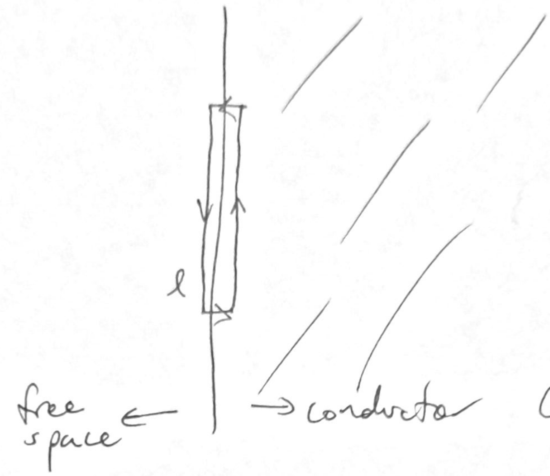
ENERGY LOSS IN GUIDES

First, let's review boundary conditions.

Take a small loop:



(assume $\epsilon \approx \epsilon_0$, $\mu \approx \mu_0$ for simplicity)

- $$\int_\ell \vec{B} \cdot d\vec{\ell} = (B_{\parallel})_{out} - (B_{\parallel})_{in}$$

$$\underset{\text{Stokes}}{=} \int_{\ell} (\vec{\nabla} \times \vec{B}) \cdot d\vec{A}$$

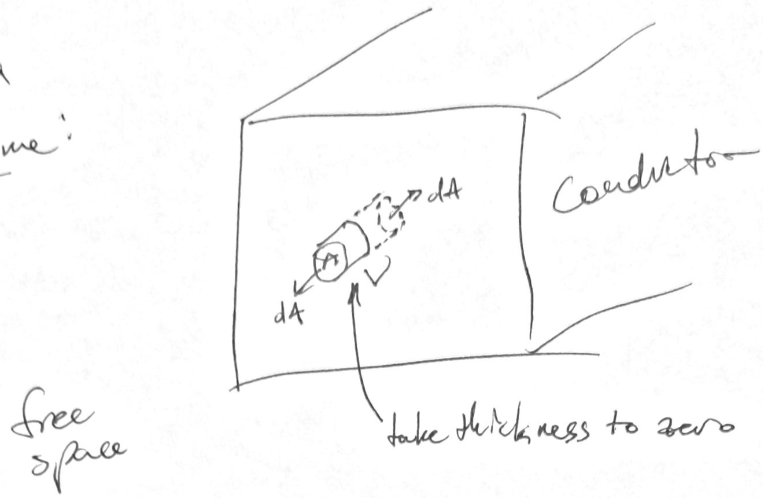
$$\underset{\text{Maxwell}}{=} \underbrace{\mu_0 \int \vec{J} \cdot d\vec{A}}_{\propto \text{ surface current}} + \underbrace{\mu_0 \epsilon_0 \int \dot{\vec{E}} \cdot d\vec{A}}_{\to 0 \text{ in the limit of vanishing area if } \vec{E} \text{ is finite.}}$$

- $$\int_\ell \vec{E} \cdot d\vec{\ell} = (E_{\parallel})_{out} - (E_{\parallel})_{in}$$

$$\underset{\text{Stokes}}{=} \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{A}$$

$$\underset{\text{Maxwell}}{=} \underbrace{- \int \dot{\vec{B}} \cdot d\vec{A}}_{\to 0 \text{ in the limit of vanishing area if } \vec{B} \text{ is finite}}$$

Now take a small volume:



- $$\int_V \vec{\nabla} \cdot \vec{E} \, dV \underset{\text{gauss}}{=} \int_{out} \vec{E} \cdot d\vec{A} + \int_{in} \vec{E} \cdot d\vec{A}$$

$$= A \left(\vec{E}_{\perp out} - \vec{E}_{\perp in} \right) \quad \text{(for small } A\text{)}$$

$$\underset{\text{maxwell}}{=} \frac{Q_{enclosed}}{\epsilon_0}$$

or $\Delta \vec{E}_\perp \propto$ surface charge density

- $$\int_V \vec{\nabla} \cdot \vec{B} \, dV \underset{\text{gauss}}{=} A \Delta \vec{B}_\perp$$

$$\underset{\text{maxwell}}{=} 0$$

In sum,

$\Delta B_{\parallel} \propto$ surf current density

$\Delta B_\perp = 0$

$\Delta E_{\parallel} = 0$

$\Delta E_\perp \propto$ surf charge density

For a perfect conductor, $\vec{B}_{in} = \vec{E}_{in} = 0$, so we find, e.g., $(\vec{B}_\perp)_{out} = 0$, $(\vec{E}_\parallel)_{out} = 0$, the usual conditions.

Now say we have an imperfect conductor, with finite conductivity: $\vec{J} = \sigma \vec{E}$ (ohm's law)

This, together with $(\Delta \vec{E}_\parallel) = 0$, says there can't be a surface current density. (Meaning a genuinely singular $\vec{J}$, a delta function at the surface.) We will find a nonsingular $\vec{J}$ inside the conductor (still close to the surface.)

So $\Delta B_\parallel = 0$, and $(\vec{\nabla} \times \vec{B})_{in} \cong \mu_0 \vec{J}_{in} = \sigma \mu_0 \vec{E}_{in}$

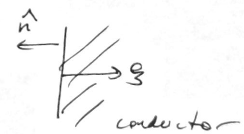
Here we drop the Maxwell term; assume the conductivity is large.

$$\Rightarrow \vec{E}_{in} \approx \frac{1}{\sigma \mu_0} \vec{\nabla} \times \vec{B}_{in}$$

and from Maxwell, $\vec{B}_{in} = \frac{-i}{\omega} \vec{\nabla} \times \vec{E}_{in}$ ($B \sim e^{-i\omega t}$ assumed)

Now let's assume slow variations of the fields in the $\parallel$ directions, compared to the $\perp$ directions. Then

$$\vec{\nabla} \approx -\hat{n} \frac{\partial}{\partial \xi}$$



and
$$\vec{E}_{in} \approx \frac{-1}{\sigma \mu_0} \hat{n} \times \partial_\xi \vec{B}_{in}$$
$$\vec{B}_{in} \approx \frac{i}{\omega} \hat{n} \times \partial_\xi \vec{E}_{in}$$

These are still small (close to the perfect conductor limit) if σ is large, at least far from the surface.

Now we substitute in and solve!

$$\vec{B}_{in} \simeq -\frac{i}{\omega}\hat{n} \times \partial_\xi \left(\frac{1}{\sigma\mu_0} \hat{n} \times \partial_\xi \vec{B}_{in} \right)$$

$$\hat{n} \times (\hat{n} \times \vec{B}_{in}) = -\vec{B}_{\parallel}$$

$$\text{so} \quad \partial_\xi^2 \vec{B}_{\parallel in} = i\omega\sigma\mu_0 \vec{B}_{\parallel in}$$

$$\text{and} \quad \vec{B}_{\perp in} = 0$$

The physically relevant solution is the one that doesn't blow up inside the conductor!

$$\vec{B}_{\parallel in} = \vec{B}_{\parallel out}\, e^{-\sqrt{i\omega\sigma\mu_0}\,\xi} \qquad \sqrt{i} = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$$

$$= \vec{B}_{\parallel out}\, e^{-\xi/\delta}\, e^{-i\xi/\delta} \qquad \delta \equiv \sqrt{2/\omega\sigma\mu_0}$$

– small for large σ –

- exponential decay beyond the "skin depth" δ.
- for $\xi = 0$, $\vec{B}_{\parallel in} = \vec{B}_{\parallel out}$ consistent w/ zero surf current.

And we can compute $\vec{E}_{in} = \frac{-1}{\sigma\mu_0}\hat{n} \times \partial_\xi \vec{B}_{in}$

Since $\vec{B}_{in} = \vec{B}_{in\parallel}$ and $(\hat{n} \times \vec{B})$ is $\perp$ to $\hat{n}$,

this is a contribution to $\vec{E}_{in\parallel}$, but not $\vec{E}_{in\perp}$

Together with $\Delta\vec{E}_{\parallel} = 0$ & $\Delta\vec{B}_{\parallel} = 0$ we find there is a small $\vec{E}_{out\parallel}$,

$$\vec{E}_{\parallel out} = \frac{-1}{\sigma\mu_0}\hat{n} \times \partial_\xi \vec{B}_{out}$$

There is also a small $(B_\perp)_{in,out}$ generated near the surface, but we won't need it.

The cool thing about this is we can use it to compute Ohmic losses in real conductors: The existence of $(\vec{B}_{\parallel})_{out}$ + Ohm's law generated $\vec{E}_{\parallel out}$, so now there can be a nonzero <u>normal</u> component to the Poynting vector!

$$\frac{dP}{d(\text{area})} = -\frac{1}{2\mu_0} \text{Re}\left[\hat{n}\cdot\left(\vec{E}\times\vec{B}^*\right)\right]$$

(if $\mu_{out} \neq \mu_0$, the formula is $-\frac{1}{2}\text{Re}\left(\hat{n}\cdot\left(E\times H^*\right)\right)$)

$$= \frac{\omega\delta}{4\mu_0}|\vec{B}_{\parallel}|^2$$

small if the skin depth is small.

This is resistive heating energy loss: Since $\vec{E}_{\parallel in}$ is now nonzero, there is a current $\vec{J} = \sigma\vec{E}_{\parallel in}$ by Ohm's law, and the analog of $P = I^2 R$ is $\frac{dP}{d\text{vol}} = \frac{1}{2\sigma}|J|^2$

We can define an effective surface current:

$$K_{eff} = \int_0^\infty J\, d\xi$$

$$= -\frac{\hat{n}\times\vec{B}_{\parallel out}}{\mu_0}\int_0^\infty \partial_\xi e^{-\xi/\delta}\, d\xi$$

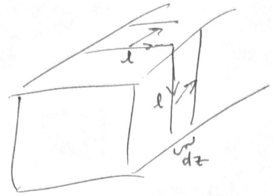
$$= \frac{1}{\mu_0}\hat{n}\times\vec{B}_{\parallel out} = \frac{1}{\mu_0}\hat{n}\times\vec{B}_{out}$$

and with $d(\text{vol}) \equiv \delta\, d(\text{area})$, $K_{eff} \sim \delta J$, one can show

$$\boxed{\frac{dP}{da} = \frac{1}{2\sigma\delta}|K_{eff}|^2}$$

so $(\sigma\delta)^{-1}$ is like Resistance/unit area.

Let's go back to our waveguides.



We can write the power loss as

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint d\ell \left|\frac{1}{\mu_0}\hat{n}\times\vec{B}\right|^2$$

We know that for TM modes, inside the guide

$$\vec{B} = \vec{B}_t = \frac{i}{(\omega^2\mu\epsilon - k^2)}\left(\mu\epsilon\omega\,\hat{z}\times\vec{\nabla}_t\psi\right)$$

$$(\psi = E_z)$$

while for TE modes,

$$\vec{B} = \psi\hat{z} + \frac{ik}{\omega^2\mu\epsilon - k^2}\vec{\nabla}_t\psi \qquad (\psi = B_z)$$

recall that for a given mode, $\omega^2\mu\epsilon - k^2 \equiv \gamma_\lambda^2$ where $\{\gamma_\lambda^2\}$ are eigenvalues of the 2D Laplace operator, and the lowest possible frequency for a mode λ is $\omega_\lambda \equiv \gamma_\lambda/\sqrt{\mu\epsilon}$. So we can write $\frac{k^2}{\mu\epsilon} = \omega^2 - \omega_\lambda^2$

$$\hat{n}\times\vec{B} = \begin{cases} \dfrac{i\omega}{\omega_\lambda^2}\dfrac{\partial\psi}{\partial n}\hat{z}, & TM \\[2ex] (\hat{n}\times\hat{z})\psi + \dfrac{i\gamma\sqrt{\omega^2-\omega_\lambda^2}}{\omega_\lambda^2}\hat{n}\times\vec{\nabla}_t\psi, & TE \end{cases}$$

And so
$$|\hat{n}\times\vec{B}|^2 = \begin{cases} \dfrac{\omega^2}{\omega_\lambda^4}\left|\dfrac{\partial\psi}{\partial n}\right|^2 & ;\ TM \\ |\psi|^2 + \dfrac{\omega^2 - \omega_\lambda^2}{\omega_\lambda^2}\,|\hat{n}\times\vec{\nabla}_t\psi|^2 & ;\ TE \end{cases}$$

using the fact that $(\hat{n}\times\hat{z}) \perp \hat{n}\times\vec{\nabla}_t$

Inserting into $\oint dl\,|\hat{n}\times\vec{B}|^2$ we can compute the losses if we know a solution ψ to $(\nabla_t^2 + \mu_0\epsilon_0\omega_\lambda^2)\psi = 0$.

We can estimate the behavior by noting that ∇_t means transverse derivatives of ψ are of order $\sqrt{\mu_0\epsilon_0\omega_\lambda^2}\,\psi$. Thus,

eg for TM modes, $|\hat{n}\times\vec{B}|^2 \sim \dfrac{\omega^2}{\omega_\lambda^4}(\mu_0\epsilon_0\omega_\lambda^2)|\psi|^2$

and
$$-\frac{dP}{dz} \sim \frac{1}{2\sigma\delta}\,\frac{\omega^2\mu_0\epsilon_0}{\mu_0^2\omega_\lambda^2}\left(\frac{\text{circumference}}{\text{area}}\right)\int d(\text{area})\,|\psi|^2$$

By computing the $\hat{z}$ component of the poynting vector (see Jackson pg 363) one can show that without losses the power shooting down the guide is $P_{TM} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\left(\frac{\omega}{\omega_\lambda}\right)^2\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}\int dA\,|\psi|^2$

"field squared" as usual

$$\Rightarrow \quad \frac{dP}{dz} = -2\beta_\lambda^{TM} P \quad \text{with}$$
$$\beta_\lambda^{TM} \approx \sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{1}{\sigma\delta}\left(\frac{\text{circumf}}{2\text{ area}}\right)\frac{1}{\sqrt{1-\omega_\lambda^2/\omega^2}}$$

"attenuation const"

with sol$^{\underline{n}}$ $P(z) = P_0\,e^{-2\beta_\lambda^{TM} z}$

Jackson writes $\frac{1}{\delta} = \frac{1}{\delta_\lambda}\sqrt{\omega/\omega_\lambda}$ with $\omega \to \omega_\lambda$ in δ_λ

The TE modes have an extra term in β_λ^{TE}, see Jackson 8.63.

The minimum attenuation for TM modes is at $\omega = \sqrt{3}\,\omega_\lambda$, and grows like $\sqrt{\omega}$ at large ω.

For microwaves in copper, $\beta_\lambda \sim 10^{-4}\omega_\lambda/c$, lose an $O(1)$ fraction of the power in a few hundred meters.