Chapter 10 - lecture 1

Now, after months & months in the E&M sweat-shop, we are finally ready to take a look at the fully-loaded Maxwell equations:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Dynamic equations (the fields are coupled) with the source terms ρ, $\vec{J}$ included. We have a few difficulties to deal with. One is that nothing goes faster than the speed of light, so at some place away from the sources, knowledge of what the sources are doing is delayed. This must be taken into account when calculating potentials (for example) at a place away from the sources. Another difficulty is that the equations for the potentials are now more complicated, due to the presence of all the terms.

Nevertheless, it is still often easier to find potentials first and then calculate the fields, so we don't want to just throw them out. We can simplify them using the guage freedom inherent in the equations. This means choosing $\vec{\nabla} \cdot \vec{A}$ to have a form which is suited for the dynamic equations. Way back when in magnetostatics days, we chose $\vec{\nabla} \cdot \vec{A} = 0$ (The coulomb guage) because this choice simplified analysis of certain magnetostatic problems. By choosing $\vec{\nabla} \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$ (The Lorentz Guage) we can decouple the equations with the potentials $\vec{A}$ and V.

Once we have chosen a guage, decoupled the equations for the potentials, adjusted for the delay in knowing what the sources are doing and written the field equations (pant!) we can consider the case of a moving point charge. We'll wind up the chapter with this special case, and find the potentials + fields due to a moving point charge.

First things first, finding the equations for the potentials.

Before: $\vec{\nabla} \cdot \vec{B} = 0 \implies \vec{B} = \vec{\nabla} \times \vec{A}$

Still true.

Before: $\vec{\nabla} \times \vec{E} = 0 \longrightarrow \vec{E} = -\vec{\nabla} V$

Not true, in general.

Now: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$

$$\nabla \times \underbrace{\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right)} = 0$$

If the curl of a vector is zero, that vector can be represented as the gradient of a scalar.

So: $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$

or, $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

We obtained expressions for the fields in terms of potentials using the homogeneous Maxwell's equations. To get the equations giving the potentials due to sources, we take these results and plug into the equations with the source terms.

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \quad \rightarrow \quad \boxed{\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad \rightarrow$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right)$$

use identity
$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$$\boxed{\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}}$$

Now we have those nasty looking coupled equations for the potentials, so it is time to invoke the magic of our guage freedom.

Let's start with the vector potential, $\vec{A}$. If $\vec{A} \rightarrow \vec{A}'$, it is only OK if $\vec{A}'$ gives the same field $\vec{B}$. Also, since $\vec{A}'$ is a vector, it may be related to $\vec{A}$ by adding an appropriate vector to $\vec{A}$. In other words:

$$\left.\begin{aligned} \vec{\nabla} \times \vec{A} &= \vec{B} \\ \nabla \times \vec{A}' &= \vec{B} \end{aligned}\right\} \text{Both } \vec{A} + \vec{A}' \text{ give the same field}$$

$$\vec{A} + \vec{\alpha} = \vec{A}' \qquad \text{Adding } \vec{\alpha} \text{ to } \vec{A} \text{ makes } \vec{A}'$$

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\alpha} = \vec{B} + \vec{\nabla} \times \vec{\alpha}$$

Thus, $\vec{\nabla} \times \vec{\alpha} = 0$, and so we know $\vec{\alpha}$ can be written as the gradient of a scalar.

$$\vec{\alpha} = \vec{\nabla} \lambda$$

We have to be a little more careful with the scalar potential, because it is related to the vector potential

$$-\vec{\nabla} V = \vec{E} + \frac{\partial \vec{A}}{\partial t}$$

So, a shift in $\vec{A}$ to $\vec{A}'$ <u>necessarily</u> requires that V change. V can be related to the new V' by addition of some scalar. $V' = V + \beta$

$$-\vec{\nabla} V' = \vec{E} + \frac{\partial \vec{A}'}{\partial t}$$

$$-\nabla(V + \beta) = \vec{E} + \frac{\partial \vec{A}}{\partial t} + \frac{\partial(\vec{\nabla}\lambda)}{\partial t}$$

then $\Longrightarrow$ $\beta = -\frac{\partial \lambda}{\partial t}$

We may make the following <u>guage transformation</u> of the potentials without changing the fields

$$\boxed{\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\lambda \\ V' &= V - \frac{\partial \lambda}{\partial t} \end{aligned}}$$

Now to decouple the potentials.

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\epsilon_0$$

Choosing $\vec{\nabla}\cdot\vec{A}$ to be a function of V will put the above equation in terms of V only. But what will it do to the other coupled equation? which is ->

$$\nabla^2 A - \mu_0\epsilon_0 \frac{\partial^2 A}{\partial t^2} - \nabla\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0 J$$

If this is $-\mu_0\epsilon_0\frac{\partial V}{\partial t}$ then this equation will depend only on $\vec{A}$.

OK, then.

$$\vec{\nabla}\cdot\vec{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t} \quad \Big\} \text{ Lorentz Guage}$$

The decoupled equations are

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2 A - \mu_0\epsilon_0\frac{\partial^2 A}{\partial t^2} = -\mu_0 J$$