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Math 0

# Lesson 6 Solutions

**1) Proposition:** Let $r \neq 1$. Show that $\sum_{j=0}^{n} r^j = \frac{1-r^{n+1}}{1-r}$.

**Discussion:** Since our summation is indexed by real numbers and it seems that we can relate $k$ to $k+1$ using exponent properties, a proof of induction would be useful here.

- Identify $A(n)$: $A(n)$ is the statement that $\sum_{j=0}^{n} r^j = \frac{1-r^{n+1}}{1-r}$
- Base Case: We'll show that A(0) is true which makes sense because the summation of $r^0$ is just equal to 1
- Inductive Step: We'll assume that for $k \in \mathbb{N}$,

$$\sum_{j=0}^{k} r^j = \frac{1-r^{k+1}}{1-r}$$

  We will show that $A(k+1)$ is also true by showing that

$$\sum_{j=0}^{k+1} r^j = \frac{1-r^{(k+1)+1}}{1-r}$$

**Proof:** We will use a proof by induction to prove the statement $A(n)$ given by $\sum_{j=0}^{n} r^j = \frac{1-r^{n+1}}{1-r}$ for all $n \in \mathbb{N}$.

First, we'll prove the base case $A(0)$ to be true. Computing $A(0)$, we get

$$\sum_{j=0}^{0} r^j = \frac{1-r^{0+1}}{1-r} = \frac{1-r}{1-r} = 1$$

Since $r^0 = 1$, we can see that $A(0)$ is true.

Now, we'll start with the inductive step. We'll assume for $k \in \mathbb{N}$, $A(k)$ is true. That is,

$$\sum_{j=0}^{k} r^j = \frac{1-r^{k+1}}{1-r}$$

We will show that $A(k+1)$ is true by showing that

$$\sum_{j=0}^{k+1} r^j = \frac{1-r^{(k+1)}+1}{1-r}$$

Looking at the left side of the $A(k+1)$ statement, we can use our inductive assumption to show the following:

$$\sum_{j=0}^{k+1} r^j = (\sum_{j=0}^{k} r^j) + r^{k+1} = (\frac{1-r^{k+1}}{1-r}) + r^{k+1} = \frac{1-r^{k+1}}{1-r} + \frac{r^{k+1}(1-r)}{1-r} = \frac{1-r^{k+1}+r^{k+1}-r\cdot r^{k+1}}{1-r} =$$

$$\frac{1-r^{(k+1)+1}}{1-r}$$

Thus, we have used our inductive assumption to show that $A(k+1)$ is true.

By induction, we now know that the statement $A(n)$ given by $\sum_{j=0}^{n} = \frac{1-r^{n+1}}{1-r}$ is true for all $n \in \mathbb{N}$.

$\square$

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**2)** Consider $f(x) = \frac{1}{1-x}$

**a)** Computer the first several derivatives of $f$ and use them to conjecture a pattern for $f^{(n)}(x)$.

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| $n$ | $f^{(n)}(x)$ |
|---|---|
| 0 | $(1-x)^{-1}$ |
| 1 | $-(1-x)^{-2}$ |
| 2 | $2(1-x)^{-3}$ |
| 3 | $-6(1-x)^{-4}$ |
| 4 | $24(1-x)^{-5}$ |

Conjecture: $f^{(n)}(x) = (-1)^n n!(1-x)^{-(n+1)}$

**b) Proposition:** Let $x \neq 1$. Show that $f^{(n)}(x) = (-1)^n n!(1-x)^{-(n+1)}$ for $n \in \mathbb{N}$

**Discussion:** Since the proposition is indexed by natural numbers and it seems that we can use factorial and exponent properties to relate $k$ to $k+1$, a proof by induction would work well here.

- Identify $A(n)$: $A(n)$ is the statement that $f^{(n)}(x) = (-1)^n n!(1-x)^{-(n+1)}$
- Base Case: We'll show that $A(0)$ is true by using the fact that the 0-th derivative of a function is just the function itself
- Inductive Step: We'll assume that for some $k \in \mathbb{N}$,
$$f^{(k)}(x) = (-1)^k k!(1-x)^{-(k+1)}$$
We will show that $A(k+1)$ is true by showing that
$$f^{(k+1)}(x) = (-1)^{(k+1)}(k+1)!(1-x)^{-((k+1)+1)}$$

**Proof:** We will used a proof by induction to prove the statement $A(n)$ given by $f^{(n)} = (-1)^n n!(1-x)^{-(n+1)}$ for $n \in \mathbb{N}$.

First, we'll show the base case $A(0)$ to be true. Computing $A(0)$, we get

$$f^0(x) = (-1)^0 0!(1-x)^{-(0+1)}$$

This is equivalent to $f(x) = \frac{1}{1-x}$ which means $A(0)$ is true.

Now, we'll start with the inductive step by assuming that for some $k \in \mathbb{N}$, $A(k)$ is true. That is,

$$f^{(k)}(x) = (-1)^k k!(1-x)^{-(k+1)}$$

We will show that $A(k+1)$ is true by showing that

$$f^{(k+1)}(x) = (-1)^{(k+1)}(k+1)!(1-x)^{-((k+1)+1)}$$

Looking at the left hand side of the $A(k+1)$ statement, we can use our inductive assumption to show the following

$$\begin{aligned} f^{(k+1)}(x) = \frac{d}{dx}f^{(k)}(x) &= \frac{d}{dx}(-1)^k k!(1-x)^{-(k+1)} \\ &= (-1)^k k!\frac{d}{dx}(1-x)^{-(k+1)} \\ &= (-1)^k k! \cdot -(k+1)(1-x)^{-(k+1)-1} \\ &= (-1)(-1)^k(k+1)k!(1-x)^{-((k+1)+1)} \\ &= (-1)^{k+1}(k+1)!(1-x)^{-((k+1)+1)} \end{aligned}$$

Thus, we have used our inductive assumption to show that $A(k+1)$ is true.

By induction, we now know that the statement $A(n)$ given by $f^{(n)}(x) = (-1)^n n!(1-x)^{-(n+1)}$ is true for all $n \in \mathbb{N}$.

$\square$

**3) Proposition:** Let $x > -1$. Show that $(1+x)^n \geq 1 + nx$ where $n \in \{m \in \mathbb{N} \mid m \geq 1\}$

**Discussion:** Since the inequality can be said to be indexed by natural numbers and it seems that we can use the combination of inequalities and exponent properties to relate $k$ to $k+1$.

- Identify $A(n)$: $A(n)$ is the statement $(1+x)^n \geq 1 + nx$
- Base Case: We will show that A(1) is true because some quantity $a$ will always be greater than or equal to $a$
- Inductive step: We'll assume that for $k \geq 1$ such that $k \in \mathbb{N}$,

  $$(1+x)^k = 1 + kx$$

  We will show that $A(k+1)$ is also true by showing that

  $$(1+x)^{k+1} = 1 + (k+1)x$$

**Proof:** We will use a proof by induction to prove the statement $A(n)$ given by $(1+x)^n \geq 1 + nx$ where $x > -1$ and $n$ is all the integers such that $n \geq 1$.

First we'll prove the base case $A(1)$ to be true. Computing $A(1)$, we get

$$(1+x)^1 \stackrel{?}{=} 1 + (1)x$$
$$1 + x \geq 1 + x$$

Since we created a logically true statement, we can see that $A(1)$ is true.

Now, we'll start with the inductive step by assuming that for $k \geq 1$ such that $k \in \mathbb{N}$, $A(k)$ is true, that is,

$$(1+x)^k \geq 1 + kx$$

We will show that $A(k+1)$ is true by showing that

$$(1+x)^{k+1} \geq 1 + (k+1)x$$

Looking at the left side of the $A(k+1)$ statement, we can use our inductive assumption to show the following:

$$(1+x)^{k+1}$$
$$(1+x)^k \cdot (1+x)$$
$$(1+x)^k \cdot (1+x) \geq (1+kx) \cdot (1+x)$$
$$(1+x)^k \cdot (1+x) \geq 1 + x + kx + kx^2$$
$$(1+x)^k \cdot (1+x) \geq 1 + x + kx + kx^2 \geq 1 + x + kx$$
$$(1+x)^k \cdot (1+x) \geq 1 + x + kx$$
$$(1+x)^{k+1} \geq 1 + (k+1)x$$

Thus, we have used our inductive assumption to show that $A(k+1)$ is true.

By induction, we now know that the statement $A(n)$ given by $(1+x)^n \geq 1 + nx$ where $x > -1$ is true for all $n \geq 1$ such that $n \in \mathbb{N}$.

$\square$