Black-Litterman Model

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Introduction

Black-Litterman model is an asset allocation model that was first developed in 1990 at Goldman Sachs by Fischer Black and Robert Litterman after whom it was named. It was an attempt to modify the existing framework for asset allocation that was established by Harry Markowitz, known as the Mean-Variance Analysis or Modern portfolio theory (See in Appendix 3). The key improvement that Black-Litterman model provides is that it addresses the views of the portfolio manager about the portfolio providing an additional qualitative input that adjusts the expected returns. The contribution to expected return of each of the portfolio asset about which a view is expressed is balanced against its contribution to overall portfolio risk.

The key assumptions of the Black-Litterman model is that the asset returns are assumed to follow normal distribution. The investor begins with equilibrium assumptions that the asset allocation of an investor should be proportional to the market values of the assets and if the views of the investor are not in compliance with it then the Black-Litterman model can provide the adjusted neutral weights (adjusted for investor's view).

The views of the investor can be absolute or relative to other assets in the portfolio. These inputs come with a level of confidence that affirm how they feel about these views. It essentially follows a Bayesian approach (See in Appendix 5) to develop a probability for the expected return with market returns as the prior distribution and when combined with investor's view will give the posterior distribution of the expected returns.

Data Analysis

1). Sample Selection

We select 135 stocks covering all industries from S&P 500 and download their adjusted close prices dating from Dec 31st, 1992 to Oct 31st, 2018. For the completeness of the data, the 135 stocks we choose have been in the S&P 500 through all the 26 years, which leads to the survivor bias problem. To solve the problem, we select the value-weighted portfolio of these 135 stocks as our benchmark, instead of S&P 500. As BL requires historical data with a period of at least 10 years, the data from 1993 to 2002 is used as the initial historical data window, while back-testing period ranges from 2003 to 2018. Meanwhile, we compare with equal-weighted portfolio and traditional mean variance optimization portfolio.

2). Specifying the Views

In this section, we describe the process of specifying the investors views on the estimated mean and variance. We define the combination of the investors views as the conditional distribution. First, by construction we require each view to be unique and uncorrelated with the other views. This will give the conditional distribution the property that the covariance matrix will be diagonal, with all off-diagonal entries equal to 0. We constrain the problem this way in order to improve the stability of the results and to simplify the problem. Second, we require views to be fully invested, either the sum of weights in a view is zero (relative view) or is one (an absolute view). In addition, it is actually possible for the views to conflict, the mixing process will merge the views based on the confidence in the views and the confidence in the prior.

According to the momentum method, we create a 135x135 identity matrix as our pick matrix and regard average rate of return of last 6 moths as our views to these stocks. To specify the variance of the view, we choose to use proportional to the variance of the prior. For this method, we need to assume that the variance of the views will be proportional to the variance of the asset returns, just as the variance of the prior distribution is. As a result, the variance of the views is

$$\Omega = \mathrm{diag}(\mathrm{P}(\tau\Sigma)P^T)$$

This specification of the variance, or uncertainty, of the views essentially equally weights the investor's views and the market equilibrium weights. $\tau\sum$ is analogous to the standard error. By including τ in the expression, the posterior estimate of the returns becomes independent of τ as well.

3). Using the Formula

By putting the numbers into the following Black-Litterman master formula (See in Appendix 1):

$$\hat{\Pi} = \Pi + \tau\Sigma P^T[(P\tau\Sigma P^T) + \Omega]^{-1}[Q - P\Pi]$$

$$\mathrm{M} = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}$$

The first formula is to calculate the mean value of the posterior, while the second is to calculate the covariance matrix of the posterior. Then we gain weights of the assets through mean-variance optimizer and build the portfolio.

4). Operation Repetition

We repeat Step 2 and 3 every twenty-one trading days, that is to say, we balance our portfolio every month.

5). Gaining the Results

The Black-Litterman line is gained from what we have done through the past four steps. As for the other three lines, one is gained by putting historical data into the mean-variance optimizer, one is for value weighted, and the other is for equally weighted. (See in Appendix 4)

Conclusion

We choose to use a logarithmic scale to show our results to find the relative performances between the portfolios. We can easily find that the orange one is relatively better in these four lines, especially before the financial crisis (2007-2008). When the market volatility is low, the performance of Black-Litterman is pretty good. However, when the market volatility is getting higher, the performance of our model will get much worse.

According to the table in the Appendix 4, we can get that the mean of expected return of Black-Litterman is higher than mean-variance model's mean. And the Sharpe ratio and Jensen's alpha are also a little higher than mean-variance model, and at the same time the beta is a little lower than mean-variance model. However, the performance of our Black-Litterman model is still not ideal.

The first reason is that there are some problems with the views we specify. Our views are still specified from historical data. If we use financial models, econometric models or machine learning instead, we can get a much better result.

Then our omega is still proportional to the variance of the prior. Actually, if we choose a better method to get the omega, just like the Information Ratio or information coefficient, we can make the views confidence more precise.

Third, our samples are selected from S&P 500, so the samples have the trends to focus on large-cap stocks, which cannot represent the whole market. The assumption of the Black-Litterman model is built on equilibrium return, so this issue can also cause some errors.

Finally, the key assumption of Black-Litterman is that all assets returns follow the normal distribution. As a result, if we introduce copula-opinion pooling approach into Black-Litterman model, we can improve our model's performance.

Appendix 1

Black-Litterman Formula

The equation to derive the implied returns, Π, of the assets in a portfolio. Assuming there are N-assets in the portfolio, Π will be a Nx1 vector.

$$\mathbf{\Pi = \delta\Sigma\omega} \qquad (1)$$

where

δ = Risk aversion coefficient. It can either be an arbitrary assumption or can be given by $\delta = (E(r) - r_f)/\sigma^2$

E(r) = Return of the market portfolio (a portfolio that includes all the assets in the market or any other index benchmark that the investor decides to choose)

r_f = Risk free market rate

σ^2 = Variance of the market portfolio

Σ = A covariance matrix of the assets (NxN matrix)

ω = Weights of assets according to their market capitalization

After deriving the assets' implied returns, then we can compute the expected return, E(R) which is a Nx1 vector, of the assets under the Black-Litterman model with the following equation.

$$\mathbf{E(R) = [(\tau\,\Sigma)^{-1} + P^T\,\Omega P]^{-1}\,[(\tau\,\Sigma)^{-1}\,\Pi + P^T\,\Omega Q]}$$

τ = A scalar number indicating the uncertainty of the market returns

P = A matrix with investors views; each row a specific view of the market and each entry of the row represents the weights of each assets (KxN matrix)

Q = The expected returns of the portfolios from the views described in matrix P (Kx1 vector)

Ω = A diagonal covariance matrix with entries of the uncertainty within each view (KxK matrix)

Σ and Π as described in previous section.

The posterior variance matrix, $\mathbf{M = [(\tau\Sigma)^{-1} + P^T\,\Omega^{-1}P]^{-1}}$ will be used to compute the new covariance matrix $\mathbf{\Sigma_p = \Sigma + M}$ considering the additional variance resulting from the investor views. With the new covariance matrix, we can then calculate the new portfolio weights, using equation (1) with new Σ_p and solving for ω instead of Π this time.

Appendix 2

Literature Review

Black and Litterman (1992) show how to combine statistical information on asset returns with investor's views (private information/beliefs) in the framework of Markowitz portfolio optimization. They contribute to the asset management literature in two distinct ways.

Firstly, they postulate that there exists an equilibrium portfolio with which one can associate an equilibrium distribution of asset returns. This equilibrium assumption, which follows from the Capital Asset Pricing Model, is used to replace the most unstable parameter of returns, the mean vector, with a vector reverse-engineered from the market portfolio. The prior distribution summarizes neutral information and is significantly less sensitive to estimation errors than estimates purely based on time-series analysis since it utilizes a directly observable quantity – the market portfolio.

The second contribution of Black and Litterman (1992) is the process that twists the prior distribution according to investor's views/opinions (private information). Views are represented as uncertain predictions about returns of combinations of assets. An application of a Bayesian argument gives a returns distribution that is subsequently used in a standard Markowitz optimization procedure. The Black–Litterman model considerably improves statistical properties of portfolio recommendations and allows for intuitive incorporation of private information.

Satchell and Scowcroft (2000) presented details of Bayesian portfolio construction procedures which have become known in the asset management industry as Black-Litterman models. The authors explained the construction, presented some extensions and argued that those models were valuable tools for financial management.

Giacometti, et al (2007) improve the classical Black-Litterman model by applying more realistic models for asset returns (the normal, the t-student, and the stable distributions) and by using alternative risk measures (dispersion-based risk measures, value at risk, conditional value at risk). Results are reported for monthly data and goodness of the models are tested through a rolling window of fixed size along a fixed horizon. They find that incorporation of the views of investors into the model provides information as to how the different distributional hypotheses can impact the optimal composition of the portfolio.

Martellini and Ziemann (2007) extend the Black-Litterman Bayesian approach to portfolio construction in the presence of non-trivial preferences about higher moments of asset return distributions has a particular application to active style allocation decisions in hedge fund investing. Results in that paper suggest that the systematic implementation of active style allocation decisions can add significant value in a hedge fund portfolio, provided implementation of a sound investment process to account for non-normality and parameter uncertainty in hedge fund return distributions.

Beach and Orlov (2007) provide an application of the Black-Litterman methodology to portfolio management in a global setting. The novel feature of that paper relative to the extant literature on Black-Litterman methodology is that they use GARCH-derived views as an input into the Black-Litterman model. The returns on their portfolio surpass those of portfolios that rely on market equilibrium weights or Markowitz-optimal allocations. They thereby illustrate how the Black-Litterman model can be put to work in designing global investment strategies.

Sujin and Jaewook (2016) construct a low-risk portfolio that responds to low-risk anomalies in the Korean market using the Black–Litterman framework. They use three machine-learning predictive and traditional time-series models to predict the volatility of assets listed in the Korean Stock Price Index 200 (KOSPI 200) and select the best-performing one. Then, they use the model to classify assets into high- and low-risk groups and create a Black–Litterman portfolio that reflects the investor's view where low-risk stocks outperform high-risk stocks. The experiment shows that reflecting the low-risk view in the market equilibrium portfolio improves profitability and that this view dominates the market portfolio.

Using ideas from the Black–Litterman methodology, Andrzej and Jan (2018) design numerical methods (with variance reduction techniques) for the inverse portfolio optimization that extracts statistical information from historical data in a stable way. They introduce a quantitative model for stating investor’s views and blending them consistently with the market information. The theory is complemented by efficient numerical methods with the implementation distributed in the form of publicly available R packages. They conduct practical tests, which demonstrate significant impact of the choice of distributions on optimal portfolio weights to the extent that the classical Black–Litterman procedure cannot be viewed as an adequate approximation.

Appendix 3

Comparison to MPT

Modern portfolio theory (MPT), or mean-variance analysis, is a mathematical framework for assembling a portfolio of assets such that the expected return is maximized for a given level of risk. It is a formalization and extension of diversification in investing, the idea that owning different kinds of financial assets is less risky than owning only one type. Its key insight is that an asset's risk and return should not be assessed by itself, but by how it contributes to a portfolio's overall risk and return. It uses the variance of asset prices as a proxy for risk.

MPT assumes that investors are risk averse, meaning that given two portfolios that offer the same expected return, investors will prefer the less risky one. Thus, an investor will take on increased risk only if compensated by higher expected returns. Conversely, an investor who wants higher expected returns must accept more risk. The exact trade-off will be the same for all investors, but different investors will evaluate the trade-off differently based on individual risk aversion characteristics. The implication is that a rational investor will not invest in a portfolio if a second portfolio exists with a more favorable risk-expected return profile – i.e., if for that level of risk an alternative portfolio exists that has better expected returns.

In general:

- Expected return:

$$\mathrm{E}(R_p) = \sum_i w_i \, E(R_i)$$

Where R_p is the return on the portfolio, R_i is the return on asset, and p_{ij} is the weighting of component asset i.

- Portfolio return variance:

$$\sigma_p^2 = \sum_i w_i^2 \, \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j p_{ij}$$

Where σ is the standard deviation of the periodic returns on the asset, and p_{ij} is the correlation coefficient between the returns on assets i and j.

- Portfolio return volatility:

$$\sigma_p = \sqrt{\sigma_p^2}$$

Despite its theoretical importance, critics of MPT question whether it is an ideal investment tool, because its model of financial markets does not match the real world in many ways.

The risk, return, and correlation measures used by MPT are based on expected values, which means that they are mathematical statements about the future. In practice, investors must substitute predictions based on historical measurements of asset return and volatility for these values in the equations. Very often such expected values fail to take account of new circumstances that did not exist when the historical data were generated.

More fundamentally, investors are stuck with estimating key parameters from past market data because MPT attempts to model risk in terms of the likelihood of losses, but says nothing about why those losses might occur. The risk measurements used are probabilistic in nature, not structural. This is a major difference compared to many engineering approaches to risk management.

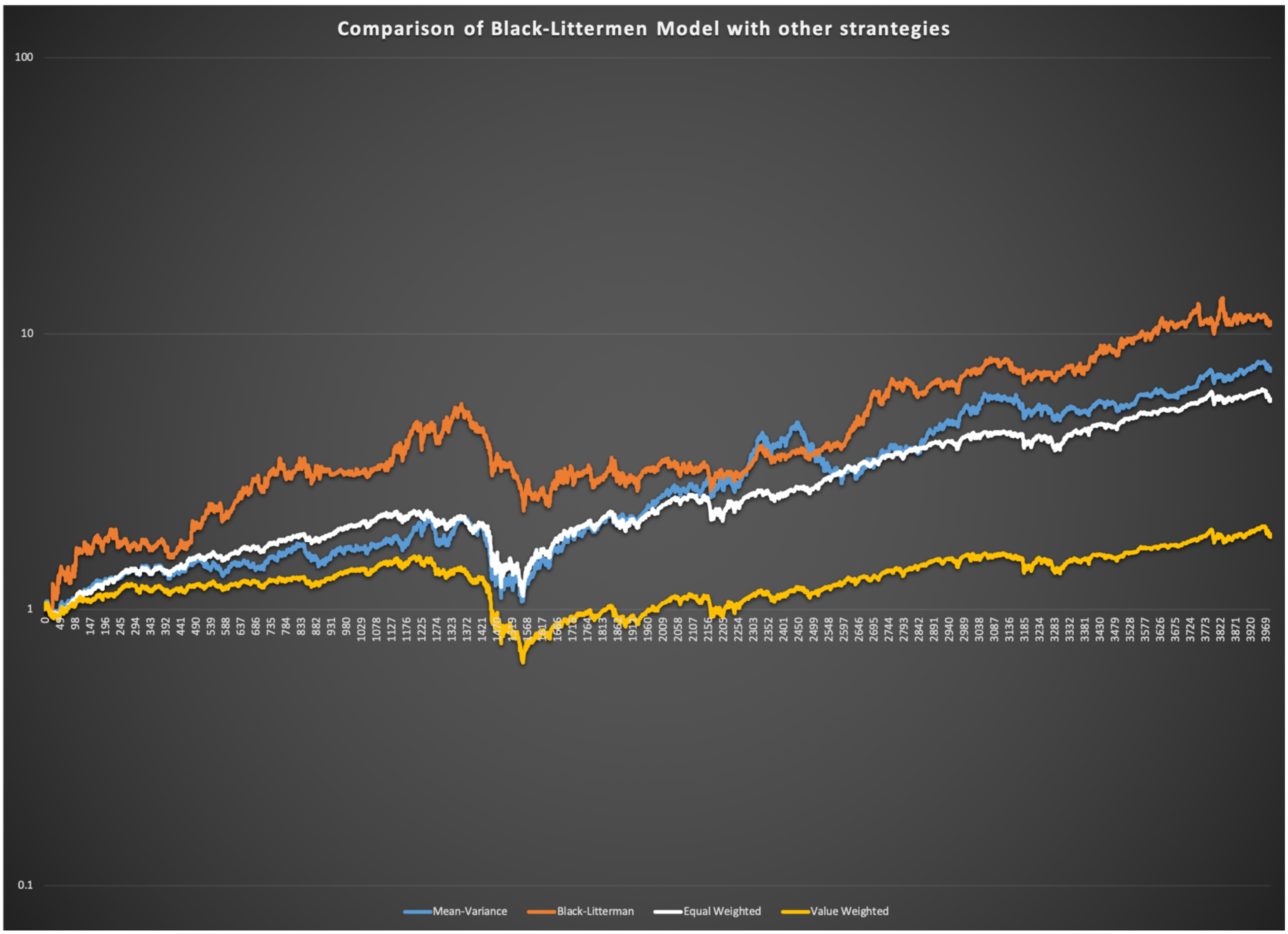
Mathematical risk measurements are also useful only to the degree that they reflect investors' true concerns—there is no point minimizing a variable that nobody cares about in practice. In particular, variance is a symmetric measure that counts abnormally high returns as just as risky as abnormally low returns. The psychological phenomenon of loss aversion is the idea that investors are more concerned about losses than gains, meaning that our intuitive concept of risk is fundamentally asymmetric in nature. There many other risk measures (like coherent risk measures) might better reflect investors' true preferences.

Modern portfolio theory has also been criticized because it assumes that returns follow a Gaussian distribution. Already in the 1960s, Benoit Mandelbrot and Eugene Fama showed the inadequacy of this assumption and proposed the use of stable distributions instead. Stefan Mittnik and Svetlozar Rachev presented strategies for deriving optimal portfolios in such settings.

However, comparing to MPT, Black-Litterman asset allocation model is a sophisticated portfolio construction method that overcomes the problem of unintuitive, highly-concentrated portfolios, input-sensitivity, and estimation error maximization. These three related and well-documented problems with mean-variance optimization are the most likely reasons that more practitioners do not use MPT mentioned above, in which return is maximized for a given level of risk. The Black-Litterman model uses a Bayesian approach to combine the subjective views of an investor regarding the expected returns of one or more assets with the market equilibrium vector of expected returns (the prior distribution) to form a new, mixed estimate of expected returns.

Appendix 4

Graph



Data

	E(R)-MV	E(R)-BL	SD-MV	SD-BL
2003	0.285827	0.547484	0.200992	0.398407
2004	0.142904	0.141977	0.153854	0.232144
2005	-0.02013	0.427473	0.149925	0.256755
2006	0.078694	0.003925	0.168421	0.260248
2007	0.28657	0.435222	0.191374	0.329621
2008	-0.52667	-0.33359	0.451836	0.409763
2009	0.433896	-0.07594	0.280138	0.375821
2010	0.266235	0.096783	0.201668	0.300655
2011	0.136451	-0.12863	0.247735	0.254938
2012	0.196694	0.19718	0.276882	0.173646
2013	0.084026	0.575728	0.220088	0.262597
2014	0.30405	0.11314	0.180378	0.179125
2015	-0.02246	-0.03652	0.19991	0.19695
2016	0.041349	0.272789	0.143136	0.225296
2017	0.265402	0.170975	0.100121	0.209966
2018	0.073887	-0.00555	0.159457	0.282348
Mean	0.12728	0.151724	0.22284	0.281123

	Sharpe-MV	Sharpe-BL	Alpha-MV	Alpha-BL	Beta-MV	Beta-BL
2003	0.51911	0.918646	0.090093	0.342883	1.100769	1.163507
2004	0.703098	0.461986	0.109685	0.108381	1.18918	1.141938
2005	-0.22124	1.614133	-0.02965	0.430841	1.117887	1.549482
2006	-0.11376	-0.36092	-0.03488	-0.12815	1.314933	1.68577
2007	1.084681	1.080731	0.207812	0.342667	0.992878	1.415537
2008	0.068807	0.547065	0.076753	0.10106	1.076823	0.792889
2009	1.143795	-0.504	0.315994	-0.17184	1.054762	0.782546
2010	0.888155	0.032133	0.173036	-0.01941	1.110541	1.528783
2011	0.54896	-0.50633	0.133397	-0.13112	0.905001	0.925415
2012	0.408564	0.654265	0.085197	0.113287	1.426132	1.00494
2013	-0.5738	1.391541	-0.1286	0.328279	1.012356	1.198783
2014	1.13106	0.073177	0.200796	-0.0017	1.043162	1.198421
2015	0.16203	0.09303	0.038002	0.007545	1.0736	0.858622
2016	-0.22503	0.884306	-0.02833	0.219144	0.929616	0.63915
2017	1.189441	0.11745	0.124479	-0.06608	0.956178	1.737622
2018	0.353272	-0.08185	0.056047	-0.01949	0.975517	1.311638
Mean	0.393296	0.398707	0.087143	0.112071	1.063492	1.001855

Appendix 5

A Quick Introduction to Bayes Theory

Bayes theory states

$$\mathrm{P(A|B)} = \frac{P(B|A)P(A)}{P(B)}$$

P(A|B) The conditional (or joint) probability of A, given B Also known as the posterior distribution. We will call this the posterior distribution.

P(B|A) The conditional probability of B given A. Also know as the sampling distribution. We will call this the conditional distribution.

P(A) The probability of A. Also known as the prior distribution. We will call this the prior distribution.

P(B) The probability of B. Also known as the normalizing constant.

When actually applying this formula and solving for the posterior distribution, the normalizing constant will disappear into the constants of integration so from this point on we will ignore it.

A general problem in using Bayes theory is to identify an intuitive and tractable prior distribution. One of the core assumptions of the Black-Litterman model (and Mean-Variance optimization) is that asset returns are normally distributed. For that reason, we will confine ourselves to the case of normally distributed conditional and prior distributions. Given that the inputs are normal distributions, then it follows that the posterior will also be normally distributed. When the prior distribution and the posterior have the same structure, the prior is known as a conjugate prior. Given interest there is nothing to keep us from building variants of the Black-Litterman model using different distributions, however the normal distribution is generally the most straight forward.

Another core assumption of the Black-Litterman model is that the variance of the prior and the conditional distributions about the actual mean are known, but the actual mean is not known. This case, known as “Unknown Mean and Known Variance” is well documented in the Bayesian literature. This matches the model which Theil uses where we have an uncertain estimate of the mean, but know the variance.