Magnetostatics and the Biot-Savart law

Steady currents are the source of magnetostatic fields. Superposition holds for magnetic field, so contributions from small sections of steady current (Idl') may be added up to get the net magnetic field at any point in space. The dependence of $\vec{B}$ on the distance $\mathscr{r}$ to the current element, and the value of the current I, was experimentally determined (just as was the dependence of $\vec{E}$ on $\mathscr{r}$ and q). This is expressed as the Biot-Savart law:

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, dl' \tag{1}$$

The Biot-Savart law may also be written for surface currents and volume currents:

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

Doing some examples is the best way to get a feeling for the Biot-Savart law.

Magnetic field at distance s from a wire segment with steady current I

As an example of calculating the magnetic field with the Biot-Savart law, consider a segment of uniform current along the y-axis, as shown in Fig. 1. A point P is located a distance s above the y-axis in the y-z plane. What is the magnetic field at P due to the current in the segment?

Direction:

First, consider the direction of the field contribution from different sections of the wire segment. Since both the current Idl' and the distance $\vec{\mathscr{r}}$ to P are in the plane of the page, the direction of their cross-product must be perpendicular to the plane of the page. The RHR gives the direction of $\vec{B}$ to be out of the page ($\hat{x}$ direction) for any section of the wire. Or, arguing another way, Idl' is in the $\hat{y}$ direction, while $\vec{\mathscr{r}}$ can be resolved into an $\hat{y}$ and a $\hat{z}$ component. Then $\hat{y} \times \hat{y} = 0$ while $\hat{y} \times \hat{z} = \hat{x}$, so the cross-product $Idl' \times \vec{\mathscr{r}}$ is in the $\hat{x}$ direction. Also note that the right-hand-rule for wires gives the same result; if you put your thumb in the direction of the current, your fingers curl around the wire so that they are coming out of the page at point P.

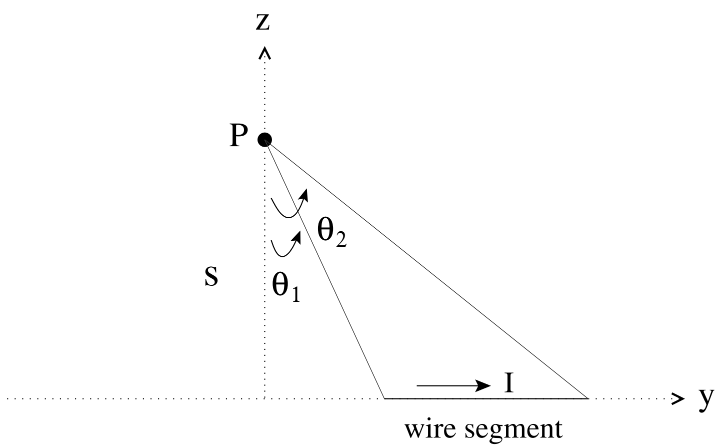

Figure 1: Current carrying wire segment. One end of the segment makes an angle θ_1 between s and $\mathscr{r}_1$, while the other end of the segment makes an angle θ_2 between s and $\mathscr{r}_2$.

Begin evaluation of the Biot-Savart integral by finding $\vec{\mathscr{r}}$, $\mathscr{r}$ and $\hat{\mathscr{r}}$. In general terms, the distance between the charge and the observer is $\vec{\mathscr{r}} = \vec{r} - \vec{r'}$. In this case, the distance from the origin to P is $\vec{r} = s\hat{z}$. The position of an element of current Idl' is $\vec{r'} = y'\hat{y}$.

$$
\begin{aligned}
\vec{\mathscr{r}} &= \vec{r} - \vec{r'} \\
&= s\hat{z} - y'\hat{y}
\end{aligned}
$$

The distance between the observer and the element of charge is then given by:

$$
\begin{aligned}
\mathscr{r} &= \sqrt{\left(\vec{r} - \vec{r'}\right)^2} \\
&= \sqrt{(s\hat{z} - y'\hat{y})^2} \\
&= \sqrt{s\hat{z} \cdot s\hat{z} - 2(s\hat{z} \cdot y'\hat{y}) + y'\hat{y} \cdot y'\hat{y}} \\
&= \sqrt{s^2 + (y')^2}
\end{aligned}
$$

And the unit vector,

$$\begin{aligned}\hat{\mathscr{r}} &= \frac{\vec{\mathscr{r}}}{\mathscr{r}} \\ &= \frac{s\hat{z} - y'\hat{y}}{\sqrt{s^2 + (y')^2}}\end{aligned}$$

The differential current element, $Id\vec{l'}$, goes along the wire segment, which lies on the y-axis. Then, $Id\vec{l'} = Idy'\hat{y}$. Take the cross-product with the unit vector, $Idy'\hat{y} \times \hat{\mathscr{r}}$, to get the numerator of the Biot-Savart integrand.

$$Idy'\hat{y} \times \hat{\mathscr{r}} = Idy'\hat{y} \times \frac{(s\hat{z} - y'\hat{y})}{\sqrt{s^2 + (y')^2}} = Isdy'\frac{\hat{y} \times \hat{z}}{\sqrt{s^2 + (y')^2}} = \frac{Isdy'\hat{x}}{\sqrt{s^2 + (y')^2}}$$

So, the Biot-Savart integral for the magnetic field at point P is given by:

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{Id\vec{l'} \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 Is}{4\pi}\int_{y_1}^{y_2} \frac{dy'}{(s^2 + (y')^2)^{\frac{3}{2}}}\,\hat{x}$$

Evaluating the integral:

$$\begin{aligned}\vec{B} &= \frac{\mu_0 Is\,\hat{x}}{4\pi}\int_{y_1}^{y_2} \frac{dy'}{(s^2 + (y')^2)^{\frac{3}{2}}} \\ \vec{B} &= \frac{\mu_0 Is}{4\pi}\left[\left.\frac{y}{s^2\sqrt{(y^2 + s^2)}}\right|_{y_1}^{y_2}\right]\hat{x} = \frac{\mu_0 I}{4\pi s}\left[\frac{y_2}{\sqrt{(y_2^2 + s^2)}} - \frac{y_1}{\sqrt{(y_1^2 + s^2)}}\right]\hat{x}\end{aligned}$$

Check the limit as $y_1 \to -\infty$ ***and*** $y_2 \to \infty$***:***

If $y_1 \to -\infty$, and $y_2 \to +\infty$ then the segment of current becomes an infinite line of current. Start with Eq. 2 for the field from a short segment of current between end points y_1 and y_2:

$$B = \frac{\mu_0 I}{4\pi s}\left[\frac{y_2}{\sqrt{(y_2^2 + s^2)}} - \frac{y_1}{\sqrt{(y_1^2 + s^2)}}\right] \tag{2}$$

Divide the top and bottom of the first fraction by $|y_2|$ and the top and bottom of the second fraction by $|y_1|$ to give the following,

$$B = \frac{\mu_0 I}{4\pi s}\left[\frac{\frac{y_2}{|y_2|}}{\sqrt{\left(\left(\frac{y_2}{y_2}\right)^2+\left(\frac{s}{y_2}\right)^2\right)}} - \frac{\frac{y_1}{|y_1|}}{\sqrt{\left(\frac{y_1}{y_1}\right)^2+\left(\frac{s}{y_1}\right)^2}}\right]$$

Now taking the appropriate limits:

$$\lim_{y_2\to+\infty}\frac{\frac{y_2}{|y_2|}}{\sqrt{\left((1)^2+\left(\frac{s}{y_2}\right)^2\right)}} = 1 \qquad \lim_{y_1\to-\infty}\frac{\frac{y_1}{|y_1|}}{\sqrt{\left((1)^2+\left(\frac{s}{y_1}\right)^2\right)}} = -1$$

So that, as expected for an infinite line of current,

$$\lim_{y_{2/1}\to\pm\infty} B = \frac{\mu_0 I}{4\pi s}\left[1-(-1)\right] = \frac{\mu_0 I}{2\pi s}$$

Magnetic field near a wire segment, Take 2:

Now let's do the exact same problem, but instead of classically finding $\vec{\mathscr{r}}$, $\mathscr{r}$ and $\hat{\mathscr{r}}$, parameterize the integrand of the Biot-Savart integral in terms of the angles in Fig. 1 and Fig. 2.

The cross-product $Id\vec{l'}\times\hat{\mathscr{r}} = Idl'\sin\left(\theta+\frac{\pi}{2}\right)|\hat{\mathscr{r}}|$, as shown in Fig. 2. The angle θ is defined as the angle at point P between s and $\mathscr{r}$. Use the following trigonometric identity,

$$\sin\left(\theta+\frac{\pi}{2}\right) = \sin(\theta)\cos\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{2}\right)\cos(\theta) = \cos(\theta)$$

to write $Idl'\sin\left(\theta+\frac{\pi}{2}\right) = Idl'\cos(\theta)$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{Id\vec{l'}\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\int\frac{Idl'\cos(\theta)}{\mathscr{r}^2}$$

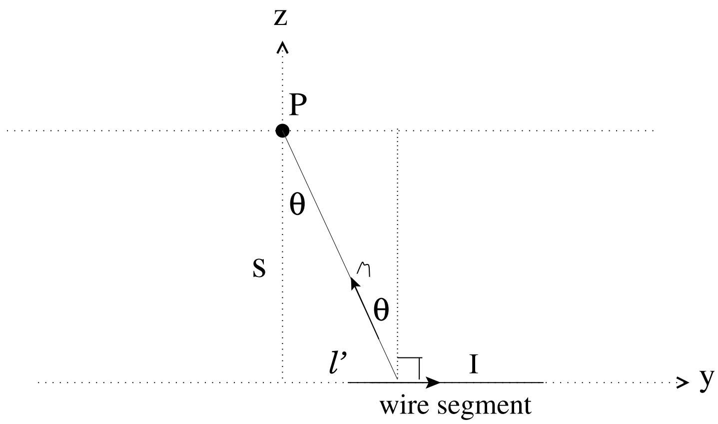

Figure 2: Depiction of the angle between a section of the current carrying wire segment, Idl', and the $\hat{\mathscr{r}}$ direction. That angle is $\theta + \frac{\pi}{2}$.

The distance $\mathscr{r}$ between P and a segment of current, Idl', may be obtained from the geometry of the figure:

$$\cos(\theta) = \frac{s}{\mathscr{r}}$$

$$\mathscr{r} = \frac{s}{\cos(\theta)}$$

So that

$$\frac{1}{\mathscr{r}^2} = \frac{\cos^2(\theta)}{s^2}$$

An expression for dl' in terms of θ is still needed. Since the distance from the origin to the current element is l', and the distance from the origin to point P is s, then $l'/s = \tan(\theta)$. Differentiate this,

$$\frac{dl'}{d\theta} = s\frac{d}{d\theta}(\tan(\theta)) = s\frac{d}{d\theta}\left(\frac{\sin(\theta)}{\cos(\theta)}\right)$$

Then use the product rule,

$$
\begin{aligned}
\frac{dl'}{d\theta} &= s\frac{d}{d\theta}\left(\frac{\sin(\theta)}{\cos(\theta)}\right) \\
&= s\left[\sin(\theta)\frac{d}{d\theta}\left(\frac{1}{\cos(\theta)}\right) + \frac{1}{\cos(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\right)\right] \\
&= s\left[\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos(\theta)}{\cos(\theta)}\right] = s\left[\frac{\sin^2(\theta)+\cos^2(\theta)}{\cos^2(\theta)}\right] \\
dl' &= \frac{s}{\cos^2(\theta)}\,d\theta
\end{aligned}
$$

Then putting it all together to calculate the field,

$$
\begin{aligned}
B &= \frac{\mu_0 I}{4\pi}\int \frac{dl'}{\mathscr{r}^2}\left(\cos(\theta)\right) \\
&= \frac{\mu_0 I}{4\pi}\int \left(\frac{\cos^2(\theta)}{s^2}\right)\left(\frac{s}{\cos^2(\theta)}\right)\cos(\theta)\,d\theta \\
&= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos(\theta)\,d\theta
\end{aligned}
$$

Finally,

$$
B = \frac{\mu_0 I}{4\pi s}\left[\sin(\theta_2) - \sin(\theta_1)\right] \tag{3}
$$

Note that:

$$
\sin(\theta_1) = \frac{y_1}{\sqrt{(y_1^2 + s^2)}}
$$

and similarly for $\sin(\theta_2)$. So, Eq. 3 is identical to Eq. 2. For the wire to become infiinite, $\theta_1 \to -\frac{\pi}{2}$ and $\theta_2 \to +\frac{\pi}{2}$. In that case, Eq. 3 becomes:

$$
B = \frac{\mu_0 I}{4\pi s}\left[1 - (-1)\right] = \frac{\mu_0 I}{2\pi s}
$$

as expected.

Magnetic field at the center of a current loop

As another example of calculating the magnetic field with the Biot-Savart law, let's find the field at the center of a current loop. Put the current loop in the x-y plane, centered on the origin.

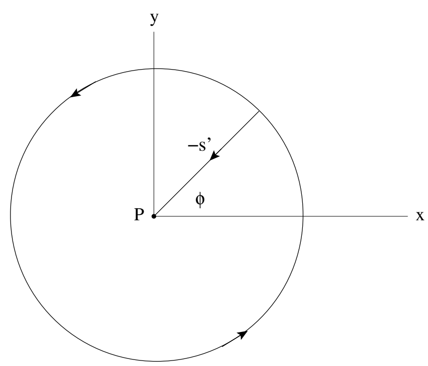

Figure 3: Current loop of radius R in the x-y plane, carrying current I. Point P is in the center of the loop, at the origin.

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l'} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \longrightarrow \qquad \text{BS to find field}$$

In this case, the field is being sought at the origin, so the location of the observation of the field is $r = 0$. All current elements are on the circular current loop, so $\vec{r'} = R\hat{s}$. Then,

$$\vec{\mathscr{r}} = \vec{r} - \vec{r'} = 0 - R\hat{s}$$

$$\mathscr{r} = R$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = -\hat{s}$$

All current segments, $I d\vec{l'}$, are on the circular loop. Taking the current to be flowing counterclockwise as shown in the figure, then the direction of $I d\vec{l'}$ is $\hat{\phi}$.

$$I d\vec{l'} = I ds\, \hat{\phi} = I R\, d\phi\, \hat{\phi}$$

Since $\hat{\phi} \times -\hat{s} = \hat{z}$, the direction of $\vec{B}$ at the center of the circle is $\hat{z}$, and the field is,

$$\vec{B} = \frac{\mu_0 I}{4\pi R} \int_0^{2\pi} d\phi\, \hat{z} = \frac{\mu_0 I}{2R} \hat{z}$$

Magnetic field above the center of a circular loop of current

The Biot-Savart law is also used to find the magnetic field a distance z above the center of a circular current-carrying loop. Let the loop have radius R and carry current I.

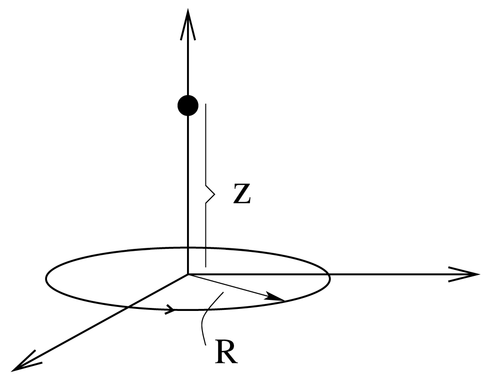

Figure 4: Current loop of radius R in the x-y plane, carrying current I.

Begin evaluation of the Biot-Savart integral by finding $\vec{\mathscr{r}}$, $\mathscr{r}$ and $\hat{\mathscr{r}}$. In general terms, the distance between the charge and the observer is $\vec{\mathscr{r}} = \vec{r} - \vec{r'}$. In this case, the distance from the origin to P is $\vec{r} = z\hat{z}$. An element of current Idl' may be anywhere on the circular loop, $\vec{r'} = R\cos(\phi)\hat{x} + R\sin(\phi)\hat{y}$. Then, the expression for $\vec{\mathscr{r}}$ is,

$$
\begin{aligned}
\vec{\mathscr{r}} &= \vec{r} - \vec{r'} \\
&= z\hat{z} - R\hat{s} \\
&= z\hat{z} - R\cos(\phi)\hat{x} - R\sin(\phi)\hat{y}
\end{aligned}
$$

Now it is possible to get $\mathscr{r}$,

$$
\begin{aligned}
\mathscr{r} &= \sqrt{\left(\vec{r} - \vec{r'}\right)^2} \\
&= \sqrt{(z\hat{z} - R\hat{s}) \cdot (z\hat{z} - R\hat{s})}
\end{aligned}
$$

The cross-terms of the dot product include $\hat{z} \cdot \hat{s}$ which is zero since the unit vectors are mutually orthogonal. This leaves the following,

$$
\begin{aligned}
\mathscr{r} &= \sqrt{(z\hat{z}\cdot z\hat{z}) + (R\hat{s}\cdot R\hat{s})} \\
&= \sqrt{z^2 + R^2}
\end{aligned}
$$

And the unit vector,

$$
\begin{aligned}
\hat{\mathscr{r}} &= \frac{\vec{\mathscr{r}}}{\mathscr{r}} \\
&= \frac{z\hat{z} - R\hat{s}}{\sqrt{z^2 + R^2}}
\end{aligned}
$$

Now, let's get the direction by considering the numerator of the integral expression for the field, and also by referring to Fig. 5 for a physical picture.

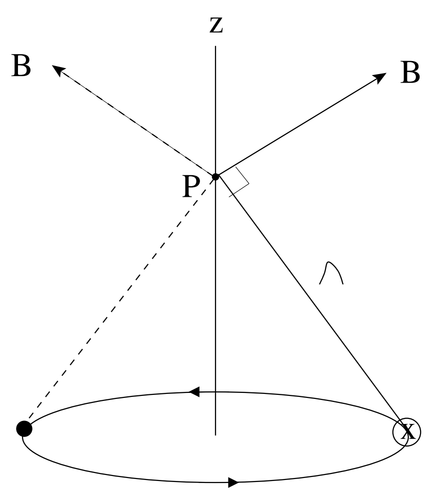

Figure 5: Magnetic field directions at point P from two points on a current loop of radius R in the x-y plane, carrying current I. The point on the loop labeled with an 'x' denotes the location where the current is going directly into the page, whereas the dot across from the 'x' denotes the location where the current is directed out of the page.

The direction of field contributions from certain segments of the current ring is shown graphically in Fig. 5. Looking at the figure, consider two current segments at opposite sides of the ring; current in the right segment is directed into the page as indicated by the 'x', and the current in the left segment is directed out of the page as indicated by the dot. Consider the right segment with current flowing into the page; $\vec{\mathscr{r}}$ lies in the plane of the page directed from the 'x' to point P. Since the field generated by this element of current is in the direction of $Id\vec{l} \times \vec{\mathscr{r}}$, it is perpendicular to both vectors. That is, in the plane of the page (perpendicular to $Id\vec{l}$) and at a right angle to $\vec{\mathscr{r}}$. The

$\vec{B}$ field vector is shown as the solid (not dashed) B vector in figure. Similarly, the left segment with current flowing out of the page produces a contribution to $\vec{B}$ shown as the dashed B vector in the figure. Note that these two contributions to the total field add in the $\hat{z}$ direction, but are opposed in the $\hat{s}$ direction. So, we expect the analytic result to be in the $\hat{z}$ direction.

The numerator in the integrand of the Biot-Savart law (repeated below) gives the direction of the field.

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l'} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

In this case, $Id\vec{l'}$ is a segment of current along the circular loop. Taking the current to be flowing counterclockwise as shown in the figure, then the direction of $Id\vec{l'}$ is $\hat{\phi}$.

$$Id\vec{l'} = Ids\,\hat{\phi} = IR\,d\phi\,\hat{\phi}$$

Then the numerator of the Biot-Savart integral is the following,

$$\begin{aligned} & IR\,d\phi\,\hat{\phi} \times \hat{\mathscr{r}} \\ & = IR\,d\phi\,\hat{\phi} \times \frac{z\hat{z} - R\hat{s}}{\sqrt{z^2 + R^2}} \\ & = \frac{IzR\,d\phi\,\hat{s} + IR^2\,d\phi\,\hat{z}}{\sqrt{z^2 + R^2}} \end{aligned}$$

So the entire Biot-Savart integral is then,

$$\begin{aligned} \vec{B} &= \frac{\mu_0 I}{4\pi} \int \frac{zR\,d\phi\,\hat{s}}{(z^2 + R^2)^{\frac{3}{2}}} + \frac{\mu_0 I}{4\pi} \int \frac{R^2\,d\phi\,\hat{z}}{(z^2 + R^2)^{\frac{3}{2}}} \\ \vec{B} &= \left(\frac{\mu_0 I}{4\pi}\right) \frac{zR}{(z^2 + R^2)^{\frac{3}{2}}} \int_0^{2\pi} d\phi\,\hat{s} + \left(\frac{\mu_0 I}{4\pi}\right) \frac{R^2\,\hat{z}}{(z^2 + R^2)^{\frac{3}{2}}} \int_0^{2\pi} d\phi \\ \vec{B} &= \frac{\mu_0 I z R}{4\pi\,(z^2 + R^2)^{\frac{3}{2}}} \int_0^{2\pi} d\phi\,\hat{s} + \frac{\mu_0 I R^2}{2\,(z^2 + R^2)^{\frac{3}{2}}}\hat{z} \qquad (4) \end{aligned}$$

where the constants have been pulled out of the integrals, and the second integral evaluated.

The first cannot be evaluated as is, since $\hat{s}$ is not a constant and cannot be pulled out of an integral. In order to calculate the first integral of Eq. 4, $\hat{s}$ will be written in terms of the Cartesian unit vectors, $\hat{s} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$. Then the first integral is,

$$B\hat{s} = \frac{\mu_0 I z R \hat{x}}{4\pi\left(z^2+R^2\right)^{\frac{3}{2}}} \int_0^{2\pi} \cos(\phi)\, d\phi + \frac{\mu_0 I z R \hat{y}}{4\pi\left(z^2+R^2\right)^{\frac{3}{2}}} \int_0^{2\pi} \sin(\phi)\, d\phi = 0$$

So, $B\hat{z}$ is the only non-zero component of $\vec{B}$ after integration over the entire current loop, as expected. Note that if $z = 0$ then point P is in the center of the ring, and the expression for B simplifies,

$$\begin{aligned} \left(z^2+R^2\right)^{\frac{3}{2}} &\longrightarrow R^3 \\ B = \frac{\mu_0 I R^2}{2\left(z^2+R^2\right)^{\frac{3}{2}}}\hat{z} &\longrightarrow B = \frac{\mu_0 I}{2R}\hat{z} \end{aligned}$$

This is consistent with the earlier result for $\vec{B}$ in the center of a circular current ring.

Summarizing the results of these notes on the Biot-Savart law:

$B = \frac{\mu_0 I}{4\pi s}\left[\frac{y_2}{\sqrt{(y_2^2+s^2)}} - \frac{y_1}{\sqrt{(y_1^2+s^2)}}\right]$	B field from a wire segment
$B = \frac{\mu_0 I}{4\pi}\left[\sin(\theta_2) - \sin(\theta_1)\right]$	B field from a wire segment
$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$	B field from an infinite wire
$\vec{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{\frac{3}{2}}}$	B above the center of a wire loop
$\vec{B} = \frac{\mu_0 I}{2R}\hat{z}$	B at the center of a wire loop