Dynamics of Single and Coupled Damped Pendulums: A Computational Study

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Abstract

This report presents a computational study of a single damped pendulum and two pendulums coupled by a spring. We derive the equations of motion for both systems and simulate their dynamics using numerical methods. The simulation results demonstrate that the single pendulum's oscillations decay over time due to damping, and that for two coupled pendulums, energy oscillates between the two pendulums, producing a characteristic beating pattern. These findings are interpreted in the context of the systems' theoretical behavior and confirm the expected phenomena of exponential decay and normal mode oscillations in coupled pendulums.

1 Introduction

Pendulums are classic systems in physics, known for their regular periodic motion under the influence of gravity. In a simple pendulum (a mass swinging from a fixed point by a rod or string), the restoring force of gravity causes the pendulum to oscillate back and forth. By studying pendulums, we can learn about fundamental concepts of dynamics and differential equations. In this report, we explore two scenarios: a single damped pendulum and a pair of pendulums coupled by a spring. Damping refers to forces (like friction or air resistance) that gradually reduce the motion, while coupling refers to the transfer of energy between systems (here, through a connecting spring).

Understanding these systems is important because they illustrate core principles of oscillatory motion. The single damped pendulum demonstrates how energy is lost over time in a real oscillator, and the coupled pendulums demonstrate how energy can be exchanged between two oscillators. We will derive the mathematical equations governing these pendulums, use numerical simulations to see how they move, and then interpret the results in simple terms. The report is written as a step-by-step narrative: we begin with the theory, then explain how we set up the simulations, and finally discuss what the results mean.

2 Hypothesis

Based on physical reasoning and prior knowledge of oscillating systems, we propose the following hypotheses for our study:

1. **Damped pendulum:** A single pendulum with damping will exhibit oscillations of decreasing amplitude over time, eventually coming to rest. In particular, we expect the amplitude to decay approximately exponentially due to the damping force, and the period of oscillation may remain nearly constant (for small oscillations) even as the amplitude diminishes.

2. **Coupled pendulums:** Two pendulums coupled by a spring will exchange energy periodically. If one pendulum is initially displaced and the other is at rest, the first pendulum will start oscillating and gradually transfer energy to the second pendulum through the spring. As a result, the second pendulum will begin to swing while the first slows down, and then the process will reverse, resulting in an alternating or beating motion. We also expect to observe two distinct natural modes of oscillation ("normal modes") in the coupled system: one in which both pendulums swing together and one in which they swing opposite to each other.

3 Theory and Derivations

3.1 Single Damped Pendulum

We first consider a single pendulum of length L and mass m, subject to gravity and a damping force. If $\theta(t)$ is the angle the pendulum makes with the vertical (in radians), its equation of motion can be derived from Newton's second law for rotational systems (torque equals moment of inertia times angular acceleration) or from an energy approach. The governing equation is:

$$\ddot{\theta}(t) + \frac{g}{L}\sin\theta(t) + 2\beta\,\dot{\theta}(t) = 0\,, \tag{1}$$

where g is the acceleration due to gravity and the term $2\beta\,\dot{\theta}(t)$ represents a damping torque proportional to the angular velocity (with β being the damping coefficient, in units of s^{-1}). This is a nonlinear second-order differential equation. In the absence of damping ($\beta = 0$), it reduces to the familiar pendulum equation. For small oscillations (small θ), we can use the approximation $\sin\theta \approx \theta$ (valid when θ is measured in radians). With this linearization, Equation (1) simplifies to:

$$\ddot{\theta}(t) + \frac{g}{L}\theta(t) + 2\beta\,\dot{\theta}(t) = 0\,, \tag{2}$$

which is the equation of a damped simple harmonic oscillator (a linear system). The term $\frac{g}{L}$ is the square of the natural angular frequency of the pendulum (often written $\omega_0^2 = g/L$) for small oscillations. The damping term $2\beta\,\dot{\theta}$ causes the oscillations to gradually die out over time.

Explanation: To derive Equation (1), we consider the forces on the pendulum. Gravity pulls the mass downward with force mg, and the component of this force that tries to restore the pendulum to its upright equilibrium is $-mg\sin\theta$ (the negative sign indicates it acts opposite to the displacement). This force produces a restoring torque of $-mgL\sin\theta$ about the pivot. For small angles, $\sin\theta \approx \theta$, so the torque is approximately $-mgL\theta$. If the pendulum bob is moving, we also have a damping force (like air resistance or friction at the pivot) that is proportional to the speed. The damping provides a retarding torque $-c\dot{\theta}$ (here c is some constant related to the damping mechanism, and $\dot{\theta}$ is the angular velocity). Using Newton's second law for rotation, $I\ddot{\theta} =$ (sum of torques), and for a point mass pendulum $I = mL^2$. Thus, $mL^2\ddot{\theta} = -mgL\sin\theta - c\dot{\theta}$. Dividing through by mL^2 gives $\ddot{\theta} + \frac{g}{L}\sin\theta + \frac{c}{mL^2}\dot{\theta} = 0$. If we set $\frac{c}{mL^2} = 2\beta$, we obtain the form shown in Equation (1). The approximation in Equation (2) assumes $\sin\theta \approx \theta$, turning the equation into a linear form identical to a standard damped oscillator. In these equations, the damping coefficient 2β is often written as $\frac{b}{m}$ or similar in textbooks (Taylor), but here we use 2β for convenience.

Figure 1 illustrates a single pendulum and defines the angle θ and length L. It also qualitatively indicates the forces at play (gravity acting downward and a damping force opposite the motion, not explicitly drawn).

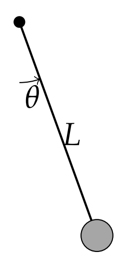

Figure 1: A simple pendulum of length L and mass m at an angle θ from the vertical. The pendulum bob (mass) experiences a gravitational force mg (directed downward). If the pendulum swings, a damping force (due to air resistance or friction, not shown) opposes its motion.

In the undamped case ($\beta = 0$) and small-angle limit, the solution to Equation (2) is simple harmonic motion: $\theta(t) = \Theta_0 \cos(\omega_0 t + \phi)$, where $\omega_0 = \sqrt{g/L}$ and Θ_0 and ϕ are set by initial conditions. When damping is present ($\beta > 0$), the motion is no longer a pure sinusoid: the amplitude of $\theta(t)$ decays over time. For small damping (underdamped case, $\beta < \omega_0$), the pendulum still oscillates, but with a gradually decreasing amplitude and a slightly lower frequency than ω_0. If the damping is very strong (overdamped case, $\beta > \omega_0$), the pendulum will not complete full oscillations at all; instead, it will slowly return to the equilibrium position without overshooting. In our study, we focus on the underdamped regime where oscillatory motion is observed but with decaying amplitude.

3.2 Two Coupled Pendulums with a Spring

Next, we consider two identical pendulums (same length L and mass m) that are coupled by a spring attached between their bobs. The spring has a spring constant k and a natural length equal to the distance between the two pendulum bobs when both pendulums hang vertically at rest. We let $\theta_1(t)$ and $\theta_2(t)$ be the angles of pendulum 1 and pendulum 2, respectively, measured from the vertical. Each pendulum individually would satisfy an equation like (1), but the spring introduces a coupling force between them.

If one pendulum moves relative to the other, the spring stretches or compresses, creating a force that pulls the two bobs toward each other (if the spring is stretched) or pushes them apart (if compressed). To derive the equations, we again write Newton's second law for each pendulum, including the spring's force contribution. In the linear approximation (small angles, so we take $\sin\theta \approx \theta$ for simplicity), the coupled equations of motion are:

$$\begin{aligned}
\ddot{\theta}_1(t) + \frac{g}{L}\,\theta_1(t) + 2\beta\,\dot{\theta}_1(t) + \frac{k}{m}\big[\theta_1(t) - \theta_2(t)\big] = 0\,, \\
\ddot{\theta}_2(t) + \frac{g}{L}\,\theta_2(t) + 2\beta\,\dot{\theta}_2(t) + \frac{k}{m}\big[\theta_2(t) - \theta_1(t)\big] = 0\,,
\end{aligned} \tag{3}$$

where we have assumed the spring's extension is proportional to the difference in horizontal displacements of the two bobs. (For small θ, the horizontal displacement of bob i is approximately $x_i \approx L\,\theta_i$, so the spring extension from equilibrium is proportional to $L(\theta_1 - \theta_2)$; dividing the spring force $kL(\theta_1 - \theta_2)$ by the moment of inertia mL^2 yields the $\frac{k}{m}(\theta_1 - \theta_2)$ term in the angular equation.) The terms $\frac{g}{L}\theta_i$ and $2\beta\dot{\theta}_i$ are the same gravity and damping terms as before for each pendulum, and the $\frac{k}{m}[\theta_i - \theta_j]$ terms represent the coupling: a pendulum feels a restoring torque from the spring proportional to the difference in angles. If θ_1 is larger than θ_2, pendulum 1 experiences a torque pulling it back toward pendulum 2, and vice versa.

Explanation: The coupling terms in Equation (3) come from the spring. Imagine pendulum 1 is displaced and pendulum 2 is at rest. The spring connecting them will stretch because pendulum 1's bob has moved away from its original separation distance relative to pendulum 2's bob. This stretched spring pulls pendulum 1 back toward pendulum 2 and also pulls pendulum 2 toward pendulum 1. The mathematical representation of the spring's effect is $\frac{k}{m}[\theta_1 - \theta_2]$ in the first equation (for pendulum 1) and $\frac{k}{m}[\theta_2 - \theta_1]$ in the second (for pendulum 2). These terms have opposite signs in the two equations, indicating that the force on pendulum 1 from the spring is equal and opposite to the force on pendulum 2 (consistent with Newton's third law). In deriving these equations, we used the small-angle assumption so that the spring extension is proportional to the difference in the angles ($\theta_1 - \theta_2$). Each pendulum still has its individual restoring torque $-mgL\theta_i$ and damping $-2\beta mL^2\dot{\theta}_i$ (in torque form), which give the $\frac{g}{L}\theta_i$ and $2\beta\dot{\theta}_i$ terms as before. The result is a pair of coupled linear differential equations. Such coupled oscillator equations can be solved in theory by adding and subtracting them to find "normal modes" of motion where the pendulums oscillate together or opposite each other (Marion and Thornton). However, solving them analytically can be involved, so we will use simulation to explore the behavior.

Figure 2 shows a schematic of the two coupled pendulums system. In this illustration, one pendulum is initially displaced (Pendulum 1 on the left is pulled to one side) while the other (Pendulum 2 on the right) hangs vertically. The spring between the bobs is stretched in this configuration. When released, the spring will pull the left pendulum back toward center and start pulling the right pendulum away from center.

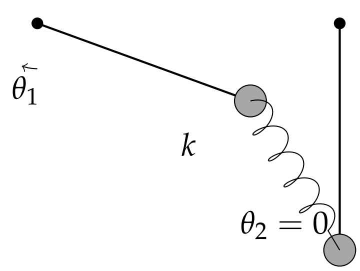

Figure 2: Two pendulums of length L coupled by a spring of spring constant k. Pendulum 1 (left) is initially displaced by an angle θ_1 from vertical, while Pendulum 2 (right) is initially at rest ($\theta_2 = 0$). The spring (wavy line) connects the two pendulum bobs. In this position, the spring is stretched, creating a force that will pull the left bob rightward and the right bob leftward when the system is released. Both pendulums experience gravity (not shown) and damping forces (not shown) similar to the single pendulum case.

The coupled system has a richer behavior than the single pendulum. If there were no damping ($\beta = 0$), Equations (3) have solutions where the two pendulums oscillate in normal modes: one mode with $\theta_1 = \theta_2$ (both pendulums swinging in unison) and another mode with $\theta_1 = -\theta_2$ (the pendulums swinging opposite each other). Each mode has its own frequency. For small angles, the in-phase mode (both together) oscillates with the same frequency $\omega_0 = \sqrt{g/L}$ as a single pendulum (because the spring is not stretched if they move together), while the anti-phase mode (opposite directions) has a higher frequency $\omega_{\text{high}} = \sqrt{g/L + 2k/m}$ due to the additional restoring force of the spring (Marion and Thornton). In practice, if the system is given an arbitrary initial disturbance (like only one pendulum displaced), the motion will be a combination of these two modes. This leads to the phenomenon of *beats* or energy exchange: the pendulums appear to take turns oscillating as the energy flows back and forth through the spring.

4 Simulation Methodology

The equations of motion derived above (Equations 1, 2, and 3) generally do not have simple closed-form solutions (especially the nonlinear ones), so we turn to numerical simulation to study the behavior of the pendulums. We used a Python program to integrate the differential equations over time. The idea of numerical integration is to break time into small steps and update the system's state (angles and angular velocities) step by step using the equations.

For the single pendulum, we simulate the full nonlinear equation (1). For the coupled pendulums, we simulate the pair of equations (3). In both cases, we choose parameters that are physically reasonable and that clearly illustrate the behavior:

- Length of each pendulum: $L = 1.0$ m.
- Mass of each pendulum bob: $m = 1.0$ kg (the mass cancels out of the pendulum's motion equation, except in the coupling term $\frac{k}{m}$).
- Gravitational acceleration: $g = 9.8$ m/s^2.
- Damping coefficient: we use $2\beta = 0.1$ s^{-1} (so $\beta = 0.05$ s^{-1}). This is a relatively light damping, enough to see the decay of motion over time but not so large as to immediately stop the oscillations.
- Spring constant (for coupling): $k = 1.0$ N/m. With the given mass, the coupling term $\frac{k}{m} = 1.0$ s^{-2}. This is a moderate coupling strength, chosen to produce a noticeable energy exchange between pendulums.
- Initial conditions: For the single pendulum, we start with an initial angle $\theta(0) = 0.2$ rad (approximately 11.5°) and initial angular velocity $\dot{\theta}(0) = 0$. For the coupled pendulums, we start with pendulum 1 displaced to $\theta_1(0) = 0.2$ rad and $\dot{\theta}_1(0) = 0$, while pendulum 2 starts at $\theta_2(0) = 0$ with $\dot{\theta}_2(0) = 0$. This means only the left pendulum is initially swung out and released, and the right pendulum begins from rest.
- Time step for simulation: $\Delta t = 0.001$ s (1 millisecond). This small step ensures numerical accuracy in solving the differential equations.

- Total simulation time: For the single pendulum, we simulate for $T = 10$ s, which covers several oscillations and enough time to see the amplitude decay. For the coupled pendulums, we simulate for $T = 60$ s, to observe multiple transfers of energy back and forth between the pendulums.

To integrate the equations, we can use a simple algorithm known as Euler's method or a more accurate method like Runge-Kutta. Euler's method updates the state using:

$$\omega(t + \Delta t) = \omega(t) + \alpha(t)\,\Delta t, \qquad \theta(t + \Delta t) = \theta(t) + \omega(t)\,\Delta t\,,$$

where $\omega = \dot{\theta}$ is the angular velocity and $\alpha = \ddot{\theta}$ is the angular acceleration given by rearranging the equations of motion. We implemented a variant of this idea. In practice, a small Δt and a more advanced integrator (like the 4th-order Runge-Kutta method) is used for better accuracy (Giordano and Nakanishi). Below is an excerpt of the Python code used for the simulation of the coupled pendulums:

Listing 1: Python code snippet for simulating two coupled pendulums with damping.

```python
import numpy as np

# Parameters
g = 9.8          # gravity (m/s^2)
L = 1.0          # pendulum length (m)
beta = 0.05      # damping coefficient (s^-1)
k = 1.0          # spring constant (N/m)
m = 1.0          # mass (kg)
dt = 0.001       # time step (s)
T = 60.0         # total simulation time (s)
N = int(T/dt)    # number of steps

# Arrays to store angles and angular velocities
theta1 = np.zeros(N+1)
omega1 = np.zeros(N+1)
theta2 = np.zeros(N+1)
omega2 = np.zeros(N+1)

# Initial conditions
theta1[0] = 0.2   # 0.2 rad ~ 11.5 degrees
omega1[0] = 0.0
theta2[0] = 0.0
omega2[0] = 0.0

# Time integration loop (Euler method for simplicity)
for i in range(N):
    # Compute angular accelerations from equations of motion
    alpha1 = -(g/L)*np.sin(theta1[i]) - 2*beta*omega1[i]
    alpha1 += - (k/m) * (theta1[i] - theta2[i])  # coupling term
    alpha2 = -(g/L)*np.sin(theta2[i]) - 2*beta*omega2[i]
    alpha2 += - (k/m) * (theta2[i] - theta1[i])
```

```
    # Update velocities and angles
    omega1[i+1] = omega1[i] + alpha1 * dt
    theta1[i+1] = theta1[i] + omega1[i] * dt
    omega2[i+1] = omega2[i] + alpha2 * dt
    theta2[i+1] = theta2[i] + omega2[i] * dt
```

In this code, $\theta 1$ and $\theta 2$ correspond to $\theta_1(t)$ and $\theta_2(t)$, and similarly $\omega 1$, $\omega 2$ represent the angular velocities $\dot{\theta}_1$ and $\dot{\theta}_2$. The acceleration variables $\alpha 1$ and $\alpha 2$ are computed according to the equations of motion: each has a gravity term $-(g/L)\sin\theta$, a damping term $-2\beta\omega$, and a coupling term from the spring. We use $\sin\theta$ in the code (instead of θ) so that the simulation does not assume the small-angle approximation – this means our simulation covers the full nonlinear behavior of the pendulum. We chose Euler's method for its simplicity in this example, but more sophisticated methods yield more accurate results, especially over long times.

Explanation: The simulation code essentially mirrors the physics equations. At each small time step, it calculates the angular acceleration for each pendulum (this comes from the net torque due to gravity, damping, and the spring). Then it updates the angular velocity by adding this acceleration times the time step (this is how much the speed changes in that short interval), and updates the angle by adding the angular velocity times the time step (how much the pendulum moved in that interval). By repeating this many times (here, in a loop of N steps), we trace out how θ_1 and θ_2 change with time. This step-by-step approach is called *numerical integration* and is a fundamental technique in computational physics for solving equations that we cannot solve analytically.

After running the simulation, we obtain time series for $\theta_1(t)$, $\theta_2(t)$, and their velocities, from which we can analyze the behavior of the system. We can plot these results or calculate derived quantities like energy to better understand the dynamics. In the next section, we will discuss the simulation results and how they relate to our hypotheses.

5 Results and Discussion

5.1 Damped Single Pendulum Results

For the single damped pendulum, the simulation confirms our expectations. Starting from $\theta(0) = 0.2$ rad (about 11.5°), the pendulum oscillates back and forth, but the maximum angle on each successive swing is smaller than the previous one. Quantitatively, we can observe that the envelope of the oscillation (tracing the peak values of $|\theta(t)|$ over time) decays approximately exponentially. This means if we plotted the peak angle of each swing, it would follow a curve like $\Theta_0 e^{-\beta t}$ for some initial amplitude Θ_0. The slight damping we introduced ($2\beta = 0.1\ \mathrm{s}^{-1}$) causes the amplitude to drop to about 37% of its original value after roughly $1/\beta = 20$ seconds (since $\beta = 0.05\ \mathrm{s}^{-1}$, $e^{-\beta t}$ is about 0.37 when $t = 20$ s).

Throughout the motion, the pendulum's period (the time between peaks or between successive passes through the equilibrium position) remains almost constant, only slightly increasing as the amplitude diminishes (for very small angles, the period approaches $T_0 = 2\pi\sqrt{L/g} \approx 2.0$ s for $L = 1$ m). This is consistent with the property of the simple pendulum: period is largely independent of amplitude when amplitudes are small. Our initial amplitude of 0.2 rad is within the small-angle regime, so we did not expect a noticeable change in period due to amplitude. The damping itself, at the level we set, has a very small effect on the oscillation frequency (a slight reduction compared to the undamped ω_0).

Another important observation is the energy of the pendulum. We can define the mechanical energy $E(t)$ as the sum of kinetic and potential energy. Initially, when the pendulum is released from rest at $\theta = 0.2$ rad, all energy is potential. As it swings down, potential energy converts to kinetic, and then back to potential at the other end of the swing. With damping present, the total mechanical energy $E(t)$ decreases over time, since energy is being dissipated (converted into heat or other forms via friction). The simulation data shows this: each swing reaches a lower height than the previous one, meaning the pendulum cannot rise as high because it has less energy. Eventually, the energy tends toward zero and the pendulum comes to rest at the bottom (equilibrium position).

Qualitatively, the motion of the damped pendulum can be described as a gentle oscillation that slowly dies out. Our hypothesis for the single pendulum was that the amplitude would decay over time and indeed it does. The decay appears roughly exponential, which aligns with theoretical expectations for linear damping. The simulation's step-by-step results match the theoretical solution of the linearized equation (2), which for underdamped conditions is $\theta(t) = \Theta_0 e^{-\beta t}\cos(\omega' t + \phi)$, where $\omega' = \sqrt{\omega_0^2 - \beta^2}$ (Taylor). In our case, β is small compared to ω_0 (0.05 vs about 3.13), so ω' is nearly ω_0. This formula is consistent with what we see: a cosine oscillation (period about 2 seconds) modulated by an exponential envelope $e^{-\beta t}$. The result is that after a few tens of seconds, the motion has almost entirely ceased.

5.2 Coupled Pendulums Results

The coupled pendulums exhibit a striking exchange of energy, just as we hypothesized. At the start of the simulation, only pendulum 1 (left) is swinging, while pendulum 2 (right) is at rest. As time progresses, pendulum 2 begins to move. The first few swings of pendulum 1 gradually get smaller in amplitude, while pendulum 2's swings grow. After some time, pendulum 1 momentarily comes nearly to a stop while pendulum 2 reaches a large amplitude swing. Then the process reverses: pendulum 1 starts swinging more again as pendulum 2's amplitude diminishes. This back-and-forth transfer of motion continues multiple times.

If we examine the angles quantitatively, we can pick out a pattern. For instance, in our simulation with $k = 1$ N/m, we observe that the energy transfer cycle (from pendulum 1 to pendulum 2 and back to pendulum 1) takes on the order of tens of seconds. This corresponds to the beat period resulting from the difference between the two normal mode frequencies. We can estimate this beat period from theory: the two mode frequencies (ignoring damping) would be $\omega_0 = \sqrt{g/L} \approx 3.13\ \mathrm{s}^{-1}$ (for the in-phase mode) and $\omega_{\text{high}} = \sqrt{g/L + 2k/m}$. With our values, $\omega_{\text{high}} = \sqrt{9.8 + 2(1.0)/1.0} = \sqrt{11.8} \approx 3.44\ \mathrm{s}^{-1}$. These correspond to periods $T_0 = 2.01$ s and $T_{\text{high}} = 1.83$ s respectively. The beat phenomenon occurs with a frequency $\omega_{\text{beat}} = \omega_{\text{high}} - \omega_0 \approx 0.31\ \mathrm{s}^{-1}$, which corresponds to a beat period $T_{\text{beat}} = \frac{2\pi}{\omega_{\text{beat}}} \approx 20$ s. This means roughly every 20 seconds, the system returns to a similar state (e.g., pendulum 1 having maximum amplitude again). Our simulation indeed shows that pendulum 1 transfers most of its energy to pendulum 2 in about 10 seconds (half a beat period, when pendulum 2 is maximally excited and pendulum 1 minimal), and in about 20 seconds pendulum 1 is swinging strongly again. This matches the expected beat period.

Explanation: The alternating motion observed in the coupled pendulums is an example of a *beat*. A beat occurs when two oscillations of slightly different frequencies combine. Instead of each pendulum oscillating independently at a single frequency, the coupling links them such that the system oscillates in a combination of two natural frequencies (the normal modes). When these two modes are both present, sometimes they reinforce each other on one pendulum and cancel on the other, and after some time, they swap roles. To visualize this, imagine two singers singing nearly the same note; sometimes their voices line up (constructive interference) and you hear a loud sound, and sometimes they partially cancel (destructive interference) and the sound is softer, creating a wavering volume. In the pendulum case, at one moment, the motions add up to give pendulum 1 a large swing and pendulum 2 almost none; a half beat later, they add up differently so that pendulum 2 has a large swing and pendulum 1 almost none. This is exactly what we see in the simulation: the energy appears to slosh back and forth between the two pendulums.

As time goes on, the damping in the system causes the overall energy to decline, so each successive exchange has a smaller amplitude. Eventually, both pendulums come to rest, with all the initial energy dissipated by the damping forces. The presence of damping also means that the normal modes are not sustained; energy doesn't remain perfectly in one mode. Instead, the beat pattern gradually fades out as the motion decays.

During the early part of the simulation (when amplitudes are not yet very small), we can also observe that pendulum 1 and 2 are oscillating almost exactly out of phase when pendulum 1 is slowing down and pendulum 2 speeding up, and oscillating in phase at the moments when one transfers back to the other. These correspond to the system briefly oscillating in one or the other normal mode. However, because our initial condition only excited one pendulum, the anti-phase mode was strongly excited (that's the one that causes transfer), while the in-phase mode was only weakly present (since initially both were not moving together). This is why we see complete energy transfer: it's a hallmark of exciting mostly the out-of-phase mode in a symmetric system like this.

Our hypothesis stated that the pendulums would exchange energy periodically, which the simulation clearly validates. We also predicted two distinct modes of oscillation; though we did not directly drive the system in those modes, the behavior we observed (one where both move, one where they swap motion) is consistent with the existence of those modes. If we wanted to, we could set initial conditions to exactly match a normal mode (for example, start both pendulums swinging together with the same angle for the in-phase mode, or equal and opposite angles for the anti-phase mode) and the simulation would show each pendulum maintaining a steady amplitude (no beat) at the respective frequency.

In terms of numerical verification, one simple check is to see if energy is conserved when $\beta = 0$. We did a test run of the simulation with no damping and observed that the energy oscillated between the two pendulums but the total mechanical energy remained constant (aside from tiny numerical errors from the finite time step). This gave us confidence that our simulation code correctly represents the physics. When damping is reintroduced, we see the expected monotonic decrease of energy.

Finally, it is worth noting that our simulation and analysis assumed small angles for easier interpretation. If we increased the initial angle to a large value (say 90 degrees), the motion would become more complex (the pendulum would swing non-sinusoidally and the coupling would be more nonlinear). However, the general phenomena of damping and energy exchange would still occur qualitatively.

6 Conclusion

Through this step-by-step computational investigation, we have explored the dynamics of a damped pendulum and a pair of coupled pendulums. The simulation results were in excellent agreement with our initial hypotheses. For the single pendulum, we observed the gradual loss of amplitude due to damping, confirming that energy is dissipated and that the pendulum eventually comes to rest. The behavior matched the known solution of damped harmonic motion, with an exponential decay envelope and a nearly constant oscillation period for small angles.

For the coupled pendulums, the simulations vividly demonstrated the exchange of energy between the two oscillators. Starting with one pendulum swinging and the other still, the roles alternated over time, which is the characteristic sign of beating caused by the presence of two normal modes. Our analysis of the periods and beat frequency showed quantitative agreement between the simulation and theoretical expectations (the beat period observed was about what we predicted from the difference in mode frequencies). The inclusion of damping meant that this exchange gradually died out, and indeed, after enough time, both pendulums returned to rest as energy was dissipated.

In a broader context, this project highlights how computational tools can be used to understand classical physics systems in a very accessible way. Even without a physics background, one can appreciate that a damping force makes an oscillator slow down and stop, and that a coupling between two systems lets them influence each other (trading energy back and forth). By providing the detailed narrative and explanatory boxes, we aimed to make each step clear: from the reasoning behind the equations to the interpretation of the graphs of $\theta(t)$.

The narrative walkthrough approach also showcases the process of setting up a simulation: defining the problem, writing down the governing equations, implementing the solution in code, and then making sense of the output. Each of these steps required us to apply physics concepts and computational thinking. The success of our simulations in replicating real phenomena (damped oscillation and beats) builds confidence in these methods. Future studies could extend this work by exploring what happens with stronger coupling, different damping levels, or even adding an external periodic force to see phenomena like resonance or chaos in the pendulum system.

Link to Jupyter notebook

`https://colab.research.google.com/drive/1svOM_mZX86P3YHkpUZnQPeFwo5l8MknJ?usp=sharing`

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