Waveguides

In the discussion of electromagnetic waves propagating through free space, the waves were taken to be plane waves. Plane waves traveling in the $\hat{z}$ direction have no x or y dependence, i.e.

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}$$

$$\tilde{B}(z,t) = \tilde{B}_0 e^{i(kz-\omega t)}$$

In other words, upon examination of a cross-section to the propagation direction (a plane parallel to the xy plane), at one instant in time, $\tilde{E}$ + $\vec{B}$ have the same magnitude everywhere in that plane.

The fundamental difference of guided waves is that they are no longer constant in magnitude at a given cross-section to the propagation direction at an instant in time. This difference is due to boundary conditions that must be met at the walls of the waveguide.

Now,
$$\tilde{E}(x,y,z,t) = \tilde{E}_0(x,y)\, e^{i(kz-\omega t)}$$
$$\tilde{B}(x,y,z,t) = \tilde{B}_0(x,y)\, e^{i(kz-\omega t)}$$

$$\tilde{E}_0 = E_{x0}(x,y)\,\hat{x} + E_{y0}(x,y)\,\hat{y} + E_z(x,y)\,\hat{z}$$

$$\tilde{B}_0 = B_{x0}(x,y)\,\hat{x} + B_{y0}(x,y)\,\hat{y} + B_z(x,y)\,\hat{z}$$

Note that $\vec{E}$ + $\vec{B}$ need no longer be solely transverse to the direction of propagation. For plane waves this was true because the only surviving term in Maxwell's equations $\vec{\nabla}\cdot\vec{E} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$ were $\frac{\partial \hat{B}_{0z} e^{i(kz-\omega t)}}{\partial z} = 0$ and $\frac{\partial \hat{E}_{0z} e^{i(kz-\omega t)}}{\partial z} = 0$ forcing $\tilde{E}_{0z} = \tilde{B}_{0z} = 0$

Not only is it possible to have non-zero B_z + E_z , it turns out that one of these must be non-zero to meet boundary conditions.

Griffiths supports this with the following argument:

① Suppose both E_z & B_z are zero.
Then $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \times \vec{E} = 0$
which means $\vec{E} = -\vec{\nabla} V$, $\nabla^2 V = 0$

② The surface of the waveguide must be an equipotential, and cannot be a minimum or maximum (Laplace says!). So, the potential is constant throughout the waveguide. —> Electric field is zero everywhere.

The upshot of all this is specific solutions for the fields must be found by applying boundary conditions. Not only that, since we need to solve for each component of the field, it is more involved to derive the needed equations. Since we are assuming a wave that propagates in the $\hat{z}$ direction, we expect to be able to derive <u>wave equations</u> for E_z & B_z. The other components of $\vec{E}$ & $\vec{B}$ are related through Maxwell's equations.

After working with Maxwell's equations we have uncoupled equations for E_z & B_z:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z = 0$$

Application to TE waves in a rectangular wave guide. TE waves $\rightarrow$ $E_z = 0$, so we work with the wave equation for B_z.

1. Solve for $B_z(x,y)$ using separation of variables, get the general solution

2. Apply the boundary conditions

$$B_x(x=0,y) = B_x(x=a,y) = 0$$

$$B_y(x,y=0) = B_y(x,y=b) = 0$$

($B_\perp$ is continuous)

to determine the arbitrary constants

① (In more detail)

$$B_z(x,y) = X(x)\,Y(y)$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{\substack{-k_x^2 \\ \text{constant}}} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{\substack{-k_y^2 \\ \text{constant}}} + \underbrace{\left[(\omega/c)^2 - k^2\right]}_{-k_x^2 - k_y^2 + (\omega/c)^2 - k^2 = 0} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad \rightarrow \quad \frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

$$X = A \sin(k_x x) + B \cos(k_x x)$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad \rightarrow \quad \frac{\partial^2 Y}{\partial y^2} + k_y^2 Y = 0$$

$$Y = C \sin(k_y y) + D \cos(k_y y)$$

The solutions to Maxwell's equations gave us :

1.) $$B_x = \frac{i}{(\omega/c)^2 - k^2} \, k \, \frac{\partial B_z}{\partial x}$$

2.) $$B_y = \frac{i}{(\omega/c)^2 - k^2} \, k \, \frac{\partial B_z}{\partial y}$$

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$$B_x = \frac{i}{(\omega/c)^2 - k^2} \, k \, Y \left[A\cos(k_x x) - B\sin(k_x x) \right]$$

at $x = 0$, $B_x = 0 \;\rightarrow\; A = 0$

at $x = a$, $B_x = 0 \;\rightarrow\; k_x = \frac{m\pi}{a}$ $(m = 0, 1, 2 \ldots)$

$$B_y = \frac{i}{(\omega/c)^2 - k^2} \, k \, X \left[C\cos(k_y y) - D\sin(k_y y) \right]$$

at $y = 0$, $B_y = 0 \;\rightarrow\; C = 0$

at $y = b$, $B_y = 0 \;\rightarrow\; k_y = \frac{n\pi}{b}$ $(n = 0, 1, 2 \ldots)$

$$\Longrightarrow B_z(x, y) = B_0 \cos(m\pi x/a) \cos(n\pi y/b)$$

We have even more information:

$$k = \sqrt{(\omega/c)^2 - \pi^2\left[(m/a)^2 + (n/b)^2\right]}$$

If $\omega < c\pi\sqrt{(m/a)^2 + (n/b)^2} \equiv \omega_{mn}$

the k is imaginary, no propagation