0: Functions on Trees

For all T $\in$ `Tree`, `size(T)` $\leq 2^{\texttt{height(T)}} - 1$

<u>Lemma:</u> $2^a + 2^b \leq 2 * 2^{max(a,b)}$

Proof. We prove this lemma by cases:

<u>$a = b$:</u>

$$
\begin{aligned}
2^a + 2^b &\overset{?}{\leq} 2 * 2^{max(a,b)} \\
2^a + 2^a &\overset{?}{\leq} 2 * 2^{max(a,a)} && \text{[this is true because of our assumption of } a = b\text{]} \\
2 * 2^a &\overset{?}{\leq} 2 * 2^a \\
1 &\leq 1
\end{aligned}
$$

<u>$a < b$:</u>

$$
\begin{aligned}
2^a + 2^b &\overset{?}{\leq} 2 * 2^{max(a,b)} \\
2^a + 2^b &\overset{?}{\leq} 2 * 2^b \\
2^{a-b} + 1 &\overset{?}{\leq} 2 \\
\frac{2^a}{2^b} &\overset{?}{\leq} 1 \\
2^a &\leq 2^b && \text{[this is true because of our assumption of } a < b\text{]}
\end{aligned}
$$

A similar argument is made for $a > b$, thus our lemma is true. ☐

We now prove the original statement by structural induction. Let P(`T`) be the original claim.

<u>Base Case:</u> `T = Nil`: `size(Nil)` $= 0 \leq 0 = 2^0 - 1 = 2^{\texttt{height(T)}} - 1$

<u>Inductive Hypothesis:</u> For some `L, R` $\in$ `T`, suppose P(`L`) and P(`R`)

<u>Induction Step:</u> We want to show P(`T`) where `T` = `Tree(`x`, L, R)`.

$$
\begin{aligned}
\texttt{size(T)} &= \texttt{1 + size(L) + size(R)} && \text{[definition of } \texttt{size(T)}\text{]} \\
&\leq 1 + (2^{\texttt{height(L)}} - 1) + (2^{\texttt{height(R)}} - 1) && \text{[by I.H.]} \\
&= 2^{\texttt{height(L)}} + 2^{\texttt{height(R)}} - 1 \\
&\leq 2 * 2^{max(\texttt{height(L)},\texttt{height(R)})} - 1 && \text{[by the lemma]} \\
&= 2^{1+max(\texttt{height(L)},\texttt{height(R)})} - 1 \\
&= 2^{\texttt{height(T)}} - 1 && \text{[definition of } \texttt{height(T)}\text{]} \\
\texttt{size(T)} &\leq 2^{\texttt{height(T)}} - 1
\end{aligned}
$$

Thus we have proven the original claim by structual induction. ☐