Physics 580: Quantum Mechanics I

Summary of σ Matrices

In the eigen-basis of σ_z:

$$\sigma_1 \equiv \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 \equiv \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 \equiv \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. $\sigma_j \sigma_k = \delta_{jk} I + i\,\epsilon_{jk\ell}\sigma_\ell$, where I is the 2×2 identity matrix and the repeated index ℓ is summed (Einstein summation convention).

2. For any $\vec{a}, \vec{b} \in \mathbb{R}^3$,
$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})I + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}.$$

3. $[\sigma_j, \sigma_k] = 2\,i\,\epsilon_{jk\ell}\,\sigma_\ell$.

4. $\{\sigma_j, \sigma_k\} = 2\,\delta_{jk} I$, where I is the 2×2 identity matrix.

5. For any unit vector $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we have $(\hat{n}\cdot\sigma)^2 = I$. Hence, the eigenvalues of $\hat{n}\cdot\sigma$ are ± 1. The corresponding eigenvectors are:
$$\psi_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \quad \psi_- = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}.$$

6. Eigenvectors of σ_x:
$$\psi_{x,\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

7. Eigenvectors of σ_y:
$$\psi_{y,\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

8. For any unit vector $\hat{n} \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$,
$$e^{i(\hat{n}\cdot\vec{\sigma})\theta/2} = \cos\frac{\theta}{2}\ I + i\,(\hat{n}\cdot\vec{\sigma})\sin\frac{\theta}{2}.$$
In particular, if we set $\hat{n} = (\hat{x}+\hat{z})/\sqrt{2}$ and $\theta = \pi$, then
$$e^{i(\hat{n}\cdot\vec{\sigma})\pi/2} = iH\,, \quad \text{where} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

H is called the Hadamard matrix and plays an important role in building quantum circuits. In this course, you will learn that iH is the fundamental representation of rotating spin-$\frac{1}{2}$ fermions around $(\hat{x}+\hat{z})/\sqrt{2}$ by angle π. You will also learn about the surprising fact that rotation by 2π yields $(iH)^2 = -I$ not I.

9. For any $\vec{x} \in \mathbb{R}^3$, unit vector $\hat{n} \in \mathbb{R}^3$, and $\theta \in \mathbb{R}$,

$$e^{-i(\hat{n}\cdot\vec{\sigma})\theta/2}(\vec{x}\cdot\vec{\sigma})e^{i(\hat{n}\cdot\vec{\sigma})\theta/2} = (\vec{x}\cdot\vec{\sigma})\cos\theta - \vec{x}\cdot(\hat{n}\times\vec{\sigma})\sin\theta + (\vec{x}\cdot\hat{n})(\hat{n}\cdot\vec{\sigma})(1-\cos\theta).$$

In particular,

$$(-iH)\sigma_x(iH) = H\sigma_x H = \sigma_z.$$

In this course, you will learn that this is the quantum analog of rotating $\hat{x}$ around the diagonal $\hat{n} = (\hat{x}+\hat{z})/\sqrt{2}$ by angle π.

10. A linear map $T : \mathbb{C}^n \to \mathbb{C}^n$ is called an involution if $T^2 = I_n$, where I_n is the identify matrix.

Exercise: Show that the eigenvalues of an involution are either 1 or -1.

Exercise: Show that an involution is Hermitian iff it is unitary.

σ_x, σ_y, σ_z and H are Hermitian involutions and, thus, are also unitary. Their eigenvalues are $+1$ and -1.