Chapter 12 - Lecture 1

Reference : "Spacetime Physics" Taylor/Wheeler
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Some definitions

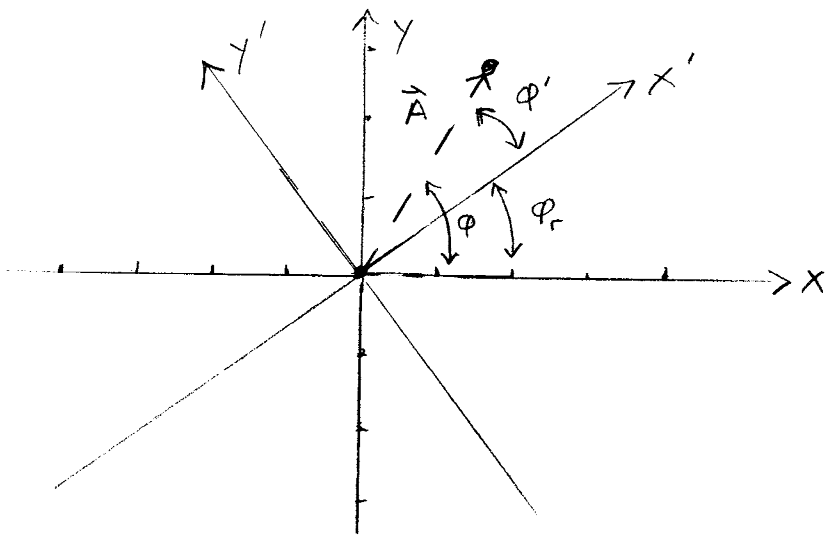
Inertial reference frame - A system at rest or moving with constant velocity (Newton's first law holds).

Special theory of relativity -

1) The laws of physics are the same in all inertial frames
2) The speed of light in vacuum is the same for all inertial observers

The following discussion is extracted from the first chapter of Taylor/Wheeler. This builds an understanding of Lorentz transformations (which transform coordinates from one inertial system to another) by comparing to rotational transformations (which transform spatial coordinates from one coordinate system to another).

Suppose there are two coordinate systems, whose origins coincide, but are rotated with respect to each other.



The components of $\vec{A}$ will have different coordinates in the two systems, (x, y) and (x', y'). However, even though the coordinates differ, the length of vector $\vec{A}$ will be the same (invarient) in both systems, providing the units of length are the same along all axes. If the distance along the x-direction is measured in meters, but along the y-direction is measured in miles, the y distance must be converted from miles to meters using a scale factor. This scale factor is the same regardless of which coordinate system you happen to be in. Once the scaling is done,

$$\text{distance} = \text{invarient} = \left[(\Delta x)^2 + (\Delta y)^2\right]^{1/2}$$
$$= \left[(\Delta x')^2 + (\Delta y')^2\right]^{1/2}$$

The components $A_{x'}$, $A_{y'}$ can be written in terms of A_x, A_y with a covarient transformation. (See Griffiths page 10).

$$A_{x'} = A\cos\varphi' = A\cos(\varphi - \varphi_r)$$
$$= A(\cos\varphi\cos\varphi_r + \sin\varphi\sin\varphi_r)$$
$$= \cos\varphi_r\, A_x + \sin\varphi_r\, A_y$$

similarly, $A_{y'} = -\sin\varphi\, A_x + \cos\varphi_r\, A_y$

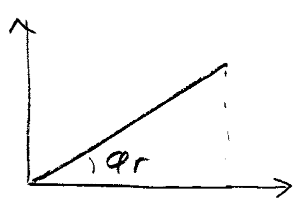
$$\begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix} = \begin{pmatrix} \cos\varphi_r & \sin\varphi_r \\ -\sin\varphi_r & \cos\varphi_r \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

This convenient form for the transformation comes about because angles are additive

$$\varphi' = \varphi - \varphi_r$$

The transformation could also be written in terms of the slopes of the lines, but since slopes are not additive, the form is more complicated.

The tangent of an angle is the slope of the line:



$$\tan\varphi_r = S_r = \text{slope}$$

We need $\cos\varphi_r$ and $\sin\varphi_r$
These can be found with trigonometry

$$\cos^2(\varphi_r) + \sin^2(\varphi_r) = 1$$

$$\cos^2(\varphi_r) = 1 - \sin^2(\varphi_r) = 1 - \tan^2(\varphi_r)\cos^2(\varphi_r)$$

solving for $\cos(\varphi_r)$ ->

$$\cos(\varphi_r) = \frac{1}{[1 + \tan^2(\varphi_r)]^{1/2}} = \frac{1}{[1 + S_r^2]^{1/2}}$$

multiply both sides by $\tan(\varphi_r)$ ->

$$\sin(\varphi_r) = \frac{\tan(\varphi_r)}{[1 + \tan^2(\varphi_r)]^{1/2}} = \frac{S_r}{[1 + S_r^2]^{1/2}}$$

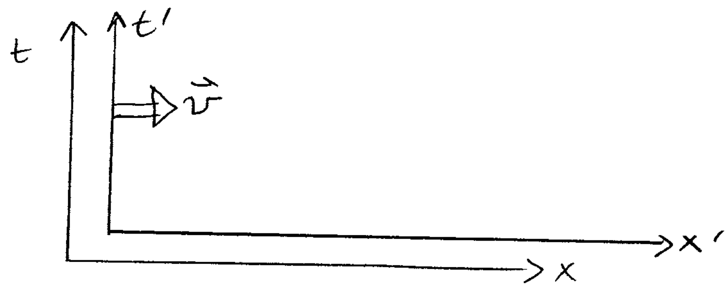
In terms of slope, the rotational transformations become the following:

$$A_{x'} = \cos(\varphi_r) A_x + \sin(\varphi_r) A_y$$
$$A_{y'} = -\sin(\varphi_r) A_x + \cos(\varphi_r) A_y$$

$$A_{x'} = [1 + S_r^2]^{-1/2} A_x + S_r [1 + S_r^2]^{-1/2} A_y$$

$$A_{y'} = -S_r [1 + S_r^2]^{-1/2} A_x + [1 + S_r^2]^{-1/2} A_y$$

Now consider two inertial frames of reference, the rocket frame moving to the right along the x-axis at speed v.

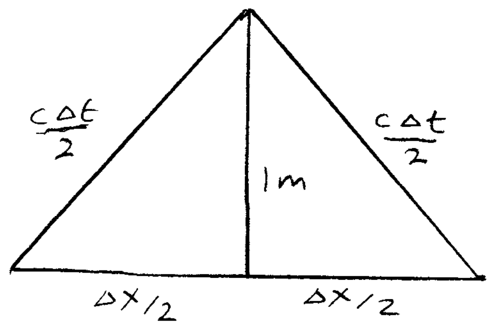


The components describing the interval between events will have different coordinates in the two systems; (ct, x) in S and (ct', x') in S'. However, even though the coordinates differ, the displacement 4-vector (invarient interval) will be the same. $I \equiv (\Delta x)^2 - (c\Delta t)^2 = (\Delta x')^2 - (c\Delta t')^2$
The t distance must be converted from seconds to meters using a scale factor (c), so that both coordinates have the same units. This scale factor is the same regardless of which coordinate system you happen to be in.

The components x, ct can be written in terms of x', ct' with a covarient transformation.

Invarience of the interval $(\Delta x)^2 - (c\Delta t)^2$ is a consequence of the two postulates of special relativity, that the laws of physics are the same in all inertial frames and that the speed of light in vacuum is the same for all inertial observers. The former implies length as measured in directions perpendicular to the direction of relative motion is the same in different inertial frames. As Griffiths suggests, suppose a guy in a moving train leans out the window and paints a line on a wall parallel to the train, at some height above the ground. This height must have the same measure in both the train and lab frames, since there is no way to distinguish one inertial frame from another. In the direction of relative motion, displacement is entangled with time.

Let a flash of light be emitted from the floor of the railway car reflected by a mirror one meter directly above, returning to the point of emission. In the railway car frame, $\Delta x' = 0$ where x is the direction of motion of the car with respect to ground. In the lab (ground) frame, the light pulse moves by Δx, since the detector in the car has moved Δx. The situation is depicted on the next page.



Light has the same speed in all frames. For a perpendicular distance of one meter, since we have a right triangle;

$$(1)^2 + (\Delta x/2)^2 = \left(\frac{c\Delta t}{2}\right)^2 \rightarrow (\Delta x)^2 - (c\Delta t)^2 = -4$$

No matter how fast the train goes, this relation must always hold. While Δx increases if the train goes faster, $c\Delta t$ must increase in the same proportion.

Note that $\cancel{(\Delta x')^2}^{\to 0} - (c\Delta t')^2 = -(c\Delta t')^2 = -(2m)^2$

$$\rightarrow \quad (\Delta x')^2 - (c\Delta t')^2 = (\Delta x)^2 - (c\Delta t)^2$$

This interval is an invarient.

Now find out how components Δx, $c\Delta t$ can be written in terms of components $\Delta x'$, $c\Delta t'$.

We need to find the coefficients of the transformation.

$$x = A x' + H\,(ct')$$

$$ct = K x' + D\,(ct')$$

As Taylor/Wheeler do, we'll use pion decay to help pin them down. In the frame moving with the velocity of the pion, the pion is at rest. If it is created at the origin, it stays there, $\Delta x' = 0$. The relative velocity of the pion frame to the lab frame is

$$\frac{\Delta x}{c\Delta t} = \frac{v}{c}$$

Call this β. Then $\Delta x = \beta c \Delta t$, $\Delta x' = 0$

The invarient interval between the creation and decay of the pion is given by,

$$(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - 0$$

$$(\cancel{c}\Delta t)^2 - (\beta \cancel{c}\Delta t)^2 = (\cancel{c}\Delta t')^2$$

solve for Δt,

$$\Delta t = \frac{\Delta t'}{[1-\beta^2]^{1/2}} = \gamma\,\Delta t'$$

Moving clocks run slow!

Also, now we know that the heretofor unknown coefficient D is γ. Since $\Delta x = \beta c \Delta t$, then $\Delta x = \beta c \gamma \Delta t'$. So, we also know $H = \beta\gamma$. To find the remaining coefficients A and K, consider a general interval where $\Delta x' \neq 0$, $\Delta t' \neq 0$.

Invarient interval $(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - (\Delta x')^2$

$$\underbrace{(K\Delta x' + \gamma c \Delta t')^2}_{(c\Delta t)^2} - \underbrace{(A\Delta x' + \beta\gamma c\Delta t')^2}_{(\Delta x)^2} = (c\Delta t')^2 - (\Delta x')^2$$

$$\gamma^2(1-\beta^2)\ (c\Delta t')^2 + 2\gamma(K - A\beta)\,\Delta x' c\Delta t' + (K^2 - A^2)\,\Delta x'^2$$

$$= (c\Delta t')^2 - (\Delta x')^2$$

Thus, $K - A\beta = 0$ or $K = A\beta$

and $(K^2 - A^2) = A^2(1-\beta^2) = \frac{1}{\gamma^2} A^2 = 1$

Then $A = \gamma$ and $K = \beta\gamma$

We have the inverse Lorentz transformations

$$\Delta x = (1-\beta^2)^{-1/2}\,\Delta x' + \beta(1-\beta^2)^{-1/2}\,(c\Delta t')$$

$$c\Delta t = \beta(1-\beta^2)^{-1/2}\,\Delta x' + (1-\beta^2)^{-1/2}\,(c\Delta t')$$

The Lorentz transformations (non-inverse) are;

$$\Delta x' = (1-\beta^2)^{-1/2}\,\Delta x - \beta(1-\beta^2)^{-1/2}(c\Delta t)$$

$$c\Delta t' = -\beta(1-\beta^2)^{-1/2}\,\Delta x' + (1-\beta^2)^{-1/2}(c\Delta t)$$

To get the velocity addition law, consider a bullet fired in the rocket frame. The bullet has velocity $\beta' = \Delta x'/c\Delta t'$ in the rocket frame, The rocket is moving with relative velocity β_r to the lab frame. What is the velocity of the bullet in the lab frame ($\beta = \Delta x/c\Delta t$)?

$$\beta = \frac{\Delta x}{c\Delta t} = \frac{(1-\beta_r^2)^{-1/2}\,\Delta x' + \beta_r(1-\beta_r^2)^{-1/2}(c\Delta t')}{\beta_r(1-\beta_r^2)^{-1/2}\,\Delta x' + (1-\beta_r^2)^{-1/2}(c\Delta t')}$$

multiply top & bottom by $\dfrac{(1-\beta_r^2)^{+1/2}}{c\Delta t'}$

Then,

$$\beta = \frac{\frac{\Delta x'}{c\Delta t'} + \beta_r}{\beta_r\frac{\Delta x'}{c\Delta t'} + 1} = \frac{\beta' + \beta_r}{1 + \beta'\beta_r}$$

multiplying both sides by c $\Rightarrow$ $v = \dfrac{v' + u}{1 + \frac{v'u}{c^2}}$

Just as a rotational transformation has a simpler form when written in terms of the angle of rotation than in terms of the slopes, so also the Lorentz transformations can be written in a simpler form. To go from describing a rotation in terms of slopes to describing it in terms of angle, we used slope $= \tan\varphi$. For the Lorentz transformation we'll use $\beta = \tanh(\varphi)$. This matches up with the forms of the invarients:

$$\Delta x^2 + \Delta y^2 = \text{constant} \qquad \text{Equation of a circle}$$

$$\Delta x^2 - (c\Delta t)^2 = \text{constant} \qquad \text{Equation of hyperbola}$$

If $\beta = \tanh(\varphi)$, then $(1-\beta^2)^{-1/2}$ is

$$\frac{1}{\left(1 - \frac{\sinh^2\varphi}{\cosh^2\varphi}\right)^{-1/2}} = \frac{\cosh\varphi}{(\cosh^2\varphi - \sinh^2\varphi)^{-1/2}} = \cosh\varphi$$

$$\beta(1-\beta^2)^{-1/2} = \frac{\sinh\varphi}{\cosh\varphi}\cosh\varphi = \sinh\varphi$$

$$\Delta x = \cosh\varphi\,\Delta x' + \sinh\varphi\,(c\Delta t')$$

$$c\Delta t = \sinh\varphi\,\Delta x' + \cosh\varphi\,(c\Delta t')$$