Lecture 22 – Landau theory

LAST TIME: We discussed phase transitions, in particular behavior near a critical point
Systems near a critical point display universality in behavior
Ex: scaling laws near critical point

$$M \sim (T_c - T)^{\beta} \quad \text{and} \quad \chi \sim (T - T_c)^{-\gamma} \quad \text{for a ferromagnet}$$

This universality not only occurs for different materials

Ex: ^{3}He, CO_2, etc. for gas-liquid transition

But also for completely different physical phenomena (provided there are some commonalities like dimensionality)

Ex: gas-liquid, paramagnet-ferromagnet, normal-superconducting phase transitions

Why does this happen? There must be something general about these systems near critical points despite their differences

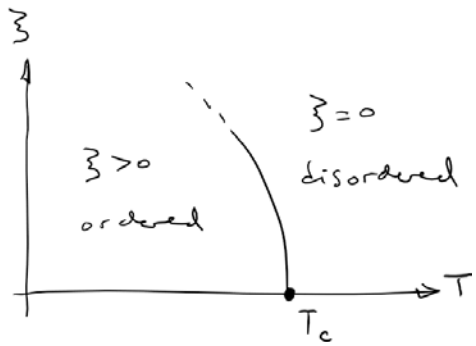
TODAY: Landau theory of phase transitions (1936)
Formulation of general theory for 2nd order phase transitions (but can be extended to 1st order)

- Phenomenological, i.e. not derived from microscopic theory or first principles (in most cases)
- Mean-field theory

KEY CONCEPT: order parameter

We know that a system minimizes its Gibbs free energy *G* (or Helmholtz free energy *F*, if *V* is constant) at equilibrium during a phase transition – minimizes with respect to what variable(s)?

Phase transitions involve transformation from disordered (high entropy *S*) phase and ordered (low *S*) phases:



The order parameter ξ is a single variable that specifies the phase

In disordered phase $\xi = 0$

In ordered phase $\xi > 0$

Ex:

Disordered phase	Ordered phase	Order parameter	Description
gas	liquid	$\xi = \rho - \rho_C$	density*
paramagnet	ferromagnet	$\xi = M$	magnetization
normal	superfluid	$\xi = \|\psi\|^2$	superfluid fraction

* ρ_C = critical density, i.e. density at the critical point

KEY CONCEPT: Landau free energy

Landau postulated that the system has a <u>Landau free energy</u> $F_L(\xi,T)$ minimized at equilibrium with respect to the order parameter:

$$\left(\frac{\partial F_L}{\partial \xi}\right)_T = 0 \quad \text{at equilibrium}$$

(Note: if there is more than 1 local minimum, the global minimum determines the equilibrium state $\xi_{eq}(T)$)

We are interested in the behavior of the system near a critical point $T \approx T_C$ where we expect ξ to be small – therefore, we can expand $F_L(\xi, T \approx T_C)$ about $\xi = 0$

For $T \approx T_C$:

$$F_L(\xi,T) = g_0(T) + g_1(T)\xi + \frac{1}{2}g_2(T)\xi^2 + \frac{1}{3}g_3(T)\xi^3 + \underbrace{\frac{1}{4}g_4(T)\xi^4}_{\uparrow\ \text{Highest order we'll need}} + \cdots$$

where the coefficients $g_i(T)$ are functions of T.

At equilibrium:

$$\left(\frac{\partial F_L}{\partial \xi}\right)_T = 0 = g_1(T) + g_2(T)\xi + g_3(T)\xi^2 + g_4(T)\xi^3$$

We can simplify/refine this expression based on our knowledge of systems undergoing phase transitions:

1. We can use symmetry arguments to claim that $F_L(\xi,T)$ must be <u>even</u> in ξ, so that odd orders vanish: i.e. $g_1 = g_3 = \cdots = 0$. (Ex: in ferromagnet phase with $B = 0$, two opposite spin states $\xi = M$ = ↑↑↑↑ and $-M$ = ↓↓↓↓ are equally likely)
   (Note: we'll see later how to generalize $F_L(\xi,T)$ when $B > 0$).

$$F_L(\xi,T)=g_0(T)+\frac{1}{2}g_2(T)\xi^2+\frac{1}{4}g_4(T)\xi^4$$

Question 1: Analyze the Landau free energy and locate its minima, identifying the conditions under which they are minima

Taking the derivative of $F_L(\xi,T)$:

$$\left(\frac{\partial F_L}{\partial \xi}\right)_T=0=g_2(T)\xi+g_4(T)\xi^3$$

which has solutions: $\xi_0=0$ and $\xi_\pm=\pm\sqrt{\frac{-g_2(T)}{g_4(T)}}$

To determine whether they are minima, look at the second derivative:

$$\left(\frac{\partial^2 F_L}{\partial \xi^2}\right)_T=g_2(T)+3g_4(T)\xi^2$$

$$=\begin{cases} g_2(T) & \text{for } \xi_0 \\ -2g_2(T) & \text{for } \xi_\pm \end{cases}$$

Therefore, ξ_0 is a minimum if $g_2(T)>0$, $\xi_\pm$ are minima if $g_2(T)<0$. Note that to ensure real solutions, we must have $g_4(T)>0$.

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2. We expect $\xi_{eq}=0$ to be the solution for $T \geq T_C$ and $\xi_{eq}\neq 0$ to be the solution for $T < T_C$. This means that $g_2(T)\propto(T-T_C)$ and must flip sign across the transition temperature.

As we are interested in behavior near T_C, we want to expand each coefficient $g_i(T)$ about $T = T_C$:

$$g_i(T)=g_i(T_C)+g_i{}'(T-T_C)+\cdots$$

The argument above means that $g_2(T_C)=0$ and

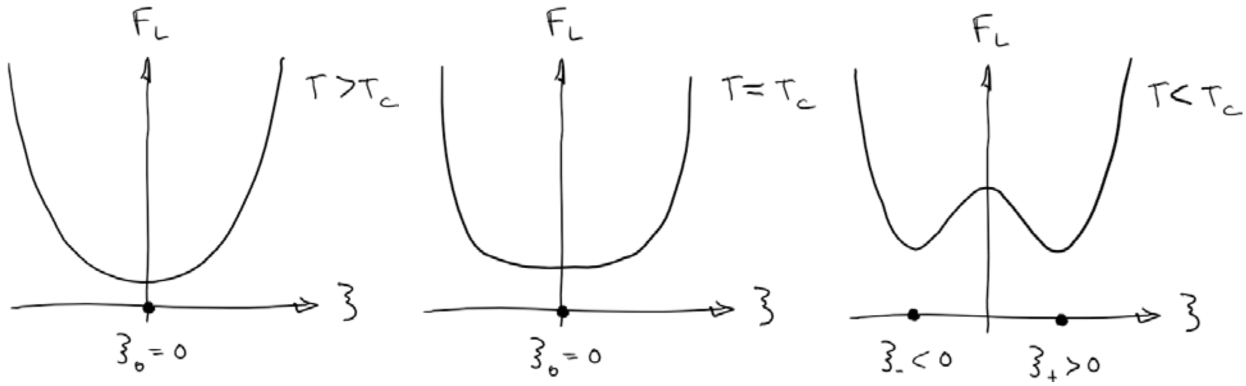
$$g_2(T)=g_2{}'(T-T_C)\equiv\alpha(T-T_C)$$

We cannot make similar statements for $g_0(T)$ or $g_4(T)$.
So, to lowest non-vanishing order in $T - T_C$ and ξ:

$$F_L(\xi,T)=g_0(T)+\frac{1}{2}\alpha(T-T_C)\xi^2+\frac{1}{4}g_4\xi^4$$

with $g_4>0$

Let's plot the Landau free energy vs. ξ:

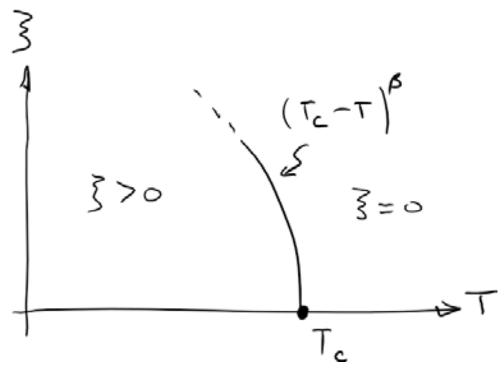


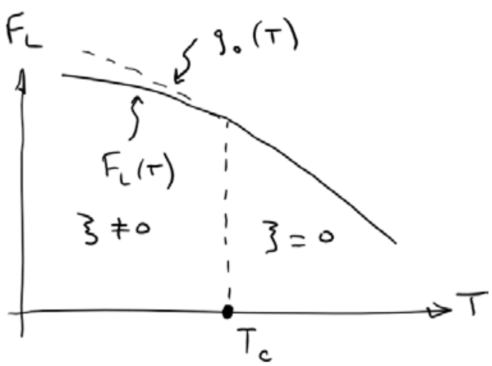
For $T < T_C$ the equilibrium solutions are:

$$\xi_\pm = \pm\sqrt{\frac{-g_2(T)}{g_4(T)}} = \pm\sqrt{\frac{\alpha(T_C - T)}{g_4}}$$

$$\sim \left(T_C - T\right)^\beta \quad \text{with } \beta = \tfrac{1}{2}$$

We get the same scaling law and critical exponent as in Lect. 21





Now let's plot the Landau free energy vs. T at the equilibrium order parameter:

$$F_L(T) = \begin{cases} g_0(T) & \text{for } T \geq T_C \\ g_0(T) - \dfrac{\alpha^2 (T_C - T)^2}{4 g_4} & \text{for } T < T_C \end{cases}$$

The system adopts the minimum free energy $F_L(T)$.

This matches the behavior of a ferromagnet at B = 0, also gas-liquid phase transition near its critical point.

Landau's postulate is that <u>all 2nd order phase transitions</u> will have this form of free energy $F_L(\xi,T)$ sufficiently close to the critical temperature – leads to universal behavior, scaling laws with universal critical exponents (α, β, γ, δ, η, and ν).

In a ferromagnet, we had a magnetic field B. How do we incorporate this in Landau theory? Simple, add an alignment energy $-MB$:

$$F_L(\xi,T) = g_0(T) + \frac{1}{2}\alpha(T-T_C)\xi^2 + \frac{1}{4}g_4\xi^4 - H\xi$$

H is a generalized external force, = B for a magnetic system, = p (pressure) for a gas or liquid

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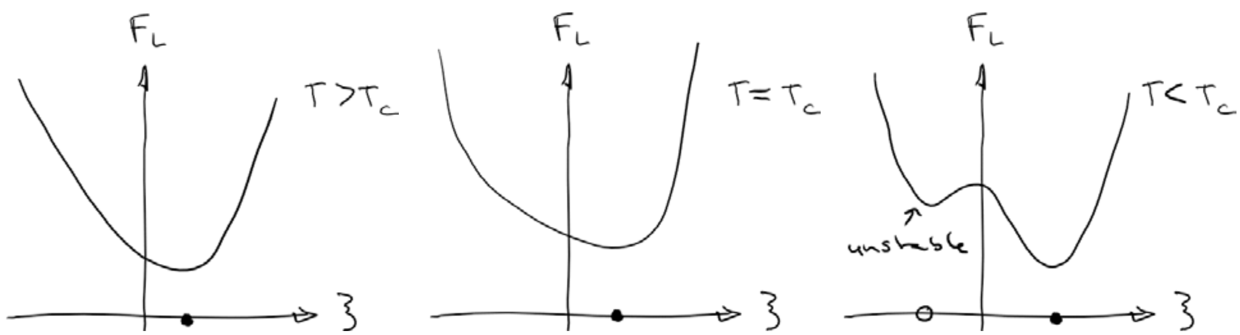
Question 2: Replot the Landau free energy vs. order parameter for $H > 0$ and $T > T_C$, $T = T_C$, and $T < T_C$, and highlight where the equilibria are.



Notice the unstable state $T < T_C$, matching what we observed in Lect. 21.

Equilibrium ξ is:

$$\left(\frac{\partial F_L}{\partial \xi}\right)_T = 0 = \alpha(T-T_C)\xi + g_4\xi^3 - H$$

To lowest order (ignoring the ξ^3 term): $\qquad \xi = \dfrac{H}{\alpha(T-T_C)}$

$$\chi = \left(\frac{\partial \xi}{\partial H}\right)_{H=0} \sim \left(T-T_C\right)^{-\gamma} \text{ with } \gamma = 1,$$

as in Lect. 21.

Summary of Landau theory

- Landau theory is phenomenological, but in some cases can be connected directly to microscopic theory (e.g. for ferromagnet-paramagnet, gas-liquid transitions)
- It gets a lot of things qualitatively right. However, because it is a mean field theory, the critical exponents it predicts don't match experiments exactly
- More sophisticated field theoretical techniques are required to get better agreement with experiments

KEY CONCEPT: Extension to 1st order phase transitions
(Note: this formalism is different than that used in K & K)

Above we excluded odd powers of ξ. What if we put some back in (assuming H = 0)?

$$F_L(\xi,T) = g_0(T) + \frac{1}{2}\alpha(T-T_0)\xi^2 + \frac{1}{3}g_3\xi^3 + \frac{1}{4}g_4\xi^4$$

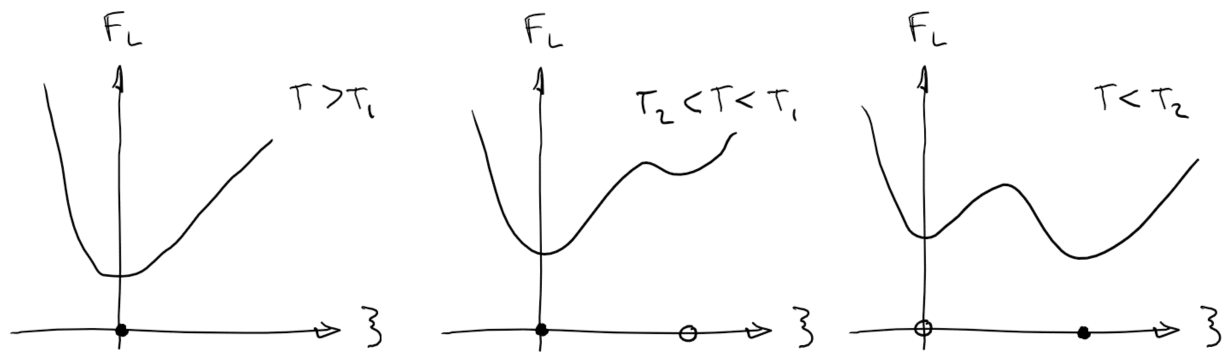
The extrema are found from

$$\left(\frac{\partial F_L}{\partial \xi}\right)_T = 0 = \alpha(T-T_0)\xi + g_3\xi^2 + g_4\xi^3$$
$$= \xi\left(\alpha(T-T_0) + g_3\xi + g_4\xi^2\right)$$

which has solutions: $\xi_0 = 0$ and $\xi_\pm = \dfrac{-g_3 \pm \sqrt{g_3^2 - 4g_4\alpha(T-T_0)}}{2g_4}$

The latter solutions are real if $g_3^2 - 4g_4\alpha(T-T_0) \geq 0$, i.e. if temperature $T < T_1 \equiv T_0 + \dfrac{g_3^2}{4g_4\alpha}$

Plotting this function at different temperatures:



Question 3: Plot the order parameter vs. *T* across the phase transition

$T > T_1$ the only minimum is at $\xi_0 = 0$

$T_2 < T < T_1$ there is a second minimum at ξ_+ but this is <u>not a global minimum</u>, so $\xi_0 = 0$ is the equilibrium

$T < T_2$ the minimum at ξ_+ becomes a <u>global minimum</u> and the equilibrium, ξ jumps discontinuously at T_2 – first order phase transition!