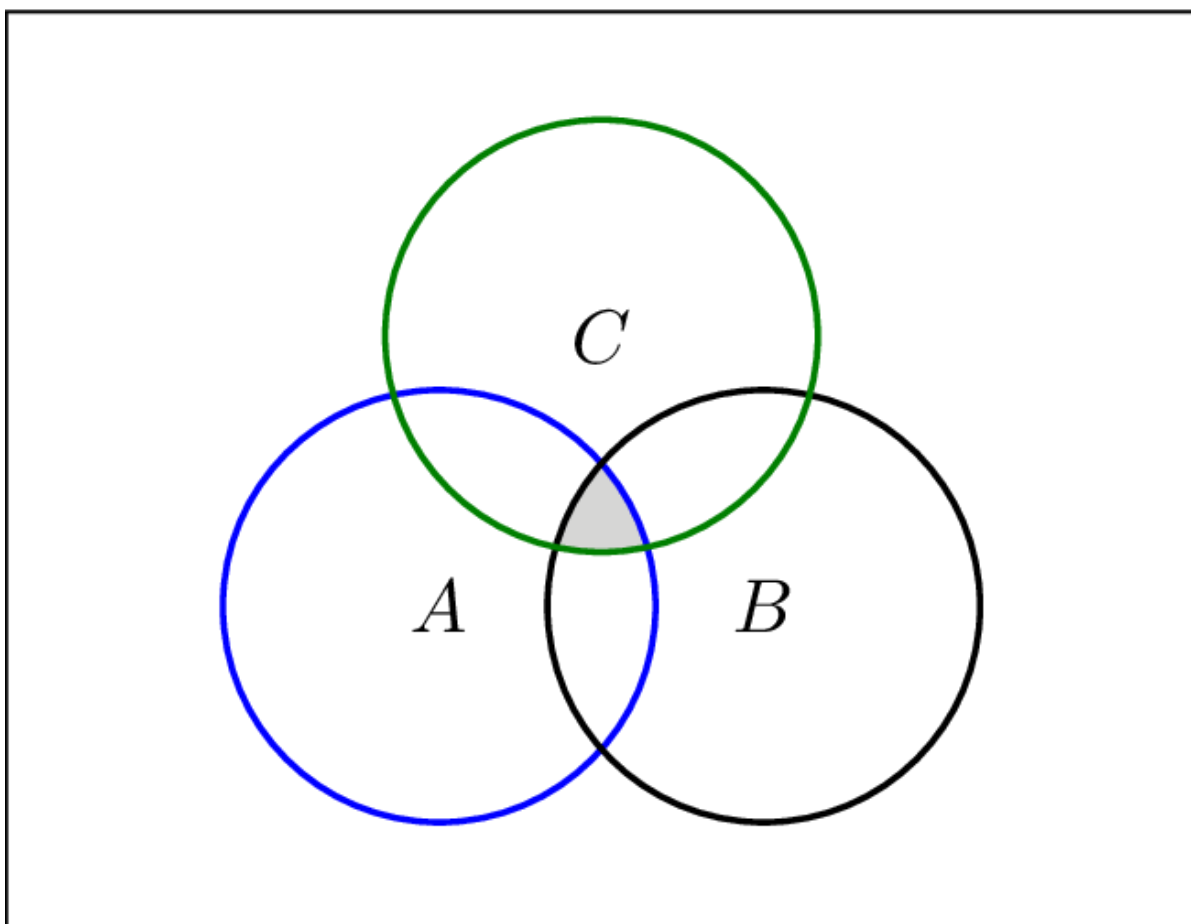
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Fig.1.6 - The shaded area shows the set $A \cap B \cap C$.

The complement of a set A , denoted by A^c or \bar{A} , is the set of all elements that are in the universal set S but are not in A . In Figure 1.7, \bar{A} is shown by the shaded area using a Venn diagram.

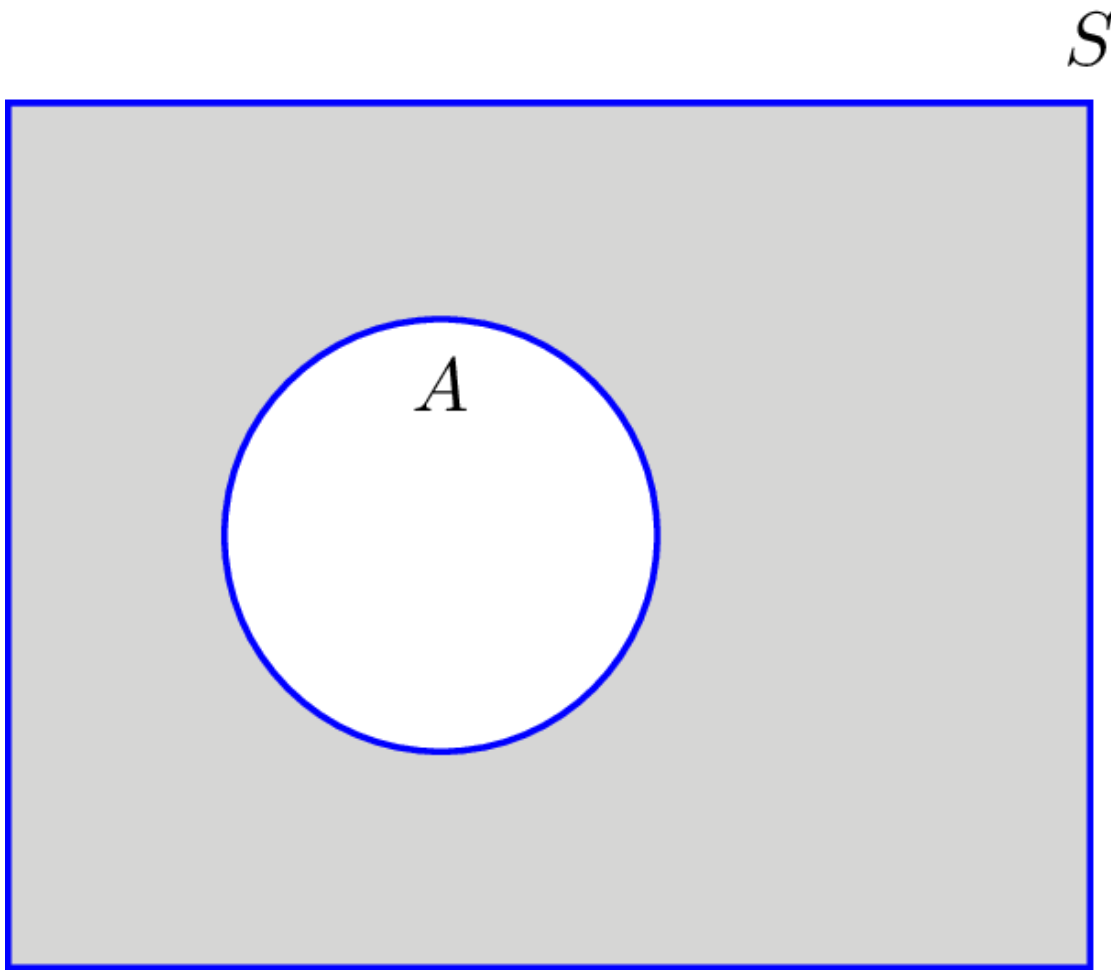


Fig.1.7 - The shaded area shows the set $\bar{A} = A^c$.

The difference (subtraction) is defined as follows. The set $A - B$ consists of elements that are in A but not in B . For example if $A = \{1, 2, 3\}$ and $B = \{3, 5\}$, then $A - B = \{1, 2\}$. In Figure 1.8, $A - B$ is shown by the shaded area using a Venn diagram. Note that $A - B = A \cap B^c$.

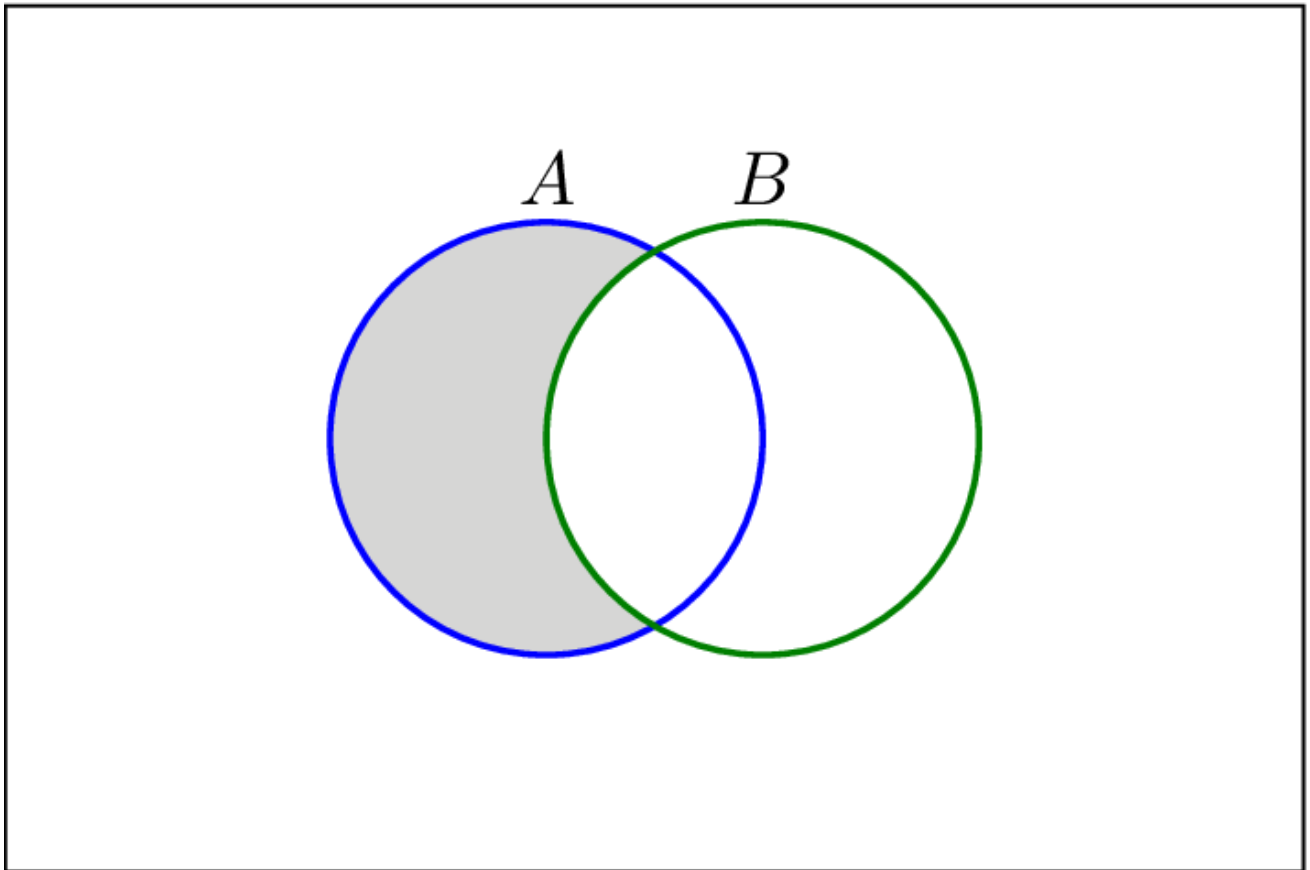


Fig.1.8 - The shaded area shows the set $A - B$.

Two sets A and B are mutually exclusive or disjoint if they do not have any shared elements; i.e., their intersection is the empty set, $A \cap B = \emptyset$. More generally, several sets are called disjoint if they are pairwise disjoint, i.e., no two of them share a common element. Figure 1.9 shows three disjoint sets.

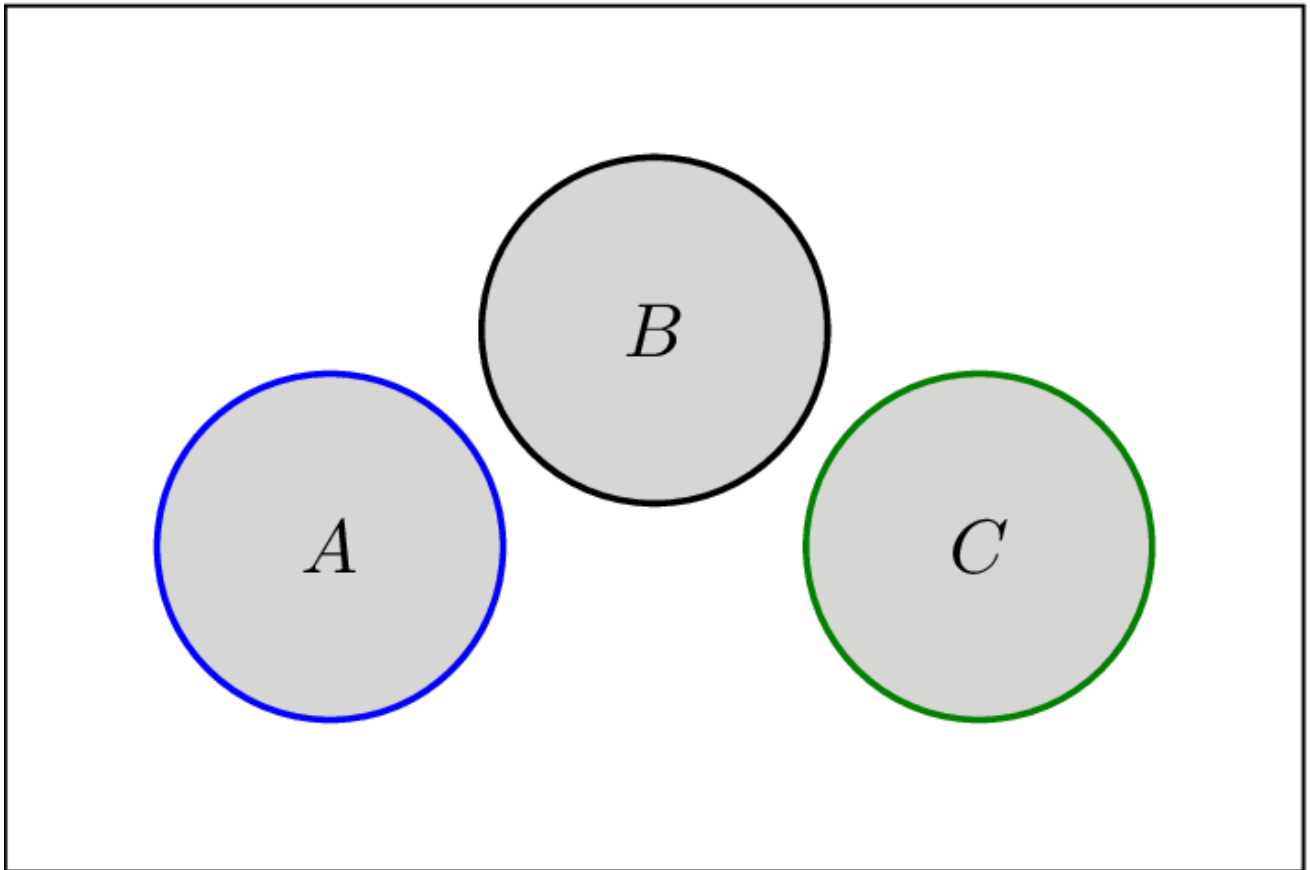


Fig.1.9 - Sets A , B , and C are disjoint.

If the earth's surface is our sample space, we might want to partition it to the different continents. Similarly, a country can be partitioned to different provinces. In general, a collection of nonempty sets A_1, A_2, \dots is a partition of a set A if they are disjoint and their union is A . In Figure 1.10, the sets A_1, A_2, A_3 and A_4 form a partition of the universal set S .

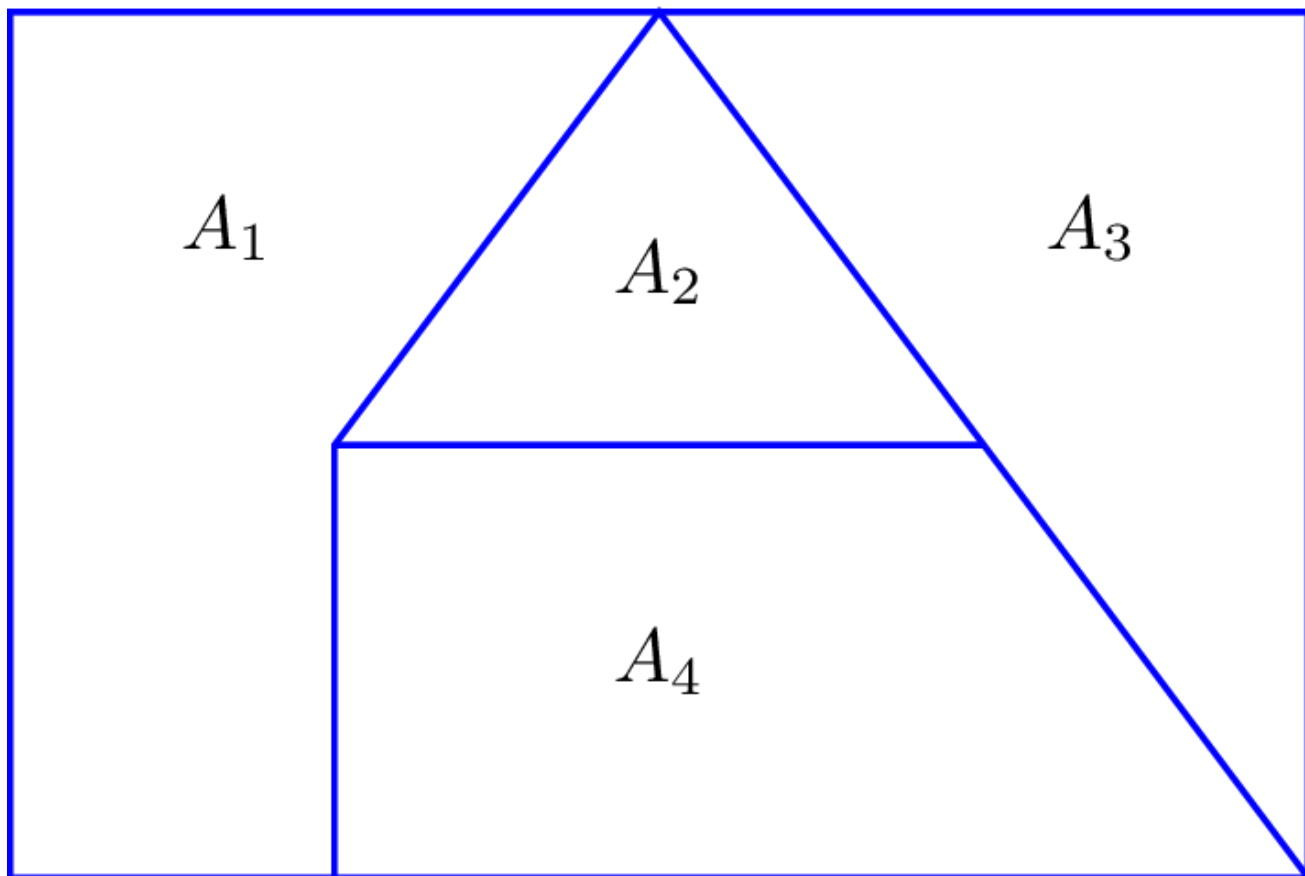
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Fig.1.10 - The collection of sets A_1, A_2, A_3 and A_4 is a partition of S .

Here are some rules that are often useful when working with sets. We will see examples of their usage shortly.

Theorem 1.1: De Morgan's law

For any sets A_1, A_2, \dots, A_n , we have

- $(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap A_3^c \dots \cap A_n^c$;
 - $(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n)^c = A_1^c \cup A_2^c \cup A_3^c \dots \cup A_n^c$.
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Theorem 1.1: Distributive law

For any sets A, B , and C we have

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
 - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
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Example 1.1

If the universal set is given by $S = \{1, 2, 3, 4, 5, 6\}$, and $A = \{1, 2\}$, $B = \{2, 4, 5\}$, $C = \{1, 5, 6\}$ are three sets, find the following sets:

- $A \cup B$
- $A \cap B$
- \overline{A}

d. \overline{B}

e. Check De Morgan's law by finding $(A \cup B)^c$ and $A^c \cap B^c$.

f. Check the distributive law by finding $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.

Solution

- a. $A \cup B = \{1, 2, 4, 5\}$.
- b. $A \cap B = \{2\}$.
- c. $\overline{A} = \{3, 4, 5, 6\}$ (\overline{A} consists of elements that are in S but not in A).
- d. $\overline{B} = \{1, 3, 6\}$.
- e. We have

$$(A \cup B)^c = \{1, 2, 4, 5\}^c = \{3, 6\},$$

which is the same as

$$A^c \cap B^c = \{3, 4, 5, 6\} \cap \{1, 3, 6\} = \{3, 6\}.$$

f. We have

$$A \cap (B \cup C) = \{1, 2\} \cap \{1, 2, 4, 5, 6\} = \{1, 2\},$$

which is the same as

$$(A \cap B) \cup (A \cap C) = \{2\} \cup \{1\} = \{1, 2\}.$$

A Cartesian product of two sets A and B , written as $A \times B$, is the set containing ordered pairs from A and B . That is, if $C = A \times B$, then each element of C is of the form (x, y) , where $x \in A$ and $y \in B$:

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{H, T\}$, then

$$A \times B = \{(1, H), (1, T), (2, H), (2, T), (3, H), (3, T)\}.$$

Note that here the pairs are ordered, so for example, $(1, H) \neq (H, 1)$. Thus $A \times B$ is not the same as $B \times A$.

If you have two finite sets A and B , where A has M elements and B has N elements, then $A \times B$ has $M \times N$ elements. This rule is called the multiplication principle and is very useful in counting the numbers of elements in sets. The number of elements in a set is denoted by $|A|$, so here we write $|A| = M$, $|B| = N$, and $|A \times B| = MN$. In the above example, $|A| = 3$, $|B| = 2$, thus $|A \times B| = 3 \times 2 = 6$. We can similarly define the Cartesian product of n sets A_1, A_2, \dots, A_n as

$$A_1 \times A_2 \times A_3 \times \dots \times A_n = \{(x_1, x_2, \dots, x_n) | x_1 \in A_1 \text{ and } x_2 \in A_2 \text{ and } \dots x_n \in A_n\}.$$

The multiplication principle states that for finite sets A_1, A_2, \dots, A_n , if

$$|A_1| = M_1, |A_2| = M_2, \dots, |A_n| = M_n,$$

then

$$|A_1 \times A_2 \times A_3 \times \dots \times A_n| = M_1 \times M_2 \times M_3 \times \dots \times M_n.$$

An important example of sets obtained using a Cartesian product is \mathbb{R}^n , where n is a natural number. For $n = 2$, we have

$$\begin{aligned}\mathbb{R}^2 &= \mathbb{R} \times \mathbb{R} \\ &= \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}.\end{aligned}$$

Thus, \mathbb{R}^2 is the set consisting of all points in the two-dimensional plane. Similarly, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and so on.

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