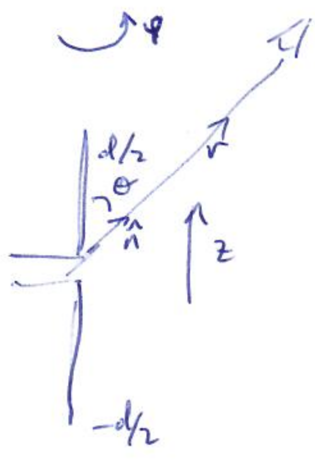
Half & Full Wave Linear Center-fed Antennae

Previously, we considered a linear center-fed antenna with the approximation that the charge density along the antenna was constant (in z). This is a bad approximation if d (antenna size) $\sim \lambda$.

Mathematically, it is a bit of a pain to generalize to $d \sim \lambda$. For a thin antenna, one uses the requirement that E_z (surf of antenna) $= 0$ to arrive at
$$I(z) = I \sin\left(\frac{kd}{2} - k|z|\right)$$
I'll skip the derivation. Let's take this as given and derive the fields.

$$\vec{J} \simeq I(z)\,\delta(x)\,\delta(y)\,\hat{z}$$



$$B^{\varphi}_{\text{rad zone}}(\vec{x}) = \frac{-ik\mu_0 e^{ikr}}{4\pi r} \int_{-d/2}^{d/2} dz\, I(z) \sin\theta\, e^{-ikz\cos\theta}$$

inserting $I(z)$ from above,

$$B^{\varphi}_{\omega} = \frac{-ik\mu_0 e^{ikr} I_0}{4\pi r}\int_{-d/2}^{d/2} \sin\left(\frac{kd}{2} - k|z|\right)\sin\theta\, e^{-ikz\cos\theta}\, dz$$

combine

The integrand can be rewritten using trig identities.

$$= \frac{-ik\mu_0 e^{ikr} I_0}{4\pi r}\left(\frac{2}{k\sin\theta}\right)\left(\cos\left(\frac{dk}{2}\cos\theta\right) - \cos\left(\frac{dk}{2}\right)\right)$$

$E \perp B \perp \hat{n}$ and $|E| = c|B|$

So we get $\vec{E} = c\vec{B}\times\hat{n}$ as usual and

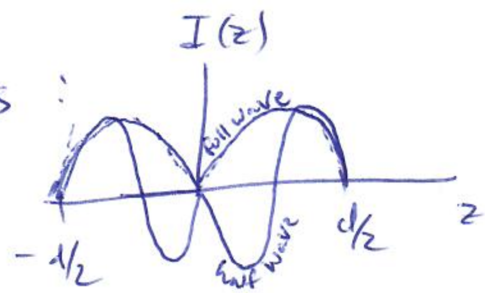
$$\frac{dP}{d\Omega} = \frac{r^2}{2\mu_0}\mathrm{Re}\left(\vec{E}\times\vec{B}^*\right) \propto \frac{\left(\cos\left(\frac{dk}{2}\cos\theta\right) - \cos\left(\frac{dk}{2}\right)\right)^2}{\sin^2\theta}$$

A rather involved formula for the angular dependence!

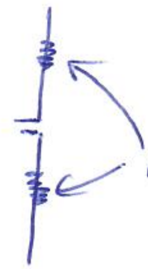
Two special cases: (1) "half wave" - $d = \lambda/2$ or $\frac{dk}{2} = \frac{\pi}{2}$

(2) "full wave" - $d = \lambda$ or $\frac{dk}{2} = \pi$.

In these cases:



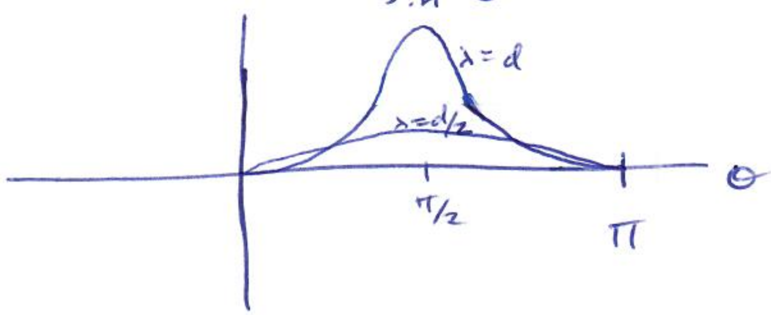
So eg for the full wave the current peaks at



And

$$\frac{dP}{d\Omega} \propto I_0^2 \cdot \begin{cases} \dfrac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} & d = \lambda/2 \\[2ex] \dfrac{(\cos(\pi\cos\theta)+1)^2}{\sin^2\theta} & d = \lambda \end{cases}$$

$$= 4\,\frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$



@ 90° the intensity from full-wave is 4x half-wave, but drops more rapidly as we depart from 90°.

We've now done a deep dive into the theory of radiation in free space. From here there are several directions we could develop:

- Material effects (beyond linear dielectrics) - nonlinear materials, plasmas, superconductors, ...
- Boundary effects - guided waves, cavity resonators, ...
- Theoretical generalizations of electrodynamics - magnetic monopoles, massive photons, confining phases, p-form electrodynamics, lattice models, ...

Since a lot of our work so far has addressed the question "where is the radiation going?", it's only natural to continue that theme and discuss guided waves - how do we use boundaries to channel waves where we want them to go?

Guided Waves

For $\vec{E}, \vec{B} \sim e^{-i\omega t}$, we have in the absence of ρ, j:

$$\vec{\nabla} \times \vec{E}_\omega = i\omega \vec{B}_\omega \qquad \vec{\nabla} \cdot \vec{B}_\omega = 0$$
$$\vec{\nabla} \times \vec{B}_\omega = -i\mu\epsilon\omega \vec{E}_\omega \qquad \vec{\nabla} \cdot E_\omega = 0$$

Some uniform medium w/ μ, ϵ.

Taking additional $\nabla \times$, these can be separated:

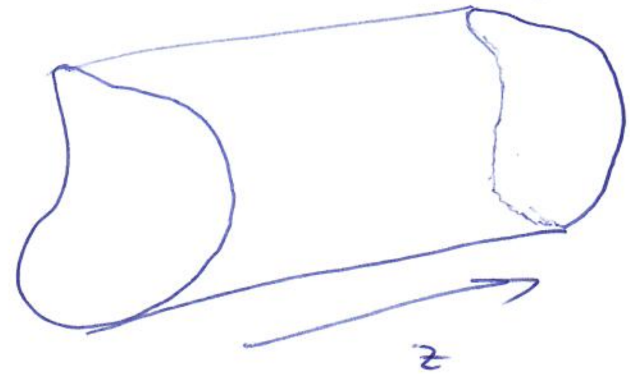
$$(\nabla^2 + \mu\epsilon\omega^2) \begin{Bmatrix} \vec{E}_\omega \\ \vec{B}_\omega \end{Bmatrix} = 0.$$

I will sometimes drop the subscript ω in the following. In free space, we would solve these to get plane waves. Now, we'll assume some cavity boundary conditions.

Specifically, we'll look at <u>perfect conductors</u>. These are defined by ① $\hat{n} \times \vec{E}\big|_S = 0$, where S is a conducting surface with normal $\hat{n}$, and ② $\hat{n} \cdot \vec{B}\big|_S = 0$.

① says $E_\parallel = 0$ @ surface S. If there were an $E_\parallel$ charges would flow to cancel it.
② is because if B had a $\perp$ comp and was time varying, charges could again flow ($F = qv \times B$) to cancel it.

To go further and solve the Maxwell eqs, we must specify a surface. A simple case is a tube — a surface with a translation symmetry:



Because of the translation symmetry in z, it's useful to fourier transform in that direction too:

$$\vec{E}(t,x,y,z) \rightarrow \vec{E}_{\omega k}(x,y)\, e^{-i\omega t \pm ikz}$$

$$\vec{B}(t,x,y,z) \rightarrow \vec{B}_{\omega k}(x,y)\, e^{-i\omega t \pm ikz}$$

Then the wave eqs become

$$\Big[\underbrace{\nabla_t^2}_{\partial_x^2 + \partial_y^2} + (\mu\epsilon\omega^2 - k^2)\Big] \begin{Bmatrix} \vec{E} \\ \vec{B} \end{Bmatrix} = 0$$

"transverse"

Likewise it's convenient to separate polarizations:

$$\vec{E} = \vec{E}_z + \vec{E}_t$$

$$\vec{B} = \vec{B}_z + \vec{B}_t$$

$$\vec{E}_z = \hat{z} E_z, \quad \vec{E}_t = (\hat{z} \times \vec{E}) \times \hat{z}, \quad \text{same for } \vec{B}.$$

Let's go back to the original form of the Maxwell eqs for $\vec{E}_\omega(x,y,z)$ and $\vec{B}_\omega(x,y,z)$ and rewrite them in two steps,

first $\vec{E}_\omega, \vec{B}_\omega \to \vec{E}_{t,\omega} + \vec{E}_{z,\omega}, \ \vec{B}_{t,\omega} + \vec{B}_{z,\omega}$

then $\vec{E}_{t,\omega}, \vec{E}_{z,\omega}, \vec{B}_{t,\omega}, \vec{B}_{z,\omega} \to \left(\vec{E}_{t,\omega k}, \vec{E}_{z,\omega k}, \vec{B}_{t,\omega k}, \vec{B}_{z,\omega k}\right) e^{ikz}$

can get e^{-ikz} by $k \to -k$

(suppressing the ω subscript for brevity)

$$\left(\vec{\nabla}\times\vec{E} - i\omega\vec{B}\right)_t = 0 \quad \Rightarrow \quad \partial_z \vec{E}_t + i\omega\hat{z}\times\vec{B}_t = \vec{\nabla}_t E_z \quad ①$$

$\hat{z}\cdot\vec{E}_z \equiv E_z$

$$\left(\vec{\nabla}\times\vec{E} - i\omega\vec{B}\right)_z = 0 \quad \Rightarrow \quad \hat{z}\cdot\left(\vec{\nabla}_t\times\vec{E}_t\right) = i\omega B_z \quad ②$$

$\hat{z}\cdot\vec{B}_z \equiv B_z$

$$\left(\vec{\nabla}\times\vec{B} + i\mu\epsilon\omega\vec{E}\right)_t = 0 \quad \Rightarrow \quad \partial_z \vec{B}_t - i\mu\epsilon\omega\hat{z}\times\vec{E}_t = \vec{\nabla}_t B_z \quad ③$$

$$\left(\vec{\nabla}\times\vec{B} + i\mu\epsilon\omega\vec{E}\right)_z = 0 \quad \Rightarrow \quad \hat{z}\cdot\left(\vec{\nabla}_t\times\vec{B}_t\right) = -i\mu\epsilon\omega E_z \quad ④$$

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \Rightarrow \quad \vec{\nabla}_t\cdot\vec{E}_t = -\partial_z E_z \quad ⑤$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}_t\cdot\vec{B}_t = -\partial_z B_z \quad ⑥$$

Next, plugging in e^{ikz} z-dependence for the fields (suppressing the $k\omega$ subscript for brevity)

(1) $\longrightarrow$ $ik\vec{E}_t + i\omega\hat{z}\times\vec{B}_t = \vec{\nabla}_t E_z$

(2) $\longrightarrow$ unchanged

(3) $\rightarrow$ $ik\vec{B}_t - i\mu\epsilon\omega\hat{z}\times\vec{E}_t = \vec{\nabla}_t B_z$

(4) $\rightarrow$ unchanged

(5) $\rightarrow$ $\vec{\nabla}_t\cdot\vec{E}_t = -ikE_z$

(6) $\rightarrow$ $\vec{\nabla}_t\cdot\vec{B}_t = -ikB_z$

If E_z and B_z are known, we can insert (3) into $ik\cdot(1)$ to write

$$(ik)^2\vec{E}_t + i\omega\hat{z}\times\underbrace{(\vec{\nabla}_t B_z + i\mu\epsilon\omega\hat{z}\times\vec{E}_t)}_{ik\vec{B}_t} = \vec{\nabla}_t E_z$$

and solve for $\vec{E}_t$ (using $\hat{z}\times(\hat{z}\times\vec{E}_t) = -\vec{E}_t$):

$$\left[(ik)^2 - (i\omega)^2\mu\epsilon\right]\vec{E}_t = ik\vec{\nabla}_t E_z - i\omega\hat{z}\times(\vec{\nabla}_t B_z)$$

(7) $$\Rightarrow \vec{E}_t = \frac{i}{(\omega^2\mu\epsilon - k^2)}\left(k\vec{\nabla}_t E_z - \omega\hat{z}\times\vec{\nabla}_t B_z\right)$$

(8) similarly $$\vec{B}_t = \frac{i}{(\omega^2\mu\epsilon - k^2)}\left(k\vec{\nabla}_t B_z + \mu\epsilon\omega\,\hat{z}\times\vec{\nabla}_t E_z\right).$$

But wait... $\mu\epsilon = \frac{1}{c^2}$ and by now we've trained ourselves to set $\omega = ck$, so the denominator would vanish... actually we couldn't divide by it in the first place...

what's going on??

Go back to the wave eqs:

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0$$

k was just for z direc - did not include info about transverse variations

If $k^2 = \mu\epsilon\omega^2$, then

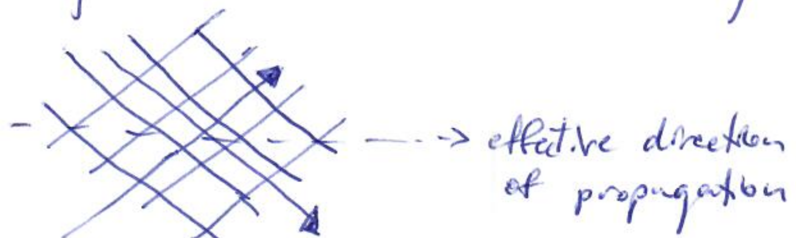
$$\nabla_t^2 E_z = 0$$
$$\nabla_t^2 \vec{E}_t = 0$$
$$\nabla_t^2 B_z = 0$$
$$\nabla_t^2 \vec{B}_t = 0$$

But E_z is a scalar that vanishes on the closed boundary of the tube. The only solution to the Laplace eq consistent with these boundary conditions is $E_z = 0$ (otherwise E_z would have some max or min inside the cylindrical tube, violating Laplace.) Likewise $\vec{\nabla}_t B_z = 0$ implies $B_z = \text{const}$, and only $B_z = 0$ is consistent with (6).
So $k^2 = \mu\epsilon\omega^2 \Rightarrow E$ and B are completely transverse $(E_z = B_z = 0)$ and $\vec{E}_t$ solves a 2D electrostatics problem, $\vec{\nabla}_t \cdot \vec{E}_t = 0$ (by eq (5))

and $\vec{\nabla}_t \times \vec{E}_t = 0$ (by eq (2))

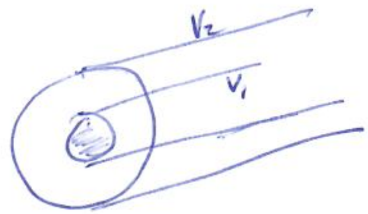
and $\vec{B}_t = \pm\frac{1}{\epsilon}\,\hat{z}\times\vec{E}_t$ (by eq (3))

These are called TEM (transverse electromagnetic) modes. Unlike in free space, transverse waves are not automatic and in fact are rather special in waveguides. The basic reason is the boundary conditions allow us to have standing waves in the transverse directions - which would look like superpositions of plane waves in free space - and as a result waves can propagate effectively in a longitudinal direction with a longitudinal component.



The TEM modes are the special cases that don't look like this. Furthermore there are no TEM modes if we just have a single tube: the 2D electrostatics problem can be written in terms of a scalar potential $\vec{E}_t = \vec{\nabla}_t V$, $\nabla_t^2 V = 0$ and the only solutions with $V =$ const on the conductor (equipotential!) are $V =$ const everywhere, so $\vec{E}_t = 0$.

To get a nonzero TEM mode you need two equipotential surfaces @ diff potentials:



The annulus between surface 1 & 2 supports nontrivial $\vec{E}_t = \vec{E}_{radial}$. Like a coax cable - TEM modes are dominant modes of propagation in these cables.

Returning to the case $c^2k^2 \neq \omega$, we see that there is only nonzero solution to ⑦ and ⑧ if the wave is not transverse!

And in this case we get the transverse fields from ⑦ and ⑧, so we really just have to solve

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right] E_z = 0$$
$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right] B_z = 0$$

with BCs

$$E_z\Big|_{surf} = 0$$

$$\frac{\partial B_z}{\partial n}\Big|_{surf} = 0.$$

(The BC for B_z comes from dotting $\hat{n}$ into (3):
$$ik \underbrace{B_\perp\Big|_s}_{=0 @ S} - i\mu\epsilon\omega \underbrace{E_\parallel\Big|_s}_{=0 @ S} = \partial_n B_z\Big|_s$$)

Since these separate nicely into eigenvalue problems for E_z and B_z (given ω (or k)), solve the eq & BC to get a spectrum of allowed k (or ω))

it is convenient to focus on:

$$\text{TM waves:} \quad B_z = 0 \text{ everywhere}, \quad E_z\big|_s = 0$$

$$\text{TE waves:} \quad E_z = 0 \text{ everywhere}, \quad \frac{\partial B_z}{\partial n}\Big|_s = 0$$

We write the general problem as

$$(\nabla_t^2 + \gamma^2)\psi = 0$$

$$\text{with } \gamma^2 = \mu\epsilon\omega^2 - k^2$$

$$\text{and} \quad \underset{(\text{TM})}{\psi\big|_s = 0}, \quad \text{or} \quad \underset{(\text{TE})}{\partial_n \psi\big|_s = 0}$$

$$E_t = \pm\frac{ik}{\gamma^2}\vec{\nabla}_t\psi, \quad \text{or} \quad B_t = \pm\frac{ik\mu}{\gamma^2}\vec{\nabla}_t\psi$$

In general, if $\gamma^2 > 0$, ψ will be oscillatory, and $\nabla_t^2\psi = -\gamma^2\psi$ will have solutions consistent with the BCs for some discrete spectrum of modes, γ_λ^2, $\lambda = 1, 2, 3 \ldots$ with ψ_λ forming an orthonormal set. Given ω, $k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$ are the allowed wavenumbers, up to $\gamma_{\lambda_{max}} \leq \sqrt{\mu\epsilon}\,\omega$. For larger λ, k_λ is imaginary and the waves attenuate in z rather than propagate.

Convenient to choose ω s.t.h. $\lambda_{max} = 1$ - only 1 propagating mode!