PHYS 427 - Thermal and Statistical Physics - Discussion 09

April 1, 2025

Last time, we started to study systems of non-interacting identical particles, also known as *quantum gases*. In particular, we studied the bosonic quantum gas. Today, we will discuss the fermionic quantum gas. This happens to be a pretty good model for the free electrons in a metal[1].

Recall that fermions are particles that obey the **Pauli exclusion principle**: no two identical fermions can occupy the same quantum state[2].

Furthermore, recall that the **spin-statistics theorem** (which we won't derive in this course) tells us that fermions are particles with half-integer spin. The electron (spin-1/2) is a fermionic fundamental particle[3].

Finally, recall that in the grand canonical ensemble one can compute average energy U and particle number N by taking the following partial derivatives of the grand partition function.

$$N = \frac{1}{\beta}\left(\frac{\partial \ln \mathcal{Z}}{\partial \mu}\right)_\beta \qquad\qquad U - \mu N = -\left(\frac{\partial \ln \mathcal{Z}}{\partial \beta}\right)_\mu . \tag{1}$$

[1]In the last part of the last problem in discussion 7, we used a *classical* model for the free electrons in a metal. We said it was not a good approximation, but that we would do better soon. Now we are doing better.

[2]For example, if we put a bunch of non-interacting identical fermions into a one-dimensional harmonic well, then no two particles can ever have the same value of n.

[3]There are also composite particles that are fermions, like a lithium-6 atom (spin-1/2). Protons and neutrons are spin-1/2 composite fermions made of quarks. The Delta baryon is a more exotic composite particle made of quarks. It has spin-3/2, which is a half integer, so it's a fermion too.

1. **Non-interacting fermions**: Consider a system of identical non-interacting fermions. (This is a good model for the free electrons roaming around inside a metal.) Each fermion can occupy many different **single-particle quantum states** which are labelled by an index $m = 0, 1, 2, \ldots$. The energy of a fermion in the single-particle state m will be denoted ε_m.

   In order to specify a microstate state of the whole system, we just have to specify N_m, the number of particles occupying the m^{th} single-particle state. Then, for example, the energy of the microstate is $\sum_m \varepsilon_m N_m$. Note that N_m can only be 0 or 1, because of the Pauli exclusion principle.

   (a) Show that the grand partition function becomes

   $$\mathcal{Z} = \prod_m \left(1 + e^{-\beta(\varepsilon_m - \mu)}\right). \tag{2}$$

   (b) Using the result from part (a), show that the number of particles in the system is

   $$N = \sum_m f(\varepsilon_m), \tag{3}$$

   where

   $$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \tag{4}$$

   is the **Fermi-Dirac** distribution function.

   (c) Sketch the Fermi-Dirac distribution function at some positive temperature.

   i. Is the function consistent with the Pauli-exclusion principle?

   ii. What happens to the graph of the function if you change the value of the chemical potential?

   iii. Which single-particle states have a greater than 50% chance of being occupied?

   (d) Using the result from part (a), show that the energy of the system is

   $$U = \sum_m \varepsilon_m f(\varepsilon_m). \tag{5}$$

   (e) Suppose you have actually created a gas of N^* non-interacting fermions in the lab which is in equilibrium at a temperature T (perhaps these are electrons in a metal). Convince yourself that you could use (3) to solve for μ (although in practice you would need to do it numerically, or make further approximations as in the next problem).

   Note that you could plug this μ into (5), hence determining $U(T, N^*)$.

   This gives another perspective on the meaning of μ. If you have the ability to choose (or measure) the number of particles in your system, μ can be thought of as a value which is tuned such that (3) correctly gives the chosen (or measured) number of particles in the system. Then all other thermodynamic quantities could in principle be expressed in terms of the number of particles instead of μ. However, it is typically much more convenient to continue working with μ.

2. **Non-interacting ultra-relativistic gas**: Consider a collection of massive spin–3/2 particles in three dimensions confined to a square of volume $V = L^3$. Suppose the particles are moving very close to the speed of light; in this so-called "ultra-relativistic" limit, the relation between a particle's energy and wavevector is $\varepsilon_{\vec{k}} = c\hbar|\vec{k}|$, where c is the speed of light and the wavevector is quantized as $\vec{k} = \pi\vec{n}/L$ with $n_x, n_y, n_z = 1, 2, \ldots, \infty$.

   (a) Show that when L is sufficiently large we can approximate a sum over single-particle eigenstates by an integral over energy, i.e. show

   $$\sum_{m_s=-3/2}^{3/2} \sum_{\vec{k}} F(\varepsilon_{\vec{k}}) \to \frac{2}{\pi^2} \frac{L^3}{(\hbar c)^3} \int_0^\infty d\varepsilon\, \varepsilon^2 F(\varepsilon) \tag{6}$$

   for any function F. It is customary to define the so-called *density of states*

   $$\mathcal{D}(\varepsilon) = \frac{2}{\pi^2} \left(\frac{L}{\hbar c}\right)^3 \varepsilon^2, \tag{7}$$

   so that the result can be written as

   $$\sum_{m_s=-3/2}^{3/2} \sum_{\vec{k}} F(\varepsilon_{\vec{k}}) \to \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) F(\varepsilon). \tag{8}$$

   Explain why $\mathcal{D}(\varepsilon)$ is called the density of states.

   Note: you have calculated the density of states of ultrarelativistic massive spin-3/2 particles in 3 dimensions. For other dispersion relations in different number of dimensions, the expression for $\mathcal{D}(\varepsilon)$ will look different from (7).

   (b) Consider cooling down the system to absolute zero, $T = 0$. By inspecting the Fermi-Dirac distribution function, convince yourself that all single-particle states with energy less than μ are occupied, while all single-particle states with energy greater than μ are unoccupied. The Fermi energy ε_F is defined as the highest-energy single-particle state occupied at $T = 0$. In other words, $\varepsilon_F = \mu(T = 0)$.

   (c) Using (3) and (8), compute the Fermi energy as a function of the particle density $n = N/V$. *Hint: remember $T = 0$; use the result from (b) to simplify the integral. Answer: $\varepsilon_F = \hbar c(3\pi^2 n/2)^{1/3}$*

   (d) Compute the ground state energy of the system using (5) and (8). Write your answer in terms of the number of particles and Fermi energy. *Answer: $U = \frac{4}{3} N\varepsilon_F$. Does this make physical sense?*

   (e) Using $p = -(\partial F/\partial V)_{T,N}$ and noting that $F = U$ when $T = 0$, use your answer from part (d) to compute the pressure of the fermi gas at absolute zero. *Answer: $p = U/3V$.*

   Note: it may seem weird for the gas to have non-zero pressure at absolute zero temperature, because we expect things to stop "jiggling" at $T = 0$. The remnant pressure at $T = 0$ is called the *degeneracy pressure*, and it is ultimately a consequence of the Pauli exclusion principle. The degeneracy pressure of cold systems of identical fermions is the reason large numbers of particles can form long-lived macroscopic objects. This includes compact astrophysical objects (white dwarfs, neutron stars), which would otherwise collapse into black holes under gravity, but also the ordinary materials found in this room.