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Math 0

Lesson 3 Solutions

1) a) Proposition: Assuming $m \neq 0$ and b are real numbers, when $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$, f is a bijection.

Dicussion: To prove that f is a bijection, we need to prove that f is both an injection and a surjection.

To prove that f is an injection, we will start by assuming that $f(s_1) = f(s_2)$ and use algebraic manipulation on that expression to show that $s_1 = s_2$.

To prove that f is surjective, we will start by letting $y \in \mathbb{R}$. We will set $f(x) = y$ and solve for x. Then, we will show that $x \in \mathbb{R}$ thus making f a surjection.

Proof: To prove that f is a bijection, we need to prove that f is an injection and a surjection.

For injectivity, we will let $f(s_1) = f(s_2)$ and show that $s_1 = s_2$. Since $f(s_1) = f(s_2)$, we can say that $m(s_1) + b = m(s_2) + b$. Subtracting b from both sides, we get $m(s_1) = m(s_2)$, and dividing m from both sides, we get $s_1 = s_2$. When $f(s_1) = f(s_2)$, $s_1 = s_2$, so f is an injection.

For surjectivity, let $y \in \mathbb{R}$. Let's start by considering $f(x) = y$ and solve for x.

$$f(x) = y$$

$$mx + b = y$$

$$mx = y - b$$

$$x = \frac{y - b}{m}$$

Since $x = \frac{y-b}{m} \in \mathbb{R}$, f is surjective. In other words, for $y \in \mathbb{R}$ we have shown that x is the pre-image of y ($f^{-1}(y) = x$), thus proving that f is surjective.

Now that we've proven that f is injective and surjective, we have proven that f is a bijection.

b) Proposition: f being a bijection makes it invertible, so find f^{-1} and show that it is an inverse.

Discussion: Since f is a bijection, we can find its' inverse f^{-1} by inverting the domain and co-domain and show that f^{-1} is an inverse by demonstrating that $f^{-1}(f(x)) = x$.

Proof: Inverse functions invert the domain and co-domain, so $f^{-1}(x)$ can be found by solving $x = mf^{-1}(x) + b$. Subtracing b from both sides, we get that $x - b = mf^{-1}(x)$, and dividing m from both sides gives us $f^{-1}(x) = \frac{x-b}{m}$.

To show that $f^{-1}(x)$ is an inverse, we'll show that $f^{-1}(f(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}(mx + b) = \frac{(mx + b) - b}{m} = \frac{mx}{m} = x$$

. Thus, $f^{-1}(x)$ is an inverse.

2) Proposition: Given that $\gamma, \rho \in \mathbb{R}$ such that $\rho \cdot \gamma \neq 1$ and $f : \mathbb{R} - \{-\rho\} \to \mathbb{R} - \{\gamma\}$, let

$$f(x) = \frac{\gamma x + 1}{x + \rho}$$

.

Show that f is a bijection.

Discussion: To show that f is a bijection, we need to show that f is injective and surjective.

To show f is an injection, we will start by letting $f(s_1) = f(s_2)$. We will use algebraic manipulation of this expression to show that $s_1 = s_2$.

To show that f is surjective, let $y \in \mathbb{R} - \{\gamma\}$. We will solve $f(x) = y$ for x and show that $x \in \mathbb{R} - \{-\rho\}$ by showing how $x \neq -\rho$ (since $-\rho$ is the only value excluded from the domain, if we can show that $x \neq -\rho$, then x must be in the domain). Now, we have shown x is a pre-image of y and therefore f is surjective.

Proof: To prove that f is a bijection, we need to show that f is an injection and surjection.

To show injectivity, we will let $f(s_1) = f(s_2)$ and show that $s_1 = s_2$. Let's work with $f(s_1) = f(s_2)$.

$$f(s_1) = f(s_2)$$
$$\frac{\gamma s_1 + 1}{s_1 + \rho} = \frac{\gamma s_2 + 1}{s_2 + \rho}$$
$$(\gamma s_1 + 1)(s_2 + \rho) = (\gamma s_2 + 1)(s_1 + \rho)$$
$$\gamma s_1 s_2 + \gamma\rho s_1 + s_2 + \rho = \gamma s_1 s_2 + \gamma\rho s_2 + s_1 + \rho$$
$$\gamma\rho s_1 + s_2 = \gamma\rho s_2 + s_1$$
$$\gamma\rho s_1 - s_1 = \gamma\rho s_2 - s_2$$
$$s_1(\gamma\rho - 1) = s_2(\gamma\rho - 1)$$
$$s_1 = s_2$$

We have shown that when $f(s_1) = f(s_2)$, $s_1 = s_2$. Thus, f is injective.

To show f is a surjection, we will let $y \in \mathbb{R} - \{\gamma\}$. Let's solve $f(x) = y$ for x.

$$f(x) = y$$
$$\frac{\gamma x + 1}{x + \rho} = y$$
$$\gamma x + 1 = y(x + \rho)$$
$$\gamma x + 1 = xy + \rho y$$
$$\gamma x - xy = \rho y - 1$$
$$x(\gamma - y) = \rho y - 1$$
$$x = \frac{\rho y - 1}{\gamma - y}$$

To show that $x \in \mathbb{R} - \{-\rho\}$, we will show that $x \neq -\rho$. We'll start by assuming $x = -\rho$ and show how a contradition arises.

$$x = -\rho$$
$$\frac{\rho y - 1}{\gamma - y} = -\rho$$
$$\rho y - 1 = -\rho(\gamma - y)$$
$$\rho y - 1 = \rho y - \rho\gamma$$
$$-1 = -\rho\gamma$$
$$\rho\gamma = 1$$

This contradicts the original given condition that $\rho\gamma \neq 1$, so our original statement, $x = -\rho$ is false. Thus, $x \neq -\rho$ and therefore $x \in \mathbb{R} - \{-\rho\}$.

We have shown that x is a pre-image of y (both x and y are within the domain and range of f, respectively) such that $f(x) = y$. Thus, f is surjective.

Since f is injective and surjective, we have shown that f is a bijection.

3) Proposition: Let S, T, and R be sets, and let $f : S \to T$ and $g : T \to R$ be functions. If $g \circ f$ is injective, then f is injective.

Discussion:
What we know: $(g \circ f)$ is injective. Thus, if $(g \circ f)(s_1) = g \circ f(s_2)$, then we know that $s_1 = s_2$.

What we want: To prove that f is injective, we will need to show that when $f(s_1) = f(s_2)$, $s_1 = s_2$.

What we'll do: We'll start by looking at $g \circ f$ being injective. Since it is injective, when we input $(g \circ f)(s_1) = (g \circ f)(s_2)$, we know that the inputs are equal: $s_1 = s_2$. Now, if we input $f(s_1)$ and $f(s_2)$, we get from the injectivity of $g \circ f$, $(g \circ f)(f(s_1) = (g \circ f)(s_2))$, that $f(s_1) = f(s_2)$. Thus we now have two equalities that when put together make f injuctive by definition.

Proof: We will show that f is injective by showing that $s_1 = s_2$, for $s_1, s_2 \in S$, and $f(s_1) = f(s_2)$.

We'll start by looking at $g \circ f$ being injective. Thus, when $(g \circ f)(s_1) = (g \circ f)(s_2)$, we can say that $s_1 = s_2$. Similarly, when $(g \circ f)(f(s_1)) = (g \circ f)(f(s_2))$, we can say that $f(s_1) = f(s_2)$. Since $f(s_1) = f(s_2)$ and $s_1 = s_2$, we can say that f is injective.

□

4) Let $C([0, 1])$ be the set of all real, continuous functions on the interval $[0, 1]$. That is,

$$C([0,1]) = \{f \mid f : [0,1] \to \mathbb{R} \text{ is a continuous function}\}.$$

Thus, an element of the set $C([0, 1])$ is simply a function $f(x)$ that is continuous on $[0, 1]$. Furthermore, consider the function $\varphi : C([0, 1]) \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(x)\,dx.$$

a) Proposition: $\forall a \in \mathbb{R}$, there exists a pre-image $f \in C([0, 1])$ such that $\varphi(f) = a$, so φ is surjective.

Discussion: To prove that a general function is surjective, we need to start with some arbitrary $y \in Y$, solve for $f(x) = y$, and check whether $x \in X$. To prove φ is surjective, we'll start by letting $a \in \mathbb{R}$. From there, we'll solve the equation $\varphi(f) = a$ and present a valid solution for f. If $f \in C([0, 1])$, then we have proven the surjectity of φ.

Proof: Let $a \in \mathbb{R}$. We'll start by considering $\varphi(f) = a$ and solving for f.

$$\int_0^1 f(x)\,dx = a$$

$$F(1) - F(0) = a$$

where $F(x)$ is an antiderivative of $f(x)$

From this, we can see that one possible solution is $f(x) = a$. Since $f(x) = a \in C([0, 1])$, we know that the φ is subjective since we started with $a \in \mathbb{R}$ and showed that there was a valid pre-image $f \in C([0, 1])$ such that $\varphi(f) = a$.

b) Proposition: φ is not injective

Discussion: To prove that φ is not injective, we need to find two distinct functions, $f, g \in C([0, 1])$, such that $\varphi(f) = \varphi(g)$. Put another way, if φ is injective, then when we start with $f \neq g$, $\varphi(f) \neq \varphi(g)$. If $\varphi(f) = \varphi(g)$, then we know φ is not injective.

Proof: To prove that φ is not injective, let $f(x) = 1$ and $g(x) = 2x$ such that $f, g \in C([0, 1])$. Note that $f \neq g$. We'll evaulate $\varphi(f)$ and $\varphi(g)$.

$$\varphi(f) = \int_0^1 1\,dx = x|_0^1 = 1 - 0 = 1$$

$$\varphi(g) = \int_0^1 2x\, dx = x^2|_0^1 = 1 - 0 = 1$$

. From this, we see that $\varphi(f) = \varphi(g)$ and since $f \neq g$, we have proven that φ is not injective.

$\square$