Transition to Mathematical Proofs
Chapter 5 - Complex Numbers Assignment

Instructions: For the below questions, show all of your work. For the proofs, be sure that you

(i) write a complete proof in full English sentences;
(ii) if hand-writing, write legibly and clearly.

Note: Discussion sections are no longer required. You may, of course, include them in your assignments, as they may help the grader give more helpful feedback.

Question 1. Similar to how we obtained the double-angle formulae in the notes, use the Euler equation to show the two angle-sum formulae hold:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha;$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Question 2.

(a) Show that $|z| = \mathrm{Re}(z)$ if and only if z is a non-negative real number.

(b) Show that $(\overline{z})^2 = z^2$ if and only if z is purely real or purely imaginary (i.e., its real part is 0).

Question 3. The modulus of a complex number is, in many ways, a generalization of the absolute value of a real number. Here, we give another property of the modulus that the absolute value of a real number already enjoys.
If $z, w \in \mathbb{C}$, show that

$$|z \cdot w| = |z| \cdot |w|$$

in the following two ways:

(a) By using the Cartesian form $z = a + bi$ and $w = c + di$ for the complex numbers z and w.

(b) By using the polar form $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ for the complex numbers z and w.

Question 4. Below, we will prove a remarkable fact about real polynomials using complex numbers. For the parts below, let $z = a + bi$ and $w = c + di$ be complex numbers.

(a) Show that $\overline{z+w} = \overline{z} + \overline{w}$.

(b) Show that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

(c) Use (b) to show that $\overline{z^n} = (\overline{z})^n$ for any natural number $n \in \mathbb{N}$.

(d) Consider the following polynomial $p(z)$ with *real coefficients*:

$$p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0,$$

where each α_i is a real number. Show that if a complex number w is a root to the above polynomial with real coefficients, then its conjugate $\overline{w}$ is also a root to the same polynomial. That is, use (a) - (c) to show that if $p(w) = 0$, then $p(\overline{w}) = 0$.