Using $\int_0^{\pi} \sin^2\theta (\sin\theta \, d\theta) = \int_{-1}^{1} (1-\cos^2\theta) \, d\cos\theta$

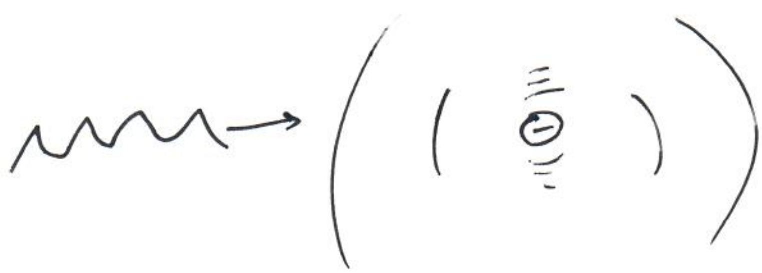
$$= 2 - \frac{2}{3} = \frac{4}{3},$$

We obtain the Larmor formula,

$$\boxed{P = \frac{2e^2a^2}{4\pi\epsilon_0 \cdot 3c^3}} \qquad \left(\frac{e^2a^2}{6\pi\epsilon_0 c^3}\right)$$

This is the power radiated by the charge at some t' when its acceleration was $|\vec{a}|$. It passes through the sphere at $t = t' + R/c$.

Thomson scattering



an electromagnetic wave strikes a free electron, causing it to accelerate and radiate.

the acceleration is $\vec{a} = e\vec{E}/m_e$

(Newton's law + coulomb's law)

So the total power radiated is $\frac{2e^2a^2}{4\pi\epsilon_0 \cdot 3c^3} = \frac{2e^4|\vec{E}|^2}{3m_e^2c^3 \cdot 4\pi\epsilon_0}$

The incident power flux is $\left|\frac{E\times B}{\mu_0}\right| = \frac{|E|^2}{\mu_0 c}$

(units of power/area)

So the ratio is $\frac{2e^4\mu_0}{4\pi\epsilon_0\cdot 3m_e^2c^2} = \underbrace{\underbrace{\frac{8\pi e^4}{3m_e^2 c^4}}_{\text{cgs units}}\cdot\frac{1}{(4\pi\epsilon_0)^2}}_{\text{SI units}} \equiv \sigma_T$

"Thomson cross section"

What does it mean?

$\left(\text{"size of scatterer"}\right)\times\left(\text{incident }\frac{\text{power}}{\text{area}}\right) = \left(\text{total radiated power}\right)$

So the cross section, intuitively, is how big the electron looks to the incoming EM wave.

Numerically: (use cgs units where $e=1$)

$$(m_e c^2) \cong 1/2 \text{ MeV}$$

$$\hbar c \cong 200 \text{ MeV}\cdot\underbrace{\text{fermi}}_{10^{-15}\,m}$$

$$e^2/\hbar c \cong 1/137$$

So $\sigma_T \simeq \frac{8\pi}{3}\left(\frac{200}{137}\right)^2\frac{1}{(1/2)^2}\text{ fermi}^2 \sim (10\text{ fermi})^2$

more precisely $6.65\times10^{-25}\text{ cm}^2$

The cross sectional area of an atom is about 10^{-16} cm^2 ($(1\,\text{Å})^2$) and the area of a proton is about 10^{-26} cm^2. The Thomson xsec is close to "1 barn" or the area of a uranium nucleus.

Scattering by a bound charge

Light is incident on an electron bound to other charges. (solid, molecular gas, ...)

Appx:

- harmonic potential (small perturbation)
- damped (some loss mechanism, not re-radiation which is v. nonlinear)
- nonrelativistic (neglect forces due to $\vec{B}$)
  $F = qE + v\times B$, $B \sim E/c$, so
  $F \sim qE + \underbrace{\cancel{\beta} B}_{0}$

$$\vec{E} = E_0 \cos(kz - \omega t)\,\hat{x}$$

$$\rightarrow \quad E_0 \cos(\omega t)\,\hat{x} \quad \text{@ the origin}$$

EOM: $$m\ddot{x} = \underbrace{-m\omega_0^2 x}_{\substack{\text{restoring}\\ \text{harmonic}\\ \text{force}}} - \underbrace{\gamma m \dot{x}}_{\substack{\text{frictional}\\ \text{damping}}} + \frac{qE_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

(we expect motion along the x direction alone in the limit that we ignore $\vec{B}$)

This is damped driven harmonic motion.

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

$$A = \frac{qE_0}{2m(\omega_0^2 - \omega^2 + i\gamma\omega)}$$

$$B = A^* \quad \text{(since } x \text{ is real)}$$

After rewriting A & B as $\rho e^{\pm i\varphi}$, find

$$x = \frac{qE_0 \cos(\omega t - \varphi)}{m\sqrt{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}}, \quad \varphi = \tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

(we could add transients of E_0, but these die off due to friction)

- Resonance at $\omega = \omega_0$
- Amplitude maximum at $\partial_\omega \left((\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2 \right) = 0$

$$\Rightarrow 4\omega^3 - 4\omega_0^2 \omega + 2\gamma^2 \omega = 0$$

$$\Rightarrow \omega^2 = \omega_0^2 - \gamma^2/2$$

$$\omega = \omega_0 \sqrt{1 - \gamma^2/2\omega_0^2}$$

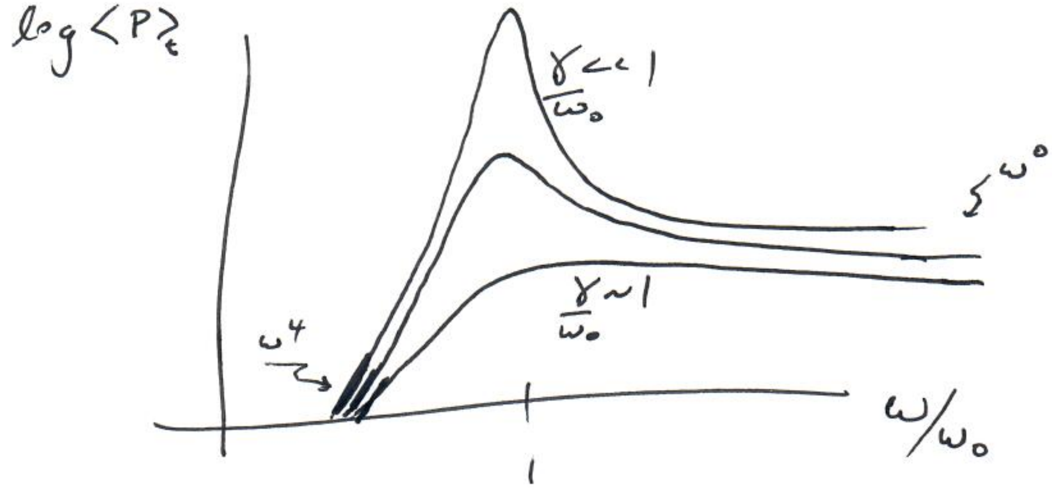
- Instantaneous accel $-\omega^2 x(t)$ is maximized @ $\partial_\omega \left(\frac{\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} \right) = 0$

$$\Rightarrow \omega = \frac{\omega_0}{\sqrt{1 - \gamma^2/2\omega_0^2}}$$

Since we made a nonrelativistic assumption, we can get total power from Larmor:

$$P = \frac{1}{(4\pi\epsilon_0)^2} \times \frac{8\pi e^4 \epsilon_0}{3 m^2 c^3} \underbrace{\left[\frac{\omega^4 E_0^2 \cos^2(\omega t - \varphi)}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \right]}$$

$$\sim \begin{cases} E_0^2 \left(\frac{\omega}{\omega_0}\right)^4, & \omega \ll \omega_0 \\ E_0^2, & \omega \gg \omega_0 \end{cases}$$



Rayleigh scattering: $\omega \ll \omega_0$.

$$\sigma_{Ray} = \frac{\text{scattered power}}{\text{incident power flux}}$$

$$\cong \frac{\left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{q\omega}{\omega_0}\right)^4 \frac{8\pi\epsilon_0}{3} \frac{E_0^2}{m^2 c^3}}{E_0^2 / c\mu_0}$$

(scatterer of charge q)

$$= \left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{q\omega}{\omega_0}\right)^4 \frac{8\pi}{3} \frac{1}{(mc^2)^2}$$

while for $\omega \gg \omega_0$,

$$\sigma_{total} = \frac{\frac{1}{(4\pi\epsilon_0)^2} q^4 \frac{8\pi\epsilon_0}{3} \frac{E_0^2}{m^2 c^3}}{E_0^2 / c\mu_0}$$

$$= \frac{1}{(4\pi\epsilon_0)^2} \frac{8\pi q^4}{3(mc^2)^2} = \sigma_T$$

at high frequency the charge acts as if it's free.

Frequency Distribution

Energy / Area / frequency interval?

$$\frac{dP}{d\Omega} = \frac{|\vec{E}(t)|^2}{\mu_0 c} R^2 \qquad \text{for } R \text{ large cf. source}$$

Fourier transform!

$$\tilde{\vec{E}}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \vec{E}(t)\, dt$$

$$\frac{d\tilde{P}(\omega)}{d\Omega} = \int_{-\infty}^{\infty} e^{-i\omega t} \frac{dP(t)}{d\Omega}\, dt$$

How is $\tilde{P}(\omega)$ related to $\tilde{\vec{E}}(\omega)$?

$$\frac{d\tilde{P}(\omega)}{d\Omega} = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \left(\frac{R^2}{\mu_0 c}\right) \left(\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{i\omega' t} \tilde{\vec{E}}(\omega')\right)^{*} \cdot \left(\int_{-\infty}^{\infty} \frac{d\omega''}{2\pi} e^{i\omega'' t} \tilde{\vec{E}}(\omega'')\right)$$

do the t integral:

$$\int_{-\infty}^{\infty} dt\, e^{-i(\omega + \omega' - \omega'')t} = 2\pi\, \delta(\omega + \omega' - \omega'')$$

$$\Rightarrow \quad \frac{R^2}{\mu_0 c} \int_{-\infty}^{\infty} \frac{d\omega'\, d\omega''}{(2\pi)^2} 2\pi\, \delta(\omega + \omega' - \omega'')\, \tilde{\vec{E}}(\omega')^{*} \cdot \tilde{\vec{E}}(\omega'')$$

$$= \frac{R^2}{\mu_0 c} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \tilde{\vec{E}}(-\omega') \cdot \tilde{\vec{E}}(\omega' + \omega)$$

↑ using $\tilde{\vec{E}}(\omega)^{*} = \tilde{\vec{E}}(-\omega)$ as you can readily verify, since $\vec{E}(t)$ is real.

(example of the convolution theorem)

$$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} \frac{dP}{d\Omega}\, dt = \frac{d\tilde{P}(\omega=0)}{d\Omega}$$

So $$\frac{dE}{d\Omega} = \frac{R^2}{\mu_0 c} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{\vec{E}}(-\omega) \cdot \tilde{\vec{E}}(\omega)$$ (just relabelling $\omega' \to \omega$)

or $$\frac{dE}{d\Omega\, d\omega} = \frac{R^2}{\mu_0 c} \left| \tilde{\vec{E}}(\omega) \right|^2$$

This is an example of Parseval's Theorem:

if $g(y)$ is the fourier transform of $f(x)$,

then $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |g(y)|^2\, dy$$

In your homework you'll work out an example, the spectrum emitted by nonrelativistic bremsstrahlung.