Chapter 11 - Radiation

To get radiation, there must be accelerating charges and changing currents. Static distributions cannot produce radiation.

Power radiated:

$$P_{rad} \equiv \lim_{r \to \infty} \oint \vec{S} \cdot d\vec{a} = \lim_{r \to \infty} \frac{1}{\mu_0} \oint (\vec{E} \times \vec{B}) \cdot d\vec{a}$$

Radiated power is the energy/time transported to infinity. Since surface area goes as r^2, then $\vec{E} \times \vec{B}$ must decrease <u>no more rapidly</u> than $\frac{1}{r^2}$ to have any power radiate away to infinity. Supposing $\vec{E}$ + $\vec{B}$ fall off in the same power of r, then neither can fall off faster than r, and still have radiated power. Lets check out the expressions for the fields that were derived in Chapter 10.

For a general distribution of charges:

$$E(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{r}\,',t_r)}{\eta^2}\hat{\eta} + \underbrace{\frac{\dot{\rho}(\vec{r}\,',t_r)}{c\eta}\hat{\eta} - \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c^2\eta}}_{\text{These terms can be associated with radiation.}}\right]d\tau'$$

static term falls off too rapidly

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}\,',t_r)}{\eta^2} + \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c\eta}\right]\times\hat{\eta}\,d\tau'$$

static term falls off too rapidly

This term can be associated with radiation

So says Jefimenko. What about the Liénard - Wiechert guys? For a point charge, the general expression for the fields are:

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\eta}{(\vec{\eta}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\eta}\times(\vec{u}\times\vec{a})\right]$$

too fast

acceleration term can make radiation

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\eta}\times\vec{E}(\vec{r},t)$$

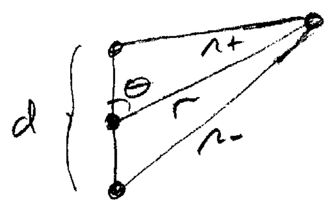
OK – so now what? As usual it is useful to start with a very simple radiating system to develop 'intuition.' (simple being a relative term.) The oscillating dipole. After that, we'll attack the more general case of radiation from an arbitrary source.

Time for another flashback – the static dipole. What did we do?

1. Write the expression for the potential summing the contribution from each charge:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

2. Re-write $r_\pm$ using geometry of the problem and the law of cosines.



$$r_\pm^2 = r^2 + (d/2)^2 \mp r d\cos\theta$$

3. Take the case far from the dipole, $r \gg d$. This allows a binomial expansion of $\frac{1}{r_\pm}$ to give $V = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$

Remember, if $r >> d$ does not hold, the two charges don't look as much like a dipole, but more like two charges.

Now, for the oscillating dipole (let's say harmonically oscillating):

$$q = q_0 \cos \omega t$$

$$\vec{p}(t) = p_0 \cos(\omega t)\, \hat{z} \quad , \quad p_0 \equiv q_0 d \qquad \left\{ \text{dipole moment} \right.$$

Way out at point P we need to use the retarded time:

① $$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q_0 \cos[\omega(t - \imath_+/c)]}{\imath_+} - \frac{q_0 \cos[\omega(t - \imath_-/c)]}{\imath_-} \right\}$$

② Geometry & the law of cosines:

$$\imath_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2} = r\sqrt{1 \mp \frac{d}{r}\cos\theta + \overset{\text{small}}{\cancel{\left(\frac{d}{2r}\right)^2}}}$$

③ We still want a <u>dipole</u>, $r >> d$

This will get us a binomial expansion - but in addition we must handle the argument of the cosine functions.

(4) We can approximate our way out of this difficulty:

$r >> d$ Dipole

$\frac{c}{\omega} >> d \implies \lambda >> d$ Structure of dipole is small compared to λ

$r >> \frac{c}{\omega} \implies r >> \lambda$ Radiation zone

Far, far away where far $\equiv$ many λ

$(\lambda = c/f \approx c/\omega)$

These are suited to our desire to study radiation, anyway.

(5) Once V is obtained, find $\vec{A}$.

This will get us the fields

Since $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Binomial Expansions of $r_\pm$, $\frac{1}{r_\pm}$:

$$\frac{1}{r_\pm} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \quad \Big\}\ \text{For denominators}$$

$$r_\pm \approx r\left(1 \mp \frac{d}{2r}\cos\theta\right) \quad \Big\}\ \text{For argument of cosines}$$

so far only $r >> d$ assumed

Let $A \equiv \frac{d}{2r}\cos\theta$

$$V = \frac{q_0}{4\pi\epsilon_0 r}\Big\{(1+A)\cos[\omega t - \omega/c\, r(1-A)]$$
$$- (1-A)\cos[\omega t - \omega/c\, r(1+A)]\Big\}$$

The portion in curly brackets is:

$$(1+A)\cos\left[\omega(t - r/c) + \frac{\omega r}{c}A\right] + (A-1)\cos\left[\omega(t-r/c) - \frac{\omega r}{c}A\right]$$

let $\alpha \equiv \omega(t - r/c)$, $\beta \equiv \frac{\omega r}{c}A$

$$(1+A)\cos(\alpha+\beta) + (A-1)\cos(\alpha-\beta)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Giving:

$$(1+A)[\cos\alpha\cos\beta - \sin\alpha\sin\beta]$$

$$+ (A-1)[\cos\alpha\cos\beta + \sin\alpha\sin\beta]$$

$$= 2A\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$

$$V(\vec{r},t) = \frac{q_0}{2\pi\epsilon_0 r}\left[\frac{d}{2r}\cos\theta\cos[\omega(t-r/c)]\cos\left[\frac{\omega d}{2c}\cos\theta\right]\right.$$

$$\left. - \sin[\omega(t-r/c)]\sin\left[\frac{\omega d}{2c}\cos\theta\right]\right]$$

(4)

Now use approximation 2, $d << \lambda$

or, $d << \frac{c}{\omega}$ $\Rightarrow$ $\frac{\omega d}{2c}\cos\theta$ is small

$$\sin(\theta_{small}) \approx \theta_{small}, \quad \cos(\theta_{small}) \approx 1$$

Then,

$$V(\vec{r},t) = \frac{q_0}{2\pi\epsilon_0 r}\left[\frac{d}{2r}\cos\theta\cos[\omega(t-r/c)] - \frac{\omega d}{2c}\cos\theta\sin[\omega(t-r/c)]\right]$$

$$= \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left[\frac{1}{r}\cos[\omega(t-r/c)] - \omega/c\,\sin[\omega(t-r/c)]\right]$$

when $\omega \to 0$ (no motion)

$$V \to \frac{p_0\cos\theta}{4\pi\epsilon_0 r^2}$$ same as the stationary case

Using approximation 3, $r \gg \lambda$;

$$V(\vec{r},t) = -\frac{p_0\omega\cos\theta}{4\pi\epsilon_0 rc}\sin[\omega(t-r/c)]$$

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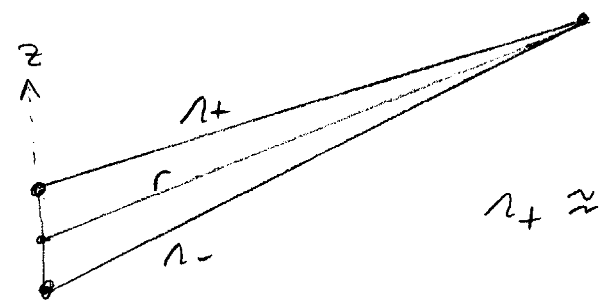
Now we need $\vec{A}$, $\vec{A} = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{\vec{I}\,dz}{\imath}$

$$I = \frac{dq}{dt} = -q_0\omega\sin[\omega t]$$

Inside the integral, $t \to t_r$ and the retarded time is slightly different from different locations on the current element.

But wait! We'll cheat again, since we are in the far field limit.



$$\mathscr{r}_+ \approx r \approx \mathscr{r}_-$$

Choose the average distance, r, to represent the distance, and replace integration over z with a factor of d. $\int dz = d$

$$\vec{A} = \frac{\mu_0}{4\pi} \; -q_0 \omega \sin[\omega(t - r/c)] \; \frac{1}{r} \; d \; \hat{z}$$

$$= -\frac{\mu_0 p_0 \omega \sin[\omega(t - r/c)]}{4\pi r} \; \hat{z}$$

We have the potentials, now for the fields:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

Choose spherical coordinates

$$V(r,\theta) \rightarrow \vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

Let $K \equiv \frac{p_0 \omega}{4\pi\epsilon_0 c}$ $\Rightarrow$ $V = -\frac{K\cos\theta}{r}\sin[\omega(t-r/c)]$

$$\vec{\nabla} V = -K\cos\theta \frac{\partial}{\partial r}\left(\frac{\sin[\omega(t-r/c)]}{r}\right)\hat{r} - \frac{K\sin[\omega(t-r/c)]}{r^2}\frac{\partial \cos\theta}{\partial \theta}\hat{\theta}$$

$$= \cancel{\frac{K\cos\theta \sin[\omega(t-r/c)]}{r^2}\hat{r}} - \frac{K\cos\theta\left[-\omega/c\right]\cos[\omega(t-r/c)]}{r}\hat{r}$$

$$+ \cancel{\frac{K\sin[\omega(t-r/c)]\sin\theta}{r^2}}\hat{\theta}$$

$\Uparrow$ dominant term

$$\vec{A} = -\frac{k}{rc}\sin[\omega(t-r/c)]\,[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}]$$

$$\vec{A}(r,\theta,t) = A_r(r,\theta,t)\,\hat{r} + A_\theta(r,\theta,t)\,\hat{\theta}$$

$$\frac{\partial\vec{A}}{\partial t} = -\frac{k\omega}{rc}\cos\theta\cos[\omega(t-r/c)]\,\hat{r} + \frac{k\omega}{rc}\sin\theta\cos[\omega(t-r/c)]\,\hat{\theta}$$

$$\vec{\nabla}\times\vec{A} = \vec{\nabla}\times[A_r\hat{r} + A_\theta\hat{\theta}]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}(A_r)\right]\hat{\varphi}$$

$$= \frac{1}{r}\left[\frac{k}{c}\sin\theta\frac{\partial}{\partial r}\left(\sin[\omega(t-r/c)]\right) + \frac{k}{rc}\sin[\omega(t-r/c)]\frac{\partial\cos\theta}{\partial\theta}\right]\hat{\varphi}$$

$$= \frac{k}{rc}\hat{\varphi}\left[\underbrace{\sin\theta(-\omega/c)\cos[\omega(t-r/c)]}_{\Uparrow\ \text{dominant term}} - \cancel{\frac{\sin[\omega(t-r/c)]\sin\theta}{r}}\right]\hat{\varphi}$$

Now, we can write the fields

$$\bar{E} = -\nabla \bar{V} - \frac{\partial \vec{A}}{\partial t} = -\frac{k\omega}{rc} \sin\theta \cos[\omega(t - r/c)]\, \hat{\theta}$$

$$+ \cancel{\frac{k\omega}{rc} \cos\theta \cos[\omega(t - r/c)]\, \hat{r}} - \cancel{\frac{k\omega}{rc} \cos\theta \cos[\omega(t - r/c)]\, \hat{r}}$$

$$\vec{E} = -\frac{\rho_0 \mu_0 \omega^2}{4\pi} \left(\frac{\sin\theta}{r} \right) \cos[\omega(t - r/c)]\, \hat{\theta}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\rho_0 \mu_0 \omega^2}{4\pi c} \left(\frac{\sin\theta}{r} \right) \cos[\omega(t - r/c)]\, \hat{\varphi}$$

As Griffiths points out, these field equations "represent monochromatic spherical waves of frequency ω traveling in the radial direction at the speed of light. $\vec{E}$ & $\vec{B}$ are in phase, mutually perpendicular, and transverse; the ratio of their amplitudes is $E_0/B_0 = c$."

Now we can calculate $\vec{S}$, the $\frac{\text{energy}}{(\text{time})(\text{area})}$ of the radiated fields.

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \frac{E^2}{c}$$

$$= \frac{\mu_0}{c} \left(\frac{p_0 \omega^2}{4\pi} \right)^2 \left(\frac{\sin\theta}{r} \right)^2 \cos^2[\omega(t - r/c)] \, (\hat{\theta} \times \hat{\phi})$$

$$= \frac{p_0^2 \omega^4 \mu_0}{16\pi^2 c} \frac{\sin^2\theta}{r^2} \cos^2[\omega(t-r/c)] \, \hat{r}$$

Average this, $\langle s \rangle$, over one cycle to get the intensity. Everything in the expression is a constant but $\cos^2[\omega(t-r/c)]$. The average of $\cos^2 \varphi(t)$ over one cycle is $\frac{1}{2}$.

$$\langle s \rangle = \frac{p_0^2 \omega^4 \mu_0}{32\pi^2 c} \frac{\sin^2\theta}{r^2} \hat{r}$$

Note that for $\theta = 0$ or $\theta = \pi$, $\langle s \rangle = 0$