Subham Sahoo
Teacher: Professor Pelayo
Math 0

Lesson 1 Solutions

1) Let $m \neq 0$ and b be real numbers. Show that there exists a unique x such that $mx + b = 0$.

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Propsition: $\exists! x \; mx + b = 0$

Discussion: First we will show that there is an x that solves $mx + b = 0$ by solving the equation algebraically. Then we will show that it's the only solution by assuming there are are two solutions, x and y, and show that $x = y$.

Proof. Solving $mx+b = 0$ algebraically, first we subtract b from each side to get $mx = -b$. Diving m by both sides, we get that $x = \frac{-b}{m}$. Plugging this x back into $mx + b = 0$ yields $m(\frac{-b}{m}) + b = -b + b = 0$, so we see this x satisfies $mx + b = 0$. To show that this x is unique, let's assume x and y both satisfy $mx + b = 0$. If $mx + b = 0$ and $my + b = 0$, then $mx + b = my + b$. Subtracting b from both sides, $mx = my$, and when we divide both sides by m, we get that $x = y$. Thus, there exists a unique x that satisfies $mx + b = 0$. $\square$

2) Prove the following biconditional statement.

Let x be a real number. $-1 \leq x \leq 1$ if and only if $x^2 \leq 1$.

In proving this, it may be helpful to note that $-1 \leq x \leq 1$ is equivalent to $-1 \leq x$ and $x \leq 1$.

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Propsition: $-1 \leq x \leq 1 \iff x^2 \leq 1$

Discussion: We'll break up this biconditional statement into $p \Rightarrow q$: "If x is between -1 and 1, then x^2 is less than or equal to 1", and $q \Rightarrow p$: "If x^2 is less than or equal to 1, then x is between -1 and 1 (inclusive)."

The first statement can be proven easily by recognizing that p is a conjunction: $-1 \leq x \cap x \leq 1$. Since p is assumed to be true, each part of the conjuction is also true. As such, we can work with each piece of the conjuction and prove that q is true for each piece, thus being true for p. Put another way, if we can show that $(-1 \leq x \Rightarrow x^2 \leq 1) \cap (x \leq 1 \Rightarrow x^2 \leq 1)$, then we prove that $p \Rightarrow q$.

The second statement is a little trickier to prove since the hypothesis gives us information about x^2 and not x. The best way to get information about x would be to use the contrapositive i.e. $\neg p \Rightarrow \neg q$: "If x is not between -1 and 1 (exclusive), then $x^2 > 1$." That is, we need to prove that $(x < -1 \cup x > 1) \Rightarrow x^2 > 1$. Similar to the first statement, since the hypothesis is a disjunction assumed to be true, each part of the disjunction can also be assumed to be true. We can show that $\neg q$ is true for each part of the disjunction, which proves $\neg p \Rightarrow \neg q$, proving the second statement by proving the contrapositive.

Proof: To prove the biconditional statement, we'll prove two conditional statements

The first statement $p \Rightarrow q$: "If x is between -1 and 1, then x^2 is less than or equal to 1", can be proven by proving that q is true for each part of the conjunction $(-1 \leq x \cap x \leq 1)$ in the hypothesis. For the first part of the conjuction, we assume that $-1 \leq x$. Mulipling each side of the inequality by -1 flips the inequality to $-x \leq 1$. Multiplying $-x \leq 1$ by itself won't change the sign and will instead yield $-x * -x \leq 1 * 1$ which is equivalent to $x^2 \leq 1$. For the second part of the conjunction, we assume that $1 \leq x$. Multiplying the inequality by itself won't change the sign and yields $1 * 1 \leq x * x$ which is equivalent to $x^2 \leq 1$. This proves the first statement.

The second statement, $q \Rightarrow p$: "If x^2 is less than or equal to 1, then x is between -1 and 1 (inclusive)", can be proven by proving the contrapositive, $\neg p \Rightarrow \neg q$: "If x is not between -1 and 1 (exclusive), then $x^2 > 1$." Put another way, we can prove that $(x < -1 \cup x > 1) \Rightarrow x^2 > 1$ by applying the same logic we did to the first statement i.e. proving $\neg q$ to be true with each part of the disjunction. For the first part of the disjunction, we assume that $x < -1$. Multipling each side of the inequality by -1 flips the inequality to $-x > 1$. Multipling $-x > 1$ by itself won't change the sign of the inequality and yields $-x * -x > 1 * 1$ which is equivalent to $x^2 > 1$. For the second part of the disjunction, we assume that $x > 1$. Multiplying this inequality by itself yields $x * x > 1 * 1$ which is equivalent to $x^2 > 1$. This proves the contrapositive $\neg p \Rightarrow \neg q$ which proves $q \Rightarrow p$.

Since both $p \Rightarrow q$ and $q \Rightarrow p$ are proven to be true, the biconditinal statement "$-1 \leq x \leq 1$ if and only if $x^2 \leq 1$" is true.

$\square$

3) Two whole numbers are said to have the same parity if they are both even or both odd. Prove the following biconditional statement:

Let m and n be whole numbers. m and n have the same parity if and only if $m + n$ is even.

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Proposition: m and n have the same parity $\iff$ $(m + n) \equiv 0 \pmod 2$

Discussion: To prove the biconditional statement, we need to prove the two statements $p \Rightarrow q$: "If m and n have the same parity, then $m+n$ is even." and $q \Rightarrow p$: "If $m+n$ is even, then m and n have the same parity."

The first statement is relatively easy to prove because we're already assuming that m and n have the same parity. We can look at two cases: m and n are both even, or m and n are both odd. If m and n are both even, then we can represent them as $2k$ and $2l$, respectively, where $k \in \mathbb{N}$ and $l \in \mathbb{N}$. Adding $2k$ and $2l$ together yields $2k + 2l = 2(k + l)$ which is divisible by 2, therefore making the sum even. If m and n are both odd, then we can represent them as $2k + 1$ and $2l + 1$, respectively, where $k \in \mathbb{N}$ and $l \in \mathbb{N}$. Adding $2k + 1$ and $2l + 1$ yields $2k + 2l + 2$ which is divisible by 2, therefore making the sum even. In both cases where m and n had the same parity, $m + n$ was even, therefore proving $p \Rightarrow q$.

The second statement is a little tricker to prove in its current form because its difficult to extract information about m and n from $m + n$, so it'd be useful to prove the contrapositive instead. The contrapositive is $\neg p \Rightarrow \neg q$: "If m and n don't have the same parity, then $m + n$ is odd". This is much easier to prove. We can represent m and n as $2k$ and $2l + 1$ where $k \in \mathbb{N}$ and $l \in \mathbb{N}$. The order of this assignment doens't matter, as long as both numbers have different parities, because addition is commutative. Adding $2k$ and $2l + 1$ yields $2k + 2l + 1$ which will always be odd. This proves the contrapositive which then proves $q \Rightarrow p$.

Since we proved both conditional statements, the biconditional statement "If m and n have the same parity, then $m + n$ is even" is proven true.

Proof: To prove the biconditional statement, we will prove two conditional statements.

The first statement $p \Rightarrow q$: "If m and n have the same parity, then $m + n$ is even" can be proven by analyzing the two cases that arise from assuming the hypothesis to be true. If m and n have the same parity, then either m and n are both odd or both even.

- ***m* and *n* are even:** Since both are even, we can represent them as $2k$ and $2l$, respectively, where $k \in \mathbb{N}$ and $l \in \mathbb{N}$. Adding these two expressions yields $m + n = 2k + 2l = 2(k + l)$. No matter what k and l are, multiplying $k + l$ by 2 makes it divisible by two ($\frac{m+n}{2} = \frac{2(k+l)}{2} = k + l \equiv 0 \pmod 2$) and therefore even. This proves that if m and n are even, then $m + n$ is even.
- ***m* and *n* are odd:** If m and n are odd, then they can be represented as $2k+1$ and $2l+1$, respectively, where $k \in \mathbb{N}$ and $l \in \mathbb{N}$. Adding these two expressions yields $m + n = 2k + 1 + 2l + 1 = 2k + 2l + 2 = 2(k + l + 1)$. If we divide the sum by 2, $\frac{m+n}{2} = \frac{2(k+l+1)}{2} = k + l + 1 \equiv 0 \pmod 2$, then we can see that is is divisible by 2 and therefore even. This proves that when m and n are odd, then $m + n$ is even.

Now that we've proved both cases, we can say that the first statement $p \Rightarrow q$ is true.

The second statement $q \Rightarrow p$ will be proven by proving the contrapositive statement $\neg p \Rightarrow \neg q$: "If m and n don't have the same parity, then $m+n$ will be odd.' Since m and n are of different parities, one of them is odd and one of them is even. That means that they can be represented as $2k$ and $2l+1$, where $k \in \mathbb{N}$ and $l \in \mathbb{N}$, in no particular order because addition is commutative (and it doesn't matter whether m or n is odd). Adding these two yields $m+n = 2k+2l+1$. When diving this sum by 2, $\frac{m+n}{2} = \frac{2k+2l+1}{2} \equiv 1 \pmod 2$, it's clear that the sum is not divisible by two and therefore odd. This proves the contrapositive $\neg p \Rightarrow \neg q$ which then proves $q \Rightarrow p$.

Now that we have proved that $p \Rightarrow q$ and $q \Rightarrow p$, we have proved the biconditional statement $p \iff q$ that "m and n have the same parity if and only if $m+n$ is even."

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4) Use proof by contrapositive to prove the following conditional statement.

Let m and n be whole numbers. If $m * n$ is odd, then m and n are both odd.

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Proposition: $m * n \equiv 1 \pmod 2 \Rightarrow m \equiv 1 \pmod 2 \cap n \equiv 1 \pmod 2$

Discussion: Since our hypothesis gives us information about $m * n$ and not m or n, it would be useful to work with the contrapositive $\neg q \Rightarrow \neg p$ which states "If either m or n is even, then $m * n$ is even." This statement is much easier to prove because if we're assuming one of the numbers to be even, then it can be written as $2k$ where $k \in \mathbb{N}$. Irregardless of n's parity, the resulting product will be divisible by two and therefore even. Proving the contrapositive proves the original statement.
We will assume: m or n is even
We will prove: $m * n$ is even.

Proof: To prove the statement, we will prove the contrapositive $\neg q \Rightarrow \neg p$ that states "If either m or n is even, then $m * n$ is even." We're assuming that at least one of the numbers is even, so let's take m to be even for now. If m is even, then it can be represented as $2k$ where $k \in \mathbb{N}$. Multiplying m with n yields $m * n = 2k * n = 2 * (k * n)$. Dividing this expression with 2, $\frac{m*n}{2} = \frac{2*(k*n)}{2} = k * n \equiv 0 \pmod 2$, shows that not only do we get a product that is divisible by 2 and therefore even, but that the parity of n does not matter. This process can recreated again if we take just n to be even or both m and n to be even. This proves the contrapositive $\neg q \Rightarrow \neg p$ which then proves the original statement that "If $m * n$ is odd, then m and n are both odd."

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5) We will investigate the following statement:

Every odd whole number can be written as the difference of two perfect squares.

(a) For the odd whole numbers n = -3,-1,1,3,5,7,9, write n as the difference of two perfect squares.

(b) Use any pattern that you found in (a) to help you write a proof of our statement.

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a)

n	Difference of squares
-3	$1^2 - (-2)^2$
-1	$0^2 - (-1)^2$
1	$1^2 - 0^2$
3	$2^2 - 1^2$
5	$3^2 - 2^2$
7	$4^2 - 3^2$
9	$5^2 - 4^2$

b) Proposition: Every odd whole number can be written as the difference of two perfect squares.

Discussion: After filling out the table with a few odd numbers and their corresponding difference of perfect squares, I notices that all the perfect squares were created from consecutive whole numbers. Even the negative odd numbers continue this progression if you rewrite the second number as a negative (gets turned positive when squaring it). This pattern is what I honed in on to show that every odd number can be written as a difference of perfect squares, and I will show this pattern by demonstrating how $2k+1$, where $k \in \mathbb{N}$, can be rewritten algebraically as a difference of perfect squares.

We will assume: We're starting an odd whole number
We will prove: Every odd whole number can be expressed as a difference of perfect squares

Proof: Since we're starting with an odd whole number, we can express that as $2k+1$ where $k \in \mathbb{N}$. The expression can be rewritten as follows: $2k+1 = 2k+1+k^2-k^2$. After reordering the terms a bit, we get $2k+1 = k^2+2k+1-k^2$. The first three terms on the right side of the equation are the expanded form of the square of a binomial, so it can be condensed to $2k+1 = (k+1)^2 - k^2$. From this form, it becomes evident that $2k+1$ (the odd number) is written as the difference between the square of $k+1$ and k. This proves that every odd whole number can be written as the difference between two perfect squares.

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