Neutrino Oscillations

A nice application of QM to current physics research is neutrino oscillations. Here I'll illustrate the main idea.

A bit of neutrino physics

The neutrino is a weakly interacting subatomic particle which was first discovered in nuclear decays like

electron anti-neutrino.

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

(neutron) (proton) + (electron)

It is interesting that the neutrino was not detected directly, but its presence had to be inferred by the fact that energy and momentum had to be conserved. That's another story...

After further study, we now know there are 3 kinds (flavors) of neutrinos:

ν_e (electron neutrino)

ν_μ (muon neutrino)

ν_τ (tau neutrino).

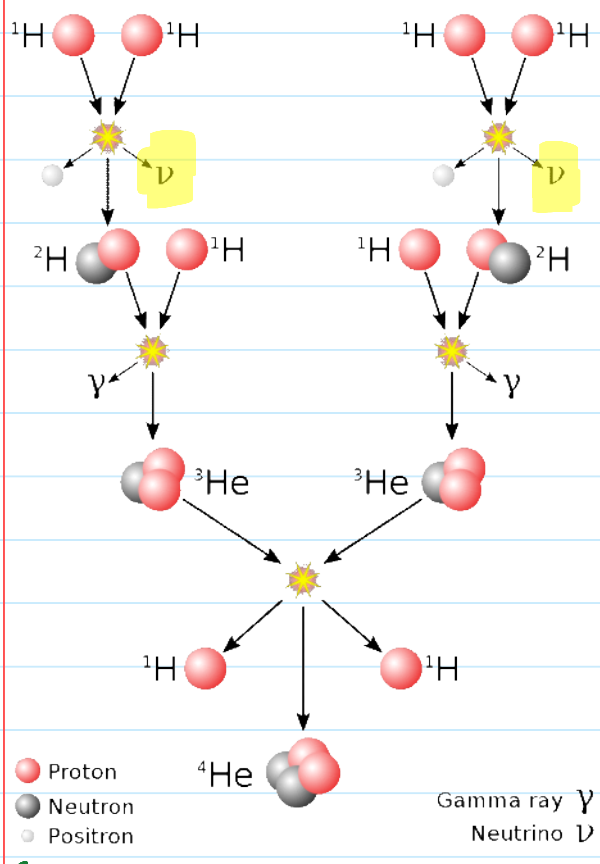
The others show up in reactions like $\pi^+ \longrightarrow \mu^+ + \nu_\mu$

(pion) (pion) (anti-muon) (muon neutrino)

Neutrinos are incredibly abundant. About 100 trillion (!!) pass through your body every second. However, they are very weakly interacting. You have about a 25% chance of a neutrino interacting in your body over your lifetime. [Note: this is not harmful]

Neutrino Oscillations

<u>Solar Neutrino Problem</u> - Neutrinos are produced in the nuclear reactions in the sun.



[Source: Wikipedia]

From the reaction

$$p + p \rightarrow {}^2_1d + e^+ + \nu_e$$

Now, we understand nuclear physics, and we can measure how much energy is emitted by the sun. So, we have a good expectation of how many neutrinos we expect to arrive at Earth every second.

Experiments in 1960s → today were done to try to measure the neutrino flux from the sun. They continued to find only about 1/3 of the expected neutrinos. A <u>huge</u> deficit!

How can this be? The idea for the resolution came from B. Pontecorvo, along with Maki, Nakagawa, and Sakata.
Perhaps the neutrino quantum states (which are produced in the sun and detected on Earth), ν_e, ν_μ, ν_τ are <u>not</u> the same as the states which propagate through space, call them ν_1, ν_2, ν_3.

In other words, the <u>weak interaction basis</u> is not the same as the energy basis. Let's see how this plays out... we'll assume for simplicity that there are only <u>two</u> neutrinos. The 3-flavor case goes through the same way, but is messier.

Neutrino Oscillations

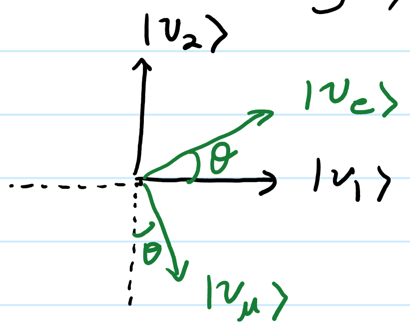
Suppose the electron neutrino $|\nu_e\rangle$ is a <u>combination</u> of the "mass" eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$
(energy)

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$$

So that, if I could somehow make a mass measurement, I'd have a probability $\cos^2\theta$ to get m_1 and $\sin^2\theta$ to get m_2.

Similarly: $|\nu_\mu\rangle = \sin\theta|\nu_1\rangle - \cos\theta|\nu_2\rangle$

θ is called the "mixing angle". It's a parameter and doesn't really have a geometric interpretation. But it's kind of like rotating your coordinate system



So, we can write this in matrix form

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

If $\theta = 0$, there is no mixing $|\nu_e\rangle = |\nu_1\rangle$, $|\nu_\mu\rangle = -|\nu_2\rangle$

if $\theta = 90°$, there is also no mixing $|\nu_e\rangle = |\nu_2\rangle$, $|\nu_\mu\rangle = |\nu_1\rangle$

Neutrino Oscillations

Poll Q: Recall that $\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$

where $|\nu_1\rangle$, $|\nu_2\rangle$ are energy eigenstates with energy E_1, E_2. Suppose an electron neutrino is created in the sun, so $|\psi(0)\rangle = |\nu_e\rangle$

What is $|\psi(t)\rangle$?

A.) $|\psi(t)\rangle = |\psi(0)\rangle e^{-iE_1 t/\hbar}$

C.) $|\psi(t)\rangle = \cos\theta\, e^{\frac{-iE_1 t}{\hbar}} |\nu_1\rangle + \sin\theta\, e^{\frac{-iE_2 t}{\hbar}} |\nu_2\rangle$

C.) $|\psi(t)\rangle = |\psi(0)\rangle e^{\frac{-iE_2 t}{\hbar}}$

D.) $|\psi(t)\rangle = \sin\theta\, e^{\frac{-iE_1 t}{\hbar}} |\nu_1\rangle - \cos\theta\, e^{\frac{-iE_2 t}{\hbar}} |\nu_2\rangle$

Poll Q: What is the probability of measuring an electron neutrino at a later time t?

A.) $\langle \nu_1 | \psi(t)\rangle$

B.) $\langle \nu_e | \psi(t)\rangle$

C.) $|\langle \nu_1 | \psi(t)\rangle|^2$

D.) $|\langle \nu_e | \psi(t)\rangle|^2$

Neutrino Oscillations

$$P_{\nu_e}(t) = |\langle \nu_e | \psi(t) \rangle|^2 = |C_{\nu e}|^2$$

$$C_{\nu e} = \left(\langle \nu_1 | \cos\theta + \langle \nu_2 | \sin\theta \right)\left(\cos\theta\, e^{\frac{-iE_1 t}{\hbar}} |\nu_1\rangle + \sin\theta\, e^{\frac{-iE_2 t}{\hbar}} |\nu_2\rangle \right)$$

$$C_{\nu e} = \cos^2\theta\, e^{-iE_1 t/\hbar} + \sin^2\theta\, e^{-iE_2 t/\hbar}$$

$$P_{\nu e}(t) = |C_{\nu e}|^2 =$$

$$\left(\cos^2\theta\, e^{-iE_1 t/\hbar} + \sin^2\theta\, e^{-iE_2 t/\hbar} \right)\left(\cos^2\theta\, e^{\frac{iE_1 t}{\hbar}} + \sin^2\theta\, e^{\frac{iE_2 t}{\hbar}} \right)$$

$$P_{\nu e}(t) = |C_{\nu e}|^2 =$$

$$\left(\cos^2\theta\, e^{-iE_1 t/\hbar} + \sin^2\theta\, e^{-iE_2 t/\hbar} \right)\left(\cos^2\theta\, e^{\frac{iE_1 t}{\hbar}} + \sin^2\theta\, e^{\frac{iE_2 t}{\hbar}} \right)$$

$$= \cos^4\theta + \sin^4\theta + \cos^2\theta \sin^2\theta \left[e^{\frac{i(E_2 - E_1)t}{\hbar}} + e^{\frac{-i(E_2 - E_1)t}{\hbar}} \right]$$

$$= \underbrace{1 - \tfrac{1}{2}\sin^2(2\theta)}_{\text{trig identity}} + \underbrace{\left[\tfrac{1}{2}\sin(2\theta)\right]^2}_{\sin(2\theta) = 2\sin\theta\cos\theta} \left[\underbrace{2 \cdot \cos\left(\tfrac{\Delta E t}{\hbar}\right)}_{e^{ix} + e^{-ix} = 2\cos(x)} \right]$$

$$P_{\nu_e}(t) = 1 - \tfrac{1}{2}\sin^2(2\theta)\left[\underbrace{1 - \cos\left(\tfrac{\Delta E t}{\hbar}\right)}_{1 - \cos(x) = 2 \cdot \sin^2\left(\frac{x}{2}\right)} \right]$$

$$\boxed{P_{\nu e}(t) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta E t}{2\hbar}\right)}$$

So, if an electron neutrino is created in the sun, the probability that you measure an electron neutrino later on depends on

- The mixing angle θ
- The energy difference $\Delta E = E_2 - E_1$
- Time.

Let's put this formula in a form which is commonly used:

$$E = \sqrt{(pc)^2 + (mc^2)^2} \qquad \text{(Relativistic energy)}$$

Let's assume the neutrinos are very relativistic so $E \gg mc^2$

$\varepsilon = \dfrac{mc^2}{pc}$ is a small parameter

$$E = \sqrt{(pc)^2 + (pc)^2\varepsilon^2} = pc\sqrt{1+\varepsilon^2} \approx pc\left(1 + \frac{\varepsilon^2}{2}\right)$$

What shows up in the formula is $E_2 - E_1 = \Delta E$. Write it as:

Note: p is the same for both neutrinos $|\nu_1\rangle$ and $|\nu_2\rangle$. This is the momentum of the state which is produced or detected. Their masses are different, so their energies are different.

$$\text{So,} \quad E_2 - E_1 = pc\left(1 + \frac{\varepsilon_2^2}{2}\right) - pc\left(1 + \frac{\varepsilon_1^2}{2}\right)$$

$$\Delta E = \frac{pc}{2}\left(\varepsilon_2^2 - \varepsilon_1^2\right) = \frac{pc}{2}\left[\frac{m_2^2c^4}{p^2c^2} - \frac{m_1^2c^4}{p^2c^2}\right]$$

$$= \frac{c^3}{2p}\underbrace{\left[m_2^2 - m_1^2\right]}$$

in literature, this is usually called

$\Delta m^2 \equiv m_2^2 - m_1^2$

Note it is $\Delta(m^2)$ not $(\Delta m)^2$.

Neutrino Oscillations

$$\Delta E = \frac{c^3}{2p}\Delta m^2$$

Note this is a small parameter of order ε^2

To leading order, then $E_1 = E_2 = pc$

so we can replace $p = E/c$ where $E = E_1 = E_2$

to leading order. Of course E_1 is not exactly equal to E_2, but the difference is small, and our formula already contains the small parameter ε^2, so we can neglect any other differences between E_1 & E_2.

$$\Delta E = \frac{\Delta m^2 c^4}{2E}$$

Finally, if the neutrinos are very relativistic, then $v \approx c$ and so the distance they travel is $L = c \cdot t \rightarrow t = L/c$. Put it all together:

$$P_{\nu e}(L) = 1 - \sin^2(2\theta)\sin^2\left(\frac{1}{2\hbar}\cdot\frac{\Delta(m^2)c^4}{2E}\cdot\frac{L}{c}\right)$$

$$\boxed{P_{\nu e}(L) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta(m^2)c^3 L}{4E\hbar}\right)}$$

If an electron neutrino is created at some point, this is the probability that you'll detect an electron neutrino at a distance L from the creation point.

Neutrino Oscillations

How does this solve the solar neutrino problem? The experiments on earth were designed to measure ν_e. They did not detect ν_μ. if $P_{\nu e}(L) \approx 1/3$, you have a 33% chance of measuring an electron neutrino and a ~67% chance of measuring a muon neutrino But, if your detector can't measure muon neutrinos, it looks like a deficit where $\frac{2}{3}$ of the original neutrinos are missing!

Of course, the real world has 3 neutrinos.
These days, current research is being done at Fermilab & other places to measure precisely the mixing angles θ_{12}, θ_{13}, θ_{23} and the mass squared differences

$$m_2^2 - m_1^2 , \quad m_3^2 - m_1^2 , \quad m_3^2 - m_2^2 .$$