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Math 0

Lesson 4 Solutions

1) Proposition: Let $a, b \in \mathbb{Z}$. $4 \mid a^2 - b^2 \iff a$ and b are of the same parity.

Discussion: To prove the proposition, we need to prove that $p \Rightarrow q$: "$4 \mid a^2 - b^2 \Rightarrow a$ and b are of same parity", and $q \Rightarrow p$: "a and b are the same parity $\Rightarrow 4 \mid a^2 + b^2$".

To prove the first statement, we'll prove the contrapositive since that gives us information about a and b. Defining a and b as $2m$ and $2n+1$ (the order doesn't matter since they just have to be of different parity) where $m, n \in \mathbb{Z}$. From there, we can plug that in to $a^2 - b^2$ and see how we get an expression resulting in $4x + 1$ (we will show $x \in \mathbb{Z}$). This makes it not divisibile by 4 which make the first statement true.

To prove the second statement, we'll look at two cases. When a and b are both even, we can rewrite them as $2m$ and $2n$ where $m, n \in \mathbb{Z}$ and simplify $a^2 - b^2$ to get an expression that is divisible by 4. A very similar process is applied when a and b are both odd, except now, they're defined as $2m + 1$ and $2n + 1$ where $m, n \in \mathbb{Z}$.

Proof: To prove that "$4 \mid a^2 - b^2 \iff a$ and b are of the same parity",
we will need to prove the two conditional statements $p \Rightarrow q$: "$4 \mid a^2 - b^2 \Rightarrow a$ and b are of the same parity" and $q \Rightarrow p$: "a and b are of the same parity $\Rightarrow 4 \mid a^2 - b^2$".

To prove $p \Rightarrow q$, we will prove the contrapositive $\neg q \Rightarrow \neg p$ which states "a and b are not of the same parity $\Rightarrow 4 \nmid a^2 - b^2$". Since a and b are not of the same parity, we can define them as $a = 2m+1$ and $b = 2n$ where $m, n \in \mathbb{Z}$. Thus,

$$a^2 - b^2 = (2m + 1)^2 - (2n)^2 = 4m^2 + 4m + 1 - 4n^2 = 4(m^2 + m - n^2) + 1$$

. Since $m, n \in \mathbb{Z}$, we can say that $m^2 + m - n^2 \in \mathbb{Z}$. Since we have shown that $a^2 - b^2$ is in the form of $4x + 1$, we have shown that $4 \nmid a^2 - b^2$. We have now proven the contrapositive which means we have proven the original statement $p \Rightarrow q$.

To prove $q \Rightarrow p$, we'll start by looking at two cases.

- **a and b are even:** We can rewrite a and b as $a = 2m$ and $b = 2n$ where $m, n \in \mathbb{Z}$. Thus,

  $$a^2 - b^2 = (2m)^2 - (2n)^2 = 4m^2 - 4n^2 = 4(m^2 - n^2)$$

  Since $m, n \in \mathbb{Z}$, $m^2 - n^2 \in \mathbb{Z}$. Therefore, when a and b are even, $4 \mid a^2 - b^2$

- **a and b are odd:** We can rewrite a and b as $a = 2m + 1$ and $b = 2n + 1$ where $m, n \in \mathbb{Z}$. Thus,

  $$a^2 - b^2 = (2m+1)^2 - (2n+1)^2 = 4m^2 + 4m + 1 - 4n^2 - 4n - 1 = 4m^2 + 4m - 4n^2 - 4n = 4(m^2 + m - n - n^2)$$

  Since $m, n \in \mathbb{Z}$, $m^2 + m - n - n^2 \in \mathbb{Z}$. Therefore, when a and b are odd, $4 \mid a^2 - b^2$

By reaching the same conclusion at the end of both cases, we have proved the original staement $q \Rightarrow p$ by showing how when a and b are of the same parity, $4 \mid a^2 - b^2$.

Now that we've proved both $p \Rightarrow q$ and $q \Rightarrow p$, we have proved $p \iff q$ which states that $4 \mid a^2 - b^2 \iff a$ and b are of the same parity.

$\square$

2) a) Proposition: Let $a \in \mathbb{Z}$. Show $3 \mid a \iff 3 \mid a^2$.

Discussion: To prove the proposition, we need to prove "$3 \mid a \Rightarrow 3 \mid a^2$" and "$3 \mid a^2 \Rightarrow 3 \mid a$".

To prove the first statement, we'll start by recognizing that since $3 \mid a$, $a = 3k$ where $k \in \mathbb{Z}$. Now, we can look at a^2 and see that it produces a form that is divisible by 3, and so the first statement is true.

To prove the second statement, we will prove the contrapositive: "$3 \nmid a \Rightarrow 3 \nmid a^2$" since that gives us information about a. Since $3 \nmid a$, we can write a as $a = 3m + 1$ or $a = 3m + 2$ where $m \in \mathbb{Z}$. We can take each expression of a and square it to get an expression that isn't divisible by 3, thus proving the second statement.

Proof: To prove $3 \mid a \iff 3 \mid a^2$, we need to prove that "$3 \mid a \Rightarrow 3 \mid a^2$" and "$3 \mid a^2 \Rightarrow 3 \mid a$".

To prove the first statement, since $3 \mid a$ there is some $k \in \mathbb{Z}$ such that $a = 3k$. Thus, $a^2 = (3k)^2 = 9k^2 = 3(3k^2)$. Since $k \in \mathbb{Z}$, we can say that $3k^2 \in \mathbb{Z}$. Thus, $3 \mid a^2$ proving the first statement.

To prove the second statement, we will prove the contrapositive: "$3 \nmid a \Rightarrow 3 \nmid a^2$". Since $3 \nmid a$, there is some $k \in \mathbb{Z}$ such that $a = 3k + 1$ or $a = 3k + 2$. Let's look at both ways of expressing a:

$$a^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$$

$$a^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$$

Since $k \in \mathbb{Z}$, we know that $3k^2 + 2k \in \mathbb{Z}$ and $3k^2 + 4k + 1 \in \mathbb{Z}$. Since we're able to write a^2 in the form of $3x + 1$ or $3x + 2$ (where $x \in \mathbb{Z}$), we can say that $3 \nmid a^2$ as desired, proving the contrapositive. Thus, we have proven the original second statement.

Now that we've proven that "$3 \mid a \Rightarrow 3 \mid a^2$" and "$3 \mid a^2 \Rightarrow 3 \mid a$', we can say that we've proven $3 \mid a \iff 3 \mid a^2$. $\square$

b) Proposition: $\sqrt{3}$ is irrational

Discussion: We'll use a proof by contradiction and start by assuming that $\sqrt{3}$ is rational and can be expressed as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and p and q share no common divisors. We'll use the conclusion from (a) to show that p and q have a common divisor of 3 which contradicts the original statement of them having no common divisors proving that $\sqrt{3}$ is irrational.

Proof: Assume, to the contrary, that $\sqrt{3}$ is rational. We can express it as

$$\sqrt{3} = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$ and they share no common divisors.

From here, we can square both sides to get

$$3 = \frac{p^2}{q^2}$$

which can be written as $3q^2 = p^2$. Since p^2 is written as 3 times an integer ($q \in \mathbb{Z}$ so $q^2 \in \mathbb{Z}$), we know that $3 \mid p^2$. From (a), we then know that $3 \mid p$. If $3 \mid p$, when we can write p as $3k$ for some $k \in \mathbb{Z}$. Thus,

$$p^2 = 3q^2$$

$$(3k)^2 = 3q^2$$

$$9k^2 = 3q^2$$

$$3k^2 = q$$

Since we were able to write q as the product of 3 and another integer ($k \in \mathbb{Z}$ so $k^2 \in \mathbb{Z}$), we know that $3 \mid q^2$. From (a), we then know that $3 \mid q$. Since $3 \mid p$ and $3 \mid q$, p and q share a common divisor of 3 which contradicts the original assumption of p and q having no divisors in common.

Thus, our initial assumption of $\sqrt{3}$ being rational must be false, so $\sqrt{3}$ is indeed irrational.

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3) **Proposition:** Let $a, b \in \mathbb{R}$. Show $a + b \in \mathbb{Q} \Rightarrow a \in \mathbb{R} - \mathbb{Q}$ or $b \in \mathbb{Q}$

Discussion: To prove the propsotion, we will prove the contrapositive so that we have information about a and b. The contrapositive states that "If a is rational and b is irrational, then $a + b$ is irrational". Put another way, we need to prove "$a \in \mathbb{Q}$ and $b \in \mathbb{R} - \mathbb{Q} \Rightarrow a + b \in \mathbb{R} - \mathbb{Q}$".

We'll start by letting $a \in \mathbb{Q}$ and $b \in \mathbb{R} - \mathbb{Q}$ and use a proof of contradiction. We'll assume that $a + b$ is rational and show how a contradiction arises.

Proof: We will prove the proposition by proving the contrapositive that states "$a \in \mathbb{Q}$ and $b \in \mathbb{R} - \mathbb{Q} \Rightarrow a + b \in \mathbb{R} - \mathbb{Q}$".

Assume, to the contrary, that $a + b \in \mathbb{Q}$. Since $a \in \mathbb{Q}$, it's additive inverse $-a$ exists and $-a \in \mathbb{Q}$. Since $-a$ and $a + b$ are rational numbers, their sum is also a rational number. Thus,

$$(-a) + (a + b) = -a + a + b = b$$

is rational which contradicts the irrationality of b which means our assumption of $a + b \in \mathbb{Q}$ was false. Thus, $a + b \in \mathbb{R} - \mathbb{Q}$. This proves the contrapositive which then proves our original statement: "If $a + b$ is rational, then a is irrational or b is rational".

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