Axiom 2.1.1 [Principle of Mathmatical Induction]. *If a set $S \subseteq \mathbb{N}$ is such that the following two conditions hold,*

1. $1 \in S$.
2. *For each* $k \in \mathbb{N}$, *if* $k \in S$, *then* $k + 1 \in S$.

then $S = \mathbb{N}$.

Axiom 2.2.3 [Least Upper Bound Property]. *A nonempty subset $S \subset \mathbb{R}$ that is bounded above always has a least upper bound.*

Theorem 2.2.5 [Greatest Lower Bound Property]. *A nonempty subset $S \subset \mathbb{R}$ that is bounded below always has a greatest lower bound.*

Theorem 2.2.6 [Archimedean Property]. *The set $\mathbb{N} \subset \mathbb{R}$ has no upper bound in $\mathbb{R}$. That is, $\mathbb{N}$ is not bounded above.*

Corollary 2.2.7 [Archimedean Property]. *For every $\epsilon > 0$, then there exists an $n \in \mathbb{N}$ satisfying $0 < \frac{1}{n} < \epsilon$.*

Corollary 2.2.9 [Density of Rationals and Irrationals]. *For every a, $b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ and $s \notin \mathbb{Q}$ satisfying $a < r < b$ and $a < s < b$.*

Absolute Value 1. [Positive Definiteness:] $|x| \geq 0$, with $|x| = 0$ if and only if $x = 0$,

2. [Positive Homogeneity:] $|xy| = |x||y|$, and

3. [Triangle Inequality:] $|x + y| \leq |x| + |y|$.

Idenity

1. $|x - a| < \delta$ if and only if $x \in (a - \delta, a + \delta)$,

2. $0 < |x - a| < \delta$ if and only if $x \in (a - \delta, a + \delta) \setminus \{a\}$,

3. $|x - a| \leq \delta$ if and only if $x \in [a - \delta, a + \delta]$.

Theorem 2.3.4 [Triangle Inequality]. For all x, y, z ∈ R, $|x - y| \leq |x - z| + |z - y|$.

Corollary 2.3.5 [Triangle Inequality II]. For all x, y ∈ R, $|x + y| \leq |x| + |y|$.

Corollary 2.3.6 [Triangle Inequality III]. For all x, y ∈ R, $||x| - |y|| \leq |x - y|$.

THEOREM **3.1.4.** *Suppose that* $\lim_{n\to\infty} a_n = L$ *and that* $\lim_{n\to\infty} a_n = M$. *Then* $L = M$.

Theorem 3.1.5. Every convergent sequence is bounded

Theorem 3.2.2 [Monotone Convergence Theorem]. If a sequence {an} is monotonic and bounded, then it converges.

Corollary 3.2.3. A monotonic sequence converges if and only if it is bounded

THEOREM **3.3.2** [GEOMETRIC SERIES TEST]. *A geometric series* $\sum_{n=0}^{\infty} r^n$ *converges if and only if* $|r| < 1$.

Theorem 3.3.3. Assume that a1 ≤ a2 ≤ a3 ≤ · · · and 0 ≤ b1 ≤ b2 ≤ b3 ≤ · · · ; that is, assume that {an} and {bn} are non-decreasing sequences. Assume also that an ≤ bn for each n ∈ N. Then {an} converges if {bn} converges.

Theorem 3.4.2. Suppose that {an} converges to L. Let {ank } be a subsequence of {an}. Then {ank } converges to L.

Lemma 3.4.5 [Peak Point Lemma]. Every sequence has a monotone subsequence.

Theorem 3.4.6 [Bolzano-Weierstrass Theorem]. Every bounded sequence has a convergent subsequence

THEOREM **3.5.2.** *Assume that* $\{a_n\}$ *and* $\{b_n\}$ *are sequences with* $b_n \neq 0$ *for all* $n \in \mathbb{N}$. *Suppose* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and equals* L. *Also suppose that* $\lim_{n\to\infty} b_n$ *exists and equals* 0. *Then* $\lim_{n\to\infty} a_n$ *exists and equals* 0.

THEOREM **3.6.1** [SQUEEZE THEOREM]. *Assume that* $a_n \le b_n \le c_n$, *and*

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n.$$

Then $\{b_n\}$ *converges and* $\lim_{n\to\infty} b_n = L$.

Theorem 3.7.2. Any convergent sequence {an} is Cauchy.

Lemma 3.7.3. If {an} is Cauchy, then {an} is bounded

LEMMA **3.7.4.** *If* $\{a_n\}$ *is Cauchy, and if* $\{a_{n_k}\}$ *is a subsequence of* $\{a_n\}$ *with* $\lim_{k\to\infty} a_{n_k} = L$, *then* $\{a_n\}$ *converges and* $\lim_{n\to\infty} a_n = L$.

Theorem 3.7.5 [Completeness Theorem]. If a sequence {an} is Cauchy, then {an} converges.

THEOREM **4.1.2.** *Assume that* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} f(x) = M$. *Then* $L = M$.

THEOREM **4.1.6** [SEQUENTIAL CHARACTERIZATION OF LIMITS]. *Suppose that* $f : S \to \mathbb{R}$, *where* $S \subseteq \mathbb{R}$. *Also suppose that* $a \in \mathbb{R}$ *is such that there exists an open interval* I *containing* a *with* $I\setminus\{a\} \subset S$. *Then the following are equivalent.*

1. $\lim_{x\to a} f(x)$ *exists and is equal to* L.

2. *Every sequence* $\{x_n\} \subset S$ *such that* $x_n \neq a$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = a$ *has the property that* $\{f(x_n)\}$ *is a sequence that converges to* L.

Partial Fraction Expansion of $\sqrt{2}$.

Let $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{1+a_n}$.

Herron's Method for Calculating Square Roots.

Let $a > 0$. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{a}{a_n})$ for all $n \in \mathbb{N}$.

Decimal Expansion

defined $a_1, a_2, \ldots, a_n$ and $s_1, s_2, \ldots, s_n$, then we define $a_{n+1} = \max\{k \in \{0, 1, 2, \ldots, 9\} \mid \frac{k}{10^{n+1}} \leq \alpha - s_n\}$ and

$$s_{n+1} = \sum_{i=1}^{n+1} \frac{a_i}{10^i}$$

Golden Ratio

$$a_n = \frac{F(n+1)}{F(n)}$$

Compact sequence

A subset K of $\mathbb{R}$ is called *compact* if every sequence $\{x_n\}$ in K has a subsequence $\{x_{n_k}\}$ that converges to some $x_0 \in K$. Show that K is

Nested Interval Theorem:

Let $\{[a_n, b_n]\}$ be a sequence of closed intervals such that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. Prove that $\bigcap [a_n, b_n] \neq \emptyset$.