Electromagnetic radiation in materials

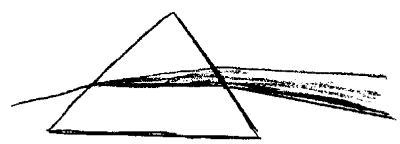
Electromagnetic waves in materials may be subject to

- dispersion
- absorption

Absorbtion is when energy is taken out of the em wave by the material, and so the em wave is attenuated, perhaps even extinguished. If absorption is present, the wave number k will be complex $(\tilde{k})$, the imaginary part of k describing the attenuation of the wave.

Dispersion is when different frequency components of incoming radiation propagate with different velocities through the material. If the index of refraction, n, is a function of frequency $n \to n(\omega)$ the medium it describes is dispersive.

The most well-known example of dispersion is light passing through a prism. Different colors (frequencies) of the light are refracted by different angles at the interfaces of the material (corresponding to the different propagation speeds in the material), thus the light spreads out into a rainbow of colors.

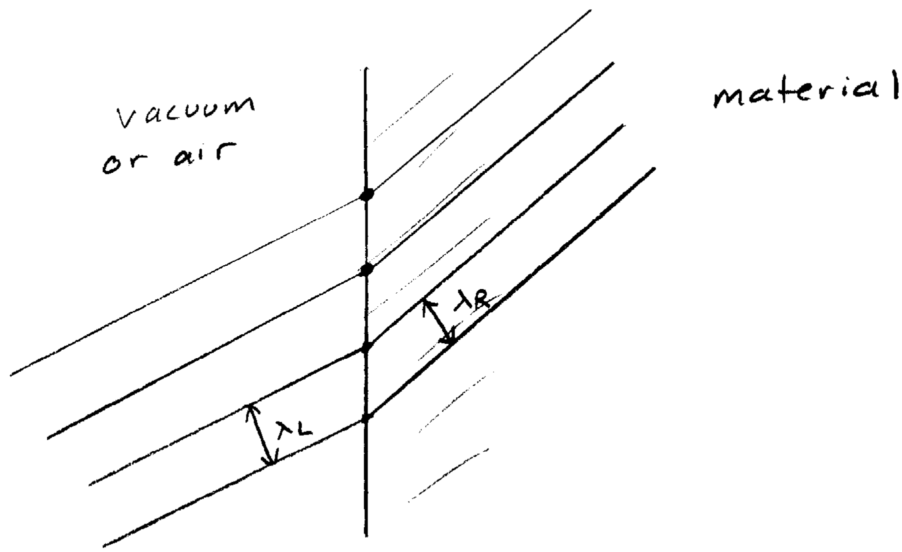


We will investigate dispersion further with some help from our friends.

1. Feynman's model of em radiation at an interface. He gets an expression for $n(\omega)$ by considering how the material modifies an incoming electric field.

2. J.R. Pierce's example of how not to suffer dispersion of an am modulated signal.

Feynman's simple model for index of refraction



The bends

1) Frequency is always the same $\omega_L = \omega_R$
2) $\lambda f = v \longrightarrow \lambda_L = \frac{c}{f}$, $\lambda_R = v/f$

   If $v < c$ the $\lambda_R < \lambda_L$

3) The spacing of the wavefronts must match at the boundary, so the wave is refracted as it enters the material.

The mechanism

When an em wave impinges on the material, it moves the electrons in the material up & down. The moving charges produce em radiation. The transmitted wave is the sum of incident wave and wave from moving charges. The sum is shifted in phase compared to the incident wave. This shift in phase is equivalent to having a different speed of propagation in the material.

So, we can model the transmitted wave by saying the incident wave travels more slowly through the material <u>or</u> by examining the radiation produced by oscillating charges in the material. These end up having similar forms, and can be compared to determine the index of refraction.

① Wave goes slow through material

Δt = additional time to go through plate

$$= \frac{\Delta z}{v} - \frac{\Delta z}{c} = \frac{n\Delta z}{c} - \frac{\Delta z}{c} = (n-1)\frac{\Delta z}{c}$$

$$E_T = E_0 \, e^{i\omega[(t-\Delta t) - z/c]}$$

$$E_T = E_0 e^{-i\omega\Delta t} e^{i\omega[t - z/c]} = e^{-i\omega\Delta t} E_I$$

To compare this result (phase shift) with the sum of two em waves that we'll get from the oscillating charge model, do a small angle approximation on the phase shift term

$$e^{-i\omega\Delta t} \approx 1 - i(\omega\Delta t) = 1 - i\omega(n-1)\frac{\Delta z}{c}$$

Then:

$$E_T = E_I \underbrace{- \frac{i\omega(n-1)\Delta z}{c} E_I}_{\text{contribution from oscillating charges in material}}$$

(2) Transmitted wave is modified by oscillating charges

$$\underbrace{m\left(\frac{d^2x}{dt^2} + \omega_0^2 x\right)}_{\text{atoms are little oscillators}} = \underbrace{q_e \tilde{E}_{0I} e^{i\omega t}}_{\text{driving force}}$$

The driven harmonic oscillator has solution

$$x = \frac{q_e \tilde{E}_{0I}}{m(\omega_0^2 - \omega^2)} e^{i\omega t}$$

(Try solution $x = Ae^{i\omega t}$)

What field is produced by a sheet of charges that all move together?

$$\Delta E = \underbrace{\frac{-\eta q_e}{2\epsilon_0 c}}_{\text{negative constant}} \Big[\underbrace{i\omega \frac{q_e \tilde{E}_{0I}}{m(\omega_0^2 - \omega^2)} e^{i\omega(t - z/c)}}_{\text{velocity at retarded time}} \Big]$$

$\eta = N\Delta z$

$N = \#/\text{volume}$

We'll just use this for now...

Comparing the two results:

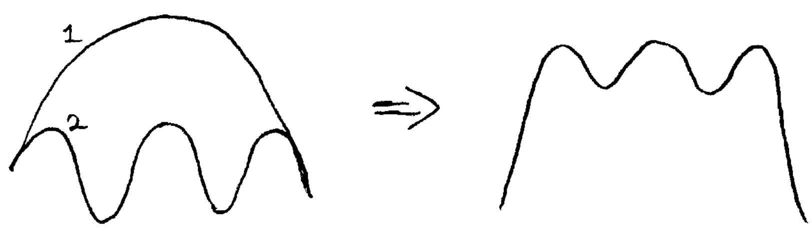
$$n = 1 + \frac{N q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)}$$

Dispersion curves

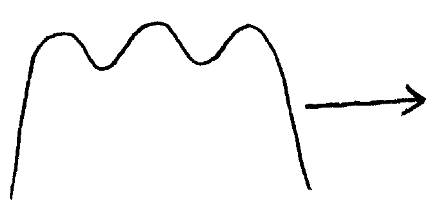
A common way to describe dispersion in a medium is via a dispersion curve.

The phase velocity of a wave of given frequency is $v = \omega / K$

Suppose we make a pulse with 2 different frequency components:



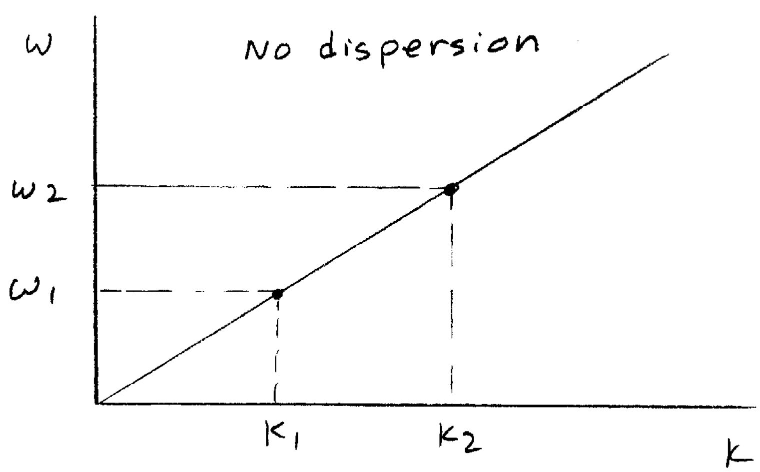
As long as these have the same phase velocity, the two components will travel together and the pulse will retain its shape.



$$v = \frac{\omega_1}{K_1} = \frac{\omega_2}{K_2}$$

Otherwise, the components will slide away and the pulse will disperse.

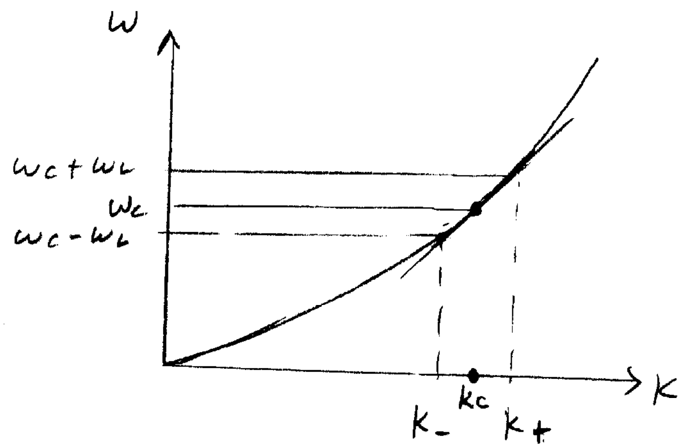
The drawings stink, but hopefully you get the idea. A plot of ω versus k is called a dispersion curve. When there is no dispersion in a medium, the ω/k curve will be a straight line with slope $v = \omega/k$.



$$s = \text{slope} = \omega/k = \omega_1/k_1 = \frac{\omega_2}{k_2}$$

Example from Pierce

When a signal is broadcast using am modulation, how is it that the signal does not disperse or become distorted at the receiving end? One reason is that the modulating (information) signal band is much lower in frequency than the carrier. Then the slope of the dispersion curve is essentially constant over the entire range of the signal.



$\omega_L = \Delta\omega$, $\quad S = \dfrac{\Delta\omega}{\Delta k} = \text{slope}$

Consider a single low frequency component of the information signal. After modulation

$$S = \cos(\omega_L t)\cos(\omega_c t)$$

$$= \tfrac{1}{2}\left[\cos(\omega_c+\omega_L)t + \cos(\omega_c-\omega_L)t\right]$$

k_c is the wave number corresponding to ω_c

The wave number corresponding to $\omega_c + \omega_L$ is k_+ where

$$K_+ = k_c + \omega_L/s \quad , \quad s = \text{slope of dispersion curve at } k_c$$

$$K_- = k_c - \omega_L/s = k_c - \frac{\Delta\omega}{\Delta\omega/\Delta k} = k_c - \Delta k$$

After the two frequèncy components have traveled a distance z, the signal S given on the previous page at $z = 0$ becomes:

$$S = \tfrac{1}{2}\Big[\cos[(\omega_c + \omega_L)t - k_+ z] + \cos[(\omega_c - \omega_L)t - k_- z]\Big]$$

$$= \tfrac{1}{2}\Big[\cos[(\omega_c t - k_c z) + (\omega_L t - \omega_L/s\, z)]$$

$$+ \cos[(\omega_c t - k_c z) - (\omega_L t - \omega_L/s\, z)]\Big]$$

$$= \cos[\omega_c t - k_c z] \cos[\omega_L t - \omega_L/s\, z]$$

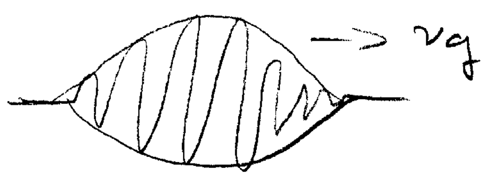
The high frequency carrier wave has phase velocity $v = \omega_c/k_c$. The modulating wave has velocity

$$v_g = \omega_L / \frac{\omega_L}{s} = s$$

slope of tangent line on dispersion curve at k_c

All the frequency components of the modulating function have velocity v_g, as long as the slope of the dispersion curve doesn't change significantly over the frequency range of the modulating function.

Suppose we modulate a high frequency carrier with a low frequency pulse. The <u>energy</u> of the modulated signal is in the pulse.



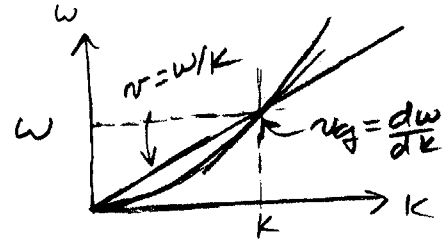
Then v_g is the velocity at which the energy is flowing. v_g is called the <u>group velocity</u>.

$$v_g \equiv \frac{d\omega}{dK}$$

Group velocity is the slope of a tangent line to the dispersion curve

$$v = \omega / K$$

Phase velocity is the slope from (K, ω) to the origin.



Electromagnetic waves in conductors

In a conductor there may be unbound charges which flow when an electric field is present $\vec{J}_f = \sigma \vec{E}$. Maxwell's equations are then written:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \text{(no \underline{static} free charge, goes to surface)}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu \underbrace{\sigma \vec{E}}_{\vec{J}_f}$$

The presence of this term modifies the wave equations:

$$\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \sigma \frac{\partial \vec{E}}{\partial t}$$

$$-\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \sigma \frac{\partial \vec{E}}{\partial t}$$

$$\nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \sigma \frac{\partial \vec{E}}{\partial t}$$

similarly,

$$\vec{\nabla} \times \frac{\partial \vec{E}}{\partial t} = -\frac{\partial^2 B}{\partial t^2}$$

$$\vec{\nabla} \times \left[\left(\frac{1}{\mu\epsilon} \vec{\nabla} \times \vec{B} \right) - \frac{\sigma}{\epsilon} \vec{E} \right] = -\frac{\partial^2 B}{\partial t^2}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \mu\sigma \left(\vec{\nabla} \times \vec{E} \right) - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$-\nabla^2 \vec{B} = -\mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu\epsilon \frac{\partial^2 B}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

Note that whenever dealing with em <u>waves</u>

$$\nabla \rightarrow ik$$

$$\nabla^2 \rightarrow (ik)^2 = -k^2$$

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$

$$\frac{\partial^2}{\partial t^2} \rightarrow -\omega^2$$

Let's check that $\vec{\nabla} \to i\vec{k}$:

$$\vec{\nabla}\left(\tilde{A}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\tilde{A}e^{i(k_x x + k_y y + k_z z - \omega t)}\right)$$

symbolize $\tilde{A}e^{i(k_x x + k_y y + k_z z - \omega t)}$ as $\tilde{A}e^{i\theta(r,t)}$

$$\vec{\nabla}(\tilde{A}e^{i\theta}) =$$

$$\hat{x}\,\tilde{A}e^{i\theta}\frac{\partial\theta(r,t)}{\partial x} + \hat{y}\,\tilde{A}e^{i\theta}\frac{\partial\theta(r,t)}{\partial y} + \hat{z}\,\hat{A}e^{i\theta}\frac{\partial\theta(r,t)}{\partial z}$$

$$= \tilde{A}e^{i\theta}\left[\hat{x}(ik_x) + \hat{y}\,i(k_y) + \hat{z}(ik_z)\right]$$

$$= i(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z)\tilde{A}e^{i\theta}$$

$$= i\vec{k}\tilde{A}e^{i\theta} = (i\vec{k})\left(\tilde{A}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)$$

Now, lets check $\nabla^2 \to -k^2$

$$\vec{\nabla}\left[\vec{\nabla}\left(\tilde{A}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)\right] = \vec{\nabla}\cdot\left[i\vec{k}\left(\tilde{A}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)\right]$$

$$= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(i(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z)\left(\tilde{A}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)\right)$$

This is

$$\frac{\partial}{\partial x}\left(ik_x \tilde{A} e^{i\theta}\right) + \frac{\partial}{\partial y}\left(ik_y A e^{i\theta}\right) + \frac{\partial}{\partial z}\left(ik_z A e^{i\theta}\right)$$

$$= ik_x \tilde{A} e^{i\theta} \frac{\partial \theta}{\partial x} + ik_y \tilde{A} e^{i\theta} \frac{\partial \theta}{\partial y} + ik_z \tilde{A} e^{i\theta} \frac{\partial \theta}{\partial z}$$

$$= \tilde{A} e^{i\theta} \left\{ ik_x (ik_x) + ik_y (ik_y) + ik_z (ik_z) \right\}$$

$$= \left(-k_x^2 - k_y^2 - k_z^2\right) \tilde{A} e^{i\theta}$$

$$= -k^2 \tilde{A} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\Rightarrow \quad \nabla^2 \to -k^2$$

Now, use these replacements in the wave equation:

$$\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

From the wave equation:

$$(i\tilde{k})^2 B = \mu\epsilon(-\omega^2)B + \mu\sigma(-i\omega)B$$

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\omega\sigma\mu$$

$\tilde{k}$ in general is complex, and we see that in this case $\tilde{k}$ must be complex since there is a dissipation term in the wave equation.

$$\tilde{k} = [\mu\epsilon\omega^2 + i\omega\sigma\mu]^{1/2}$$

Put $\mu\epsilon\omega^2 + i\omega\sigma\mu$ in polar form to facilitate taking the root.

$$a + ib \;\to\; (a^2+b^2)^{1/2}\, e^{i\tan^{-1} b/a}$$

$$\tilde{k} = [(\mu\epsilon\omega^2)^2 + (\omega\sigma\mu)^2]^{1/4}\, e^{i/2\,\tan^{-1}\left(\frac{\omega\sigma\mu}{\mu\epsilon\omega^2}\right)}$$

Now go back to cartesian form

Since $\theta = \tan^{-1}\left(\frac{\omega\sigma\mu}{\mu\epsilon\omega^2}\right) = \tan^{-1}\left(\frac{\sigma}{\epsilon\omega}\right)$

$$\sin\theta = \frac{\sigma}{(\sigma^2 + (\epsilon\omega)^2)^{1/2}} \quad , \quad \cos\theta = \frac{\epsilon\omega}{(\sigma^2 + (\epsilon\omega)^2)^{1/2}}$$

$$e^{i\theta/2} = \cos(\theta/2) + i\sin(\theta/2)$$

$$\sin(\theta/2) = \frac{1}{\sqrt{2}}(1 - \cos\theta)^{1/2} = \frac{1}{\sqrt{2}}\left(1 - \frac{\epsilon\omega}{(\sigma^2 + (\epsilon\omega)^2)^{1/2}}\right)^{1/2}$$

$$\cos(\theta/2) = \left[1 - \sin^2\theta/2\right]^{1/2}$$

$$= \left[1 - \tfrac{1}{2}(1 - \cos\theta)\right]^{1/2}$$

$$= \frac{1}{\sqrt{2}}\left[1 + \cos\theta\right]^{1/2} = \frac{1}{\sqrt{2}}\left(1 + \frac{\epsilon\omega}{(\sigma^2 + (\epsilon\omega)^2)^{1/2}}\right)^{1/2}$$

Then:

$$\mathrm{Re}(\tilde{k}) = \sqrt{\mu\omega}\left[(\epsilon\omega)^2 + \sigma^2\right]^{1/4} \frac{1}{\sqrt{2}}\left(1 + \frac{\epsilon\omega}{(\sigma^2 + (\epsilon\omega)^2)^{1/2}}\right)^{1/2}$$

$$= \sqrt{\frac{\mu\omega}{2}}\left\{\left((\epsilon\omega)^2 + \sigma^2\right)^{1/2} + \epsilon\omega\right\}^{1/2}$$

$$= \omega\sqrt{\frac{\mu\epsilon}{2}}\left\{\sqrt{1 + \left(\sigma/\epsilon\omega\right)^2} + 1\right\}^{1/2}$$

$$\operatorname{Im}(\tilde{k}) = \omega \sqrt{\frac{\mu\epsilon}{2}} \left\{ \sqrt{1+(\sigma/\epsilon\omega)^2} - 1 \right\}^{1/2}$$

<u>If we have</u>

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)} = \tilde{E}_0 \underbrace{e^{-\operatorname{Im}(\tilde{k})z}}_{\text{attenuation of wave}} e^{i(\operatorname{Re}(\tilde{k})z-\omega t)}$$

Remember the RLC circuit?

$$i(t) = \tilde{A} \; e^{-\frac{R}{2L}t} \cos\left[\left\{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right\}^{1/2} t\right]$$

The greater the magnitude of $\operatorname{Im}(\tilde{k})$ the faster an em wave attenuates in the conductor. The <u>skin depth</u> is defined as the distance it takes to reduce the wave amplitude by a factor of $1/e \approx .37$.

$$d \equiv \frac{1}{\operatorname{Im}(\tilde{k})} \equiv \text{skin depth}$$

So, it takes energy to get the free charges in a conductor to flow in currents, and that energy can be taken from em radiation interacting with the material. The energy is absorbed from the radiation, and the em waves attenuate.

But, even when there is no free charge (insulator) the material can absorb energy from em radiation via motion of the bound charges. We now know that in this circumstance k will be complex, and also $n(\omega)$ and $\epsilon(\omega)$ will also then be complex. We can cover this situation by writing the wave equation;

$$\nabla^2 \tilde{E} = \tilde{\epsilon}\mu_0 \frac{\partial^2 \tilde{E}}{\partial t^2}, \quad \text{where } \tilde{E} = \tilde{E}_0 e^{i(\tilde{k}z - \omega t)}$$

giving the required attenuated solution

$$\tilde{E} = \tilde{E}_0 e^{-Im(\tilde{k})z} e^{i(Re(\tilde{k})z - \omega t)}$$

Now, we need to find $\tilde{\epsilon}$, and hence $\tilde{k}$ since $\tilde{k} \equiv \sqrt{\tilde{\epsilon}\mu_0}\,\omega$.

Way back in chapter 4 we learned that ϵ, the permitivity, describes how an applied electric field interacts with the charges embedded in the material In a non-conducting material, ϵ is a measure of how strongly bound charges couple to externally applied fields. A material with large permittivity becomes more polarized in the presence of an external field than a material with lower permitivity.

The dipole moment of an individual dipole is given by:

$$\tilde{p} = q\tilde{x}$$

separation of charges

charge

In the case of materials with linear response, the polarization (dipole moment per unit volume) is proportional to $\tilde{E}$:

$$\tilde{P} = \epsilon_0 \tilde{\chi}_e \tilde{E}$$

where $\tilde{\epsilon} = \epsilon_0 (1 + \tilde{\chi}_e) = \epsilon_0 \tilde{\epsilon}_r$

Also, $\tilde{P} = Nf\,\tilde{p}$

Polarization

dipole moment of one dipole

$N = \frac{\#\ \text{molecules}}{\text{volume}}$, f = fraction of N electrons that actually get polarized

$$\tilde{P} = Nf\,q\,\tilde{x} = \epsilon_0\,\tilde{\chi}_e\,\tilde{E}$$

We have a way of obtaining $\tilde{\chi}_e$ (and thus $\tilde{\epsilon}$), we just need $\tilde{x}$. This individual displacement can be found by modeling the motion of the charges as harmonic motion <u>with damping</u> this time. Damping must be included in the model in order to have a mechanism for energy absorption.

$$F = ma = \text{driving force}$$

$$m(\ddot{x} + \gamma\dot{x} + \omega_0^2 x) = q\tilde{E}$$

$\tilde{x} = \tilde{x}_0 e^{i\omega t}$

Damping term (1st derivative)

$\tilde{E} = \tilde{E}_0 e^{i\omega t}$
applied field written as complex field

Solving for $\tilde{x}$;

$$m(-\omega^2 - i\gamma\omega + \omega_0^2)\tilde{x} = q\tilde{E}$$

$$\tilde{x} = \frac{q/m}{(\omega_0^2 - \omega^2) - i\gamma\omega}\tilde{E}$$

Then $$\tilde{p} = \frac{Nq^2}{m}\frac{f}{(\omega_0^2 - \omega^2) - i\gamma\omega}\tilde{E}$$

If there is more than one mode of oscillation, we generalize this as:

$$\tilde{p} = \frac{Nq^2}{m}\sum_k \frac{f_k}{(\omega_k^2 - \omega^2) - i\gamma_k\omega}\tilde{E}$$

sum over modes

<u>Now</u> we can identify $\tilde{\chi}_e$

$$\chi_e = \frac{Nq^2}{m\epsilon_0}\sum_k \frac{f_k}{(\omega_k^2 - \omega^2) - i\gamma_k\omega}$$

$$\tilde{\epsilon}_r = 1 + \tilde{\chi}_e = 1 + \frac{Nq^2}{m\epsilon_0}\sum_k \frac{f_k}{(\omega_k^2 - \omega^2) - i\gamma_k\omega}$$

$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\ \omega = \sqrt{\mu_0 \epsilon_0 \tilde{\epsilon}_r}\ \omega = \omega/c \sqrt{\tilde{\epsilon}_r}$$

$$n = \frac{c}{\omega} Re(\tilde{k})$$

$$\tilde{E} = \tilde{E}_0\, e^{-Im(\tilde{k})z}\ e^{i(Re(\tilde{k})z - \omega t)}$$

The <u>intensity</u> of the wave goes as E^2, so the attenuation of the intesity, or energy, goes as $\alpha \equiv 2\, Im(\tilde{k})$

We must find the real and imaginary part of $\tilde{k}$ ($\sqrt{\tilde{\epsilon}_r}$) to calculate any quantitative result. For the special case when $\chi_e \ll 1$ we can make the approximation:

$$\sqrt{\tilde{\epsilon}_r} = \sqrt{1 + \tilde{\chi}_e} = (1 + \tilde{\chi}_e)^{1/2} \approx 1 + \tfrac{1}{2}\tilde{\chi}_e$$

Then

$$\left(\tfrac{c}{\omega}\right) Re(\tilde{k}) = (c/\omega)(\omega/c)\left(1 + \frac{Nq^2}{2m\epsilon_0} \sum_k Re\left\{\frac{f_k}{(\omega_k^2 - \omega^2) - i\gamma_k \omega}\right\}\right.$$

In general, if

$$\tilde{d} = \frac{1}{a - ib}$$

$$Re(\tilde{d}) = Re\left\{\frac{1}{a-ib} \; \frac{a+ib}{a+ib}\right\} = Re\left\{\frac{a+ib}{a^2+b^2}\right\}$$

$$Re(\tilde{d}) = \frac{a}{a^2+b^2} \quad , \quad Im(\tilde{d}) = \frac{b}{a^2+b^2}$$

So,

$$n = 1 + \frac{Nq^2}{2m\epsilon_0} \sum_K \frac{f_K(\omega_K^2 - \omega^2)}{(\omega_K^2 - \omega_0^2)^2 + \gamma_K^2 \omega^2}$$

$$\alpha = \frac{Nq^2\omega^2}{m\epsilon_0 c} \sum_K \frac{f_K \gamma_K}{(\omega_K^2 - \omega_0^2)^2 + \gamma_K^2 \omega^2}$$