Maxwell and others believed the failure of his equations to obey Galilean relativity was due to a medium, the "luminiferous aether," through which waves propagated. Only inertial frames at rest wrt the ether would obey Maxwell's eq.

Attempts to measure the velocity of Earth relative to the ether failed.

This led Einstein in 1905 to propose that Galilean relativity was only an approximate symmetry of nature, to be replaced by another 10 parameter group of coordinate transf., the Poincaré group, at velocities $\simeq c$, and of which the Galilean xfms are a first appx in a $|v|/c$ expansion.

The principle of special relativity: the laws of nature are invariant under Lorentz xforms.

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, x^{\beta} + a^{\alpha}$$

$\alpha, \beta = 0,1,2,3$

$x^0 = t$

$x^{1,2,3} = x^i$

here $c=1$ so that all coords have dim of length.

with a^{α}, $\Lambda^{\alpha}{}_{\beta}$ constants and

$$\Lambda^{\alpha}{}_{\beta}\, \Lambda^{\gamma}{}_{\delta}\, \eta_{\alpha\gamma} = \eta_{\beta\delta}$$

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

write in matrix mult. form: $(\Lambda^T)_{\beta}{}^{\alpha}\, \eta_{\alpha\gamma}\, \Lambda^{\gamma}{}_{\delta}$

These xforms leave invariant the proper time

$$d\tau^2 \equiv -\eta_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}$$

$$dx'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, dx^{\beta}$$

So
$$\begin{aligned} d\tau'^2 &= -\eta_{\alpha\beta}\, dx'^{\alpha}\, dx'^{\beta} \\ &= -\eta_{\alpha\beta}\, \Lambda^{\alpha}{}_{\gamma}\, \Lambda^{\beta}{}_{\delta}\, dx^{\gamma}\, dx^{\delta} \\ &= -\eta_{\gamma\delta}\, dx^{\gamma}\, dx^{\delta} \\ &= d\tau^2 \end{aligned}$$

Maxwell's eqs are Lorentz invariant, although we need a bit of machinery to show it completely. For the moment let's just look at the wave equation, $\Box A^\mu = 0$.

$$\Box = \partial_t^2 - \partial_{x_i}^2 = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}$$

under a Lorentz transformation,

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x^{\nu'}}{\partial x^\mu} \frac{\partial}{\partial x^{\nu'}} \quad \text{(chain rule)}$$

$$\text{and} \quad x^{\nu'} = \Lambda^\nu{}_\mu x^\mu$$

$$\text{so} \quad \frac{\partial x^{\nu'}}{\partial x^\mu} = \Lambda^\nu{}_\mu$$

$$\Rightarrow \Box = \eta^{\mu\nu} \frac{\partial}{\partial x^{\rho'}} \frac{\partial}{\partial x^{\sigma'}} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$$

$$= \left(\Lambda \eta^{-1} \Lambda^T\right)^{\rho\sigma} \frac{\partial}{\partial x^{\rho'}} \frac{\partial}{\partial x^{\sigma'}}$$

$$= \eta^{\rho\sigma} \frac{\partial}{\partial x^{\rho'}} \frac{\partial}{\partial x^{\sigma'}} = \Box'$$

$\Rightarrow$ the wave eq takes the same form.

A light wave crest 

moves with velocity $\left|\frac{dx}{dt}\right| = c = 1$

so $d\tau^2 = -\eta_{\alpha\beta}\, dx^\alpha dx^\beta = dt^2 - d\vec{x}\cdot d\vec{x}$

$= 0.$

In another Lorentz frame $d\tau' = d\tau = 0$ so

$\left|\frac{d\vec{x}'}{dt'}\right| = 1$ as well. Speed of light is

the same in all frames.

The a^α are just translations. The subset of xforms w/ $a^\alpha = 0$ are the 6-param "homogeneous" Lorentz group.

They have a subgroup: $\Lambda^i{}_j = R^i{}_j$

$\Lambda^i{}_0 = \Lambda^0{}_i = 0$

$\Lambda^0{}_0 = 1$

<u>Ordinary Rotations</u> $R^T R = \mathbb{I}$.

The 3 rot^n & 4 transl are the same as in the Galilean group. The difference is the boosts.

Suppose a particle is at rest in x^α and has velocity $\bar{V}$ in $x^{\alpha'}$.

$dx'^\alpha = \Lambda^\alpha{}_\beta \, dx^\beta$ in some time dt. Since

$dx^i = 0$, $\quad dx'^i = \Lambda^i{}_0 \, dt$,

$$dt' = \Lambda^0{}_0 \, dt$$

$$\frac{dx'^i}{dt'} = v^i = \Lambda^i{}_0 / \Lambda^0{}_0$$

Also, $\Lambda^T \eta \Lambda = \eta$ means

$$\Lambda^i{}_0 \Lambda^i{}_0 \cancel{\eta_{ii}}^{\,1} + \Lambda^0{}_0 \Lambda^0{}_0 \cancel{\eta_{00}}^{\,-1} = \cancel{\eta_{00}}^{\,-1}$$

or $(\Lambda^i{}_0)^2 - (\Lambda^0{}_0)^2 = -1$

$$(\Lambda^0{}_0)^2 (v^{i\,2} - 1) = -1$$

$$\Lambda^0{}_0 = \sqrt{\frac{1}{1-v^2}}$$

and $\Lambda^i{}_0 = \Lambda^0{}_0 v_i = \gamma v_i$ $\quad \gamma \equiv \Lambda^0{}_0 = \frac{1}{\sqrt{1-v^2}}$

The other $\Lambda^i{}_j$ are not uniquely determined because if $\Lambda^\alpha{}_\beta$ boosts a particle to velocity v, then so does $\Lambda^\alpha{}_\gamma R^\gamma{}_\delta$ where $R^\gamma{}_\delta$ is a rotation. Convenient choice:

$$\Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{|v|^2}, \quad \Lambda^0{}_j = \gamma v_j$$

<u>Time dilation</u>: A clock is anything that measures the passage of time. Clocks are physical, they run on mechanical or electrodynamic etc eqn.

If a clock is at rest it will click at some interval $(dt, d\vec{x} = 0)$. The proper time between clicks is $dt^2 = d\tau^2$. Another obs who sees the clock moving @ $\vec{v}$ observes clicks at $(dt', d\vec{x}' = \vec{v}\,dt')$, $d\tau'^2 = dt'^2 - d\vec{x}'^2$

Since $d\tau^2 = d\tau'^2$, $\quad dt^2 = dt'^2 - dt'^2 v^2$

$$\boxed{dt' = dt/\sqrt{1-v^2} = \gamma dt}$$

If we decide that special relativity supercedes Galilean relativity, how should we update Newton's mechanics?

We seek a version of $F = ma$ that looks the same in any inertial frame.

- We saw that the proper time τ is Lorentz invariant and reduces to the coordinate time t in a frame at rest wrt the body of interest.
- We defined that the coordinates $(t, x^i) \equiv x^\mu$ transform like a "4-vector" $\Lambda^\mu{}_\nu x^\nu = x'^\mu$.

If we could identify a "4-force" f^μ then

$$f^\mu = m \frac{\partial^2 x^\mu}{\partial \tau^2}$$

would be invariant.

Such an f^μ must reduce to $(F^0 = 0, F^i)$ if the particle is instantaneously at rest wrt observer, where F^i is the ordinary nonrelativistic force. $F^0 = \frac{\partial^2 t}{\partial t^2} = \frac{\partial 1}{\partial t} = 0.$

then $f^\mu = \Lambda^\mu{}_\nu(V)\, F^\nu$ is the 4-force in another frame where the particle has velocity V.

Knowing f^μ, Newton's laws become 4 ODE's for $x^\alpha(\tau)$. There is a constraint: τ must really be the proper time, or

$$\eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial\tau}\frac{\partial x^\beta}{\partial\tau} = -1.$$

If this holds at the initial time, (where we provide $\frac{\partial x^\alpha}{\partial\tau}$!) it holds at later times: $\partial_\tau\left(\eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial\tau}\frac{\partial x^\beta}{\partial\tau}\right) = 2\eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial\tau}\frac{\partial^2 x^\beta}{\partial\tau^2}$

This vanishes because it is Lorentz invariant and in a frame where the particle is at rest it reduces to $2\eta_{00}\frac{\partial t}{\partial\tau}\underbrace{\cancel{\frac{\partial^2 t}{\partial\tau^2}}}_{0 \text{ if } t=\tau} = 0$

$u^\mu \equiv \frac{\partial x^\mu}{\partial \tau}$ 4-velocity $u^\mu u_\nu = -1$

$a^\mu \equiv \frac{\partial^2 x^\mu}{\partial \tau^2}$ 4-accel

$p^\mu = m u^\mu$ 4-momentum

$\Rightarrow f^\mu = \frac{\partial p^\mu}{\partial \tau}$

Since $d\tau = \sqrt{dt^2 - d\vec{x}^2} = dt\sqrt{1-v^2} = dt/\gamma$

particle coordinate velocity

The spatial components of p^μ are $m \frac{dx^i}{dt}\gamma = \gamma m \vec{v}$

"$= 3$-momentum"

$= m\vec{v} + \mathcal{O}(v^3)$

The time component is $m\frac{dt}{d\tau} = m\gamma = E$ "energy"

the Lorentz invariant $p^\mu p^\nu \eta_{\mu\nu} = -(m\gamma)^2 + (\gamma m)^2 \vec{v}\cdot\vec{v}$

$= -m^2$

"invariant mass"

p^0 is the energy: $m\gamma = \underset{\text{rest mass energy}}{\underline{m}} + \underset{\text{nonrel KE}}{\underline{\tfrac{1}{2}mv^2}} + \mathcal{O}(v^4)$

These reduce to ordinary $\vec{p}$ & E in the low velocity limits. If they are conserved in one frame, they are conserved in all Lorentz frames.