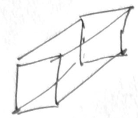
<u>Resonant Cavities</u> (microwaves & higher frequencies)



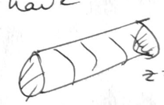
Now will (mostly) close the box.

- store EM energy @ resonant freq in standing waves
- bandpass filter
- extract microwaves for transmission etc.
- manipulate bunches of charged particles.

Mathematically, the only change is that instead of traveling waves in the z direction, now we have standing waves there, too.

$\boxed{TM}$: $E_z = \psi(x,y)\, e^{i(kz-\omega t)} \longrightarrow \psi(x,y)\,(A\sin kz + B\cos kz) \times e^{-i\omega t}$

A & B determined by BCs. Say that we have perfectly conducting caps at $z = 0, d$



Then $E_t(z=0,d) = 0$

and $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{\nabla}_t\cdot\vec{E}_t = -\dfrac{\partial E_z}{\partial z}$

$\Rightarrow$ @ $z=0,d$, we require $\dfrac{\partial E_z}{\partial z} = 0$

So $A = 0$ and $k = \dfrac{\pi p}{d}$, $p = 0, 1, 2 \ldots$

or $E_z = \psi(x,y)\cos\left(\dfrac{\pi p z}{d}\right)$.

Following our previous derivation for wave guides, but using $E_z \sim \cos kz \sim e^{ikz} + e^{-ikz}$ instead of just $e^{\pm ikz}$, one finds $\vec{E}_t = \frac{1}{2}\left(\frac{ik}{\gamma^2}\vec{\nabla}_t\psi\, e^{ikz} + (k\to -k)\right)$

$$= -\frac{k}{\gamma^2}\sin kz\, \vec{\nabla}_t\psi$$

$$= \frac{-p\pi}{d\gamma^2}\sin\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi$$

and $\vec{B}_t = \frac{i\epsilon\omega}{\mu_0\gamma^2}\cos\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi$.

For $\boxed{TE}$, $B_z = \psi(x,y)\,(A\sin kz + B\cos kz)$

and $B_\perp = 0$ @ $z=0,d \Rightarrow B_z = 0$ @ $z=0,d$

$\Rightarrow B=0,\ k=\frac{\pi p}{d}$

$\Rightarrow B_z = \psi(x,y)\sin\left(\frac{\pi p z}{d}\right)$.

$p=1,2,3$

($p=0$ is trivial)

transverse fields again follow from small modifications of the waveguide derivations: $\sin kz = \frac{1}{2i}(e^{ikz}-e^{-ikz})$, so

$$\vec{B}_t = \frac{1}{2i\gamma^2}\left(ik\,\vec{\nabla}_t\psi\, e^{ikz} - (k\to -k)\right)$$

$$= \frac{k}{\gamma^2}\cos(kz)\,\vec{\nabla}_t\psi$$

$$= \frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi$$

and $\vec{E}_t = \frac{-i\omega}{\gamma^2}\sin\left(\frac{\pi p z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi$

We still have to solve the Helmholtz eq / Laplace eigenvalue problem,

$$\nabla_t^2\psi = -\gamma^2\psi$$

$$\gamma^2 = \mu_0\epsilon_0\omega^2 - \left(\frac{p\pi}{d}\right)^2$$

We can view this as: for every longitudinal standing mode (p) and every transverse standing mode (γ_λ) there is a frequency $\omega_{\lambda p}^2 = \frac{1}{\mu_0\epsilon_0}\left[\gamma_\lambda^2 + \left(\frac{p\pi}{d}\right)^2\right]$

(This is just like the particle in a box in stat mech with relativistic dispersion relation: $\omega^2 = k_x^2 + k_y^2 + k_z^2$, $\vec{k} = \frac{\pi\vec{n}}{L}$)

Simple case:

TM: $E_z = \psi(\rho,\phi)\cos\left(\frac{\pi p z}{d}\right)$, $\nabla_t^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$

let $\psi(\rho,\phi) = \chi(\rho)\,e^{\pm im\phi}$

Then $\chi'' + \frac{1}{\rho}\chi' - \frac{m^2}{\rho^2}\chi = -\gamma^2\chi$

Bessel's eq! The regular solution is $\propto J_m(\gamma\rho)$.

We need $E_{\parallel}(\rho=R) = 0$, so in particular $E_z(\rho=R) = 0$

Therefore $E_z = E_0\cos\left(\frac{\pi p z}{d}\right)e^{\pm im\phi}J_m(\gamma_{mn}\rho)$

where γ_{mn} defined by $J_m(\gamma_{mn}R) = 0$.

The roots of J are tabulated and indexed by $n = 1, 2, 3, \ldots$

$$\omega_{mnp}^2 = \frac{1}{\mu_0\epsilon_0}\left(\gamma_{mn}^2 + \frac{p^2\pi^2}{d^2}\right)$$

lowest frequency occurs for $m=0$, $n=1$, $p=0$ "TM_{010}"

numerically, $\omega_{010} \hat{=} \dfrac{2.4}{\sqrt{\mu\epsilon}\, R}$ independent of d !

Not tunable by a simple piston.

TE: same story, except now $B_z \sim \psi$ and

at $\rho = R$ we have the boundary condition $\dfrac{\partial B_z}{\partial n} \Rightarrow \dfrac{\partial \psi}{\partial \rho} = 0$

Therefore $B_z = B_0 \sin\left(\dfrac{\pi p z}{d}\right) e^{\pm i m \phi} J_m(\gamma_{mn} \rho)$

where now $J_m'(\gamma_{mn} R) = 0$

again roots of J' are extensively tabulated.

m still runs over $0, 1, 2 \ldots$

n, p over $1, 2, 3 \ldots$

It turns out that $TE_{1,1,1}$ is the lowest frequency (roots of J_m' for $m=1$ are smaller than for $m=0$)

Since $p=1$, this frequency is d-dependent;

$$\omega_{111} = \frac{1.84}{\sqrt{\mu\epsilon}\, R} \left(1 + 2.9 \frac{R^2}{d^2}\right)^{1/2}$$

For $d \gtrsim 2R$ $\omega_{111}^{TE} < \omega_{010}^{TM}$ and this frequency can be isolated from all the rest.