Complex Fourier Series

We are going to go through the following progression:

Fourier series $\longrightarrow$ Complex Fourier Series $\longrightarrow$ Fourier Transform

Last semester, our studies of expansions with orthogonal functions included the Fourier series expansion for a repeat interval L,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{L}\right) \tag{1}$$

To go to the more compact complex Fourier series, use the following Euler relations:

$$\begin{aligned} \cos\left(\frac{2\pi n x}{L}\right) &= \frac{1}{2}\left(e^{i\frac{2\pi n x}{L}} + e^{-i\frac{2\pi n x}{L}}\right) \\ \sin\left(\frac{2\pi n x}{L}\right) &= \frac{1}{2i}\left(e^{i\frac{2\pi n x}{L}} - e^{-i\frac{2\pi n x}{L}}\right) \end{aligned} \tag{2}$$

Substituting Eq. 2 into Eq. 1 and using $-i = \frac{1}{i}$,

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)e^{i\frac{2\pi n x}{L}} + \sum_{n=1}^{\infty}(a_n + ib_n)e^{-i\frac{2\pi n x}{L}} \\ &= \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{i\frac{2\pi n x}{L}} = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{ink_0 x} \end{aligned} \tag{3}$$

where $k_0 \equiv \frac{2\pi}{L}$, and the summation is now from $-\infty$ to ∞ instead of 0 to ∞ to pick up both the positive and negative exponentials. The new coefficient $\tilde{c}_n$ is complex, and maps to the a_n, b_n as follows,

$$c_0 = \frac{1}{2}a_0$$

$$\tilde{c}_{-|n|} = \frac{1}{2}(a_{|n|} + ib_{|n|})$$

$$\tilde{c}_{|n|} = \frac{1}{2}(a_{|n|} - ib_{|n|})$$

The coefficients $\tilde{c}_n$ may be found by multiplying both sides of Eq. 3 by the orthogonal function, e^{-imk_0x}. The basis functions are complex, and their complex conjugates are the orthogonal functions that provide an orthogonality condition.

Reminder: $e^a e^b = e^{(a+b)}$

Finding $\tilde{c}_n$: what doesn't work for exponential Fourier series

Using the basis functions themselves rather than their complex conjugates does not result in a valid orthogonality condition. In this case, when $n = m$ the 'orthogonality' integral is zero instead of a constant value as required. Demonstrating this:

$$\begin{aligned}
\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{ink_0x} e^{ink_0x}\,dx &= \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{i2nk_0x}\,dx \\
&= \frac{1}{2nk_0}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{i2nk_0x}\,d(2nk_0x) \\
&= \frac{1}{2nk_0}\left[\int_{-2\pi n}^{2\pi n} \cos{(u)}\,du + i\int_{-2\pi n}^{2\pi n} \sin{(u)}\,du\right] \\
&= 0
\end{aligned}$$

Finding $\tilde{c}_n$: what does work for exponential Fourier series

Now instead, use the complex conjugate of the basis functions for the orthogonality integral. First, check that the orthogonality integral yields a constant when $m = n$.

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{i(n-n)k_0 x}\, dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx = L$$

While for the case $m \neq n$: the orthogonality integral is the following:

$$\begin{aligned}
\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{i(n-m)k_0 x}\, dx &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left((n-m)k_0 x\right) dx + i \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left((n-m)k_0 x\right) dx \\
&= \frac{1}{(n-m)k_0}\left[\sin\left(\frac{2\pi(n-m)x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}} - i\cos\left(\frac{2\pi(n-m)x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}\right] \\
&= 0
\end{aligned}$$

The sine terms vanish because when evaluated at the limits, their arguments are some integer multiple of π. The two cosine terms cancel, because they are the same when evaluated at each limit.

Reminder: $\cos(-\theta) = \cos(\theta)$
Also, The Kroneker Delta is defined as:

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

The orthogonality relation for exponential functions is then,

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{ink_0 x} e^{-imk_0 x}\, dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases} \tag{4}$$

Or,

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{ink_0 x} e^{-imk_0 x}\, dx = L\delta_{nm}$$

Using Eq. 4 with Eq. 3 the coefficients of the complex Fourier series are then:

$$\tilde{c}_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-ink_0 x} dx$$

Fourier transforms - extending to an infinite interval

One limitation of using a Fourier series is that there must be a characteristic interval upon which the function repeats. That would be the 'wavelength' or $k_0 = \frac{2\pi}{\lambda_0}$ for functions of position, and the 'period' or $\omega_0 = \frac{2\pi}{T}$ for functions of time. Some functions are not periodic, so the interval of those functions would be infinite. There is no finite repeat period. Extending the interval of a Fourier series to be infinite results in the Fourier transform expressions, which are suitable for non-repetitive functions.

Begin with the complex Fourier series expressions, both for a function of time and position.

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{ink_0x} = \sum_{n=-\infty}^{\infty} e^{ink_0x} \left[\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-ink_0x} dx \right] \\
f(t) &= \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{in\omega_0t} = \sum_{n=-\infty}^{\infty} e^{in\omega_0t} \left[\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0t} dt \right]
\end{aligned}
$$

Extending the time interval means allowing the period to become infinite:

$T \to \infty$	extend the interval
$\frac{2\pi}{T} \to dw$	$T \to \infty$, so, $\omega_0 = \frac{2\pi}{T}$ becomes infinitesimal
$n\omega_0 \to \omega$	discrete harmonics $\to$ continuous function
$\sum_{-\infty}^{\infty} \to \int_{-\infty}^{\infty}$	integrate (not sum) a continuous function

Similarly extending the spatial interval means allowing the wavelength (L here) to become infinite:

$$
\begin{array}{ll}
L \to \infty & \text{extend the interval} \\[1em]
\dfrac{2\pi}{L} \to dk & L \to \infty\text{, so, } k_0 = \dfrac{2\pi}{L} \text{ becomes infinitesimal} \\[1em]
nk_0 \to k & \text{discrete harmonics} \to \text{continuous function} \\[1em]
\displaystyle\sum_{-\infty}^{\infty} \to \int_{-\infty}^{\infty} & \text{integrate (not sum) a continuous function}
\end{array}
$$

Applying these to the summations for $f(t)$ and $f(x)$,

$$
\sum_{n=-\infty}^{\infty} e^{in\omega_0 t}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-in\omega_0 t}dt\right] \longrightarrow \int_{-\infty}^{\infty} e^{i\omega t}\left[\frac{d\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right]
$$
$$
\sum_{n=-\infty}^{\infty} e^{ink_0 x}\left[\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-ink_0 x}dx\right] \longrightarrow \int_{-\infty}^{\infty} e^{ikx}\left[\frac{dk}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\right]
$$

Finally, rearranging a bit:

$$
f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right]e^{i\omega t}d\omega \tag{5}
$$
$$
f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\right]e^{ikx}dk \tag{6}
$$

Notice that the portion of Eq. 5 in square brackets is a function of frequency only, since the time dependence is integrated away. That is,

$$
F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{7}
$$

Similarly, the portion of Eq. 6 in square brackets is a function of wave number (k) only, since the position dependence is integrated away. That is,

$$
F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{8}
$$

These functions, $F(\omega)$ and $F(k)$ are the Fourier Transforms of $f(t)$ and $f(x)$ respectively.

Fourier Transforms

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

Using Eq. 7 in Eq. 5 and Eq. 8 in Eq. 6, the *Inverse Fourier Transforms* are written:

Inverse Fourier Transforms

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ikx}dk$$