

# I Chapter 1

## 1 Preliminaries: Set theory and categories

1.1. Locate a discussion of Russel's paradox, and understand it.

*Solution.* Recall that, in naive set theory, any collection of objects that satisfy some property can be called a set. Russel's paradox can be illustrated as follows. Let  $R$  be the set of all sets that do not contain themselves. Then, if  $R \notin R$ , then by definition it must be the case that  $R \in R$ ; similarly, if  $R \in R$  then it must be the case that  $R \notin R$ .

1.2. ▷ Prove that if  $\sim$  is an equivalence relation on a set  $S$ , then the corresponding family  $\mathcal{P}_\sim$  defined in §1.5 is indeed a partition of  $S$ ; that is, its elements are nonempty, disjoint, and their union is  $S$ . [§1.5]

*Solution.* Let  $S$  be a set with an equivalence relation  $\sim$ . Consider the family of equivalence classes w.r.t.  $\sim$  over  $S$ :

$$\mathcal{P}_\sim = \{ [a]_\sim \mid a \in S \}$$

Let  $[a]_\sim \in \mathcal{P}_\sim$ . Since  $\sim$  is an equivalence relation, by reflexivity we have  $a \sim a$ , so  $[a]_\sim$  is nonempty. Now, suppose  $a$  and  $b$  are arbitrary elements in  $S$  such that  $a \not\sim b$ . For contradiction, suppose that there is an  $x \in [a]_\sim \cap [b]_\sim$ . This means that  $x \sim a$  and  $x \sim b$ . By transitivity, we get that  $a \sim b$ ; this is a contradiction. Hence the  $[a]_\sim$  are disjoint. Finally, let  $x \in S$ . Then  $x \in [x]_\sim \in \mathcal{P}_\sim$ . This means that

$$\bigcup_{[a]_\sim \in \mathcal{P}_\sim} [a]_\sim = S,$$

that is, the union of the elements of  $\mathcal{P}_\sim$  is  $S$ . ■

1.3. ▷ Given a partition  $\mathcal{P}$  on a set  $S$ , show how to define a relation  $\sim$  such that  $\mathcal{P} = \mathcal{P}_\sim$ . [§1.5]

*Solution.* Define, for  $a, b \in S$ ,  $a \sim b$  if and only if there exists an  $X \in \mathcal{P}$  such that  $a \in X$  and  $b \in X$ . We will show that  $\mathcal{P} = \mathcal{P}_\sim$ .

1. ( $\mathcal{P} \subseteq \mathcal{P}_\sim$ ). Let  $X \in \mathcal{P}$ ; we want to show that  $X \in \mathcal{P}_\sim$ . We know that  $X$  is nonempty, so choose  $a \in X$  and consider  $[a]_\sim \in \mathcal{P}_\sim$ . We need to show that  $X = [a]_\sim$ . Suppose  $a' \in X$  (it may be that  $a' = a$ .) Since  $a, a' \in X$ ,  $a \sim a'$ , so  $a' \in [a]_\sim$ . Now, suppose  $a' \in [a]_\sim$ . We have  $a' \sim a$ , so  $a' \in X$ . Hence  $X = [a]_\sim \in \mathcal{P}_\sim$ , so  $\mathcal{P} \subseteq \mathcal{P}_\sim$ .

2. ( $\mathcal{P}_\sim \subseteq \mathcal{P}$ ). Let  $[a]_\sim \in \mathcal{P}_\sim$ . From exercise I.1.1 we know that  $[a]_\sim$  is non-empty. Suppose  $a' \in [a]_\sim$ . By definition, since  $a' \sim a$ , there exists a set  $X$  such that

$a, a' \in X$ . Hence  $[a]_{\sim} \subseteq X$ . Also, if  $a, a' \in X$  (not necessarily distinct) then  $a \sim a'$ . Therefore,  $\mathcal{P}_{\sim} \subseteq \mathcal{P}$ , and with 1. we get that the sets  $\mathcal{P}$  and  $\mathcal{P}_{\sim}$  are equal. ■

**1.4.** How many different equivalence relations can be defined on the set  $\{1, 2, 3\}$ ?

*Solution.* From the definition of an equivalence relation and the solution to problem I.1.3, we can see that an equivalence relation on  $S$  is equivalent to a partition of  $S$ . Thus the number of equivalence relations on  $S$  is equal to the number of partitions of  $S$ . Since  $\{1, 2, 3\}$  is small we can determine this by hand:

$$\begin{aligned}\mathcal{P}_0 &= \{ \{1, 2, 3\} \} \\ \mathcal{P}_1 &= \{ \{1\}, \{2\}, \{3\} \} \\ \mathcal{P}_2 &= \{ \{1, 2\}, \{3\} \} \\ \mathcal{P}_3 &= \{ \{1\}, \{2, 3\} \} \\ \mathcal{P}_4 &= \{ \{1, 3\}, \{2\} \}\end{aligned}$$

Thus there can be only 5 equivalence relations defined on  $\{1, 2, 3\}$ . ■

**1.5.** Give an example of a relation that is reflexive and symmetric but not transitive. What happens if you attempt to use this relation to define a partition on the set? (Hint: Thinking about the second question will help you answer the first one.)

*Solution.* For  $a, b \in \mathbf{Z}$ , define  $a \diamond b$  to be true if and only if  $|a - b| \leq 1$ . It is reflexive, since  $a \diamond a = |a - a| = 0 \leq 1$  for any  $a \in \mathbf{Z}$ , and it is symmetric since  $a \diamond b = |a - b| = |b - a| = b \diamond a$  for any  $a, b \in \mathbf{Z}$ . However, it is not transitive. Take for example  $a = 0, b = 1, c = 2$ . Then we have  $|a - b| = 1 \leq 1$ , and  $|b - c| = 1 \leq 1$ , but  $|a - c| = 2 > 1$ ; so  $a \diamond b$  and  $b \diamond c$ , but not  $a \diamond c$ .

When we try to build a partition of  $\mathbf{Z}$  using  $\diamond$ , we get "equivalence classes" that are not disjoint. For example,  $[2]_{\diamond} = \{1, 2, 3\}$ , but  $[3]_{\diamond} = \{2, 3, 4\}$ . Hence  $\mathcal{P}_{\diamond}$  is not a partition of  $\mathbf{Z}$ . ■

**1.6.** Define a relation  $\sim$  on the set  $\mathbf{R}$  of real numbers by setting  $a \sim b \iff b - a \in \mathbf{Z}$ . Prove that this is an equivalence relation, and find a 'compelling' description for  $\mathbf{R}/\sim$ . Do the same for the relation  $\approx$  on the plane  $\mathbf{R} \times \mathbf{R}$  by declaring  $(a_1, b_1) \approx (a_2, b_2) \iff b_1 - a_1 \in \mathbf{Z}$  and  $b_2 - a_2 \in \mathbf{Z}$ . [§II.8.1, II.8.10]

*Solution.* Suppose  $a, b, c \in \mathbf{R}$ . We have that  $a - a = 0 \in \mathbf{Z}$ , so  $\sim$  is reflexive. If  $a \sim b$ , then  $b - a = k$  for some  $k \in \mathbf{Z}$ , so  $a - b = -k \in \mathbf{Z}$ , hence  $b \sim a$ . So  $\sim$  is symmetric. Now, suppose that  $a \sim b$  and  $b \sim c$ , in particular that  $b - a = k \in \mathbf{Z}$  and  $c - b = l \in \mathbf{Z}$ . Then  $c - a = (c - b) + (b - a) = l + k \in \mathbf{Z}$ , so  $a \sim c$ . So  $\sim$  is transitive.

An equivalence class  $[a]_{\sim} \in \mathbf{R}/\sim$  is the set of integers  $\mathbf{Z}$  transposed by some real number  $\epsilon \in [0, 1)$ . That is, for every set  $X \in \mathbf{R}/\sim$ , there is a real number  $\epsilon \in [0, 1)$  such that every  $x \in X$  is of the form  $k + \epsilon$  for some integer  $k$ .

Now we will show that  $\approx$  is an equivalence relation over  $\mathbf{R} \times \mathbf{R}$ . Supposing  $a_1, a_2 \in \mathbf{R} \times \mathbf{R}$ , we have  $a_1 - a_1 = a_2 - a_2 = 0 \in \mathbf{Z}$ , so  $(a_1, a_2) \approx (a_1, a_2)$ . If we also suppose that  $b_1, b_2, c_1, c_2 \in \mathbf{R} \times \mathbf{R}$ , then symmetry and transitivity can be shown as well:  $(a_1, a_2) \approx (b_1, b_2) \implies b_1 - a_1 = k$  for some integer  $k$  and  $b_2 - a_2 = l$  for some integer  $l$ , hence  $a_1 - b_1 = -k \in \mathbf{Z}$  and  $a_2 - b_2 = -l \in \mathbf{Z}$ , so  $(b_1, b_2) \approx (a_1, a_2)$ ; also if  $(a_1, a_2) \approx (b_1, b_2)$  and  $(b_1, b_2) \approx (c_1, c_2)$ , then  $(b_1, b_2) - (a_1, a_2) = (k_1, k_2) \in \mathbf{Z} \times \mathbf{Z}$  as well as  $(c_1, c_2) - (b_1, b_2) = (l_1, l_2) \in \mathbf{Z} \times \mathbf{Z}$ , so  $(c_1, c_2) - (a_1, a_2) = (c_1, c_2) - (b_1, b_2) + (b_1, b_2) - (a_1, a_2) = (k_1 + l_1, k_2 + l_2) \in \mathbf{Z} \times \mathbf{Z}$ . Thus  $\approx$  is an equivalence relation.

The interpretation of  $\approx$  is similar to  $\sim$ . An equivalence class  $X \in \mathbf{R} \times \mathbf{R}/\approx$  is just the 2-dimensional integer lattice  $\mathbf{Z} \times \mathbf{Z}$  transposed by some pair of values  $(\epsilon_1, \epsilon_2) \in [0, 1) \times [0, 1)$ . ■

## 2 Functions between sets

**2.1.** How many different bijections are there between a set with  $n$  elements and itself?

*Solution.* A function  $f : S \rightarrow S$  is a graph  $\Gamma_f \subseteq S \times S$ . Since  $f$  is bijective, then for all  $y \in S$  there exists a unique  $x \in S$  such that  $(x, y) \in \Gamma_f$ . We can see that  $|\Gamma_f| = n$ . Since each  $x$  must be unique, all the elements  $x \in S$  must be present in the first component of exactly one pair in  $\Gamma_f$ . Furthermore, if we order the elements  $(x, y)$  in  $\Gamma_f$  by the first component, we can see that  $\Gamma_f$  is just a permutation on the  $n$  elements in  $S$ . For example, for  $S = \{1, 2, 3\}$  one such  $\Gamma_f$  is:

$$\{(1, 3), (2, 2), (3, 1)\}$$

Since  $|S| = n$ , the number of permutations of  $S$  is  $n!$ . Hence there can be  $n!$  different bijections between  $S$  and itself. ■

**2.2.** ▷ Prove statement (2) in Proposition 2.1. You may assume that given a family of disjoint subsets of a set, there is a way to choose one element in each member of the family. [§2.5, V3.3]

**Proposition 2.1.** Assume  $A \neq \emptyset$ , and let  $f : A \rightarrow B$  be a function. Then

- (1)  $f$  has a left-inverse if and only if  $f$  is injective; and
- (2)  $f$  has a right-inverse if and only if  $f$  is surjective.

*Solution.* Let  $A \neq \emptyset$  and suppose  $f : A \rightarrow B$  is a function.

(  $\implies$  ) Suppose there exists a function  $g$  that is a right-inverse of  $f$ . Then  $f \circ g = \text{id}_A$ . Let  $b \in B$ . We have that  $f(g(b)) = b$ , so there exists an  $a = g(b)$  such that  $f(a) = b$ . Hence  $f$  is surjective.

(  $\impliedby$  ) Suppose that  $f$  is surjective. We want to construct a function  $g : B \rightarrow A$  such that  $f(g(a)) = a$  for all  $a \in A$ . Since  $f$  is surjective, for all  $b \in B$  there is an  $a \in A$  such that  $f(a) = b$ . For each  $b \in B$  construct a set  $\Lambda_b$  of such pairs:

$$\Lambda_b = \{ (a, b) \mid a \in A, f(a) = b \}$$

Note that  $\Lambda_b$  is non-empty for all  $b \in B$ . So that we can choose one pair  $(a, b)$  ( $a$  not necessarily unique) from each set in  $\Lambda = \{ \Lambda_b \mid b \in B \}$  to define  $g : B \rightarrow A$ :

$$g(b) = a, \text{ where } a \text{ is in some } (a, b) \in \Lambda_b$$

Now,  $g$  is a right-inverse of  $f$ . To show this, let  $b \in B$ . Since  $f$  is surjective,  $g$  has been defined such that when  $a = g(b)$ ,  $f(a) = b$ , so we get that  $f(g(b)) = (f \circ g)(b) = b$ , thus  $g$  is a right-inverse of  $f$ . ■

**2.3.** Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.

*Solution.* (1) Suppose  $f : A \rightarrow B$  is a bijection, and that  $f^{-1} : B \rightarrow A$  is its inverse. We have that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ . Hence  $f$  is the left- and right-inverse of  $f^{-1}$ , so  $f^{-1}$  must be a bijection. ■

(2) Let  $f : B \rightarrow C$  and  $g : A \rightarrow B$  be bijections, and consider  $f \circ g$ . To show that  $f \circ g$  is injective, let  $a, a' \in A$  such that  $(f \circ g)(a) = (f \circ g)(a')$ . Since  $f$  is a bijection,  $f(g(a)) = f(g(a')) \implies g(a) = g(a')$ . Also, since  $g$  is a bijection,  $g(a) = g(a') \implies a = a'$ . Hence  $f \circ g$  is injective. Now, let  $c \in C$ . Since  $f$  is surjective, there is a  $b \in B$  such that  $f(b) = c$ . Also, since  $g$  is surjective, there is an  $a \in A$  such that  $g(a) = b$ ; this means that there is an  $a \in A$  such that  $(f \circ g)(a) = c$ . So  $f \circ g$  is bijective.

**2.4.** ▷ Prove that 'isomorphism' is an equivalence relation (on any set of sets.) [§4.1]

*Solution.* Let  $S$  be a set. Then  $\text{id}_S$  is a bijection from  $S$  to itself, so  $S \cong S$ . Let  $T$  be another set with  $S \cong T$ , i.e. that there exists a bijection  $f : S \rightarrow T$ . Since  $f$  is a bijection, it has an inverse  $f^{-1} : T \rightarrow S$ , so  $T \cong S$ . Finally, let  $U$  also be a set, and assume that there exists bijections  $f : S \rightarrow T$  and  $g : T \rightarrow U$ , i.e. that  $S \cong T$  and  $T \cong U$ . From exercise I.2.3 we know that the composition of bijections is itself a bijection. This means that  $g \circ f : S \rightarrow U$  is a bijection, so  $S \cong U$ . Hence  $\cong$  is an equivalence relation. ■

**2.5.** ▷ Formulate a notion of *epimorphism*, in the style of the notion of *monomorphism* seen in §2.6, and prove a result analogous to Proposition 2.3, for epimorphisms and surjections.

*Solution.* A function  $f : A \rightarrow B$  is an *epimorphism* if and only if for all sets  $Z$  and all functions  $b' : Z \rightarrow B$ , there is a function  $a' : Z \rightarrow A$  such that  $f \circ a' = b'$ . Now we will show that  $f$  is a surjection if and only if it is an epimorphism.

( $\implies$ ) Suppose that  $f : A \rightarrow B$  is surjective. Let  $Z$  be a set and  $b' : Z \rightarrow B$  a function. We need to construct a function  $a' : Z \rightarrow A$  such that  $f \circ a' = b'$ . Fix  $z \in Z$ . Suppose  $b = b'(z) \in B$ . Since  $b \in B$  and  $f$  is surjective, there exists an  $a \in A$  such  $f(a) = b$ . Now, define  $a'(z) = a$  this way for each  $z \in Z$ . Then  $f \circ a'(z) = b'(z)$  for all  $z \in Z$ , so  $f \circ a' = b'$ . Hence  $f$  is an epimorphism.

( $\impliedby$ ) Suppose that  $f$  is an epimorphism. Let  $b' : B \rightarrow B$  be a bijection. Since  $f$  is an epimorphism, there is a function  $a' : B \rightarrow A$  such that  $f \circ a' = b'$ . Let  $b \in B$ . Since  $b'$  is a bijection, there is a unique element  $y \in B$  such that  $b'(y) = b$ . Furthermore, we have that  $(f \circ a')(y) = b$ . In other words,  $a = a'(y)$  is an element in  $A$  such that  $f(a) = b$ . Hence  $f$  is surjective, as required. ■

**2.6.** With notation as in Example 2.4, explain how any function  $f : A \rightarrow B$  determines a section of  $\pi_A$ .

*Solution.* Let  $f : A \rightarrow B$  and let  $\pi_A : A \times B \rightarrow A$  be such that  $\pi_A(a, b) = a$  for all  $(a, b) \in A \times B$ . Construct  $g : A \rightarrow A \times B$  defined as  $g(a) = (a, f(a))$  for all  $a \in A$ . The function  $g$  can be thought of as ‘determined by’  $f$ . Now, since  $(\pi_A \circ g)(a) = \pi_A(g(a)) = \pi_A(a, f(a)) = a$  for all  $a \in A$ ,  $g$  is a right inverse of  $\pi_A$ , i.e.  $g$  is a section of  $\pi_A$  as required. ■

**2.7.** Let  $f : A \rightarrow B$  be any function. Prove that the graph  $\Gamma_f$  of  $f$  is isomorphic to  $A$ .

*Solution.* Recall that sets  $\Gamma_A$  and  $A$  are *isomorphic*, written  $\Gamma_A \cong A$ , if and only if there exists a bijection  $g : \Gamma_A \rightarrow A$ . Let’s construct such a function  $g$ , defined to be  $g(a, b) = a$ . Keep in mind that here  $(a, b) \in \Gamma_f \subseteq A \times B$ .

Let  $(a', b'), (a'', b'') \in \Gamma_f$  such that  $f(a', b') = f(a'', b'')$ . For contradiction, suppose that  $(a', b') \neq (a'', b'')$ . Since  $f(a', b') = a' = a'' = f(a'', b'')$ , it must be that  $b' \neq b''$ . However, this would mean that both  $(a', b')$  and  $(a', b'')$  are in  $\Gamma_f$ ; this would mean that  $f(a') = b' \neq b'' = f(a')$ , which is impossible since  $f$  is a function. Hence  $g$  is injective.

Let  $a' \in A$ . Since  $f$  is a well-defined function with  $A$  as its domain, there must exist a pair  $(a', b') \in \Gamma_f$  for some  $b' \in B$ , in particular that  $g(a', b') = a'$ ; thus  $g$  is surjective, so it is a bijection. ■

**2.8.** Describe as explicitly as you can all terms in the canonical decomposition (cf. §2.8) of the function  $\mathbf{R} \rightarrow \mathbf{C}$  defined by  $r \mapsto e^{2\pi i r}$ . (This exercise matches one

previously. Which one?)

*Solution.* Let  $f : \mathbf{R} \rightarrow \mathbf{C}$  be as above. The first piece in the canonical decomposition is the equivalence relation  $\sim$  defined as  $x \sim x' \iff f(x) = f(x')$ , i.e.  $[x]_{\sim}$  is the set of all elements in  $\mathbf{R}$  that get mapped to the same element in  $\mathbf{C}$  by  $f$  as  $x$ .

The second piece is the set  $\mathcal{P}_{\sim}$ . This set is the set of all equivalence classes of  $\mathbf{R}$  over equality up to  $f$ . Note that, since  $f(x) = e^{2\pi ix} = \cos(2\pi x) + i \sin(2\pi x)$ ,  $f$  is periodic with period 1. That is,  $f(x) = e^{2\pi ix} = e^{2\pi ix + 2\pi i} = e^{2\pi i(x+1)} = f(x+1)$ . In other words, we can write  $\mathcal{P}_{\sim}$  as,

$$\mathcal{P}_{\sim} = \{ \{ r + k \mid k \in \mathbf{Z} \} \mid r \in [0, 1) \subseteq \mathbf{R} \},$$

and it is here when we notice uncanny similarities to exercise 1.6 where  $x \sim y$ , for  $x, y \in \mathbf{R}$ , if and only if  $x - y \in \mathbf{Z}$ , in which we could have written  $\mathcal{P}_{\sim}$  in the same way.

Now we will explain the mysterious  $\tilde{f} : \mathcal{P}_{\sim} \rightarrow \text{im} f$ . This function is taking each *equivalence class*  $[x]_{\sim}$  over the reals w.r.t.  $\sim$  and mapping it to the element in  $\mathbf{C}$  that  $f$  maps each element  $x' \in [x]_{\sim}$  to; indeed, since  $x \sim x'$  is true for  $x, x' \in \mathbf{R}$  if and only if  $f(x) = f(x')$ , we can see that for any  $x \in \mathbf{R}$ , for all  $x' \in [x]_{\sim}$ , there exists a  $c \in \mathbf{C}$  such that  $f(x') = c$ . To illustrate with the equivalence class over  $\mathbf{R}$  w.r.t.  $\sim$  corresponding to the element  $0 \in \mathbf{R}$ , we have  $[0]_{\sim} = \{ \dots, -2, -1, 0, 1, 2, \dots \}$ . We can see that  $e^{-4\pi i} = e^{-2\pi i} = e^{0\pi i} = 1 = e^{2\pi i} = e^{4\pi i}$ , etc; so the function would map  $[0]_{\sim} \mapsto 1 \in \mathbf{C}$ , and so on. Furthermore, we can see that  $\tilde{f}$  is surjective, since for  $y$  to be in  $\text{im} f$  is to say that there is an  $x \in \mathbf{R}$  such that  $f(x) = y$ ; so there must be an equivalence class  $[x]_{\sim}$  which is mapped to  $y$  by  $\tilde{f}$ .

Finally, the simple map from  $\text{im} f \rightarrow \mathbf{C}$  that simply takes  $c \mapsto c$ . This can be thought of as a potential “expansion” of the domain of  $\tilde{f}$ . It is obviously injective, since (trivially)  $c \neq c' \implies c \neq c'$ . However, it may not be surjective: for example,  $2 \in \mathbf{C}$  is not in  $\text{im} f$  as it is defined above.

**2.9.** ▷ Show that if  $A' \cong A''$  and  $B' \cong B''$ , and further  $A' \cap B' = \emptyset$  and  $A'' \cap B'' = \emptyset$ , then  $A' \cup B' \cong A'' \cup B''$ . Conclude that the operation  $A \sqcup B$  is well-defined up to *isomorphism* (cf. §2.9) [§2.9, 5.7]

*Solution.* Let  $A', A'', B', B''$  be sets as described above. Since  $A' \cong A''$  and  $B' \cong B''$ , we know there exists respective bijections  $f : A' \rightarrow A''$  and  $g : B' \rightarrow B''$ . Now, we wish to show that  $A' \cup B' \cong A'' \cup B''$ . Define a function  $h : A' \cup B' \rightarrow A'' \cup B''$  such that  $h(x) = f(x)$  if  $x \in A'$  and  $g(x)$  if  $x \in B'$ .

We will now show that  $h$  is a bijection. Let  $y \in A'' \cup B''$ . Then, since  $A'' \cap B'' = \emptyset$ , either  $y \in A''$  or  $y \in B''$ . Without loss of generality suppose that  $y \in A''$ . Then, since  $f : A' \rightarrow A''$  is a bijection, it is *surjective*, so there exists an  $x \in A' \subseteq A' \cup B'$  such that

$h(x) = f(x) = y$ . So  $h$  is surjective. Now, suppose that  $x \neq x'$ , for  $x, x' \in A' \cup B'$ . If  $x, x' \in A'$ , then since  $f$  is injective and  $h(x) = f(x)$  for all  $x \in A'$ , then  $h(x) \neq h(x')$ . Similarly for if  $x, x' \in B'$ . Now, without loss of generality if  $x \in A'$  and  $x' \in B'$ , then  $h(x) = f(x) \neq g(x') = h(x')$  since  $A'' \cap B'' = \emptyset$ . Hence  $h$  is a bijection, so  $A' \cup B' \cong A'' \cup B''$ .

Since these constructions of  $A', A'', B', B''$  correspond to creating “copies” of sets  $A$  and  $B$  for use in the disjoint union operation, we have that disjoint union is a well-defined function *up to isomorphism*. In particular, since  $\cong$  is an equivalence relation, we can consider  $\sqcup$  to be well-defined from  $\mathcal{P}_{\cong}$  to  $A' \cup B'$ . ■

**2.10.** ▷ Show that if  $A$  and  $B$  are finite sets, then  $|B^A| = |B|^{|A|}$ . [§2.1, 2.11, I.4.1]

*Solution.* Let  $A$  and  $B$  be sets with  $|A| = n$  and  $|B| = m$ , with  $n, m$  being non-negative integers. Recall that  $B^A$  denotes the set of functions  $f : A \rightarrow B$ . Now, if  $A = B = \emptyset$  or  $A = \emptyset$  and  $|B| = 1$ , we get one function, the empty function  $\Gamma_f = \emptyset$ , and  $0^0 = 1^0 = 1$ . If  $|A| = |B| = 1$ , then we get the singleton function  $\Gamma_f = \{(a, b)\}$ , and  $1^1 = 1$ . If  $A \neq \emptyset$  and  $B = \emptyset$ , then no well-defined function can exist from  $A$  to  $B$  since there will be no value for the elements in  $A$  to take; this explains  $|B^A| = |B|^{|A|} = 0^{|A|} = 0$ .

Suppose that  $B \neq \emptyset$  and  $B$  is finite. We will show inductively that  $|B^A| = |B|^{|A|}$ . First, suppose that  $|A| = 1$ . Then there are exactly  $|B|$  functions from  $A$  to  $B$ : if  $B = \{b_1, b_2, \dots, b_m\}$ , then the functions are  $\{(a, b_1)\}, \{(a, b_2)\}$ , etc. Hence  $|B^A| = |B|^{|A|} = |B|$ . Now, fix  $k \geq 2$ , and assume that  $|B^A| = |B|^{|A|}$  for all sets  $A$  such that  $|A| = k - 1$ . Suppose that  $|A| = k$ . Let  $a \in A$ . (We can do this since  $|A| = k \geq 2$ .) Then, by the inductive hypothesis, since  $|A \setminus \{a\}| = k - 1$ ,  $|B^{(A \setminus \{a\})}| = |B|^{|A|-1}$ . Let  $F$  be the set of functions from  $A \setminus \{a\}$  to  $B$ . Then, for each of those functions  $f \in F$ , there is  $|B|$  “choices” of where to assign  $a$ : one choice for each element in  $B$ . Hence,  $|B^A| = |B| |B|^{|A|-1} = |B|^{|A|}$  as required. ■

**2.11.** ▷ In view of Exercise 2.10, it is not unreasonable to use  $2^A$  to denote the set of functions from an arbitrary set  $A$  to a set with 2 elements (say  $\{0, 1\}$ ). Prove that there is a bijection between  $2^A$  and the *power set* of  $A$  (cf. §1.2). [§1.2, III.2.3]

*Solution.* Let  $S = \{0, 1\}$ , and consider  $f : \mathcal{P}(A) \rightarrow 2^A$ , defined as

$$f(X) = \{(a, 1) \text{ if } a \in X, \text{ and } (a, 0) \text{ otherwise}\}$$

We will show that  $f$  is bijective. Let  $g \in 2^A$ . Then  $f$  is a function from  $A$  to  $S$ . Let  $A_1 = \{a \in A \mid g(a) = 1\}$ . Then  $A_1$  is a set such that  $A_1 \in \mathcal{P}(A)$ , and  $f(A_1) = g$ . Hence  $f$  is surjective. Now, suppose that  $X, Y \subseteq A$  and  $f(X) = f(Y)$ . Then, for all  $a \in A$ ,  $a \in X \iff f(X)(a) = 1 \iff f(Y)(a) = 1 \iff a \in Y$ . Hence  $f$  is injective, so  $2^A \cong \mathcal{P}(A)$ . ■