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Math 0

Lesson 2 Solutions

1) In the notes, we proved one of DeMorgan's Set Theory laws. Prove the remaining one. That is, prove the following statement:

Let S and T be sets. Then
$$\overline{S \cap T} = \overline{S} \cup \overline{T}.$$

....

Proposition: $\overline{S \cap T} = \overline{S} \cup \overline{T}$

Discussion:

What we want: To prove the proposition we will prove two subset inclusions: $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$ and $\overline{S} \cup \overline{T} \subset \overline{S \cap T}$

What we'll do: Both subset inclusions will be proven using one of DeMorgan's logic laws that states $\neg(p \wedge q) = \neg p \vee \neg q$

Proof: To prove that $\overline{S \cap T} = \overline{S} \cup \overline{T}$, we need to prove the subset inclusions $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$ and $\overline{S} \cup \overline{T} \subset \overline{S \cap T}$.

To prove the first inclusion, we will assume $x \in \overline{S \cap T}$. This can be rewritten as $x \notin S \cap T$ which means that it's not true that $x \in S$ and $x \in T$. Using Demorgan's Logic Laws, this is logically equivlent to $x \notin S$ or $x \notin T$. This can be rewritten as $x \in \overline{S}$ or $x \in \overline{T}$ which shows that $x \in \overline{S} \cup \overline{T}$. Thus, $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$.

To prove the second inclusion, we will assume $x \in \overline{S} \cup \overline{T}$. This means that $x \in \overline{S}$ or $x \in \overline{T}$ which can be rewritten as $x \notin S$ or $x \notin T$. This means that it's not true that $x \in S$ or $x \in T$. Using DeMorgan's logic laws, this is logically equivalent to $x \notin S$ and $x \notin T$. This can be rewritten as $x \in \overline{S}$ and $x \in \overline{T}$, and since x is in the intersection, $x \in \overline{S \cap T}$. Thus, $\overline{S} \cup \overline{T} \subset \overline{S \cap T}$.

Knowing both of these subset inclusions to be true, we know that our two sets are equal: $\overline{S \cap T} = \overline{S} \cup \overline{T}$. $\square$

2) In the notes, we proved one distributive law. Prove the remaining one. That is, prove the following statement:

Let S, T, and R be sets. Then,
$$S \cap (T \cup R) = (S \cap T) \cup (S \cap R).$$

....

Proposition: $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$

Discussion:
What we want: To prove the proposition by proving two subset inclusions: $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$ and $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$

What we'll do: To prove the first inclusion $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$, we will assume that $x \in S \cap (T \cup R)$. This means we can assume that $x \in S$ and $x \in (T \cup R)$ to be true. Since both parts must be true for the hypothesis to be true, at least one part of the union portion must be true i.e. either $x \in T$ or $x \in R$. Combining this with the fact that $x \in S$, we will show that $x \in (S \cap T)$ or $x \in (S \cap R)$.
To prove the second inclusion $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$, we will assume that $x \in (S \cap T) \cup (S \cap R)$. Since this is an union assumption, we will look at two cases: $x \in S \cap T$ and $x \in S \cap R$. In the first case, we'll

show that $x \in T$ naturally results in $x \in T \subset T \cup R$, therefore, $x \in (T \cup R)$. Since $x \in S$, we can combine these two to yield the inclusion. A very similar process can be taken for the second case.

Proof: To prove that $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$, we will prove two subset inclusions: $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$ and $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.

For the first inclusion, we will assume $x \in S \cap (T \cup R)$. Thus, we know that $x \in S$ and $x \in T \cup R$. For the hypothesis to be true, each part of it must be true. Since the second part of the statement is a union, two cases arise: $x \in T$ or $x \in R$. Combining this with the first part of the hypothesis, we get the following:

- $x \in S$ **and** $x \in T$: In this case, we're assuming that $x \in T$. Since $x \in S$, and $x \in T$, it can be said that $x \in S \cap T$.

or

- $x \in S$ **and** $x \in R$: In this case, we're assuming that $x \in R$. With a similar logic to the previous bullet point, it can be shown that $x \in S \cap R$.

Since both cases started by taking the possibilities of a union, the results of the cases can be joined with a union resulting in $x \in (S \cap T) \cup (S \cap R)$. This proves the first inclusion that $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$.

For the second inclusion, we will assume $x \in (S \cap T) \cup (S \cap R)$. Since this is a union hypothesis, we will look at two cases:

- $x \in (S \cap T)$: Since $x \in (S \cap T)$, $x \in S$ and $x \in T$. This can be rewritten as $x \in S$ and $x \in T \subset (T \cup R)$. Since x is in both sets, x is in the intersection of the two sets: $x \in S \cap (T \cup R)$. Thus, $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.
- $x \in (S \cap R)$: Since $x \in (S \cap R)$, $x \in S$ and $x \in R$. This can be rewritten as $x \in S$ and $x \in R \subset (R \cup T)$. Since x is in both sets, it is in the intersection of both sets: $x \in S \cap (R \cup T)$. Thus, $(S \cap T) \cup (S \cap R) \subset S \cap (R \cup T)$.

In both cases, we obtained the same conclusion thus proving the inclusion $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.

Since we have now shown both inclusions to be true, we can say the sets are equal: $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$.

$\square$

3) Let A, B, C, and D be sets. Show that if $A \subset B$ and $C \subset D$, then $A \times C \subset B \times D$.

....

Proposition: $A \subset B \cap C \subset D \Rightarrow A \times C \subset B \times D$

Discussion: We can start by defining $(x, y) \in A \times C$. Since $x \in A$ and $A \subset B$, $x \in B$. Similarly, since $y \in C$ and $C \subset D$, $y \in D$. Thus, $(x, y) \in B \times D$. This shows that $A \times C \subset B \times D$.

Proof: To prove that $A \times C \subset B \times D$, let's start by supposing that $(x, y) \in A \times C$. This means that $x \in A$ and $y \in C$. Since $x \in A$ and $A \subset B$, $x \in B$. Similarly, since $y \in C$ and $C \subset D$, $y \in D$. Thus, we can now state that $(x, y) \in B \times D$. Since we started with an element (x, y) in $A \times C$ and showed that it is also in $B \times D$, we have shown that $A \times C \subset B \times D$. Thus, we have proven that "If $A \subset B$ and $C \subset D$, then $A \times C \subset B \times D$".

$\square$

4) Let A, B, and C be sets. Show that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

....

Proposition: $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Discussion:

What we want: We will prove the proposition by proving two subset inclusions:

$$A \times (B \cap C) \subset (A \times B) \cap (A \times C)$$

and

$$(A \times B) \cap (A \times C) \subset A \times (B \cap C)$$

What we'll do: To prove the first inclusion, we'll take $(x, y) \in A \times (B \cap C)$. This gives us that $x \in A$ and $y \in (B \cap C)$. Since $y \in B$ and $y \in C$, we can combine this $x \in A$ to get that $(x, y) \in A \times B$ and $(x, y) \in A \times C$. This can then be further combined to prove the first inclusion.

To prove the second inclusion, we'll take $(x, y) \in (A \times B) \cap (A \times C)$. This means $(x, y) \in A \times B$ which then yields $x \in A$ and $y \in B$. Similarly, $(x, y) \in A \times C$ which then yields $x \in A$ and $y \in C$. Since y is in both B and C, we can state that $y \in B \cap C$. Now we can see that $(x, y) \in A \times (B \cap C)$ which proves the second inclusion.

Proof: To prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$, we need to prove two subset inclusions:

$$A \times (B \cap C) \subset (A \times B) \cap (A \times C)$$

and

$$(A \times B) \cap (A \times C) \subset A \times (B \cap C)$$

To prove the first inclusion $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$, let $(x, y) \in A \times (B \cap C)$. Thus, $x \in A$ and $y \in B \cap C$. Since $y \in B \cap C$, $y \in B$ and $y \in C$. Since $x \in A$ and $y \in B$, $(x, y) \in A \times B$. Similarly, since $x \in A$ and $y \in C$, $(x, y) \in A \times C$. Since (x, y) is in both sets, it is in the intersection of the two sets such that $(x, y) \in (A \times B) \cap (A \times C)$. Thus, $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$ proving the first subset inclusion.

To prove the second inclusion $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$, let $(x, y) \in (A \times B) \cap (A \times C)$. Thus, $(x, y) \in A \times B$ which means $x \in A$ and $y \in B$. Similarly, $(x, y) \in A \times C$ which means $x \in A$ and $y \in C$. Since $y \in B$ and $y \in C$, $y \in B \cap C$. x and y can now be combined to write $(x, y) \in A \times (B \cap C)$. Thus, $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$ proving the second subset inclusion.

Now that we've proven the two subset inclusions, we can say the sets are equal: $A \times (B \cap C) = (A \times B) \cap (A \times C)$. $\square$

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5) Let A, B, and C be sets. Show that

$$A \times (B \cup C) = (A \times B) \cup (A \cup C)$$

.

. . . .

Proposition: $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Discussion: *What we want:* To prove the proposition by proving two subset inclusions:

$$A \times (B \cup C) \subset (A \times B) \cup (A \times C)$$

and

$$(A \times B) \cup (A \times C) \subset A \times (B \cup C)$$

What we'll do: To prove the first inclusion, we'll start by assuming that $(x, y) \in A \times (B \cup C)$ which means $x \in A$ and $y \in B \cup C$. Since $y \in B \cup C$, $y \in B$ or $y \in C$. Combining this with $x \in A$, we get that $(x, y) \in A \times B$ or $(x, y) \in A \times C$. Combining these two statements yields the first inclusion.

To prove the second inclusion, assume that $(x, y) \in (A \times B) \cup (A \times C)$. This means that $(x, y) \in A \times B$ or $(x, y) \in A \times C$. This can be broken down into $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$. Using the distributive law $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$ we can rewrite the previous expressions as $x \in A$ and $y \in B \cup C$. From there, it's easy to see that $(x, y) \in A \times (B \cup C)$ proving the second inclusion.

Proof: To prove the first inclusion $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$, let $(x, y) \in A \times (B \cup C)$. This means $x \in A$ and $y \in B \cup C$. Since $y \in B \cup C$, $y \in B$ or $y \in C$. Each of the statements about y can be combined with $x \in A$ to yield $(x, y) \in A \times B$ or $(x, y) \in A \times C$. This can be condensed as $(x, y) \in (A \times B) \cup (A \times C)$ which proves the inclusion $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.

To prove the second inclusion $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$, let $(x, y) \in (A \times B) \cup (A \times C)$. Thus, $(x, y) \in A \times B$ or $(x, y) \in A \times C$. Put another way, $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$. Using a distributive law, we can condense the expressions as $x \in A$ and $y \in B \cup C$. From there, we can combine the expressions into $(x, y) \in A \times (B \cup C)$. Thus, $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$ proving the second inclusion.

Now that we've proven both inclusions, we have proven that the two sets are equal: $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$\square$