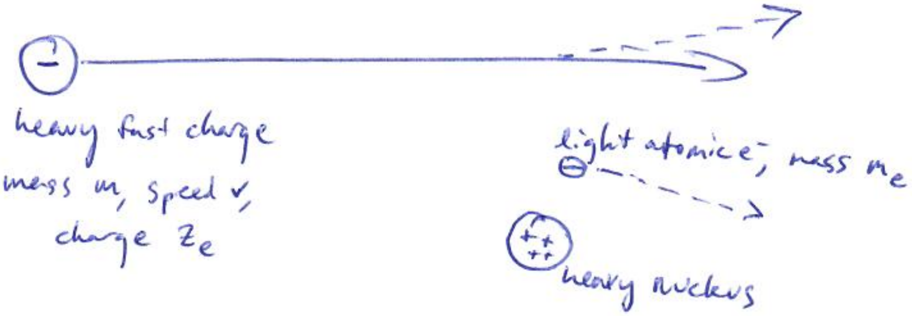
Modifications required by Quantum Mechanics



We've been looking at this problem ⇑. We computed

$$\frac{dE}{dx} \simeq \frac{1}{(4\pi\epsilon_0)^2} 4\pi \underbrace{N}_{\text{atoms}/\text{vol}} \underbrace{Z_1}_{\text{electrons}/\text{atom}} \frac{z^2 e^4}{m_e v^2} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

We neglected the electron binding energy (& atomic wavefunction) treating it as a free point charge. This is ok if the energy transfer is large compared to the quantum mechanical binding energy - ie, we computed the <u>ionization</u> energy loss, assuming small deflections ($b_{min} = ze^2/\gamma m_e v^2$) $(\times 1/4\pi\epsilon_0)$ and fast collisions ($b_{max} \simeq \gamma v/\omega$).

As in our discussion of bremsstrahlung, Quantum mechanics can modify b_{min} in some regimes.

Uncertainty principle: $\Delta p \, \Delta x \geq \hbar/2$.

We computed a classical momentum transfer

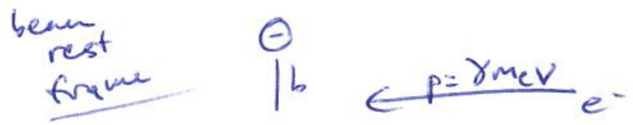
$\Delta p = \dfrac{2ze^2}{4\pi\epsilon_0 b v}$ with classical impact parameter b.

The classical appx requires $(\Delta p)(b) = \dfrac{2ze^2}{4\pi\epsilon_0 v} \geq \hbar/2$

or $\dfrac{4ze^2}{4\pi\epsilon_0 \hbar v} \gtrsim 1$. If this is violated,

we should use a different estimate to get b_{min}.

The beam particle "sees" an electron coming at it longitudinally:



In order for this momentum to make sense as "mostly longitudinal," the spread in transverse momenta around zero needs to be smaller than it:

$$(\text{typical } P_{transverse}) \lesssim (\gamma m_e v)$$

The quantum uncertainty in $P_{transverse}$ is $\mathcal{O}(\hbar/b_{min})$ if we demand that the e^- is localized transversely on scales of order b_{min} (to be determined.)

So, we require $\hbar / b_{min} < \gamma m_e v$

or $b_{min} = \hbar / \gamma m_e v$.

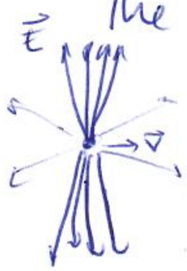
We could do the same argument for the e^- rest frame and find $b_{min} = \hbar / \gamma \underbrace{m}_{\text{beam mass}} v$

but since we assumed $m > m_e$ the requirement $b_{min} = \hbar / \gamma m_e v$ is the more stringent of the two, and we must take it.

So, if $\frac{Z e^2}{4\pi\epsilon_0 \hbar v}$ is small, we must switch from the classical $b_{min} = Ze^2 / 4\pi\epsilon_0 \gamma m_e v$ to the quantum $b_{min} = \hbar / \gamma m_e v$.

$$\Rightarrow \left(\frac{b_{min}}{b_{max}} \right)_{quantum} \approx \frac{\gamma^2 m_e v^2}{\hbar \omega}$$

ω ← a typical atomic frequency.

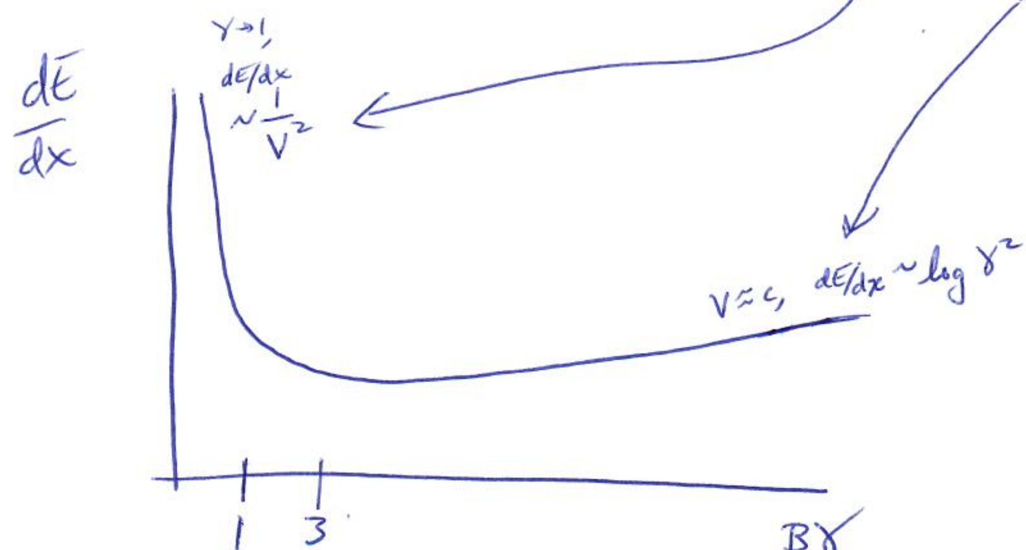


The two powers of γ in this:

(1) The transverse $\vec{E}$ field grows with γ. Therefore as γ increases we can increase the impact parameter and still get a high frequency kick

(2) energy transfer goes up, so $b_{min} \sim 1/\gamma$

So we get $\dfrac{dE}{dx} = \dfrac{4\pi N' Z z^2 e^4}{(4\pi\epsilon_0)^2 m_e v^2} \ln\left(\dfrac{\gamma^2 m_e v^3}{\hbar\omega}\right)$



This is useful in experiments: energy deposit / unit length is almost independent of energy for relativistic particles.

The rise at small v is (part of) why ionization >> brem for nonrelativistic particles.

The stopping power is numerically $\mathcal{O}(1)$ $(\mathrm{MeV}/\mathrm{cm}) \times \left(\frac{1}{g/cm^3}\right)$

Density effects

So far we've ignored the fields of other atoms near the beam particle. In reality the polarization of the material in the vicinity of the beam will reduce the effectiveness of energy transfer.

Let's see what happens if we put in a dielectric constant.

A long time ago, we wrote down the Lorentz gauge wave equations for the scalar & vector potential:

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \rho/\epsilon_0$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{d t^2} - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

These were in vacuum. In a non-magnetic medium with dielectric constant ϵ, there are two changes:

1. polarization reduces the field from a point charge $\rho/\epsilon_0 \rightarrow \rho/\epsilon$

2. phase velocity $c \rightarrow 1/\sqrt{\epsilon\mu_0} \equiv \bar{c}$

The wave equations become:

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \rho/\epsilon$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Let's define a 4D Fourier transform:

$$F(x,t) = \int_{-\infty}^{\infty} \frac{d^3k\, d\omega}{(2\pi)^4}\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}\, \tilde{F}(\vec{k},\omega)$$

Plugging in the transforms for $\rho, \vec{J}, \varphi$, and $\vec{A}$, we find

$$\left(-\frac{\omega^2}{c^2} + k^2\right)\tilde{\varphi}(k,\omega) = \frac{1}{\epsilon}\tilde{\rho}(k,\omega)$$

$$\left(-\frac{\omega^2}{c^2} + k^2\right)\tilde{\vec{A}}(k,\omega) = \mu_0 \tilde{\vec{J}}(k,\omega)$$

Now let's assume our $\approx$ const velocity point ze-charge beam particle is the source:

$$\rho = ze\,\delta^{(3)}(\vec{x}-\vec{v}t)$$

$$\vec{J} = \vec{v}\,ze\,\delta^{(3)}(\vec{x}-\vec{v}t) = \vec{v}\rho$$

$$\tilde{\rho}(\vec{k},\omega) = \int d^3x\,dt\; e^{-i(kx-\omega t)}\rho(x,t)$$

$$= \int dt\; e^{-i(\vec{k}\cdot\vec{v}t-\omega t)}\,ze$$

$$= ze\,\delta(\vec{k}\cdot\vec{v}-\omega)$$

similarly $\tilde{\vec{J}}(\vec{k},\omega) = \vec{v}\,ze\,\delta(\vec{k}\cdot\vec{v}-\omega) = \vec{v}\,\tilde{\rho}(k,\omega)$

So the wave eqs can be solved algebraically in Fourier space:

$$\tilde{\varphi}(\vec{k},\omega) = \frac{ze\,\delta(\vec{k}\cdot\vec{v}-\omega)}{\epsilon\left(k^2-\omega^2/\bar{c}^2\right)}$$

$$\tilde{\vec{A}}(k,\omega) = \mu\frac{ze\vec{v}\,\delta(\vec{k}\cdot\vec{v}-\omega)}{\left(k^2-\omega^2/\bar{c}^2\right)} = \vec{v}\,\frac{1}{\bar{c}^2}\,\tilde{\varphi}$$

Let us also allow for some dependence in the dielectric constant and phase velocity on the frequency:

$$\epsilon, \bar{c} \to \epsilon(\omega),\; \bar{c}(\omega) = \frac{1}{\sqrt{\epsilon(\omega)\mu_0}}$$

(Why is $\epsilon = \epsilon(\omega)$? Think of a single harmonically bound electron. $\ddot{\vec{x}} + \Gamma \dot{\vec{x}} + \omega_0^2 \vec{x} = -\frac{e}{m} \vec{E}$ is its Equation of motion. The Fourier transform is algebraic & can be solved:

$$\vec{x}(\omega) = \frac{-e/m \, \vec{E}(\omega)}{(\omega_0^2 - \omega^2) + i\Gamma\omega}$$

Now the polarization comes from summing over electron displacements: $\vec{P} = -\underbrace{NZe \times \vec{x}(\omega)}_{\text{dipole moment / unit vol}} = \underbrace{(\epsilon(\omega) - \epsilon_0)\,\tilde{E}(\omega)}_{\text{linear dielectric assumption)}}$

e^-/unit vol

$$\text{So} \quad \epsilon(\omega) = \epsilon_0 + \frac{NZe^2}{m} \cdot \epsilon_0 \cdot \frac{1}{(\omega_0^2 - \omega^2) - i\omega\Gamma}$$

at least, in the harmonic model. It reflects <u>resonance</u> and <u>absorption</u> (Γ). Since the applied fields are time dependent and the electrons are bound, it makes sense that the response will have some frequency dependence beyond that of the applied fields.)

From $\varphi, \vec{A}$ we can easily get the EM fields:

$$\vec{E} = -\vec{\nabla}\varphi - \dot{\vec{A}}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\Rightarrow \tilde{\vec{E}}(k,\omega) = -i\vec{k}\,\tilde{\varphi}(\vec{k},\omega) + i\omega\,\tilde{\vec{A}}(\vec{k},\omega)$$

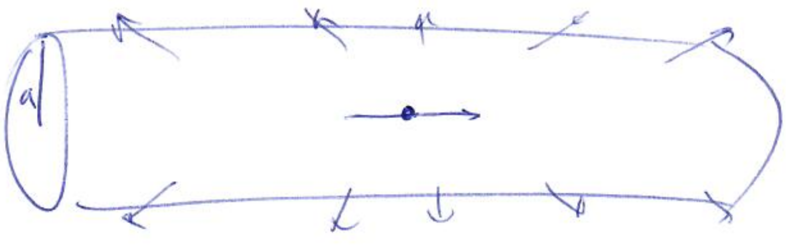
$$\tilde{\vec{B}}(k,\omega) = +i\vec{k}\times\tilde{\vec{A}}(\vec{k},\omega)$$

and for the point charge,

$$\tilde{\vec{E}} = -i\vec{k}\,\tilde{\varphi} + \frac{i\omega\vec{v}}{c^2}\,\tilde{\varphi}$$

$$\tilde{\vec{B}} = \frac{i\vec{k}\times\vec{v}}{c^2}\,\tilde{\varphi}$$

Now let's relate these to the energy loss.



Compute the electromagnetic energy flow through a cylinder of radius a around the beam particle:

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{1}{v}\frac{dE}{dt} = \frac{a}{v}\int_{-\infty}^{\infty} 2\pi\,(S_2)\,dx$$

cylindrical symmetry

$$S_2 = \frac{-1}{\mu_0}E_1 B_3$$

Integrating over all x at one time is equivalent to integrating over all time at 1 x since the particle is in constant velocity motion. So

$$\left(\frac{dE}{dx}\right)_{b=a} = -\frac{2\pi a}{\mu_0 \cancel{v}} \cancel{v} \int_{-\infty}^{\infty} E_1 B_3 \, dt.$$

Now convert to a frequency integral:

$$-\frac{2\pi a}{\mu_0} \int_{-\infty}^{\infty} dt \left(\int \frac{d^3k \, d\omega}{(2\pi)^4} e^{i(k_y a - \omega t)} \tilde{E}_1(\vec{k}, \omega) \right) \left(\int \frac{d^3k' d\omega'}{(2\pi)^4} e^{i(k_y' a - \omega' t)} \tilde{B}_3 \right)$$

here we choose our fixed point to be $(0, a, 0)$

$\hat{j} a$

$$= -\frac{2\pi a}{\mu_0} \int \frac{d\omega}{2\pi} \underbrace{\left(\int \frac{d^3k}{(2\pi)^3} e^{ik_y a} \tilde{E}_1(k, \omega) \right)}_{\equiv \tilde{E}_1(\omega)} \underbrace{\left(\int \frac{d^3k'}{(2\pi)^3} e^{ik_y' a} \tilde{B}_3(k, -\omega) \right)}_{= \tilde{B}_3(-\omega)}$$

Math Fact #1

$$\left[\begin{array}{l} \text{Since } \int d\omega \, e^{-i\omega t} \tilde{B}_3(\omega) \text{ has to be } \underline{\text{real}}, \\ \quad = \int d\omega \, e^{i\omega t} \tilde{B}_3^*(-\omega) = \int d\omega \, e^{-i\omega t} \tilde{B}_3^*(\omega) \\ \quad \text{So } \tilde{B}_3(-\omega) = \tilde{B}_3^*(\omega) \end{array} \right]$$

Math Fact #2

Furthermore

$$\int_{-\infty}^{\infty} d\omega\ \tilde{E}_1(\omega)\tilde{B}_3^*(\omega) = \int_0^{\infty} d\omega\,\tilde{E}_1(\omega)\tilde{B}_3^*(\omega) + \int_{-\infty}^{0} d\omega\ \tilde{E}_1(\omega)\tilde{B}_3(\omega)^*$$

$$= \int_0^{\infty} d\omega\left(\tilde{E}_1(\omega)\tilde{B}_3^*(\omega) + \tilde{E}_1(-\omega)\tilde{B}_3(-\omega)^*\right)$$

$$= \int_0^{\infty} d\omega\left(\tilde{E}_1\tilde{B}_3^* + \tilde{E}_1^*\tilde{B}_3\right)$$

$$= 2\,\mathrm{Re}\int_0^{\infty} d\omega\ \tilde{E}_1\tilde{B}_3^*.$$

So we need $-\frac{4\pi a}{\mu_0}\mathrm{Re}\int_0^{\infty}\frac{d\omega}{2\pi}\tilde{E}_1\tilde{B}_3^*$

It's a bit of work to get $\tilde{E}_1(\omega)$ and $\tilde{B}_3(\omega)$. Jackson does it around 13.30. Here's the result: More Bessels!

$$\tilde{E}_1(\omega) = \frac{-i z e \omega}{2\pi^2 \epsilon v^2}\left(1 - v^2/c^2\right)K_0(\lambda a)$$

$$\tilde{E}_2(\omega) = \frac{z e}{2\pi^2 \epsilon v}\lambda K_1(\lambda a)$$

$$\tilde{B}_3(\omega) = \frac{v\tilde{E}_2}{\tilde{c}^2}$$

$$\lambda = \frac{\omega}{v}\sqrt{1 - v^2/\tilde{c}^2}$$

So $\left.\frac{dE}{dx}\right|_{b>a} = -\frac{4\pi}{\mu_0} \mathrm{Re} \int_0^\infty \frac{d\omega}{2\pi} \left[\left(\frac{-iz^2e^2\omega}{(2\pi^2)^2 \epsilon^2 v^2 c^{-2}} \right) \left(1 - v^2/c^{-2}\right) K_0(\lambda a)\, \lambda^* a \times K_1(\lambda^* a) \right]$

The integrand is imaginary if ϵ is real and $\beta^2\epsilon < 1$. Then the result vanishes. We need the complex (absorptive) part of the dielectric function. That turns out to correspond to optical frequencies, so that typically $|\lambda a| \ll 1$ when λ is "very complex". Then we can expand the Bessels:

$$K_0(x) \simeq -\ln(x/2) + \ldots$$
$$K_1(x) \simeq 1/x + \ldots$$

Then the integral can be done, using the harmonic model for ϵ. The result is (in the relativistic limit)

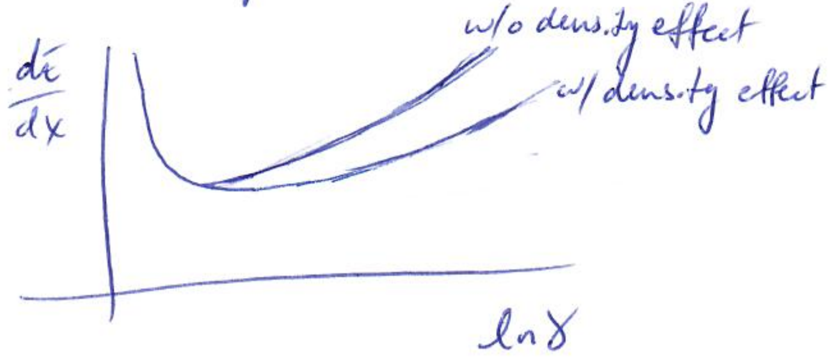
$$\left.\frac{dE}{dx}\right)_{b>a} = \frac{1}{(4\pi\epsilon_0)^2} \frac{4\pi N Z e^2 (ze)^2}{mc^2} \ln\left(\frac{1.12c}{a\,\omega_p}\right)$$

$$\omega_p^2 = 4\pi N Z e^2/m$$

The energy loss only depends on the electron density, not on atomic details. Same for all materials with similar densities!

If we rewrite our previous expression, the log was $\ln\left(\frac{1.12\,\gamma c}{a\langle\omega\rangle}\right)$ where $b_{min} \leftrightarrow a \sim 1/\gamma$.

This goes like $\ln \gamma^2$. With the density effect, we find no $\langle\omega\rangle$ dependence & $\ln \gamma$.



Experimentally, this screening effect is observed & curves rise like $\ln \gamma$.