Chapter 8 - continued

So far, we have seen local conservation of charge described by a continuity equation;

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$$

Local conservation of energy described by a continuity equation;

$$\frac{\partial (u_{mech} + u_{em})}{\partial t} = -\vec{\nabla} \cdot \vec{S}$$

So, we might expect a continuity equation for local conservation of momentum;

$$\frac{\partial (\vec{p}_{mech} + \vec{p}_{em})}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T}$$

To verify this holds, and to identify p_{em} and $\overleftrightarrow{T}$, the same procedure as was used for the energy equation may be followed.

The term $\frac{\partial \vec{\wp}_{mech}}{\partial t}$ describes the total electromagnetic force acting on charges per volume. Script $\vec{\wp}$ is the momentum density, as opposed to straight $\vec{p}$, which is the symbol for momentum. $\frac{\partial \vec{\wp}_{mech}}{\partial t}$ can be calculated because we know the force that electromagnetic fields exert on charges. If the result of this calculation can be prodded into the form of a continuity equation,

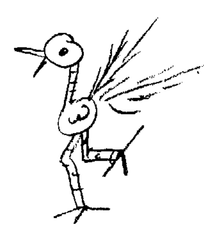
$$\frac{\partial \vec{\wp}_{mech}}{\partial t} = -\frac{\partial \vec{\wp}_{em}}{\partial t} + \vec{\nabla} \cdot \overleftrightarrow{T}$$

then local conservation holds (as we expect it must) and $\vec{\wp}_{em}$ and $\overleftrightarrow{T}$ can be identified.

Energy (and energy density) is a scalar quantity. Momentum (and momentum density) is a vector. The momentum density continuity equation is a vector equation, and so the term describing momentum density flow out of the volume, $\vec{\nabla} \cdot \overleftrightarrow{T}$, must also be a vector for consistency.

The divergence of a vector, $\vec{\nabla} \cdot \vec{v}$, is a scalar. Thus, $\overleftrightarrow{T}$ is not a vector, but a tensor (of 2<u>nd</u> rank), which is why it carries a double-headed rather than single-headed arrow on top. A vector, or vector operator, dotted into a tensor (of 2nd rank) results in a vector. The <u>momentum flux density</u> $\overleftrightarrow{T}$ is also called the 'Maxwell stress tensor'. Normally, we get tensor and tensor as the semester progresses, this time we get to start out tensor right away.

<u>Digression on tensors</u> (starts on next page)



filler bird

Introductory comments on tensors using material from the 'Feynman lectures on Physics' chapter 31. Feynman says; 'The mathematics of tensors is particularly useful for describing properties of substances which vary in direction - . .' Such a substance is called anisotropic. Elastic, optical, electrical, and magnetic properties of solids, viscous liquids, crystals, and magnetised plasmas may be anisotropic.

Flashback to Griffiths chapter 4. (He slipped a tensor in on you on p. 184) Anyway, way back in the olden days, we discussed cases where the polarization of a material was proportional to the applied electric field:

$$\vec{P} = \underbrace{\epsilon_0 \chi_e}_{\text{Feynman calls this } \alpha} \vec{E}$$

$$\vec{P} = \alpha \vec{E}$$

For $\vec{P} = \alpha \vec{E}$, α is a scalar and the implication is that whatever the direction of the applied field, $\vec{E}$, the material responds in the same way: $\vec{P}$ is in the direction of $\vec{E}$ in a proportion given by α. This isotropic situation is not always the case, and in some crystals (for example) it is easier to polarize in some directions than others. In the general case (Griffiths, p.184),

$$P_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$

$$P_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$

$$P_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z$$

The nine coefficients α_{ij} comprise the polarizability tensor, and give the description for how the polarization depends on the applied field for an anisotropic material, allowing for any orientation of applied field relative to the material.

The nine coefficients of a tensor are typically specified in square brackets;

$$\begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}$$

When a tensor is used to relate one vector quantity to another, shorthand notation (using this case as an example) can be used:

$$P_i = \sum_j \alpha_{ij} E_j$$

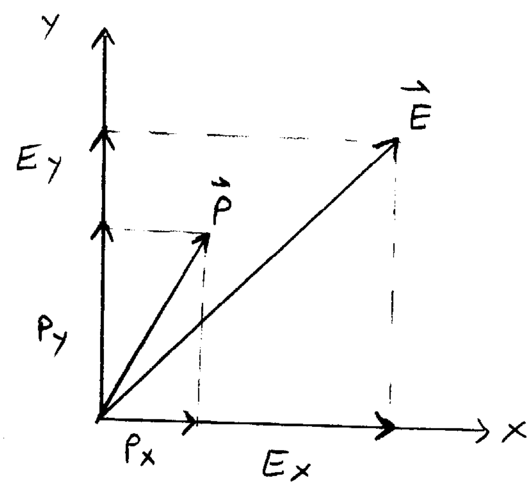
i can be x, y or z

j is summed over $x, y,$ and z

For example, to get the x component of the polarization, let $i = x$:

$$P_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$

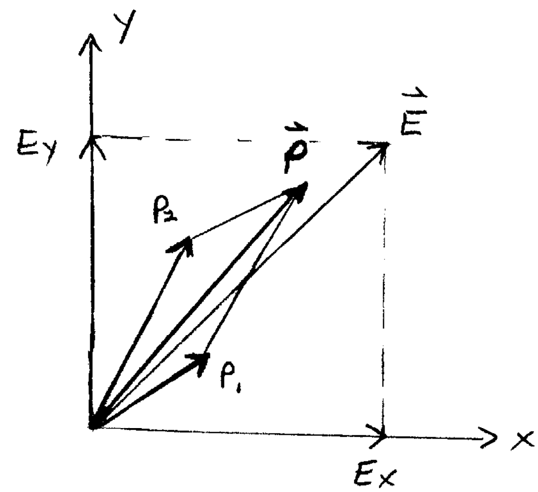
Take Feynman's 2D example. Apply $\vec{E}$ at 45° to x-axis in the xy plane, the magnitude of the projections of the field E_x and E_y are equal. Now let the material be easier to polarize in the y direction:



Now the polarization vector is <u>not</u> in the same direction as the applied $\vec{E}$ field.

Even this picture is a special case, because E_y produced a polarization in the same direction, along the y-axis. In general, the axes of polarization of the material may be rotated with respect to the chosen reference axes.

Then, <u>each</u> component of $\vec{E}$, lets take E_y for example, would produce a polarization with both an x and y component. See the figure on the next page.



P_1 induced by E_x

P_2 induced by E_y

$\vec{P} = \vec{P}_1 + \vec{P}_2$

$\vec{P}$ is the polarization resulting from $\vec{E}$

$$P_x = \alpha_{xx} E_x + \alpha_{xy} E_y = P_{1x} + P_{2x}$$

$$P_y = \alpha_{yx} E_x + \alpha_{yy} E_y = P_{1y} + P_{2y}$$

$\vec{P}_1$ in diagram (caused by E_x)

$\vec{P}_2$ in diagram (caused by E_y)

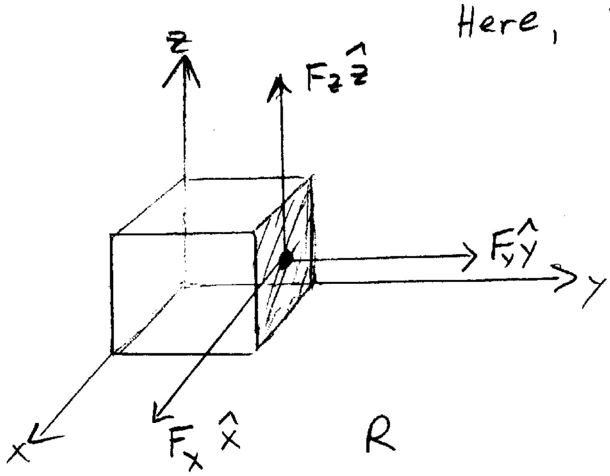
<u>Or, in vectors</u>

$$\vec{P} = P_x \hat{x} + P_y \hat{y}$$

$$= (P_{1x} + P_{2x})\hat{x} + (P_{1y} + P_{2y})\hat{y}$$

$$= (\alpha_{xx} E_x + \alpha_{xy} E_y)\hat{x} + (\alpha_{yx} E_x + \alpha_{yy} E_y)\hat{y}$$

Now, before getting back to $\overleftrightarrow{T}$, the Maxwell Stress tensor, let's talk about another stress tensor, one which describes the stresses within an elastic material. Let's call this one $\overleftrightarrow{R}$. Unlike the polarizability tensor, it does not map one vector into another, but describes the forces per area (any area you choose) within a material. A tensor is needed since a force in a certain direction, let's say the x-direction for example, can act on areas perpendicular to all three coordinate axes. Or, conversely, choosing a particular area, say perpendicular to the y-axis, forces in $\hat{x}$, $\hat{y}$ or $\hat{z}$ can act on that surface.



Here, $F_y\hat{y}$ is a pressure force

F_z, F_x are shear forces

$$R_{yy} = \frac{F_y}{\Delta x \Delta z} = \frac{F_y}{\Delta a_y} \quad , \quad R_{zy} = \frac{F_z}{\Delta x \Delta z} = \frac{F_z}{\Delta a_y}$$

and so on...