A brief look at relativistic kinematics

A few things you've seen..

The principle of relativity - The laws of physics are the same in all inertial frames

The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source

Time dialation $\quad \Delta t = \gamma \overline{\Delta t}$

The time between events appears longer in a stationary frame than in a moving frame. Example; suppose a fast moving particle decays. If in the particle frame (moving frame) the particle decays in its characteristic lifetime, $\tau_{particle}$, it will take a longer time to decay in the lab frame (stationary frame). Such an accelerator as a Muon collider could not be considered otherwise.

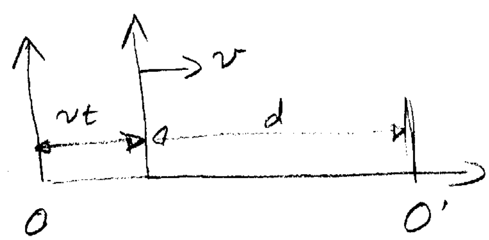
Overbars mean "moving frame" in these notes

$\overline{\Delta t} \equiv$ time in moving frame

Length contraction $\Delta x = \frac{1}{\gamma} \overline{\Delta x}$

A meter stick moving fast looks shorter than a meter to a stationary observer.

Now we'll move on. Lorentz transformations tell how to map the coordinates of events from one inertial frame to another. Lets follow Griffiths and get the Lorentz transformation for space & time.



The origins of two systems, one moving with velocity v with respect to the other, observe events O (happening when the origins of the systems coincide) and O', happening later.

The observer in the unprimed frame sees the distance between events as:

$$x = vt + d = vt + \frac{\bar{x}}{\gamma}$$

Or, solving for $\bar{x}$, $\bar{x} = \gamma(x - vt)$

Similarly, if an observer in the moving frame sees the distance between events as

$$\bar{x} = x/\gamma - v\bar{t}$$

(unprimed frame moves in negative direction wrt primed frame)

$$x = \gamma(\bar{x} + v\bar{t})$$

We may use these relations to express t in terms of overbar variables, or vice vs.

$$x' = \gamma[\underbrace{\gamma(\bar{x} + v\bar{t})}_{x} - vt]$$

Solve for t (and similarly for $\bar{t}$):

$$t = \gamma(\bar{t} + v/c^2\,\bar{x})$$

$$\bar{t} = \gamma(t - v/c^2\,x)$$

Define $x^0 \equiv ct$, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$

Why do this? Notice that $x^0 \equiv ct$ has units of length, just as x, y, z do.

① Now we may write the Lorentz transformation (and inverse) in convenient matrix notation.

② x^0, x^1, x^2, x^3 are components of a 4-vector. The four dimensional scalar product, defined for a & b as

$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$, has the same value in all inertial systems. The scalar product of a 4-vector with itself is a Lorentz invarient.

Let's do ① first. Take the equations for system S in terms of S' variables, when the S' frame is moving in the positive x' direction at speed v:

$$x^1 = \gamma(\bar{x}^1 + \beta(c\bar{t})) = \gamma(\bar{x}^1 + \beta\bar{x}_0)$$

$$x^2 = \bar{x}_2 \quad , \quad x^3 = \bar{x}_3$$

$$ct = \gamma(c\bar{t} + \beta\bar{x}^1) \rightarrow x_0 = \gamma(\bar{x}_0 + \beta\bar{x}^1)$$

The Lorentz transformation (reverse) in matrix notation is:

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix}$$

The regular Lorentz transformation matrix:

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Lorentz transformation matrix maps coordinates from one inertial frame to another.

Now for 2. Let's show that $\sum_{\mu=0}^{3} a_\mu a^\mu$, the 4 dimensional scalar product is invarient for the spacetime vector.

$$\sum_{\mu=0}^{3} \bar{x}_0 \bar{x}^0 = -\bar{x}^0\bar{x}^0 + \bar{x}^1\bar{x}^1 + \bar{x}^2\bar{x}^2 + \bar{x}^3\bar{x}^3$$

using the Lorentz transformation, this is:

$$-(\gamma x^0 + \beta\gamma x^1)^2 + (\beta\gamma x^0 + \gamma x^1)^2 + (x^2)^2 + (x^3)^2$$

$$= -(\gamma^2 (x^0)^2 + \cancel{2\beta\gamma^2 x^1 x^0} + \beta^2\gamma^2 (x^1)^2) + (\beta^2\gamma^2 (x^0)^2$$

$$+ \cancel{2\beta\gamma^2 x^0 x^1} + \gamma^2 (x^1)^2) + (x^2)^2 + (x^3)^2$$

$$= -\gamma^2(1-\beta^2)(x^0)^2 + \gamma^2(1-\beta^2)(x^1)^2 + (x^2)^2 + (x^3)^2$$

$$= -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

Ta-da!

Next on the agenda, energy and momentum. There is a momentum 4-vector, $p^0 \equiv E/c$, and p^1, p^2, p^3 are the usual components of the momentum vector that we know and love. Notice that although p^0 is written in terms of the energy, E/c has units of momentum.

Einstein said, among many other things, that a particle at rest ($\vec{p}_{particle} = 0$) has energy $E_{rest} = m_o c^2$. Many useful relations

m_o ← rest mass

can be obtained from this ($E_{rest} = m_o c^2$) and the fact that E/c is the zeroth component of the momentum four vector.

For example, transform the momentum 4-vector for a particle in the rest frame to another frame:

$$\begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{m_o c^2}{c} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E/c = \gamma m_o c$$

$$\boxed{E = \gamma m_o c^2}$$ ← General expression for total particle energy

$$p^1 = \gamma\beta \frac{m_o c^2}{c}$$

$$p = \gamma \left(\frac{v}{\cancel{c}}\right) \cancel{c}\, m_0$$

$$\boxed{p = \gamma m_0 v}$$

General expression for total particle energy

The highest energy accelerator at Fermilab accelerates protons up to ~ 1 TeV.

The storage ring for the Advanced light Source at Argonne circulates electrons at 7 GeV.

What is the γ of the particles in these machines? How fast are they going compared to the speed of light?

FNAL TeVatron :

$$\gamma \approx \frac{\gamma m_0 c^2}{m_0 c^2} = \frac{\text{total energy}}{\text{rest energy}} = \frac{1 \times 10^3 \text{ GeV}}{.938 \text{ GeV}}$$

$$= 1066$$

APS ring :

$$\gamma \approx \frac{7 \text{ GeV}}{.511 \times 10^{-3} \text{ GeV}} = 13,699$$

Now for $\beta = \sqrt{1-(1/\gamma)^2}$

FNAL Tevatron:

$$\beta = \left[1 - \left(\frac{1}{1066}\right)^2\right]^{1/2} = .99999956\ c$$

APS ring:

$$\beta = \left[1 - \left(\frac{1}{13,699}\right)^2\right]^{1/2} = .999999997\ c$$

A useful formula can be obtained by equating the Lorentz invarient as written in the center-of-mass frame (net momentum is zero) to the invarient as written in another frame.

$$-\left(E/c\right)^2 + p^2 = -\left(\frac{E_{cm}}{c}\right)^2 + 0$$

$$\frac{E^2}{c^2} = p^2 + \left(\frac{m_o c^2}{c}\right)^2 = p^2 + m_o^2 c^2$$

$$E^2 = p^2 c^2 + m_o^2 c^4$$

$$E = \sqrt{p^2c^2 + m_o^2 c^4}$$

Notice that massless particles (photons) then have:

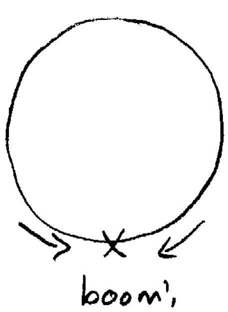
$$E = \sqrt{p^2c^2 + 0} = pc$$

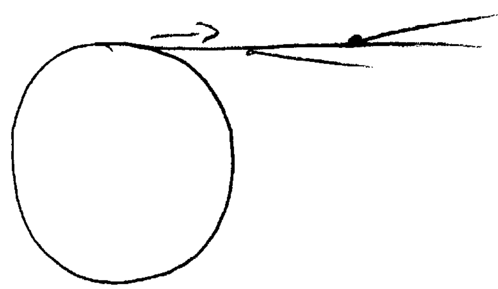
Beam physicists at high energy facilities often approximate the particle momentum as E/c, because the correction due to the rest energy is so small.

Did you ever wonder why machines used for particle physics research are often colliders?

collider

Fixed target operations





The energy available for the collision is much greater.

Lets compare the (invarient) center-of-mass energies for a 900 GeV collider vs. a 900 GeV fixed target experiment.

Invarient $\left(\frac{E_{cm}}{c}\right)^2 = \left[\left(\frac{E_{tot}}{c}\right)^2 - p_{tot}^2\right]$

$$E_{cm} = \left[E_{tot}^2 - p_{tot}^2 c^2\right]^{1/2}$$

For the collider:

$p_{tot} = 0$

$E_{tot} = 900\ \text{GeV} + 900\ \text{GeV} = 1.8\ \text{TeV}$

$E_{cm} = 1.8\ \text{TeV}$

For the fixed target experiment:

proton fixed in target $\begin{cases} p_2 = 0 \\ E_2 = m_0 c^2 \end{cases}$

accelerated proton $\begin{cases} E_1 = T_1 + m_0 c^2, \quad T_1 = KE \approx 900\ \text{GeV} \\ p_1 = ? \end{cases}$

$$E_1^2 = p_1^2 c^2 + m_0^2 c^4$$

$$\rightarrow p_1 = \left[(E_1/c)^2 - m_0^2 c^2\right]^{1/2}$$

$$E_{total}^2 = (T_1 + 2m_0c^2)^2$$

$$(p_{total})^2 = p_1^2 = (E_1/c)^2 - m_0^2c^2$$

$$= \frac{(T_1 + m_0c^2)^2}{c^2} - m_0^2c^2$$

$$= \left(\frac{T_1}{c}\right)^2 + 2T_1m_0 + \cancel{m_0^2c^2} - \cancel{m_0^2c^2}$$

$$\frac{E_{cm}}{c} = \left[\cancel{\left(\frac{T_1}{c}\right)^2} + 4T_1m_0 + 4m_0^2c^2 - \cancel{\left(\frac{T_1}{c}\right)^2} - 2T_1m_0\right]^{1/2}$$

$$E_{cm} = \left[2T_1m_0c^2 + 4(m_0c^2)^2\right]^{1/2}$$

$$= [2(900)(.938) + 4(.938)^2]^{1/2}$$

$$\simeq 41 \text{ GeV}$$

$$(E_{cm})_{collider} >> (E_{cm})_{\substack{fixed\\target}}$$

... but its a lot harder to get collisions in a collider, because the protons in a beam have much more empty space between them than the protons in a target.