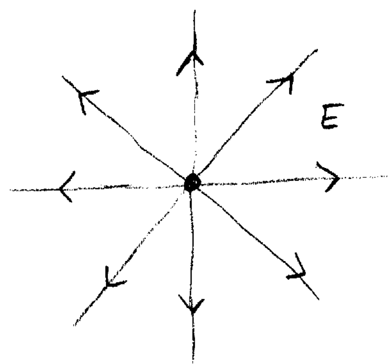
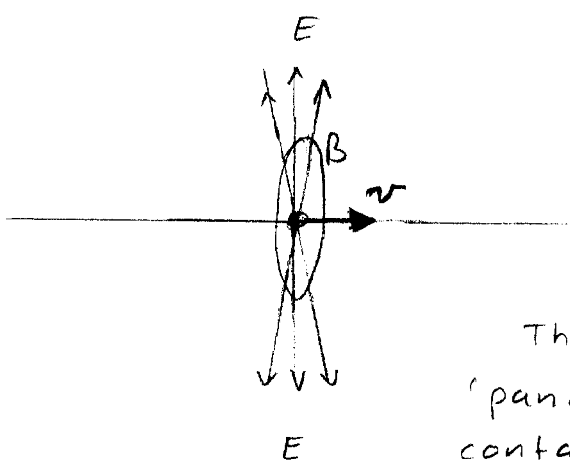
Chapter 10, Lecture 4

A stationary point charge has Electric field lines radiating out in all directions (no preference) and no magnetic field at all.



If the charge moves with significant velocity, there is a magnetic field and the $\vec{E}$ field lines are distorted compared to the stationary case.



The electric field 'pancakes' toward the plane containing the charge and perpendicular to the motion.

chapter 10, lecture 4 continued

Once again, shamelessly borrowing from Feynman, we'll consider the simplest case of a charge moving with constant velocity along the x-axis. The field equations can be cross-examined for corroboration of the story on the previous page.

From the last lecture:

$$V = \gamma \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}}$$

$$\vec{A} = \frac{\vec{v}}{c^2} V$$

Since $\vec{v} = v\hat{x}$, A_x is the only non-zero component of $\vec{A}$.

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$E_y = -\frac{\partial}{\partial y} V = -\gamma \frac{q}{4\pi\epsilon_0} \cdot -\tfrac{1}{2} \frac{2y}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}$$

$$= \frac{\gamma q}{4\pi\epsilon_0} \frac{y}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

Similarly for E_z

$$E_x = -\frac{\partial}{\partial x} V - \frac{\partial A_x}{\partial t}$$

$$= \frac{-\gamma q}{4\pi\epsilon_0} \left(-\frac{1}{2}\right) \frac{\gamma^2 2(x-vt)}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

$$- \frac{v}{c^2} \frac{\gamma q}{4\pi\epsilon_0} \left(-\frac{1}{2}\right) \frac{\gamma^2 2(x-vt)(-v)}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

$$= \frac{\gamma q}{4\pi\epsilon_0} \frac{(x-vt)\cancel{(1-v^2/c^2)\gamma^2}}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

$$= \frac{\gamma q}{4\pi\epsilon_0} \frac{x-vt}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

$$\vec{B} = \vec{\nabla} \times A_x \hat{x} = \frac{\partial A_x}{\partial z} \hat{y} - \frac{\partial A_x}{\partial y} \hat{z}$$

$$\left.\begin{aligned} B_y &= v/c^2 \, \frac{\partial V}{\partial z} = -v/c^2 \, E_z \\ B_z &= -v/c^2 \, \frac{\partial V}{\partial y} = v/c^2 \, E_y \\ B_x &= 0 \end{aligned}\right\} \quad \vec{B} = \vec{v}/c^2 \times \vec{E}$$

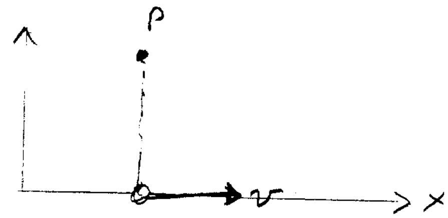
Check the strength of the field

① At right angles to the motion

② Directly ahead and behind the charge

At right angles to the motion, $x = vt$, so

$x - vt = 0$



$$\vec{E} = \gamma \frac{q}{4\pi\epsilon_0} \left\{ \frac{y\hat{y} + z\hat{z}}{(y^2 + z^2)^{3/2}} \right\} \Rightarrow |E| = \gamma \frac{q}{4\pi\epsilon_0} \frac{1}{(y^2+z^2)}$$

$|E|$ has the form of the coulomb field of a stationary charge $\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$ <u>except</u> enhanced by a factor of γ.

Directly ahead (or behind) the charge, $y = z = 0$

$$|E| = E_x = \gamma \frac{q}{4\pi\epsilon_0} \frac{x - vt}{[\gamma^2 (x-vt)^2]^{3/2}}$$

$$= \frac{1}{\gamma^2} \frac{q}{4\pi\epsilon_0} \frac{1}{(x-vt)^2}$$

Again, E has the form of the coulomb field of a stationary charge <u>except</u> decreased by a factor of $\frac{1}{\gamma^2}$.

The case of a charge moving at constant velocity helps visualize what happens to the fields. However, the general case where the charge may be accelerating is very important. Without acceleration, there cannot be radiation, fields that make it to the far zone.

So, the general case of a charge moving in an arbitrary way must be considered. As usual the fields are calculated from the potentials.

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

where

$$V = \frac{1}{4\pi\epsilon_0} \frac{q c}{\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v}} \quad , \quad \vec{A} = \vec{v}/c^2 \, V$$

$$\vec{\nabla} V = \frac{q c}{4\pi\epsilon_0} \frac{1}{(\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v})^2} \left[c^2 \vec{\nabla} t_r + \vec{\nabla}(\vec{\mathscr{r}} \cdot \vec{v}) \right]$$

using $\vec{\nabla} \mathscr{r} = -c \vec{\nabla} t_r$

$\vec{\nabla}(\vec{\mathscr{r}} \cdot \vec{v})$ has 4 terms, using product rule 4 (front cover)

$$\vec{\nabla}(\vec{\mathscr{r}} \cdot \vec{v}) = \underset{①}{\vec{\mathscr{r}} \times \vec{\nabla} \times \vec{v}} + \underset{②}{\vec{v} \times \vec{\nabla} \times \vec{\mathscr{r}}} + \underset{③}{(\vec{\mathscr{r}} \cdot \vec{\nabla}) \vec{v}} + \underset{④}{(\vec{v} \cdot \vec{\nabla}) \vec{\mathscr{r}}}$$

These 4 terms are put into other forms, (see next page) so that each is written in terms of $\vec{\nabla} t_r$.

Term ①, $\vec{\mathscr{r}} \times \vec{\nabla} \times \vec{v} = -\vec{a}\,(\vec{\mathscr{r}} \cdot \vec{\nabla} t_r) + \vec{\nabla} t_r\,(\vec{\mathscr{r}} \cdot \vec{a})$

use chain rule on $\vec{\nabla} \times \vec{v} \;\rightarrow\; -\vec{a} \times \vec{\nabla} t_r$

use triple product rule on $-(\vec{\mathscr{r}} \times \vec{a} \times \vec{\nabla} t_r)$

$$\vec{A} \times \vec{B} \times \vec{C} = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(A \cdot \vec{B})$$

Term ② $\vec{v} \times \vec{\nabla} \times \vec{\mathscr{r}} = \vec{v}\,(\vec{v} \cdot \vec{\nabla} t_r) - v^2 \vec{\nabla} t_r$

$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r) \;\rightarrow\; \vec{\nabla} \times \vec{\mathscr{r}} = \cancelto{0}{\vec{\nabla} \times \vec{r}} - \vec{\nabla} \times \vec{w}(t_r)$

use chain rule on $-\vec{\nabla} \times \vec{w}(t_r) \;\rightarrow\; -\vec{v} \times \vec{\nabla} t_r$

use triple product rule on $\vec{v} \times \vec{v} \times \vec{\nabla} t_r$

Term ③, $(\vec{\mathscr{r}} \cdot \vec{\nabla})\,\vec{v} = \vec{a}\,(\vec{\mathscr{r}} \cdot \vec{\nabla} t_r)$

use chain rule on $(\vec{\mathscr{r}} \cdot \vec{\nabla})\,\vec{v}$

Term ④, $(\vec{v} \cdot \vec{\nabla})\,\vec{\mathscr{r}} = \vec{v} - \vec{v}\,(\vec{v} \cdot \vec{\nabla} t_r)$

Use chain rule on $(\vec{v} \cdot \vec{\nabla})\,\vec{\mathscr{r}}$;

Note: $\mathscr{r}$ has <u>both</u> t_r <u>and</u> r dependence

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\imath c - \vec{\imath}\cdot\vec{v})^2} \left[\vec{v} + (c^2 - v^2 + \vec{\imath}\cdot\vec{a}) \vec{\nabla} t_r \right]$$

$$\vec{\nabla} t_r = -\frac{\vec{\imath}}{\imath c - \vec{\imath}\cdot\vec{v}}$$ (see longer version of this derivation)

$$\frac{\partial \vec{A}}{\partial t} = \frac{q}{c 4\pi\epsilon_0} \frac{\partial}{\partial t} \left(\frac{\vec{v}}{\imath c - \vec{\imath}\cdot\vec{v}} \right)$$

$$= \frac{q}{4\pi\epsilon_0 c} \frac{1}{(\imath c - \vec{\imath}\cdot\vec{v})^2} \left\{ (\imath c - \vec{\imath}\cdot\vec{v}) \frac{d\vec{v}}{dt} - c\vec{v} \frac{\partial \imath}{\partial t} + \vec{v} \frac{\partial(\vec{\imath}\cdot\vec{v})}{\partial t} \right\}$$

Apply chain rule liberally to portion in brackets

Also use:

$$\frac{\partial \imath}{\partial t_r} = \frac{-\vec{\imath}\cdot\vec{v}}{\imath} \quad ; \quad \frac{\partial \vec{\imath}}{\partial t_r} = -\vec{v}$$

$$\frac{\partial t}{\partial t_r} = \frac{\imath c}{\imath c - \vec{\imath}\cdot\vec{v}} \quad ; \quad \frac{\partial \vec{v}}{\partial t_r} = \vec{a}$$

Now put it all together,

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \rightsquigarrow$$

$$\vec{E} = \frac{q\,r}{4\pi\epsilon_0}\,\frac{1}{(\vec{r}\cdot\vec{u})^3}\left\{\underbrace{(c^2-v^2)\,\vec{u}}_{\substack{\text{velocity}\\ \text{or}\\ \text{coulomb}\\ \text{field}}} + \underbrace{\vec{r}\times(\vec{u}\times\vec{a})}_{\substack{\text{acceleration}\\ \text{or}\\ \text{radiation}\\ \text{field}}}\right\}$$

$$\vec{u} \equiv c\hat{r} - \vec{v}$$

$$\vec{B} = \frac{1}{c^2}\vec{\nabla}\times(\vec{v}\,V) = \frac{1}{c^2}\left[V(\vec{\nabla}\times\vec{v}) - \vec{v}\times\vec{\nabla}V\right]$$

$$\vec{B} = \frac{1}{c}\,\hat{r}\times\vec{E}$$

The derivation with all steps was scanned in separately for the gung-ho.