Today's outline - January 12, 2023

- State space of a qubit
- Bloch sphere
- Bases, measurement & phases
- Quirk basics

Reading Assignment: Reiffel: 3.2-3.3 Wong: 4.2.3-4.3.2

Homework Assignment #01:
due Thursday, January 19, 2023

Qubit review

A qubit is a two-state quantum system that can be modeled as a superposition of two linearly independent states called the basis of the space in which the qubit exists

$$|q\rangle = a|0\rangle + b|1\rangle, \qquad a = |a|e^{i\alpha}, \qquad b = |b|e^{i\beta}$$

Basis vectors must be orthonormal, that is for the basis $\{|\beta_1\rangle, |\beta_2\rangle\}$, the inner product must be $\langle\beta_i|\beta_j\rangle = \delta_{ij}$

The ket, $|\alpha\rangle$, corresponds to a column vector, α, in linear algebra while a bra $\langle\alpha|$ is its conjugate transpose, $\alpha^\dagger$, a row vector

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad \langle\alpha| = \begin{pmatrix} \overline{a_1} & \cdots & \overline{a_n} \end{pmatrix}$$

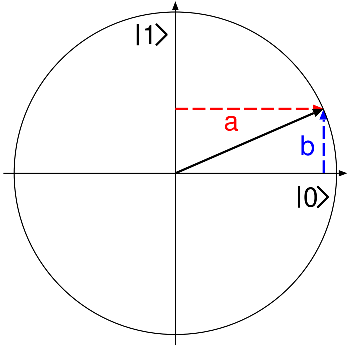
In the standard basis, $\{|0\rangle, |1\rangle\}$, the vector $|v\rangle = a|0\rangle + b|1\rangle$ is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Qubit representation

A qubit is represented by a generalized quantum superposition of two orthonormal states, where a and b are complex constants

Two qubits are identical if related by a complex constant with modulus 1



$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad 1 = |a|^2 + |b|^2$$

$$a = |a|e^{i\alpha}, \quad b = |b|e^{i\beta}$$

$$a|0\rangle + b|1\rangle \equiv e^{i\theta}(a'|0\rangle + b'|1\rangle)$$

The global phase, $e^{i\theta}$ of a qubit cannot be measured but the relative phase, $e^{i\phi}$ can

$$\frac{b}{a} = \frac{|b|}{|a|}e^{i(\beta-\alpha)} = \frac{|b|}{|a|}e^{i\phi}$$

Changing the relative phase in a superposition changes the superposition itself

$$a|0\rangle + b|1\rangle \neq |a||0\rangle + e^{i\phi}|b||1\rangle$$

Qubit complex plane

Starting with the general representation of a qubit

$$|\psi\rangle = e^{i\phi}|a||0\rangle + |b||1\rangle$$

we define four additional special orthogonal single qubit states

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |\nwarrow\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |\nearrow\rangle$$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|\bar{i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

The $\{|+\rangle, |-\rangle\}$ basis is also called the Hadamard basis and is sometimes represented by $\{|\nwarrow\rangle, |\nearrow\rangle\}$

These qubits can be mapped onto the complex plane by defining the mapping

$$a|0\rangle + b|1\rangle \mapsto \alpha = \frac{b}{a}$$

Which results in the mappings

$$\alpha \mapsto \frac{1}{\sqrt{1+|\alpha|^2}}|0\rangle + \frac{\alpha}{\sqrt{1+|\alpha|^2}}|1\rangle$$

$$|+\rangle \mapsto +1, \quad |-\rangle \mapsto -1, \quad |i\rangle \mapsto +i, \quad |\bar{i}\rangle \mapsto -i, \quad |0\rangle \mapsto 0, \quad |1\rangle \mapsto ??$$

the problem with $|1\rangle$ can be solved by extending the complex plane

Extended complex plane

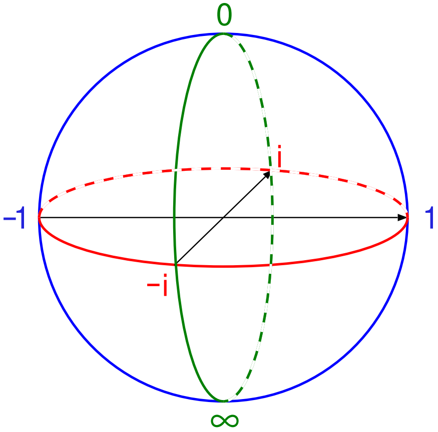
The qubit basis vectors which are properly mapped can be drawn on a unit circle in the complex plane

By adding an extra point called ∞ and defining the mapping: $|1\rangle \mapsto \infty$

Since each of the qubit basis vectors are normalized they have a magnitude of 1 the extended complex plane can be mapped to a sphere of radius 1

The general qubit can also be represented as a function of θ and ϕ

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix}$$



This maps an arbitrary single qubit state to a point on the surface of the Bloch sphere

Spherical coordinates & the Bloch sphere

Given the spherical representation of a general qubit, the three basis sets can easily be mapped onto the surface of the Bloch sphere

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$$

$$|0\rangle = 1|0\rangle + 0|1\rangle \quad \longmapsto \quad \theta = 0, \phi = 0$$

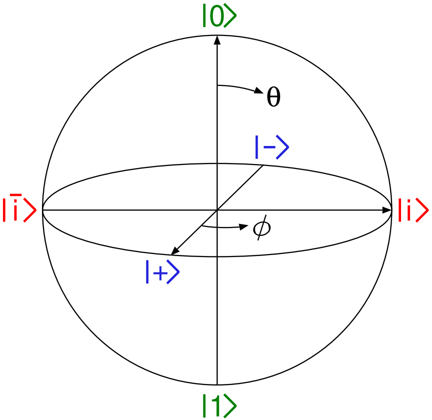
$$|1\rangle = 0|0\rangle + 1|1\rangle \quad \longmapsto \quad \theta = \pi, \phi = 0$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad \longmapsto \quad \theta = \frac{\pi}{2}, \phi = 0$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \quad \longmapsto \quad \theta = \frac{\pi}{2}, \phi = \pi$$

$$|i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \quad \longmapsto \quad \theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}$$

$$|\bar{i}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right) \quad \longmapsto \quad \theta = \frac{\pi}{2}, \phi = \frac{3\pi}{2}$$



The points in the interior of the Bloch sphere have meaning for quantum information processing

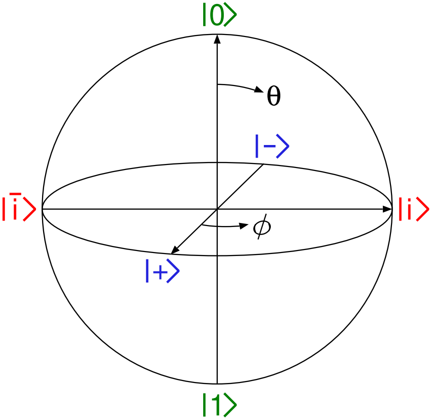
Stereographic projection & the Bloch sphere

An alternative model is that of the stereographic projection which posits that $\alpha = s + it$ is complex

$$(s,t) \mapsto \left(\frac{2s}{|\alpha|^2+1}, \frac{2t}{|\alpha|^2+1}, \frac{1-|\alpha|^2}{|\alpha|^2+1} \right)$$

Each of the 6 qubit basis states are mapped as

$$
\begin{aligned}
|0\rangle &\longmapsto (0,0,1) \\
|1\rangle &\longmapsto (0,0,\infty) \longmapsto (0,0,-1) \\
|+\rangle &\longmapsto (1,0,0) \\
|-\rangle &\longmapsto (-1,0,0) \\
|i\rangle &\longmapsto (0,1,0) \\
|\bar{i}\rangle &\longmapsto (0,-1,0)
\end{aligned}
$$



Bases

The standard basis is $\{|0\rangle, |1\rangle\}$ but any two "kets" which lie opposite to each other on the Bloch sphere may be used as a "basis" for measurement

Consider the qubit

$$|q\rangle = \tfrac{2}{3}|0\rangle + \tfrac{1-2i}{3}|1\rangle$$

When measured, the probability of getting $|0\rangle$ is the modulus squared of the coefficient of $|0\rangle$

$$P_0 = \left|\tfrac{2}{3}\right|^2 = \tfrac{4}{9}$$

The probability of getting $|1\rangle$ is thus

$$P_1 = \left|\tfrac{1-2i}{3}\right|^2 = \tfrac{1-2i}{3}\tfrac{1+2i}{3} = \tfrac{1+4}{9} = \tfrac{5}{9}$$

The qubit can be expressed in terms of the $|+\rangle, |-\rangle$ basis by converting to the standard basis

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|+\rangle + |-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + \cancel{|1\rangle}) + \tfrac{1}{\sqrt{2}}(|0\rangle - \cancel{|1\rangle}) = \tfrac{2}{\sqrt{2}}|0\rangle \;\longrightarrow\; |0\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|+\rangle - |-\rangle = \tfrac{1}{\sqrt{2}}(\cancel{|0\rangle} + |1\rangle) - \tfrac{1}{\sqrt{2}}(\cancel{|0\rangle} - |1\rangle) = \tfrac{2}{\sqrt{2}}|1\rangle \;\longrightarrow\; |1\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

Alternate bases

$$|q\rangle = \tfrac{2}{3}|0\rangle + \tfrac{1-2i}{3}|1\rangle, \qquad |0\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \qquad |1\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

$$|q\rangle = \tfrac{2}{3}\tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + \tfrac{1-2i}{3}\tfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = \tfrac{3-2i}{3\sqrt{2}}|+\rangle + \tfrac{1+2i}{3\sqrt{2}}|-\rangle$$

Thus the probabilities of finding $|q\rangle$ in the $|+\rangle$ and state are

$$P_+ = \tfrac{3-2i}{3\sqrt{2}}\tfrac{3+2i}{3\sqrt{2}} = \tfrac{9+4}{18} = \tfrac{13}{18}, \qquad P_- = \tfrac{1+2i}{3\sqrt{2}}\tfrac{1-2i}{3\sqrt{2}} = \tfrac{1+4}{18} = \tfrac{5}{18}$$

Similarly for the $|i\rangle, |\bar{i}\rangle$ basis

$$|i\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \qquad |\bar{i}\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$|0\rangle = \tfrac{1}{\sqrt{2}}(|i\rangle + |\bar{i}\rangle), \qquad |1\rangle = \tfrac{-i}{\sqrt{2}}(|i\rangle - |\bar{i}\rangle)$$

$$|q\rangle = \tfrac{2}{3}\tfrac{1}{\sqrt{2}}(|i\rangle + |\bar{i}\rangle) + \tfrac{1-2i}{3}\tfrac{-i}{\sqrt{2}}(|i\rangle - |\bar{i}\rangle) = \tfrac{-i}{3\sqrt{2}}|i\rangle + \tfrac{4+i}{3\sqrt{2}}|\bar{i}\rangle$$

$$P_i = \tfrac{-i}{3\sqrt{2}}\tfrac{+i}{3\sqrt{2}} = \tfrac{1}{18}, \qquad P_{\bar{i}} = \tfrac{4+i}{3\sqrt{2}}\tfrac{4-i}{3\sqrt{2}} = \tfrac{16+1}{18} = \tfrac{17}{18}$$

Global phases

Apply a global phase shift $|q\rangle \longrightarrow |q\rangle e^{i\theta}$ then compute the probabilities again

$$e^{i\theta}|q\rangle = e^{i\theta}\left(\tfrac{2}{3}|0\rangle + \tfrac{1-2i}{3}|1\rangle\right) = e^{i\theta}\tfrac{2}{3}|0\rangle + e^{i\theta}\tfrac{1-2i}{3}|1\rangle$$

$$P_0 = \left|e^{i\theta}\tfrac{2}{3}\langle 0|0\rangle\right|^2 = \left(\cancel{e^{-i\theta}}\tfrac{2}{3}\right)\left(\cancel{e^{i\theta}}\tfrac{2}{3}\right) = \tfrac{4}{9}$$

$$P_1 = \left|e^{i\theta}\tfrac{1-2i}{3}\langle 1|1\rangle\right|^2 = \left(\cancel{e^{-i\theta}}\tfrac{1+2i}{3}\right)\left(\cancel{e^{i\theta}}\tfrac{1-2i}{3}\right) = \tfrac{1+4}{9} = \tfrac{5}{9}$$

Computing the probabilities in the $|+\rangle, |-\rangle$ basis

$$e^{i\theta}q_0 = e^{i\theta}\tfrac{3-2i}{3\sqrt{2}}|+\rangle + e^{i\theta}\tfrac{1+2i}{3\sqrt{2}}|-\rangle$$

$$P_+ = \left|e^{i\theta}\tfrac{3-2i}{3\sqrt{2}}\langle +|+\rangle\right|^2 = \left(e^{-i\theta}\tfrac{3+2i}{3\sqrt{2}}\right)\left(e^{i\theta}\tfrac{3-2i}{3\sqrt{2}}\right) = \tfrac{9+4}{18} = \tfrac{13}{18}$$

$$P_- = \left|e^{i\theta}\tfrac{1+2i}{3\sqrt{2}}\langle -|-\rangle\right|^2 = \left(e^{-i\theta}\tfrac{1-2i}{3\sqrt{2}}\right)\left(e^{i\theta}\tfrac{1+2i}{3\sqrt{2}}\right) = \tfrac{1+4}{18} = \tfrac{5}{18}$$

Global phases do not change the qubit and have no physical significance

Relative phases

A relative phase applied to the same qubit gives a different answer depending on the basis used for measurement

$$|q'\rangle = e^{i\theta}\tfrac{2}{3}|0\rangle + \tfrac{1-2i}{3}|1\rangle = e^{i\theta}\tfrac{2}{3\sqrt{2}}(|+\rangle + |-\rangle) + \tfrac{1-2i}{3\sqrt{2}}(|+\rangle - |-\rangle)$$

$$= \left(e^{i\theta}\tfrac{2}{3\sqrt{2}} + \tfrac{1-2i}{3\sqrt{2}}\right)|+\rangle + \left(e^{i\theta}\tfrac{2}{3\sqrt{2}} - \tfrac{1-2i}{3\sqrt{2}}\right)|-\rangle$$

$$P_0 = \left|e^{i\theta}\tfrac{2}{3}\langle 0|0\rangle\right|^2 = \tfrac{4}{9}, \qquad P_1 = \left|\tfrac{1-2i}{3}\langle 1|1\rangle\right|^2 = \tfrac{1+4}{9} = \tfrac{5}{9}$$

$$P_+ = \left|\left(e^{i\theta}\tfrac{2}{3\sqrt{2}} + \tfrac{1-2i}{3\sqrt{2}}\right)\right|^2 = \cancel{e^{-i\theta}}\tfrac{2}{3\sqrt{2}}\cancel{e^{i\theta}}\tfrac{2}{3\sqrt{2}} + e^{-i\theta}\tfrac{2}{3\sqrt{2}}\tfrac{1-2i}{3\sqrt{2}} + e^{i\theta}\tfrac{2}{3\sqrt{2}}\tfrac{1+2i}{3\sqrt{2}} + \tfrac{1+2i}{3\sqrt{2}}\tfrac{1-2i}{3\sqrt{2}}$$

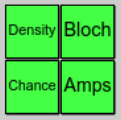
$$= \tfrac{4}{18} + \tfrac{5}{18} + e^{-i\theta}\tfrac{2}{3\sqrt{2}}\tfrac{1-2i}{3\sqrt{2}} + e^{i\theta}\tfrac{2}{3\sqrt{2}}\tfrac{1+2i}{3\sqrt{2}} = \tfrac{1}{2} + \tfrac{1}{9}\left(e^{i\theta} + e^{-i\theta}\right) + \tfrac{i}{9}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= \tfrac{1}{2} + \tfrac{2}{9}(\cos\theta - \sin\theta)$$

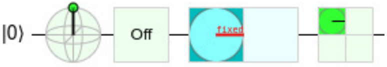
$$P_- = \left|\left(e^{i\theta}\tfrac{2}{3\sqrt{2}} - \tfrac{1-2i}{3\sqrt{2}}\right)\right|^2 = \cancel{e^{-i\theta}}\tfrac{2}{3\sqrt{2}}\cancel{e^{i\theta}}\tfrac{2}{3\sqrt{2}} - e^{-i\theta}\tfrac{2}{3\sqrt{2}}\tfrac{1-2i}{3\sqrt{2}} - e^{i\theta}\tfrac{2}{3\sqrt{2}}\tfrac{1+2i}{3\sqrt{2}} + \tfrac{1+2i}{3\sqrt{2}}\tfrac{1-2i}{3\sqrt{2}}$$

$$= \tfrac{1}{2} - \tfrac{1}{9}\left(e^{i\theta} + e^{-i\theta}\right) - \tfrac{i}{9}\left(e^{i\theta} - e^{-i\theta}\right) = \tfrac{1}{2} - \tfrac{2}{9}(\cos\theta - \sin\theta)$$

Quirk: States and displays



Quirk displays



The Chance display gives the probability of the qubit being a $|1\rangle$

The Bloch display is a visualization of the Bloch sphere representation of the qubit

The Amps display gives the amplitude and relative phase of the $|0\rangle$ and $|1\rangle$ components of the qubit

The Density display describes the density matrix of the qubit including any superpositions

Each of the 6 orthogonal basis vectors of the Bloch sphere displays differently using these four displays

$$|0\rangle = 1|0\rangle + 0|1\rangle \qquad |1\rangle = 0|0\rangle + 1|1\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad |\bar{i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$