PROBABILITY THEORY

Session 2 - solutions

Warm up questions

(a) No; unless we know some extra information that can help us to calculate the joint distribution (e.g. knowing that the random variables are independent).

(b) $P(X = 100) = 0$

(c) No. E.g. it is not true for $E[X^2] = E[X]^2$. However, it is true if $g(x)$ is a linear function.

(d) No.

(e) it will not change.

(f) $E[X]$, $E[X^2]$

(g) Yes.

Problem 1

(b) $P(X > 1/2) = 1 - P(X \leq 1/2) = 1 - F(1/2) = 1 - \frac{1}{4} == 3/4$

(c) $P(2 < X \leq 4) = F(4) - F(2) = 1 - 11/12 = 1/12$

(d) $P(X < 3) = F(3) = 11/12$

(e) In order to answer 4.4(e) we are using the explanation from problem 4.5 (see below): $P(X = 1) = F(1+) - F(1-) = 2/3 - 1/2 = 1/6$

Problem 2

First, we determine the exact value of λ using the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$:

$$\int_{-\infty}^{\infty} f(x)dx = \lambda \int_{-\infty}^{\infty} e^{-\frac{x}{100}} dx = \left[-100\lambda e^{-\frac{x}{100}}\right]_0^{\infty} = 100\lambda$$

Thus,

$$\lambda = \frac{1}{100}$$

Probability that a computer will function between 50 and 150 hours before breaking down:

$$P(50 \leq X \leq 150) = \int_{50}^{150} \frac{1}{100} e^{-\frac{x}{100}} dx = \left(-e^{-\frac{x}{100}}\right) |_{50}^{150} =$$

$$e^{1/2} - e^{-3/2} \simeq 0.3834$$

Probability that a computer will function less than 100 hours:

$$P(X \leq 100) = \int_{0}^{100} \frac{1}{100} e^{-\frac{x}{100}} dx = \left(-e^{-\frac{x}{100}}\right) |_{0}^{100} =$$

$$1 - e^{-1} \simeq 0.6321$$

Problem 3

from (a) we get $X = \{0, 1, 2, 3\}$
from (b) we get $P(X = 1) = P(X = 2)$
from (c) we get $P(X = 0) = P(X = 3)$
from (d) we get $P(X = 1) = \frac{1}{2}P(X = 0)$
Let $P(X = 1) = x$. Since the sum of all probabilities should give us 1, we have $x + x + 2x + 2x = 1$. Then $x = 1/6$ and we have the following pmf:

$P(X = 0) = 1/3$, $P(X = 1) = 1/6$, $P(X = 2) = 1/6$, $P(X = 3) = 1/6$.

Problem 4

The probability density function of X can be calculated as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{\infty} xe^{-(x+y)}dy = xe^{-x}\int_0^{\infty} e^{-y}dy =$$
$$xe^{-x}\left[-e^{-y}\right]_0^{\infty} = xe^{-x}$$

Thus,

$$f_X(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & otherwise \end{cases}$$

The probability density function of Y can be calculated as follows:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = e^{-y}\int_0^{\infty} xe^{-x}dx = \left[e^{-x}(-x-1)\right]_0^{\infty} e^{-y} = e^{-y}$$

Here are the details of integral calculations[1]:

$$\int xe^{-x}dx = -\int xde^{-x} = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$$

Thus,

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & otherwise \end{cases}$$

Noticing that $f(x,y) = f_X(x)f_Y(y)$, we can conclude that random variables X and Y are independent.

[1] use formula $\int udv = uv - \int vdu$

Problem 5

If $f(x)$ is a pdf, then it should satisfy the following equality:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Thus,

$$c\int_{o}^{\infty} e^{-2x}dx = 1$$

$$c\left(-\frac{1}{2}e^{-2x}\right)\Big|_{0}^{\infty} = 1$$

$$c \cdot \frac{1}{2} = 1$$

Answer: $c = 1$

$P(X > 2) = \int_2^\infty 2e^{-2x}dx = (-e^{-2x})|_2^\infty = e^{-4}$

Problem 6

a)

$$E[(2+4X)^2] = E[4+16X+16X^2] =$$

$$E[4] + 16E[X] + 16E[X^2] =$$

$$4 + 16 \cdot 2 + 16 \cdot 8 = 164$$

b)

$$E[x^2 + (X+1)^2] = E[4+16X+16X^2] =$$

$$E[1] + 2E[X] + 2E[X^2] =$$

$$1 + 2 \cdot 2 + 2 \cdot 8 = 21$$

Problem 7

In order to determine the constants a and b, we use the following equations:

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 (a + bx^2)dx = a + \frac{1}{3}b$$

$$\frac{3}{5} = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 (ax + bx^3)dx = \frac{1}{2}a + \frac{1}{4}b$$

Thus, we have a linear system of two equations with two unknowns:

$$\left\{ \begin{array}{lcl} a + b/2 & = & \frac{6}{5} \\ a + b/3 & = & 1 \end{array} \right.$$

Solving the system, we obtain $a = 0.6$ and $b = 1.2$.

Problem 8

a) Find the marginal probability distributions:

Marginal probability distribution for X_1

$$P(X_1 = 0) = \frac{1}{8} + \frac{1}{16} = \frac{3}{16}$$

$$P(X_1 = 1) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

$$P(X_1 = 2) = \frac{3}{16} + \frac{1}{8} = \frac{5}{16}$$

$$P(X_1 = 3) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

Marginal probability distribution for X_2

$$P(X_2 = 1) = \frac{1}{8} + \frac{1}{16} + \frac{3}{16} + \frac{1}{8} = \frac{1}{2}$$

$$P(X_2 = 2) = \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2}$$

b) Find expectation, variance and covariance for X_1 and X_2

$$E[X_1] = 0 \cdot \frac{3}{16} + 1 \cdot \frac{1}{8} + 2 \cdot \frac{5}{16} + 3 \cdot \frac{3}{8} = 1.875$$

$$E[X_1^2] = 0^2 \cdot \frac{3}{16} + 1^2 \cdot \frac{1}{8} + 2^2 \cdot \frac{5}{16} + 3^2 \cdot \frac{3}{8} = 4.750$$

$$Var[X_1] = E[X_1^2] - E[X_1]^2 = 4.750 - (1.875)^2 = 1.234$$

$$E[X_2] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 1.5$$

$$E[X_2^2] = 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2} = 2.5$$

$$Var[X_2] = E[X_2^2] - E[X_2]^2 = 0.25$$

$$E[X_1 X_2] = \frac{47}{16}$$

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2] = \frac{47}{16} - \frac{45}{16} = \frac{1}{8}$$

Problem 9

There are two ways to solve this problem. First approach: there exists a formula to calculate expectaton, when we know cdf

$$E[X] = \int_0^\infty (1 - F(x))dx$$

Substituting known $F(x)$ in this formula, we get:

$$E[X] = \int_0^4 (1 - \frac{x}{4})dx = \left(x - \frac{x^2}{8} \right)\Big|_0^4 = 2$$

Second approach: we convert cdf into pdf and use the classical

formula for expectation.

$$f(x) = \frac{dF(x)}{dx} = 1/4$$

when x lies in $(0, 4)$.

$$E[X] = \int_0^4 (x \cdot \frac{1}{4})dx = \left(\frac{x^2}{8} \right)\Big|_0^4 = 2$$

Problem 10

$$E[X] = \int_0^\infty xe^{-x}dx = [-xe^{-x}]|_0^\infty + \int_0^\infty e^{-x}dx = [-e^{-x}]|_0^\infty = 1$$

$$E[X^2] = \int_0^\infty x^2e^{-x}dx = [-x^2e^{-x}]|_0^\infty + 2\int_0^\infty xe^{-x}dx = 2$$

$$Var[X] = E[X^2] - E[X]^2 = 2 - 1^2 = 1$$

Problem 11

Suppose that you are given the cumulative distribution function F of a random variable X. Determine $P(X = 1)$.

Consider a small neighborhood around the point 1: $(1 - \varepsilon, 1 + \epsilon)$. $P(X = 1)$ can be interpreted in the following way: it is approximately equal to the probability that the random variable X is contained in the interval $(1 - \varepsilon, 1 + \epsilon)$ when ϵ is small. Formally,

$$P(X = 1) = \lim_{\epsilon \to 0} P(1 - \epsilon \leq x \leq 1 + \epsilon) = \lim_{\epsilon \to 0} (F(1 + \epsilon) - F(1 - \epsilon)) =$$

$$F(1 + 0) - F(1 - 0)$$

That is, $P(X = 1)$ is equal to the jump (or step) of the function F at the point 1.