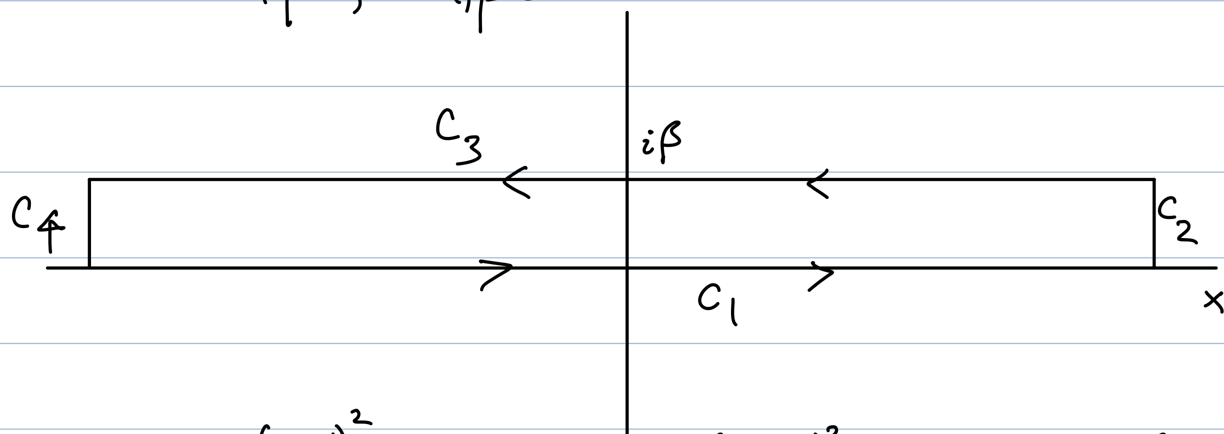
We know that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

By translating x by a real constant α, we also get

$$\int_{-\infty}^{\infty} e^{-(x-\alpha)^2}\,dx = \sqrt{\pi}.$$

What if we have $\int_{-\infty}^{\infty} e^{-(x-w)^2}\,dx$, $w \in \mathbb{C}$?

Let $w = \alpha + \beta i$, $\alpha, \beta \in \mathbb{R}$.



$$\int_{C_1+C_2+C_3+C_4} e^{-(z-w)^2}\,dz = 0 = \int_{C_1} e^{-(x-w)^2}\,dx + \int_{C_3} e^{-(x+i\beta-w)^2}\,dx$$

$$+ \int_{C_2}(\quad) + \int_{C_4}(\quad) \to 0$$

Exercise

$$\Rightarrow \boxed{\forall\, w \in \mathbb{C}, \int_{-\infty}^{\infty} e^{-(x-w)^2}\,dx = \int_{-\infty}^{\infty} e^{-(x-\alpha)^2}\,dx = \sqrt{\pi}}$$

Similarly, we have for any $\alpha > 0$,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}.$$

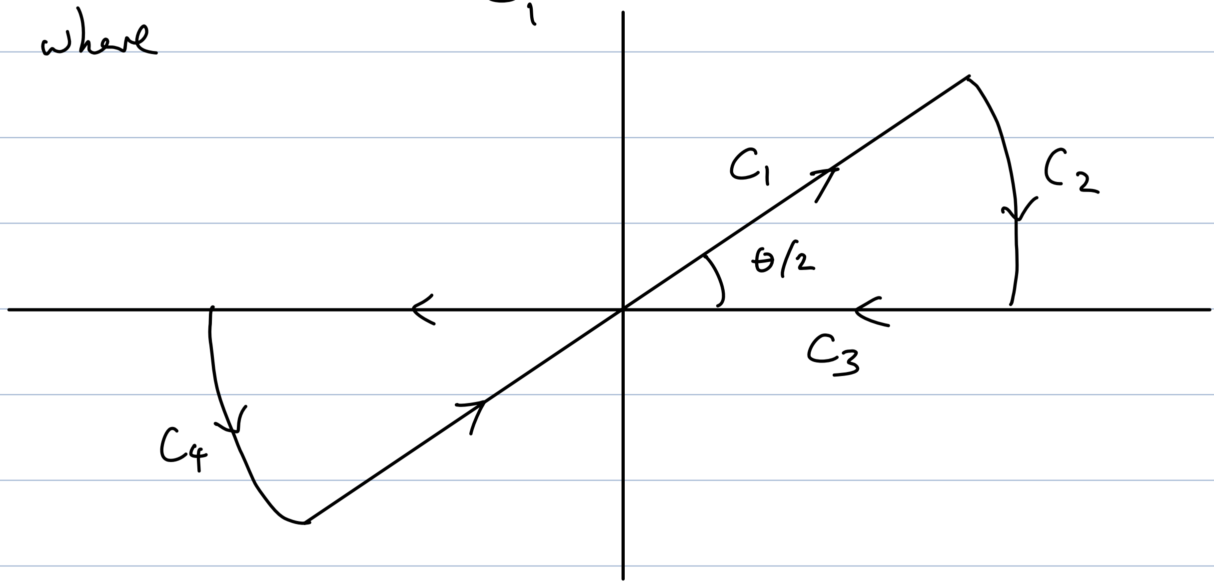
What if we have $\int_{-\infty}^{\infty} e^{-wx^2}\,dx$, $w \in \mathbb{C}$?

Let $w = \alpha e^{i\theta}$, $\alpha > 0$, $0 \leq \theta < 2\pi$.

Then, $\displaystyle\int_{-\infty}^{\infty} e^{-wx^2}\,dx = \int_{-\infty}^{\infty} e^{-\alpha(e^{i\theta/2}x)^2}\,\frac{e^{\frac{i\theta}{2}}\,dx}{e^{\frac{i\theta}{2}}}$

$$= e^{-\frac{i\theta}{2}}\int_{C_1} e^{-\alpha z^2}\,dz$$

where



$$\int_{C_1+C_2+C_3+C_4} e^{-\alpha z^2}\,dz = 0 = \int_{C_1} e^{-\alpha z^2}\,dz + \underbrace{\int_{C_2+C_4} e^{-\alpha z^2}\,dz}_{=0\,?} + \underbrace{\int_{C_3} e^{-\alpha z^2}\,dz}_{-\sqrt{\frac{\pi}{\alpha}}}$$

let $z = Re^{i\phi}$

$$\int_{C_2} e^{-\alpha z^2}\,dz = \lim_{R\to\infty}\int_0^{\frac{\theta}{2}} e^{-\alpha e^{2i\phi}R^2}\, R\, i e^{i\phi}\,d\phi$$

* This integral diverges if $\mathrm{Re}(e^{2i\phi}) < 0$, i.e. if $\mathrm{Re}(w) < 0$.

** When $\mathrm{Re}(w) \geq 0$, $\displaystyle\int_{C_2+C_4} e^{-\alpha z^2}\,dz = 0$. (why?)

Hence, $\boxed{\text{for } \mathrm{Re}(w) \geq 0,\ \int_{-\infty}^{\infty} e^{-wx^2}\,dx = \sqrt{\frac{\pi}{w}}}$