Notes on Commutator Identities

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1 Preliminaries

Let A, B, C be operators on a Hilbert space or square matrices.

Theorem 1.1. *If $[A, B]$ commutes with A, then for any power series $f(A)$,*

$$[B, f(A)] = [B, A] \frac{df(A)}{dA}.$$

Proof. Exercise. □

Theorem 1.2 (Jacobi Identity)**.** $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$

Proof. Expanding all commutators shows that all terms cancel. □

2 Baker-Campbell-Hausdorff (BCH) Formulae

2.1 Lemmas

Let A and B be operators on a Hilbert space.

Definition 2.1 (Adjoint action)**.** *If B is invertible, the Adjoint action $\mathrm{Ad}_B(A)$ of B on A is the conjugation operation defined by*

$$\mathrm{Ad}_B(A) = BAB^{-1}.$$

Definition 2.2 (adjoint action)**.** *The adjoint action $\mathrm{ad}_B(A)$ of B on A is the commutator operation defined by*

$$\mathrm{ad}_B(A) = [B, A].$$

Lemma 2.1.

$$\mathrm{Ad}_{e^B}(A) = e^{\mathrm{ad}_B}(A).$$

That is, Taylor expanding the right hand side,

$$\boxed{e^B A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}_B)^n(A) = A + [B, A] + \frac{1}{2}[B, [B, A]] + \frac{1}{3!}[B, [B, [B, A]]] + \cdots}$$

Proof. Consider the 1-parameter family $S(A, B, t) = \mathrm{Ad}_{e^{tB}}(A)$ of transformations. We can expand $S(A, B, t)$ as a formal power series as follows:

$$S(A, B, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f_n(A, B). \tag{2.1}$$

Note that $f_0(A,B) = A$. Taking derivative of $S(A,B,t)$ with respect to t, we get

$$\frac{dS(A,B,t)}{dt} = \mathrm{Ad}_{e^{tB}}([B,A]) \equiv S([B,A],B,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f_n(\mathrm{ad}_B(A),B). \tag{2.2}$$

But, taking derivative of (2.1) with respect to t yields,

$$\frac{dS(A,B,t)}{dt} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} f_n(A,B) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f_{n+1}(A,B).$$

Matching the coefficients of t^n with those in (2.2) yields

$$f_{n+1}(A,B) = f_n(\mathrm{ad}_B(A),B) = f_{n-1}(\mathrm{ad}_B(\mathrm{ad}_B(A),B) = \cdots = f_0((ad_B)^{n+1}(A),B) = (ad_B)^{n+1}(A).$$

Setting $t = 1$ proves the lemma. □

Corollary 2.1. *For any positive integer n,*

$$\boxed{\mathrm{Ad}_{e^B}(A^n) \equiv e^B A^n e^{-B} = \left[e^{\mathrm{ad}_B}(A)\right]^n}.$$

Proof. Since $e^{-B}e^B = 1$, we have

$$e^B A^n e^{-B} = \left[e^B A e^{-B}\right]^n = \left[e^{\mathrm{ad}_B}(A)\right]^n,$$

where we have applied Lemma 2.1 in the last step. □

Corollary 2.2.

$$\boxed{\mathrm{Ad}_{e^B}(e^A) \equiv e^B e^A e^{-B} = \exp[e^{\mathrm{ad}_B}(A)]}.$$

Thus,

$$e^B e^A = \exp[e^{\mathrm{ad}_B}(A)] e^B.$$

Proof. Expanding e^A in power series and applying Corollary 2.1 to each term proves that $e^B e^A e^{-B} = \exp[e^{\mathrm{ad}_B}(A)]$. □

Lemma 2.2. *Let $G(t)$ be an operator valued function of t, and denote $G' = dG/dt$.*

$$\boxed{e^{-G}\frac{de^G}{dt} = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_{-G})^n G'}{(n+1)!}}.$$

Proof. First, we note that

$$\frac{de^G}{dt} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{dG^n}{dt} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n-1} G^m G' G^{n-m-1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{G^n G' G^m}{(n+m+1)!}$$

But,

$$\frac{1}{(n+m+1)!} = \frac{1}{n!m!} B(n+1, m+1)$$

where the Beta function $B(\alpha, \beta)$ is given by

$$B(\alpha, \beta) = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} dz = \int_0^1 (1-z)^{\alpha-1} z^{\beta-1} dz.$$

Hence,

$$\frac{de^G}{dt} = \int_0^1 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (1-z)^n z^m \frac{G^n G' G^m}{n! m!} dz = \int_0^1 e^{(1-z)G} \, G' \, e^{zG} \, dz.$$

Multiplying the last equation by e^{-G} on the left and applying Lemma 2.1, we get

$$e^{-G} \frac{de^G}{dt} = \int_0^1 e^{-zG} \, G' \, e^{zG} \, dz = \int_0^1 e^{\mathrm{ad}_{-zG}} G' \, dz = \sum_{n=0}^{\infty} \int_0^1 \frac{(\mathrm{ad}_{-zG})^n G'}{n!} \, dz = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_{-G})^n G'}{(n+1)!}.$$

□

2.2 BCH Formulae

Theorem 2.3 (BCH Version 1)**.** *Suppose $[A, B]$ commutes with both A and B, then*

$$\boxed{e^A e^B = e^{A+B+\frac{1}{2}[A,B]}}.$$

Proof. Define $G(t)$ via the equation

$$e^{G(t)} = e^{tA} e^{tB}.$$

Taking derivative of $e^{G(t)}$ with respect to t, we get

$$\frac{de^{G(t)}}{dt} = Ae^{tA}e^{tB} + e^{tA}Be^{tB} = (A + e^{tA}Be^{-tA})G(t). \tag{2.3}$$

Applying Lemma 2.1 to $e^{tA}Be^{-tA}$ and noting that $(ad_A)^n B = 0$ for any $n > 1$, we get

$$\frac{de^{G(t)}}{dt} = (A + B + [A, B]) \, G(t). \tag{2.4}$$

Now, note that $A + B + [A, B]$ commutes with $t(A + B) + \frac{1}{2}t^2[A, B]$ and that

$$\frac{d}{dt}\left(t(A + B) + \frac{1}{2}t^2[A, B]\right) = A + B + [A, B].$$

Thus, together with the initial condition $G(0) = 1$, the unique solution to (2.4) is thus

$$G(t) = e^{t(A+B)+\frac{1}{2}t^2[A,B]}.$$

Setting $t = 1$ proves the theorem. □

Theorem 2.4 (BCH Version 2)**.** *In general, we have*

$$\boxed{e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}(\operatorname{ad}_A^2 B + \operatorname{ad}_B^2 A) - \frac{1}{24}\operatorname{ad}_B(\operatorname{ad}_A)^2 B + \cdots\right)}.$$

Proof. Multiplying (2.3) on the left by $e^{-G(t)}$ and applying Lemma 2.1, we get

$$e^{-G(t)}\frac{de^{G(t)}}{dt} = e^{-G(t)}(A + e^{tA}Be^{-tA})\,G(t) = e^{-tB}Ae^{tB} + B = e^{-t\operatorname{ad}_B}A + B. \tag{2.5}$$

Applying Lemma 2.2 to the left hand side now yields

$$\sum_{n=0}^{\infty}\frac{(\operatorname{ad}_{-G})^n G'}{(n+1)!} = e^{-t\operatorname{ad}_B}A + B. \tag{2.6}$$

Expanding $G(t) = tG_1 + t^2G_2 + t^3G_3 + t^4G_4\cdots$, we get

$$G'(t) = G_1 + 2tG_2 + 3t^2G_3 + 4t^3G_4\cdots.$$

Matching the coefficients of t^n in (2.6) yields:

$$\begin{aligned}
t^0: &\quad G_1 = A + B\\
t^1: &\quad 2G_2 = -t\operatorname{ad}_B(A) = [A,B] \Rightarrow G_2 = \frac{1}{2}[A,B]\\
t^2: &\quad 3G_3 - \frac{1}{2}[G_1,G_2] = \frac{1}{2}(ad_B^2 A \Rightarrow G_3 = \frac{1}{12}(\operatorname{ad}_A^2 B + \operatorname{ad}_B^2 A)\\
t^3: &\quad 4G_4 - [G_1,G_3] + \frac{1}{6}[G_1,[G_1,G_2]] = -\frac{1}{6}\operatorname{ad}_B^3 A \Rightarrow G_4 = -\frac{1}{24}\operatorname{ad}_B(\operatorname{ad}_A)^2 B.
\end{aligned}$$

One can similarly compute the higher order correction terms. Setting $t = 1$ proves the theorem. ☐

(N.B. In computing the t^3 term, one needs to use the formula $\operatorname{ad}_B(\operatorname{ad}_A)^2 B = -\operatorname{ad}_A(\operatorname{ad}_B)^2 A$, which follows from the Jacobi identity.)