Radiation from Charge Densities and Currents

Our whole focus so far has been on the application of the Jefimenko equations in one form or another to the physics of point charges. It's kind of shocking how much intricate physics is in such a "simple" system!

Now we're going to switch gears and look at more general sources.

To simplify some equations we'll work a lot in the frequency domain!

$$\rho(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$

$ same for $\vec{J}$.

We'll focus on single frequency components

$$\rho(x,t) = \rho(x) e^{-i\omega t}$$
$$\vec{J}(x,t) = \vec{J}(x) e^{-i\omega t}$$

(Since all the equations are linear, we can just take the Re(·) at the end to get back real fields and sources.)

$$\vec{E}(\bar{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{ \frac{\hat{r}}{r^2}\,[\rho(x',t')]_{ret} + \frac{\hat{r}}{cr}\,\partial_t\,[\rho(x',t')]_{ret} - \frac{1}{c^2 r}\,\partial_t\,[\vec{j}(x't')]_{ret} \right\}$$

$$\vec{B}(\bar{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \left\{ [\vec{j}\,(x't')]_{ret} \times \frac{\hat{r}}{r^2} + \partial_t\,[\vec{j}\,(x't')]_{ret} \times \frac{\hat{r}}{cr} \right\}$$

where, as always, $t'_{ret} = t - r/c$, $r = |\bar{x} - \bar{x}'|$

If we plug in $\vec{E}(x,t) = \vec{E}(x)e^{-i\omega t}$,

$$\begin{aligned}[\rho(x't')]_{ret} &= \rho(x')e^{-i\omega t'_{ret}} \\ &= \rho(x')\,e^{-i\omega t + i\omega r/c} \\ &\equiv \left(\rho(x')e^{i\underset{\uparrow\, k\equiv\omega/c}{k}r}\right)e^{-i\omega t},\end{aligned}$$

etc,

Then we find

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{ \left(\frac{\hat{r}}{r^2} - \frac{i\omega\hat{r}}{cr}\right)\rho(x')e^{ikr} + \frac{i\omega}{c^2 r}\,\vec{j}(x')e^{ikr} \right\}$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \left\{ (\vec{j}(x')\times\hat{r})e^{ikr}\left(\frac{1}{r^2} - \frac{i\omega}{cr}\right) \right\}$$

All the "ret" stuff reduces just to the phase delays e^{ikr}.

Also, outside the sources (regions where $\rho(x) = j(x) = 0$),

The fields satisfy $\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \partial_t \vec{E}$ (Ampere - Maxwell)

$$\text{So} \quad \vec{\nabla} \times \left(\vec{B}(\vec{x}) e^{-i\omega t} \right) = \frac{1}{c^2} \partial_t \left(\vec{E}(\vec{x}) e^{-i\omega t} \right)$$

$$\Rightarrow \vec{\nabla} \times \vec{B}(\vec{x}) = -\frac{i\omega}{c^2} \vec{E}(\vec{x})$$

$$\text{or} \quad \boxed{\vec{E}(\vec{x}) = \frac{ic}{k} \vec{\nabla} \times \vec{B}(\vec{x})}$$

(away from sources, single Fourier mode)

So far we have been quite general.

Now consider the case where the source is of typical size d and we're interested in long wavelengths $(\lambda = 2\pi/k) \gg d$. Then we can separate space into three regions:

Near zone: $d \ll r \ll \lambda$

Induction zone: $d \ll r \sim \lambda$

Far/Radiation zone: $d \ll \lambda \ll r$

The fields in the induction zone are generally complicated, so let's focus on the near and far zones.

In the near zone, $e^{ikr} = e^{2\pi i (r/\lambda)} \longrightarrow 1.$

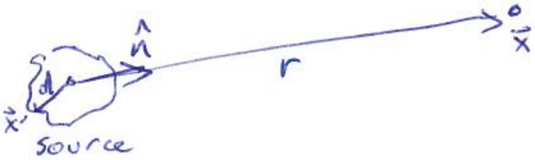
and $\frac{1}{r^2} \gg \frac{\omega}{cr} \left(= \frac{2\pi}{\lambda r} = \frac{1}{kr} \right)$

So we can approximate

(Near zone) $\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left(\frac{\hat{r}}{r^2} \rho(\vec{x}') + \underbrace{\frac{2\pi i}{\lambda r} \frac{\vec{J}(\vec{x}')}{c}}_{} \right)$ ↖ keep this in case $j \gg \rho$

$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{1}{r^2} (\vec{J} \times \hat{r})$

In the far zone, if we are far enough away to drop $1/r^2$ terms and use $|\vec{x} - \vec{x}'| \approx r - \hat{n} \cdot \vec{x}'$ for some fixed, fiducial r,



and $\hat{r} \approx \hat{n}$,

Then the radiation fields for the single fourier mode are

(far zone) $\vec{E}(\vec{x}) = \frac{-ike^{ikr}}{4\pi\epsilon_0 r} \int d^3x' \left(\hat{n}\rho - \vec{j}/c \right) e^{-ik\hat{n}\cdot\vec{x}'}$

$\vec{B}(\vec{x}) = \frac{-ik\mu_0 e^{ikr}}{4\pi r} \int d^3x' \, (\vec{j} \times \hat{n}) \, e^{-ik\hat{n}\cdot\vec{x}'}$

(Note that here we have redefined r so that it is a characteristic constant distance from the source blob to the observer. Sorry … ran out of letters!)

Now $\hat{n}\cdot\vec{x}'$ is of order d, the source size.

So <u>if</u> we agree to focus on $\lambda \gg d$, then

$$\exp(ik\,\hat{n}\cdot\vec{x}') \approx \sum_{p=0}^{N} \frac{(ik\hat{n}\cdot\vec{x}')^p}{p!}$$ where we truncate the series at some N.

So, $\vec{E}$ and $\vec{B}$ in the far zone involve integrals of the form

$$\int d^3x'\, \rho(\vec{x}')\left(\frac{\hat{n}\cdot\vec{x}'}{\lambda}\right)^p$$

and

$$\int d^3x'\, \vec{J}(x')\left(\frac{\hat{n}\cdot\vec{x}'}{\lambda}\right)^p$$

These are of decreasing importance as p increases, if $d/\lambda < 1$. This expansion is called the "multipole" expansion.

Example : Electric Dipole Radiation

The charge conservation equation is

$$\vec{\nabla}\cdot\vec{J} = -\partial_t \rho$$

For a single fourier mode, then,

$$\vec{\nabla}\cdot\vec{J}(x) = i\omega\rho(x)$$

We can use this to replace $\vec{J}$'s with ρ's as such:

Some arbitrary vector-function

Note $$\int d^3\vec{x}'\ \vec{x}'\,(\vec{\nabla}'\cdot\vec{V}(x'))$$

$$= \int d^3\vec{x}'\ (x_1', x_2', x_3')\left(\partial_{x_1'}V_1 + \partial_{x_2'}V_2 + \partial_{x_3'}V_3\right)$$

$$\underset{(IBP)}{=} -\int d^3\vec{x}'\ \left(V_1\,\partial_{x_1'}x_1',\ V_2\,\partial_{x_2'}x_2',\ V_3\,\partial_{x_3'}x_3'\right)$$

(with all other terms vanishing if V is compactly supported)

$$= -\int d^3\vec{x}'\ \vec{V}(x')$$

Applying this to: $\int d^3x'\ \vec{J}(x')$

$$= -\int d^3x'\ \vec{x}'(\vec{\nabla}'\cdot\vec{J})$$

$$= -i\omega\int d^3x'\ \vec{x}'\rho(\vec{x}') \quad \text{(charge conservation)}$$

we find (for neutral sources):

0 for neutral sources, $\int\rho = 0$

$$\vec{E}(\vec{x}) = \frac{-ik\,e^{ikr}}{4\pi\epsilon_0 r}\int d^3x'\left[\rho(x)\,\hat{n}\,(\cancel{1} - ik\hat{n}\cdot\vec{x}) + \frac{i\omega}{c}\,\vec{x}'\rho(x')\right]$$

(far field, single fourier mode)

+ (higher order)

where, in the second term, we expanded $\exp(-ik\hat{n}\cdot\vec{x}')$ to <u>zeroth</u> order ($\rightarrow 1$) and replaced $\int\vec{J}\,d^3x'$ by the formula above.

$$= \frac{-k^2e^{ikr}}{4\pi\epsilon_0 r}\left(\left(\left(\int d^3x'\ \vec{x}'\,\rho(x')\right)\times\hat{n}\right)\times\hat{n}\right)$$

Here we used $\vec{x}' - \hat{n}(\hat{n}\cdot\vec{x}') = \vec{x}'_\perp = -(\vec{x}' \times \hat{n}) \times \hat{n}$

So that the integral is $\int d^3x' \, \vec{x}' \rho(x') \equiv \vec{P}$,

the electric dipole moment.

So $$\vec{E}(x) = \frac{k^2 e^{ikr}}{4\pi\epsilon_0 r} (\hat{n} \times \vec{P}) \times \hat{n}$$

and $$\vec{B}(x) = \frac{k^2 e^{ikr} c\mu_0}{4\pi r} (\hat{n} \times \vec{P})$$ using similar methods

(so, as expected for radiation, $|B| = |E|/c$ and $\vec{B} \perp \vec{E}$.)

These are the radiation fields for an electric dipole moment oscillating with frequency ω.

The power radiated is

$$\frac{dP}{d\Omega} = r^2 \vec{S} = \frac{r^2}{\mu_0}(\vec{E} \times \vec{B})\cdot\hat{n}$$ but we have to

be careful because we have given single fourier components above. Let's take the real parts and average over a period $T = 2\pi/\omega$:

$$\frac{1}{T}\int_0^T dt\left(\mathrm{Re}\,(\vec{E}(x)\,e^{-i\omega t}\right)\times\left(\mathrm{Re}\,(\vec{B}(x)\,e^{-i\omega t})\right)$$

$$= \frac{1}{T}\int_0^T dt\,\left((\mathrm{Re}\,\vec{E})\cos\omega t+(\mathrm{Im}\,\vec{E})\sin\omega t\right)\times\left((\mathrm{Re}\,\vec{B})\cos\omega t+(\mathrm{Im}\,\vec{B})\sin\omega t\right)$$

$$= \frac{1}{2}(\mathrm{Re}\,\vec{E})\times(\mathrm{Re}\,\vec{B})+\frac{1}{2}(\mathrm{Im}\,\vec{E})\times(\mathrm{Im}\,\vec{B})$$

$$= \frac{1}{2}\,\mathrm{Re}\,(\vec{E}\times\vec{B}^*)$$

So

$$\frac{dP}{d\Omega} = \frac{r^2}{2\mu_0}\,\mathrm{Re}(\vec{E}\times\vec{B}^*)\cdot\hat{n}$$

$$= \frac{r^2}{2\mu_0}\left(\frac{k^4\mu_0 c}{(4\pi)^2\epsilon_0 r^2}\right)\mathrm{Re}\left[\left((\hat{n}\times\vec{P})\times\hat{n}\right)\times(\hat{n}\times\vec{P}^*)\right]\cdot\hat{n}$$

$$\underbrace{((\vec{A}\times\hat{n})\times(\vec{A}^*))\cdot\hat{n}}_{=\,|\hat{n}\times\vec{P}|^2 \text{ since } \vec{A}\perp\hat{n}}$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{ck^4}{8\pi}\,|\hat{n}\times\vec{P}|^2$$

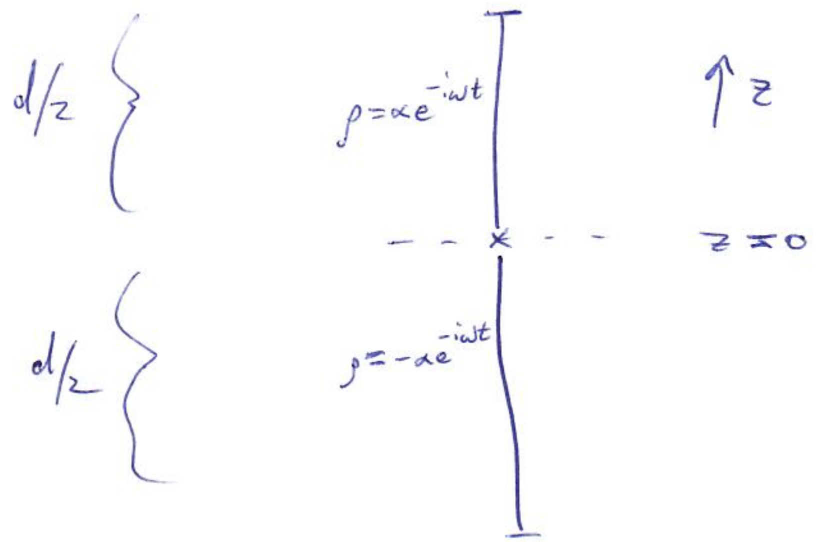
$$= \frac{1}{4\pi\epsilon_0}\,\frac{ck^4}{8\pi}\,|\vec{P}|^2\sin^2\theta$$

if all the components of $\vec{P}$ have the same phase.

Then the total power is $\int d\Omega\,\frac{dP}{d\Omega}$

$$P = \frac{1}{4\pi\epsilon_0}\,\frac{ck^4}{3}\,|\vec{P}|^2$$

A physical example is a low frequency, center-fed linear antenna:



We model each half as loaded with a constant charge density oscillating in time. The density on the two halves is opposite so the whole antenna is neutral. We can imagine it is driven by an oscillating potential at $z=0$.

Since we're on one dimension effectively,

$$i\omega\rho = \vec{\nabla}\cdot\vec{j} \text{ becomes}$$

$$\pm i\omega\alpha \cancel{e^{-i\omega t}} = \frac{dI^{\pm}}{dz}\cancel{e^{-i\omega t}}$$ on the top half (+) and bottom half (-)

Integrating, $\overset{\pm}{I}(z) = \pm i\alpha\omega z + I(0)$

or $I(z) = i\alpha\omega|z| + I(0)$

Current can't flow out the ends, so $I(\pm d/2) = 0$

$\Rightarrow \quad I(0) = -i\alpha\omega d/2 \equiv I_0$

so we write $I(z) = I_0\left(1 - \frac{2|z|}{d}\right)$

The dipole moment of the antenna is

$$p = \int_{-d/2}^{d/2} z\,\rho(z)\,dz = \int_{-d/2}^{d/2} -\frac{I_0}{i\omega}\cdot\frac{2}{d}\left(\frac{d|z|}{dz}\right)z\,dz$$

$$= \frac{2iI_0}{d\omega}\,(d/2)^2$$

$$= \frac{iI_0 d}{2\omega}$$

So
$$\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0}\,\frac{ck^4}{8\pi}\,\frac{I_0^2 d^2}{4\omega^2}\,\sin^2\theta$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{I_0^2}{32\pi c}\cdot(kd)^2\,\sin^2\theta$$

and the total power is

$$P = \frac{1}{4\pi\epsilon_0}\,\frac{I_0^2(kd)^2}{12c}$$

The power increases like the square of the frequency ($k^2 \sim \omega^2$). No surprise: the radiation fields are proportional to accelerations of charges, and for fixed "velocity" I_0, the acceleration is like velocity · frequency. Power ~ fields^2, so we get $I_0^2 \omega^2$.

Equivalent circuit

Recall that the power absorbed by a resistor is $P = \underset{\text{current}}{I} V$. This comes from doing work on a bunch of charges:

$$dW = (Ne)\,\vec{E}\cdot\vec{dx}$$

$$\begin{aligned} P = \frac{dW}{dt} &= (Ne)\,\vec{E}\cdot\vec{v} \\ &= \vec{E}\cdot\vec{I}\,dx \\ &= \text{"}IV\text{" for a linear problem} \end{aligned}$$

For
$$\vec{E} = \mathrm{Re}\left(\vec{E}_0 e^{-i\omega t}\right) \quad \text{and}$$
$$\vec{I} = \mathrm{Re}\left(\vec{I}_0 e^{-i(\omega t + \underset{\substack{\uparrow \\ \text{relative phase}}}{\varphi})}\right)$$

the power will be

$$P = I_0 E_0 \ \mathrm{Re}\left(e^{-i\omega t}\right) \mathrm{Re}\left(e^{-i(\omega t+\varphi)}\right)$$

$$= I_0 E_0 \cos(\omega t) \cos(\omega t + \varphi)$$

$$= \frac{I_0 E_0}{2} \left[\cos\varphi + \cos(2\omega t + \varphi)\right]$$

The work done on a charge over one cycle is

$$W = \frac{I_0 E_0}{2} \int_0^{2\pi/\omega} \left(\cos\varphi + \underbrace{\cos(2\omega t + \varphi)}_{\text{integrates to zero}}\right) dt$$

$$= \frac{I_0 E_0}{2} \frac{2\pi}{\omega} \cos\varphi = \frac{\pi I_0 E_0}{\omega} \cos\varphi$$

The relationship between currents and applied fields for the basic circuit elements is:

$I \propto E$ (resistor)

$I \propto \int E\,dt$ (inductor)

$I \propto dE/dt$ (capacitor)

You may be more familiar with writing it the other way around:

$V = IR$ (resistor)

$V = L\, dI/dt$ (inductor)

$V = 1/C \int I\, dt$ (capacitor)

When $I = I_0 \mathrm{Re}(e^{-i\omega t})$ we have $\pm \pi/2$ phase shifts between current and voltage for the inductor and capacitor, because e.g.

$$\mathrm{Re}\left(\partial_t \, e^{-i\omega t}\right) = \omega \sin \omega t$$
$$= \omega \cos(\omega t - \pi/2)$$

since $\cos \varphi = \cos(\pi/2) = 0$ for inductors & capacitors, we see that only the resistive component (with $\cos(\varphi{=}0) = 1$) absorbs power.

Since $\langle P \rangle = \frac{1}{2} I_0 V = \frac{1}{2} I_0^2 R$ for a resistor w/ current $I = I_0 e^{-i\omega t}$, we can identify an "effective resistance" for the antenna:

$$P = \frac{1}{4\pi\epsilon_0} \frac{I_0^2 (kd)^2}{12c} = \frac{1}{2} I_0^2 R$$

$$\Rightarrow \quad R = \frac{1}{4\pi\epsilon_0} \frac{(kd)^2}{6c}$$

If $k \sim 1/\mathrm{m}$ (GHz frequencies) and $d \sim \mathrm{cm}$ (cellphone antenna size) then $R \sim 10^{-3}\ \Omega$

(actual cellphone antennas are more complicated and transmit at higher powers $\sim 0.1 - 1$ W)

The charge density in the upper half of our linear antenna was

$$\rho = \frac{2 i I_0}{\omega d} e^{-i\omega t}$$

$$= \frac{2 I_0}{\omega d} e^{-i\omega t + i\pi/2}$$

$$= \frac{2 I_0}{\omega d} e^{-i(\omega t - \pi/2)}$$

i.e. it lags the applied current. As a result the current-voltage relationship for the antenna is mostly capacitive, except for the small resistive component due to radiation.

$\Rightarrow$ antennas are like $\overset{\text{non-ideal}}{\text{capacitors}}$ in an equivalent circuit model.

We've assumed that the antenna is short compared to the wavelength, $d \ll \lambda$. This allowed the $\rho = \text{const}$ approx in the antenna. In another Lecture or so we'll come back and generalize to larger antennas.