0: Adjacent Integers

Prove that for all integers n, $n(n+1)$ is even.

We proceed by cases:

- n is even: If n is even, then by definition, there exists a $k \in \mathbb{Z}$ such that $n = 2k$. We can now perform the following calculations.

$$\begin{aligned} n(n+1) =& 2k(2k+1) && \text{[definition of even]} \\ n(n+1) =& 2 * k(2k+1) \\ n(n+1) =& 2(2k^2+k) \\ 2|n&(n\text{+}1) && \text{[definition of |]} \end{aligned}$$

  By the definition of a factor, 2 is a factor of $n(n+1)$ making $n(n+1)$ even if n is even.

- n is odd: If n is even, then by definition, there exists a $k \in \mathbb{Z}$ such that $n = 2k+1$. We can now perform the following calculations.

$$\begin{aligned} n(n+1) =& (2k+1)(2k+2) && \text{[definition of odd]} \\ n(n+1) =& 2 * (k+1)(2k+1) \\ 2|n&(n\text{+}1) && \text{[definition of |]} \end{aligned}$$

  By the definition of a factor, 2 is a factor of $n(n+1)$ making $n(n+1)$ even if n is odd.

Thus, we have shown that $n(n+1)$ is even when n is odd or even, so $n(n+1)$ is even for all integers n.

1: Odd Squares

Prove that if k is an odd integer, $8|k^2-1$.

By the definition of an odd number, there exists a $l \in \mathbb{Z}$ such that $k = 2l+1$. We can now perform the following calculations:

$$\begin{aligned}
&k^2-1 = (2l+1)^2-1 && \text{[definition of odd]}\\
&k^2-1 = 4l^2+4l+1-1 &&\\
&k^2-1 = 4l^2+4l &&\\
&k^2-1 = 4*l(l+1) && [l(l+1) \text{ is even by our proof in the previous problem}]\\
&k^2-1 = 4*2q && [\text{by definition of even numbers, there exists a } q \in \mathbb{Z} \text{ s.t. } l(l+1) = 2q]\\
&k^2-1 = 8q &&\\
&8|k^2{-}1 && [\text{definition of } |\ ]
\end{aligned}$$

By the definition of a factor, 8 is a factor of k^2-1 given that k is an odd integer, so we have proven the claim.