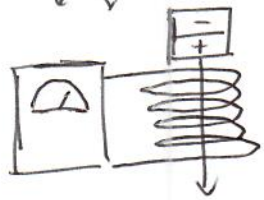
Vacuum Maxwell Eq

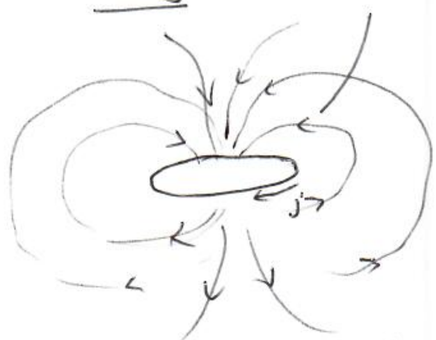
$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$ Gauss  $E \sim q/r^2\epsilon_0$

$$\vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$$ Faraday 

$$\vec{\nabla}\cdot\vec{B} = 0$$ No mag monopoles (yet!)

$$\underline{\vec{\nabla}\times\vec{B}} - \underline{\underline{\mu_0\epsilon_0\partial_t\vec{E}}} = \underline{\mu_0\vec{j}}$$ Ampere - Maxwell

radiation



electromagnet

$$\frac{d\vec{p}}{dt} = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$ Lorentz force

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ϵ_0 : vacuum permittivity. How well a charge creates $\vec{E}$.

μ_0 : vacuum permeability: How well a current creates $\vec{B}$.

We'll come back to the displacement fields $\vec{D}$ & $\vec{H}$ later.

If the sources do not extend to infinity, we may write

$$\vec{E} = \underset{\substack{\uparrow \\ \text{just a convention}}}{-}\vec{\nabla}\phi + \vec{\nabla}\times\vec{F} \qquad \text{w.l.o.g.}$$

Helmholtz thm — simplest proof uses Fourier transform — see wikipedia

ϕ is a scalar under 3D rotations & $\vec{F}$ is a vector.

similarly $\vec{B} = -\vec{\nabla} S + \vec{\nabla}\times\vec{A}$ $\qquad$ S scalar, $\vec{A}$ vector

Then: $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \nabla^2 S = 0$

$\Rightarrow S = 0$ if fields $\xrightarrow[\infty]{r} 0$

And: Faraday $\Rightarrow \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \nabla^2\vec{F}}_{\nabla\times(\vec{\nabla}\times\vec{F})} = \vec{\nabla}\cancel{\mathring{S}}^{\,0} - \nabla\times\mathring{\vec{A}}$

So $\vec{\nabla}\times\vec{F} = -\mathring{\vec{A}} + \vec{\nabla}\mathring{\lambda}$ for scalar λ.

We can absorb $\mathring{\lambda}$ into ϕ and write

$$\vec{E} = -\vec{\nabla}\phi - \mathring{\vec{A}}$$
$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \text{w.l.o.g.}$$

ϕ & $\vec{A}$ are the scalar & vector potentials.

The presence of λ means the decomposition is not unique. The transformations

$$\phi \to \phi + \dot{\lambda}$$
$$\bar{A} \to \bar{A} + \bar{\nabla}\lambda$$

"gauge symmetry"

leave $\vec{E}$ & $\vec{B}$ unique.

Only $\vec{E}$ & $\vec{B}$ are observable locally – as well as nonlocal objects like $\oint_c A_\mu dx^\mu$ –

So this gauge symmetry is not a real symmetry in the sense of taking one solution to the EOM into another, physically distinct solution. Rather, it means there are many equivalent descriptions of the same electromagnetic theory and we are free to choose λ in different problems to simplify calculations.

Inserting the potentials into the Gauss & Ampere laws,

$$-\nabla^2\phi - \partial_t \vec{\nabla}\cdot\vec{A} = \rho/\epsilon_0$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} - \mu_0\epsilon_0\left(-\vec{\nabla}\dot{\phi} - \ddot{\vec{A}}\right) = \mu_0 j$$

A convenient gauge is to choose λ so that

$$\mu_0\epsilon_0\dot{\phi} - \vec{\nabla}\cdot\vec{A} = 0 \qquad \text{"Lorenz gauge"}$$

Then the eq above simplify to:

$$\mu_0\epsilon_0\ddot{\phi} - \nabla^2\phi = \rho/\epsilon_0$$

$$\mu_0\epsilon_0\ddot{\vec{A}} - \nabla^2\vec{A} = \mu_0 j$$

(Decoupling $\vec{A}$ & ϕ)

It is convenient to define 4-vectors

$$A^\mu = \left(\tfrac{1}{c}\phi, \vec{A}\right)$$

$$J^\mu = (c\rho, \vec{j}) \qquad \text{with } c^2 \equiv \frac{1}{\mu_0\epsilon_0}$$

and $\Box \equiv \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)$

then $\boxed{\Box A^\mu = \mu_0 J^\mu}$

$\partial_\mu A^\mu = 0$ $\Rightarrow$ Lorenz gauge EOM for A^μ

This is the (inhomogeneous, ~~or~~ "sourced") wave equation.

$\Rightarrow$ Radiation.

Plug in to definitions of $\vec{E}$ and $\vec{B}$:

$$\Box \vec{E} = -\vec{\nabla}(\Box \phi) - \partial_t(\Box \vec{A})$$

$$= \underbrace{-\nabla(\rho/\epsilon_0) - \partial_t(\mu_0 \vec{j})}_{\text{effective source}}$$

and
$$\Box \vec{B} = \vec{\nabla} \times (\Box \vec{A})$$

$$= \underbrace{\nabla \times (\mu_0 \vec{j})}_{\text{effective source}}$$

Again sourced wave equations, with sources given by derivatives of ρ and $\vec{j}$. Gauge invariant

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Well, let's solve!

We have an equation of the form

$$L \varphi = f$$

L is a 2^{nd} order linear partial differential operator ($\Box$)

φ is a function of spacetime we want to know

f is a given "source" function of spacetime

Obviously $\varphi = L^{-1} f$!

Although Maxwell's equations were a triumph of matching theory to experiment, they were philosophically troubling in the late 19^{th} century. The reason goes back to Galileo and Newton.

The business of physics is to predict the future. Given the state of affairs now, what will we have in ten milliseconds? ten billion years? Newton made the solution quantitative! e.g.

$$M_N \frac{d^2 \vec{x}_N}{dt} = \vec{F}_N \quad \text{for a system of particles}$$

with $$\vec{F}_N = G \sum_M \frac{m_N m_M (\vec{x}_M - \vec{x}_N)}{|\vec{x}_M - \vec{x}_N|^3}$$ for gravitation.

These equations respect the Principle of Galilean Relativity. They are exactly the same for any observer using coordinates related by the Galilean Group:

$$\vec{x}' = R\vec{x} + \vec{v}t + \vec{d}$$
$$t' = t + T$$

- $\vec{v}, \vec{d}$ are real constant 3-vectors (6 param)
- R is a real orthogonal 3×3 matrix (3 param) $(R^T R = \mathbb{I})$
- T is a real constant (1 param)

A 10-parameter group

The R's rotate coordinates. We can change to a rotated set of coordinates

$$x'^i = \sum_j R^i{}_j x^j$$

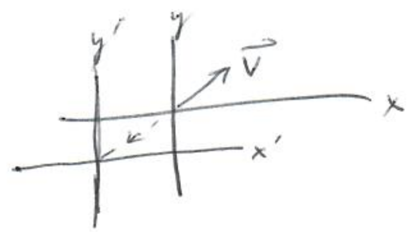
This is just a linear xform, or $\left(R^i{}_j \right) \left(x^j \right)$

"Einstein notation": we drop Σ, with understanding that repeated indices are summed.

When we rotate the coordinates, the force vector also rotates, $F'^i = R^i{}_j F^j$. Easy to check this manifestly for the law of gravitation above. (The denominator $|\vec{x}_M - \vec{x}_N|^3 \equiv \left[(x_M^i - x_N^i)(x_M^j - x_N^j)\delta_{ij} \right]^{\frac{3}{2}}$ depends only on distances, which should be unaffected by rotations - try to show it mathematically!)

So we find $m \dfrac{d^2 \vec{x}'}{dt^2} = \vec{F}'$ - the same form in the rotated system.

The transformations parametrized by $\vec{v}$: $\vec{x}' = \vec{x} + \vec{v}t$ are called Galilean boosts. $\vec{x}'$ is moving relative to $\vec{x}$ with velocity $-\vec{v}$



if originally aligned @ $t=0$, after some time Δt, the origin $x'=y'=0$ is located at $\vec{x} = -\vec{v}t$.

The transformations $\vec{d}$ and T are spacetime <u>translations</u>.

Maxwell's equations are invariant under spacetime translations and rotations, but <u>not</u> Galilean boosts.

Under a Galilean boost any velocity $\vec{u} = \frac{d\vec{x}}{dt}$ becomes $\vec{u}' = \vec{u} + \vec{v}$. But Maxwell's equations

are not invariant:

$$\frac{1}{c^2}\partial_t^2 A_\mu - \nabla^2 A_\mu = 0 \quad \text{in vacuum}$$

$$\rightarrow \left(\frac{\partial}{\partial t'} - \vec{v}\cdot\frac{\partial}{\partial \vec{x}'}\right)\left(\frac{\partial}{\partial t'} - \vec{v}\cdot\frac{\partial}{\partial \vec{x}'}\right)A_\mu - c^2\nabla^2 A_\mu = 0$$