Chapter 7 - Electrodynamics

The underlying goal of this chapter is to formulate the complete, coupled Maxwell's equations describing electrodynamics. This entails showing how a changing magnetic field induces an electric field, and how a changing electric field induces a magnetic field. Then the Maxwell's equations are:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

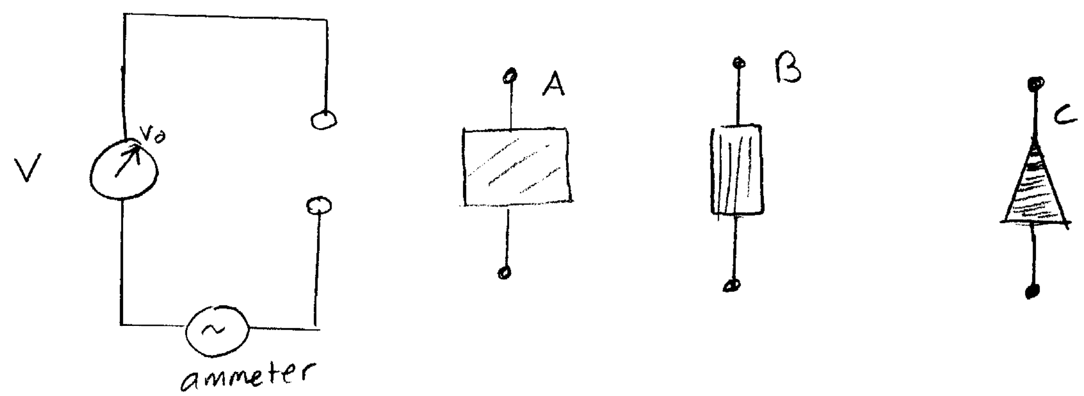
$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

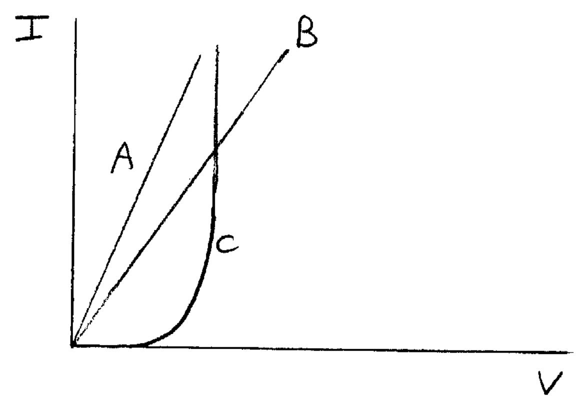
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Circuits are a convenient place (as well as true to history) to explore the coupling of $\vec{E}$ + $\vec{B}$. The circuits here may be considered quasistatic, meaning that E + B do not fluctuate too rapidly, and for reasonably slow changes the the methods of statics may be applied.

Suppose we built a test stand where we could control the voltage across a pair of terminals and, once a device was plugged in, measure the current flow:



Many materials plugged into the test stand would give a linear dependence of measured current on applied voltage.



Those devices, such as A + B shown in the graph above are resistors, and obey Ohm's law, $V = IR$, or equivalently, $J = \sigma E$.

$$J = \underset{\text{conductivity}}{\sigma} E$$

The conductivity of perfect conductors $\rightarrow \infty$

$$E = \overset{\text{resistivity}}{\rho} J \qquad \text{smaller scale}$$

$$V = -\int E \cdot dl \qquad I = \int J \cdot da$$

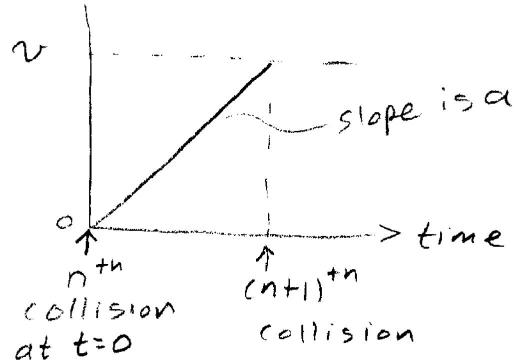
$$V = R I \qquad \text{larger scale}$$

The classical Drude model for conductivity gives an intuition for the behavior of resistive materials, although quantum mechanics is needed.

$$J = n \left[\frac{\text{molecules}}{\text{vol}}\right] f \left[\frac{\text{free electrons}}{\text{molecule}}\right] q \left[\frac{\text{charge of}}{\text{electron}}\right] v_{avg} \left[m/s\right]$$

The model assumes the electrons are experiencing coulombic collisions with the ions in the material, but accelerate with constant acceleration between collisions due to the applied field. The velocity in the expression for J gives the average velocity obtained between collisions.

It is this drift velocity which is responsible for the overall motion through the potential difference (and hence the current).



$$v_{avg} = \tfrac{1}{2} a t_{coll}$$

The acceleration is due to the applied electric field:

$$a = F/m = \frac{qE}{m}$$

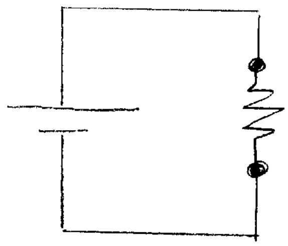
The time between collisions is the mean free path divided by the thermal velocity:

$$t_{coll} = \lambda / v_{th}$$

Then:

$$J = \underbrace{\left(\frac{n f \lambda q^2}{2 m v_{th}} \right)}_{\sim \sigma} E$$

How is a voltage maintained across the terminals of our test stand (or any other circuit)?



There is a voltage drop $V = -\int \vec{E} \cdot d\vec{\ell}$ across the resistor in the test stand.

How are the current & voltage maintained?

$$eV = \text{energy} \rightarrow V = \frac{\text{energy}}{\text{charge}}$$

$$W = \text{energy} = \int \vec{F} \cdot d\vec{\ell} = \int q\vec{E} \cdot d\vec{\ell}$$

$$\mathcal{E} = \text{emf} = \frac{\text{energy}}{\text{charge}} = \int \left(\frac{q\vec{E}}{q}\right) \cdot d\vec{\ell} = \int \vec{f} \cdot d\vec{\ell}$$

where f = force/charge.

$$\mathcal{E} = \int (f_s + E) \cdot d\ell$$

$$\mathcal{E}_{\substack{\text{around}\\ \text{circuit}}} = \oint (\vec{f_s} + \vec{E}) \cdot d\vec{\ell} = \oint \vec{f_s} \cdot d\vec{\ell} + \cancelto{0}{\oint \vec{E} \cdot d\vec{\ell}}$$

$\vec{f_s}$ = local source of energy/chrge, such as battery

$\vec{E}$ = The electric field causing charges throughout the circuit to move. $\vec{E}$ communicates the existance of the source, and serves to engender a uniform current with no pileup of charge.

If current in a conductor is uniform and does not change with time,

$$\frac{\partial \rho}{\partial t} = 0$$

Then, by the continuity equation,

$$\vec{\nabla} \cdot \vec{J} = 0$$

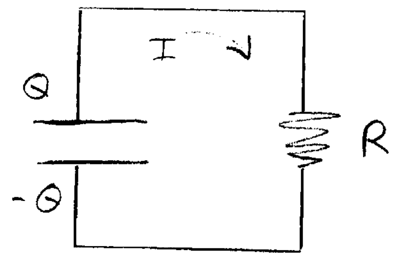
If σ is constant, then,

$$\vec{\nabla} \cdot \vec{J} = \sigma \vec{\nabla} \cdot \vec{E} = 0$$

and the electric field within the conductor is zero.

In quasi-static situations, fields do not penetrate into the conducting wires/bus of a circuit.

Problem 7.2



Initial voltage on capacitor, V_0

capacitor: $q = CV$

resistor: $V = iR$

a) Conservation of energy $\Delta V_{loop} = 0$

$$q/C + iR = 0$$

$$\frac{dq}{dt} + \frac{1}{RC} q = 0$$

$$\int \frac{dq}{q} = -\frac{1}{RC} \int dt$$

$$\ln q/c = -t/RC \quad \longrightarrow \quad e^{\ln q/c} = e^{-t/RC}$$

↑ constant of integration

$$q = c\, e^{-t/RC} \qquad \text{at } t=0,\ q = Q = CV_0$$

$$q(t) = CV_0 e^{-t/RC} = Q e^{-t/RC}$$

$$I(t) = \frac{dq(t)}{dt} = CV_0\, e^{-t/RC} \left(\frac{-1}{RC}\right) = -\frac{V_0}{R} e^{-t/RC}$$

problem 7.2 continued

b) W (energy stored) $= \int_0^Q (q/c)\,dq = \frac{1}{2}\frac{Q^2}{C}$

$$W = \tfrac{1}{2} Q^2/c = \tfrac{1}{2} C V^2$$

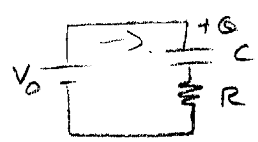
$$P = \frac{dW}{dt} \rightarrow W = \int P\,dt = \int I^2 R\,dt$$

$$W = R \int_0^\infty (V_0/R)^2 e^{-2t/RC}\,dt$$

note: $\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$

$$W = \frac{V_0^2}{R^{\cancel{2}}} \left(\frac{R C}{2}\right) \cancel{R} = \tfrac{1}{2} C V_0^2$$

c) Charging ...



$$V_0 - Q/C - iR = 0$$

$$\frac{dq}{dt} + \frac{1}{RC} q = \frac{V_0}{R}$$

This is a first order, linear differential equation.

Let's solve it from two different perspectives.

#1) The solution will be the sum of the general solution to the homogenrous equation (RHS = 0) plus a particular solution for the equation with the source term (RHS ≠ 0). The general solution will have arbitrary constant(s) to be determined using initial conditions. A first order differential equation will have one arbitrary constant, a second order differential equation will have two arbitrary constants. The particular solution will have no arbitrary constants. Caution: the initial conditions must be applied to the total solution to determine the arbitrary constants.

Homogenious equation:

$$\frac{dq}{dt} + \frac{1}{RC} q = 0$$

We already know the solution from part a)

$$q = a e^{-t/RC}$$

arbitrary constant

problem 7.2, continued

Inhomogenious equation:

$$\frac{dq}{dt} + \frac{1}{RC} q = \frac{V_0}{R}$$

To solve, pick a trial solution which has the form of the source term and its derivative. Source term is a constant, derivative zero. Then,

$$q_{trial} = K$$

$$\cancel{\frac{dK}{dt}}^{\,0} + \frac{1}{RC} K = \frac{V_0}{R} \quad \Rightarrow \quad K = CV_0$$

Total solution:

$$q = ae^{-t/RC} + CV_0$$

Apply initial conditions:

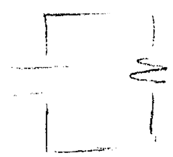
$$\text{At } t = 0, \quad q = 0 \quad \Rightarrow \quad a = -CV_0$$

$$q = CV_0 (1 - e^{-t/RC})$$

#2) The method of solution is the same, but with a different perspective. The complete response of the circuit is the sum of the natural (transient) response plus the forced (driven) response.

$$q = q_n + q_f$$

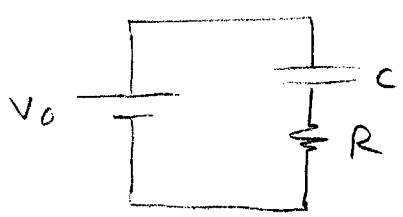
The natural response takes into account that the circuit may store energy, but doesn't consider applied emf. Then, the natural response is that of an undriven circuit, that of part a)



$$q_n = a e^{-t/RC}$$

a to be determined from initial conditions

The forced response is the steady-state response without transient effects:



Long after $t = t_0$, the capacitor is fully charged and acts like an 'open'.

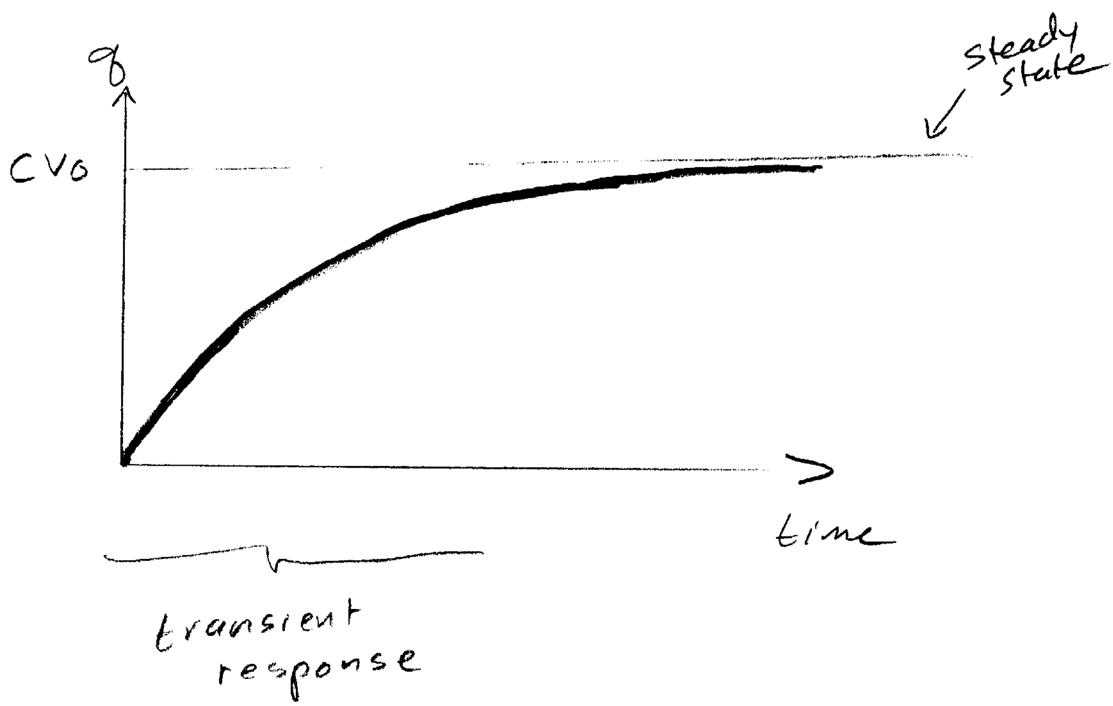
Since no current is flowing, the voltage across the capacitor is V_0, and the charge is $q = CV_0$. Then, $q_f = CV_0$.

The total solution is

$$q = a e^{-t/RC} + CV_0$$

At $t=0$, $q=0 \Rightarrow a = -CV_0$

$$q = CV_0 (1 - e^{-t/RC})$$



d) $i(t) = \frac{dq(t)}{dt} = -CV_0 e^{-t/RC}\left(-\frac{1}{RC}\right) = \frac{V_0}{R} e^{-t/RC}$

$$W_\varepsilon = \int P\,dt = \int_0^\infty V_0\, i(t)\,dt = V_0 \int_0^\infty \frac{V_0}{R} e^{-t/RC} dt$$

$$W_\varepsilon = \frac{V_0^2}{R}(RC) = CV_0^2$$

problem 7.2

The heat delivered to the resistor is:

$$W_{th} = \int_0^{\infty} P\,dt = \int_0^{\infty} i^2(t)\,R\,dt = \left(\frac{V_0}{R}\right)^2 R \int_0^{\infty} e^{-2t/RC}\,dt$$

$$= \frac{V_0^2}{\cancel{R}}\left(\frac{\cancel{R}C}{2}\right) = \frac{1}{2} C V_0^2$$

We also know the final stored energy of the capacitor is $W_c = \frac{1}{2} C V_0^2$. Thus, half the work done by the voltage source is dissipated as thermal energy in the resistor, and half is converted to stored energy (in the form of an electric field) in the capacitor.