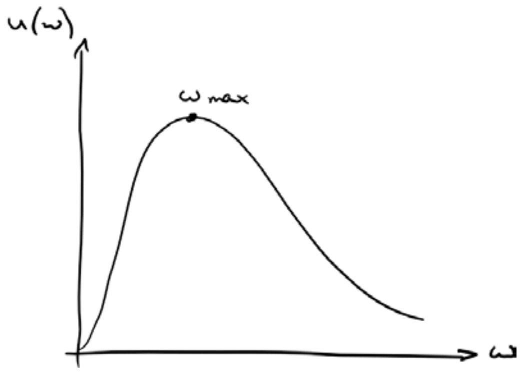
Lecture 24 – Thermal radiation

PREVIOUSLY: Thermal photon gases and the Planck distribution (Lect. 13)



A cavity of volume V containing a photon gas at temperature T contains EM radiation with the following spectrum

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\beta\hbar\omega}-1}$$

peaked at $\hbar\omega_{\max} = 2.82 k_B T$.

The total energy density is

$$u_{tot} = \frac{\pi^2}{15}\frac{\left(k_B T\right)^4}{\left(\hbar c\right)^3} = \alpha T^4$$

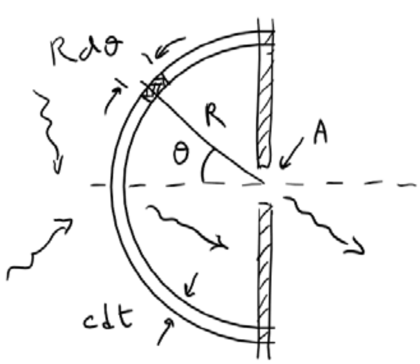
TODAY: These are the properties of a photon gas inside a cavity. We want to know the properties of photons emitted by an object at temperature T, and how radiation can transfer energy and put two bodies in thermal equilibrium.

Imagine a cavity with a small hole of size A. Photons emitted have the same spectrum as photons inside the cavity. What is the rate of emission? Define the energy flux density J_u as the rate energy is emitted per unit area (units W/m^2). This is also called the intensity (of the emitted light).

By dimensional analysis, we expect:

$$J_u = c u_{tot} \times (\text{geometrical factor})$$

Now let's derive this exactly

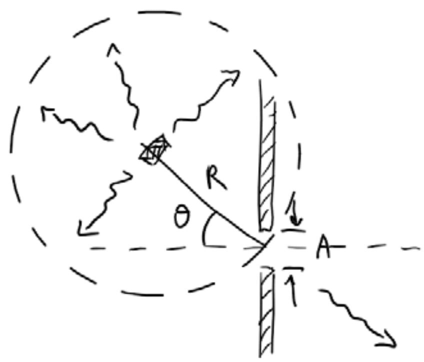


Photons that escape the cavity at time t, in time interval dt, came from volume element on hemispherical shell some time R/c before

The energy in a volume element in the shell is:

$$\frac{U}{V} R^2 \sin\theta d\theta d\phi c dt$$

Not all the energy from this volume element escapes, because not all photons point in the right direction



Photons are emitted isotropically. Therefore, they have an equal chance to land anywhere on the sphere of area $4\pi R^2$

The cross-sectional area of the hole is $A\cos\theta$, so the fraction of photons that escape is

$$\frac{A\cos\theta}{4\pi R^2}$$

So, the bit of energy that escapes in a time interval dt is

$$dU_{escape} = \int_{hemisphere} \frac{A\cos\theta}{4\pi \cancel{R^2}}\frac{U}{V}\cancel{R^2}\sin\theta d\theta d\phi c dt$$

$$= \frac{Acdt}{4\pi}\frac{U}{V}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\underbrace{\int_0^{\pi/2}\sin\theta\cos\theta d\theta}_{1/2} = \frac{1}{4}Acdt\underbrace{\frac{U}{V}}_{u_{tot}}$$

The power emitted is $P_{emit} = \dfrac{dU_{escape}}{dt} = \dfrac{1}{4}Acu_{tot}$ and the energy flux density is:

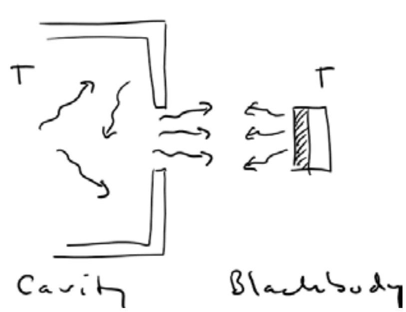
$$J_u = \frac{P_{emit}}{A} = \frac{1}{4}cu_{tot} = \frac{\pi^2}{60}\frac{(k_B T)^4}{\hbar^3 c^2} = \sigma_B T^4$$

where the Stefan-Boltzmann constant is $\sigma_B = 5.67\times 10^{-8}\,\dfrac{\mathrm{W}}{\mathrm{m}^2\cdot\mathrm{K}^4}$

This describes the flow of energy out of a cavity, but what about an object at temperature T radiating in space (e.g. the sun is not a cavity with a hole in it!)?

Equations turn out to be exactly the same!

KEY CONCEPT: blackbody radiation
Consider an object that absorbs all EM radiation (i.e. reflects no radiation, hence "black") and emits EM radiation according to its temperature T – that object is said to be a blackbody



A blackbody emits radiation the same as a cavity with a hole.

Proof: imagine a cavity with a small hole of size A facing a blackbody of same size A. Both are at the same temperature T, i.e. in thermal equilibrium with each other

- Each body emits photons and absorbs each other's photons
- Since they are at equilibrium, there can be no net energy flow. They must absorb the same fraction of the other's photons, and they must emit the same flux of photons. Otherwise there would be net energy flow.

$$J_{cavity} = J_{blackbody} = \sigma_B T^4$$

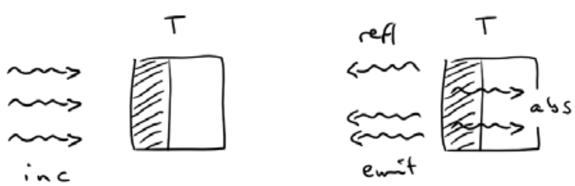
It is not only the total energy flux that is balanced. Detailed balance ensures that the energy flux at each frequency ω must be the same. Therefore the spectra $u(\omega)$ of a cavity and a blackbody are identical. (Imagine repeating the experiment, placing a filter between the cavity and blackbody that lets through frequency band ω to $\omega+d\omega$. The same argument applies.)

Blackbody summary

- Absorbs all light (no reflections, hence "black")
- Emits radiation according to Planck law and its temperature T
- Radiates as much energy as it absorbs: $J_{emit} = J_{absorb}$

KEY CONCEPT: Kirchhoff law

What about bodies that reflect light, i.e. objects that are not ideal blackbodies?



Go back to the previous situation. Imagine that for every 3 photons emitted by cavity that hit object, 1 is reflected back and 2 are absorbed (at one frequency ω)

To ensure net energy flux is 0, the object must emit 2 photons

So, we can write

$$\underset{\text{absorptivity (here 2/3)}}{P_{abs}(\omega) = \underset{\uparrow}{a(\omega)} P_{inc}(\omega)} \qquad \underset{\text{emissivity (here 2/3)}}{P_{emit}(\omega) = \underset{\uparrow}{e(\omega)} P_{inc}(\omega)} \qquad \underset{\text{reflectivity (here 1/3)}}{P_{refl}(\omega) = \underset{\uparrow}{r(\omega)} P_{inc}(\omega)}$$

Question 1: From energy balance considerations, relate *a*(*ω*), *e*(*ω*), and *r*(*ω*) to each other.

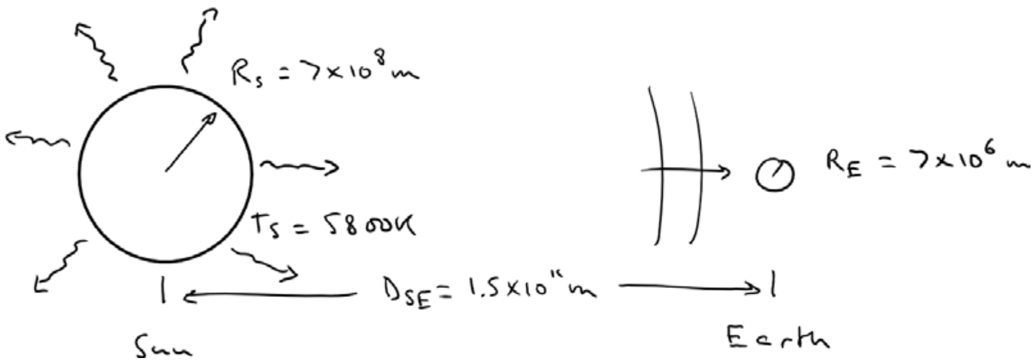
Since the 2 photons absorbed by the object must be emitted, $P_{emit}(\omega) = P_{abs}(\omega)$ and $e(\omega) = a(\omega)$
Also, by energy balance, the 3 incident photons are balanced by the 1 photon reflected and 2 photons emitted, so $P_{emit}(\omega) + P_{refl}(\omega) = P_{inc}(\omega)$ and

$$a(\omega) = e(\omega) = 1 - r(\omega)$$

This is called the Kirchhoff law. In words, it states that

- A good absorber is also a good emitter (e.g. a blackbody has $a = e = 1$)
- A good reflector is a poor absorber and emitter (e.g. $a = e = 0$)

Ex: The sun and the earth



The sun is well approximated by a blackbody. The radiated power at the sun's surface is:

$$\begin{aligned} P_S &= J_S 4\pi R_S^2 = \sigma_B 4\pi R_S^2 T_S^4 \\ &= 5.67\times10^{-8}\,\mathrm{Wm^{-2}K^{-4}} \cdot 4\pi \cdot (7\times10^8\,\mathrm{m})^2 (5800\mathrm{K})^4 \\ &= 3.9\times10^{26}\,\mathrm{W} \end{aligned}$$

$\uparrow$

"Solar luminosity"

Question 2: Find an expression for the energy flux density of solar radiation (i.e. the intensity of solar light) at the earth's atmosphere.

As light from the sun radiates outward, the power is spread over the surface area of a sphere of radius equal to the distance the light traveled. At the earth, the energy flux density is

$$J_{S,E} = \frac{P_S}{4\pi D_{SE}^2} = \frac{3.9\times10^{26}\,\mathrm{W}}{4\pi(1.5\times10^{11}\,\mathrm{m})^2} = 1370\,\mathrm{W/m^2}$$

This is the intensity of sunlight at the earth's atmosphere, called the "solar constant".

The incident power at the earth = $J_{S,E} \times$ (cross-sectional area of the earth): $P_{inc} = J_{S,E}\pi R_E^2$

At steady-state, there is no net energy flow into the earth, so the earth must emit radiation out. Assuming the earth is also an ideal blackbody ($a = e = 1$)

$$P_{abs} = P_{inc} = J_{S,E}\pi R_E^2 = P_{emit} = J_E 4\pi R_E^2 = \sigma_B 4\pi R_E^2 T_E^4$$

$$T_E = \left(\frac{J_{S,E}}{4\sigma_B}\right)^{1/4} = \left(\frac{1370\,\mathrm{W/m^2}}{4\cdot 5.67\times 10^{-8}\,\mathrm{W\cdot m^{-2}\cdot K^{-4}}}\right)^{1/4} = 279\,\mathrm{K}$$

or

$$T_E = \left(\frac{\cancel{\sigma_B}\cancel{4\pi} R_S^2 T_S^4}{4\,\cancel{\sigma_B}\cancel{4\pi} D_{SE}^2}\right)^{1/4} = \left(\frac{R_S}{4D_{SE}}\right)^{1/2} T_S = \left(\frac{7\times 10^6}{4\cdot 1.5\times 10^{11}}\right)^{1/2} 5800\,\mathrm{K} = 279\,\mathrm{K},$$

Not bad! Actual number is ~15 °C = 288 K

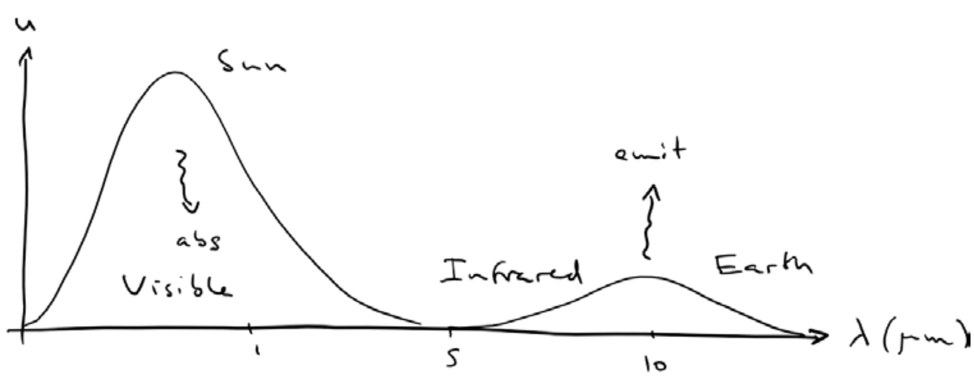
(Note that this is a situation where the sun and the earth are in steady-state, i.e. no net energy flux, but <u>not in thermal equilibrium</u>, since they are at different temperatures.)

Ultimately, the earth is not an ideal blackbody. It reflects ~30% of sunlight, so $P_{abs} = aP_{inc} = P_{emit}$ with a = 0.7 = e, and

$$T_E = \left(\frac{aJ_{S,E}}{4\sigma_B}\right)^{1/4} = \left(\frac{0.7\cdot 1370\,\mathrm{W/m^2}}{4\cdot 5.67\times 10^{-8}\,\mathrm{W\cdot m^{-2}\cdot K^{-4}}}\right)^{1/4} = 255\,\mathrm{K}$$

which is a frigid –18 °C!

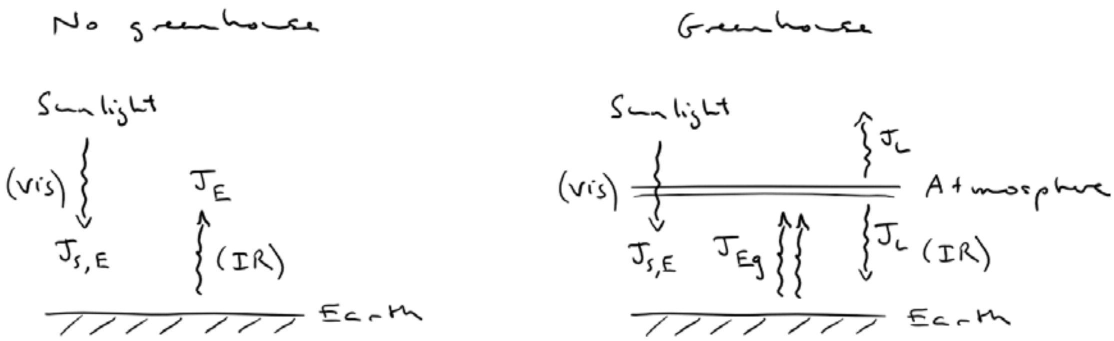
This simple model is incorrect. The problem is that a = e are strong functions of ω.
The Wein displacement law, $\hbar\omega_{\max} = 2.82k_BT$, tells us solar radiation is peaked in the near infrared (~880 nm), while earth's radiation is peaked in the far infrared (~10 µm) because $T_E \ll T_S$.



Earth absorbs visible light from the sun (some is reflected by the clouds), <u>but</u> has to re-emit it as IR light (because $T_E \ll T_S$). The problem is that the atmosphere absorbs IR light – this leads to the <u>greenhouse effect</u>.

Simple model of the greenhouse effect (from K & K):

Treat the atmosphere as a single layer above the earth's surface – transparent to visible light, absorbs IR light



The incident energy flux from the sun a the earth is still $J_{S,E}$. With greenhouse effect, the earth now emits energy flux J_{Eg} back into space and the atmosphere layer emits energy flux J_L into space <u>and</u> back to earth

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Question 3: Assuming the earth and atmosphere both attain a steady-state and using energy balance, derive an expression for the energy flux emitted by the earth J_{Eg} in terms of the solar flux $J_{S,E}$

At steady-state, the flux from space into the atmosphere must balance the flux out of the atmosphere into space: $J_{S,E} = J_L$ (and temperature of the atmosphere is 255 K)

Similarly, the flux out into the earth from space <u>and</u> from the atmosphere balance the flux out of the earth: $J_{S,E} + J_L = J_{Eg}$

Therefore $J_{Eg} = 2J_{S,E}$, compared to $J_E = J_{S,E}$ in the absence of the greenhouse effect.

Therefore,

$$\frac{J_{Eg}}{J_E} = 2 = \left(\frac{T_{Eg}}{T_E}\right)^4$$

$$T_{Eg} = 2^{1/4} T_E = 2^{1/4} \cdot 255\,\text{K} = 303\,\text{K} \quad (= 30\ \text{°C, a bit hot...})$$