Chapter 6, magnetic fields in matter

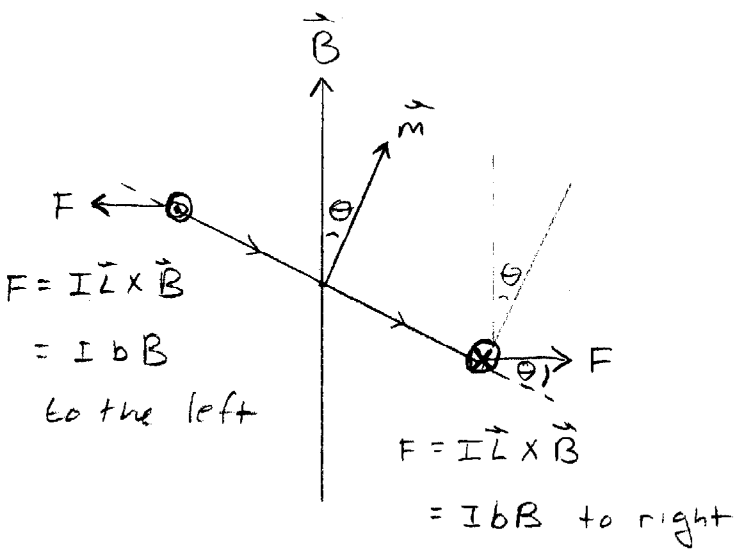
3 classifications of magnetic materials

Paramagnetic
Diamagnetic
Ferromagnetic
} You figure it out (Lots of info in Feynman)

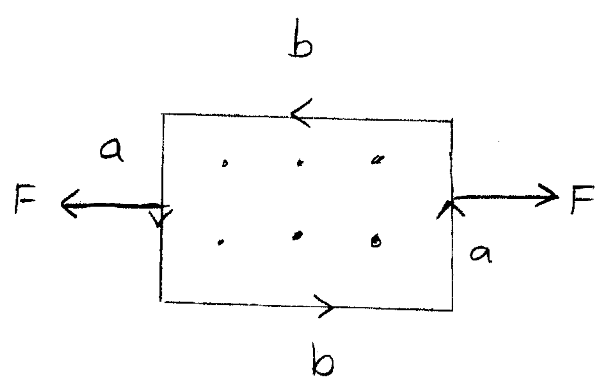
Griffiths begins by considering single dipoles, forces and torques acting on them. Then, goes on to consider materials in bulk:

Square current loop in a uniform field:

SIDE VIEW



TOP VIEW



The force on sides of length b is not completely parallel to moment arm. There <u>is</u> a torque.

Force on sides of length a is parallel to moment arm ($F \parallel r$). So, no torque, $\tau = \vec{r} \times \vec{F}$

The net torque is twice the torque from one side (the direction of both torques is out of the paper in the 'side view' diagram.

$$\tau = 2\tau_{1\,side} = 2(\vec{r}\times F) = 2(a/_2 \sin\theta\, I b B)$$

$$\tau = (Iab)\sin\theta\ B = \vec{m}\times\vec{B}$$

This has the same form as the torque on an electric dipole from an $\vec{E}$ field (as it must).

The potential also has the same form:

$$U = -\vec{m}\cdot\vec{B}$$

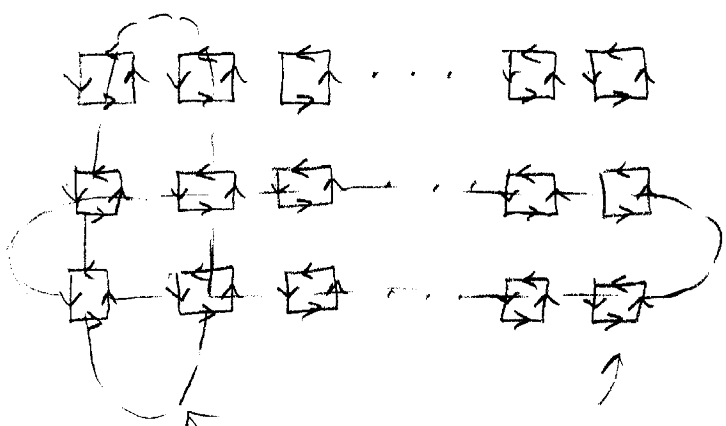
$$\vec{F} = -\vec{\nabla} U = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

Note that this shows that a dipole with dipole moment $\vec{m}$ must be immersed in a magnetic field with a non-zero gradient to experience a net force.

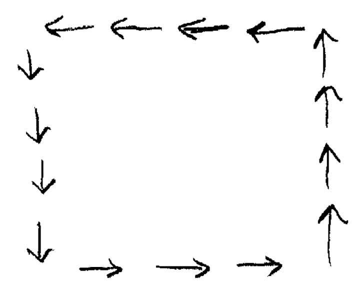
A uniform field can torque a dipole around, but cannot translate it, as there is no net force.

What if there are lots of magnetic dipoles together in a material? The magnetized material can be described as having a bound surface current, and perhaps a bound volume current. The Griffiths cartoons crudely re-visited:

Consider the case with no volume bound current, surface current only:

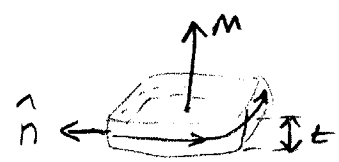


currents cancel inside material



Net current on surface

Let's find k_b in terms of M:



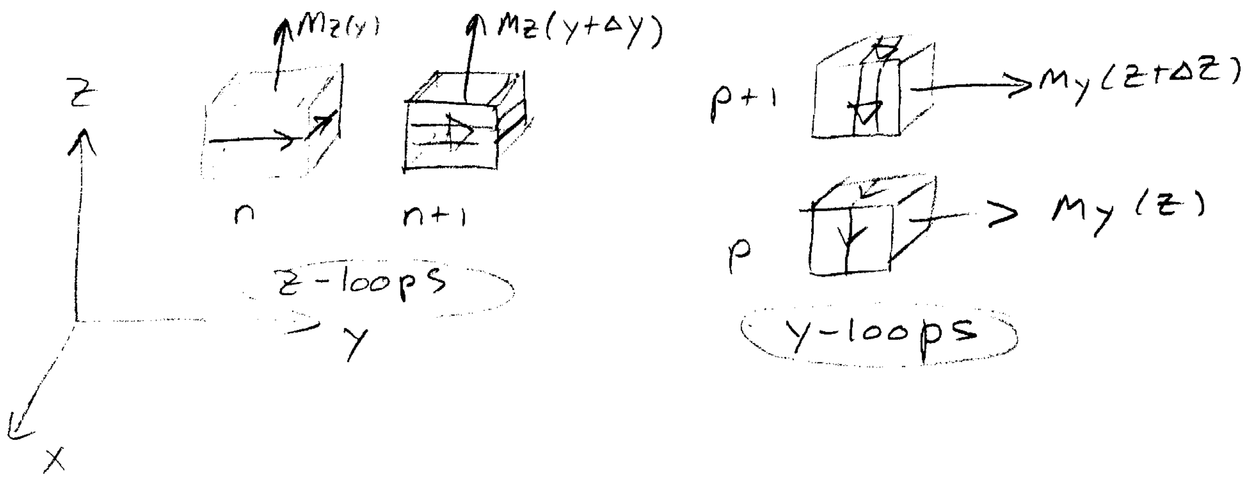
$$k_b = \frac{I_b}{t}$$

$$M = \frac{\text{dipole moment}}{\text{volume}} = \frac{I\,(\text{area})}{(\text{area})\,t} = \frac{I_b}{t}$$

It is only the component of current parallel to the surface (corresponding to an $\vec{M}$ perpendicular to the surface) which adds to the surface current. Or, in math talk,

$$\vec{K}_b = \vec{M} \times \hat{n}$$

Now consider the case when there is volume bound current.



Net current in $\hat{x}$ direction will occur if $\vec{M}_z$ changes with y, or $\vec{M}_y$ changes with z. For the pictures above:

z-loops: $I_{tot} = I_{\substack{loop \\ n+1}}\,\hat{x} - I_{\substack{loop \\ n}}\,\hat{x} = I_{bigger}\,\hat{x} - I_{smaller}\,\hat{x}$

y-loops: $I_{tot} = -I_{\substack{loop \\ p+1}}\,\hat{x} + I_{\substack{loop \\ p}}\,\hat{x} = -I_{bigger}\,\hat{x} + I_{smaller}\,\hat{x}$

z-loops: $\frac{I_{loop}}{\Delta z} = M_z$, or $I_{loop} = M_z \Delta z$

thickess in direction $\perp$ to loop area

y-loops; $\frac{I_{loop}}{\Delta y} = M_y$, or $I_{loop} = M_y \Delta y$

$$I_x\,(z\text{-loops}) = (I_{(n+1)} - I_n)\hat{x} = [M_z(y+\Delta y) - M_z(y)]\Delta z$$

$$I_x\,(y\text{-loops}) = (I_{p+1} - I_p)\hat{x} = [-M_y(z+\Delta z) + M_y(z)]\Delta y$$

$$I_x\,(z\text{-loop}) + I_x\,(y\text{-loop}) = [M_z(y+\Delta y) - M_z(y)]\Delta z + [M_y(z+\Delta z) + M_y(z)]\Delta y$$

$$I_x\text{total} = \frac{\partial M_z}{\partial y}\Delta y\Delta z - \frac{\partial M_y}{\partial z}\Delta z\Delta y$$

$$I_x = \left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right)\Delta y\Delta z$$

We have I in terms of M. If we write I in terms of J_b, then finally we can get J_b in terms of M.

$$I_x = J_{bx}\Delta y\Delta z$$

$$\Rightarrow \quad J_{b_x} = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \quad \Rightarrow \quad \vec{J} = \vec{\nabla}\times\vec{M}$$

Now, to beat a dead horse, let's get those results by finding the vector potential due to a bulk of dipoles.

$$\vec{A}_{dipole} = \frac{\mu_0}{4\pi r^2}\, \vec{m} \times \hat{r}$$

For lots of dipoles, let $m \to M\,d\tau$ and integrate over space.

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{M} \times \hat{\imath}}{\imath^2}\, d\tau$$

$$\frac{\hat{\imath}}{\imath^2} = \nabla'\left(\frac{1}{\imath}\right) \qquad \text{see chapter 1}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{m} \times \nabla'(1/\imath)\, d\tau$$

Use product rule (7): $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} f)$

$$\vec{M} \times \vec{\nabla}'\left(\frac{1}{\imath}\right) = -\vec{\nabla}' \times \left(\frac{\vec{M}}{\imath}\right) + \frac{1}{\imath}\left(\vec{\nabla}' \times \vec{M}\right)$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{(\vec{\nabla}' \times \vec{M})}{\imath}\, d\tau' - \frac{\mu_0}{4\pi} \int \vec{\nabla}' \times (\vec{M}/\imath)\, d\tau'$$

For the second integral, use

$$-\int (\vec{\nabla} \times \vec{v})\, d\tau = \int \vec{v} \times d\vec{a} \qquad \text{(more on this later)}$$

$$\int \vec{\nabla}' \times (\vec{M}/r)\, d\tau' \to -\int \frac{\vec{M}}{r} \times d\vec{a}' = -\int \frac{\vec{M} \times \hat{n}\, da'}{r}$$

Then:

$$A = \frac{\mu_0}{4\pi} \int \frac{(\vec{\nabla}' \times \vec{M})\, d\tau'}{r} + \frac{\mu_0}{4\pi} \int \frac{\vec{M} \times \hat{n}\, da'}{r}$$

$\vec{\nabla}' \times \vec{M}$ plays the role of J_b

$\vec{M} \times \hat{n}$ plays the role of K_b

$$A = \frac{\mu_0}{4\pi} \int \frac{J_b\, d\tau'}{r} + \frac{\mu_0}{4\pi} \int \frac{K_b\, da'}{r}$$

$$\boxed{\begin{aligned} J_B &= \vec{\nabla} \times \vec{M} \\ K_b &= \vec{M} \times \hat{n} \end{aligned}}$$

More on $\int \vec{\nabla} \times \vec{v}\, d\tau = -\int \vec{v} \times d\vec{a}$

i.e. proof:

To prove this, begin with the divergence theorem:

$$\int \vec{\nabla}\cdot\vec{v}\, d\tau = \oint \vec{v}\cdot d\vec{a} \quad , \; v \text{ any vector}$$

let $\vec{v} \to \vec{v}\times\vec{c}$, $\vec{c}$ a constant vector

$$\int \vec{\nabla}\cdot(\vec{v}\times\vec{c})\, d\tau = \oint (\vec{v}\times\vec{c})\cdot d\vec{a}$$

use product rule (6)

$$\vec{\nabla}\cdot(\vec{v}\times\vec{c}) = \vec{c}\cdot(\vec{\nabla}\times\vec{v}) - \vec{v}\cdot\cancelto{0}{(\vec{\nabla}\times\vec{c})}$$

By equivalence of certain triple products:

$$(\vec{v}\times\vec{c})\cdot d\vec{a} = \vec{c}\cdot(d\vec{a}\times\vec{v}) = -\vec{c}\cdot(\vec{v}\times d\vec{a})$$

Then

$$\vec{c}\cdot\int \vec{\nabla}\times\vec{v}\, d\tau' = -\vec{c}\int \vec{v}\times d\vec{a}$$

$$\hookrightarrow \int \vec{\nabla}\times\vec{v}\, d\tau = -\int \vec{v}\times d\vec{a}$$

In finding J_b, K_b we enabled ourselves to calculate the field due to a magnetized object. Now we can go on to consider situations in which magnetized objects as well as external fields from some other source are both present.

Maxwell's equations as we first wrote them are still true:

$$\vec{\nabla} \times \underbrace{\vec{B}}_{\text{complete field}} = \mu_0 \underbrace{\vec{J}}_{\text{all current sources}}$$

For convenience, $\vec{J}$ can be separated into free currents and bound currents.

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J}_f + \vec{J}_b)$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \vec{J}_f + (\vec{\nabla} \times \vec{M})$$

$$\vec{\nabla} \times \left(\vec{B}/\mu_0 - \vec{M} \right) = J_f$$

$$\nabla \times \vec{H} = J_f$$

where $\vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{M}$

Or, think of it this way:

$$\underset{\substack{\uparrow \\ \text{total} \\ \text{field}}}{\vec{B}} = \underbrace{\mu_0 \vec{H}}_{\substack{\uparrow \\ \text{field} \\ \text{due to} \\ \text{free current} \\ \text{and } -\vec{\nabla}\cdot\vec{M}}} + \underbrace{\mu_0 \vec{M}}_{\uparrow \text{ field due to bound current}}$$

Similarly,

$$\underset{\substack{\uparrow \\ \text{total} \\ \text{field}}}{\vec{E}} = \underset{\substack{\uparrow \\ \text{field} \\ \text{due to} \\ \text{free charge} \\ \text{and } \vec{\nabla}\times\vec{P}}}{\frac{\vec{D}}{\epsilon_0}} - \underset{\uparrow \text{ field due to bound charge}}{\frac{\vec{P}}{\epsilon_0}}$$

<u>Notice</u> - If $\vec{P}$ lines up with $\vec{D}$, $\vec{E}$ is reduced. This is the usual case.

Whereas, if $\vec{M}$ lines up with $\vec{H}$, $\vec{B}$ increases. Sometimes this happens, sometimes not.

To examine this further, let's consider materials with a linear response to an applied field.

In this case, $\vec{M} \equiv \chi_m \vec{H}$

magnetic susceptibility

Then, $\vec{M}$ will line up with $\vec{H}$ if χ_m is positive, and against $\vec{H}$ when χ_m is negative. Paramagnetic materials normally have positive χ_m, whereas diamagnetic materials have negative χ_m.

To finish off with definitions pertaining to linear material,

$$B = \mu_0(\vec{H} + \vec{M}) = \mu_0(1+\chi_m)H$$

$$\mu \equiv \mu_0(1+\chi_m)$$

$$\vec{B} = \mu \vec{H}$$

Remember in the electrostatic case we had

$$\vec{D} = \epsilon \vec{E}$$

Highly annoying that in one equation the total field is on the LHS, while in the other on the RHS.

Paramagnetism and diamagnetism are magnetizations that disappear when the applied field is removed. The common, permanent magnets we are more used to are ferromagnets. Their dipoles tend to line up with each other, even in the absence of an external field. An applied field will magnetize a ferromagnetic material further by lining up dipoles that were initially unaligned (due to competition from other neighboring dipoles).

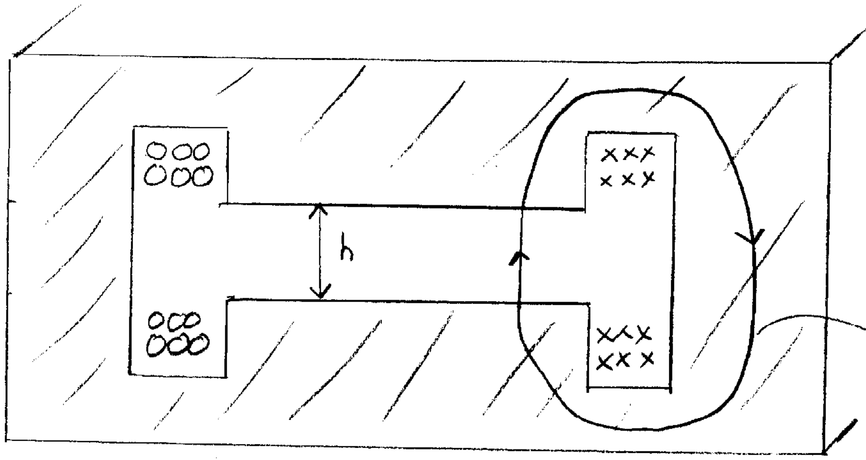
$$\vec{B} = \mu \vec{H}$$

very large in ferromagnets

For example, in Griffith's hysteresis curve (p. 281),

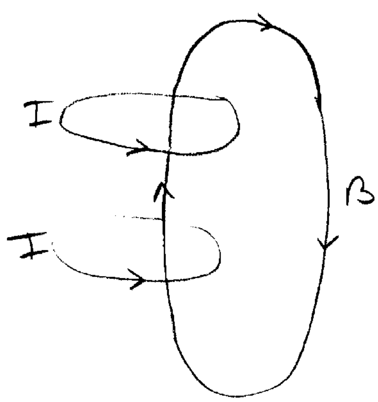
$$\mu/\mu_0 \approx 10^4$$

It is for this reason that coils in electromagnets are wrapped around ferromagnetic pole tips. Let's check it out for a dipole magnet.



Dipole Cross-section

We know from the right-hand-rule that the magnetic field from the coils is upward and uniform across the gap.



Since we don't know J_b, we must use the form of Ampere's law with $\vec{H}$.

$$\vec{\nabla} \times \vec{H} = J_f$$

$$\oint H \cdot dl = I_f$$

We have two coils passing through our Amperian loop normal to the loop.

$$I_f = 2NI$$

applied current

2 coils

N turns per coil

We have:

$$\oint \vec{H}\cdot d\hat{\ell} = \int_{\substack{\text{across}\\ \text{gap}}} \vec{H}\cdot d\hat{\ell} + \int_{\substack{\text{through}\\ \text{iron}}} H\cdot d\ell = 2NI$$

In the gap, $\mu = \mu_0$

$$\oint \vec{H}\cdot d\hat{\ell} = \int_0^h \frac{B}{\mu_0}\,dy + \int_{\substack{\text{through}\\ \text{iron}}} \frac{B}{\mu}\,d\ell$$

$B/\mu \ll \frac{B}{\mu_0}$, so the contribution from the portion of the pathlength through the iron is negligable. Then:

$$\frac{Bh}{\mu_0} = 2NI \quad \rightarrow \quad B = \frac{2\mu_0 NI}{h}$$

Notice that B increases with NI, and also increases as the gap height, h, decreases. This drives magnet apertures to be closer to the beam. Otherwise, accelerators would be larger than they already are.

Back in Chapter 3, we solved the magnetostatic scalar potential for a quadrupole magnet using the method of separation of variables. We found:

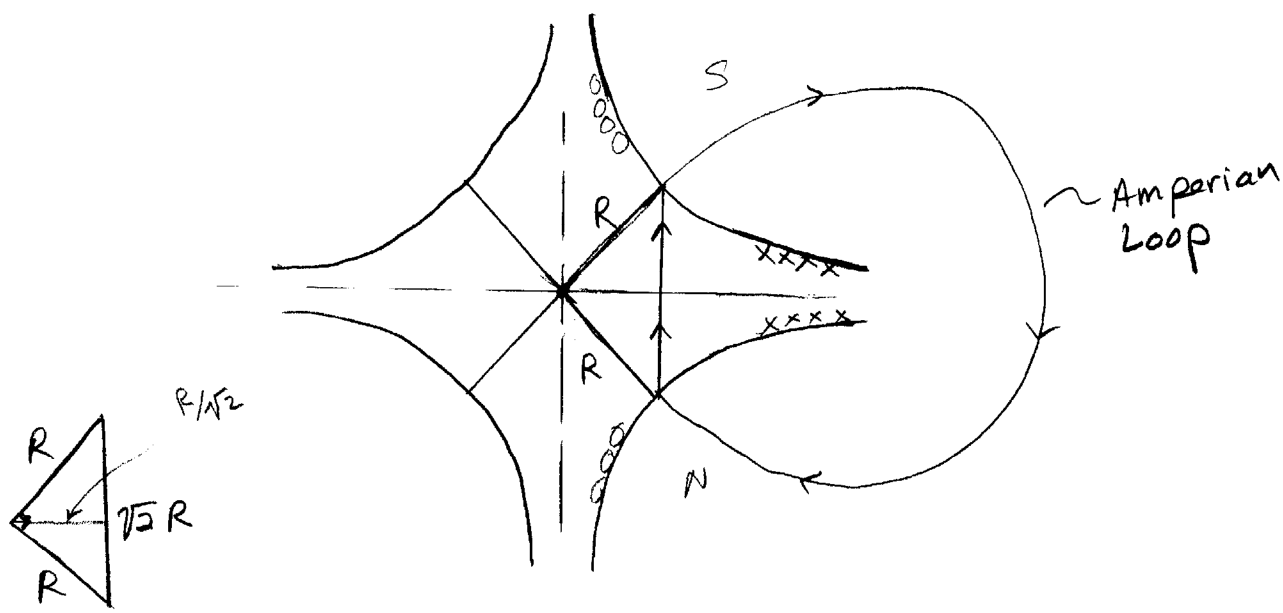
$$V_B = 2A_2 xy$$

with A_2 still undetermined, as we didn't actually say what the potential was (numerically) at the boundaries (pole surface). We let $2A_2 = B'$ since

$$\vec{B} = -\vec{\nabla} V_B$$

$$\left.\begin{aligned} \vec{B}_x &= 2A_2 y \\ \vec{B}_y &= 2A_2 x \end{aligned}\right\} \quad 2A_2 \text{ 'must be' } B'$$

Now, lets find out what that B' really is:



$$\int_{gap} \frac{\vec{B}}{\mu_0} \cdot d\vec{\ell} + \underbrace{\cancel{\int_{steel} \frac{\vec{B}}{\mu} \cdot d\vec{\ell}}}_{\text{neg}} = \underset{\text{2 coils}}{2}\,\underset{\text{N turns}}{N}\,\underset{\text{current}}{I}$$

$$\int_0^{\sqrt{2}R} \left(B'y\,\hat{x} + B'x\,\hat{y}\right) \cdot \hat{y}\,dy = B'x \int_0^{\sqrt{2}R} dy$$

$$= B'\left(R/\sqrt{2}\right)\left(\sqrt{2}R\right) = B'R^2 = 2\mu_0 NI$$

$$\Rightarrow \quad B' = \frac{2\mu_0 NI}{R^2} \quad T/m \quad \Big\} \text{ field gradient}$$

Boundary conditions at the surface of magnetic materials:

$$B_a^{\perp} - B_b^{\perp} = 0$$

$$B_a^{\parallel} - B_b^{\parallel} = \mu_0 (\vec{K} \times \hat{n})$$

These are the same as before, but beware $\vec{K}$ now may have a contribution from $\vec{K}_b$.

$$H_a^{\perp} - H_b^{\perp} = -(M_a^{\perp} - M_b^{\perp})$$

Since $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$

$$H_a^{\parallel} - H_b^{\parallel} = \vec{K}_f \times \hat{n}$$

The discontinuity in the tangential component of $\vec{H}$ is due only to free surface current.