Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors - Review

As you may already know from last semester and/or a Math methods class, there are special kets (vectors) where an operator gives you the same ket back again.

$$\hat{Q}|v\rangle = \lambda|v\rangle$$

where λ is a number, called the eigenvalue. and $|v\rangle$ is called the eigenvector (or eigenstate).

In some basis, the operator $\hat{Q}$ is represented by a matrix, and the eigenvalues/eigenvectors of that matrix are the eigenvalues/eigenvectors of $\hat{Q}$ in that basis.

Method to find the eigenvalues/eigenvectors of a matrix:

1. Find the eigenvalues from the characteristic equation

$$\det(Q - \lambda \mathbb{1}) = 0$$

determinant

matrix Q

constant λ × identity matrix

2. For each eigenvalue $\lambda_1, \lambda_2, \ldots$ plug it back in to determine the eigenvector $v_1, v_2, \ldots$

$$Qv_1 = \lambda_1 v_1$$
$$Qv_2 = \lambda_2 v_2 \quad \text{etc...}$$

v_1 is a vector w/ unknown components. Find the components using these equations

Note eigenvectors are not uniquely determined. If v_1 is an eigenvector with eigenvalue λ_1, $2v_1$, $3v_1$, $-v_1$ etc... are all eigenvectors too. There is no "right" eigenvector. Any one is ok, though we often normalize them $\langle v_1|v_1\rangle = 1$.

Eigenvalues and Eigenvectors

Griffiths Example A.1 : Find the eigenvalues & eigenvectors of

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$$

① $\det(M - \lambda \mathbb{1}) = 0$

$$\det\left(\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & -2 \\ -2i & i-\lambda & 2i \\ 1 & 0 & -(1+\lambda) \end{vmatrix} = 0$$

$$(2-\lambda)\left[(i-\lambda)(-(1+\lambda)) - 0\right] - 0\left[-2i\cdot(-(1+\lambda)) - 2i\cdot 1\right]$$
$$+(-2)\left[-2i\cdot 0 - (i-\lambda)\right] = 0$$

$$-(2-\lambda)(i-\lambda)(1+\lambda) + 2[i-\lambda] = 0$$
$$(i-\lambda)\left[(\lambda-2)(\lambda+1) + 2\right] = 0$$
$$(i-\lambda)\left[\lambda^2 - \lambda \cancel{-2} \cancel{+2}\right] = 0$$
$$(i-\lambda)\lambda[\lambda-1] = 0 \qquad \text{so} \quad \lambda = \underbrace{i, 0, +1}_{\text{eigenvalues!}}$$

$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = i$

② Get eigenvectors : $M v_3 = \lambda_3 \cdot v_3$

$$\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = i\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

I.) $2a - 2c = ia$

II.) $-2ia + \cancel{i}b + 2ic = \cancel{i}b \Rightarrow -a + c = 0 \Rightarrow \boxed{a = c}$

III.) $a - c = ic \Rightarrow 0 = ic \Rightarrow \boxed{c = 0} \Rightarrow \boxed{a = 0}$

Eigenvalues and Eigenvectors

all 3 equations are satisfied with $a = c = 0$
(b can be anything). So,

$$\lambda_3 = i \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{(chosen for simplicity)}$$

The steps are similar for getting v_2, v_3. The results are:

$$\lambda_1 = 0 \qquad v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \qquad v_2 = \begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix}$$

Now that we know & have reviewed how to calculate eigenvalues and eigenvectors, I want to start talking about the important properties of Hermitian operators. To get us started:

Poll Q: For any operator $\hat{Q}$, suppose $|\alpha\rangle$ is an eigenvector so $\hat{Q}|\alpha\rangle = \lambda|\alpha\rangle$. Which of the following is also true?

A.) $\langle\alpha|\hat{Q}^\dagger = \lambda\langle\alpha|$ B.) $\langle\alpha|\hat{Q}^\dagger = \lambda^*\langle\alpha|$

C.) $\langle\alpha|\hat{Q} = \lambda\langle\alpha|$ D.) $\langle\alpha|\hat{Q} = \lambda^*\langle\alpha|$

Eigenvalues and Eigenvectors

<u>Properties of Hermitian Operators:</u>

#1: Starting with:

$$\hat{Q}|\alpha\rangle = \lambda_1|\alpha\rangle \quad \underline{\text{and}} \quad \langle\alpha|\hat{Q}^\dagger = \lambda_1^*\langle\alpha|$$

multiply by $\langle\alpha|$ multiply by $|\alpha\rangle$

$$\langle\alpha|\hat{Q}|\alpha\rangle = \lambda_1\langle\alpha|\alpha\rangle \quad \underline{\text{and}} \quad \langle\alpha|\hat{Q}^\dagger|\alpha\rangle = \lambda_1^*\langle\alpha|\alpha\rangle$$

For a Hermitian Operator: $\hat{Q} = \hat{Q}^\dagger$ so $\langle\alpha|\hat{Q}|\alpha\rangle = \lambda_1^*\langle\alpha|\alpha\rangle$

$$\lambda_1\cancel{\langle\alpha|\alpha\rangle} = \lambda_1^*\cancel{\langle\alpha|\alpha\rangle}$$

$$\lambda_1 = \lambda_1^* \Rightarrow \lambda_1 \text{ is real}$$

$\therefore$ Hermitian Operators have <u>real</u> eigenvalues.

#2 Let $|\alpha_1\rangle$ and $|\alpha_2\rangle$ be eigenvectors with two different eigenvalues:

$$\hat{Q}|\alpha_1\rangle = \lambda_1|\alpha_1\rangle \qquad \hat{Q}|\alpha_2\rangle = \lambda_2|\alpha_2\rangle$$

real value

$$\langle\alpha_2|\hat{Q}^\dagger = \lambda_2\langle\alpha_2|$$

multiply on left by $\langle\alpha_2|$ multiply on right by $|\alpha_1\rangle$

$$\langle\alpha_2|\hat{Q}|\alpha_1\rangle = \lambda_1\langle\alpha_2|\alpha_1\rangle \qquad \langle\alpha_2|\hat{Q}^\dagger|\alpha_1\rangle = \lambda_2\langle\alpha_2|\alpha_1\rangle$$

For a Hermitian Operator, $\hat{Q} = \hat{Q}^\dagger$ so $\langle\alpha_2|\hat{Q}|\alpha_1\rangle = \lambda_2\langle\alpha_2|\alpha_1\rangle$

$$\lambda_1\langle\alpha_2|\alpha_1\rangle = \lambda_2\langle\alpha_2|\alpha_1\rangle$$

by assumption, $\lambda_1 \neq \lambda_2$. Only possibility is $\langle\alpha_2|\alpha_1\rangle = 0$

The eigenvectors of Hermitian Operators are orthogonal.

Eigenvalues and Eigenvectors of Hermitian Operators

Since physical observables are represented by Hermitian Operators, its important to know about the following properties.

For a Hermitian Operators:

1. All eigenvalues are real.
2. All eigenvectors are orthogonal.
   [subtle point: in case of degenerate eigenvalues, the eigenvectors can always be chosen to be orthogonal].
3. The eigenvectors span the space. Any vector can be expressed as a combination of the eigenvectors.

We proved the first two properties, but not the third. The last property is a little more involved to prove. See Griffiths Sec. A.6.