Today's outline - February 02, 2023

- Singly controlled transformations
- Multiply controlled operators
- Arbitrary controlled operators
- Implementing general operators
- Universally approximating gates

Reading Assignment: Reiffel: 6.1-6.3 Wong: 4.5.2-4.5.6

Homework Assignment #03:
due Tuesday, February 07, 2023

Homework Assignment #04:
due Tuesday, February 14, 2023

Singly controlled transformations

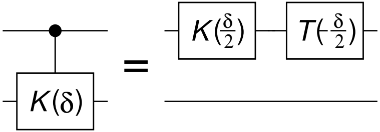
We wish to implement a controlled operator $\bigwedge Q$ where $Q = K(\delta)T(\alpha)R(\beta)T(\delta)$ and

$$K(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \qquad R(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \qquad T(\alpha) = \begin{pmatrix} e^{+i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

Because the $K(\delta)$ operator is a global phase shift it is possible to write that
$\bigwedge Q = \bigwedge K(\delta) \bigwedge (T(\alpha)R(\beta)T(\gamma)) = (\bigwedge K(\delta))(\bigwedge Q')$

The conditional phase shift, $\bigwedge K_\delta$ can be implemented using

$$\begin{aligned} \bigwedge K_\delta &= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes K(\delta) \\ &= |0\rangle\langle 0| \otimes I + e^{i\delta}|1\rangle\langle 1| \otimes I \\ &= \left(K(\tfrac{\delta}{2})T(-\tfrac{\delta}{2})\right) \otimes I \end{aligned}$$



Note that the conditional phase shift is realized by acting on the first qubit only since a phase shift changes the entire state

Singly controlled transformations (cont.)

Implementing $\bigwedge Q' = \bigwedge(T(\alpha)R(\beta)T(\gamma))$ requires defining three additional transformations

$$Q_0 = T(\alpha)R(\tfrac{\beta}{2}) = \begin{pmatrix} e^{+i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

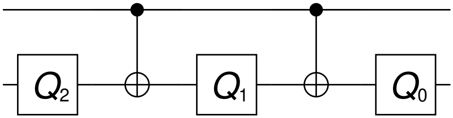
$$Q_1 = R(-\tfrac{\beta}{2})T(-\tfrac{\gamma+\alpha}{2}) = \begin{pmatrix} \cos\frac{-\beta}{2} & \sin\frac{-\beta}{2} \\ -\sin\frac{-\beta}{2} & \cos\frac{-\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i(\frac{\gamma+\alpha}{2})} & 0 \\ 0 & e^{+i(\frac{\gamma+\alpha}{2})} \end{pmatrix}$$

$$Q_2 = T(\tfrac{\gamma-\alpha}{2}) = \begin{pmatrix} e^{+i(\frac{\gamma-\alpha}{2})} & 0 \\ 0 & e^{-i(\frac{\gamma-\alpha}{2})} \end{pmatrix}$$

The assertion is that

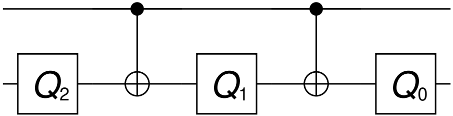
$\bigwedge Q' = (I \otimes Q_0)C_{not}(I \otimes Q_1)C_{not}(I \otimes Q_2)$,

or in graphical terms



Singly controlled transformations (cont.)

$$Q_0 = T(\alpha)R(\tfrac{\beta}{2}), \qquad Q_1 = R(-\tfrac{\beta}{2})T(-\tfrac{\gamma+\alpha}{2}), \qquad Q_2 = T(\tfrac{\gamma-\alpha}{2})$$



This circuit does the following

$$|0\rangle \otimes |x\rangle \longrightarrow |0\rangle \otimes Q_0 Q_1 Q_2 |x\rangle$$
$$|1\rangle \otimes |x\rangle \longrightarrow |1\rangle \otimes Q_0 X Q_1 X Q_2 |x\rangle$$

$$\begin{aligned} Q_0 Q_1 Q_2 &= T(\alpha)R(\tfrac{\beta}{2})R(-\tfrac{\beta}{2})T(-\tfrac{\gamma+\alpha}{2})T(\tfrac{\gamma-\alpha}{2}) \\ &= T(\alpha)T(-\tfrac{\gamma+\alpha}{2})T(\tfrac{\gamma-\alpha}{2}) = T(\alpha)T(-\alpha) = I \end{aligned}$$

but $R(\beta)R(-\beta) \equiv I$

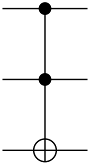
and $T(\alpha)T(\gamma) = T(\alpha+\gamma)$

$$\begin{aligned} Q_0 X Q_1 X Q_2 &= T(\alpha)R(\tfrac{\beta}{2})XR(-\tfrac{\beta}{2})T(-\tfrac{\gamma+\alpha}{2})XT(\tfrac{\gamma-\alpha}{2}) \\ &= T(\alpha)R(\tfrac{\beta}{2})XR(-\tfrac{\beta}{2})XXT(-\tfrac{\gamma+\alpha}{2})XT(\tfrac{\gamma-\alpha}{2}) \\ &= T(\alpha)R(\tfrac{\beta}{2})R(\tfrac{\beta}{2})T(\tfrac{\gamma+\alpha}{2})T(\tfrac{\gamma-\alpha}{2}) \\ &= T(\alpha)R(\beta)T(\gamma) = Q' \end{aligned}$$

but $XR(\beta)X = R(-\beta)$

and $XT(\alpha)X = T(-\alpha)$

Multiply controlled transformations



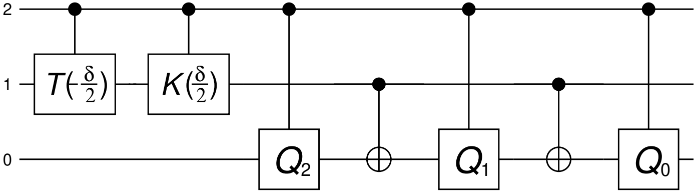
Controlled operations can be generalized to more than one control bit

$\bigwedge_k Q$ represents a $(k+1)$-qubit transformation that applied Q to the low order qubit if all of the other qubits are 1

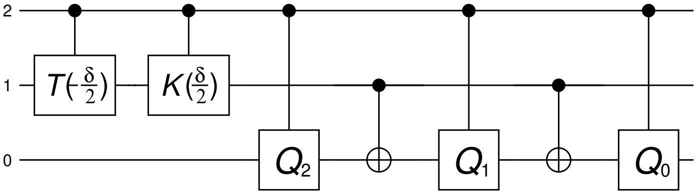
The CC_{not}, also called the Toffoli gate, $\bigwedge_2 X$ negates the last bit if the first two are 1

The arbitrary Q transformation can also be controlled by multiple qubits

The $\bigwedge_2 Q$ three-qubit gate can be obtained by adding control of the Q_0, Q_1, and Q_2 by the third qubit



Multiply controlled transformations (cont.)

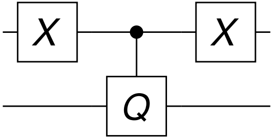


This circuit can be expanded in terms of the general phase shift and rotation gates plus C_{not}, however it requires 25 single qubit gates and 12 C_{not} gates

For a general k-qubit controlled arbitrary gate, one needs 5^k single qubit gates plus $\frac{1}{2}(5^k - 1)$ C_{not} gates which is not the most efficient implementation

Suppose we want to apply a transformation when the control qubit is 0 or a specific combination of 1's and 0's

This is possible by adding two X gates to the control bit



Arbitrary controlled transformations

These multiply controlled qubit gates will permit arbitrary circuits

Suppose we have a $(k+1)$-qubit system to which we wish to apply transformation Q on the i^{th} qubit when all the other qubits are in a specific basis state

The transformation Q is thus applied to a 2-dimensional subspace spanned by the vector with x_i and its flipped state in the standard basis, $\hat{x}_i$

$$\left\{|s_k \ldots s_{i+1} x_i s_{i-1} \ldots s_0\rangle, |s_k \ldots s_{i+1} \hat{x}_i s_{i-1} \ldots s_0\rangle\right\}, \; \hat{x}_i = x_i \oplus 1 \text{ (XOR)} \;\longrightarrow\; \left\{|x\rangle, |\hat{x}\rangle\right\} \; \hat{x} = x \oplus 2^i$$

It will be useful to define two different transformations using a k-qubit string and a single qubit transformation, Q, on a separate qubit, both of which can be represented as $\bigwedge_x^i Q$

x is a $(k+1)$-qubit string where the i^{th} qubit $|x_i\rangle$ is either $|0\rangle$ or $|1\rangle$ and the other qubits are defined as $s_k \ldots s_{i+1} s_{i-1} \ldots s_0$

If $|x_i\rangle = |0\rangle$, $Q|x_i\rangle$ is applied but if $|x_i\rangle = |1\rangle$, $XQX|x_i\rangle$ is applied

This operator has the property that: $\bigwedge_{\hat{x}}^i Q = \bigwedge_x^i \hat{Q} = \bigwedge_x^i XQX$

A 2-qubit example

A simplified example of the general $\bigwedge_x^i Q$ transformation is that of a 2-qubit system $|b_1 b_0\rangle$

Operator	Initial State	Action	Final State	Overall Effect
$\bigwedge_{10}^0 X$	$\|00\rangle$	$I\|b_0\rangle$	$\|00\rangle$	C_{not}: $\|b_1\rangle_{ctl} \rightarrow \|b_0\rangle_{tgt}$
	$\|01\rangle$	$I\|b_0\rangle$	$\|01\rangle$	
	$\|10\rangle$	$X\|b_0\rangle$	$\|11\rangle$	
	$\|11\rangle$	$XXX\|b_0\rangle$	$\|10\rangle$	
$\bigwedge_{11}^0 X$	$\|00\rangle$	$I\|b_0\rangle$	$\|00\rangle$	C_{not}: $\|b_1\rangle_{ctl} \rightarrow \|b_0\rangle_{tgt}$
	$\|01\rangle$	$I\|b_0\rangle$	$\|01\rangle$	
	$\|10\rangle$	$XXX\|b_0\rangle$	$\|11\rangle$	
	$\|11\rangle$	$X\|b_0\rangle$	$\|10\rangle$	
$\bigwedge_{00}^0 X$	$\|00\rangle$	$X\|b_0\rangle$	$\|01\rangle$	C_{not}: $\|\hat{b}_1\rangle_{ctl} \rightarrow \|b_0\rangle_{tgt}$
	$\|01\rangle$	$XXX\|b_0\rangle$	$\|00\rangle$	
	$\|10\rangle$	$I\|b_0\rangle$	$\|10\rangle$	
	$\|11\rangle$	$I\|b_0\rangle$	$\|11\rangle$	

Note that $\bigwedge_{01}^1 X$ has the effect of C_{not}: $|b_0\rangle_{ctl} \rightarrow |b_1\rangle_{tgt}$

Implementing general unitary transformations

As we have seen, any unitary transformation is just a rotation of the 2^n-dimensional vector space associated with an n-qubit system

Let $N = 2^n$ and define the standard basis as $\{|x_0\rangle, \dots, |x_{N-1}\rangle\}$ such that $|x_i\rangle$ and $|x_{i+1}\rangle$ differ only by a single bit (called Gray code)

We can define a suitable Gray code by saying that for $0 \le i \le N-2$, define j_i as the bit that differs between $|x_i\rangle$ and $|x_{i+1}\rangle$ and B_i as the shared pattern of all the the other bits in the two vectors

U_m is an operator defined as

$$U_m = \begin{pmatrix} I^{(m)} & 0 \\ 0 & V_{N-m} \end{pmatrix}$$

where $I^{(m)}$ is the $m \times m$ identity matrix and V_{N-m} is an $(N-m) \times (N-m)$ unitary matrix with $0 \le m \le N-2$

Start with $m = N-2$ at its maximum value and the smallest possible unitary matrix V_2 representing only 2 qubits

$$U_{N-2} = \begin{pmatrix} I^{(N-2)} & 0 \\ 0 & V_2 \end{pmatrix}$$

Applying this operator is identical to applying $\bigwedge_x^j V_2$ where $x = x_{N-2}$ and $j = j_{N-2}$

Generating the general unitary operator

Given the unitary matrix U_{m-1}, and the basis $\{|x_0\rangle, \dots, |x_{m-1}\rangle, \dots, |x_{N-1}\rangle\}$, the basis vector $|x_{m-1}\rangle$ is the first on which the operator has a non-trivial action since the identity matrix is $(m-1) \times (m-1)$ and $V_{N-(m-1)}$ mixes the last $N-(m-1)$ basis vectors

$$|v_{m-1}\rangle = U_{m-1}|x_{m-1}\rangle = a_{m-1}|x_{m-1}\rangle + \cdots + a_{N-1}|x_{N-1}\rangle$$

The coefficient a_{N-1} can be made real by applying a global phase shift so we need to find a unitary transformation W_m that takes $|v_{m-1}\rangle$ to $|x_{m-1}\rangle$ and does not affect basis elements $|x_0\rangle, \dots, |x_{m-1}\rangle$

This transformation will then have the property that

$$U_m = W_m U_{m-1} \longrightarrow C_m = W_m^{-1} \longrightarrow U_{m-1} = C_m U_m \longrightarrow U = U_0 = C_1 \cdots C_{N-2} U_{N-2}$$

W_m is defined iteratively starting by rewriting $|v_{m-1}\rangle$ as

$$|v_{m-1}\rangle = a_{m-1}|x_{m-1}\rangle + \cdots + c_{N-2}\cos(\theta_{N-2})e^{i\phi_{N-2}}|x_{N-2}\rangle + c_{N-2}\sin(\theta_{N-1})|x_{N-1}\rangle$$

Generating the general unitary operator (cont.)

$$
\begin{aligned}
|v_{m-1}\rangle &= a_{m-1}|x_{m-1}\rangle + \cdots + a_{N-1}|x_{N-1}\rangle \\
&= a_{m-1}|x_{m-1}\rangle + \cdots + c_{N-2}\cos(\theta_{N-2})e^{i\phi_{N-2}}|x_{N-2}\rangle + c_{N-2}\sin(\theta_{N-1})|x_{N-1}\rangle
\end{aligned}
$$

$$a_{N-2} = |a_{N-2}|e^{i\phi_{N-2}} \qquad \cos(\theta_{N-2}) = \frac{|a_{N-2}|}{c_{N-2}}$$

$$c_{N-2} = \sqrt{|a_{N-2}|^2 + |a_{N-1}|^2} \qquad \sin(\theta_{N-2}) = \frac{|a_{N-1}|}{c_{N-2}}$$

With these definitions, we can write a multiply controlled set of single qubit operators that acts on $|v_{m-1}\rangle$ to eliminate the $|x_{N-1}\rangle$ term

$$\bigwedge_{x_{N-2}}^{j_{N-2}} R(\theta_{N-2}) \bigwedge_{x_{N-2}}^{j_{N-2}} K(-\phi_{N-2})|v_{m-1}\rangle = a_{m-1}|x_{m-1}\rangle + \cdots + a'_{N-2}|x_{N-2}\rangle, \quad a'_{N-2} = c_{N-2}$$

The $K(-\phi_{N-2})$ eliminates the phase factor in front of $|x_{N-2}\rangle$ and the $R(\theta_{N-2})$ rotates amplitude from $|x_{N-1}\rangle$ to $|x_{N-2}\rangle$

Generating the general unitary operator (cont.)

The multiply controlled gate ensures that only the two basis vectors with the identical qubit pattern B_{N-2} are affected by this transformation

This same procedure is repeated for the next two lowest order qubit states until $|v_{m-1}\rangle = a'_m|x_{m-1}\rangle \equiv |x_{m-1}\rangle$ and this results in a composite operator

$$W_m = \bigwedge_{x_{m-1}}^{j_{m-1}} R(\theta_{m-1}) \bigwedge_{x_{m-1}}^{j_{m-1}} K(-\phi_{m-1}) \cdots \bigwedge_{x_{N-2}}^{j_{N-2}} R(\theta_{N-2}) \bigwedge_{x_{N-2}}^{j_{N-2}} K(-\phi_{N-2})$$

$$a_i = |a_i|e^{i\phi_i}, \quad a'_i = c_i, \quad c_i = \sqrt{|a_i|^2 + |a_{i+1}|^2}, \quad \cos\theta_i = \frac{|a_i|}{c_i}, \quad \sin\theta_i = \frac{|a'_{i+1}|}{c_i}$$

This procedure guarantees a general unitary transformation but it is exponentially expensive and therefore is of limited value

Making a practical quantum computer requires a more clever approach to take advantage of the inherent efficiency in the computations

A 3-bit example

Consider a 3-qubit system where we wish to establish a Grey code basis

$$\begin{array}{lcccccccccr} \{ & |111\rangle, & |011\rangle, & |001\rangle, & |000\rangle, & |010\rangle, & |110\rangle, & |100\rangle, & |101\rangle & \} \\ \{ & |x_0\rangle, & |x_1\rangle, & |x_2\rangle, & |x_3\rangle, & |x_4\rangle, & |x_5\rangle, & |x_6\rangle, & |x_7\rangle & \} \end{array}$$

In this case, $n = 3$, $N = 2^n = 8$, and $0 \leq m \leq N - 2 = 6$

Let's look at the U_6 and U_5 operators

$$U_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & c & d \end{pmatrix} \qquad U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & o & p & q \\ 0 & 0 & 0 & 0 & 0 & r & s & t \\ 0 & 0 & 0 & 0 & 0 & u & v & w \end{pmatrix}$$

A 3-bit example (cont.)

Our goal is to generate a universal operator

$$U = U_0 = C_1 \cdots C_6 U_6$$

Starting with the U_5 matrix, we want an operator W_6 that satisfies $W_6 U_5 = U_6$

The U_5 operator leaves all the basis vectors from $|x_0\rangle \cdots |x_4\rangle$ alone so we can write

U_5 mixes the last three basis vectors

Now rewrite the coefficients using

$a_6 = |r|e^{i\phi_6}, \qquad c_6 = \sqrt{|r|^2 + |u|^2}$

$\cos\theta_6 = \frac{|r|}{c_6}, \qquad \sin\theta_6 = \frac{|u|}{c_6}$

This eliminates the $|x_7\rangle$ term and can be repeated to eliminate the $|x_6\rangle$ term

$$U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & o & p & q \\ 0 & 0 & 0 & 0 & 0 & r & s & t \\ 0 & 0 & 0 & 0 & 0 & u & v & w \end{pmatrix}$$

$$\begin{aligned} |v_5\rangle &= U_5|x_5\rangle = o|x_5\rangle + r|x_6\rangle + u|x_7\rangle \\ &= o|x_5\rangle + c_6\cos\theta_6|x_6\rangle + c_6\sin\theta_6|x_7\rangle \end{aligned}$$

$$\bigwedge_{x_6}^{j_0} R(\theta_6) \bigwedge_{x_6}^{j_0} K(-\phi_6)|v_5\rangle = o|x_5\rangle + c_6|x_6\rangle$$

Universally approximating set of gates

The problem we encountered in trying to make a general unitary operator out of simple gates cannot be solved exactly, however the Solovay-Kitaev theorem states that there are finite sets of gates that can approximate any unitary transformation to arbitrary accuracy efficiently

If we desire accuracy to a level of 2^{-d}, there exists a polynomial $p(d)$ such that any single-qubit unitary transformation can be approximated to the desired accuracy by a sequence of no more than $p(d)$ gates

We want to find a finite set of gates that can approximate all single-qubit transformations so that with the addition of the C_{not}, we can prepare any unitary operator

Take the Hadamard and the C_{not} gates and add two phase gates $P_{\frac{\pi}{2}}$ and $P_{\frac{\pi}{4}}$

$$P_{\frac{\pi}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} = |0\rangle\langle 0| + i|1\rangle\langle 1|, \quad P_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = |0\rangle\langle 0| + e^{i\frac{\pi}{4}}|1\rangle\langle 1| = e^{i\frac{\pi}{8}}T(-\frac{\pi}{8})$$