Curvilinear coordinates (cylindrical, spherical)

A vector $\vec{A}$ in cartesian, cylindrical, spherical coordinates

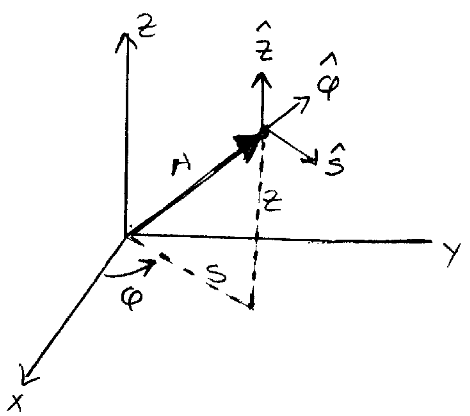
Cartesian: $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

Cylindrical: $\vec{A} = A_s \hat{s} + A_\varphi \hat{\varphi} + A_z \hat{z}$

Spherical: $\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}$

It is possible to go from one coordinate representation to another, both the vector component and the unit vectors must be transformed.

Lets take a look at a vector in the cylindrical coordinate system:



$x = s\cos\varphi$

$y = s\sin\varphi$

$z = z$

~ or ~

$s = \sqrt{x^2+y^2}$

$\varphi = \tan^{-1}(y/x)$

$z = z$

What about the unit vectors?

There are a couple ways to approach this problem,

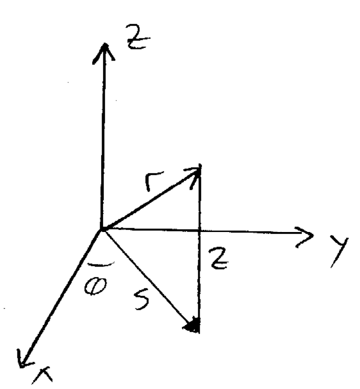
① Use geometry. Project $\hat{s}, \hat{\varphi}, \hat{z}$ (or $\hat{r}, \hat{\varphi}, \hat{\theta}$) onto cartesian unit vectors to get expressions for $\hat{s}, \hat{\varphi}, \hat{z}$ $(\hat{r}, \hat{\theta}, \hat{\varphi})$. Make sure the unit vectors are normalized (have magnitude of one).

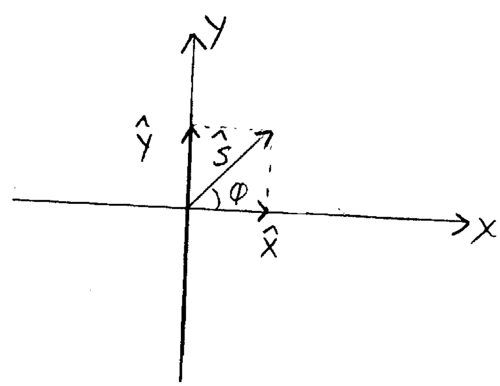
② Take the appropriate derivative of a general position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. For example, $\hat{x}$ is a unit vector pointing in the direction of increasing values of x for constant values of y & z. This is the same thing as saying

$$\hat{x} = c\,\frac{\partial \vec{r}}{\partial x}$$

where c normalizes $\frac{\partial \vec{r}}{\partial x}$ to unit magnitude.
$\left(c = 1 \text{ for } \hat{x} = c\,\frac{\partial \vec{r}}{\partial x}\right)$

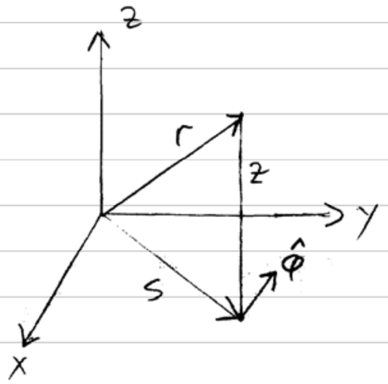
We'll check these methods out for $\hat{s}, \hat{\varphi}$ in cylindrical coordinates. The s component of a vector lies in the x-y plane, so that $\hat{s}$ also must lie in the x-y plane.

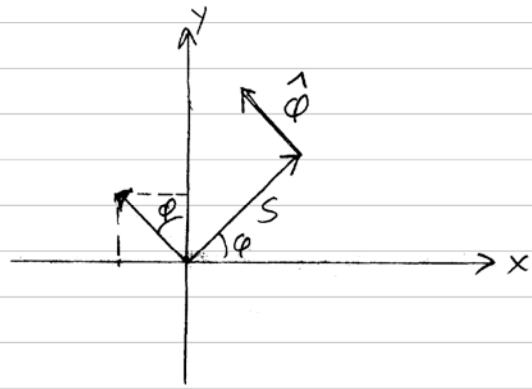




$$\hat{s} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

The $\hat{\varphi}$ unit vector also lies in the x-y plane.





$$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

Now let's check out the partial derivative method.

$$\hat{\varphi} = c\,\frac{\partial \vec{r}}{\partial \varphi} = c\left(\hat{x}\,\frac{\partial x}{\partial \varphi} + \hat{y}\,\frac{\partial y}{\partial \varphi} + \hat{z}\,\frac{\partial z}{\partial \varphi}\right)$$

$$= c\left(\hat{x}\,\frac{\partial\, s\cos\varphi}{\partial \varphi} + \hat{y}\,\frac{\partial\, s\sin\varphi}{\partial \varphi} + \hat{z}\,\frac{\partial z}{\partial \varphi}\right)$$

$$= c\left(-s\sin\varphi\,\hat{x} + s\cos\varphi\,\hat{y}\right)$$

$$\hat{\varphi}\cdot\hat{\varphi} = c^2\left[s^2\sin^2\varphi + s^2\cos^2\varphi\right]$$

$$= c^2 s^2\left(\sin^2\varphi + \cos^2\varphi\right) = c^2 s^2$$

$$\hat{\varphi}\cdot\hat{\varphi} = 1 = c^2 s^2 \longrightarrow \text{thus, } c = \frac{1}{s}$$

Since $C = 1/s$, $\quad \hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

How about $\hat{s}$?

$$\hat{s} = c\left(\hat{x}\,\frac{\partial s\cos\varphi}{\partial s} + \hat{y}\,\frac{\partial s\sin\varphi}{\partial s} + \hat{z}\,\frac{\partial z}{\partial s}\right)$$

$$= c\,(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})$$

$$\hat{s}\cdot\hat{s} = c^2\left[\cos^2\varphi + \sin^2\varphi\right] = c^2 \;\rightarrow\; c = 1$$

$$\hat{s} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

These are the same results we got geometrically.
Also, $\hat{z} = \hat{z}$ by inspection

What about $\hat{x}, \hat{y}, \hat{z}$ in terms of $\hat{s}, \hat{\varphi}, \hat{z}$?

$\hat{z} = \hat{z}$ (naturally)

To get the other relations, let's say for $\hat{x}$, we cleverly multiply $\hat{s}$ by $\cos\varphi$, $\hat{\varphi}$ by $-\sin\varphi$, and add them.

$$\hat{s}\cos\varphi = \cos^2\varphi\,\hat{x} + \cos\varphi\sin\varphi\,\hat{y}$$

$$-\hat{\varphi}\sin\varphi = \sin^2\varphi\,\hat{x} - \cos\varphi\sin\varphi\,\hat{y}$$

$$\cos\varphi\,\hat{s} - \sin\varphi\,\hat{\varphi} = (\cos^2\varphi + \sin^2\varphi)\,\hat{x} = \hat{x}$$

Dirac-delta function

In 1-D:

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

In 3-D:

$$\delta^3(r) = \delta(x)\,\delta(y)\,\delta(z)$$

$$\int_{\substack{\text{All}\\\text{space}}} \delta^3(\vec{r})\,d\tau = 1$$

$$\int_{\substack{\text{all}\\\text{space}}} f(\vec{r})\,\delta^3(\vec{r})\,d\tau = f(0)$$

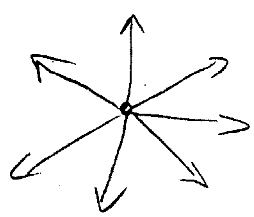
$$\int_{\substack{\text{all}\\\text{space}}} f(r)\,\delta^3(\vec{r}-\vec{a})\,d\tau = f(a)$$

What motivated the invention of such a beast?

Notice that outside the integral $\delta(r)$ is a discontinuity to end all discontinuities. Inside the integral life settles down considerably. The integral of $\overset{3}{\delta}(r)$ over all space is finite (1), and if there is a function multiplying $\overset{3}{\delta}(r)$, the delta function acts to <u>select</u> a specific evaluation of that function. So, δ acts as a selector in this case.

In a theory such as classical electrodynamics, there are discontinuities, for example a point charges representing what we can experience. The mathematical mirror of the point charge idea must reflect this lack of smoothness.

Griffiths first hints at the need for the delta function in problem 1.16. The vector function $\vec{v} = \dfrac{\hat{r}}{r^2}$ looks like field lines from a point charge.



However, the divergence calculation,

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r)$$

looks at first glance to be zero.

From the picture, this cannot be true at the origin. The trouble is that $\dfrac{\hat{r}}{r^2} \to \infty$ as $r \to 0$

So, $\dfrac{1}{r^2} \dfrac{\partial}{\partial r} (r^2 v_r) = \dfrac{1}{r^2} \dfrac{\partial}{\partial r} (r^2/r^2)$ is undefined at $r=0$.

However,

$$\int_{\substack{\text{all}\\ \text{space}}} \vec{\nabla}\cdot\vec{v}\, d\tau = \int \vec{v}\cdot d\vec{a} = \int \frac{\hat{r}}{r^2}\cdot \hat{r}\, r^2 \sin\theta\, d\theta\, d\varphi$$

$$= \int d\Omega = 4\pi \qquad \underline{\text{finite}}$$

$$\Rightarrow \qquad \nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})$$

This implies $\nabla^2\left(\frac{1}{\mathscr{r}}\right) = \nabla\cdot\nabla\left(\frac{1}{\mathscr{r}}\right) = \nabla\cdot\left(\frac{-\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = -4\pi\,\delta^3(\vec{\mathscr{r}})$

$$\boxed{\begin{aligned} \nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) &= 4\pi\,\delta^3(\mathscr{r}) \\ \nabla^2\left(\frac{1}{\mathscr{r}}\right) &= -4\pi\,\delta^3(\mathscr{r}) \end{aligned}}$$

useful to remember

$$\boxed{\vec{\mathscr{r}} = \underbrace{\vec{r}}_{\text{point of observation}} - \underbrace{\vec{r}\,'}_{\text{source point}}}$$

also useful to remember

Fun with Fourier transforms

or ~ you have to know everything! ~

kidding - I sure don't.

When looking at the response of a natural system, you could examine its behavior in time. Or, you could examine the frequency content of the response. The way people say this is often "study the time domain response" & "study the frequency domain response". Either way, (time or frequency), since you are looking at the <u>same system</u> under the <u>same circumstances</u>, the two viewpoints must be somehow related. You can go back and forth between the descriptions with a <u>Fourier transform</u>.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

function form in frequency domain

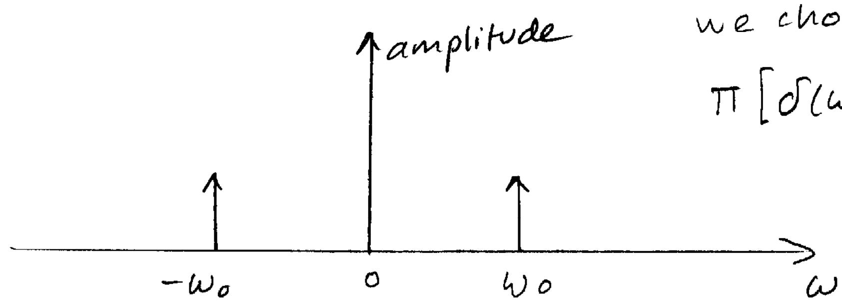
functional form in time domain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

same as above

Now we get to the part where we can see one use of the delta function.

Suppose we have a single frequency function. In the frequency domain, this might be represented by the two impulses $\delta(\omega-\omega_0)$ and $\delta(\omega+\omega_0)$.



we choose:

$$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

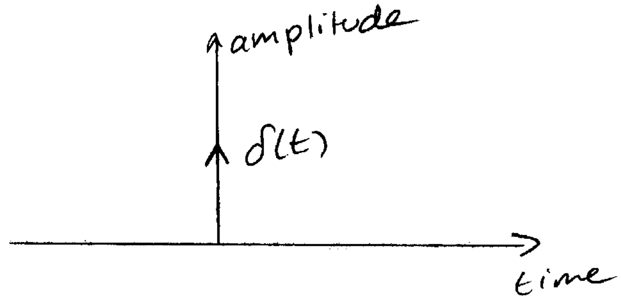
Negative ω_0 is the same <u>value</u> of frequency as ω_0, the presence of both $+\delta(\omega-\omega_0)$ and $+\delta(\omega+\omega_0)$ indicates the same behavior for positive or negative phase advance from zero. This single value of frequency, symmetric in behavior indicates a cosine-like function in the time domain. Hey, now we can check it, since we know about delta functions.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] e^{j\omega t}\, d\omega$$

$$= \frac{1}{2}\left\{\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega + \int_{-\infty}^{\infty} \delta(\omega+\omega_0)\, e^{j\omega t}\, d\omega\right\}$$

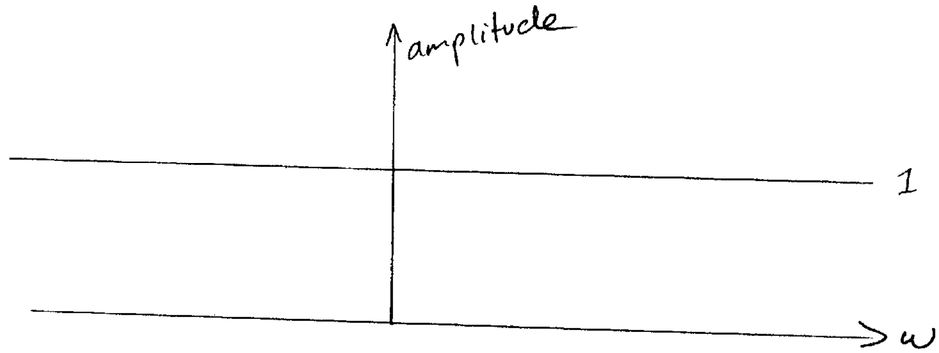
$$= \frac{1}{2}\left\{ e^{j\omega_0 t} + e^{-j\omega_0 t}\right\} = \cos(\omega_0 t) \quad \text{(yes!)}$$

How does a delta function in time (impulse) transform into frequency?



$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{-j\omega(0)} = 1$$

This is:



A perfect impulse contains all frequencies!

The implication is that if something is whacked with a perfect impulse, all frequencies are in the whack. So, a system having a certain resonant (natural) frequency will see a drive at that frequency if whacked. (This is the lay-man side of me.)

Just another example of how delta functions can serve you.