Addition of Angular Momentum

Reference: Griffiths 4.4.3, McIntyre 11.4 - 11.6

So far, we've learned that elementary particles (like the electron) carry their own angular momentum (spin). Particles have a value of s, the spin quantum number, which doesn't ever change. The spin is a property of the particle (all electrons have $s = 1/2$ always).

Now, what about the following situations:

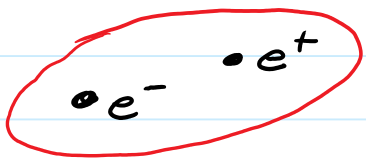
① A composite system made of 2 particles? If each particle has spin the total spin $\vec{S}_{TOT} = \vec{S}_1 + \vec{S}_2$

② A system which has both spin and orbital angular momentum the total angular momentum is $\vec{J} = \vec{L} + \vec{S}$

In both cases we need to learn how to add angular momentum quantum mechanically. [It's not as simple as just adding vectors because, remember, I can't know all 3 components of the vectors at once!].

Let's focus on case ①. Case ② is done exactly the same way.

Example: Positronium - it is possible to create a bound state of an electron and its antiparticle (positron)



"positronium"

(it is not stable ... doesn't live long).

Both the electron and the positron have spin $s = \frac{1}{2}$. What are the possible spin states $|\, s, m \rangle$ for this bound state?

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The spin state of the composite state can be specified by the spin states of the two constituents. We'll label these by $\underline{\underline{4}}$ numbers:

order matters. The first ket is particle #1.

$$|s_1, m_1\rangle |s_2, m_2\rangle$$ I like this notation using 2 kets.

Griffiths uses a single ket $|\underbrace{s_1 \overbrace{s_2\, m_1}^{\text{particle 1 quantum \#s.}} m_2}_{\text{quantum \#'s for particle 2.}}\rangle$

The operators $\hat{S}_z^{(1)}$ and $\hat{S}_z^{(2)}$ are understood to act only on the relevant particle. The superscript $^{(1)}$ or $^{(2)}$ denotes this.

poll Q: Consider the composite state formed from a spin 3 particle and a spin 2 particle: $|3221\rangle$. What is $S_z^{(1)}|3221\rangle$?

A.) $3\hbar$ C.) $\hbar$

B.) $2\hbar$ D.) 0

Let's go back to the positronium case which is two spin $\frac{1}{2}$ particles.

$$|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\,\tfrac{-1}{2}\rangle = \underbrace{|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_e}_{\text{electron}}\underbrace{|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle_p}_{\text{positron}} = \underbrace{|\uparrow\rangle_e |\downarrow\rangle_p}_{e^- \text{ spin up, } e^+ \text{ spin down.}}$$

The two kets are basically independent. So, for example if I want the inner product

$$\langle\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{-1}{2}|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}\rangle = \left({}_e\langle\uparrow|\ {}_p\langle\downarrow|\right)\left(|\uparrow\rangle_e |\uparrow\rangle_p\right)$$

$$= \langle\uparrow|\uparrow\rangle_e \langle\downarrow|\uparrow\rangle_p = 1\cdot 0 = 0$$

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We'll write these states as a single ket: $|\uparrow\downarrow\rangle = |\uparrow\rangle_e |\downarrow\rangle_p$
the order matters. The first arrow is for particle 1 (say, the e^-)
the second is particle 2 (say, the positron).

There are four possible combinations:

$$\left|\tfrac{1}{2}\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2}\right\rangle \equiv |\uparrow\uparrow\rangle$$
$$\left|\tfrac{1}{2}\tfrac{1}{2},\tfrac{1}{2}\tfrac{-1}{2}\right\rangle \equiv |\uparrow\downarrow\rangle$$
$$\left|\tfrac{1}{2}\tfrac{1}{2},\tfrac{-1}{2}\tfrac{1}{2}\right\rangle \equiv |\downarrow\uparrow\rangle$$
$$\left|\tfrac{1}{2}\tfrac{1}{2},\tfrac{-1}{2}\tfrac{-1}{2}\right\rangle \equiv |\downarrow\downarrow\rangle$$

These four states define an orthonormal basis. By including two particles I now have $\underline{\underline{4}}$ basis vectors.

$$|\uparrow\uparrow\rangle \to \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |\uparrow\downarrow\rangle \to \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |\downarrow\uparrow\rangle \to \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |\downarrow\downarrow\rangle \to \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Question: If I measure $\hat{S}^2$, the total angular momentum squared and/or $\hat{S}_z$ on one of these states, what values could I get? What are the probabilities?

We know how to attack this problem: Find the matrix elements of the operators & then find the eigenvalues & eigenvectors.

$\hat{S}_z$ is easy. Just add the components. $\hat{S}_z = \hat{S}_z^{(1)} + \hat{S}_z^{(2)}$

What are its matrix elements? For example:

$$\langle\uparrow\downarrow|\hat{S}_z|\uparrow\downarrow\rangle = {}_e\langle\uparrow| {}_p\langle\downarrow| \, \hat{S}_z^{(1)} + \hat{S}_z^{(2)} \, |\uparrow\rangle_e |\downarrow\rangle_p$$

Group e^- and e^+ terms together

$${}_e\langle\uparrow| {}_p\langle\downarrow| S_z^{(1)} |\uparrow\rangle_e |\downarrow\rangle_p + {}_e\langle\uparrow| {}_p\langle\downarrow| \hat{S}_z^{(2)} |\uparrow\rangle_e |\downarrow\rangle_p$$

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Group e^- and e^+ terms together

$${}_e\langle\uparrow|\,{}_p\langle\downarrow|\, S_z^{(1)}\,|\uparrow\rangle_e\,|\downarrow\rangle_p + {}_e\langle\uparrow|\,{}_p\langle\downarrow|\, \hat{S}_z^{(2)}\,|\uparrow\rangle_e\,|\downarrow\rangle_p$$

$$= \langle\uparrow|S_z^{(1)}|\uparrow\rangle_e\,\langle\downarrow|\downarrow\rangle_p + \langle\uparrow|\uparrow\rangle_e\,\langle\downarrow|S_z^{(2)}|\downarrow\rangle_p$$

$$= \hbar/2\cdot\langle\uparrow|\uparrow\rangle_e\,\langle\downarrow|\downarrow\rangle_p + \langle\uparrow|\uparrow\rangle_e\cdot\left(-\tfrac{\hbar}{2}\right)\langle\downarrow|\downarrow\rangle_p$$
$$= \hbar/2\cdot 1\cdot 1 - \hbar/2\cdot 1\cdot 1 = 0$$

Can you get the rest quickly? The state in the bra must match the ket by orthogonality. Only diagonal entries will be present. The value is the sum of $S_z^{(1)} + S_z^{(2)}$

$$S_z = \begin{pmatrix} \langle\uparrow\uparrow|\hat{S}_z|\uparrow\uparrow\rangle & \langle\uparrow\uparrow|\hat{S}_z|\uparrow\downarrow\rangle & \langle\uparrow\uparrow|\hat{S}_z|\downarrow\uparrow\rangle & \langle\uparrow\uparrow|\hat{S}_z|\downarrow\downarrow\rangle \\ \langle\uparrow\downarrow|S_z|\uparrow\uparrow\rangle & \langle\uparrow\downarrow|\hat{S}_z|\uparrow\downarrow\rangle & \langle\uparrow\downarrow|\hat{S}_z|\downarrow\uparrow\rangle & \langle\uparrow\downarrow|\hat{S}_z|\downarrow\downarrow\rangle \\ \langle\downarrow\uparrow|\hat{S}_z|\uparrow\uparrow\rangle & \langle\downarrow\uparrow|\hat{S}_z|\uparrow\downarrow\rangle & \langle\downarrow\uparrow|\hat{S}_z|\downarrow\uparrow\rangle & \langle\downarrow\uparrow|\hat{S}_z|\downarrow\downarrow\rangle \\ \langle\downarrow\downarrow|\hat{S}_z|\uparrow\uparrow\rangle & \langle\downarrow\downarrow|\hat{S}_z|\uparrow\downarrow\rangle & \langle\downarrow\downarrow|\hat{S}_z|\downarrow\uparrow\rangle & \langle\downarrow\downarrow|\hat{S}_z|\downarrow\downarrow\rangle \end{pmatrix}$$

$$S_z = \begin{pmatrix} \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar \end{pmatrix} = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

<u>Eigenvalues</u>: $m = \pm\hbar, 0$ matrix is diagonal. Note, $m = 0$ is a degenerate eigenvalue, there are two eigenstates with this value of S_z.

<u>Eigenvectors</u>:

$m = \hbar$, $|\uparrow\uparrow\rangle$ is already an eigenvector

$m = -\hbar$, $|\downarrow\downarrow\rangle$ " " " ".

$m = 0$?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0 \Rightarrow$$

$a = 0$
$d = 0$
b and c are unconstrained.

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$m=1 \qquad \overbrace{\qquad\qquad\qquad}^{m=0} \qquad m=-1$

Conclusion: $|\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle$ are already eigenvectors of $\hat{S}_z$ with eigenvalues $0, \pm\hbar$

There's something interesting going on with $m=0$.. why are there two states? We'll see soon...

Now, what about $\hat{S}^2$? Remember:

$$\hat{S}^2 = (\hat{\vec{S}}_{(1)} + \hat{\vec{S}}_{(2)})^2 = \hat{S}^2_{(1)} + \hat{S}^2_{(2)} + 2\,\hat{\vec{S}}_{(1)} \cdot \hat{\vec{S}}_{(2)}$$

Total spin squared

$$= \underbrace{\hat{S}^2_{(1)} + \hat{S}^2_{(2)}}_{1} + 2\Big[\underbrace{\hat{S}_x^{(1)}\hat{S}_x^{(2)} + \hat{S}_y^{(1)}\hat{S}_y^{(2)}}_{3} + \underbrace{\hat{S}_z^{(1)}\hat{S}_z^{(2)}}_{2}\Big]$$

We need to take it piece by piece...

1 First: The $\hat{S}^2$ terms.

These matrix elements are easy to calculate.

e.g. $\langle\uparrow\downarrow|\hat{S}^2_{(1)}|\uparrow\downarrow\rangle = \hbar^2\left(\frac{1}{2}\left(\frac{1}{2}+1\right)\right)\langle\uparrow\downarrow|\uparrow\downarrow\rangle = \frac{3\hbar^2}{4}$

Both particles have $s=\frac{1}{2}$, and are eigenstates of $\hat{S}^2_{(1)}$, $\hat{S}^2_{(2)}$. Matrices will be diagonal. Values are the same for all elements.

$$\hat{S}^2_{(1)} = \frac{3\hbar^3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \hat{S}^2_{(2)} = \frac{3\hbar^3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\hat{S}^2_{(1)} + \hat{S}^2_{(2)} = \frac{3\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{3\hbar^2}{2} \cdot \mathbb{1}$$

② $2\cdot\hat{S}_z^{(1)}\hat{S}_z^{(2)}$ this operator measures the product of $m_1\cdot m_2$

example: $2\langle\uparrow\downarrow|\hat{S}_z^{(1)}\hat{S}_z^{(2)}|\uparrow\downarrow\rangle = 2\cdot\langle\uparrow|\hat{S}_z^{(1)}|\uparrow\rangle_e\langle\downarrow|\hat{S}_z^{(2)}|\downarrow\rangle_p$

$= 2\cdot\left(\frac{\hbar}{2}\right)\langle\uparrow|\uparrow\rangle\cdot\left(-\frac{\hbar}{2}\right)\langle\downarrow|\downarrow\rangle = -\frac{\hbar^2}{2}$

$$2\cdot\hat{S}_z^{(1)}\hat{S}_z^{(2)} \longrightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[Basis]
$|\uparrow\uparrow\rangle$
$|\uparrow\downarrow\rangle$
$|\downarrow\uparrow\rangle$
$|\downarrow\downarrow\rangle$

③ Matrix elements of $\hat{S}_x^{(1)}\hat{S}_x^{(2)}$:

For example: $2\langle\uparrow\downarrow|\hat{S}_x^{(1)}\hat{S}_x^{(2)}|\downarrow\uparrow\rangle$

$= 2\langle\uparrow|\hat{S}_x^{(1)}|\downarrow\rangle\langle\downarrow|\hat{S}_x^{(2)}|\uparrow\rangle = 2\frac{\hbar}{2}\frac{\hbar}{2} = \frac{\hbar}{2}$

off diagonal

$$\tilde{S}_x^{(1)} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To get a non-zero entry, each arrow on left must be different from its partner on right.

call it $\hat{O}$

$$S_x^{(1)}S_x^{(2)} = \begin{pmatrix} \langle\uparrow\uparrow|\hat{O}|\uparrow\uparrow\rangle & \langle\uparrow\uparrow|\hat{O}|\uparrow\downarrow\rangle & \langle\uparrow\uparrow|\hat{O}|\downarrow\uparrow\rangle & \langle\uparrow\uparrow|\hat{O}|\downarrow\downarrow\rangle \\ \langle\uparrow\downarrow|\hat{O}|\uparrow\uparrow\rangle & \langle\uparrow\downarrow|\hat{O}|\uparrow\downarrow\rangle & \langle\uparrow\downarrow|\hat{O}|\downarrow\uparrow\rangle & \langle\uparrow\downarrow|\hat{O}|\downarrow\downarrow\rangle \\ \langle\downarrow\uparrow|\hat{O}|\uparrow\uparrow\rangle & \langle\downarrow\uparrow|\hat{O}|\uparrow\downarrow\rangle & \langle\downarrow\uparrow|\hat{O}|\downarrow\uparrow\rangle & \langle\downarrow\uparrow|\hat{O}|\downarrow\downarrow\rangle \\ \langle\downarrow\downarrow|\hat{O}|\uparrow\uparrow\rangle & \langle\downarrow\downarrow|\hat{O}|\uparrow\downarrow\rangle & \langle\downarrow\downarrow|\hat{O}|\downarrow\uparrow\rangle & \langle\downarrow\downarrow|\hat{O}|\downarrow\downarrow\rangle \end{pmatrix}$$

Only non-zero terms are in green. All entries are $\hbar^2/4$ since all entries of S_x are $\hbar/2$.

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$$2S_x^{(1)}S_x^{(2)} = \frac{\hbar^2}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\hat{S}_y^{(1)}\hat{S}_y^{(2)}$ is similar (it's off diagonal), but we need to watch signs.

For example: $\langle \uparrow\downarrow | \hat{S}_y^{(1)}\hat{S}_y^{(2)} | \downarrow\uparrow \rangle$

$$= \langle \uparrow | \hat{S}_y^{(1)} | \downarrow \rangle \underbrace{\langle \downarrow | \hat{S}_y^{(2)} | \uparrow \rangle}_{} = \frac{\hbar}{2}(-i)\cdot\frac{\hbar}{2}(i) = \frac{\hbar^2}{4}$$

off diagonal

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Again, I'll only get a non-zero entry if the arrow on the left is different than its partner on the right. The signs are tricky. If the two arrows in the ket are the same, I have $\langle \uparrow | S_y^{(1)} | \downarrow \rangle \langle \uparrow | S_y^{(2)} | \downarrow \rangle$

$$= \langle \uparrow | S_y | \downarrow \rangle^2 = (i\hbar/2)^2$$

If the arrows in the ket are different, I get $\hbar^2/4$ as above.

$$2S_y^{(1)}S_y^{(2)} = \frac{\hbar^2}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

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Put it all together:

$$\hat{S}^2 = \underbrace{\hat{S}_{(1)}^2 + \hat{S}_{(2)}^2}_{} + \underbrace{2\hat{S}_z^{(1)}\hat{S}_z^{(2)}}_{} + \underbrace{2\hat{S}_x^{(1)}\hat{S}_x^{(2)} + 2\hat{S}_y^{(1)}\hat{S}_y^{(2)}}_{}$$

$$= \underbrace{\frac{3\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{} + \underbrace{\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{} + \underbrace{\frac{\hbar^2}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{}$$

$$\boxed{\hat{S}^2 = \frac{\hbar^2}{2}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}}$$

matrix representation of $\hat{S}^2_{TOT}$ in the basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$.

(whew!)

Eigenvalues:

$$\begin{vmatrix} 2\hbar^2-\lambda & 0 & 0 & 0 \\ 0 & \hbar^2-\lambda & \hbar^2 & 0 \\ 0 & \hbar^2 & \hbar^2-\lambda & 0 \\ 0 & 0 & 0 & 2\hbar^2-\lambda \end{vmatrix}$$

$$(2\hbar^2-\lambda)\left\{(\hbar^2-\lambda)(\hbar^2-\lambda)(2\hbar^2-\lambda) - \hbar^2\hbar^2(2\hbar^2-\lambda)\right\} = 0$$

$$(2\hbar^2-\lambda)^2\left\{(\hbar^2-\lambda)^2-\hbar^4\right\} = 0$$

$$(2\hbar^2-\lambda)\left\{\cancel{\hbar^4}-2\hbar^2\lambda+\lambda^2-\cancel{\hbar^4}\right\} = 0$$

$$(2\hbar^2-\lambda)\left\{\lambda(\lambda-2\hbar^2)\right\}$$

$\lambda = 0$

$\lambda = 2\hbar^2$ [3x degenerate!]

i.e. There is one state with $s(s+1) = 0$ We call it the <u>singlet</u>

There are <u>three</u> states with $s(s+1) = 2\hbar^2 \Rightarrow s = 1$

↳ we call these the <u>triplet</u> states.

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Eigenvectors: $\boxed{\lambda = 0}$. $\hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$= a = 0, \quad d = 0, \quad b = -c$

eigenvector $= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right]$

Remember λ is the eigenvalue of $\hat{S}^2_{TOT}$

so $\lambda = 0$ means $s = 0$, $m = 0$

Apparently: $\underbrace{|0\,0\rangle}_{\text{total angular momentum } = 0.} = \frac{1}{\sqrt{2}} \left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right]$

$\lambda = 2\hbar^2$ means $s(s+1)\hbar^2 = 2\hbar^2$ i.e. these states have $s = 1$

Find eigenvectors

$$\hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 2\hbar^2 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$2a = 2a \Rightarrow$ a can be anything

$b + c = 2b \Rightarrow b = c$.

$2d = 2d \Rightarrow$ d can be anything

Orthonormal choice:

$|v_1\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad |v_2\rangle \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad |v_3\rangle \to \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$|\uparrow\uparrow\rangle$ $s = 1, m = 1$ $\quad |\downarrow\downarrow\rangle$ $s = 1, m = -1$

Total Angular Momentum Basis $\|S_{TOT}, m_{TOT}\rangle$		Composite Basis.
$\|1,1\rangle$	$=$	$\|\uparrow\uparrow\rangle$
$\|1,0\rangle$	$=$	$\frac{1}{\sqrt{2}}\left[\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle\right]$
$\|1,-1\rangle$	$=$	$\|\downarrow\downarrow\rangle$
$\|0,0\rangle$	$=$	$\frac{1}{\sqrt{2}}\left[\|\uparrow\downarrow\rangle - \|\downarrow\uparrow\rangle\right]$

triplet $S=1$

singlet $s=0$

FYI, when it comes to positronium, the triplet states are called called "ortho-positronium" and the singlet state is called "para-positronium".

Of course we could write this the other way around:

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle + |00\rangle\right]$$

$$|\downarrow\uparrow\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle - |00\rangle\right]$$

We have the answer to our question. If I measure S_z on the state $|\uparrow\downarrow\rangle$ I will get $S_z = 0$ with 100% probability

If I measure $\hat{S}^2$ on the state $|\uparrow\downarrow\rangle$ i'll get:

$$S_{TOT}^2 = 2\hbar^2 \quad (s=1) \quad \text{with 50\% probability}$$
$$= 0 \quad (s=0) \quad \text{with 50\% probability.}$$

We just added spin $\frac{1}{2}$ + spin $\frac{1}{2}$. Now let's try to generalize this...
Generally, we're interested in what values for $|SM\rangle$ are possible for the composite state.

In a very "hand-waving" sense, we could see that the largest value of M which is possible is s_1+s_2

$M \leq s_1+s_2$ — This happens when the two spins are aligned.

But this also implies $S \leq s_1+s_2$

Similarly, the lowest value of M would be $|s_2-s_1|$, this occurs when the spins are anti-aligned.
So $M \geq |s_2-s_1|$ which also implies $S \geq |s_1-s_2|$.

This turns out to be correct. Combining 2 spins s_1 and s_2, the possible values for S are

$$\boxed{S = |s_2-s_1|,\ |s_2-s_1|+1,\ \ldots\ s_1+s_2-1,\ s_1+s_2}$$

Poll Q: A spin $\frac{3}{2}$ particle and a spin 1 particle form a bound state. Which of the following are not a possible state $|SM\rangle$ for the total system?

A.) $|\frac{5}{2},\frac{1}{2}\rangle$ B.) $|\frac{1}{2},\frac{1}{2}\rangle$

C.) $|1,0\rangle$ D.) $|\frac{3}{2},\frac{3}{2}\rangle$

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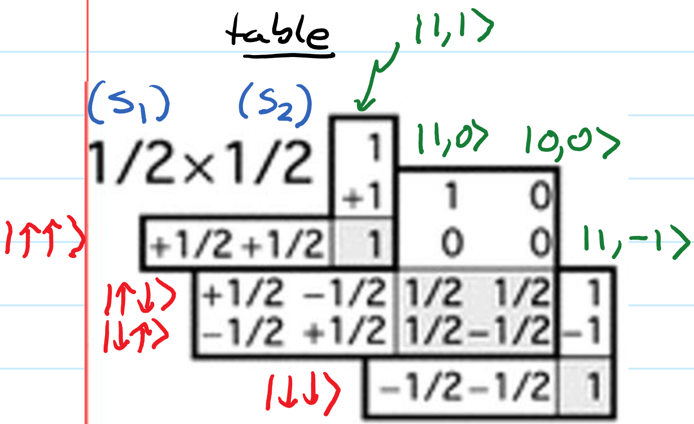
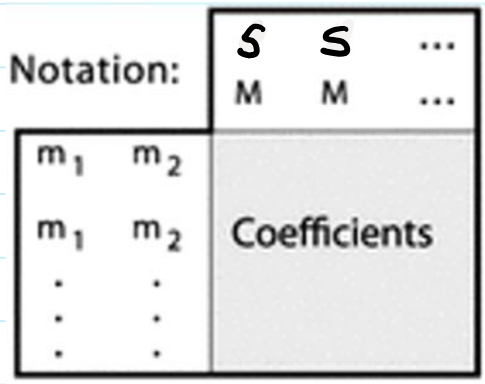
We can express the states with total angular momentum $|s,m\rangle$, or we could use the $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$ basis and we know the relationships between them.

These results can be summarized in a <u>Clebsch-Gordon table</u>





* A square root is implied over every coefficient. The (−) goes outside it

Reading <u>horizontally</u> gives the composite basis states in terms of the total spin states

$$|\downarrow\uparrow\rangle = \frac{1}{\sqrt{2}}|1,0\rangle - \frac{1}{\sqrt{2}}|0,0\rangle$$

Reading vertically gives it the other way around.

$$|00\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle$$

One can repeat the same procedure for higher spins...
Here are the results.

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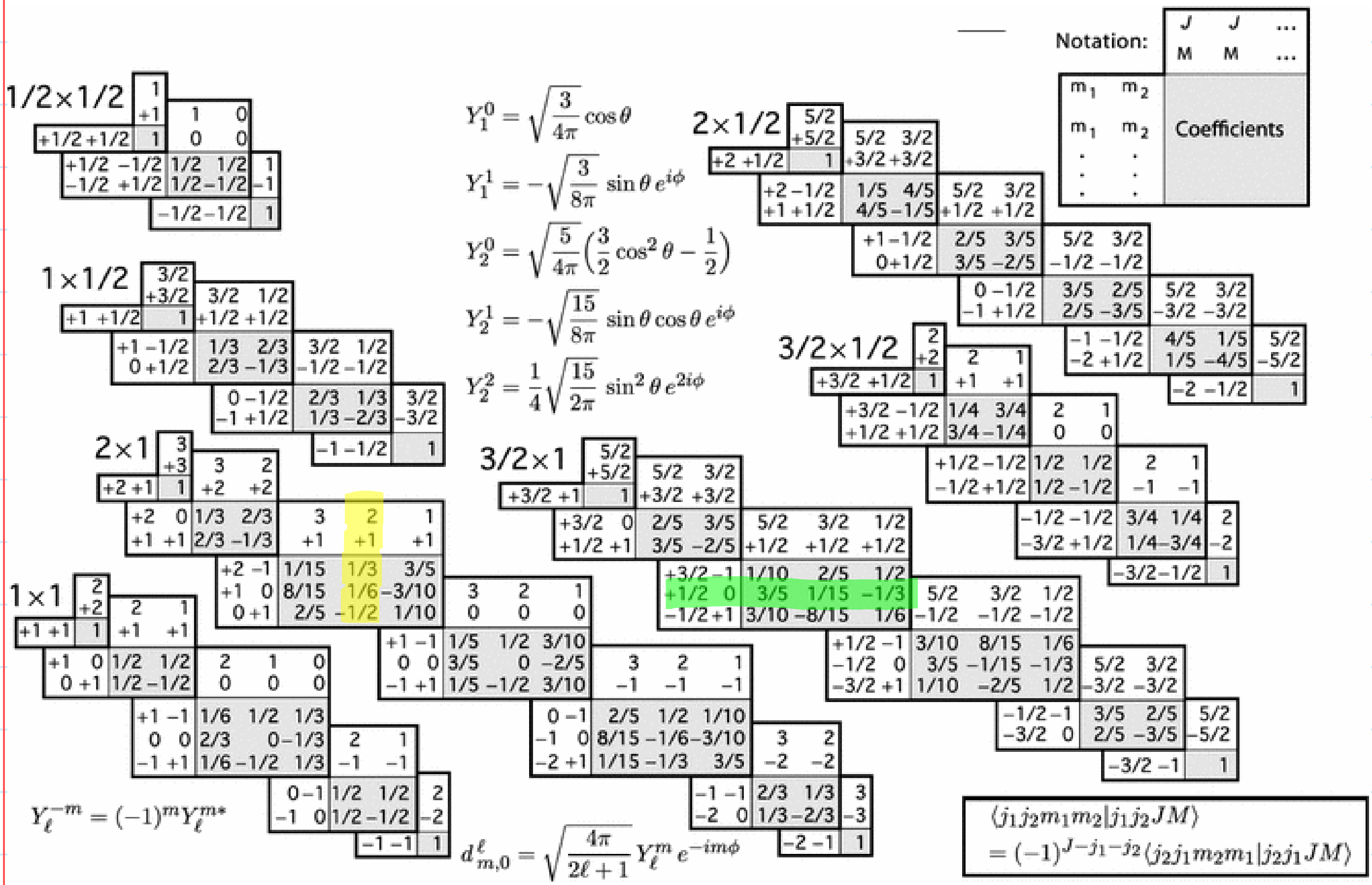
The procedure can be generalized to adding other spins:
Here is a full set of tables (!)



Example: A spin 2 particle is in a composite state with a spin 1 particle. The composite particle has $S = 2$, $m = 1$. If you measured S_z for the spin 1 particle, what's the probability that it is found to be 0?

See yellow highlight above.

$$|21\rangle = \frac{1}{\sqrt{3}}|22\rangle|1\,{-1}\rangle + \frac{1}{\sqrt{6}}|21\rangle|10\rangle - \frac{1}{\sqrt{2}}|20\rangle|11\rangle$$

Answer: $\boxed{1/6}$

Example: Two particles are in states $|\frac{3}{2},\frac{1}{2}\rangle$ and $|1,0\rangle$. They form a composite state. What's the probability that the total spin S is measured to be $\frac{3}{2}$? See green highlight:

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle|10\rangle = \sqrt{\tfrac{3}{5}}\,|\tfrac{5}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{15}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

Answer: $\boxed{1/15}$