PROBABILITY THEORY WORKSHOP

Problem 1
Let A_i be an event that ith component works. $P(A_i) = p_i$.

$P(\textit{system works}) = 1 - P(\textit{system does not work}) = 1 - P(\textit{all components down}) =$

$$= 1 - P(A_1^c \dots A_n^c) = 1 - P(A_1^c) \cdots P(A_n^c) = 1 - \prod_{i=1}^{n} (1 - p_i)$$

Problem 2

$$P(F) = 1 - P(F^c) = 1 - \prod_{i=1}^{n} P(F_i^c) = 1 - \prod_{i=1}^{n} (1 - p_i)$$

$$P(F) = 1 - 0.9 = 0.271$$

Problem 3
Let us denote as A_0 an event that an input is 0; A_1 an event that an input is 1; B_0 an event that an output is 0; and B_1 an event that an output is 1.

Using formula of total probability we can find probability that 1 is received:

$$P(B_1) = P(B_1|A_0)P(A_0) + P(B_1|A_1)P(A_1) =$$

$$= p \cdot \frac{1}{2} + (1 - p) \cdot \frac{1}{2} = \frac{1}{2}$$

Using Bayes' formula we obtain:

$$P(A_0|B_1) = \frac{P(B_1|A_0)P(A_0)}{P(B_1)} = p$$

$$P(A_1|B_1) = \frac{P(B_1|A_1)P(A_1)}{P(B_1)} = 1 - p$$

Problem 4
Let us denote E_1 an event that the first selected job is for application no. 1 and E_2 an event that the second job is for the same application. The probability we have to calculate is expressed as

$$P(E_1 \bigcap E_2) = P(E_1)P(E_2|E_1) = \frac{2}{5}\frac{1}{4} = \frac{1}{1}0$$

Problem 5
Let K be a random variable denoting the number of errors in three bits. K can take on values $0, 1, 2, 3$. The receiver can correct a single error. Thus, the probability in question is

$$P(K \geq 2) = \binom{3}{2}(0.001)^2(0.999) + \binom{3}{3}(0.001)^3 \approx 3 \cdot 10^{-6}$$

Problem 6

- (a) Let C be the event that a chip is still functioning after t sec and let G be the event that the chip is good and G^c be the event that the chip is bad. By the theorem on total probability we have

$$P(C) = P(C|G)P(G) + P(C|G^c)P(G^c) = (1-p)e^{-\alpha t} + pe^{-1000\alpha t}$$

- (b) We need to find the value of t such that $P(G|C) = 0.99$. We apply Bayes' rule:

$$P(G|C) = \frac{P(C|G)P(G)}{P(C|G)P(G) + P(C|G^c)P(G^c)} = \frac{(1-p)e^{-\alpha t}}{(1-p)e^{-\alpha t} + pe^{-1000\alpha t}} =$$

$$= \frac{1}{1 + \frac{pe^{-1000\alpha t}}{1+(1-p)e^{-\alpha t}}} = 0.99$$

Rewriting the equation we get

$$t = \frac{1}{999\alpha}\ln(\frac{99p}{1-p}) = 48$$

Problem 7
Let X be a number of working components. X is binomially distributed with parameters (n, p).

Probability for a 5-component system to function is:

$$P_1 = P(X \geq 3) = P(X=3)+P(X=4)+P(X=5) = \binom{5}{3}p^3(1-p)^2+\binom{5}{4}p^4(1-p)+\binom{5}{5}p^5$$

Probability for a 3-component system to function is:

$$P_2 = P(X \geq 2) = P(X=2) + P(X=3) = \binom{3}{2}p^2(1-p) + \binom{3}{3}p^3$$

Thus, we need to compare:

$$10p^3(1-p)^2 + 5p^4(1-p) \geq 3p^2(1-p) + p^3$$

$$P \geq 1/2$$

Problem 8

$$P(X \geq 9 | X \geq 1) = \frac{P(X \geq 9, X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 9)}{P(X \geq 1)} =$$

$$= \frac{1/(1+9)}{1/(1+1)} = \frac{1}{5}$$

$$P(X < 9 | X \geq 1) = 1 - P(X \geq 9 | X \geq 1) = 4/5$$

$$P(at\ lear\ one\ computer\ is\ wprking) = 1 - P(none\ are\ working) = 1 - (\frac{4}{5})^3 = 0.48$$

Problem 9

- (a) $P(L) = P(L|A)P(A) + P(L|B)P(B) = 0.01 \cdot 0.3 + 0.005 \cdot 0.7 = 0.0065$
- (b) $P(A|L) = P(L|A)P(A)/P(L) = 0.01 \cdot 0.3/0.0065 = 0.46$
- (c) Packets are not obliged to be routed via the same channel. Since each packet individual packet may be routed via either channel, we have independence. $P = P(L_1)P(L_2)P(L_3) = 0.0065^3 = 2.710^{-7}$

Problem 10

Let M be a random variable denoting a number of times a certain message has been transmitted. $M = \{1, 2, \ldots\}$. We derive a general formula for probability that more than k transmissions is required to receive the message:

$$P(M > k) = P(M = k+1) + P(M = k+2) + \ldots = (1-p)^k p + (1-p)^{k+1} p + \ldots =$$

$$p(1-p)^k \sum_{i=0}^{\infty} (1-p)^i = (1-p)^k$$

- (a)Thus,
$$P(M > 2) = (1 - 0.9)^2 = 10^{-2}$$
- (b) M is a geometric random variable.
$$E[M] = 1/p \approx 1.11$$

Problem 11

- (a) We use normalization condition:

$$1 = \int_{-\infty}^{+\infty} ce^{\alpha|x|} \qquad dx = 2c \int_{0}^{+\infty} e^{\alpha x} \qquad dx =$$

$$= 2c \left. \frac{e^{\alpha x}}{-\alpha} \right|_0^\infty = \frac{2c}{\alpha}$$

Thus, $c = \alpha/2$.

- (b)

$$P(|X| < v) = \frac{\alpha}{2} \int_{-v}^{v} ce^{\alpha|x|} \qquad dx = \alpha \int_{0}^{v} ce^{\alpha x} \qquad dx =$$

$$= 1 - e^{\alpha v}$$

Problem 12
X takes on values in the set $\{0, 1, \ldots, n\}$.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

X is the binomial random variable.

$$P(X \leq 1) = \binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p(1-p)^{n-1} = (1-p)^n + np(1-p)^{n-1}$$

Problem 13

$$P(Y < 0) = P(\alpha V + N < 0) = P(N < - < alphaV) = P(Z < \frac{\alpha V}{\sigma})$$

where Z is a standard normal r.v.

$$= P(Z < -\frac{10^{-2}V}{2}) = 1 - \Phi(\frac{10^{-2}V}{2}) = 10\ -3$$

Thus,

$$\frac{10^{-2}V}{2} = 3.1$$

and

$$V = 6.210^2 = 620$$

Problem 14
Let X be a r.v. corresponding to "no signal". It's distribution is normal with $\mu = 0$ and $\sigma = 0.1$. Let Y be a r.v. corresponding to "signal is present". It's distribution is normal with $\mu = 0.5$ and $\sigma = 0.1$.

- (a) $P(Y < 0.25) = \mathit{normcdf}(0.25, 0.5, 1) = 0.0062$
- (b) $P(X > 0.25) = 1 - \mathit{normcdf}(0.25, 0, 1) = 0.0062$

Problem 15
we use memoryless property of an exponential r.v.:

$$P(X > 20|X > 10) = P(X > 10$$

Problem 16
The outcome for an input port can take on the following values: "n" (no packet arrival; with probability 1/2); "a1" (packet arrival for output port 1; with probability 1/4); "a2" (packet arrival for output port 2; with probability 1/4).

$$p(0,0) = P(n,n) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$$

$$p(0,1) = P((n,a2),(a2,n)) = \frac{1}{4}\frac{1}{4} + \frac{1}{4}\frac{1}{4} = \frac{1}{8}$$

$$p(1,0) = \frac{1}{4}$$

$$p(1,1) = \frac{1}{8}$$

$$p(0,2) = \frac{1}{16}$$

$$p(1,0) = \frac{1}{16}$$

X and Y are not independent.

Problem 17

- (a) $P(X \geq 3) = 1 - P(X < 3) = 1 - P(X = 1) - P(X = 2) = 1 - 0.2 - 0.2 \cdot 0.8 = 0.64$
- (b) $P(X \leq 5) = 0.2(1 + 0.8 + 0.8^2 + 0.8^3 + 0.8^4) = 0.67232$

Problem 18

- (a) First we find pmf of Z:

$$P(Z=9)=1/2$$

$$P(Z=1)=1/2$$

Thus,

$$E[Z]=1\cdot\frac{1}{2}+9\cdot\frac{1}{2}=5$$

Alternatively, one can calculate it as

$$E[Z]=E[X^2]=\sum_k k^2P(X=k)=\frac{1}{4}\left((-3)^2+(-1)^2+1^2+3^2\right)=5$$

- (b)

$$E[Z]=E[(2X+10)^2]=E[4X^2+40X+100]=4E[X^2]+40E[X]+100=120$$

Problem 19

$$E[X^2]=Var(X)+E[X]^2=1+4=5$$

$$E[Y^2]=Var(Y)+E[Y]^2=2+9=11$$

$$E[4X^2Y+2Y^2+1]=4E[X^2]E[Y]+2E[Y^2]+1=-4\cdot5\cdot3+22+1=-37$$

Problem 20

The marginal cdf's are obtained by letting one of the variables approach infinity:

$$F_X(x)=\lim_{y\to\infty}F_{X,Y}(x,y)=1-e^{-\alpha x},\ x\geq 0$$

$$F_Y(y)=\lim_{x\to\infty}F_{X,Y}(x,y)=1-e^{-\beta y},\ y\geq 0$$

Problem 21

We will use conditional probability to make the calculations:

$$P(X=+1,Y\leq y)=P(Y\leq 0|X=+1)P(X=+1)$$

When the input is $+1$, the output is uniformly distributed in the interval $[-1,3]$:

$$P(Y\leq 0|X=+1)=\frac{y+1}{4},\ for\ -1\leq y\leq 3$$

Thus, for $y = 0$ we get

$$P(X = +1, Y \leq 0) = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$$

Problem 22
Let the interarrival times for the different sources be $X_1, \ldots, X_n$. The time until the next arrival at the multiplexer is then:

$$Z = \min(X_1, \ldots, X_n)$$

The cdf of Z is:

$$1 - F_Z(z) = P(\min(X_1, \ldots, X_n) > z) = P(X_1 > z) \cdots P(X_n > z) =$$
$$= (1 - F_{X1}(z)) \cdots (1 - F_{Xn}(z)) = e^{-(\lambda_1 + \cdots + \lambda_n)z}$$

The interarrival time is an exponential random variable with rate $\lambda_1 + \cdots + \lambda_n$.

Problem 23
Let the of each subsystem be given by $X_1, \ldots, X_n$. The time until the last subsystem fails is:

$$W = \max(X_1, \ldots, X_n)$$

The cdf of W is:

$$F_W(w) = (F_X(w))^n = (1 - e^{-\lambda w})^n$$

Problem 24

$$E[S_n] = E[X_1] + \cdots E[X_n] = n\mu$$

The covariance of pairs of independent random variables is zero, thus

$$Var(S_n) = nVar(X_j) = n\sigma^2$$

Problem 25

- (a) $P(0\ in\ t = 0.5) = \frac{(5 \cdot 0.5)^0}{0!} e^{-5 \cdot 0.5} = 0.082$
- (b) $1 - P(0\ in\ t = 0.5) = 0.918$
- (c) $P(0\ in\ t = 0.5) + P(1\ in\ t = 0.5) = \frac{(2.5)^0}{0!} e^{-2.5} + \frac{(2.5)^1}{1!} e^{-2.5} = 0.2873$