Transition to Mathematical Proofs
Chapter 3 - Functions Assignment

Instructions: For the below questions, show all of your work. For the proofs, be sure that you

(i) include a Discussion section;
(ii) write a complete proof in full English sentences;
(iii) if hand-writing, write legibly and clearly.

Question 1. Let $m \neq 0$ and b be real numbers and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$.

(a) Prove that f is a bijection.

(b) Since f is a bijection, it is invertible. Find its inverse f^{-1}, and show it is an inverse by demonstrating that

$$f^{-1}(f(x)) = x.$$

Question 2. Let $\gamma, \rho \in \mathbb{R}$ be real numbers such that $\gamma \cdot \rho \neq 1$. Let $\mathbb{R} - \{\gamma\}$ and $\mathbb{R} - \{-\rho\}$ be the set of all real numbers $\mathbb{R}$ except for γ and $-\rho$, respectively. Consider the function $f : \mathbb{R} - \{-\rho\} \to \mathbb{R} - \{\gamma\}$ given by

$$f(x) = \frac{\gamma x + 1}{x + \rho}.$$

Show that f is a bijection.

Question 3. Let S, T, and R be sets, and let $f : S \to T$ and $g : T \to R$ be functions. Show that if $g \circ f$ is injective, then f is injective.

Question 4. Let $C([0,1])$ be the set of all real, continuous functions on the interval $[0,1]$. That is,

$$C([0,1]) = \{f \mid f : [0,1] \to \mathbb{R} \text{ is a continuous function}\}.$$

Thus, an element of the set $C([0,1])$ is simply a function $f(x)$ that is continuous on $[0,1]$. Furthermore, consider the function $\varphi : C([0,1]) \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(x)\, dx.$$

(a) Show that the function φ is surjective by showing that for every $a \in \mathbb{R}$, there exists a pre-image $f \in C([0,1])$ such that $\varphi(f) = a$.

(b) Show that the function φ is not injective by finding two distinct functions $f, g \in C([0,1])$ such that $\varphi(f) = \varphi(g)$.