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Linear Algebra

1) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$\xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \xrightarrow{R_4 - 6R_1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Rank $= 3$

2) $W \to$ Symmetric 2×2 $\quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b) + (b-c)x + (c-a)x^2$

$c = b \to$ Symmetric

$T\begin{bmatrix} a & b \\ b & d \end{bmatrix} = (a-b) + (b-b)x + (c-a)x^2$
$= (a-b) + (c-a)x^2$

3) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $|A - \lambda I| = 0$

$\cancel{2-\lambda \quad -1}$

$\cancel{-1 \quad 2-\lambda}$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)^2 - 1 = 0$

$(2-\lambda)^2 = 1 \Rightarrow 2-\lambda = 1, -1$

$\lambda = 1, 3$

Eigen ~~val~~ value $= 1, 3$

$(2-\lambda)^2 = 1 \Rightarrow 4 + \lambda^2 - 4\lambda = 1$

$\lambda^2 - 4\lambda + 3 = 0$

$A^2 - 4A + 3 = 0$

$AA - 4A + 3 = 0$

$A^{-1}AA - 4A^{-1}A + 3A^{-1} = 0$

$A - 4I + 3A^{-1} = 0$

$3A^{-1} = 4I - A$

$$A^{-1} = \frac{1}{3}\left(4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{3}\left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A + 4I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix}$$

$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$|A^{-1} - \lambda I| = 0 \Rightarrow \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$

$\begin{bmatrix} \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3}-\lambda \end{bmatrix} = 0 \Rightarrow \left(\frac{2}{3}-\lambda\right)^2 - \frac{1}{9} = 0$

$\Rightarrow \left(\frac{2}{3}-\lambda\right)^2 = \frac{1}{9}$

$\frac{2}{3} - \lambda = \frac{1}{3}, \frac{-1}{3}$

$\lambda = \frac{1}{3}, 1$

Eigen Values of A^{-1} is $\frac{1}{3}, 1$

$A^{-1}X = 0 \Rightarrow \cancel{\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$

$\lambda = 1, (A^{-1} - \lambda I)X = 0 \Rightarrow \left[\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\left(\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -\frac{x}{3} + \frac{y}{3} \\ \frac{x}{3} - \frac{y}{3} \end{bmatrix} = 0$

$\frac{x}{\cancel{3}} = \frac{y}{\cancel{3}} \Rightarrow x = y$

Eigen vector $= K\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = \frac{1}{3} \Rightarrow \left(\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\left(\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\frac{x}{3} - \frac{y}{3} = 0 \quad \Big| \quad -\frac{x}{3} + \frac{y}{3} = 0$

$x = y$

Eigen vector $= k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen Value of $A + 4I = 1+4, 3+4$
$= 5, 7$

$\lambda = 5$, Eigen Vector : $[A + 4I - 5I] = 0$
$[A - I]x = 0$

$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$x - y = 0$

$x = y$

Eigen Vector $= k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 7$; $[A + 4I - 7I]X = 0 \Rightarrow (A - 3I)X = 0$

$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad -x-y=0$$

$$x = -y$$

Eigen Vector: $K \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4) $3x - 0.1y - 0.2z = 7.85$

$0.1x + 7y - 0.3z = -19.3$

$0.3x - 0.2y + 10z = 71.4$

$x = 0, y = 0 \Rightarrow z = \frac{-7.85}{0.2} = 39.25$

$x = 0, \quad 7 \times y = -19.3 + 0.3 \times 39.25$

$y = -1.075$

$0.3x - 0.2 \times -1.075 + 10 \times 39.25 = 71.4$

$x = -1071.05$

first iteration :

$3 \times -1071.05 - 0.1 \times -1.075 - 0.2z = 7.85$

$z = -16104.46$

$0.1 \times -1071.05 + 7y - 0.3 \times -16104.46$

4) $3x - 0.1y - 0.2z = 7.85$

$0.1x + 7y - 0.3z = -19.3$

$0.3x - 0.2y + 10z = 71.4$

1st iteration: $z = \frac{7.85}{-0.2} = -39.25$

$y = \frac{-19.3 + 0.3\times -39.25}{7} = -4.43$

$x = \frac{71.4 - 10\times -39.25 + 0.2\times -4.43}{0.3} = 1543.38$

2nd iteration: $\cancel{z = \frac{7.85 + 0.1\times -4.43 - 3\times 1543.38}{-0.2} = 4622.933}$

$\cancel{y = z = \frac{7.85 - 3\times 1543.38}{-0.2}}$

$z = \frac{7.85 + 0.1\times -4.43 - 3\times 1543.38}{-0.2} = 4622.933$

$y = \frac{-19.3 + 0.3\times 4622.93 - 0.1\times 1543.38}{7} = 173.32$

$x = \frac{71.4 - 10\times 4622.93 + 0.2\times 173.32}{0.3} = -153744.12$

3rd iteration: $z = \frac{7.85 + 0.1\times 173.32 - 3\times -153744.12}{-0.2} = -2306287.71$

$y = \frac{-19.3 + 0.3\times -2306287.71 - 0.1\times -153744.12}{7}$

$= -986215.43$

$x = \frac{71.4 - 10\times -2306287.71 + 0.2\times -986215.43}{0.3}$

$= 76219018.05$

5) Consistent system of equations have solution inconsistent does don't have solⁿ.

$$x+3y+2z=0$$
$$2x-y+3z=0$$
$$3x-5y+4z=0$$
$$x+17y+4z=0$$

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & -5 & 4 & 0 \\ 1 & 17 & 4 & 0 \end{bmatrix}$$

$$= \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{array} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & -20 & -2 & 0 \\ 0 & 14 & 2 & 0 \end{bmatrix} = \begin{array}{l} \\ \\ \\ R_4+2R_2 \end{array} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & -20 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -20 & -2 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & \frac{1}{10} & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{array}{l} \\ \\ R_3-7R_2 \\ \\ \end{array} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{3}{10} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

solⁿ $x=y=z=0$

8)

8) $\cancel{3x+}\, 3x - 6y + 2z = 23, \quad -4x + y - z = -15, \quad x - 3y + 7z = 16$

$$x = \frac{23 - 2z + 6y}{3}, \quad y = -15 + z + 4x, \quad z = \frac{16 - x + 3y}{7}$$

1st iteration: $x = y = z = 1$

$$x = \frac{23 - 2 + 6}{3} = 9, \quad y = -15 + 1 + 4 = -10, \quad z = \frac{16 - 1 + 3}{7} = 2.57$$

2nd iteration: $x = \dfrac{23 - 2 \times 2.57 + 6 \times -10}{3} = -14.04$

$$y = -15 + 2.57 + 4 \times \cancel{14.04}\ 9 = \cancel{68.59}\ 23.57$$

$$z = \frac{16 - 9 + 3 \times -10}{7} = -3.28$$

3rd iteration: $x = \dfrac{23 - 2 \times -3.28 + 6 \times 23.57}{3} = 56.99$

$$y = -15 + -3.28 + 4 \times -14.04 = -74.44$$

$$z = \frac{16 + 14.04 + 3 \times -74.44}{7} = -27.61$$

9) One common application of matrix application in image processing is in ~~filter~~ applying filters to images like blur, sharp, edge detection, noise reduction.

For example: If an image is too warm (more orange-yellowish tint) we can make it cooler & can subtract the R, G values and increase the B values.

10) In computer vision, linear transformation play a crucial role in manipulating and analyzing images. One such common linear transformation is affine transformation.

In affine transformation, preserves points, straight lines and planes. It includes translations, rotations, scaling and shearing.

The rotation works by:

1) Define rotation matrix: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta \to$ angle of rotation

2) After defining, we multiply the coordinates of each pixel by rotation matrix.

3) After rotation the new coordinates obtained from rotation might not align with the grid of pixels in original image. Interpolation techniques as bilinear interpolation or nearest - neighbor interpolation are used to estimate pixel values at non-integer coordinates.

4) After transformation and interpolation, we get rotated image.

1.2) $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b) + (b-c)x + (c-a)x^2$

$W =$ Symmetric matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow b = c \Rightarrow \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$(a-b) + (b-b)x + (b-a)x^2$

$\Rightarrow (a-b) + (b-a)x^2 \to$ nullity $\Rightarrow a = b$

$\Rightarrow W = \begin{bmatrix} b & b \\ b & d \end{bmatrix} \Rightarrow b\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

nullity $= 2$ $\qquad$ $\dim(A) = r(A) + \text{nullity}(A)$

$$\dim(W) = W = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \Rightarrow a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\dim(W) = 3$

$\dim(W) = r(W) + n(W)$

$3 = r + 2$

$r = 1$

6) $T(a + bx + cx^2) = (a+1) + (b+1)x + (c+1)x^2$

For linear tranformation: $T(x) + T(y) = T(x+y)$

$T(a_1 + b_1 x + c_1 x^2) = (a_1 + 1) + (b_1 + 1)x + (c_1 + 1)x^2 = T(x)$

$T(a_2 + b_2 x + c_2 x^2) = (a_2 + 1) + (b_2 + 1)x + (c_2 + 1)x^2 = T(y)$

$T((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) = a_1 + a_2 + 1 + (b_1 + b_2 + 1)x + (c_1 + c_2 + 1)x^2$
$T(x+y) =$

$T(x) + T(y) = a_1 + a_2 + 2 + (b_1 + b_2 + 2)x + (c_1 + c_2 + 2)x^2$

$T(x) + T(y) \neq T(x+y) \Rightarrow$ Not linear transformation

7) $S = \{(1,2,3), (3,1,0), (-2,1,3)\}$ of $V_3(R)$

$\dim(S) = 3 = V_3$ ✓

$$\begin{array}{c|ccc|c} 3 & 1 & 2 & 3 & 1 \\ 0 & 3 & 1 & 0 & 3 \\ 3 & -2 & 1 & 3 & -2 \end{array}$$

$9 + 3 - (18 - 6) = 12 - 12 = 0$

$\Rightarrow$ Infinite solⁿ $\Rightarrow$ linearly dependent

$\Rightarrow$ Not a basis

$D(p(x)) : \frac{d}{dx}(p(x)) = p(x)$

$D(p(x)+q(x))$, $D(p(x)) = \frac{d}{dx} p(x)$ $D(q(x)) = \frac{d}{dx} q(x)$

$D(p(x)) + D(q(x)) = \frac{d}{dx} p(x) + \frac{d}{dx} q(x)$

$= \frac{d}{\cancel{d(x)}\, dx} (p(x) + q(x))$

$= D(p(x) + q(x))$

$T(x,y) = xy$ $T((x,y)+(p,q)) = T(x+p, y+q) = (x+p)(y+q)$

$T(p,q) = pq$

$T(x,y,z) = (|x|, 0)$ $T(x+u, y+v, z+w)$

$T(u,v,w) = (|u|, 0)$ $= (|x+u|, 0)$

$T(x,y) = |x+y|$ $T(x+p, y+q) = |x+p+p+q|$

$T(p,q) = |p+q|$

4) e_1, e_2, e_3, e_4 span $\mathbb{R}^4$, Ae_1, Ae_2, Ae_3, Ae_4 span Image(A)

$T(x,y,z) = (x-y, x+2y, y+3z)$

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 3(2+1) = 9$$