Chapter 9

There are many equations which describe a certain characteristic behavior and may be applied to many different physical systems. Probably the most familiar to you so far is the harmonic oscillator equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

with solutions of the form:

$$x = A\cos(\omega t) + B\sin(\omega t)$$

We are now embarking on the investigation of another famous and ubiquetous equation, the wave equation. The wave equation has the form, in one dimension,

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

Or, in three dimensions,

$$\nabla^2 f - \frac{1}{v^2} f = 0$$

Solutions to the one dimensional wave equation have the form

$$f(z,t) = g(z - vt) + h(z + vt)$$

where $g(z-vt)$ is some function that depends on z and t in the combination $z - vt$, and similarly for $h(z+vt)$. Notice (as is the case for the harmonic oscillator) that a linear combination of solutions is also a solution to the wave equation. A sinusoidal function, such as

$$f(z,t) = A \cos[k(z - vt) + \delta]$$

has a form which satisfies the wave equation. We will find that such sinusoidal oscillations are the appropriate representation for electromagnetic radiation.

Before moving to the specific consideration of sinusoidal solutions, I repeat here the proof (Griffiths p. 366) that a general function of the form $g(z-vt)$ satisfies the wave equation.

We need to show that $\frac{\partial^2 g(z-vt)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 g(z-vt)}{\partial t^2}$

To do this we define $u \equiv z - vt$ for convenience, and do repeated applications of the chain rule:

$$\frac{\partial g}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}\frac{\partial(z-vt)}{\partial z} = \frac{dg}{du}$$

$$\frac{\partial^2 g}{\partial z^2} = \left(\frac{d\left(\frac{\partial g}{\partial z}\right)}{du}\right)\left(\frac{\partial u}{\partial z}\right) = \left(\frac{d}{du}\left(\frac{dg}{du}\right)\right)\left(\frac{\partial(z-vt)}{\partial z}\right) = \frac{d^2 g}{du^2}$$

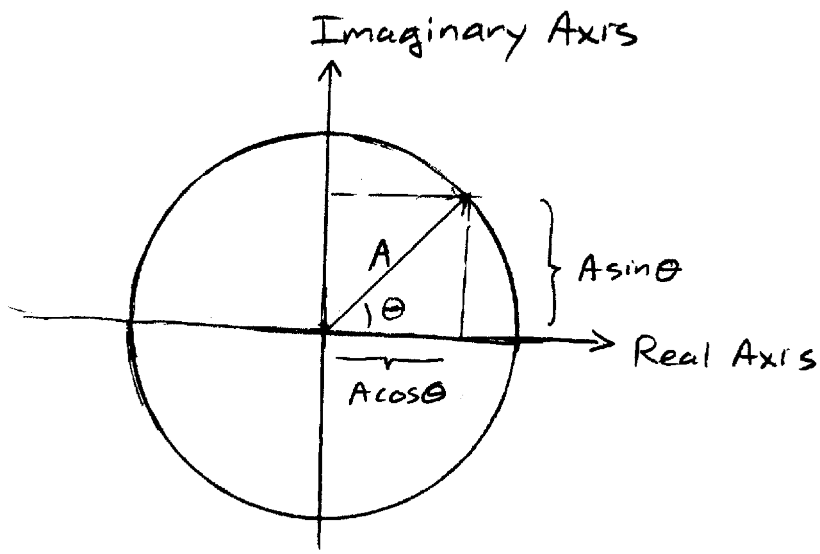
$$\frac{\partial g}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = \frac{dg}{du}\frac{\partial(z-vt)}{\partial t} = -v\frac{dg}{\partial u}$$

$$\frac{\partial^2 g}{\partial t^2} = \left(\frac{d\left(\frac{\partial g}{\partial t}\right)}{du}\right)\left(\frac{\partial u}{\partial t}\right) = \left(-v\frac{d^2 g}{du^2}\right)\left(\frac{\partial(z-vt)}{\partial t}\right) = v^2\frac{d^2 g}{du^2}$$

Comparing, we see that

$$\frac{\partial^2 g}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2}$$

A little review:



$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{Euler Relation}$$

$\mathrm{Re}[Ae^{i\theta}] = A\cos\theta$ projection on real axis

$\mathrm{Im}[Ae^{i\theta}] = A\sin\theta$ projection on imaginary axis

$i = \sqrt{-1}$

Cartesian to polar representation:

$$x + iy = re^{i\theta} \quad \left\{ \begin{array}{l} r = x^2 + y^2 \\ \theta = \tan^{-1}(y/x) \\ x = r\cos\theta \\ y = r\sin\theta \end{array} \right.$$

Sinusoidal solutions to the wave equation may be represented by a complex wave function;

$$f(z,t) = A\cos[\theta(z,t)+\delta]$$

$$\Rightarrow \quad \tilde{f}(z,t) = A e^{i\delta} e^{i\theta(z,t)}$$

$$= \tilde{A} e^{i\theta(z,t)}$$

As long as it is understood that the real part of the complex wave function is what corresponds to the actual physical system.

A few more reminders:

$$f(z,t) = A \cos[kz - kvt]$$

Suppose we stop time and look at the spatial variation of the wave at that one instant. One wavelength λ of the oscillation corresponds to a phase advance of 2π. So, $2\pi = k\lambda \rightarrow k = \frac{2\pi}{\lambda}$

k is the wave number.

Now, suppose we let time roll but look at one location, z. The time it takes for a complete oscillation at z is one period, T. One complete oscillation is a phase advance of 2π radians, so,

$$k v T = 2\pi$$

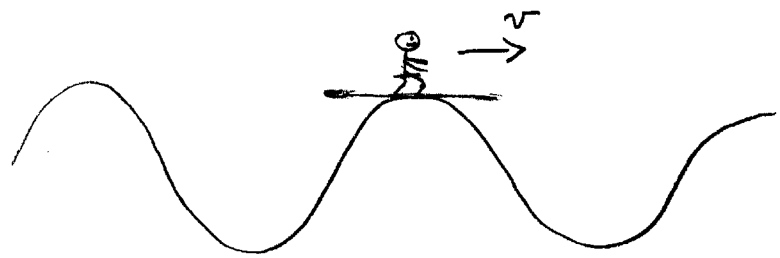
$$v = \left(\frac{2\pi}{k}\right)\left(\frac{1}{T}\right) = \frac{\lambda}{T}$$

Also, $T = 1/f = \frac{2\pi}{\omega}$ (since $\omega = 2\pi f$)

frequency (f) angular frequency (ω)

$$v = \lambda f = \omega / k$$

v is called the phase velocity. If you were surfing on a crest of the wave, you would be traveling at velocity v.



Now, we'll show that when there are no source terms ($\rho, \vec{J}$) in Maxwell's equations, that the fields ($\vec{E}, \vec{B}$) are described by wave equations.

No Sources:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad\qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

To get the wave equation (for $\vec{E}$ or $\vec{B}$) the equations must be written with <u>only</u> $\vec{E}$ or <u>only</u> $\vec{B}$. Maxwell's curl equations are first order, coupled differential equations. Taking a derivative of both sides of a curl equation (either taking the curl as Griffiths does, or the time derivative) will yield a 2nd order equation. Substitution from the other first order equation then produces an equation (2nd order) in $\vec{E}$ or $\vec{B}$ only, which has the form of a wave equation.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) = -\frac{\partial^2 \vec{B}}{\partial t^2} \qquad\qquad \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times \frac{\partial \vec{E}}{\partial t} = -\frac{\partial^2 \vec{B}}{\partial t^2} \qquad\qquad \nabla \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad\qquad \vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla(\cancelto{0}{\nabla \cdot \vec{B}}) - \nabla^2 \vec{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad\qquad \vec{\nabla}(\cancelto{0}{\vec{\nabla} \cdot \vec{E}}) - \nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \qquad\qquad \nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Three dimensional wave equations!

$$\nabla^2 f - \frac{1}{v^2} \frac{\partial^2 f}{\partial t} = 0 \quad \Rightarrow \quad \mu_0 \epsilon_0 = \frac{1}{v^2} = \frac{1}{c^2}$$

Wave equations for $\vec{E}$ & $\vec{B}$ follow directly from Maxwell's equations in the absence of sources.

The speed of propagation of these waves is:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \left[(8.85 \times 10^{-12})(4\pi \times 10^{-7})\right]^{-1/2} = 2.99 \times 10^{8}\ \mathrm{m/s}$$

Electromagnetic radiation in a vacuum travels with the speed of light! → Light is em radiation!

Using Maxwell's equations we can also find:

1) $\vec{E}$ & $\vec{B}$ oscillate transversely with respect to the direction of propagation

2) $\vec{E}$ & $\vec{B}$ are mutually perpendicular

To show 1) use $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$

Let's show that $\vec{E}$ is transverse to the direction of propagation, which we choose to be the $\hat{z}$ direction.

$$\vec{E} = \tilde{E}_{0x} e^{i(kz - \omega t)} \hat{x} + \tilde{E}_{0y} e^{i(kz-\omega t)} \hat{y} + \tilde{E}_{0z} e^{i(kz-\omega t)} \hat{z}$$

$$\vec{\nabla} \cdot \vec{E} = \underbrace{\cancelto{0}{\frac{\partial E_x}{\partial x}} + \cancelto{0}{\frac{\partial E_y}{\partial y}}}_{\tilde{E}_{0x}\ \&\ \tilde{E}_{0y}\ \text{are constant}} + \frac{\partial E_z}{\partial z} = \tilde{E}_{0z} (ik) e^{i(kz-\omega t)} = 0$$

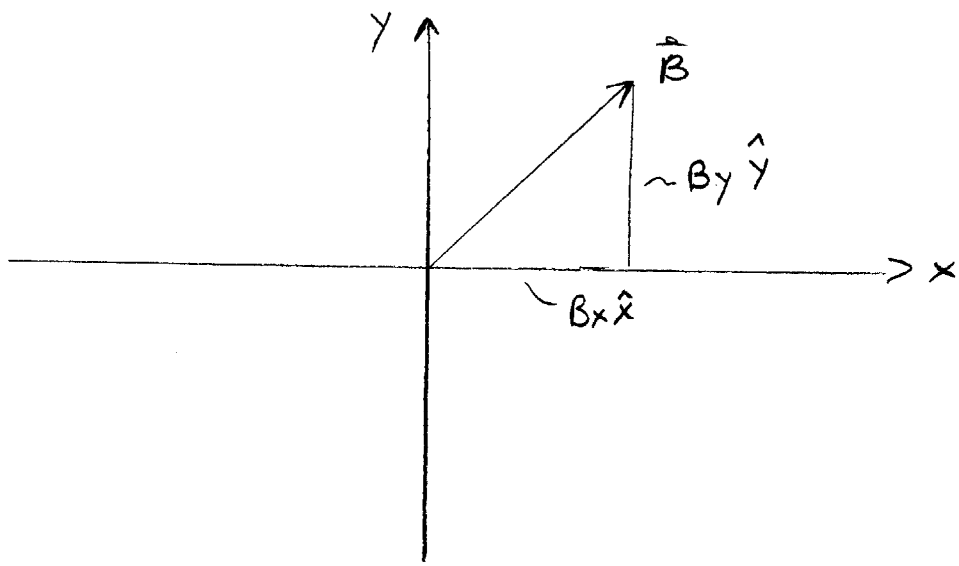
Therefore, $\tilde{E}_{0z} = 0$ otherwise $\vec{\nabla} \cdot \vec{E} = 0$ is not satisfied

Now, lets show that $\vec{E}$ and $\vec{B}$ are mutually orthogonal. Use $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\left.\begin{aligned} \vec{E} &= \tilde{E}_{ox}\, e^{i\theta}\, \hat{x} + E_{oy}\, e^{i\theta}\, \hat{y} \\ \vec{B} &= \tilde{B}_{ox}\, e^{i\theta}\, \hat{x} + \tilde{B}_{oy}\, e^{i\theta}\, \hat{y} \end{aligned}\right\} \begin{array}{l} \text{since} \\ B_{oz} = E_{oz} = 0 \end{array}$$

What the heck does this mean?

Since $\vec{E}$ (or $\vec{B}$) is transverse to the direction of propagation, we can draw an $\vec{E}$ or $\vec{B}$ vector at any instant of time in the transverse plane (xy plane in this case).



Whatever arbitrary direction $\vec{B}$ has, it may be resolved into x and y components. Sometimes $\vec{B}$ will be zero at a certain point in time and space, when $B = Re\left[\tilde{B}_0 e^{i(kz-\omega t)}\right] = 0$.

For propagation in the z-direction:

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{ox}e^{i\theta} & E_{oy}e^{i\theta} & 0 \end{vmatrix}$$

$$\theta(z,t) = kz - \omega t$$

$$\vec{\nabla}\times\vec{E} = \hat{x}\left(-E_{oy}\frac{\partial e^{i\theta}}{\partial z}\right) + \hat{y}\left(E_{ox}\frac{\partial e^{i\theta}}{\partial z}\right)$$

$$= -ikE_{oy}\hat{x} + ikE_{ox}\hat{y}$$

Meanwhile:

$$-\frac{\partial\vec{B}}{\partial t} = -B_{ox}(-i\omega)\hat{x} - B_{oy}(-i\omega)\hat{y}$$

$$= i\omega B_{ox}\hat{x} + i\omega B_{oy}\hat{y}$$

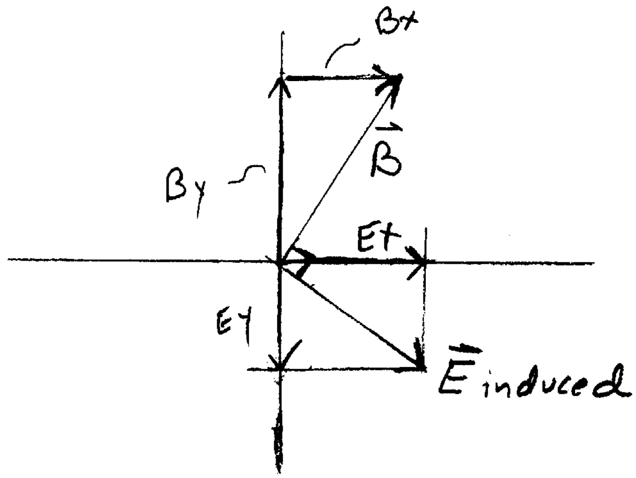
Thus, $-ikE_{oy} = i\omega B_{ox} \rightarrow B_{ox} = -k/\omega\, E_{oy}$

$ikE_{ox} = i\omega B_{oy} \rightarrow B_{oy} = k/\omega\, E_{ox}$

In summary: $\vec{B} = k/\omega\,(\hat{z}\times\vec{E})$

$$\hat{z}\times\hat{y} = -\hat{x}, \quad \hat{z}\times\hat{x} = \hat{y}$$

Graphically,



Whatever direction $\vec{B}$ has, $\vec{E}$ induced is perpendicular to $\vec{B}$. $-E_y$ is induced by B_x, E_x is induced by B_y, $\vec{E}$ is the sum of $\vec{E}_x + \vec{E}_y$.

$$\vec{E} \cdot \vec{B} = E_{ox} \cdot B_{ox} + E_{oy} \cdot B_{oy}$$

$$= (\omega/k \, B_{oy}) B_{ox} + (-\omega/k \, B_{ox})(B_{oy})$$

$$= 0$$

Generalize to any direction of propagation

$$\tilde{E}(\vec{r}, t) = \tilde{E}_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)} \, \hat{n}$$

$\hat{k}$ in direction of propagation

$\hat{n}$ in direction of $\vec{E}$

$$\tilde{B}(\vec{r}, t) = \frac{1}{c} \tilde{E}_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)} \, (\hat{k} \times \hat{n})$$

Polarization

When the electric field of an em wave oscillates in a specific direction $\hat{n}$, that vector defines a plane of vibration, and the wave is said to be linearly or plane polarized. A combination of horizontally and vertically polarized waves of the same phase will result in a plane polarized wave in direction $\hat{n}$, where;

$$\hat{n} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\theta \equiv \text{polarization angle}$$

Electromagnetic radiation from the sun is not polarized. Radiation produced by man is sometimes polarized. Radiation reflected from a surface is often plane polarized.

Another type of polarization produced by man is circular polarization. If the components of horizontally and vertically polarized waves are equal, but out of phase by 90°, the result is a circularly polarized wave.

The advantage of circular polarization is that the strength (amplitude) of the net traveling wave is constant. For example, a right-handed circularly polarized voltage wave is given by components:

$$V_x = V_o \cos(\omega t - kz)$$

$$V_y = V_o \cos(\omega t - kz - \pi/2) = V_o \sin(\omega t - kz)$$

$$\text{Net strength} = V_{net} = (V_x^2 + V_y^2)^{1/2}$$

$$V_{net} = V_o\,(\sin^2(\omega t - kz) + \cos^2(\omega t - kz))^{1/2}$$

$$V_{net} = V_o \quad \Rightarrow \quad P = \frac{V_o^2}{R}$$

The power of this circularly polarized wave is constant in time, whereas the power of a single varying sinusoid fluctuates;

$$P = \frac{V_o^2}{R}\cos^2(\omega t - kz) = \frac{V_o^2}{R}(1 + \cos 2(\omega t - kz))$$

P has a DC component plus a fluctuation at twice the frequency of the voltage oscillation.