Notes on Fourier Transformation

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1 Convention in 1D

Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function that decays rapidly at $\pm\infty$. Then,

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}} \tilde{f}(p)\, dp,$$

where the Fourier transform $\tilde{f}(p) \equiv \mathcal{F}[f](p)$ of $f(x)$ is defined by

$$\tilde{f}(p) \equiv \mathcal{F}[f](p) \equiv \int_{-\infty}^{\infty} \frac{e^{-\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}} f(x)\, dx\,.$$

We will also use the notation

$$f(x) = \mathcal{F}^{-1}[\tilde{f}](x).$$

Example 1.1 (Gaussian is invariant up to rescaling)**.** *If we have*

$$f(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

then

$$\mathcal{F}[f](p) = \sqrt{\frac{\sigma^2}{\hbar}} \exp\left(-\frac{\sigma^2 p^2}{2\hbar^2}\right).$$

If we choose $\sigma^2 = \hbar$, i.e.

$$f(x) = \exp\left(-\frac{x^2}{2\hbar}\right),$$

then

$$\tilde{f}(p) \equiv \mathcal{F}[f](p) = \exp\left(-\frac{p^2}{2\hbar}\right) = f(p).$$

2 Convention in 3D

Let $f : \mathbb{R}^3 \to \mathbb{C}$ be an integrable function that decays rapidly at $\pm\infty$. Then,

$$f(\vec{x}) = \int_{-\infty}^{\infty} \frac{e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}}{(2\pi\hbar)^{3/2}} \tilde{f}(\vec{p})\, d^3\vec{p},$$

where

$$\tilde{f}(\vec{p}) \equiv \mathcal{F}[f](\vec{p}) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}}}{(2\pi\hbar)^{3/2}} f(\vec{x})\, d^3\vec{x}\,.$$