0: Bipartite Coloring!

Prove that a graph is bipartite if and only if it is 2-colorable.

Since this is an if and only if statement, we need to prove the implication in both directions:

Proof ($\Rightarrow$): We start with a graph $G = (V, E)$ being bipartite. By the definition of bipartite, we can say that V can be partitioned into two disjoint sets A and B such that every edge connects to a vertex in A to a vertex in B.

Let a coloring function c be defined as $c : V \to \{1, 2\}$ where every vertex in A is colored 1 and every vertex in B is colored 2.

For every $(u, v) \in E$, by the definition of bipartite, $(u, v) \in E \Rightarrow u \in A \wedge v \in B$. By the construction of our coloring function, $c(u) = 1 \neq 2 = c(v)$. Since this is true for every $(u, v) \in E$, there is no $(u, v) \in E$ with $c(u) = c(v)$. Therefore, G is two-colorable by the definition of n-colorable where n=2.

Proof ($\Leftarrow$): We start with a graph $G = (V, E)$ being 2-colorable. By the definition of n-colorable where n=2, there is no $(u, v) \in E$ with $c(u) = c(v)$ where c is a coloring function defined as $c : V \to \{1, 2\}$.

Let $A = \{v | v \in V, c(v) = 1\}$ and $B = \{v | v \in V, c(v) = 2\}$. We will prove that A and B are disjoint.

Proof. We perform this proof by contradiction. Suppose that A and B are not disjoint. By definition of disjoint sets, $A \cap B \neq \emptyset$ i.e. there is an element v where $v \in A$ and $v \in B$. Since $v \in A$, $c(v) = 1$ by the construction of A. Similarly, since $v \in B$, $c(v) = 2$. This gives us that $c(v) = 1 = 2 = c(v)$ which is clearly false, so our initial assumption was wrong. Therefore, A and B are disjoint sets. $\square$

By the definition of n-colorable, $\forall (u, v) \in E, c(u) \neq c(v)$. For every pair of vertices $v_1, v_2 \in A$, by the construction of A, $c(v_1) = c(v_2)$. By the definition of n-colorable, $(v_1, v_2) \notin E$. A similar argument can be made for every pair of vertices in B. Since there are no edges within the vertices in A or B, all the edges must be connecting a vertex in A to a vertex in B.

Thus, since we were able to partition V into two disjoint sets A and B where every edge connects a vertex in A to a vertex in B, G is bipartite.

Thus, since we have proven the implication in both directions, we have proven our original claim. $\square$