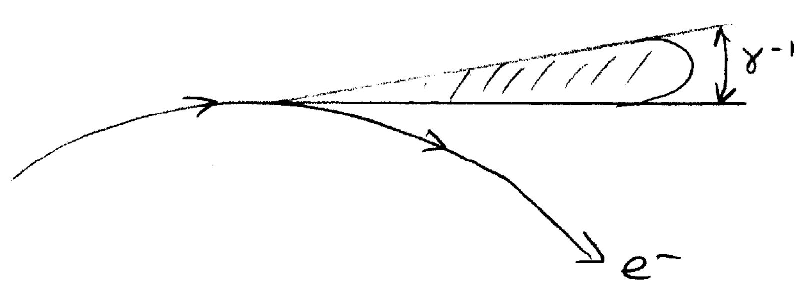
Chapter 11 - lecture 3

Now on the table for discussion is the power radiated by a point charge. We have seen that accelerating charges produce radiation, extremely important, for this allows for the transmission of signals, such as radio, cell phone, etc. Also, this enables us to know about our universe, as we rely on radiation from stars and the space around us to learn about it, and our relation to it. An important research tool is the synchrotron light source. A form of particle acceleration is to bend electron trajectories, causing photon emission (radiation). For highly relativistic electrons the radiation produced is sharply peaked near the instantaneous direction of forward motion. This radiation is used for research in many disciplines.



First, consider the case of a non-relativistic particle ($v \ll c$). To find the total power radiated, calculate the Poynting vector:

$$P = \oint \vec{S}_{rad} \cdot d\vec{a}$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \vec{E} \times \left(\frac{1}{c} \hat{r} \times \vec{E}\right)$$

using the BAC-CAB rule,

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

$$\vec{E} \times \hat{r} \times \vec{E} = \hat{r}(\vec{E} \cdot \vec{E}) - \cancel{\vec{E}(\vec{E} \cdot \hat{r})}$$

0, radiated $\vec{E}$ is $\perp$ to $\hat{r}$

$$\vec{S}_{rad} = \frac{1}{\mu_0 c} E^2 \hat{r}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{r}{(\vec{r} \cdot \vec{u})^3} \left[(c^2 - v^2)\vec{u} + \vec{r} \times (\vec{u} \times \vec{a})\right]$$

This term falls off too rapidly to produce radiation

$$\vec{r} \times \vec{u} \times \vec{a} = \vec{u}(\vec{r}\cdot\vec{a}) - \vec{a}(\vec{r}\cdot\vec{u})$$

Non-relativistic approximation:

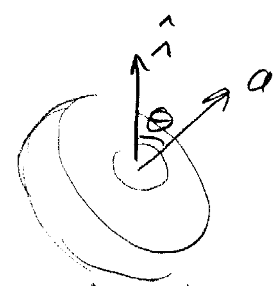
$$\vec{u} = c\hat{r} - \vec{v} \approx c\hat{r}$$

$$\vec{S}_{rad} = \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \left(\frac{r}{c^3 r^3}\right)^2 \left[c\hat{r}(r\hat{r}\cdot\vec{a}) - \vec{a}(cr)\right]^2 \hat{r}$$

$$= \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c^2 r}\right)^2 \left[a^2 - (\hat{r}\cdot\vec{a})^2\right]\hat{r}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c r^2}\left[1 - \cos^2\theta\right]\hat{r}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$$



donut

$$P = \oint \vec{S}_{rad}\cdot d\vec{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta \, d\theta$$

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}}$$

Larmor formula

Relativistic charges are power houses in comparison. The radiation is concentrated in a much tighter angular distribution. We will consider two cases, acceleration parallel to the motion and acceleration perpendicular to the motion. Both cases exist in manmade particle accelerators; linear accelerators (linacs) accelerate parallel to the motion, circular accelerators produce a centripetel force on the particles to keep them in a fixed orbit, and so have an acceleration perpendicular to the motion. For the general relativistic case;

1. The velocity $\vec{v}$ in $\vec{u}$ cannot be neglected.

   $\vec{u} = c\hat{r} - \vec{v}$

2. There must be an adjustment to the energy per time. The rate of energy flow at time t_r (what observer sees) can be related to the power radiated by the charge at t:

$$\frac{dW}{dt_r} = \frac{\partial t}{\partial t_r}\frac{\partial W}{\partial t} = \frac{\vec{r}\cdot\vec{u}}{rc}\frac{dW}{dt}$$

(See problem 10,17 for $\partial t / \partial t_r$ calculation)

Then,

$$\vec{S} = \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \left[\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\right]^2 \left[\frac{\hat{\mathscr{r}}\cdot\vec{u}}{\mathscr{r}c}\right] \left[\vec{\mathscr{r}}\times\vec{u}\times\vec{a}\right]^2 \hat{\mathscr{r}}$$

$$\mathscr{r}^2 S = \frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0}\,\frac{\mathscr{r}^3}{(\vec{\mathscr{r}}\cdot\vec{u})^5}\left[\vec{\mathscr{r}}\times\vec{u}\times\vec{a}\right]^2$$

First, consider the case where $\vec{a} \parallel \vec{v}$

$$\vec{\mathscr{r}}\times\vec{u}\times\vec{a} = \vec{\mathscr{r}}\times\left[(c\hat{\mathscr{r}}\times\vec{a}) - \cancelto{0}{(\vec{v}\times\vec{a})}\right]$$

$$= \mathscr{r}\hat{\mathscr{r}}\times c\hat{\mathscr{r}}\times\vec{a} = c\mathscr{r}\left[\hat{\mathscr{r}}\times\hat{\mathscr{r}}\times\vec{a}\right]$$

$$= c\mathscr{r}\left[\hat{\mathscr{r}}(\hat{\mathscr{r}}\cdot\vec{a}) - \vec{a}(\hat{\mathscr{r}}\cdot\hat{\mathscr{r}})\right] \qquad \text{(BAC-CAB rule)}$$

$$= c\mathscr{r}\left[\hat{\mathscr{r}}(\hat{\mathscr{r}}\cdot\vec{a}) - \vec{a}\right]$$

Then,

$$\left[\vec{\mathscr{r}}\times\vec{u}\times\vec{a}\right]^2 = (c\mathscr{r})^2\left[a^2 + (\hat{\mathscr{r}}\cdot\vec{a})^2 - 2(\vec{a}\cdot\hat{\mathscr{r}})(\hat{\mathscr{r}}\cdot\vec{a})\right]$$

$$= (c\mathscr{r})^2\left[a^2 - (\hat{\mathscr{r}}\cdot\vec{a})^2\right]$$

$$(\vec{r}\cdot\vec{u}) = \vec{r}\cdot(c\hat{r}-\vec{v}) = cr - \vec{r}\cdot\vec{v}$$

Let the motion $\vec{v}$ be along the z-axis, so that $\vec{r}\cdot\vec{v} = rv\cos\theta$, the angle between $\vec{r}$ and $\vec{v}$ is θ.

$$\vec{r}\cdot\vec{u} = cr(1 - v/c\cos\theta)$$

Reminder: $\beta \equiv v/c$, $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$

Then:

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0}\,\frac{c^2\cancel{r^5}\,a^2}{c^5\cancel{r^5}(1-\beta\cos\theta)^5}(1-\cos^2\theta)$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\,\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

Now we can find:

1. The total power, $P = \int \frac{dP}{d\Omega}\,d\Omega$

see 11.15

2. The maximum of the angular distribution of the radiation, θ_{max}
3. The opening angle $2\theta_{max}$.

Total power:

$$P = \frac{\mu_0 q^2 a^2}{16\pi^2 c} (2\pi) \int_0^{\pi} \frac{-\sin^2\theta \, d(\cos\theta)}{(1-\beta\cos\theta)^5}$$

let $\cos\theta = u$

$$-\int_1^{-1} \frac{(1-x^2)\,dx}{(1-\beta x)^5} \qquad \text{Integrate by parts}$$

Let $u = (1-x^2)$, $v = \frac{1}{4\beta(1-\beta x)^4}$

Then,

$$\int_{-1}^{1} \frac{(1-x^2)\,dx}{(1-\beta x)^5} = \frac{1}{4\beta} \cancelto{0}{\frac{(1-x^2)}{(1-\beta x)^4}} \Bigg|_{-1}^{1} - \frac{1}{4\beta} \int_{-1}^{1} \frac{-2x\,dx}{(1-\beta x)^4}$$

Integrate by parts again:

Let $u = x$, $v = \frac{1}{3\beta} \frac{1}{(1-\beta x)^3}$

Then,

$$\frac{1}{2\beta}\int_{-1}^{1}\frac{x\,dx}{(1-\beta x)^4} = \frac{1}{6\beta^2}\left\{ \left.\frac{x}{(1-\beta x)^3}\right|_{-1}^{1} - \int_{-1}^{1}\frac{dx}{(1-\beta x)^3} \right\}$$

To evaluate the integral, let $u = 1-\beta x$

$$\frac{1}{6\beta^2}\left\{ \frac{1}{(1-\beta)^3} + \frac{1}{(1+\beta)^3} + \frac{1}{\beta}\int_{1+\beta}^{1-\beta}\frac{du}{u} \right\}$$

$$= \frac{1}{6\beta^2}\left\{ \frac{2+6\beta^2}{(1-\beta^2)^3} - \frac{1}{2\beta}\left(\frac{1}{(1-\beta)^2} - \frac{1}{(1+\beta)^2}\right)\right\}$$

$$= \frac{1}{6\beta^2}\left\{ \frac{2+6\beta^2}{(1-\beta^2)^3} - \frac{1}{2\beta}\,\frac{4\beta(1-\beta^2)}{(1-\beta^2)^3}\right\}$$

$$= \frac{4}{3}\,\frac{1}{(1-\beta^2)^3} = \frac{4}{3}\gamma^6$$

The total power is $P = \dfrac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}$

The relativistic momentum of a particle is given by $p = \gamma m v$ (we'll discuss this more later). Since $v \parallel a$, the particle is accelerating.

$$\frac{dp_{\parallel}}{dt} = \frac{d(\gamma m v)}{dt} = \gamma m \dot{v} + m v \frac{d}{dt}\left(1 - v^2/c^2\right)^{-1/2}$$

$$= \gamma m \dot{v} + \frac{m v \left(-\tfrac{1}{2}\right)}{\left(1 - v^2/c^2\right)^{3/2}} \left(\frac{-2v}{c^2}\right) \dot{v}$$

$$= \gamma m \dot{v} \left(1 + \frac{\beta^2}{1 - \beta^2}\right) = \gamma m \dot{v} \left(\frac{1 - \cancel{\beta^2} + \cancel{\beta^2}}{1 - \beta^2}\right)$$

$$= \gamma^3 m \dot{v}$$

The power may be written:

$$P = \frac{\mu_0 q^2 \dot{v}^2 \gamma^6}{6 \pi c} = \frac{\mu_0 q^2}{6 \pi c m^2} \left(\frac{dp_{\parallel}}{dt}\right)^2$$

As per S.Y. Lee "Accelerator Physics" :

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For the SLAC Linac, $\frac{\Delta E}{\Delta s} \approx 20 \frac{MeV}{m}$

Note: $\Delta E = F \cdot \Delta S \rightarrow \frac{\Delta E}{\Delta S} = \frac{dp_{\parallel}}{dt}$

$$P = \frac{e^2 \mu_0}{6\pi c m^2} \left(20 \frac{MeV}{m}\right)^2$$

Wheres for circular accelerators (see problem 11.16) the power is given by

$$P = \frac{e^2 \mu_0 \gamma^2}{6\pi c m^2} \left(\frac{dp_{\perp}}{dt}\right)^2$$

$$\boxed{\begin{aligned} F_c = F_B &= \frac{dp_{\perp}}{dt} \\ \frac{\gamma m (\beta c)^2}{\rho} &= q(\beta c) B \end{aligned}}$$

$$\frac{dp_{\perp}}{dt} = \gamma m \dot{v} = \gamma m \frac{\beta^2 c^2}{\rho} = 300 \beta B[T] \left(\frac{MeV}{m}\right)$$

Radiation from circular acceleration is about γ^2 larger than from linear acceleration.