$\boxed{\text{WKB}}$ (Wentzel, Kramers, Brillouin)

Semi-classical approximation of eigenvalues & eigenvectors assuming $\hbar \ll$ action and expanding in powers of $\hbar$.

General Framework:

Global approximation to a linear diff. eq.

$$\left[\varepsilon^n \frac{d^n}{dx^n} + a_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0(x) \right] f(x) = 0 \quad (*)$$

for $0 < \varepsilon$, $\varepsilon^n \ll 1$, with initial or boundary value problem.

Seek $f(x) \sim \exp(i \underbrace{W(x,\delta)}_{\hookrightarrow \,\int (\text{algebraic fn})} / \delta)$ or Airy fn. $(\delta \to 0)$

δ depends on ε.

- W real: dispersive, oscillatory solution.
- W imaginary: dissipative solution

Expanding $W(x,\delta)$ in powers of δ,

$$f(x) \sim \exp\left[\frac{i}{\delta} \sum_{n=0}^{\infty} \delta^n W_n(x) \right] \quad (\delta \to 0). \quad (**)$$

Substitute $(**)$ into $(*)$ and solve for W_n.

$\boxed{\text{E.g. 1}}$ $\quad \varepsilon^2 f'' = -Q(x)\, f \qquad \Big[\overset{\varepsilon^2}{\hbar^2}\, \psi'' = -\overbrace{2m(E - V(x))}^{Q(x) = P(x)^2}\,\psi\Big]$

Let $W(x,\delta) = \sum_{n=0}^{\infty} \delta^n \underbrace{\int^x Y_n(y)\, dy}_{W_n(x)}$ ↳ arbitrary lower limit

Let $f(x) = \exp\left(\frac{i}{\delta} \sum_{n=0}^{\infty} \delta^n \int^x Y_n(y)\, dy\right)$

$$f'(x) = \frac{i}{\delta} \sum_{n=0}^{\infty} \delta^n Y_n(x)\, f(x)$$

$$f''(x) = \left[-\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n Y_n(x)\right)^2 + \frac{i}{\delta}\left(\sum_{n=0}^{\infty} \delta^n Y_n'(x)\right)\right] f(x)$$

$$\Rightarrow -\frac{\varepsilon^2}{\delta^2} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \delta^{n+k}\, Y_n(x) Y_k(x) + \frac{i\varepsilon^2}{\delta} \sum_{n=0}^{\infty} \delta^n Y_n'(x) = -Q(x)$$

In the limit $\delta \to 0$, the largest term on the LHS is $\frac{\varepsilon^2}{\delta^2} Y_0^2(x)$ which has to match $Q(x)$.

Hence $\varepsilon^2 \propto \delta^2$ and we can assume $\delta = \varepsilon$.

$$\Rightarrow \quad Y_0 = \pm\sqrt{Q(x)}. \qquad \boxed{W_0(x) = \pm\int^x \sqrt{Q(y)}\, dy}$$

$O(\varepsilon)$: $-2Y_1(x) Y_0(x) + iY_0'(x) = 0$

$$\Rightarrow Y_1(x) = \frac{i}{2}\frac{Y_0'(x)}{Y_0(x)}. \qquad \boxed{W_1(x) = \int^x \frac{i}{2}\left(\log Y_0(y)\right)' dy}$$

$$\Rightarrow W_1(x) = \int^x Y_1(y)\, dy = \frac{i}{4}\log Q(x) + \text{const}$$

$O(\varepsilon^2)$: $-Y_1(x)^2 - 2Y_0(x) Y_2(x) + iY_1'(x) = 0$

$$\Rightarrow Y_2 = \frac{1}{2}\frac{\left(iY_1'(x) - Y_1^2(x)\right)}{Y_0(x)} \Rightarrow \boxed{W_2(x) = \pm\int^x \left[\frac{-Q''}{8Q^{3/2}} + \frac{5}{32}\frac{Q'^2}{Q^{5/2}}\right] dy}$$

$$\Rightarrow \boxed{\begin{aligned} f(x) \sim \; & \frac{c_1}{Q^{\frac14}} \exp\left(\frac{i}{\varepsilon} \int_a^x \sqrt{Q(y)}\, dy + i\varepsilon \int_a^x \left(-\frac{Q''}{8Q^{3/2}} + \frac{5}{32}\frac{Q'^2}{Q^{5/2}} \right) dy \right) \\ & + \frac{c_2}{Q^{\frac14}} \exp\left(-\frac{i}{\varepsilon} \int_{\underset{\text{arbitrary fixed point}}{a}}^x \sqrt{Q}\, dy - i\varepsilon \int_a^x \left(\frac{-Q''}{8Q^{3/2}} + \frac{5}{32}\frac{Q'^2}{Q^{5/2}} \right) dy \right) \end{aligned}}$$

Set $\hbar = \varepsilon$. (to avoid confusion with $\mathcal{E}$)

Notice that the leading even power terms are in the exponent while the leading odd power term is providing a multiplicative correction. This observation generalizes to all order:

$$W(x) = \int^x \underbrace{\sum_{n=0}^{\infty} \hbar^{2n} Y_{2n}(y)}_{\mathcal{E}(y)} dy + \int^x \underbrace{\sum_{n=0}^{\infty} \hbar^{2n+1} Y_{2n+1}(y)}_{\mathcal{O}(y)} dy$$

$$f(x) = \exp\left(\frac{i}{\hbar} W(x)\right)$$

$$\Rightarrow f'(x) = \frac{i}{\hbar} W'(x) f(x) = \frac{i}{\hbar}\left(\mathcal{E}(x) + \mathcal{O}(x)\right) f(x)$$

$$\Rightarrow f''(x) = \left[\frac{i}{\hbar}\left(\mathcal{E}'(x) + \mathcal{O}'(x)\right) - \frac{1}{\hbar^2}\left(\mathcal{E}(x) + \mathcal{O}(x)\right)^2 \right] f(x)$$

$$\Rightarrow \quad -\left(\mathcal{E}(x)^2 + \mathcal{O}(x)^2\right) + i\hbar\mathcal{O}'(x) = -Q \quad (1) \quad (\text{even in } \hbar)$$

$$-2\mathcal{E}(x)\mathcal{O}(x) + i\hbar\mathcal{E}'(x) = 0 \quad (2) \quad (\text{odd in } \hbar)$$

$$(2) \Rightarrow \quad \mathcal{O}(x) = \frac{i\hbar\,\mathcal{E}'(x)}{2\,\mathcal{E}(x)} = \frac{i\hbar}{2}\left(\operatorname{Log}\mathcal{E}(x)\right)'$$

$$\Rightarrow e^{\frac{i}{\hbar}\int^x \mathcal{O}(y)\,dy} = e^{-\frac12 \log \mathcal{E}(x)} = \frac{1}{\sqrt{\mathcal{E}(x)}}$$

If we rename $\int^{x}\mathcal{E}(y)\,dy = S(x)$, we get

$$\boxed{f(x) = \frac{1}{\sqrt{S'(x)}} \exp\left(\frac{i}{\hbar} S(x)\right)}$$

where $S(x)$ satisfies

$$\left(\frac{dS}{dx}\right)^2 + \frac{\hbar^2}{2} \underbrace{\left[\left(\frac{S'''(x)}{S'(x)}\right) - \frac{3}{2}\left(\frac{S''}{S'}\right)^2\right]}_{=\{S,x\}\ \text{Schwarzian derivative}} = Q$$

$$\frac{1}{2m}\left(\frac{dS}{dx}\right)^2 + \frac{\hbar^2}{4m}\{S,x\} + V(x) = E$$

To leading order, this is just the Hamilton-Jacobi equation and S is the classical action.

$$S(x) = \int^{x} \sqrt{2m(E-V(y))}\, dy$$