

Chapter 4 - Electric fields in matter

so begins a fascinating and technically fruitful study of the electromagnetic properties and behavior of materials. We start with static electric fields, and history and convenience have made this partially seem like a laundry list of definitions. Just keep your mind on the underlying physics, this will help you not get queasy.

Most naturally occurring material is electrically neutral, and so does not exhibit monopole-like behavior in an electric field. Applied fields do tend to separate the centers of positive and negative charge in neutral atoms or molecules making them into dipoles (assuming the electrons are not free, as in conductors). Griffiths begins by considering the individual dipoles, which are induced (and may stretch) in an applied field. Some materials have permanent dipole characteristics even with no applied field.

Considering the single dipole, it may

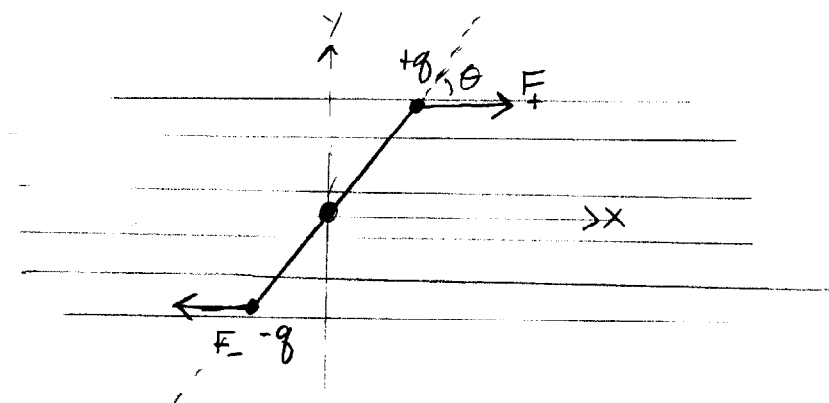
- stretch $\vec{p} = \alpha \vec{E}$, $\alpha = \text{atomic polarizability}$
- rotate $\vec{N} = \vec{p} \times \vec{E}$, torque

in an applied field.

Torque

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How much torque is there on a dipole in a uniform field?



$$\vec{\tau} = \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_-$$

These torque act to rotate dipole in the same direction. $|\vec{\tau}| = |\vec{r}_+ \times \vec{F}_+| + |\vec{r}_- \times \vec{F}_-|$

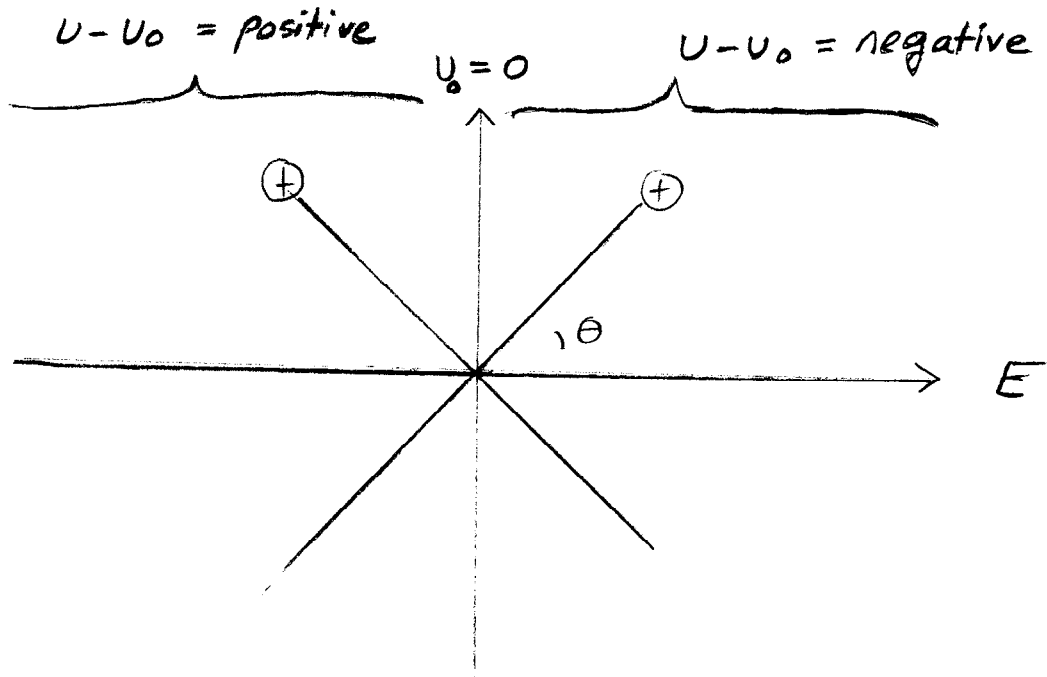
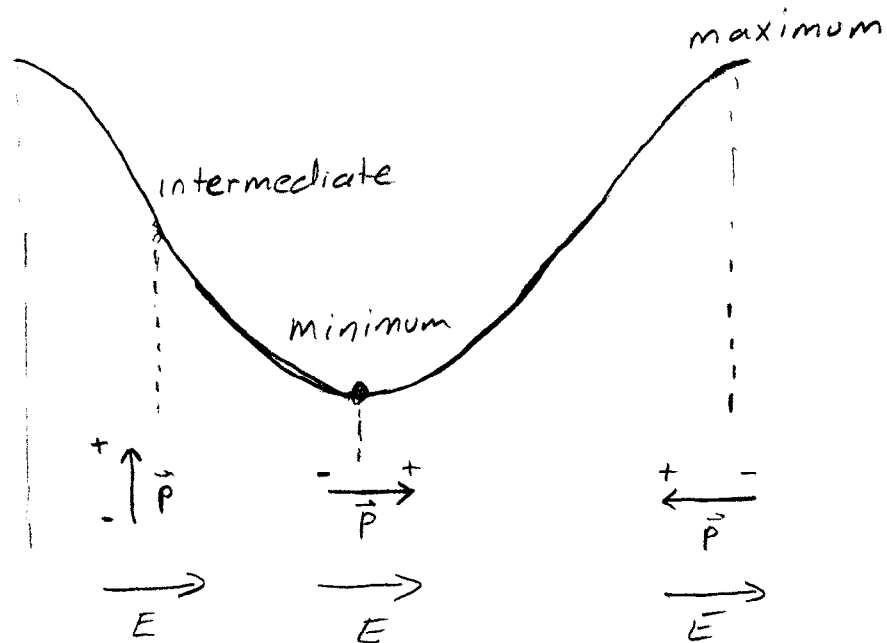
$$\vec{\tau} = (d_+ \sin \theta qE + d_- \sin \theta qE) (-\hat{z})$$

$$\vec{\tau} = qd \sin \theta E (-\hat{z}) = \vec{p} \times \vec{E}$$

What is the potential energy of the dipole in the field?

If it is free to move, it will only spin. Potential energy is associated with rotation not translation.

Potential well



$$U = \int_{\pi/2}^{\theta} r d\theta = \int_{\pi/2}^{\theta} p E \sin \theta d\theta = p E (-\cos \theta \Big|_{\pi/2}^{\theta})$$

$$= -p E \cos \theta = -\vec{p} \cdot \vec{E}$$

check it

$$-\vec{p} \cdot \vec{E} = -pE \cos \theta$$

$$\text{if } \theta \rightarrow -\theta \quad U \rightarrow -pE \cos(-\theta) = -pE \cos \theta$$

$$\text{if } \theta \rightarrow \pi - \theta \quad U \rightarrow -pE \cos(\pi - \theta) = pE \cos \theta$$

All is right with the world

Material in an \vec{E} field can contain many little dipoles, which may or may not align well.

$$\vec{P} \equiv \begin{cases} \text{dipole moment per unit volume} \\ \text{Polarization} \end{cases}$$

The charges causing polarization are bound charges. The polarization can be used to find the bound charge.

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

Bound volume charge density

$$\sigma_b = \vec{P} \cdot \hat{n}$$

Bound surface charge density

The potential due to a single dipole is given by,

$$V = \frac{1}{4\pi\epsilon_0} \frac{\hat{r} \cdot \vec{p}}{r^2}$$

So, if we have a dipole moment per unit volume, \vec{P} , we integrate over volume to get the potential.

$$V = \frac{1}{4\pi\epsilon_0} \int_V \frac{\hat{r} \cdot \vec{P}}{r^2} d\tau'$$

Use the following identities;

$$\frac{\hat{r}}{r^2} = \vec{\nabla}' \left(\frac{1}{r} \right)$$

$$\begin{aligned} \vec{\nabla}' \cdot (f \vec{v}) &= f (\vec{\nabla}' \cdot \vec{v}) + \vec{v} \cdot \vec{\nabla}' f \\ \rightarrow \vec{P} \cdot \vec{\nabla}' \left(\frac{1}{r} \right) &= \vec{\nabla}' \cdot \left(\frac{\vec{P}}{r} \right) - \frac{1}{r} (\vec{\nabla}' \cdot \vec{P}) \end{aligned}$$

Then,

$$V = \frac{1}{4\pi\epsilon_0} \int_V \vec{\nabla}' \cdot \left(\frac{\vec{P}}{r} \right) d\tau' - \frac{1}{4\pi\epsilon_0} \int \frac{\vec{\nabla}' \cdot \vec{P}}{r} d\tau'$$

↑ Apply divergence theorem on this integral

$$V = \frac{1}{4\pi\epsilon_0} \int_S \frac{\vec{P} \cdot \hat{n} da'}{r} - \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{\nabla}' \cdot \vec{P}}{r} d\tau'$$

$\vec{P} \cdot \hat{n}$ has the same form as $\vec{E} \cdot \hat{n}$

$$\vec{E} \cdot \hat{n} = \sigma / \epsilon_0 \quad \rightarrow \quad \vec{P} \cdot \hat{n} = \sigma_b \quad \text{Looks like a surface charge density}$$

$\vec{\nabla} \cdot \vec{P}$ has the same form as $\vec{\nabla} \cdot \vec{E}$

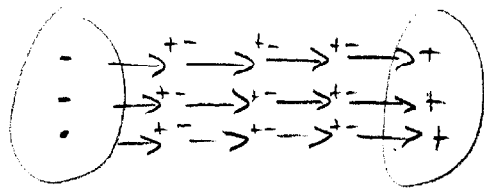
$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0 \quad \rightarrow \quad -\vec{\nabla} \cdot \vec{P} = \rho_b \quad \text{Looks like a volume charge density}$$

How do these come about in a material?

Note that a net volume charge density requires a divergence of \vec{P} (such as in this Griffiths cartoon)



A uniform \vec{P} results only in a surface charge density (as shown in this Griffiths cartoon)



Keep this in mind when solving problems.

A large class of dielectric materials are linear dielectrics. In this case the following relations hold:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E}$$

$$= \epsilon \vec{E} \quad \left\{ \begin{array}{l} \epsilon \equiv \epsilon_0 (1 + \chi_e) \\ \epsilon_r \equiv (1 + \chi_e) \end{array} \right.$$

$$= \epsilon_0 \epsilon_r \vec{E}$$

χ_e is called the electric susceptibility

ϵ_r is what you know and love (as K , most likely) as the dielectric constant.

Note the following:

$\vec{D} = \epsilon \vec{E}$ being linear, is a sort of Hooke's law approximation

\vec{E} in these expressions is the full electric field, due to any applied field and the field due to polarization.

In fact, the Maxwell's equations we know and love are always true, with \vec{E} being the total field and ρ being the total charge density:

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

$$\vec{\nabla} \times \vec{E} = 0$$

These are sometimes re-written in a form convenient for linear problems with bound charges as well as free charges. This is how:

$$(1) \quad \epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_{\text{free}} + \rho_{\text{bound}}$$

$$(2) \quad -\vec{\nabla} \cdot \vec{P} = \rho_{\text{bound}}$$

subtract (2) from (1)

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} + \vec{\nabla} \cdot \vec{P} = \rho_f + \rho_b - \rho_b$$

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

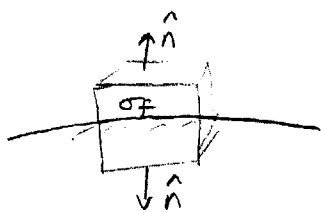
$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

The reason this final form is convenient is that the bound charge (which is often not known) doesn't enter the equation.

If symmetry is appropriate, then Gauss' law (divergence theorem) techniques may be applied to find \vec{D} from ρ_f . Once \vec{D} is known, \vec{E} & \vec{P} follow.

Note that there are a set of boundary conditions for \vec{D} , which while obtained with similar methods to those for \vec{E} , emphasize a fundamental difference between \vec{D} & \vec{E} . The tangential component of \vec{D} at either side of a boundary need not be continuous (and all that implies)!

Divergence theorem to get D^\perp b.c.



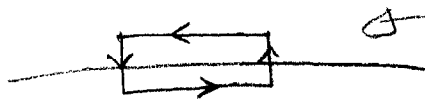
$$\int_{\text{top}} D^\perp \cdot da + \int_{\text{sides}} D^\perp \cdot da + \int_{\text{bottom}} D^\perp \cdot da = \int \sigma_f da$$

$$D_{\text{above}}^\perp - D_{\text{below}}^\perp = \sigma_f$$

What about the tangential component of \vec{D} ?

$$\int \vec{\nabla} \times \vec{D} \cdot d\vec{a} = \int \vec{\nabla} \times (\cancel{\epsilon_0 \vec{E}} + \vec{P}) \cdot d\vec{a}$$

$$\oint \vec{D} \cdot d\vec{\ell} = \oint \vec{P} \cdot d\vec{\ell}$$



Examine tangential component
No line integral contribution from sides

$$D''_{\text{above}} - D''_{\text{below}} = P''_{\text{above}} - P''_{\text{below}}$$

Suppose the boundary is the surface of a polarized material, since there is polarization only on one side, it is clear in this case

$$P''_{\text{above}} - P''_{\text{below}} \neq 0$$