Today's outline - April 06, 2023

- LOCC
- Majorization
- Examples
- Measurement of density operators
- SpinQ NMR systems

Reading assignment: Reiffel: 11.1

Opportunity for a workshop by

https://www.classiq.io/

Local operations and classical communication (LOCC)

Suppose we have a multi-partite system where different subsystems are under the control of different people with only classical communications channels between them

These individuals only have a restricted set of operations that they can perform on the system and these are called local operations with classical communications (LOCC)

The measure of entanglement of a system cannot be increased by LOCC

Consider a system with a specific decomposition given by $X = X_1 \otimes \cdots \otimes X_n$

If it is possible to convert $|\psi\rangle \in X$ into $|\phi\rangle \in X$ via a series of unitary transformations and measurements which guaranteed to succeed deterministically then the conversion is said to be done by LOCC with respect to the particular tensor decomposition

Two states are LOCC equivalent if they can be transformed into each other by LOCC

The level of entanglement cannot be increased by LOCC so an unentangled state cannot be converted to an entangled state by LOCC alone

Classifying bipartite states

It is possible to classify pure states of bipartite systems using a metric called majorization of the eigenvalues of subsystem density operators

Let a and b be two vectors in the same m dimensional space

$$a = (a_1, \dots, a_m), \quad b = (b_1, \dots, b_m)$$

The vectors $a^{\downarrow}$ and $b^{\downarrow}$ are reordered in such a way that the coefficients are arranged by magnitude

$$a^{\downarrow} = (a_1^{\downarrow}, \dots, a_m^{\downarrow}), \quad a_i^{\downarrow} \geq a_{i+1}^{\downarrow}$$

$$b^{\downarrow} = (b_1^{\downarrow}, \dots, b_m^{\downarrow}), \quad b_i^{\downarrow} \geq b_{i+1}^{\downarrow}$$

Note that $a^{\downarrow}$ and $b^{\downarrow}$ are not necessarily ordered in the same way!

b is said to majorize a ($b \succeq a$) if for each $k, 1 \leq k \leq m$

$$\sum_{j=1}^{k} a_j^{\downarrow} \leq \sum_{j=1}^{k} b_j^{\downarrow}, \quad \sum_{j=1}^{m} a_j^{\downarrow} = \sum_{j=1}^{m} b_j^{\downarrow}$$

This can be applied to density matrix eigenvalues to define LOCC equivalence and relative degrees of entanglement

Majorization & LOCC equivalence

For states $|\psi\rangle$ and $|\phi\rangle$ of a bipartite system $X = A \otimes B$, the partial density matrices with respect to subsystem A are $\rho_\psi = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_\phi = \mathrm{Tr}_B(|\phi\rangle\langle\phi|)$

The eigenvalues of ρ_ψ and ρ_ϕ are given by $\lambda^\psi = (\lambda_1^\psi, \ldots, \lambda_m^\psi)$ and $\lambda^\phi = (\lambda_1^\phi, \ldots, \lambda_m^\phi)$

It can be shown that $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$ only if $\lambda^\phi \succeq \lambda^\psi$

An unambiguous definition of LOCC equivalence is thus $\lambda^\phi \succeq \lambda^\psi$ and $\lambda^\psi \succeq \lambda^\phi$

For a bipartite system of 2 qubits let $|\psi\rangle$ and $|\phi\rangle$ be states such that

$$\lambda^\psi = (\lambda, 1-\lambda), \quad \lambda^\phi = (\mu, 1-\mu), \quad \lambda, \mu \geq \tfrac{1}{2}$$

$|\psi\rangle$ can be converted to $|\phi\rangle$ only if $\mu \geq \lambda$ so that $\lambda^\phi \succeq \lambda^\psi$

Furthermore it is clear that $\lambda^\phi \succeq \lambda^\psi$ only when the von Neumann entropy satisfies $S[\mathrm{Tr}_2(|\psi\rangle\langle\psi|)] \geq S[\mathrm{Tr}_2(|\phi\rangle\langle\phi|)]$

Thus $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$ is possible only when $|\psi\rangle$ is more, or equally as, entangled as $|\phi\rangle$

Larger bipartite systems

When bipartite systems have subsystems of more than one qubit, the situation is a bit more complex as there are states which cannot be compared for majorization

$$|\psi\rangle = \tfrac{3}{4}|0\rangle|0\rangle + \tfrac{2}{4}|1\rangle|1\rangle + \tfrac{\sqrt{2}}{4}|2\rangle|2\rangle + \tfrac{1}{4}|3\rangle|3\rangle$$

$$|\phi\rangle = \tfrac{\sqrt{8}}{4}|0\rangle|0\rangle + \tfrac{\sqrt{6}}{4}|1\rangle|1\rangle + \tfrac{\sqrt{1}}{4}|2\rangle|2\rangle + \tfrac{1}{4}|3\rangle|3\rangle$$

There is an inconsistency which prevents a a comparison

$$\lambda_1^\psi = \tfrac{9}{16} > \tfrac{1}{2} = \lambda_1^\phi$$

$$\lambda_1^\psi + \lambda_2^\psi = \tfrac{13}{16} < \tfrac{14}{16} = \lambda_1^\phi + \lambda_2^\phi$$

However, it is unambiguous that in any bipartite system, the vector for an unentangled state majorizes all others and it can be shown that a maximally entangled state is majorized by all others

Consider $|\psi\rangle$ in a bipartite system $X = A \otimes B$ where A and B have dimensions $N, M : N \geq M$

$$|\psi\rangle = \frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |\phi_i^A\rangle \otimes |\phi_i^B\rangle$$

$\{|\phi_i^A\rangle\}$ and $\{|\phi_i^B\rangle\}$ are orthonormal sets and the latter must be a basis for B so $\lambda^\phi \succeq \lambda^\psi$ must hold for all $|\phi\rangle \in A \otimes B$

Example 10.2.5

Consider the two Bell states $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

The partial density matrices for each of these states with respect to the first qubit are

$$\rho_{\Phi^+} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \rho_{\Psi^+} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

The ordered eigenvalues are thus

$$\lambda^{\Phi^+} = (\tfrac{1}{2}, \tfrac{1}{2}), \qquad \lambda^{\Psi^+} = (\tfrac{1}{2}, \tfrac{1}{2})$$

Therefore, it is easy to see that $\lambda^{\Phi^+} \succeq \lambda^{\Psi^+}$ and $\lambda^{\Psi^+} \succeq \lambda^{\Phi^+}$

$$\lambda_1^{\Phi^+} = \lambda_1^{\Psi^+} = \tfrac{1}{2}$$

$$\lambda_1^{\Phi^+} + \lambda_2^{\Phi^+} = \lambda_1^{\Psi^+} + \lambda_2^{\Psi^+} = 1$$

With von Neumann entropies of

$$S(\rho_{\Phi^+}) = -2(\tfrac{1}{2}\log_2 \tfrac{1}{2}) = 1 = S(\rho_{\Psi^+})$$

These two states are both maximally entangled and thus are LOCC equivalent and can be transformed into each other using a local operation only

In this case, applying an X transformation to the first qubit will transform $|\Phi^+\rangle \xleftrightarrow{X\otimes I} |\Psi^+\rangle$

Example 10.2.6

Can the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ be converted to $|00\rangle$ using LOCC?

We know that the partial density matrices for the two states with respect to the first qubit are

$$\rho_{\Phi^+} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \rho_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The ordered eigenvalues are thus

$$\lambda^{\Phi^+} = (\tfrac{1}{2}, \tfrac{1}{2}), \qquad \lambda^{00} = (1, 0)$$

Therefore since $\lambda^{00} \succeq \lambda^{\Phi^+}$ we see that $|00\rangle$ majorizes $|\Phi^+\rangle$

$$\lambda_1^{00} = 1 > \lambda_1^{\Phi^+} = \tfrac{1}{2}$$

$$\lambda_1^{00} + \lambda_2^{00} = \lambda_1^{\Phi^+} + \lambda_2^{\Phi^+} = 1$$

With von Neumann entropies of

$$S(\rho_{\Phi^+}) = 1, \quad S(\rho_{00}) = -1\log_2 1 - 0\log_2 0 = 0$$

Thus it is possible to convert $|\Phi^+\rangle$ to $|00\rangle$ using LOCC

This can be done by measuring the first qubit and and then applying X to each qubit if the result is $|1\rangle$ and doing nothing if the result is $|0\rangle$

Mixed bipartite systems

A mixed state of a quantum system is separable with respect to a particular tensor decomposition $V_0 \otimes \cdots \otimes V_{N-1}$ if it can be written as a probabilistic mixture of unentangled states

$$\rho = \sum_{j=1}^{m} p_j |\phi_j^{(0)}\rangle\langle\phi_j^{(0)}| \otimes \cdots \otimes |\phi_j^{(N-1)}\rangle\langle\phi_j^{(N-1)}|, \quad |\phi_j^{(i)}\rangle \in V_i, \quad p_i \geq 0, \quad \sum_{i=0}^{N-1} p_i = 1$$

For any given value of i, the $|\phi_j^{(i)}\rangle$ do not need to be orthogonal

If an mixed state cannot be written in this way it is said to be entangled

A mixed state that can be written as a probabilistic mixture of entangled states can still be separable

$$\rho = \tfrac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \tfrac{1}{2}|\Phi^-\rangle\langle\Phi^-| = \tfrac{1}{2}\left[(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)\right] + \tfrac{1}{2}\left[(|00\rangle - |11\rangle)(\langle 00| - \langle 11|)\right]$$

$$= \tfrac{1}{2}\left[2|00\rangle\langle 00| + \cancel{|00\rangle\langle 11|} + \cancel{|11\rangle\langle 00|} + 2|11\rangle\langle 11| - \cancel{|00\rangle\langle 11|} - \cancel{|11\rangle\langle 00|}\right] = \tfrac{1}{2}\left[|00\rangle\langle 00| + |11\rangle\langle 11|\right]$$

Which is separable and really just a probabilistic mixture of product states

Measuring the entropy of a bipartite mixed state is complicated and can be done in a number of different ways

Measurement of density operators

Let $|x\rangle$ be an element of an $N = 2^n$ dimensional vector space X with density operator $\rho_x = |x\rangle\langle x|$

If $|x\rangle$ is measured with an operator O that has K associated projectors P_j the result, with probability p_j is

$$\frac{P_j|x\rangle}{|P_j|x\rangle|} = \frac{1}{\sqrt{p_j}} P_j|x\rangle, \qquad p_j = \langle x|P_j|x\rangle$$

The density operator for each of these states is

$$\rho_x^j = \frac{1}{p_j} P_j|x\rangle\langle x|P_j = \frac{1}{p_j} P_j \rho_x P_j$$

The density operator for O which summarizes the possible outcomes of the measurement is

$$\rho_x^O = \sum_{j=0}^{K-1} p_j \rho_x^j = \sum_{j=0}^{K-1} P_j \rho_x P_j^\dagger$$

Let $\{|\alpha_i\rangle\}$ be an eigenbasis of the operator O that contains the result of the measurement of $|x\rangle$ by P_j as the first K elements of the N-element basis

$$|x\rangle = \sum_{j=0}^{K-1} \frac{P_j|x\rangle}{|P_j|x\rangle|} = \sum_{i=0}^{N-1} x_i|\alpha_i\rangle$$

$$x_i = \begin{cases} \sqrt{p_i} & i < K \\ 0 & i \geq K \end{cases}$$

Measurement of density operators

We can now write the density operator ρ_x in terms of this eigenbasis

$$\rho_x = |x\rangle\langle x| = \left(\sum_{i=0}^{N-1} x_i|\alpha_i\rangle\right)\left(\sum_{j=0}^{N-1} x_j|\alpha_j\rangle\right)^\dagger = \sum_{j=0}^{N-1}\sum_{i=0}^{N-1} x_i\overline{x_j}|\alpha_i\rangle\langle\alpha_j|$$

The ij^{th} entry of the matrix ρ_x in the $\{|\alpha_k\rangle\}$ basis is just $\overline{x_i}x_j$

The density operator ρ_x^O is thus

$$\rho_x^O = \sum_{j=0}^{N-1} x_j\overline{x_j}|\alpha_j\rangle\langle\alpha_j| = \sum_{j=0}^{N-1} P_j|x\rangle\langle x|P_j^\dagger$$

So ρ_x^O in the $\{|\alpha_k\rangle\}$ basis is simply ρ_x with all the off-diagonal elements set to zero

Measurement of mixed states

Let ρ be a density operator representing a mixed state which can be written as a probabilstic mixture of pure states $|\psi_i\rangle$

$$\rho = \sum_i q_i |\psi_i\rangle\langle\psi_i|$$

The outcomes of measuring the mixed state can be written as a probabilistic mixture of the density operators

$$\rho_i' = \sum_j P_j |\psi_i\rangle\langle\psi_i| P_j^\dagger$$

The density operator for the possible outcomes of measuring the mixed state ρ is

$$\rho' = \sum_i q_i \sum_j P_j |\psi_i\rangle\langle\psi_i| P_j^\dagger = \sum_j P_j \left(\sum_i q_i |\psi_i\rangle\langle\psi_i| \right) P_j^\dagger = \sum_j P_j \rho P_j^\dagger$$

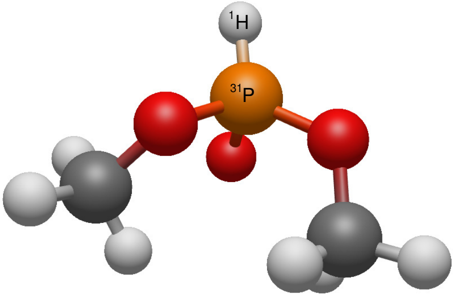
The term $P_j \rho P_j^\dagger$ is not necessarily a density operator but is positive and Hermitian with a trace $\mathrm{Tr}(P_j \rho P_j^\dagger) \leq 1$

ρ' may be viewed as a probabilistic mixture of density operators $\rho_j = P_j \rho P_j^\dagger / \mathrm{Tr}(P_j \rho P_j^\dagger)$ with weighting $p_j = \mathrm{Tr}(P_j \rho P_j^\dagger)$

$$\rho' = \sum_j p_j \rho_j = \sum_j p_j \frac{P_j \rho P_j^\dagger}{\mathrm{Tr}(P_j \rho P_j^\dagger)}$$

SpinQ Gemini

In 2021 the Chinese company SpinQ announced a commercial, low cost 2-qubit NMR quantum computer using hydrogen and phosphorous nuclei

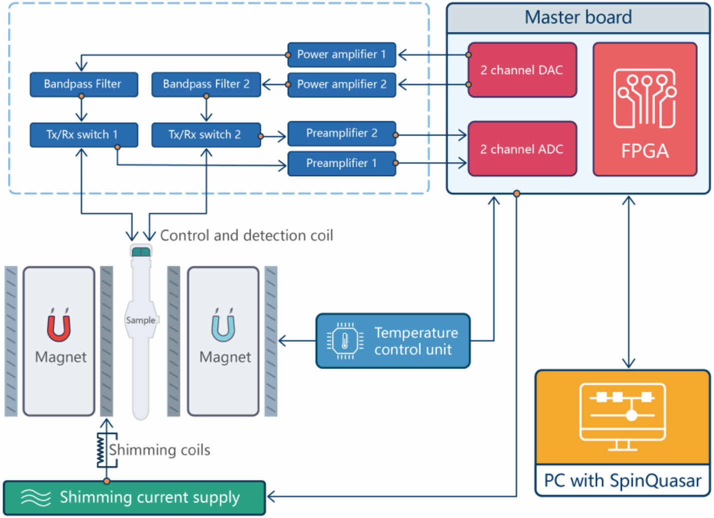


With 11 single-qubit gates and 4 two-qubit gates, it is possible to demonstrate at least 10 quantum algorithms

In 2022, the even lower cost Gemini Mini was announced along with a 3-qubit system

“SpinQ Gemini: a desktop quantum computer for education and research,” Sh.-Y. Hao et al., *arXiv* 10017 (2021).

SpinQ Gemini



"SpinQ Gemini: a desktop quantum computer for education and research," Sh.-Y. Hao et al., *arXiv* 10017 (2021).

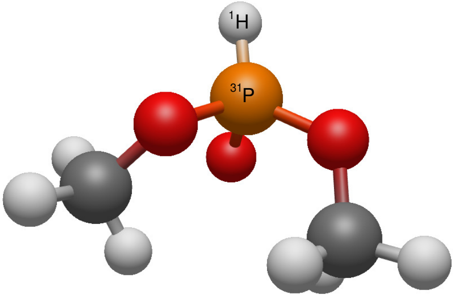
SpinQ Gemini

The two Larmor frequencies in the system are 17.2 MHz for ^{31}P and 46.2 MHz for ^{1}H

The coupling between the two spins is $J = 697.4$ Hz with interaction Hamiltonian

$$H_0 = \frac{\pi}{2} J \sigma_x^H \sigma_z^P$$

90 degree rotation gates can be implemented by 20 μs (^{31}P) and 10 μs (^{1}H) square pulses at the respective Larmor frequencies



SpinQ Gemini

With appropriate application of square pulse sequences, the following single-qubit gates can be implemented

$$X = \sigma_x \quad , \quad Y = \sigma_y \quad , \quad Z = \sigma_z$$

$$X90 = e^{-i\frac{\pi}{4}\sigma_x} \quad , \quad Z90 = e^{-i\frac{\pi}{4}\sigma_y} \quad , \quad Z90 = e^{-i\frac{\pi}{4}\sigma_z}$$

$$Rx = e^{-i\frac{\alpha}{2}\sigma_x} \quad , \quad Ry = e^{-i\frac{\alpha}{2}\sigma_y} \quad , \quad Rz = e^{-i\frac{\alpha}{2}\sigma_z}$$

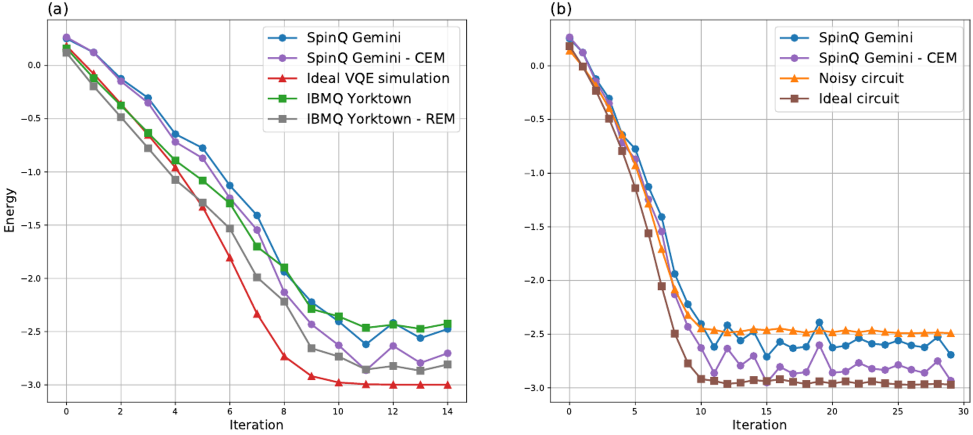
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & \bar{1} \end{pmatrix} \quad , \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

There are four two-qubit gates as well

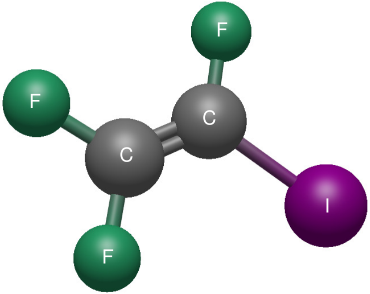
$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; CY = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \bar{i} \\ 0 & 0 & i & 0 \end{pmatrix}, \; CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \bar{1} \end{pmatrix}, \; D = e^{-itH_0}$$

SpinQ Gemini

SpinQ Gemini compared to to IBMQ (2 qubits) in the variational quantum solver algorithm to compute the ground state of a two-qubit system



SpinQ Triangulum



The three qubits are the ^{19}F atoms in the C_2F_3I molecule whose Larmor frequencies are close to each other, simplifying the RF stage of the Triangulum compared to the Gemini

"SpinQ Triangulum: a commercial three-qubit desktop quantum computer," G. Feng et al., *arXiv* 2983 (2022).