In doing Griffiths problem 9.2, we wrote a standing wave as the sum of a forward and backward traveling wave.

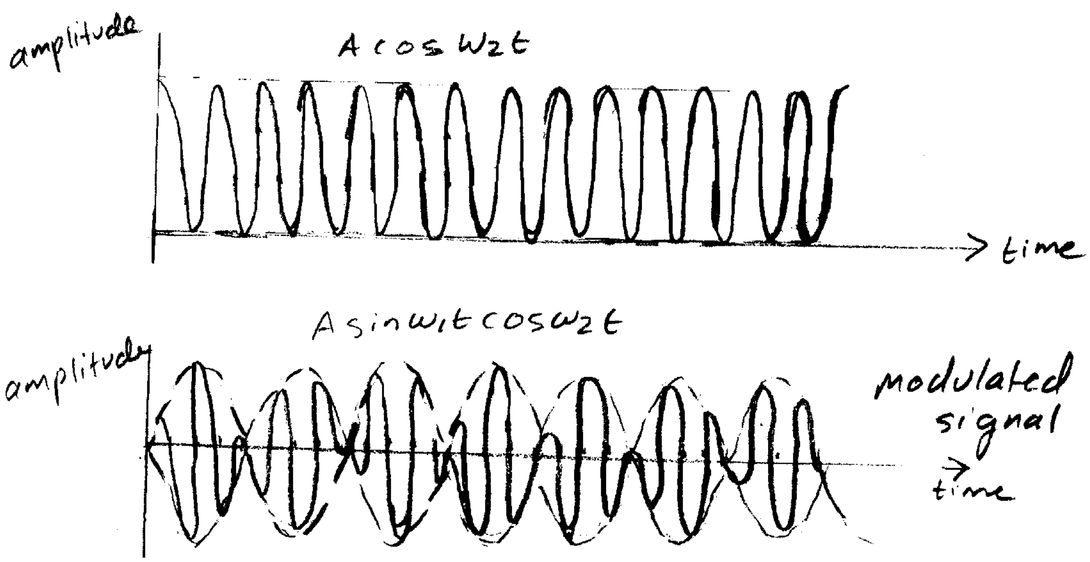
Suppose we began instead with:

$$f = A \sin \omega_1 t \cos \omega_2 t$$

This is:

$$f = \frac{A}{2}\left[\sin(\omega_1+\omega_2)t + \sin(\omega_1-\omega_2)t\right]$$

In the time domain, multiplication of a sinusoid of angular frequency ω_2 by $\sin\omega_1 t$ puts an envelope on $\cos\omega_2 t$.



(not to scale)

In the frequency domain, multiplication (in time) of a sinusoid of angular frequency ω_2 shifts the frequency of the signal to $\omega_1+\omega_2$ and $\omega_1-\omega_2$.

Lets look at this more generally. We can go from a representation of a signal in time to its representation in frequency using a Fourier transform. (We go the other way using an inverse Fourier transform)

$$f(t) \xrightarrow[\text{transform}]{\text{fourier}} F(\omega)$$

The fourier transform of $f(t)$ is given by:

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

If the Fourier transform of $f(t)$ is $F(\omega)$, then the Fourier transform of $f(t)\cos\omega_0 t$ is $\frac{1}{2}[F(\omega+\omega_0)+F(\omega-\omega_0)]$. Showing this:

$$f(t)\cos\omega_0 t = f(t)\left[\frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}\right]$$

$$= \frac{1}{2}\left\{ f(t)e^{i\omega_0 t} + f(t)e^{-i\omega_0 t} \right\}$$

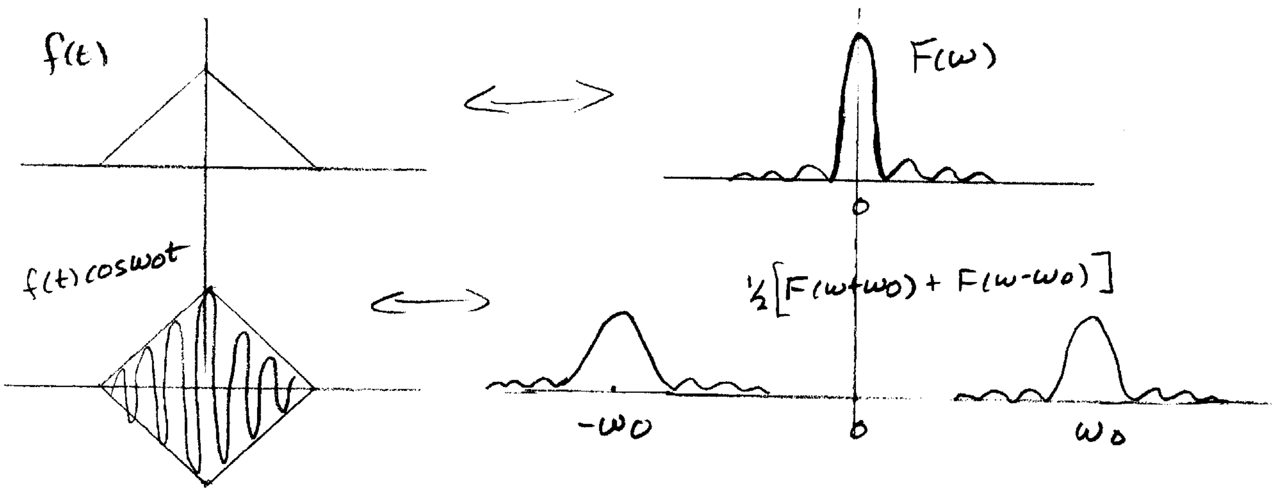
Then;

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}\int_{-\infty}^{\infty} f(t)e^{i\omega_0 t}e^{-i\omega t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} f(t)e^{-i\omega_0 t}e^{-i\omega t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(t)\,e^{-i(\omega-\omega_0)}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} f(t)\,e^{-i(\omega+\omega_0)}\,dt$$

$$= \frac{1}{2}\left[F(\omega-\omega_0) + F(\omega+\omega_0)\right]$$

As shown in your Lathi handout



With modulation, the signal in the frequency domain has the same shape, but is shifted to an upper and lower sideband by the carrier frequency.

Demodulation for detection. Multiply by the carrier again!

$$f_r = [f(t)\cos\omega_0 t]\cos\omega_0 t = f(t)\cos^2\omega_0 t$$

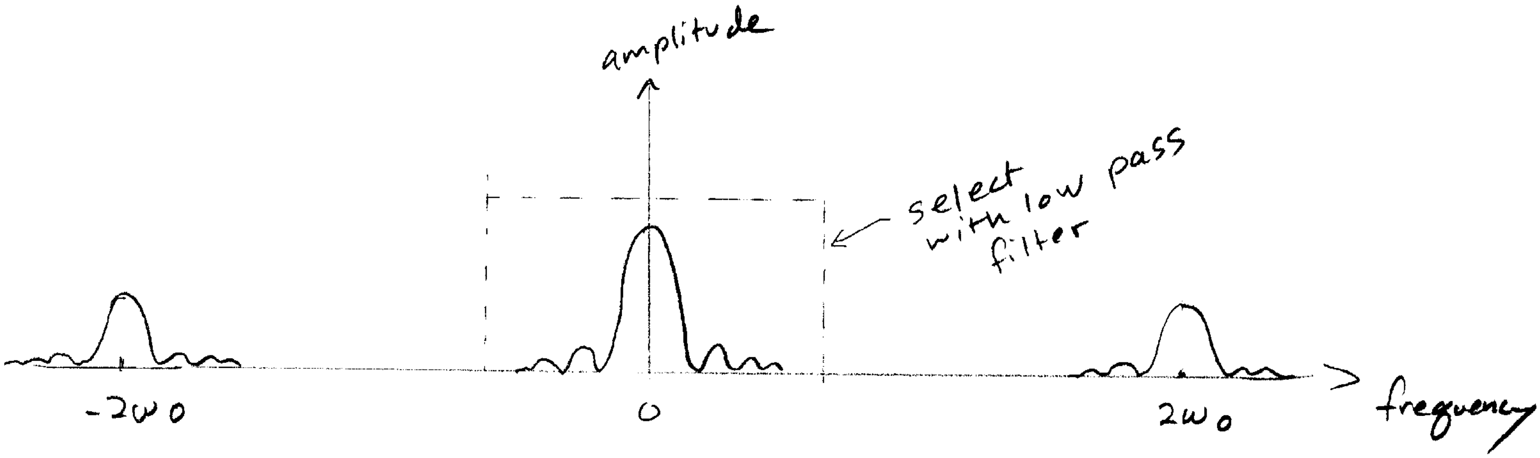
$$= \frac{1}{2} f(t)\,[1 + \cos 2\omega_0 t]$$

$$\mathcal{F}[f_r] = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} + \frac{1}{4}\int_{-\infty}^{\infty} f(t)\, e^{-i(\omega - 2\omega_0)} dt$$

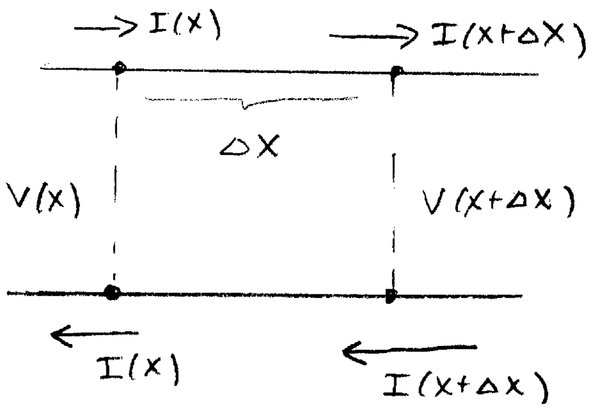
$$+ \frac{1}{4}\int_{-\infty}^{\infty} f(t)\, e^{-i(\omega + 2\omega_0)} dt$$

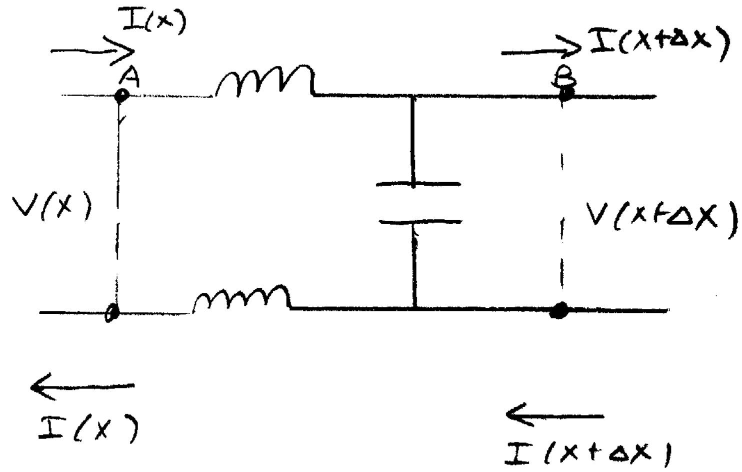
$$= \frac{1}{2} F(\omega) + \frac{1}{4}\left[F(\omega + 2\omega_0) + F(\omega - 2\omega_0)\right]$$

Now we have (see Lathi "Communication Systems" handout):



Guided waves in transmission lines





There is a magnetic field associated with current carrying conductors. Energy resides in this field, so the transmission line has an inductance per length.

$$\Delta V = V(x+\Delta x) - V(x) = -\mathcal{L}\,\Delta x \frac{\partial I}{\partial t}$$

$$\frac{\partial V}{\partial x} = -\mathcal{L}\frac{\partial I}{\partial t}$$

Since there is a change in current going from A to B in the line, there must be a capacitance between the conductors.

The charge in length Δx of the line is:

$$q = \mathcal{C}\,\Delta x\, V$$

$$\Delta I = I(x+\Delta x) - I(x) = \frac{\partial q}{\partial t} = -C\,\Delta x \frac{\partial V}{\partial t}$$

$$\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}$$

Again, we have two first-order coupled equations:

1) $$\frac{\partial V}{\partial x} = -\mathcal{L} \frac{\partial I}{\partial t}$$

2) $$\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}$$

To decouple the equations and obtain wave equations for V and I, take the derivative of 1) with respect to x, the derivative of 2) with respect to t, and combine.

$$\frac{\partial^2 V}{\partial x^2} = -\mathcal{L} \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial t}\right) = -\mathcal{L} \frac{\partial}{\partial t}\left(\frac{\partial I}{\partial x}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial x}\right) = -C \frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 V}{\partial x^2} = \mathcal{L}C \frac{\partial^2 V}{\partial t^2}$$

The characteristic velocity of voltage and current waves down the line is:

$$v = \frac{1}{\sqrt{\mathcal{L}C}}$$

The characteristic impedance of the transmission line, Z_0, is given by the relation between voltage and current, $V = Z_0 I$.

Consider a forward propagating voltage wave, $V_+(x - vt)$, and forward propagating current wave $I_+(x - vt)$. Let $u = x - vt$, and apply the chain rule to the differential equation relating voltage to current:

$$\frac{\partial V_+}{\partial x} = \frac{\partial V_+(u)}{\partial u}\frac{\partial u}{\partial x} = -\mathcal{L}\frac{\partial I_+(u)}{\partial u}\frac{\partial u}{\partial t}$$

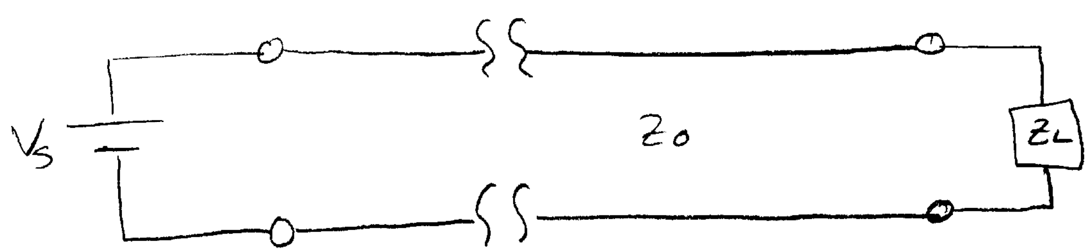
$$\frac{\partial V_+(u)}{\partial u} = -\mathcal{L}(-v)\frac{\partial I_+(u)}{\partial u} = \sqrt{\mathcal{L}/C}\;\frac{\partial I_+(u)}{\partial u}$$

$$\longrightarrow \quad V_+ = \sqrt{\mathcal{L}/C}\; I_+$$

Similarly, $\quad V_- = -\sqrt{\mathcal{L}/C}\; I_-$

Thus, the characteristic impedance is $Z_0 = \sqrt{L/C}$ for the lossless transmission line.

Now, let's attach a load and find the reflection coefficient:



We know that in the line:

$$I_+ = \frac{1}{Z_0} V_+$$

$$I_- = -\frac{1}{Z_0} V_-$$

$$I = I_+ + I_- = \frac{V_+ - V_-}{Z_0}$$

We know that at the load, the total voltage is $V_+ + V_-$ (superposition) and that $I = \frac{V}{Z_L}$ (ohm's law).

Then, matching at the boundary,

$$I = \frac{V_+ + V_-}{Z_L} = \frac{V_+ - V_-}{Z_0}$$

Solving for V_-

$$\frac{V_-}{Z_L} + \frac{V_-}{Z_0} = \frac{V_+}{Z_0} - \frac{V_+}{Z_L}$$

$$V_- = \left(\frac{Z_L Z_0}{Z_L + Z_0}\right)\left(\frac{1}{Z_0} - \frac{1}{Z_L}\right) V_+$$

$$= \frac{Z_L - Z_0}{Z_L + Z_0} V_+$$

The reflection coefficient is:

$$R = \frac{Z_L - Z_0}{Z_L + Z_0}$$ (Notice $R = 0$ when $Z_L = Z_0$)

The transmission coefficient is:

$$T = 1 - R = \frac{2 Z_0}{Z_L + Z_0}$$ (Notice $T = 1$ when $Z_L = Z_0$)

Compare this result to what Griffiths got for a wave on a string tied to another string:

$$\tilde{A}_R = \left(\frac{k_1 - k_2}{k_1 + k_2} \right) \tilde{A}_I \quad , \quad \tilde{A}_T = \left(\frac{2 k_1}{k_1 + k_2} \right) \tilde{A}_I$$

In the case of the strings, if they are the same mass;

$$v_1 = v_2 = \sqrt{T/\mu} \quad , \quad v = \omega/k \implies k_1 = k_2$$

Then all of the incident wave is transmitted at the knot (no reflection).

For the transmission line problem, we found R, T by matching the current at the boundary:

$$I = \frac{V_+ + V_-}{Z_L} = \frac{V_+ - V_-}{Z_0} \quad \left. \right\} \begin{array}{l} \text{1 eq.} \\ \text{1 unknown} \end{array}$$

For the string problem, R & T were found by matching the displacement & the derivative of displacement at the boundary:

$$f(0^-, t) = f(0^+, t) \quad ; \quad \left. \frac{\partial f}{\partial z} \right|_{0^-} = \left. \frac{\partial f}{\partial z} \right|_{0^+} \quad \left. \right\} \begin{array}{l} \text{2 eqs} \\ \text{2 unknowns} \end{array}$$