Before we continue with the fields of moving point charges, let's take stock & analyze the structure of Jefimenko's eq.

Story so far:

Maxwell eq

$\Downarrow$

Wave eq $\quad \begin{aligned} \Box\vec{E} &= -\nabla\rho/\epsilon_0 - \partial_t \mu_0 \vec{j} \\ \Box\vec{B} &= \mu_0 \vec{\nabla}\times\vec{j} \end{aligned}$ $\Bigg|$ $\Box A^{\mu} = \mu_0 J^{\mu}$ or in Lorenz gauge

$\Downarrow$

Jefimenko eq $\quad \begin{aligned} \vec{E} &= \Box^{-1}\left(-\nabla\rho/\epsilon_0 - \partial_t \mu_0 \vec{j}\right) \\ \vec{B} &= \vec{\Box}^{-1}(\mu_0 \vec{j}) \end{aligned}$

or $\quad \left.\begin{aligned} \phi &= \Box^{-1}(\rho/\epsilon_0) \\ \vec{A} &= \Box^{-1}(\mu_0 j) \end{aligned}\right\}$ "Retarded potential" Griffiths guesses from superposition principle.

Here $\Box^{-1}_{\hat{x}\hat{x}'} f(x^{\mu}) = \int d^3x'\,dt'\; G(x^{\mu}, x^{\mu'})\, f(x^{\mu'})$

$$G = \frac{\delta(t - t' - r/c)}{4\pi r} \qquad r = |\bar{x} - \bar{x}'|$$

Here we focussed on the sourced wave eq. The vacuum wave eq is $\Box\psi = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\psi = 0$

which has plane wave solutions $\psi = \psi_0 e^{\pm i(\bar{k}\cdot\bar{x} - \omega t)}$

plug in: solves $\underset{\left(-\omega^2/c^2 + |k|^2\right)\psi}{\Box\psi \overset{\shortparallel}{=} 0}$ if $\omega = c|k|$ dispersion relation

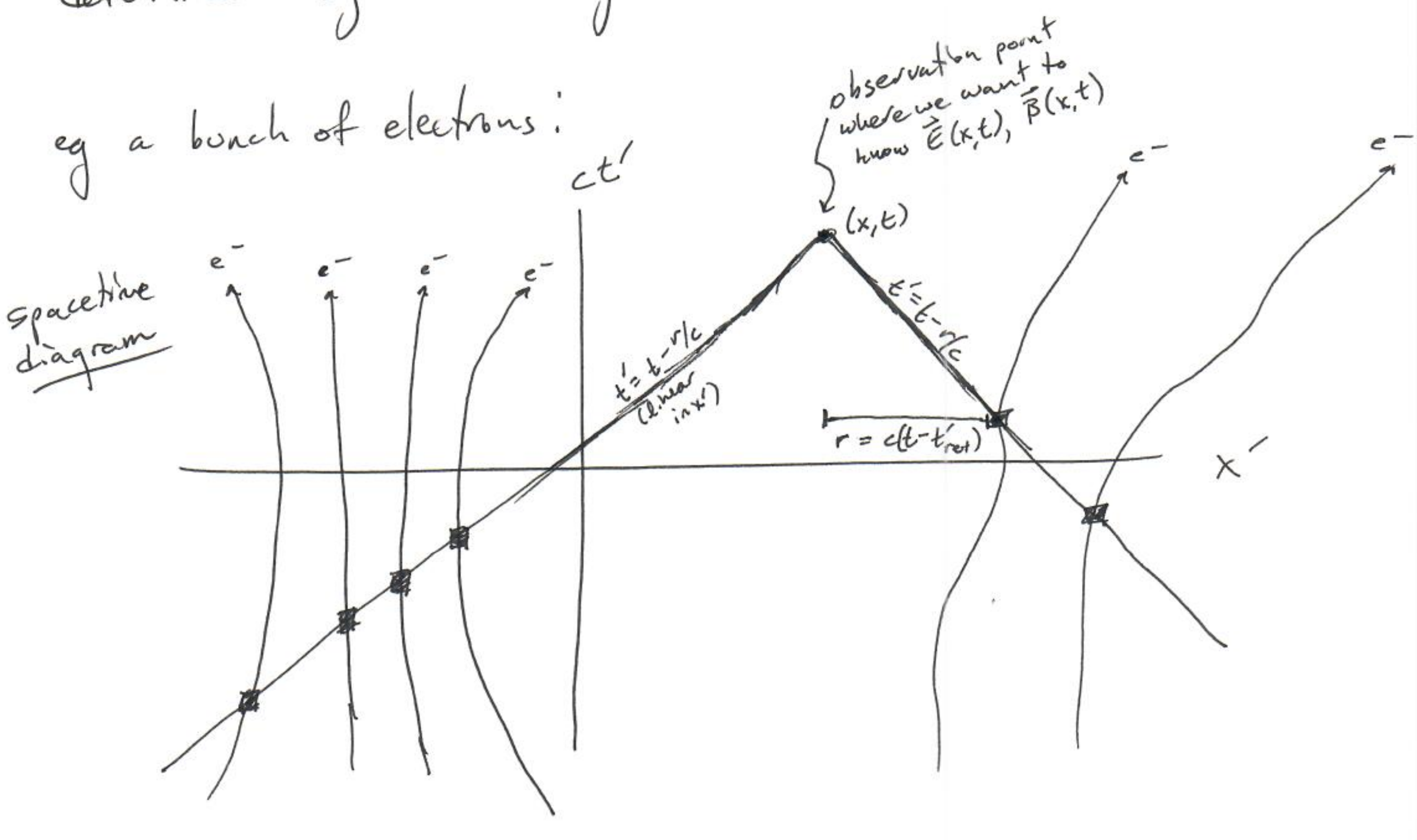
- propagation speed $= c = 1/\sqrt{\mu_0\epsilon_0}$
- propagation direction $\vec{k}$

The finite propagation speed is reflected in Jefimenko's eq by the retarded time: $t'_{ret} = t - r/c$

$$\text{eg} \quad \vec{E}(x,t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \left[\frac{\hat{r}}{r^2} \rho(\vec{x}', t'_{ret}) + \dots \right]$$

$\vec{E}, \vec{B}$ depend on all the sources $\rho, \vec{J}$ throughout all of space at a very specific set of times determined by causality.

eg a bunch of electrons:



The fields at $\vec{x}, t$ depend on what the sources were doing, when they were on the past lightcone – only these points propagate information to x,t at the speed of light.

For static sources, Jefimenko reduces to Coulomb and Biot-Savart!

$$\rho(x',t') \to \rho(x')$$
$$\vec{J}(x',t') \to \vec{J}(x')$$

$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\hat{r}}{r^2}\,\rho(x')$$

$$\vec{B}(\vec{x},t) = -\frac{\mu_0}{4\pi}\int d^3x' \, \frac{\hat{r}\times\vec{J}(x')}{r^2}$$

Jefimenko generalizes this to arbitrary time-dependent sources. The *really* interesting thing about the new terms is they fall off like $\underline{1/r}$ instead of $\underline{1/r^2}$.

In brief, Wave energy density ~ (Amplitude)^2.

At long distances, an amplitude that goes like $1/r$ can transmit finite energy

$$\underbrace{(4\pi r^2)}_{\text{Area}}\underbrace{\left(\frac{1}{r}\,\frac{1}{r}\right)}_{\text{energy/area}} \sim \text{finite}$$

We will discuss in more detail.

Back to Jefimenko. Let's evaluate on point charge sources!

$$\rho = e\,\delta^{(3)}(\vec{x}' - \vec{s}(t'))$$
$$\vec{J} = e\vec{v}\,\delta^{(3)}(\vec{x}' - \vec{s}(t'))$$

Now plug into $\vec{E}$ & $\vec{B}$ (Jefimenko):

$$\vec{E}(\bar{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{ \frac{\hat{r}}{r^2}\rho_{ret} + \frac{\hat{r}}{cr}\partial_t(\rho_{ret}) - \frac{1}{c^2 r}\partial_t(\vec{J}_{ret}) \right\}$$

Let's start with the first term.

$$\frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\hat{r}}{r^2} e \, \delta^{(3)}(\bar{x}' - \bar{s}(t'_{ret}))$$

ugh!

$$\star = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\hat{r}}{r^2} e \, \delta^{(3)}\left(\vec{x}' - \vec{s}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)$$

The $\bar{x}'$ dependence in $\bar{s}$ means $\int d^3x' \, \delta^{(3)}(\ldots)$ is not so simple.

Recall math: $\int d^3x \, h(\bar{x}) \, \delta^{(3)}(\bar{f}(\bar{x})) = ?$

change vars $\vec{x} \to \bar{f}$!

$$= \int d^3f \, \underbrace{\left| \partial f_i / \partial x_j \right|^{-1}}_{\text{Jacobian determinant}} h(\bar{x}) \, \delta^{(3)}(\vec{f})$$

$$= \sum_{\text{zeros of } \vec{f}} \frac{h(\vec{x})}{\left| \partial f_i / \partial x_j \right|}$$

So $\star$ gives

$$\frac{1}{4\pi\epsilon_0}\left(\frac{\hat{r}}{r^2} e \frac{1}{(\text{Jacobian } J)} \right)\Bigg|_{\vec{x}' = \vec{s}\left(t - \frac{|x - \vec{x}'|}{c}\right)}$$

easy to miss the Jacobian if not careful!

The Jacobian is $J = \det\left(\frac{\partial}{\partial x'^j}\left(x'^i - s^i\left(t - \frac{|\bar{x}-\bar{x}'|}{c}\right)\right)\right)$

The first term is easy: $\frac{\partial x'^i}{\partial x'^j} = \delta_{ij}$

Second term: $\frac{\partial}{\partial x'^j} s^i\left(t - \frac{|\bar{x}-\bar{x}'|}{c}\right) \underset{\text{chain rule}}{=} -\frac{1}{c} \left.\frac{\partial s^i(t')}{\partial t'}\right|_{t'=t-r/c} \frac{\partial|\bar{x}-\bar{x}'|}{\partial x'^j}$

but $\frac{\partial|\vec{x}-\vec{x}'|}{\partial x'^j} = \frac{\partial\sqrt{(\bar{x}-\bar{x}')\cdot(\bar{x}-\bar{x}')}}{\partial x'^j} = \frac{1}{|\bar{x}-\bar{x}'|}(x^j - x'^j) = (\hat{r})^j$

So $J = \det\left(\delta^{ij} - \frac{1}{c}\left.\frac{\partial s^i(t')}{\partial t'}\right|_{t'=t-r/c} (\hat{r})^j\right)$

now $\det\left(\mathbb{I}_{3\times3} - \vec{A}\,\vec{B}\right) = 1 - \vec{A}\cdot\vec{B}$

so $J = 1 - \vec{\beta}\cdot\hat{r}$ where $\vec{\beta} = \frac{1}{c}\left.\frac{\partial \bar{s}(t')}{\partial t'}\right|_{t'=t-r/c}$

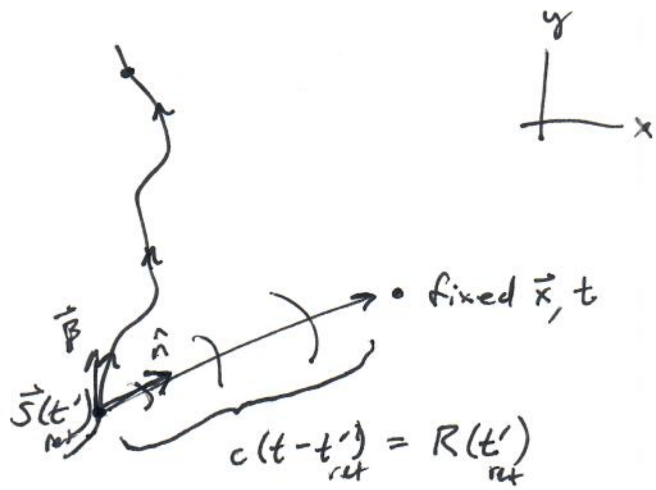
this is the lab frame velocity of the particle at the retarded time.

So the 1st term in $\vec{E}$ is $\left.\frac{1}{4\pi\epsilon_0}\left(\frac{\hat{r}}{r^2}\, e\, \frac{1}{J}\right)\right|_{\bar{x}'=s\left(t-\frac{|\bar{x}-\bar{x}'|}{c}\right)}$

$$= \frac{e}{4\pi\epsilon_0}\left(\frac{\hat{n}}{K R^2}\right)_{\text{ret}}$$

Here
- $\vec{R} \equiv \vec{x} - \vec{s}(t')$ $\left(\vec{R} = \vec{r}\,\big|_{\bar{x}' \to \vec{s}(t')}\right)$
- $\vec{R} \equiv R\hat{n}$
- $K \equiv 1 - \vec{\beta}\cdot\hat{n}$
- ret, as always, means eval @ $t' = t - R/c$

Note that this definition of the retarded time is quite implicit. We are supposed to evaluate $\vec{R}(t') = \vec{x} - \vec{s}(t')$ and $\vec{\beta}(t') = \dfrac{\partial \vec{s}(t')}{\partial t'}$ at an earlier time $t' = t - R(t')/c$, i.e. when the distance from the charge to the observation point $\underbrace{\vec{x}}_{R(t')}$ was $c(t-t')$.



the signal emitted @ t' is arriving now, at t.

Continuing with the other terms in $\vec{E}$. We still have to evaluate

$$\frac{1}{4\pi\epsilon_0} \int d^3x' \left\{ \frac{\hat{r}}{cr} \frac{\partial \rho_{ret}}{\partial t} - \frac{1}{c^2 r} \frac{\partial \vec{j}_{ret}}{\partial t} \right\}$$

plug in
$$\rho(x', t') = e\,\delta^{(3)}(\vec{x}' - \vec{s}(t'))$$
$$\vec{j}(x', t') = e\,\vec{\beta}(t')\,c\,\delta^{(3)}(\vec{x}' - \vec{s}(t'))$$

So these terms are

$$\frac{e}{4\pi\epsilon_0}\,\underset{\substack{\uparrow\\ \text{pull out } \partial_t}}{\partial_t}\left[\int d^3x' \left(\frac{\hat{r}}{cr} - \frac{\vec{\beta}(t - r/c)}{cr}\right)\delta^{(3)}(\vec{x}' - \vec{s}(t - r/c))\right]$$

The integrals can be done exactly as before, giving

$$\frac{e}{4\pi\epsilon_0}\,\partial_t\left[\left(\frac{\hat{R} - \vec{\beta}}{cR\underset{\substack{\uparrow\\ \text{here's our determinant factor again}}}{K}}\right)_{ret}\right]$$

$\vec{R}(t') = \vec{x} - \vec{s}(t')$
$K(t') = 1 - \hat{\beta}\cdot\hat{R}$
$\vec{\beta}(t') = \frac{1}{c}\frac{\partial \vec{s}(t')}{\partial t'}$
$t'_{ret} = t - R(t'_{ret})/c$

In total, for a point charge,

$$\vec{E}_{(x,t)} = \frac{e}{4\pi\epsilon_0}\left\{\left[\frac{\hat{R}}{KR^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{R} - \vec{\beta}}{KR}\right]_{ret}\right\}$$

Similarly,

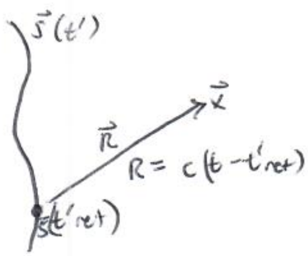
$$\vec{B}(x,t) = \frac{\mu_0 e}{4\pi}\left\{\left(\frac{c(\vec{\beta}\times\hat{R})}{KR^2}\right)_{ret} + \frac{\partial}{\partial t}\left(\frac{\vec{\beta}\times\hat{R}}{KR}\right)_{ret}\right\}$$

It is possible to evaluate the derivatives, giving a useful form in which $\vec{E}$ & $\vec{B}$ are explicitly just functions of $\vec{R}$, $\vec{\beta}$, and $\dot{\vec{\beta}}$. The algebra is a little lengthy, so we won't work it all out. Here is the result:

$$\vec{E} = \frac{e}{4\pi\epsilon_0}\left\{ \left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3 R^2}\right]_{ret} + \frac{1}{c}\left[\frac{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right]}{(1-\vec{\beta}\cdot\hat{n})^3 \; R}\right]_{ret} \right\}$$

$$\vec{B} = \frac{1}{c}\left[\hat{n}\times\vec{E}\right]_{ret} \qquad \left(\gamma = \frac{1}{\sqrt{1-\beta^2}}\right)$$

These are the fields, at $\vec{x}$ & t, generated by a point charge that had velocity $c\vec{\beta}(t')$ and accel. $c\dot{\vec{\beta}}(t')$ at $t'_{ret} = t - \frac{|\vec{x}-\vec{s}(t'_{ret})|}{c}$.



Note:
- If $\dot{\vec{\beta}} = 0$ (no accel), fields $\sim 1/R^2$
- If $\dot{\vec{\beta}} \neq 0$, there is a $1/R$ component. This is called the "radiation field"

Let $\vec{a} = \dot{\vec{\beta}}c$. Then
$$\vec{a}_\parallel \equiv (\vec{a}\cdot\hat{n})\hat{n}$$
$$\vec{a}_\perp \equiv (\vec{a} - \vec{a}_\parallel)$$

plug in $\dot{\vec{\beta}} = \frac{1}{c}(\vec{a} - \vec{a}_\perp + \vec{a}_\parallel)$

use, $a_\parallel \times n = 0$

and (algebra)
$$\vec{E}_{rad} = \left(\frac{-e\left(\frac{\vec{a}_\perp}{c} - \hat{n}\times(\vec{a}\times\vec{\beta})\right)}{4\pi\epsilon_0 c R\,(1-\vec{\beta}\cdot\hat{n})^3}\right)_{ret}$$

$$\vec{B}_{rad} = \left(\frac{\hat{n}\times\vec{E}_{rad}}{c}\right)_{ret}$$