# Chapter 12, lecture 3

Reference: "Relativity of the mind" Porter Wear Johnson, Rinton Press, ISBN 1589490541

It was found that spacetime components in one inertial frame can be transformed into another inertial frame (moving at speed $v$ with respect to the first) with a Lorentz transformation. For example, if the frame using coordinates $\bar{x}, \bar{y}, \bar{z}, \bar{ct}$ is moving at speed $v$ in the $x$ direction with respect to a frame using coordinates $x, y, z, ct$, then

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

A 4-vector is any set of four components that transform the same way as $(x^0, x^1, x^2, x^3)$ under Lorentz transformations:

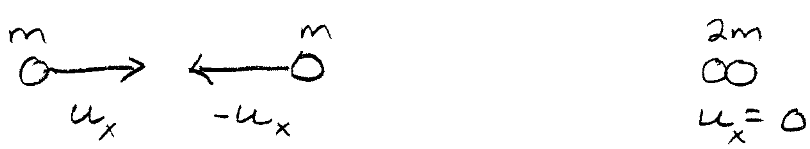
$$\bar{a}^\mu = \Lambda^\mu_\nu a^\nu$$

Ordinary velocity (as opposed to proper velocity) does not transform like a four-vector. Instead, the components are given by the velocity addition rule; transforming to an inertial frame moving at speed $v$ in the x-direction,

$$\bar{u}_x = \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{\left(1 - \frac{v u_x}{c^2}\right)} \quad , \quad u_{y(z)} = \frac{u_{y(z)}}{\left(1 - \frac{v u_{y(z)}}{c^2}\right)}$$

Also, conservation of momentum is not consistent with the principle of relativity if momentum is defined as $m\vec{u}$.

To see this, consider a completely inelastic collision of two particles of equal mass with equal and opposite velocities that hit each other head on.




In the center of momentum frame, momentum is conserved. $mu - mu = 0$

What about in a boosted frame moving at speed $v_x$? Before the collision,

$$P_{\substack{total\\ Before}} = \frac{m\,(u_x - v_x)}{\left(1 - \frac{v_x u_x}{c^2}\right)} - \frac{m\,(u_x + v_x)}{\left(1 + \frac{v_x u_x}{c^2}\right)}$$

$$= m \; \frac{-2v_x + 2\left(\frac{v_x u_x^2}{c^2}\right)}{1 - \frac{v_x^2 u_x^2}{c^4}}$$

$$P_{\substack{total\\ after}} = 2m \; \frac{-v_x}{1-(0)} = -2mv_x$$

In general, $\dfrac{-2mv_x\left(1 + \frac{u_x^2}{c^2}\right)}{1 - \left(\frac{v_x u_x}{c^2}\right)^2} \neq -2mv_x$

Proper velocity, $\eta^\mu = \dfrac{dx^\mu}{d\tau}$ where $\tau$ is the proper time, does transform as a four-vector. The proper time is the time associated with the moving object, and so it is an invarient,

$$(x^2+y^2+z^2) - (ct)^2 = -(c\tau)^2 = \text{invarient}$$

So far we have encountered the displacement four-vector, with components $x^{\mu}$, and the velocity four-vector with components $\eta^{\mu}$. The momentum four-vector has components $m\eta^{\mu}$, or, $(m\eta^0, m\eta^1, m\eta^2, m\eta^3)$ where $m\eta^1 = m\frac{dx^1}{d\tau} = \gamma m u^1$, since

the proper time is related to time in a moving frame as, $t = \gamma\tau$, so $\frac{dx'}{d\tau} = \gamma m \frac{dx'}{dt} = \gamma m u'$.

Also, $m\eta^0 = mc\frac{dt}{d\tau} = \gamma mc = \frac{E}{c}$.

Then $p = (E/c, \gamma m u^1, \gamma m u^2, \gamma m u^3)$.

Another four-vector is the current density four vector, $J^{\mu} = (c\rho, J_x, J_y, J_z)$. If a cloud of charge is moving by, the charge density is $\rho = Q/V$ and the current density is $\vec{J} = \rho\vec{u}$. The proper charge density in the rest frame of the charge is $\rho_0 = Q/V_0$. In the moving frame the rest volume is Lorentz contracted

$$V = (1 - u^2/c^2)^{1/2} V_0$$

Then, $\rho = \gamma \rho_0$ and $\vec{J} = \rho \vec{u} = \gamma \rho_0 \vec{u}$.
Then, $J^\mu$ has the same form as the proper velocity $\eta^\mu$, and so is a four-vector.

Another four-vector is the wave vector $(k_0, \vec{k}) = (\omega/c, \vec{k})$

The phase of a plane wave, $\omega t - \vec{k} \cdot \vec{r}$, is invarient. The elapsed phase is proportional to the number of wave crests that pass the observer, the number must be the same in all frames. Then $k_\mu x^\mu = k'_\mu x'^\mu$ is invarient, as are all scalar products of four-vectors. So, $k^\mu$ transforms as any four-vector,

$$k'^0 = \gamma(k^0 - \beta k^1) \qquad k'^2 = k^2$$

$$k'^1 = \gamma(k^1 - \beta k^0) \qquad k'^3 = k^3$$

for a boost in the x-direction. Suppose k makes an angle $\varphi$ with the x-axis, then

$$k'^0 = \gamma(k^0 - \beta k \cos\varphi)$$

$$\omega'/c = \gamma\, \omega/c\, (1 - \beta\cos\varphi) \qquad \text{since } k^0 = |k| = \omega/c$$

$$\nu' = \nu \frac{(1 - \beta\cos\varphi)}{(1-\beta^2)^{1/2}}$$

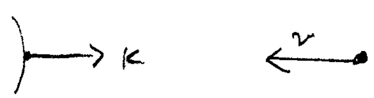
Now, suppose $\vec{k}$ is along the x-direction, parallel to $\vec{v}$. Then, $\varphi = 0$, $\cos\varphi = 1$ and $\vec{k}$ is in the same direction as the relative motion of the frames.


Then $$\nu' = \nu \, \frac{(1-\beta)}{[(1+\beta)(1-\beta)]^{1/2}} = \nu \sqrt{\frac{(1-\beta)\cancel{(1-\beta)}}{(1+\beta)\cancel{(1-\beta)}}}$$

$$\nu' = \nu \left[ \frac{1-\beta}{1+\beta} \right]^{1/2}$$

The $\nu' < \nu$, red shift

If $\vec{k}$ is antiparallel to $\vec{v}$,




then $\varphi = \pi$, $\cos\varphi = -1$

$$\nu' = \nu \sqrt{\frac{(1+\beta)}{(1-\beta)}}$$ ; $\nu' > \nu$ ; blue shift

As mentioned, the scalar product of two four-vectors, $a_\mu b^\mu$, is invarient.

The electric and magnetic fields do not transform as simply as a four-vector. A stationary charge distribution has no magnetic field in the stationary frame of reference. Yet in an inertial frame moving relative to the charge distribution, there is a magnetic field (the charges are moving in that frame). How do magnetic and electric fields transform? We'll follow Griffiths physical arguments to get the field transformations that apply to an x boost.

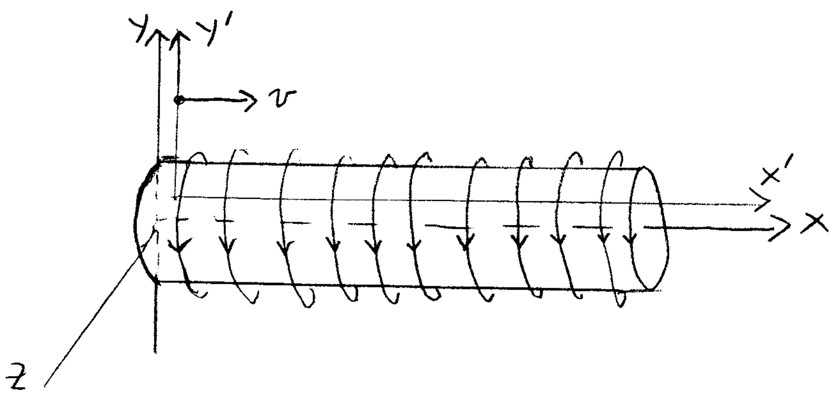
1. Consider a solenoid with a uniform $\vec{B}$ field along the x-direction to see that the magnetic field component parallel to the direction of motion does not change from one frame to another. $B'_{\parallel} = B_{\parallel}$

2. Consider a capacitor with a uniform $\vec{E}$ field to show that going from one frame to another, $E'_{\parallel} = E_{\parallel}$ and $E'_{\perp} = \gamma E_{\perp}$

3. Since a capacitor in a stationary frame has no associated magnetic field, we still don't have a transformation relation for perpendicular field components.

To remedy this we'll consider two separate moving frames with respect to the stationary capacitor. There is a magnetic field in each of these frames, since there are moving charges. So, we'll get the magnetic field transformation from one moving frame to the other.

Beginning with the solenoid, make it coaxial with the x-axis so that there is a uniform field $B_x$.



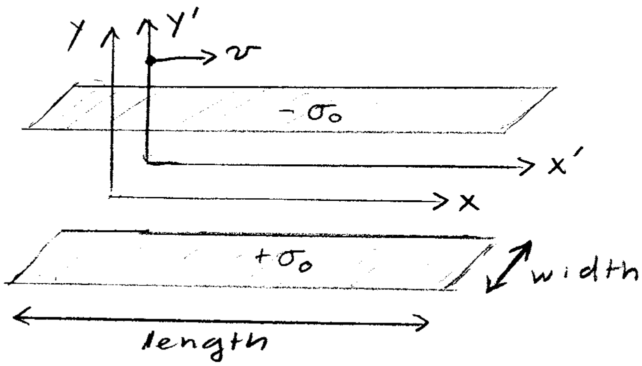

In the stationary frame, $B_x = \mu_0 n i$

$n$ is the number of turns per length, and since length contracts in a moving frame, $l' = \frac{1}{\gamma} l$, the density of turns/length will go up by a factor of $\gamma$; $n' = \gamma n$. The current $i = \frac{\Delta q}{\Delta \tau}$. The charge does not change, but the time interval increases in the moving frame, $\Delta t = \gamma \Delta \tau$. So, $i' = \frac{1}{\gamma} i$.

So, $B_x' = \mu_0 n' i' = \mu_0 (\gamma n)\left(\frac{1}{\gamma} i\right) = \mu_0 n i$

Then $B_x' = B_x$, the magnetic field component parallel to the direction of relative motion of two inertial frames does not change.

Now for the capacitor:




In the stationary frame:

$$\vec{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{y}$$

In the moving frame the charge density $\sigma$ increases compared to in the stationary frame.

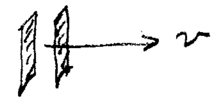
$$\sigma_0 = \frac{\text{charge}}{\text{area}} = \frac{\text{charge}}{\ell w}$$

$$\sigma = \frac{\text{charge}}{\text{area}} = \frac{\text{charge}}{\ell' w} = \frac{\text{charge}}{\left(\frac{1}{\gamma}\ell\right)w} = \gamma \frac{\text{charge}}{\ell w}$$

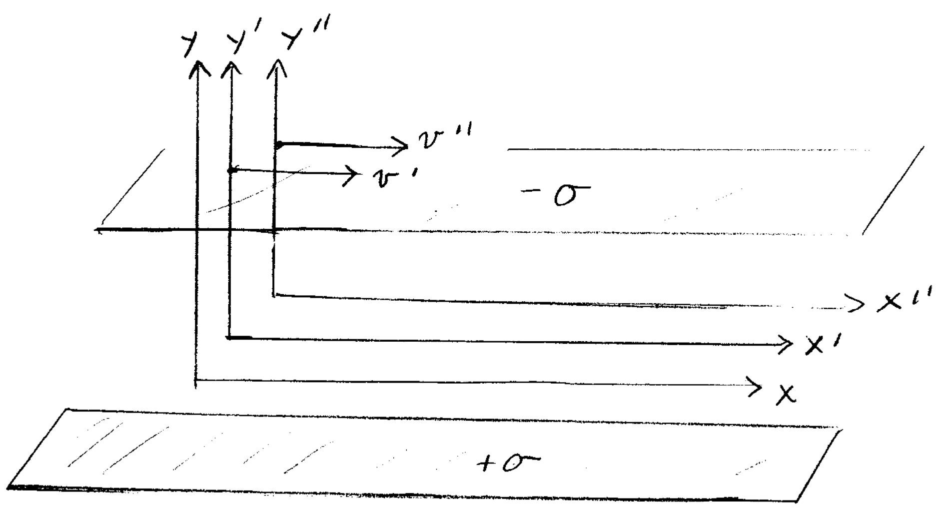
$$\vec{E}' = \frac{\sigma}{\varepsilon_0}\hat{y} = \gamma \frac{\sigma_0}{\varepsilon_0}\hat{y} = \gamma \vec{E}_0'$$

Then $E_\perp' = \gamma E$

What about $E_\parallel'$ ?

Turn the plates sideways,  to see that the charge density $\sigma = \sigma_0$ in this case. No plate dimension is contracted. Then, $E'_{\parallel} = E_{\parallel}$.

Finally, we need to find out how magnetic field components perpendicular to the direction of relative motion transform. We need two moving frames relative to the stationary capacitor.




$S_0$ is the stationary frame

$S'$ is moving at speed $v'$ in the x-direction relative to the stationary frame

$S''$ is moving at speed $v''$ in the x-direction relative to the stationary frame

$S''$ is moving at speed $v$ relative to $S'$

The electric field in the S' frame is

$$\vec{E}' = \frac{\sigma'}{\varepsilon_0}\hat{y} = \frac{\gamma'\sigma_0}{\varepsilon_0}\hat{y} \quad ; \quad \gamma' \equiv \frac{1}{\sqrt{1 - v'^2/c^2}}$$

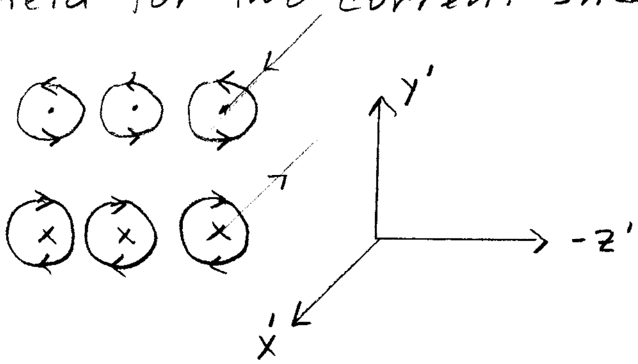
The magnetic field in S' is due to the contribution from the upper plate, which looks like a current sheet with current density $K'\hat{x}$, and a contribution from the lower plate which looks like a current sheet with current density $-K'\hat{x}$.

$$\longrightarrow \quad \vec{K}' = \sigma' v' \hat{x}$$

$$\longleftarrow \quad \vec{K}' = -\sigma' v' \hat{x}$$

current defined as the flow of positive charge

$\vec{B}'$ field for two current sheets:




$\vec{B}'$ is in the $-\hat{z}'$ direction

B' has double the magnitude due to a single current sheet

$$\vec{B}' = 2\left(\mu_0 \frac{K}{2}\right)(-\hat{z}')$$

$$= -\mu_0 K \hat{z} = -\mu_0 \sigma' v' \hat{z}'$$

In the S'' frame,

$$E_y'' = \frac{\sigma''}{\varepsilon_0} = \gamma''\frac{\sigma_0}{\varepsilon_0} = \left(\frac{\gamma''}{\gamma'}\right)\frac{\sigma'}{\varepsilon_0}$$

where $\gamma'' \equiv \dfrac{1}{\sqrt{1 - v''^2/c^2}}$

$$B_z'' = -\mu_0 \sigma'' v'' = -\mu_0 \gamma'' \sigma_0 v''$$

$$= -\mu_0 \left(\frac{\gamma''}{\gamma'}\right) \sigma' v''$$

$v''$ may be written in terms of $v'$ and $v$ using the velocity addition rule,

$$v'' = \frac{v' + v}{1 + vv'/c^2}$$

$\dfrac{\gamma''}{\gamma'}$ also needs to be put in terms of $v, v'$

So, $$\frac{\gamma''}{\gamma'} = \frac{\sqrt{1 - v'^2/c^2}}{\sqrt{1 - v''^2/c^2}}$$

$$(1 - v''^2/c^2) = 1 - \frac{(v + v')^2}{c^2(1 + vv'/c^2)^2}$$

$$1 - \frac{v''^2}{c^2} = \frac{c^2(1+vv'/c^2)^2 - (v+v')^2}{c^2(1+vv'/c^2)^2}$$

$$(1 - v''^2/c^2)^{1/2} = \frac{\left[c^2\left(1 + \frac{2vv'}{c^2} + \frac{v'^2v^2}{c^4}\right) - (v^2+v'^2+2vv')\right]^{1/2}}{c(1+vv'/c^2)}$$

$$= \frac{\left[1 + \cancel{\frac{2vv'}{c^2}} + \frac{v'^2v^2}{c^4} - v^2/c^2 - \frac{v'^2}{c^2} - \cancel{\frac{2vv'}{c^2}}\right]^{1/2}}{(1+vv'/c^2)}$$

$$\frac{\gamma''}{\gamma'} = \frac{\cancel{\left[1 - v'^2/c^2\right]^{1/2}}\,(1+vv'/c^2)}{\left[(1-v^2/c^2)\cancel{(1-v'^2/c^2)}\right]^{1/2}}$$

$$= \gamma\,(1+vv'/c^2)$$

Then, $$E_y'' = \gamma\frac{\sigma'}{\varepsilon_0} + \gamma\frac{v}{c^2}\frac{\sigma'v'}{\varepsilon_0}$$

$$= \gamma\left(E_y' - v/c^2\,\frac{1}{\mu_0\varepsilon_0}(-\mu_0\sigma'v')\right)$$

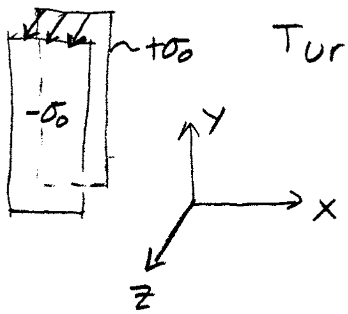
$$= \gamma(E_y' - v\,B_z')$$

and $$B_z'' = -\mu_0\gamma\,\cancel{(1+vv'/c^2)}\,\sigma'\left(\frac{v+v'}{\cancel{1+vv'/c^2}}\right)$$

$$= \gamma\left(-\mu_0\sigma'v' - \mu_0\varepsilon_0\frac{\sigma'}{\varepsilon_0}v\right)$$

$$= \gamma\,(B_z' - v/c^2\,E_y')$$

To get the remaining transformations, $E_z$ and $B_y$, turn the capacitor so that the field is along the z-axis.

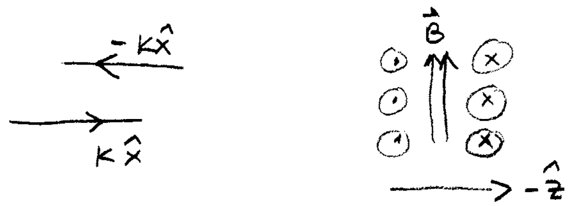



Turning the plates this way

$$E_y \to E_z$$

What happens to B?

The current densities are oriented




$$-B_z \to +B_y$$

Since E does not change sign, but B does, the transformations will be almost the same, but terms with B will switch sign. Then,

$$E_z'' = \gamma(E_z + v B_y) \qquad B_y'' = \gamma(B_y + v/c^2 E_z)$$

So, all transformations are;

$$E_x' = E_x \qquad B_x' = B_x$$

$$E_y' = \gamma(E_y - v B_z) \qquad B_y' = \gamma(B_y + v/c^2 E_z)$$

$$E_z' = \gamma(E_z + v B_y) \qquad B_z' = \gamma(B_z - v/c^2 E_y)$$

In the past, one way that weve been able to obtain the fields is by starting with the potentials. We still haven't found a four-vector representing the potential, or the four dimensional gradient. We have been using the invarient scalar product of two four vectors $a_\mu b^\mu$, where the repeated index implies summation over all components, $a_\mu b^\mu = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3$. The combination of a lower index and an upper index means $a_\mu$ and $b^\mu$ combine to form a single entity, $a_\mu b^\mu$. A vector whose index is a subscript is called a covarient vector. The sign of the first component of a four-vector is different for the contravarient and covarient vector; $a_\mu = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3)$. We run into a slight complication when defining the gradient four-vector. Using

http://en.wikipedia.org/wiki/Covarience_and_contravarience_of_vectors

to state the nature of this complication:

(next page)

For a vector to be coordinate system invarient, components of the vector must contra-vary with a change of basis to compensate for the effect of changing the basis. The vector components must vary in the opposite way (the inverse transformation) as the change of basis. For a vector operator such as the gradient, the components must co-vary with a change of basis to maintain coordinate system invarience. The components must vary by the same transformation as the change of basis.

Now use the Feynman Lectures on physics to make that statement more concrete. Assume the components of the gradient are all positive (normal contravarient convention):

$$\frac{\partial}{\partial x^{\mu}} = \left( c\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Consider a scalar function, f, that depends only on x and t. The change in f if x is held constant and a small change $\Delta t$ is made, is given by

$$\Delta f = \frac{1}{c}\frac{\partial f}{\partial t}(c\Delta t)$$

According to a moving observer,

$$\Delta f = \frac{\partial f}{\partial x'}\,\Delta x' + \frac{1}{c}\frac{\partial f}{\partial t'}\,c\Delta t'$$

Use the Lorentz transform to express $\Delta t', \Delta x'$ in terms of $\Delta t$:

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \end{pmatrix}$$

$$\left.\begin{aligned} c\Delta t' &= \gamma c\Delta t - \beta\gamma\Delta x = \gamma c\Delta t \\ \Delta x' &= -\beta\gamma c\Delta t + \gamma\Delta x = -\beta\gamma c\Delta t \end{aligned}\right\}\ \text{since } \Delta x = 0$$

Then, $\Delta f = \dfrac{\partial f}{\partial x'}(-\beta\gamma c\Delta t) + \dfrac{\partial f}{\partial(ct')}(\gamma c\Delta t)$

$$\Delta f = \left(\gamma\frac{\partial f}{\partial(ct')} - \beta\gamma\frac{\partial f}{\partial x'}\right)c\Delta t$$

compare to $\Delta f = \dfrac{\partial f}{\partial(ct)}(c\Delta t)$ on the previous page, then

$$\frac{\partial f}{\partial(ct)} = \gamma\frac{\partial f}{\partial(ct')} - \beta\gamma\frac{\partial f}{\partial x'}$$

Doing a similar procedure to find $\frac{\partial f}{\partial x}$,

$$\frac{\partial f}{\partial x} = -\beta\gamma \frac{\partial f}{\partial (ct')} + \gamma \frac{\partial f}{\partial x'}$$

These two equations look like a Lorentz transform of $\frac{\partial f}{\partial (ct')}$ and $\frac{\partial f}{\partial x'}$. <u>However</u> in going from the moving frame (primed frame) to the stationary frame (unprimed frame) we should have the <u>inverse</u> Lorentz transform. Assuming that the gradient was contravarient, we found that the coordinate transform was the inverse of what it should have been. So, all positive coordinates are defined to be the covarient operator:

$$\frac{\partial}{\partial x^\mu} = \left( \frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \partial_\mu$$

$$\frac{\partial}{\partial x_\mu} = \left( -\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \partial^\mu$$

Then $\partial_\mu J^\mu$ represents the scalar product of $J$ with the gradient vector operator.

$$\partial_\mu J^\mu = \frac{1}{c}\frac{\partial (c\rho)}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J}$$

So,

$$\partial_\mu J^\mu = 0 \quad \text{is the continuity equation}$$

Now, lets take a look at the equations for the potentials.

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0} \rho$$

If we multiply both sides of the second equation by $1/c$, it is

$$\nabla^2 (V/c) - \mu_0 \varepsilon_0 \frac{\partial^2 (V/c)}{\partial t^2} = -\mu_0 \rho c$$

Now the equations can be written

$$\underbrace{\partial_\mu \partial^\mu}_{} \vec{A} = -\mu_0 \vec{J}$$

$$\partial_\mu \partial^\mu (V/c) = \underbrace{-\mu_0 \rho c}_{}$$

$\partial_\mu \partial^\mu$ is a scalar invarient operator

$(\rho c, \vec{J})$ is a four-vector

Thus, $(V/c, \vec{A})$ must be a four-vector