Limits :

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \to \infty} \frac{\cos x}{x} = 0$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x \to 0} (1+x)^{1/x} = e$$

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x = e^k$$

$$\lim_{x \to \infty} \frac{\log x}{x^m} = 0$$

$$\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$$

$$\lim_{x \to 0} \frac{(a+x)^n - a^n}{x} = n a^{n-1}$$

$$\lim_{x \to 0} \frac{\sin^{-1} x}{x} = \frac{\tan^{-1} x}{x} = \cos x = 1$$

$$\lim_{n \to \infty} [h(x)]^{g(x)} = e^{\lim_{x \to \infty} (h(x) - 1)\, g(x)} \quad \Big\| \quad h(x) \to 0,\ g(x) \to \infty$$

Differential Calculus :

$$\frac{d}{dx} f(x) = \frac{f(x+\Delta x) + f(x)}{\Delta x}$$

1. $\frac{d}{dx}(x^n) = \underline{n x^{n-1}}$

2. $d(c) = 0$

3. $d(x) = 1$

4 $d(\sqrt{x}) = \frac{1}{2\sqrt{x}} = x^{\frac{1}{2}-1}$

5. $d(e^x) = e^x$

6. $d(a^x) = a^x \log a$

7. $d(\log x) = \frac{dx}{x}$

8 $d(\sin x) = \cos x$

9. $d(\cos x) = -\sin x$

10. $d(uv) = vu' + uv'$

11. $d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

12. $d(\tan x) = \sec^2 x$

13. $d(\sec x) = \sec x \tan x$

14. $d(\cot x) = -\operatorname{cosec}^2 x$

15. $d(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

16 $d(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

17. $d(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

18. $(\tan^{-1} x) = \frac{1}{x^2+1}$

19. $(\cot^{-1} x) = \frac{-1}{x^2+1}$

20. $(\sec^{-1} x) = \frac{-1}{x\sqrt{1-x^2}} \;||\; \frac{1}{x\sqrt{x^2-1}}$

21. $(\csc^{-1} x) = \frac{1}{x\sqrt{1-x^2}} \;||\; \frac{-1}{x\sqrt{x^2-1}}$

22. $d\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

23. $d(e^{ax}) = ae^{ax}$

$\Rightarrow e^{ax} \cdot \frac{d}{dx}(ax)$

24 $d e^{ax+b} = a e^{ax+b}$

25 $\frac{d}{dx} f(x) e^x \Rightarrow e^x [1 + f(x)]$

Integration : Unit 9

$$\int f'(x)\,dx = f(x) + c$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

$$\int e^x\,dx = e^x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos mx\,dx = \frac{\sin mx}{m} + c$$

$$\int \frac{dx}{x} = \log x + c$$

Substitution: ILATE (Inverse, Log, Algebra, Trigno, exponent)

if	subs
$a^2 - x^2$	$x = a\sin\theta$
$x^2 + a^2$	$x = a\tan\theta$
$x^2 - a^2$	$x = a\sec\theta$

S.no	d/dx	$\int$
1	$x^n\,dx$	$\frac{x^{n+1}}{n+1} + c$
2.	$1\,dx$	$x + c$
3.	$\frac{1}{2\sqrt{x}}\,dx$	$\sqrt{x} + c$
4	$\frac{-1}{x^2}\,dx$	$\frac{1}{x} + c$
5.	$\cos x\,dx$	$\sin x + c$
6.	$\sin x\,dx$	$-\cos x + c$
7	$\sec^2 x\,dx$	$\tan x + c$

	$\frac{d}{dx}$	$\int$
8.	$\sec x \tan x \, dx$	$\sec x + c$
9.	$\operatorname{cosec}^2 x \, dx$	$-\cot x + c$
10	$\operatorname{cosec} x \cot x \, dx$	$-\operatorname{cosec} x + c$
11.	$\tan x \, dx$	$\log \lvert \sec x \rvert + c$
12.	$\sec x \, dx$	$\log \lvert \sec x + \tan x \rvert + c$
13	$\cot x \, dx$	$\log \lvert \sin x \rvert + c$
14	$\operatorname{cosec} x \, dx$	$\log \lvert \operatorname{cosec} x - \cot x \rvert + c$
15	$\frac{1}{x} dx$	$\log x + c$
16.	$\frac{1}{\sqrt{1-x^2}} dx$	$\sin^{-1} x + c$
17.	$\frac{-1}{\sqrt{1-x^2}} dx$	$\cos^{-1} x + c$
18.	$\frac{1}{x^2+1} dx$	$\tan^{-1} x + c$
19	$\frac{-1}{x^2+1} dx$	$\cot^{-1} x + c$
20.	$\frac{1}{x\sqrt{1-x^2}} dx$	$\sec^{-1} x + c$
21.	$\frac{-1}{x\sqrt{1-x^2}} dx$	$\operatorname{cosec}^{-1} x + c$

d/dx $\qquad\qquad$ $\int$

22 $\dfrac{1}{a^2+x^2}\,dx$ $\qquad$ $\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$

23 $\dfrac{1}{a^2-x^2}\,dx$ $\qquad$ $\dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right|+c$

24 $\dfrac{1}{\cancel{x}+x^2-a^2}\,dx$ $\qquad$ $\dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right|+c$

25. $\dfrac{1}{\sqrt{a^2-x^2}}\,dx$ $\qquad$ $\sin^{-1}\left(\dfrac{x}{a}\right)+c$

26 $\dfrac{1}{\sqrt{x^2+a^2}}\,dx$ $\qquad$ $\log\left|x+\sqrt{x^2+a^2}\right|+c$

27 $\dfrac{1}{\sqrt{x^2-a^2}}\,dx$ $\qquad$ $\log\left|x+\sqrt{x^2-a^2}\right|+c$

28. $\sqrt{a^2-x^2}\,dx$ $\qquad$ $\dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)+c$

29 $\sqrt{x^2+a^2}\,dx$ $\qquad$ $\dfrac{x}{2}\sqrt{x^2+a^2}+\dfrac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$

30 $\sqrt{x^2-a^2}\,dx$ $\qquad$ $\dfrac{x}{2}\sqrt{x^2-a^2}-\dfrac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$

31. $e^{-ax}x^n\,dx$ $\qquad$ $\dfrac{n!}{a^{n+1}}$

32. $e^x\left[f(x)+f'(x)\right]dx$ $\qquad$ $e^x f(x)+c$

33. $\dfrac{f'(x)}{f(x)}\,dx$ $\qquad$ $\log|f(x)|+c$

34. $\left[f(x)\right]^n f'(x)\,dx$ $\qquad$ $\dfrac{\left[f(x)\right]^{n+1}}{n+1}+c$

d/dx $\qquad\qquad$ $\int$

34. for : $\qquad$ turn, ax^2+bx+c

$\int \frac{dx}{ax^2+bx+c}$, $\qquad \equiv a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$

$\int \frac{dx}{\sqrt{ax^2+bx+c}}$ $\qquad = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$

using completion squares.

and then uses

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} \quad \sin^{-1}\left(\frac{x}{a}\right)+c$$

$$\int \frac{dx}{\sqrt{x^2\pm a^2}} \quad \log\left|x+\sqrt{x^2\pm a^2}\right|+c$$

---

35. for :

$$\int \frac{px+q}{ax^2+bx+c}\,dx \quad , \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$$

(I) $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

$\hookrightarrow$ $px+q = A(2ax+b)+B$

$$\int \frac{px+q}{ax^2+bx+c}\,dx \longrightarrow \int \frac{A(2ax+b)+B}{ax^2+bx+c}\,dx$$

$$\hookrightarrow = A\int \frac{2ax+b}{ax^2+bx+c}\,dx + B\int \frac{dx}{ax^2+bx+c}$$

$$\hookrightarrow \int \frac{f'(x)\,dx}{f(x)} = \log|f(x)|$$

|| completion of squares (34)

$$\Rightarrow A\log|ax^2+bx+c| + B\int \frac{dx}{ax^2+bx+c}$$

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx \longrightarrow \int \frac{A(2ax+b)+B}{\sqrt{ax^2+bx+c}}\,dx$$

$$\hookrightarrow = A\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + B\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

$\hookrightarrow$ use:

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$$

|| use completion of squares : ax^2+bx+c (34)

$$\Rightarrow A\left(2\sqrt{ax^2+bx+c}\right) + B\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

First Fundamental Theorem:

$$F(x) = \int_a^{b} f(x)\, dx \qquad \frac{d}{dx} F(x) = f(x)$$

$$F'(x) = f(x)$$

Second Fundamental Theorem

$$\int_a^b f(x)\, dx = [F(x)]_a^b$$

$$= F(b) - F(a)$$

Some important properties:

1. $$\int_a^b f(x)\, dx = \int_a^b f(y)\, dy$$

2. $$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

3. $$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \quad \| \quad a < c < b$$

4. $$\int_a^b [\alpha f(x) + \nu\, g(x)]\, dx = \alpha \int_a^b f(x)\, dx + \nu \int_a^b g(x)\, dx$$

5. If $x = g(u)$,

$$\int_a^b f(x)\, dx \Rightarrow \int_c^d f(g(u)) \frac{dg(u)}{du}\, du$$

where, $g(c) = a$; $g(d) = b$

Integration Properties:

6. $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$\hookrightarrow \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

---

7. If $f(x)$ is even : $f(-x) = f(x)$:

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

---

8. If $f(x)$ is odd : $f(-x) = -f(x)$:

$$\int_{-a}^{a} f(x)\,dx = \emptyset$$

---

9. $$\int_0^{2a} f(x)\,dx = \int_0^a [f(x) + f(2a-x)]\,dx$$

---

10. If $f(2a-x) = f(x)$;

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

---

11. If $f(2a-x) = -f(x)$:

$$\int_0^{2a} f(x)\,dx = \emptyset$$

---

12. $$\int_0^a x\,f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx \quad \| \quad f(a-x) = f(x)$$

Integration Properties:

Extra Properties:

i) $\int_0^{\pi} g(\sin x)\, dx = 2\int_0^{\pi/2} g(\sin x)\, dx$

ii) $\int_0^{2\pi} g(\cos x)\, dx = 2\int_0^{\pi} g(\cos x)\, dx$

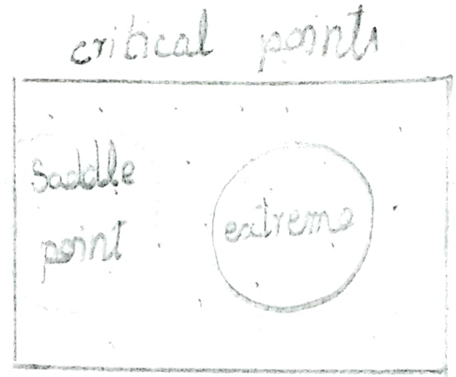
iii) If $f(a+x) = f(x)$,

$$\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$$

Double derivative test for 2 var func :

for $f(x, y)$

→ critical points → $f_x = 0$
→ $f_y = 0$



and then if :

→ $f_{xx} \cdot f_{yy} - (f_{xy})^2 = 0$ inconsistent needs investigation

→ $f_{xx} \cdot f_{yy} - (f_{xy})^2 < 0$ saddle point

→ $f_{xx} \cdot f_{yy} - (f_{xy})^2 > 0 \Rightarrow$ has extrema

↳ maxima $f_{xx} < 0$ && $f_{xx} \cdot f_{yy} - (f_{xy})^2 > 0$

dome ← maxima

bowl ← minima $f_{xx} > 0$ && $f_{xx} \cdot f_{yy} - (f_{xy})^2 > 0$

Partial Derivatives :

for $F(x_0, y_0)$, $\frac{\partial F(x_0, y_0)}{\partial x}$

$$\frac{\partial F}{\partial x} = F_x = \frac{\partial F}{\partial x}(x_0, y_0)\Big|_{x_0, y_0}$$

$$\frac{\partial F}{\partial y} = F_y = \frac{\partial F}{\partial y}(x, y)\Big|_{x_0, y_0}$$

---

for $U(x, y)$

* $U_x = \frac{\partial u}{\partial x}$; $U_y = \frac{\partial u}{\partial y}$

* $U_{xx} = \frac{\partial^2 u}{\partial x^2}$; $U_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

* $U_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$; $U_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

---

* for $U(x, y)$ $A \to \mathbb{R}^2$

$u \in U$ is harmonic in A if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad // \quad U_{xx} + U_{yy} = 0$$

Laplace Eqn

* Clairaut Theorem : If U_{xy}, U_{yx} are continuous

$$U_{xy} = U_{yx}$$

* $\sin^{-1}(x) = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \dots$

↳ provided $|x| \leq 1$

* $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

* $(1+x^n) = 1 + nx + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots$

* $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \cancel{+} - x^5 + \dots \infty$

Taylor Series:

$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{n'}(a)}{n!}(x-a)^n$ (for $x - a = 0$)

$\Rightarrow f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \frac{(x-a)^n}{n!} f^{n'}(a)$

Derivatives:

* Linearization: $L(x) = f(a) + f'(a)(x-a)$

as $\frac{dy}{dx} = f'(a) = \cancel{f(a) - f(a)}\ \frac{f(x) - f(a)}{x - a}$ //

---

* $\boxed{\cancel{L(x,y)}}$ $\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{F(x_0 + h, y_0) - F(x_0, y_0)}{h}$

* $\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{F(x_0, y_0 + k) - F(x_0, y_0)}{k}$

For ~~W~~ $U(x, y)$: $x = h(t)$
$y = g(t)$

$$\frac{dU}{dt} = \frac{\partial U}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial t}$$

For $U(x, y)$: $x = h(s, t)$
$y = g(s, t)$

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial t}$$

For $U(x, y)$

$$\frac{dU}{dx} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}\frac{dy}{dx}$$

important :

for $\left[U(x, y) : \frac{dy}{dx} = \frac{-\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} \right]$

Homogenous $U(x, y)$ if , for some const t

$$U(tx, ty) = t^P U(x, y)$$

t - const
p - degree

3- September 2021 - Dr S. Srinivas

Maclaurian series:

$$\Rightarrow f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \frac{x^n}{n!} f^{n}(0)$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{n}(0)$$

Notes:

* $$\int_0^1 f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right)$$

$k = \left(\frac{2n+1}{1}\right)$

* $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad \sum_{\substack{k=\text{odd}\\ i=0}}^{n} \frac{x^k}{k!}$$

* $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad \sum_{\substack{k=\text{even}\\ i=0}}^{n} \frac{x^k}{k!} \quad (k = 2n)$$

* $$\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \dots \qquad \sum_{\substack{k=\text{odd}\\ i=0}}^{n}$$

* $$\cancel{\ln} \; e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad \sum_{r=0}^{n} \frac{x^r}{r!}$$

* $$\ln|(1+x)| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots \qquad \sum_{r=0}^{n} (-1)^{r+1} \frac{x^{(r+1)}}{(r+1)}$$

provided $-1 \le |x| \le 1$

If Homogenous of degree p,

Euler's Theorem,

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} = p\left[F(x,y)\right]$$

so ..... Taylor Series

* for $U(x,y)$

$$\frac{dy}{\partial x} = -\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}}$$

* for $U(x,y)$, Linearization at (a,b)

$\Rightarrow L(x,y) = U(a,b) + (x-a)\,U_x(a,b)$

1° - linear $+ U(\cancel{a},b) + (y-b)\,U_y(a,b) + \ldots\ldots$

$\Rightarrow f(x,y) = f(a,b) + (x-a) f_x(a,b)$

$U = f$ $\quad + (y-b) f_y(a,b)$

2° ↳ quadratic $\quad + \frac{(x-a)^2}{2} f_{xx}(a,b)$

$+ (x-a)(y-b) f_{xy}(a,b)$

$* (x-a)(y-b) \cancel{f_{yx}} (a,b)$

$+ \frac{(y-b)^2}{2} f_{yy}(a,b)$

Taylor Series

→ for $f(x)$ type at (a)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n'}(a)}{n!} (x-a)^n \quad \text{for } x-a = 0$$

$$\Rightarrow f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \frac{(x-a)^n}{2!} f^{n'}(a)$$

→ Taylor Series at origin $a=0$

↳ Maclaurian series

↳ $$f(x) = \sum_{n=0}^{\infty} \frac{f^{n'}(0)}{n!} (x)^n$$

↳ $$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \frac{x^n}{n!} f^{n'}(0)$$

→ for $f(x,y)$ type at (a,b)

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right)^n f(a,b)$$

↳ $$f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$

$$+ \frac{1}{2!} \left[(x-a)^2 f_{xx} + 2(x-a)(y-b) f_{xy} + (y-b)^2 f_{yy} \right] \Big|_{(a,b)}$$

$+ \dots\dots$

→ Taylor series at origin $(0,0)$ ⇒ Maclaurian Series

↳ $$\boxed{f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right)^n f(0,0)}$$

Notes :

* Taylor series comes from fine tuning Linearization.

$$L(x) = f(a) + \cancel{f'(a)}\ (x-a)\, f'(a)$$

$$L(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$

---

* **Alternate notation**

$$f_x(a,b) = f_x \big|_{(a,b)}$$

$x = a+h$
$y = b+k$ ~~so~~ here so

$$f(a+h, b+k) = f(a,b) + (h f_x + k f_y)\big|_{(a,b)}$$
$$+ \frac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)\Big|_{a,b}$$
$$+ \dots \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{a,b}$$

---

* **Linear** = 1°

$$f(x,y) = \left[f + (x-a) f_x + (y-b) f_y\right]\Big|_{(a,b)} + \dots$$

$$\Rightarrow f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \dots$$

↳ 3 terms

* **Quadratic** = 2°

$$f(x,y) = \left[f + (x-a) f_x + (y-b) f_y + \frac{(x-a)^2}{2} f_{xx} + (x-a)(y-b) f_{xy} + (y-b)^2 f_{yy}\right]\Bigg|_{(a,b)}$$
$$+ \dots$$

6 terms ↵

Hyperbolic funcs:

~~some~~ identities:

1. $\cosh^2 x - \sinh^2 x = 1$

2. $\tanh^2 x - 1 = \text{sech}^2 x$

3. $\coth^2 x - 1 = -\text{cosech}^2 x$

4. $\sinh 2x = 2 \sinh x \cosh x$

5. $\cosh 2x = 1 + \sinh^2 x$

$= 2\cosh^2 x - 1$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

---

derivatives

1. $\frac{d}{dx} \sinh x = \cosh x$

2. $\frac{d}{dx} \cosh x = +\sinh x$

3. $\frac{d}{dx} \tanh x = \text{sech}^2 x$

4. $\frac{d}{dx} \text{sech}\, x = -\text{sech}\, x \tanh x$

5. $\frac{d}{dx} \coth x = -\text{cosech}^2 x$

6. $\frac{d}{dx} \text{cosech}\, x = -\coth x \, \text{cosech}\, x$

↱ Derivative under Integral

~~Double~~ Integrals containing parameter Leibnitz rule:

↳ The Leibnitz rule for Integral

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \left[\frac{\partial}{\partial t} f(x,t)\right] dx$$

Lagrange Multiplier
– Constrained Optimization

– Dr. Srinivas

1. $f(\ldots) = k$
2. $g(\ldots) = C$
3. $F = f(\cdot) + \lambda\,(g(\cdot) - c)$
4. $\nabla_{x,y,\lambda} F = 0$
5. $g(\ldots) - C = 0$

   ↳ $\frac{\partial F}{\partial z}$ original constraint

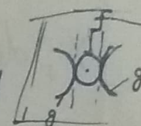
   ↳ $\nabla_{x,y,\lambda}$

6. Solve and eval $f(x_0, y_0)$

Mine:

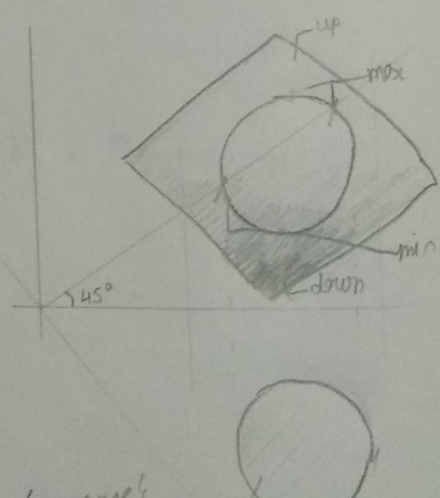
1. $f(\ldots)$ ← opt $g(\ldots) = c$ ← constraint
2. $\nabla f = \lambda \nabla g$
3. Solve $g(\ldots) = C$ with 2.
4. and find stationaries
5. Solve & eval $f(x, y)$

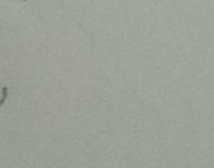
idea: f shape & g shape have same tan at extremas



Drawbacks:

* Can't determine nature of stationary pnt (max, min or saddle?)
* have to sub & find manually





Lagrange's works on shadow

The Jacobian Determinant J

* To transform co-ordinates
* To know if 2 funcs (x, y, z) are related $J = 0$
* if $x = g(u, v)$ || $y = h(u, v)$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

* $J(u, v) = \dfrac{\partial (x, y)}{\partial (u, v)}$ || up $\downarrow$ down (x, y) || ~~right $\rightarrow$ left~~ left $\rightarrow$ right (u, v)

* Important results: $(x^2+y^2)\, \partial x\, \partial y \rightarrow r\, \partial r\, \partial \theta$

The Jacobian Properties

1) if $J(x, y) = \dfrac{\partial(u, v)}{\partial(x, y)}$

$J'(u, v) = \dfrac{\partial(x, y)}{\partial(u, v)}$ $\Big\}$ $JJ' = 1$

2) if u_1, u_2, u_3 aren't given explictly in x_1, x_2, x_3.

but connected as $f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

$f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

$f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$

then,

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \Big/ \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}$$

3) The Chain rule:

$$\frac{\partial(u, v)}{\partial(r, s)} \frac{\partial(r, s)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)}$$

$$\Rightarrow \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix} \Rightarrow \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \Rightarrow \frac{\partial(u, v)}{\partial(x, y)}$$

4) If $u_1\ u_2\ u_3$ are funcs of $x_1\ x_2\ x_3$ then the necessary and sufficient condition for existence of a fun R of form $f(u_1\ u_2\ u_3) = \emptyset$ is

$$J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = \emptyset$$

Problems :

① if $x = r\cos\theta \ || \ y = r\sin\theta$ then $J(r, \theta)$

$$J(r,\theta) \Rightarrow \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \Rightarrow \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\Rightarrow r(\cos^2\theta) + r(\sin^2\theta) = r(\sin^2\theta + \cos^2\theta) = \underline{\underline{r(1)}}$$

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② For cyllindrical co-ords r, θ, z $J(r, \theta, z) = ?$

$x = r\cos\theta \ || \ y = r\sin\theta \ || \ z = z$

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & \emptyset \\ \sin\theta & r\cos\theta & \theta \\ 0 & 0 & 1 \end{vmatrix}$$

$$J(r,\theta,z) = 1(r\cos^2\theta + r\sin^2\theta) \Rightarrow r(1) = \underline{\underline{r}}$$