Synchrotron Radiation

There are still a couple of generalizations we can make. 1st, we can consider more general motion. 2nd, we might like to know the frequency distribution of radiation emitted by relativistic accelerating charges. (So far we have just discussed the total power and some angular distributions.)

Previously, we saw that the total power radiated is $P \propto \frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \frac{d\vec{p}}{d\tau}\cdot\frac{d\vec{p}}{d\tau} - \beta^2\left(\frac{d|\vec{p}|}{d\tau}\right)^2$.

Compare circular & linear accel:

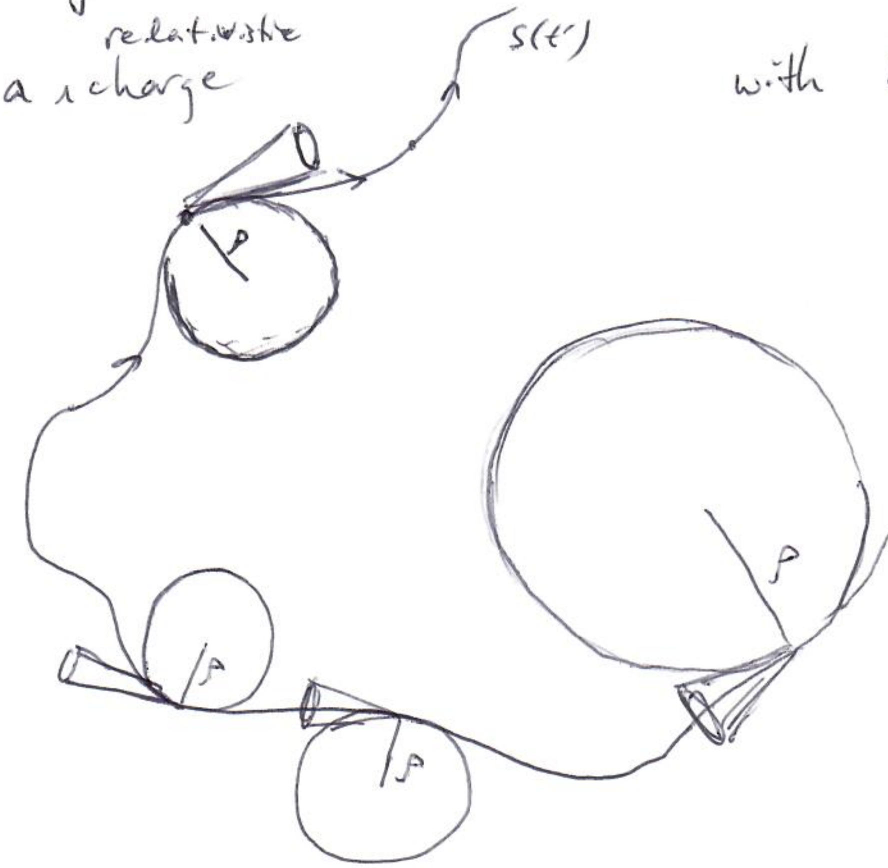
linear has $\left|\frac{d\vec{p}}{d\tau}\right| = \frac{d|p|}{d\tau}$, $P_- \sim (1-\beta^2)\left|\frac{d\vec{p}}{d\tau}\right|^2$

circular has $\left|\frac{d\vec{p}}{d\tau}\right| \gg \frac{d|p|}{d\tau} = 0$, $P_0 \sim \left|\frac{d\vec{p}}{d\tau}\right|^2$

For a given applied force, $\frac{P_0}{P_-} = \gamma^2$

This means we can neglect parallel forces when considering radiation from highly relativistic charges in arbitrary motion.

More precisely, the instantaneous radiation emission from a relativistic charge with $\vec{a}_\perp$ and $\vec{a}_\parallel$



is mainly due to $\vec{a}_\perp$, which satisfies $a_\perp = \frac{v^2}{\rho} \approx \frac{c^2}{\rho}$ (rel)

for path radius of curvature ρ (at that instant.)

So the emission spectrum is just like that of a particle moving in a circle of that radius.

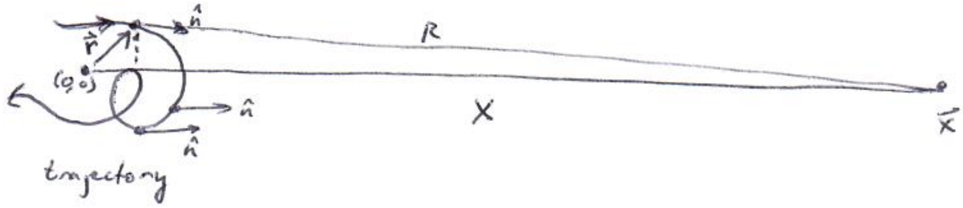
It is strongly peaked in a cone of size $\theta \sim 1/\gamma$ in the instantaneous forward direction: like a searchlight.

Time to work out the frequency distribution.

We need the Fourier transform of the electric field:

$$\tilde{\vec{E}}(\vec{x}, \omega) = \frac{e}{4\pi\epsilon_0 c} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \left[\frac{\hat{n} \times \left([\hat{n} - \vec{\beta}] \times \dot{\vec{\beta}} \right)}{(1 - \vec{\beta}\cdot\hat{n})^3 R} \right]_{ret}$$

We will assume the observer is always far from the source charge. Then we can make the following appx: Define a fiducial origin close to the charge,



and let the distance to the charge and the observer be $|\vec{r}|$ and x, respectively. Then we can approximate $\hat{n}$ as a constant and $R(t') \simeq x - \hat{n}\cdot\vec{r}(t')$. Here $x \gg |\hat{n}\cdot\vec{r}|$.

So $\quad t'_{ret} = t - \dfrac{R(t'_{ret})}{c} \approx t - \dfrac{x}{c} + \dfrac{\hat{n}\cdot\vec{r}(t'_{ret})}{c}$

and (take d/dt'_{ret} on both sides) $\quad 1 \approx \dfrac{dt}{dt'_{ret}} + \hat{n}\cdot\vec{\beta}(t'_{ret}) \quad$ using $\dfrac{d\vec{r}}{dt'} = c\vec{\beta}$

or $\quad \dfrac{dt}{dt'_{ret}} = 1 - \hat{n}\cdot\vec{\beta}(t'_{ret}) \quad$ (same as our previous Doppler factor)

Now we change integration variables from $t \to t'_{ret}$.

$$\vec{\tilde{E}}(\vec{x},\omega) = \frac{e}{4\pi\epsilon_0 c}\int_{-\infty}^{\infty} dt'_{ret}\ \frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\vec{\beta}\cdot\hat{n})^2 R}\ e^{-i\omega\left(t'_{ret}+\frac{x}{c}-\frac{\hat{n}\cdot\vec{r}}{c}\right)}$$

for notational brevity, relabel t'_{ret} as t going forward - it's just an integration variable:

$$\vec{\tilde{E}}(\vec{x},\omega) = \frac{e\, e^{-i\omega\frac{x}{c}}}{4\pi\epsilon_0 c}\int_{-\infty}^{\infty} dt\ \frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\vec{\beta}\cdot\hat{n})^2 R}\ e^{-i\omega\left(t-\frac{\hat{n}\cdot\vec{r}}{c}\right)}$$

Next we use the $\hat{n}\approx$ const appx to write as a total derivative

$$\frac{d}{dt}\left(\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right) \simeq \frac{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right]}{(1-\vec{\beta}\cdot\hat{n})^2}$$

work it out explicitly & use vector identities to prove.

$$\vec{\tilde{E}}(\vec{x},\omega) = \frac{e\, e^{-i\omega x/c}}{4\pi\epsilon_0 c}\int_{-\infty}^{\infty} dt\ \frac{d}{dt}\left\{\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right\}\underbrace{\left(\frac{e^{-i\omega\left(t-\frac{\hat{n}\cdot\vec{r}}{c}\right)}}{x-\hat{n}\cdot\vec{r}(t)}\right)}$$

Now integrate by parts to move $\frac{d}{dt}$ onto

$$\frac{d}{dt}\left(\frac{e^{-i\omega\left(t-\frac{\hat{n}\cdot\vec{r}}{c}\right)}}{x-\hat{n}\cdot\vec{r}}\right) \cong \left(\underbrace{\frac{-i\omega(1-\hat{n}\cdot\vec{\beta})}{x-\hat{n}\cdot\vec{r}}}_{O(1/R)} + \underbrace{\frac{\hat{n}\cdot\vec{\beta}}{(x-\hat{n}\cdot\vec{r})^2}}_{O(1/R^2),\ \text{drop}}\right) e^{-i\omega(t-\hat{n}\cdot\vec{r}/c)}$$

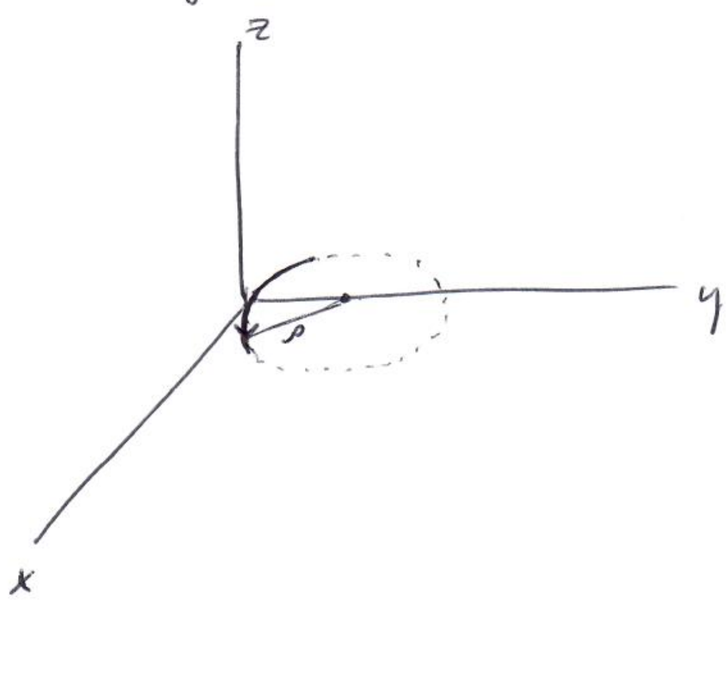
$$\simeq \frac{-i\omega}{R}\left(1-\hat{n}\cdot\vec{\beta}\right) e^{-i\omega(t-\hat{n}\cdot\vec{r}/c)}$$

The factors of $1-\vec{\beta}\cdot\hat{n}$ cancel and we are left with

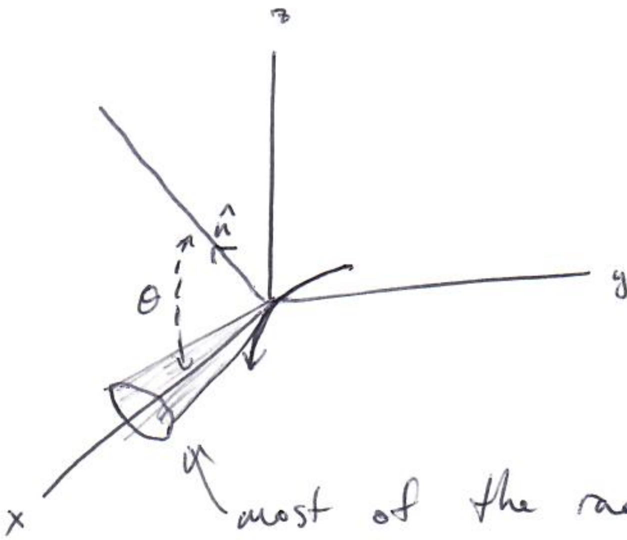
$$\tilde{\vec{E}}(\omega,\vec{x}) = \frac{i\omega e\, e^{-i\omega x/c}}{4\pi\epsilon_0 c}\int_{-\infty}^{\infty} dt\, \frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{R(t)}\, e^{-i\omega(t-\hat{n}\cdot\vec{r}/c)} .$$

To go further we need some geometry. Recall the searchlight picture. We argued that relativistic sources radiate mostly in a beam in the direction of motion and with intensity $\approx$ that of a charge in circular motion with radius = instantaneous radius of curvature.

Define coords such that the particle is at the origin @ $t=0$, moving in an appx circle in the xy plane of radius ρ:



Place the observer in the xz plane:



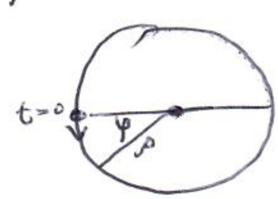
most of the radiation goes in this cone, so only small-θ observers will see much.

Let $\hat{\epsilon}_{\parallel} = \hat{y}$ be parallel to the acceleration at $t=0$

$\hat{\epsilon}_{\perp} \equiv \hat{n} \times \hat{\epsilon}_{\parallel}$ be $\perp$ to accel & observer at $t=0$

$= \cos\theta\,\hat{z} - \sin\theta\,\hat{x}$

Over a time t, the charge moves through an angle $\varphi \approx vt/\rho$



So $$\hat{n} \times (\hat{n} \times \vec{\beta}) = |\beta| \Big(\underbrace{\sin\theta\,\hat{z} + \cos\theta\,\hat{x}}_{\hat{n}}\Big) \times \Big[\Big(\underbrace{\sin\theta\,\hat{z} + \cos\theta\,\hat{x}}_{\hat{n}}\Big) \times \Big(\underbrace{\cos\tfrac{vt}{\rho}\,\hat{x} + \sin\tfrac{vt}{\rho}\,\hat{y}}_{\hat{\beta}}\Big)\Big]$$

(working out the algebra...) $$= |\beta| \left(-\sin\frac{vt}{\rho}\,\hat{\epsilon}_{\parallel} + \cos\frac{vt}{\rho}\,\sin\theta\,\hat{\epsilon}_{\perp}\right)$$

and the argument of the exponential is

$$-i\omega\left(t - \frac{\hat{n}\cdot\vec{r}}{c}\right) = -i\omega\left(t - \frac{\rho\cos\theta\sin\frac{vt}{\rho}}{c}\right)$$

$$\left(\text{using } \hat{n}\cdot\vec{r} = (\sin\theta\,\hat{z} + \cos\theta\,\hat{x})\cdot(\rho\sin\tfrac{vt}{\rho}\hat{x} - \rho\cos\tfrac{vt}{\rho}\hat{y}) = \rho\cos\theta\,\sin\frac{vt}{\rho}\right)$$

$$\tilde{\vec{E}}(\omega,\vec{x}) = \frac{i\omega e\, e^{-i\omega x/c}}{4\pi\epsilon_0 c}\int_{-\infty}^{\infty} dt\; \frac{|\beta|\left(-\sin\frac{vt}{\rho}\hat{\epsilon}_{\parallel} + \cos\frac{vt}{\rho}\sin\theta\,\hat{\epsilon}_{\perp}\right)}{x - \rho\cos\theta\sin\frac{vt}{\rho}}\, e^{-i\omega(t - \frac{\rho}{c}\cos\theta\sin\frac{vt}{\rho})}$$

The searchlight is only "on" for small-θ observers in a small time window around $t=0$ $\left(\frac{vt}{\rho} \lesssim \theta_{max} \sim \frac{1}{\gamma} \rightarrow t \lesssim \frac{\rho}{c\gamma}\right)$

Here "small" means $O\left(\frac{1}{\gamma}\right) = O\left(\frac{m}{E}\right)$.

For a 10 GeV electron, $\gamma = \frac{10^4 \text{ MeV}}{\frac{1}{2}\text{ MeV}} \sim 10^4$

If $\rho \sim 10$ m this is a time of order 10^{-12} s. (In fact for the observer the time length is even shorter, because of the Doppler effect - the charge is moving toward us almost as fast as the radiation it emits! This results

The upshot of this is we can expand in small t and θ in the integrand. The way to do it is to treat $\beta \approx 1 - \frac{1}{2\gamma^2}$, $\theta \sim \mathcal{O}(\frac{1}{\gamma})$, $t \sim \mathcal{O}(\frac{1}{\gamma})$.

For the exponent, we get $\omega\left(t - \frac{\hat{n}\cdot\vec{r}}{c}\right) \approx \frac{\omega}{2}\left[\left(\frac{1}{\gamma^2} + \theta^2\right)t + \frac{c^2}{3\rho^2}t^3\right]$

at order $1/\gamma^3$. (leading order)

For the $\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{R}$ term, we get

$$\approx \quad -\frac{ct}{\rho x}\hat{\epsilon}_{\parallel} + \frac{\theta}{x}\hat{\epsilon}_{\perp}$$

at order $1/\gamma$. (leading order)

So we have to evaluate stuff like

$$\tilde{\vec{E}}_{\parallel}(\omega,\vec{x}) \propto \frac{c}{\rho x}\int_{-\infty}^{\infty} dt\; t\, e^{-i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t + \frac{c^2}{3\rho^2}t^3\right]}$$

$$\tilde{\vec{E}}_{\perp}(\omega,\vec{x}) \propto \frac{\theta}{x}\int_{-\infty}^{\infty} dt\; e^{-i\omega\left[\left(\frac{1}{\gamma^2}+\theta^2 t + \frac{c^2}{3\rho^2}t^3\right]}$$

(It looks weird that we did a small t appx but then integrate over <u>all</u> t. This is allowed because at high frequencies the integrand oscillates rapidly and averages to zero. At low frequencies the approx breaks down, but also little power is carried off, so the error is unimportant. See Jackson p. 678 footnote for detail.)

Anytime you see "$\int e^{i(ax+bx^3)}$" think

Airy function or Bessel function.

The result, and recalling $\frac{dE}{d\Omega\, d\omega} = \frac{2\pi R^2}{\mu_0 c} |\tilde{\vec{E}}(\omega)|^2$,

is

$$|\tilde{\vec{E}}_{\parallel}| = \frac{\omega e}{4\pi\epsilon_0 \cancel{c}} \frac{\cancel{c}}{\rho x} \frac{\rho^2}{c^2}\left(\frac{1}{\gamma^2}+\theta^2\right)\frac{2}{\sqrt{3}} K_{2/3}\left[\frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\right]$$

$$|\tilde{\vec{E}}_{\perp}| = \frac{\omega e}{4\pi\epsilon_0 c} \frac{\theta}{x} \frac{\rho}{c} \sqrt{\frac{1}{\gamma^2}+\theta^2}\, \frac{2}{\sqrt{3}} K_{1/3}\left[\frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\right]$$

which we square and add to get the energy per unit solid angle per unit frequency

Here $K_a(z)$ is the modified bessel function of order a.

Oof! What to make of this? Use asymptotics:

for $a \neq 0$, $$K_a(z) \sim \begin{cases} \frac{\Gamma(a)}{2}\left(\frac{2}{z}\right)^a, & z \ll 1 \\ \sqrt{\frac{\pi}{2z}}\, e^{-z}\left(1 + \mathcal{O}(1/z)\right), & z \gg 1, a \end{cases}$$

if $\dfrac{\omega\rho}{3c\gamma^3}$ is $\gtrsim 1$, both bessels are

exponentially small for all θ. Jackson

defines a cutoff "critical frequency" ω_c with an

extra $1/2$: $\omega_c = \frac{3}{2}\gamma^3\left(\frac{c}{\rho}\right)$

for larger frequencies, radiation is negligible.

at low frequencies, and $\theta = 0$ for simplicity,

$\omega K_{2/3}\left(\frac{\omega}{2\omega_c}\right) \propto \omega^{1/3}$, $\omega K_{1/3}\left(\frac{\omega}{2\omega_c}\right) \propto \omega^{2/3}$

But only contributes at $\theta = 0$, so $\dfrac{dE}{d\omega d\Omega} \propto \omega^{2/3}$

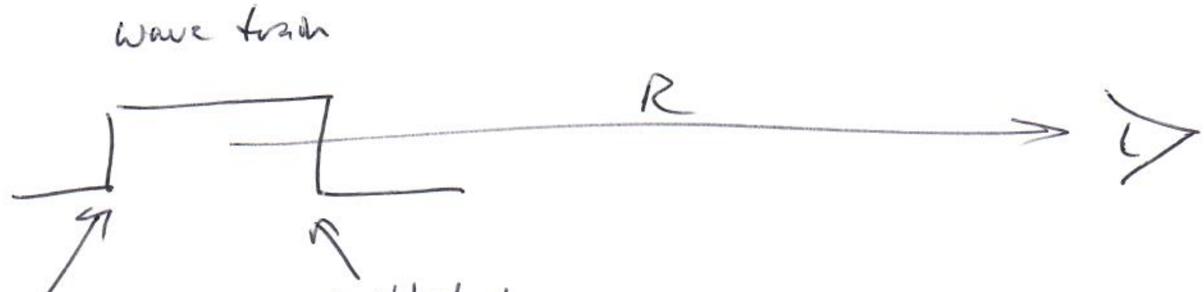
All together, at $\theta = 0$, we get

grows w/ γ^2

$$\frac{dE}{d\omega d\Omega} \sim \frac{1}{(4\pi\epsilon_0)^2}\frac{3e^2\gamma^2}{c^6 x^2} \times \begin{cases} 2^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)^2\left(\frac{\omega}{\omega_c}\right)^{2/3}, & \omega \ll \omega_c \\ \frac{\pi\omega}{\omega_c} e^{-\omega/\omega_c}, & \omega \gg \omega_c \end{cases}$$

Physics of ω_c:

$\omega_0 = c/\rho$ is the natural frequency of the circular motion. $\omega_c \sim \gamma^3 \omega_0$. One factor of γ is the emission time for a cone sweeping over a fixed observer. The other two factors of γ are Doppler.



emitted at
$t = \rho/c\gamma$,
$x = \beta c t$
$\simeq ct - \frac{ct}{2\gamma^2}$
$= \rho\left(\frac{1}{\gamma} - \frac{1}{2\gamma^3}\right)$;
travels $R - x \simeq R - \frac{\rho}{\gamma} + \frac{\rho}{2\gamma^3}$

emitted at
$t = 0$
$x = 0$;
travels R

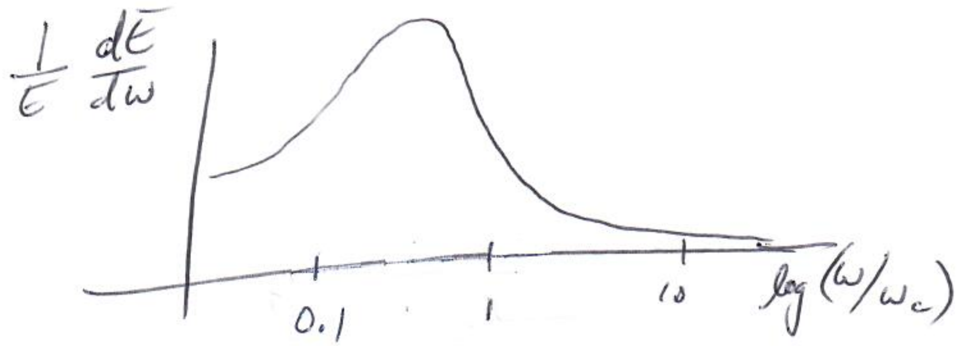
The front arrives at time $= R/c$.
The back arrives at time $= \frac{R-x}{c} + t$

$$= \frac{R}{c} - \cancel{\frac{\rho}{c\gamma}} + \frac{\rho}{2c\gamma^3} + \cancel{\frac{\rho}{c\gamma}}$$

$$= \frac{R}{c} + \frac{\rho}{2c\gamma^3}$$

Difference: $\mathcal{O}(1/\gamma^3)$

If you integrate over angles, you get the total energy / frequency bin. Here's a plot!



This is called synchrotron radiation. It is ubiquitous in nature, and useful in the lab for tailored light sources. For $\rho \sim 100$ m, $\gamma \sim 10^4$ (GeV-scale electrons) ω_c corresponds to keV x-rays useful in biology & condensed matter.