

PHYS 427 - Thermal and Statistical Physics - Discussion 08 - Solutions

1. **Phonons in one dimension:** In lecture, you saw that the Debye model accurately predicts the heat capacity of solids for the high and low temperature limits. You showed that, in 3D, $C_V \propto T^3$ at low temperature. In this problem, you will find the dependence of the heat capacity on the temperature for a one-dimensional system.

- (a) What is the density of states in one dimension? How does it differ from $D(\omega)$ in 3D? How many phonon polarizations are there?
- (b) Calculate the internal energy of this system for the low temperature and the high temperature limits. Find the Debye temperature in the process.
- (c) Finally, calculate the heat capacity at constant volume, C_V , for each limit.

By definition, the density of states is the number of modes with frequencies between ω and $\omega + d\omega$. It is then given by

$$D(\omega)d\omega = \# \text{ polarizations} \times \frac{\text{volume in k-space with } k = \omega/c_s}{\text{volume in k-space per state}}. \quad (1)$$

(a) In one dimension, there is only the longitudinal direction for the atoms to oscillate in, so there is only 1 phonon polarization mode. Additionally, the "volume" in one dimension is simply a length. The length in k-space with $k = \omega/c_s$ for some ω is just dk . Then

$$D(\omega)d\omega = 1 \times \frac{dk}{\pi/L} = \frac{L}{\pi c_s} d\omega. \quad (2)$$

Note that $D(\omega)$ for the 1D case is constant for any ω . This is a peculiarity of the one-dimensional case, and in general, like in the 3D model done in class, $D(\omega)$ does depend on some power of ω .

Why did we not divide the length element dk by 2, if we were dividing the volume element by 8 in 3D? We divide it by 8 in the 3D case because we had changed variables from cartesian to spherical coordinates, but needed to keep only the positive octant of the sphere. In one dimension, we are keeping the cartesian coordinates and already only integrating over positive $k = \pi/L, 2\pi/L, 3\pi/L, \dots$. Therefore, we do not need to divide by anything.

(b) There are $1 \times N$ phonon modes, where N is the number of atoms in the system. Then

$$U = \sum_{n=1}^N \frac{\hbar\omega_n}{e^{\beta\hbar\omega_n} - 1} = \int_0^{\omega_D} d\omega D(\omega) \frac{\hbar\omega_n}{e^{\beta\hbar\omega_n} - 1} \quad (3)$$

$$= \frac{\hbar L}{\pi c_s} \int_0^{\omega_D} \frac{\omega}{e^{\beta\hbar\omega} - 1} d\omega. \quad (4)$$

We must find the Debye frequency ω_D such that the total number of modes adds up to N :

$$\int_0^{\omega_D} d\omega D(\omega) = N. \quad (5)$$

Recall that $\omega_D = c_s k_D$. The integral is the length (volume in one dimension) in “ ω ”-space that includes N modes. We know that there is one mode per π/L length in k -space, so it is convenient to work directly with momentum and convert back to ω later. Then

$$N = \frac{k_D}{\pi/L} = \frac{L}{\pi c_s} \omega_D \implies \omega_D = \frac{c_s \pi}{L} N. \quad (6)$$

To solve the integral

$$\int_0^{\omega_D} \frac{\omega}{e^{\beta \hbar \omega} - 1} d\omega, \quad (7)$$

we first change variables to $x = \beta \hbar \omega$:

$$\frac{(k_B T)^2}{\hbar^2} \int_0^{x_D} \frac{x}{e^x - 1} dx. \quad (8)$$

The upper bound can be written as a ratio of temperatures:

$$x_D = \beta \hbar \omega_D = \frac{\hbar c_s \pi N}{L k_B T} = \frac{\Theta_D}{T}, \quad (9)$$

where $\Theta_D = \frac{\hbar c_s \pi N}{L k_B}$ is the Debye temperature. Putting everything together,

$$U = \frac{\hbar L}{\pi c_s} \frac{(k_B T)^2}{\hbar^2} \int_0^{\frac{\Theta_D}{T}} \frac{x}{e^x - 1} dx. \quad (10)$$

Now we take the limits. In the high temperature limit, $x = \frac{\hbar \omega}{k_B T} \rightarrow 0$, and

$$\lim_{x \rightarrow 0} \frac{x}{e^x - 1} = 1. \quad (11)$$

So we are just integrating over a constant, and we get

$$U_{\text{high } T} = \frac{\hbar L}{\pi c_s} \frac{(k_B T)^2}{\hbar^2} \frac{\Theta_D}{T} = N k_B T. \quad (12)$$

In the low temperature limit, on the other hand, $x_D = \Theta_D/T \rightarrow \infty$, so we get

$$U_{\text{low } T} = \frac{\hbar L}{\pi c_s} \frac{(k_B T)^2}{\hbar^2} \int_0^\infty \frac{x}{e^x - 1} dx \quad (13)$$

$$= \frac{\hbar L}{\pi c_s} \frac{(k_B T)^2}{\hbar^2} \frac{\pi^2}{6} \propto T^2 \quad (14)$$

(you can just look up the value of the integral, which is now completely dimensionless).

(c) Finally, we can calculate the heat capacity $C_V = \left(\frac{\partial U}{\partial T}\right)_V$:

$$C_{V, \text{high } T} = N k_B, \quad (15)$$

$$C_{V, \text{low } T} = \frac{L \pi}{3 c_s} \frac{k_B^2}{\hbar} T. \quad (16)$$

2. **Non-interacting bosons:** Consider a system of identical non-interacting bosons. The bosons can be in many different **single-particle quantum states** which are labelled by an index $m = 0, 1, 2, \dots$. The energy of a single boson in the single-particle state m will be denoted ε_m .

In order to specify a microstate state of the whole system, we just have to specify N_m , the number of particles in the m^{th} single-particle state. Then, for example, the energy of the microstate is $\sum_m \varepsilon_m N_m$. Note that this scheme for specifying microstates automatically takes into account the fact that the bosons are indistinguishable.

Let ε_0 denote the lowest energy level (it may be degenerate).

- (a) Show the grand partition function is

$$\mathcal{Z} = \prod_m \frac{1}{1 - e^{-\beta(\varepsilon_m - \mu)}} \quad (17)$$

as long as the chemical potential obeys $\mu < \varepsilon_0$. What goes wrong if $\mu > \varepsilon_0$?

- (b) Using the result from part (a), show that the number of particles in the system is

$$N = \sum_m f(\varepsilon_m), \quad (18)$$

where

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} \quad (19)$$

is called the **Bose-Einstein** distribution function.

- (c) Sketch the Bose-Einstein distribution function at some positive temperature. Which single-particle state(s) contains the most particles on average?
- (d) Using the result from part (a), show that the energy of the system is

$$U = \sum_m \varepsilon_m f(\varepsilon_m). \quad (20)$$

- (e) Suppose you have actually created a gas of N^* non-interacting bosons in the lab which is in equilibrium at a temperature T . Briefly explain how you could use (18) to solve for μ (although in practice you would need to do it numerically or make further approximations).

Note that you could plug this μ into (20), hence determining $U(T, N^*)$.

This gives another perspective on the meaning of μ . If you have the ability to choose (or measure) the number of particles in your system, μ can be thought of as a value which is tuned such that (18) correctly gives the chosen (or measured) number of particles in the system. Then all other thermodynamic quantities could in principle be expressed in terms of the number of particles instead of μ . However, it is generally much more convenient to continue working with μ .

To be concrete, we can imagine that the system is a collection of non-interacting identical bosons trapped in a one-dimensional harmonic potential. This is routinely done in atomic, molecular and optical physics laboratories. They carefully point a bunch of lasers at a point in space, and manage to create a roughly harmonic potential well (a "trap" with potential $U(x) \sim x^2$). Then they insert bosonic atoms into that trap. Actually, they create a three-dimensional harmonic trap, but we will assume it is one-dimensional for simplicity.

The energy eigenstates of a single particle the trap are, as we know, are labelled by m and have energies $\varepsilon_m = \hbar\omega(m + 1/2)$. We refer to these energy eigenstates as "single-particle states". If we put a bunch of bosons into the trap, they will become distributed between the single-particle states. And because they are bosons (not fermions), there is no limit on how many bosons can be in each single-particle state. A question we would like to answer is, "*how* do the bosons distribute themselves between the single-particle states, on average"? More precisely, what is the average number of bosons we should expect to find in the single-particle state m ?

With this as our motivation, let us start to address the problem at hand. It is very convenient to imagine¹ that the system of bosons in a trap is not isolated, but is in fact in contact with a thermal/particle reservoir. That is, we imagine that there is a large reservoir which can exchange energy with the system in the form of heat, and which can also exchange bosons with the system. In other words, we work in the grand canonical ensemble.

When we take a snapshot of the system ("system" will always refer to the bosons in the trap, not including the reservoir), then we can find it in one of many different microstates i . These microstates differ in their energy E_i (because the system can exchange energy with the reservoir) but also in their particle number N_i (because the system can exchange particles with the reservoir). If the temperature and chemical potential of the reservoir are T and μ , respectively, then

$$p_i = \frac{e^{-\beta(E_i - \mu N_i)}}{\mathcal{Z}} \quad (21)$$

is the probability to find the system to be in the microstate i . As with the canonical ensemble, our first objective is to compute the grand canonical partition function

$$\mathcal{Z} = \sum_i e^{-\beta(E_i - \mu N_i)}, \quad (22)$$

because from \mathcal{Z} we can compute nice things like the total average energy and the total average number of particles by taking derivatives:²

$$N = \frac{1}{\beta} \left(\frac{\partial \ln \mathcal{Z}}{\partial \mu} \right)_\beta \quad U - \mu N = - \left(\frac{\partial \ln \mathcal{Z}}{\partial \beta} \right)_\mu. \quad (23)$$

In order to evaluate (22) for our system of trapped non-interacting bosons, we need to think about how to represent the microstate i . The key is this: if I tell you that there are N_0 particles in the single-particle state $m = 0$ and N_1 particles in the single-particle

¹See footnote 3 in discussion 7 for a bit more discussion of this point.

²You can prove these formulas by starting with the definitions $U = \sum_i p_i E_i$ and $N = \sum_i p_i N_i$ and inserting p_i —try it!

state $m = 1$ and N_2 particles in the single-particle state $m = 2$ and so on, then I will have completely specified the microstate of the system. In other words, specifying the microstate i is equivalent to specifying the values of all the *occupation numbers* N_m :

$$i \leftrightarrow \{N_0, N_1, N_2, \dots\}. \quad (24)$$

So when we see \sum_i , we should replace that with a summation over all the possible values of N_0 , all the possible values of N_1 , etc. In principle we could find any number of bosons in the m th single-particle state—they are bosons, so there is no exclusion principle!—so N_m ranges from 0 to ∞ . That is,

$$\sum_i \rightarrow \sum_{N_0=0}^{\infty} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \dots \quad (25)$$

Also, we can write

$$E_i = \sum_m N_m \varepsilon_m \quad N_i = \sum_m N_m. \quad (26)$$

Note there is a clash of notation here, because I made a poor choice earlier. N_i refers to the number of bosons in the system when it's in microstate i , whereas N_m refers to the number of bosons in the single-particle state m .

(a) Finally, we can start to put these pieces together to compute \mathcal{Z} . We have

$$\mathcal{Z} = \sum_i e^{-\beta(E_i - \mu N_i)} \quad (27)$$

$$= \left(\sum_{N_0=0}^{\infty} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \dots \right) e^{-\beta(\sum_m N_m \varepsilon_m - \mu \sum_m N_m)} \quad (28)$$

$$= \left(\sum_{N_0=0}^{\infty} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \dots \right) e^{\sum_m [-\beta(\varepsilon_m - \mu) N_m]} \quad (29)$$

$$= \left(\sum_{N_0=0}^{\infty} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \dots \right) e^{\sum_m [-\beta(\varepsilon_m - \mu) N_m]} \quad (30)$$

$$= \left(\sum_{N_0=0}^{\infty} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \dots \right) e^{-\beta(\varepsilon_0 - \mu) N_0 - \beta(\varepsilon_1 - \mu) N_1 - \beta(\varepsilon_2 - \mu) N_2 - \dots} \quad (31)$$

$$= \left(\sum_{N_0=0}^{\infty} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \dots \right) e^{-\beta(\varepsilon_0 - \mu) N_0} e^{-\beta(\varepsilon_1 - \mu) N_1} e^{-\beta(\varepsilon_2 - \mu) N_2} \dots \quad (32)$$

$$= \left(\sum_{N_0=0}^{\infty} e^{-\beta(\varepsilon_0 - \mu) N_0} \right) \left(\sum_{N_1=0}^{\infty} e^{-\beta(\varepsilon_1 - \mu) N_1} \right) \left(\sum_{N_2=0}^{\infty} e^{-\beta(\varepsilon_2 - \mu) N_2} \right) \dots \quad (33)$$

In the last step, I used the fact that $e^{-\beta(\varepsilon_0 - \mu) N_0}$ doesn't depend on N_1 or N_2 , etc., to slide that factor through the \sum_{N_1} , \sum_{N_2} , etc. And similarly for $e^{-\beta(\varepsilon_1 - \mu) N_1}$ and so on.

Each of the infinite sums in parentheses can now be evaluated as long as $\mu < \varepsilon_0$. In this case, $e^{-\beta(\varepsilon_0 - \mu)} < 1$ and $e^{-\beta(\varepsilon_1 - \mu)} < 1$ (because $\varepsilon_1 \geq \varepsilon_0$) and so on, and we can use the geometric series result $\sum_{N=0}^{\infty} x^N = 1/(1-x)$ which is valid whenever $|x| < 1$. We get

$$\mathcal{Z} = \frac{1}{1 - e^{-\beta(\varepsilon_0 - \mu)}} \frac{1}{1 - e^{-\beta(\varepsilon_1 - \mu)}} \frac{1}{1 - e^{-\beta(\varepsilon_2 - \mu)}} \dots \quad (34)$$

$$= \prod_m \frac{1}{1 - e^{-\beta(\varepsilon_m - \mu)}}. \quad (35)$$

Phew! Let's take stock. We didn't actually need to assume the trap was harmonic. That was just to guide your intuition. We did assume that the particles were identical bosons (we allowed N_m to be anything from 0 to ∞), and we assumed that they were non-interacting (so that $E_i = \sum_m N_m \varepsilon_m$). So this result is valid for any collection of non-interacting identical bosons distributed among any collection of single-particle states ε_m . We will derive a similar result for fermions in the next discussion.

(b) From (23),

$$N = \frac{1}{\beta} \left(\frac{\partial}{\partial \mu} \ln \mathcal{Z} \right)_{\beta} \quad (36)$$

$$= -\frac{1}{\beta} \left[\frac{\partial}{\partial \mu} \sum_m \ln(1 - e^{-\beta(\varepsilon_m - \mu)}) \right]_{\beta} \quad (37)$$

$$= -\frac{1}{\beta} \sum_m \frac{1}{1 - e^{-\beta(\varepsilon_m - \mu)}} (0 - e^{-\beta(\varepsilon_m - \mu)} \beta) \quad (38)$$

$$= \sum_m \frac{1}{e^{\beta(\varepsilon_m - \mu)} - 1} \quad (39)$$

$$= \sum_m f(\varepsilon_m). \quad (40)$$

(c) Figure 1 shows the Bose-Einstein distribution function at a few different (positive) temperatures. One can see the function is larger for smaller energies (it diverges as $\varepsilon \rightarrow \mu$). Notice, from Eq. 40, the function $f(\varepsilon_m)$ tells us the average number of particles in state m ; hence, the lower a state's energy level, the more particles it will contain.

In particular the ground state (which could be degenerate) is the most occupied state.

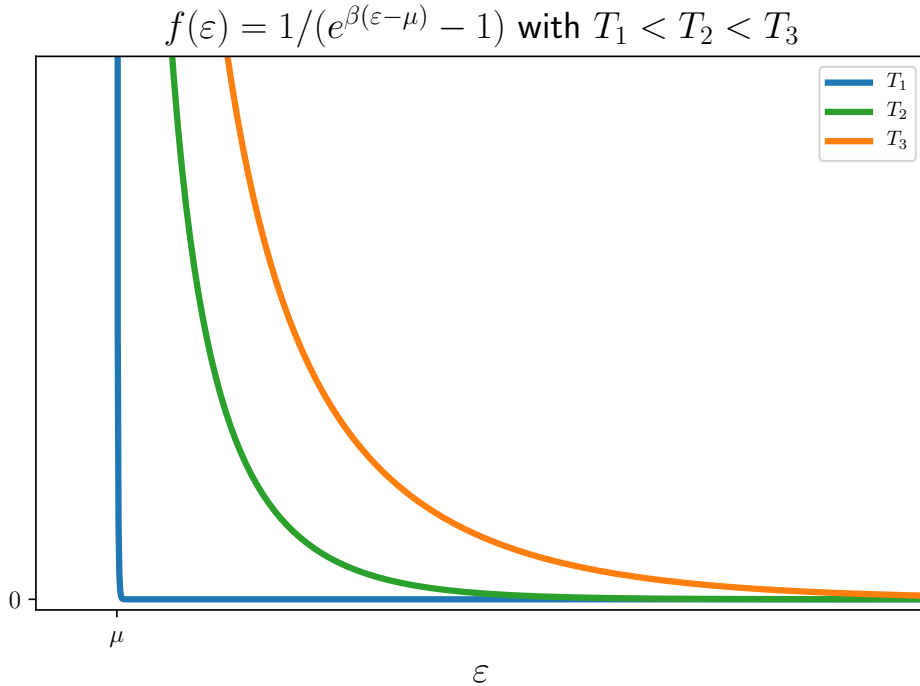


Figure 1

(d) From (23),

$$U = - \left(\frac{\partial}{\partial \beta} \ln \mathcal{Z} \right)_\mu + \mu N \quad (41)$$

$$= \left[\frac{\partial}{\partial \beta} \sum_m \ln(1 - e^{-\beta(\varepsilon_m - \mu)}) \right]_\mu + \mu \sum_m f(\varepsilon_m) \quad (42)$$

$$= \sum_m \left[\frac{1}{1 - e^{-\beta(\varepsilon_m - \mu)}} (0 + (\varepsilon_m - \mu) e^{-\beta(\varepsilon_m - \mu)}) + \mu f(\varepsilon_m) \right] \quad (43)$$

$$= \sum_m \left[(\varepsilon_m - \mu) \frac{1}{e^{\beta(\varepsilon_m - \mu)} - 1} + \mu f(\varepsilon_m) \right] \quad (44)$$

$$= \sum_m (\varepsilon_m - \mu + \mu) f(\varepsilon_m) \quad (45)$$

$$= \sum_m \varepsilon_m f(\varepsilon_m) \quad (46)$$

3. **Non-interacting ultra-relativistic gas:** Consider a collection of massive spin-1 particles in two dimensions confined to a square of area $A = L^2$. Suppose the particles are moving very quickly; in this so-called "ultra-relativistic" limit, the relation between a particle's energy and wavevector is $\varepsilon_{\vec{k}} = c\hbar|\vec{k}|$, where c is the speed of light and the wavevector is quantized as $\vec{k} = \pi\vec{n}/L$ with $n_x, n_y = 1, 2, \dots, \infty$.

- (a) Show that when L is sufficiently large we can approximate a sum over single-particle eigenstates by an integral over energy, i.e. show

$$\sum_{m_s=-1}^1 \sum_{\vec{k}} F(\varepsilon_{\vec{k}}) \rightarrow \frac{3}{2\pi} \frac{L^2}{(\hbar c)^2} \int_0^\infty d\varepsilon \varepsilon F(\varepsilon) \quad (47)$$

for any function F . It is customary to define the so-called *density of states*

$$\mathcal{D}(\varepsilon) = \frac{3}{2\pi} \left(\frac{L}{\hbar c} \right)^2 \varepsilon, \quad (48)$$

so that the result can be written as

$$\sum_{m_s=-1}^1 \sum_{\vec{k}} F(\varepsilon_{\vec{k}}) \rightarrow \int_0^\infty d\varepsilon \mathcal{D}(\varepsilon) F(\varepsilon). \quad (49)$$

Explain why $\mathcal{D}(\varepsilon)$ is called the density of states.

Note: you have calculated the density of states of ultrarelativistic massive spin-1 particles in 2 dimensions. For other dispersion relations in different number of dimensions, the expression for $\mathcal{D}(\varepsilon)$ will look different from (48).

- (b) Consider the special case where $\mu \rightarrow 0$. Using your result from part (a), compute the energy of the system as a function of T in the infinite-area limit.
- First, *without evaluating the integral all the way*, show that $U(T) \propto T^3$.
 - Then evaluate the integral to find the exact constant of proportionality, namely

$$U(T) = \frac{3}{2\pi} \frac{L^2}{(\hbar c)^2} (k_B T)^3 \Gamma(3) \zeta(3), \quad (50)$$

where the gamma function is defined as

$$\Gamma(s) = \int_0^\infty dt \, t^{s-1} e^{-t} \quad (51)$$

and the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (52)$$

Hint: At some point in the calculation, use $\frac{1}{1-e^{-x}} = \sum_{n=0}^{\infty} e^{-nx}$. Why is this true?

(a) We actually worked through this procedure in discussion 4! Recalling the wavevectors are quantized as $\vec{k} = (\pi/L)\vec{n}$ we have that the smallest spacing between two wavevectors is $\Delta k_i = \pi/L$ for $i = x, y$. Then inserting a fancy version of 1 gives

$$\sum_{m_s=-1}^1 \sum_{\vec{k}} F(\varepsilon_{\vec{k}}) = 3 \sum_{\vec{k}} F(\varepsilon_{\vec{k}}) \times 1 \quad , \quad 1 = \frac{L}{\pi} \Delta k_x \cdot \frac{L}{\pi} \Delta k_y \quad (53)$$

$$= 3 \left(\frac{L}{\pi} \right)^2 \sum_{k_x} \Delta k_x \sum_{k_y} \Delta k_y F(\varepsilon_{\vec{k}}) \quad (54)$$

$$\rightarrow 3 \left(\frac{L}{\pi} \right)^2 \int_0^\infty dk_x \int_0^\infty dk_y F(\varepsilon_{\vec{k}}) \quad , \quad \text{as } L \rightarrow \infty \quad (55)$$

The passage to the integral in the last line is possible because, as $L \rightarrow \infty$ we have $\Delta k_x, \Delta k_y \rightarrow 0$. So this is just the definition of an integral by a sequence of Riemann sums with decreasing interval size.

If we then use the fact that the dispersion is **isotropic** (i.e. $\varepsilon_{\vec{k}}$ only depends on the magnitude of \vec{k} , not its direction) we can see that $\varepsilon(k_x, k_y) = \varepsilon(k_x, -k_y) = \varepsilon(k_x, -k_y) = \varepsilon(-k_x, -k_y)$, because in each case the argument has the same magnitude. Then we can extend the integral to all of \mathbb{R}^2 :

$$\sum_{m_s=-1}^1 \sum_{\vec{k}} F(\varepsilon_{\vec{k}}) \rightarrow 3 \left(\frac{L}{2\pi} \right)^2 \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y F(\varepsilon_{|\vec{k}|}) \quad (56)$$

$$= 3 \left(\frac{L}{2\pi} \right)^2 2\pi \int_0^\infty dk \, k F(\varepsilon_k) \quad (57)$$

where in the last line we went to polar coordinates and integrated over the polar angle. Making a change of variables to $\varepsilon = c\hbar k$ gives the final result:

$$\sum_{m_s=-1}^1 \sum_{\vec{k}} F(\varepsilon_{\vec{k}}) \rightarrow \int_0^\infty d\varepsilon \, \mathcal{D}(\varepsilon) F(\varepsilon) \quad (58)$$

where

$$\mathcal{D}(\varepsilon) = \frac{3}{2\pi} \frac{L^2}{(\hbar c)^2} \varepsilon. \quad (59)$$

If you trace back through the derivation, you can see that $\mathcal{D}(\varepsilon)d\varepsilon$ is equal to the number of quantum states whose energies lie within the interval $(\varepsilon, \varepsilon + d\varepsilon)$. That's why $\mathcal{D}(\varepsilon)$ is

called the density of states. It's useful precisely because it appears in (58). The right-hand side is almost always easier to evaluate than the left-hand side.

(b)(i) From question 1, we know the energy is given by

$$U = \sum_{m_s=-1}^1 \sum_{\vec{k}} f(\varepsilon_{\vec{k}}) \varepsilon_{\vec{k}} \quad (60)$$

$$\sim \int_0^\infty d\varepsilon \mathcal{D}(\varepsilon) f(\varepsilon) \varepsilon \quad (61)$$

$$= \frac{3}{2\pi} \frac{L^2}{(\hbar c)^2} \int_0^\infty d\varepsilon \frac{\varepsilon^2}{e^{\beta\varepsilon} - 1}, \quad \text{since } \mu = 0. \text{ Now let } x = \beta\varepsilon \quad (62)$$

$$= \frac{3}{2\pi} \frac{L^2}{(\hbar c)^2} (k_B T)^3 \int_0^\infty dx \frac{x^2}{e^x - 1} \quad (63)$$

The integral over x doesn't depend on any of the parameters of the problem. It's just some pure numerical factor. The interesting part is the other stuff. We see that the energy of the system scales as $U \propto T^3$.

(b)(ii)

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = \int_0^\infty dx e^{-x} x^2 \frac{1}{1 - e^{-x}} \quad (64)$$

$$= \int_0^\infty dx x^2 e^{-x} \sum_{n=0}^\infty e^{-nx} \quad \text{geometric series, note } e^{-x} < 1 \quad (65)$$

$$= \sum_{n=0}^\infty \int_0^\infty dx x^2 e^{-(n+1)x} \quad (66)$$

$$= \sum_{m=1}^\infty \int_0^\infty dx x^2 e^{-mx} \quad m = n + 1 \quad (67)$$

$$= \sum_{m=1}^\infty \frac{1}{m^3} \int_0^\infty du u^2 e^{-u} \quad u = mx \quad (68)$$

$$= \zeta(3) \Gamma(3). \quad (69)$$

The Gamma function is simply $\Gamma(3) = 2! = 2$ and the Riemann zeta function is $\zeta(3) \approx 1.202$. Hence the energy of the system is

$$U = \frac{3\zeta(3)}{\pi} \frac{L^2 (k_B T)^3}{(\hbar c)^2} \approx \frac{3.606}{\pi} \frac{L^2 (k_B T)^3}{(\hbar c)^2}. \quad (70)$$

To emphasize, usually it's not necessary (or very interesting) to know the exact value of the numerical prefactor. Often we are more interested in *how* U depends on T , which we figured out in (i) with much less effort.