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Math 0

Lesson 5 Solutions

1) Proposition: Show using Euler's equation that the two angle-sum formulae hold:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Discussion: We'll expand $e^{i(\alpha+\beta)}$ and $e^{i\alpha}e^{i\beta}$. We'll then equate the imaginary and real parts of both equations to show that the identities hold.

Proof: Consider $e^{i(\alpha+\beta)}$. This can be rewritten as $e^{i\alpha}e^{i\beta}$ using the general properties of exponents.

Let's expand $e^{i(\alpha+\beta)}$:

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

Now, let's expand $e^{i\alpha}e^{i\beta}$:

$$e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta =$$
$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

Since the above expressions are all equal, we can write

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

For the above equation to be true, the real and imaginary parts must be equal. Thus, we get the double-angle forumlae:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$\square$

2) a) Proposition: $|z| = \mathrm{Re}(z) \iff z \in R_0^+$

Dicussion: To prove the biconditional statement, we'll need to prove that $|z| = \mathrm{Re}(z) \Rightarrow z \in R_0^+$ and $z \in R_0^+ \Rightarrow |z| = \mathrm{Re}(z)$.

To prove the first statement, we'll prove the contrapositive to get information about z. Since $z \notin R_0^+$, we can take $z = -a$ and $z = a + bi$ (we're looking at both values to cover all possible values of z outside of R_0^+). We can find the magnitude of both z values and show that it is not equal to $\mathrm{Re}(z)$.

To prove the second statement, since $z \in R_0^+$, we can say that there is some $a \in \mathbb{R}_0^+$ such that $z = a$. From there, we can find $|z|$ and show that it is equal to a which is the real part of z, thus proving the second statement.

Now that we've proven both statements, we can say that the original biconditional statement is true.

Proof: To prove that $|z| = \mathrm{Re}(z) \iff z \in R_0^+$, we will need to prove that $|z| = \mathrm{Re}(z) \Rightarrow z \in R_0^+$ and $z \in R_0^+ \Rightarrow |z| = \mathrm{Re}(z)$.

To prove the first statement $|z| = \mathrm{Re}(z) \Rightarrow z \in R_0^+$, we'll prove the contrapositive which states that $z \notin R_0^+ \Rightarrow |z| \neq \mathrm{Re}(z)$. Since $z \notin R_0^+$, there are two general possibilities for z. $z_1 = -a$ or $z_2 = b + ci$ where $a \in R_0^+$ and $b, c \in \mathbb{R}$. The real parts are $\mathrm{Re}(z_1) = -a$ and $\mathrm{Re}(z_2) = b$. Now, let's look at both cases:

$$|z_1| = \sqrt{(-a)^2} = \sqrt{a^2} = a \neq \mathrm{Re}(z_1)$$

$$|z_2| = \sqrt{b^2 + c^2} \neq \text{Re}(z_2)$$

. In both cases, we have shown $|z| \neq \text{Re}(z)$ as desired, thus proving the contrapositive and the orginal statement.

To prove the second statement $z \in R_0^+ \Rightarrow |z| = \text{Re}(z)$, we'll start by letting $z = a$ for some $a \in R_0^+$. Thus, $\text{Re}(z) = a$. Now, let's look at $|z|$:

$$|z| = \sqrt{a^2} = a = \text{Re}(z)$$

Thus, we have shown that $|z| = \text{Re}(z)$ as desired, proving the second statement.

Now that we've proven both conditional statements, we can say that we've proven the biconditional statement "$|z| = \text{Re}(z)$ if and only if z is a non-negative real number."

□

b) Proposition: $(\overline{z})^2 = z^2$ if and only if z is purely real or purely imaginary.

Dicussion: To prove the proposition, we need to prove two conditional statements "If $(\overline{z})^2 = z^2$, then z is purely real or purely imaginary." and "If z is purely real or purely imaginary, then $(\overline{z})^2 = z^2$".

To prove the first statement, we will prove the contrapositive since that gives us information about z. The contrapositive states that "If z is not purely imaginary and not purely real, then $(\overline{z})^2 \neq z^2$" which is equivalent to saying "If z is a complex number with real and imaginary parts, then $(\overline{z})^2 \neq z^2$". We can show this by letting $z = a + bi$ for some $a, b \in \mathbb{R}$ and show how $(\overline{z})^2 \neq z^2$.

To prove the second statement, we'll look at two cases: z is purely real, and z is purely imaginary and show how in each case $(\overline{z})^2 = z^2$.

Proof: To prove the biconditional statement, we will need to prove two conditional statements: "If $(\overline{z})^2 = z^2$, then z is purely real or purely imaginary" and "If z is purely real or purely imaginary, then $(\overline{z})^2 = z^2$."

To prove the first statement "If $(\overline{z})^2 = z^2$, then z is purely real or purely imaginary", we will prove the contrapositive which states that "If z is a complex number with real and imaginary parts, then $(\overline{z})^2 \neq z^2$." Since z has real and imaginary parts, we can write $z = a + bi$ where $a, b \neq 0$ and $a, b \in \mathbb{R}$. From this, we then know that $z^2 = a^2 - b^2 + 2abi$, Thus,

$$(\overline{z})^2 = (a - bi)^2 = a^2 + b^2 - 2abi \neq z^2$$

Thus, we have shown that $(\overline{z})^2 \neq z^2$, as desired, proving the contrapositive. Since we have proved the contrapositive, we have proven the first statement.

To prove the second statement "If z is purely real or purely imaginary, then $(\overline{z})^2 = z^2$", we'll start by looking at two cases: z is purely real or z is purely imaginary.

- **z is purely real:** If z is purely real, then $z = a + 0i$ where $a \in \mathbb{R}$. Thus,

  $$(\overline{z})^2 = (a - 0i)^2 = (a)^2 = z^2$$

  Thus, we have shown that when z is purely real, $(\overline{z})^2 = z^2$.

- **z is purely imaginary:** If z is purely imaginary, then $z = 0 + bi$ where $b \in \mathbb{R}$. Thus,

  $$(\overline{z})^2 = (0 - bi)^2 = (-bi)^2 = (-1)^2(bi)^2 = (bi)^2 = z^2$$

  Thus, we have shown that when z is purely imaginary, $(\overline{z})^2 = z^2$.

We have shown in both cases that $(\overline{z})^2 = z^2$ as desired, so we have proven the second statement.

Now that we've proven both conditional statements, we have proven the statement "$(\overline{z})^2 = z^2$ if and only if z is either purely real or purely imaginary".

□

3) a) Proposition: If $z = a + bi$ and $w = c + di$, then $|z \cdot w| = |z| \cdot |w|$

Dicussion: We'll show that the equation is true by just plugging in the cartesian forms and questioning whether the equation is true. Once we show that a true statement emerges, we'll know the statement is true.

Proof:

$$|z \cdot w| \stackrel{?}{=} |z| \cdot |w|$$

$$|(a + bi) \cdot (c + di)| \stackrel{?}{=} |a + bi| \cdot |c + di|$$

$$|ac + adi + bci - bd| \stackrel{?}{=} |a + bi| \cdot |c + di|$$

$$\sqrt{(ac - bd)^2 + (ad + bc)^2} \stackrel{?}{=} \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2}$$

$$\sqrt{(ac)^2 - 2abcd + (bd)^2 + (ad)^2 + 2abcd + (bc)^2} \stackrel{?}{=} \sqrt{(a^2 + b^2)(c^2 + d^2)}$$

$$(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2 \stackrel{?}{=} (a^2 + b^2)(c^2 + d^2)$$

$$a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \stackrel{?}{=} a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

$$0 = 0$$

Thus we have shown that when $z = a + bi$ and $w = c + di$, $|z \cdot w| = |z| \cdot |w|$.

□

b) Proposition: If $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$, then $|z \cdot w| = |z| \cdot |w|$

Dicussion: We'll show that the equation is true by just plugging in the polar forms and and questioning whether the equation is true. Once we show that a true statement emerges, we'll know the statement is true. We'll make use of the fact that if $z = re^{i\theta}$, then $|z| = r$.

Proof:

$$|z \cdot w| \stackrel{?}{=} |z| \cdot |w|$$

$$|r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}| \stackrel{?}{=} |r_1 e^{i\theta_1}| \cdot |r_2 e^{i\theta_2}|$$

$$|(r_1 \cdot r_2) e^{i\theta_1 + i\theta_2}| \stackrel{?}{=} r_1 \cdot r_2$$

$$|(r_1 \cdot r_2) e^{i(\theta_1 + \theta_2)}| \stackrel{?}{=} r_1 \cdot r_2$$

$$r_1 \cdot r_2 = r_1 \cdot r_2$$

Thus we have shown that when $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$, then $|z \cdot w| = |z| \cdot |w|$.

□

4) For all the following parts, let $z = a + bi$ and $w = c + di$.

a) Proposition: $\overline{z + w} = \overline{z} + \overline{w}$

Discussion: We'll show that the equation is true by just plugging in the cartesian forms of z and w and and questioning whether the equation is true. Once we show that a true statement emerges, we'll know the statement is true.

Proof:

$$\begin{gathered}
\overline{z + w} \stackrel{?}{=} \overline{z} + \overline{w} \\
\overline{a + bi + c + di} \stackrel{?}{=} \overline{a + bi} + \overline{c + di} \\
\overline{(a + c) + (b + d)i} \stackrel{?}{=} (a - bi) + (c - di) \\
(a + c) - (b + d)i \stackrel{?}{=} a + c - bi - di \\
(a + c) - (b + d)i = a + c - (b + d)i
\end{gathered}$$

Thus we have shown that $\overline{z + w} = \overline{z} + \overline{w}$.

$\square$

b) Proposition: $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

Discussion: Similar to (a), we'll show that the equation holds by just pluggin in the cartesian forms ofr z and w and and questioning whether the equation is true. Once we show that a true statement emerges, we'll know the statement is true.

Proof:

$$\begin{gathered}
\overline{z \cdot w} \stackrel{?}{=} \overline{z} \cdot \overline{w} \\
\overline{(a + bi) \cdot (c + di)} \stackrel{?}{=} \overline{a + bi} \cdot \overline{c + di} \\
\overline{ac + adi + bci + i^2bd} \stackrel{?}{=} (a - bi) \cdot (c - di) \\
\overline{(ac - bd) + (ad + bc)i} \stackrel{?}{=} ac - adi - bci + i^2bd \\
(ac - bd) - (ad + bc)i = (ac - bd) - (ad + bc)i
\end{gathered}$$

Thus, we have shown that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

$\square$

c) Proposition: For $n \in \mathbb{N}$, $\overline{z^n} = (\overline{z})^n$

Discussion: We'll prove this statement by rewriting $\overline{z^n}$ as $\overline{z^{n-1} \cdot z}$ and $(\overline{z})^n$ as $(\overline{z})^{n-1} \cdot (\overline{z})$. Since z^{n-1} and z are different complex numbers, we can see from (b) that the inital statement holds.

Proof: To start, let's rewrite $\overline{z^n} \stackrel{?}{=} (\overline{z})^n$ as

$$\overline{z^{n-1} \cdot z} = (\overline{z})^{n-1} \cdot (\overline{z})$$

using our exponent properties. Since z^{n-1} and z are just different complex numbers, we can see from (b) that the original equation holds true. Thus, $\overline{z^n} = (\overline{z})^n$.

$\square$

d) Proposition: Consider the polynomial $p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-2} + \cdots + \alpha_i z + \alpha_0$ where $\alpha_i \in \mathbb{R}$. If $p(w) = 0$ such that $w \in \mathbb{C}$, show that $p(\overline{w}) = 0$.

Discussion: This problem seems daunting at first, but all it comes down to is generalizing the statements we proved in (a) - (c). The best way to show this is to look at a simpler case. Let $f(z) = \alpha_i z^2 + \alpha_i z + \alpha_0$ where $\alpha_i \in \mathbb{R}$, and let $x \in \mathbb{C}$ be a root to $f(z)$ such that $f(x) = 0$. Let's plug in $\overline{x}$. We get that $f(\overline{x}) = \alpha_2(\overline{x})^2 + \alpha_1\overline{x} + \alpha_0$. Using (c), we can rewrite this as $f(\overline{x}) = \alpha_2\overline{x^2} + \alpha_1\overline{x} + \alpha_0$. Since α_i is a real constant, when multipled with a complex number, it'll just give another complex number. Similarly, raising a complex number to a natural number. just gives us another complex number. That means that $\alpha_2\overline{x^2}$ and $\alpha_1\overline{x}$ is just another complex number. Thus, we can use (a) to rewrite it as $f(\overline{x}) = \overline{\alpha_2 x^2 + \alpha_i x + \alpha_0}$. Now, we recognize that $f(\overline{x}) = \overline{f(x)}$. Since $f(x) = 0$ and $\overline{0} = 0$, $f(\overline{x}) = 0$. This can be applied to any polynomial like f, in our case p, without loss of generality.

Note that while (b) wasn't directly applied, it was needed to prove (c) which is directly applied.

Proof: We'll start by looking at $p(z)$ and generally applying (a) - (c)

$$p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$$

$$p(w) = \alpha_n w^n + \alpha_{n-1} w^{n-1} + \cdots + \alpha_1 w + \alpha_0 = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p(\overline{w}) = \alpha_n(\overline{w})^n + \alpha_{n-1}(\overline{w})^{n-1} + \cdots + \alpha_1\overline{w} + \alpha_0$$

Using (c), we can go through each term and rewrite it:

$$p(\overline{w}) = \alpha_n\overline{w^n} + \alpha_{n-1}\overline{w^{n-1}} + \cdots + \alpha_1\overline{w} + \alpha_0$$

Since α_i is a real constant, when multipled with a complex number, it'll just give another complex number. Similarly, raising a complex number to a natural number just gives us another complex number.
Thus, using (a) in a general sense (if we can apply it two terms, we can apply to n terms W.L.O.G.)

$$p(\overline{w}) = \overline{\alpha_n w^n + \alpha_{n-1} w^{n-1} + \cdots + \alpha_1 w + \alpha_0}$$

$$p(\overline{w}) = \overline{p(w)}$$

$$p(\overline{w}) = \overline{0}$$

$$p(\overline{w}) = 0$$

Thus, we have proven that when we have a real polynomial using complex numbers (say f) and $f(w) = 0$, then $f(\overline{w}) = 0$ as well.

$\square$