Dispersion

Awhile ago, we discussed a simple toy "harmonic binding" model for a frequency-dependent dielectric constant. Let's examine in more detail the physics of dispersion, generalizing this model slightly.

Previous model:

heavy fixed nucleus

$$m\left(\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right) = -e\vec{E}(t)$$

electron mass; electron acceleration; damping constant; natural frequency, harmonic appx; assume vibration amplitude is small so that $E \approx$ spatially constant!

- assume $\mu = \mu_0$
- neglect magnetic forces (nonrelativistic appx)
- assume $\vec{E}(t) = \vec{E}_\omega e^{-i\omega t}$. Then $\vec{x}(t) = \vec{x}_\omega e^{-i\omega t}$

Then the contribution of this electron to the dipole moment is

$$\vec{p}_\omega = -e\vec{x}_\omega = \frac{e^2}{m}\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)^{-1}\vec{E}_\omega$$

If there are $\underline{N \text{ atoms / unit volume}}$ with $\underline{Z\ e^-/\text{atom}}$, and there are f_i e^-/atom with binding/natural frequency ω_i and damping constant γ_i, then

$$\epsilon(\omega) = \epsilon_0(1+\chi_e) = \epsilon_0 + \frac{\vec{P}_{tot}}{\vec{E}_{net}}$$

and making the appx $\vec{E}_{net} \approx \vec{E}_{applied}$ for simplicity (this can be corrected if needed, but fine for dilute systems)

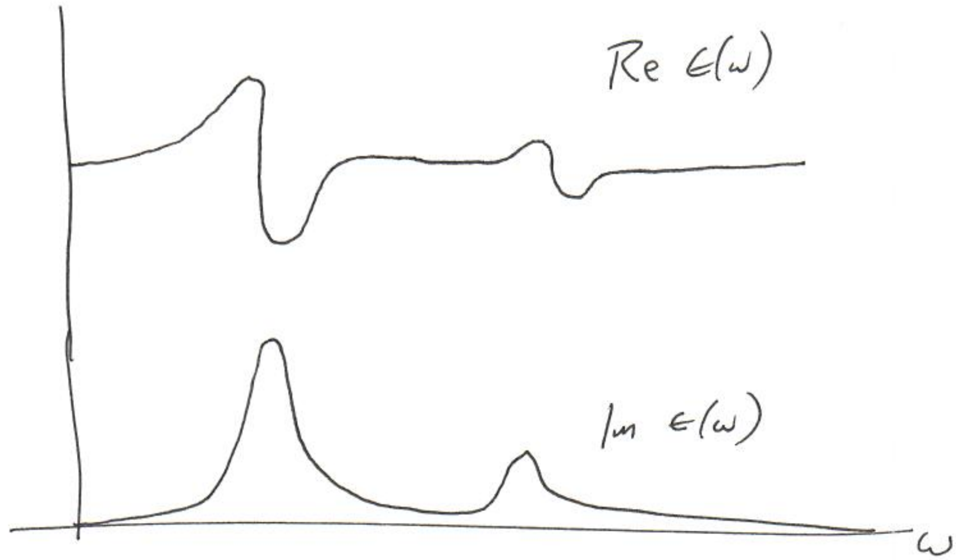
we find

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\vec{P}_{tot,\omega}}{\vec{E}_\omega \epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_i f_i \left(\omega_i^2 - \omega^2 - i\omega\gamma_i\right)^{-1}$$

$$\text{with} \quad \sum_i f_i = Z$$

Neglect γ_i for a moment. Then $\frac{1}{\omega_i^2-\omega_0^2} < 0$ for $\omega > \omega_i$ and $\frac{1}{\omega_i^2-\omega_0^2} > 0$ for $\omega < \omega_i$. Therefore, at low frequencies, below $\underline{\text{all}}$ ω_i (if there is a gap, so that the smallest ω_i is > 0), we have $\epsilon(\omega)/\epsilon_0 > 1$.

As ω increases, we accrue more and more negative terms until eventually $\epsilon(\omega)/\epsilon_0 < 1$.



- near resonances, Re ϵ crosses zero and Im ϵ spikes.
- "Normal dispersion": Re(ϵ) increases with ω
- "Anomalous dispersion": Re (ϵ) decreases with ω — happens near resonances.

Dispersion Relation for EM waves

$$\omega = \bar{c}(\omega) k = \frac{1}{\sqrt{\mu_0 \epsilon(\omega)}} k \quad \text{or} \quad k = \omega\sqrt{\mu_0 \epsilon(\omega)}$$ (this is why we are talking about $\epsilon(\omega)$!)

let $k = \beta + i\alpha/2$. Then $\frac{1}{2}(k^2 + k^{*2}) = \beta^2 - \alpha^2/4$

$$= \frac{1}{2}(\omega^2 \mu_0)(\epsilon + \epsilon^*)$$

$$= \frac{\omega^2}{c^2} \mathrm{Re}(\epsilon/\epsilon_0)$$

and $\frac{1}{2}(k^2 - k^{*2}) = \alpha\beta = \frac{\omega^2}{c^2} \mathrm{Im}(\epsilon/\epsilon_0)$

for small α, $\alpha \approx \frac{\mathrm{Im}\,\epsilon}{\mathrm{Re}\,\epsilon} \times \beta$

and $\beta \approx \sqrt{\mathrm{Re}\,\epsilon/\epsilon_0}\, \omega/c$

in general, $\alpha = \frac{\omega}{c}\sqrt{2}\sqrt{|\epsilon| - \mathrm{Re}\,\epsilon}$ is nonzero <u>either</u> if $\mathrm{Im}\,\epsilon \neq 0$ ~ absorption

<u>or</u> $\mathrm{Re}\,\epsilon < 0$.
~ reflection

So the wave behaves as

$$e^{i(kz-\omega t)} = e^{i(\beta z-\omega t)}\, e^{-\alpha z/2}$$

for small α $\left\{ \begin{array}{l} \text{with phase velocity } \omega/\beta \simeq \dfrac{c}{\sqrt{\mathrm{Re}\,\epsilon/\epsilon_0}} \\ \text{and attenuation scale } 2/\alpha \simeq \dfrac{2\,\mathrm{Re}\,\epsilon}{\beta\,\mathrm{Im}\,\epsilon} \\ \quad \text{(or absorption)} \end{array} \right.$

frequency dependence is what is sometimes referred to as dispersion

If there are free electrons, we can model them as a term with vanishing resonant frequency, $\omega_i \to 0$. Then $\epsilon(\omega)$ is singular as $\omega \to 0$.

$$\epsilon(\omega) = \underset{\substack{\text{contribution}\\ \text{of all}\\ \text{bound } e^-}}{\epsilon_b(\omega)} + \underbrace{i\,\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}_{\equiv\, \epsilon_{free}}$$

free e^-/atom

The Ampere-Maxwell equation for a material with $\epsilon = \epsilon_b$ is

$$\vec{\nabla}\times\vec{B}_\omega = -i\omega\mu_0\epsilon_b\vec{E}_\omega + \mu_0\vec{J}_\omega$$

If we use Ohm's law, $\vec{J}_\omega = \sigma_\omega\vec{E}_\omega$, then

$$\vec{\nabla}\times\vec{B}_\omega = -i\mu_0\omega\left(\epsilon_b + \frac{i\sigma_\omega}{\omega}\right)\vec{E}_\omega$$

We can identify the conductivity term $\frac{i\sigma_\omega}{\omega}$ with ϵ_{free} and $\sigma = \dfrac{f_0 N e^2}{m(\gamma_0 - i\omega)}$. In other words, we can model the dispersive properties of the material as <u>either</u> a complex dielectric constant ϵ, or a complex dielectric constant ϵ_b + a freq-dependent conductivity.

— Drude Model —

High frequencies

for $\omega \gg \max(\omega_i)$, $\epsilon(\omega)/\epsilon_0 \approx 1 - \omega_p^2/\omega^2$

where $\omega_p^2 = NZe^2/\epsilon_0 m$ = "plasma frequency."

(at high frequencies, all e^- respond like they're free, as in a plasma.)

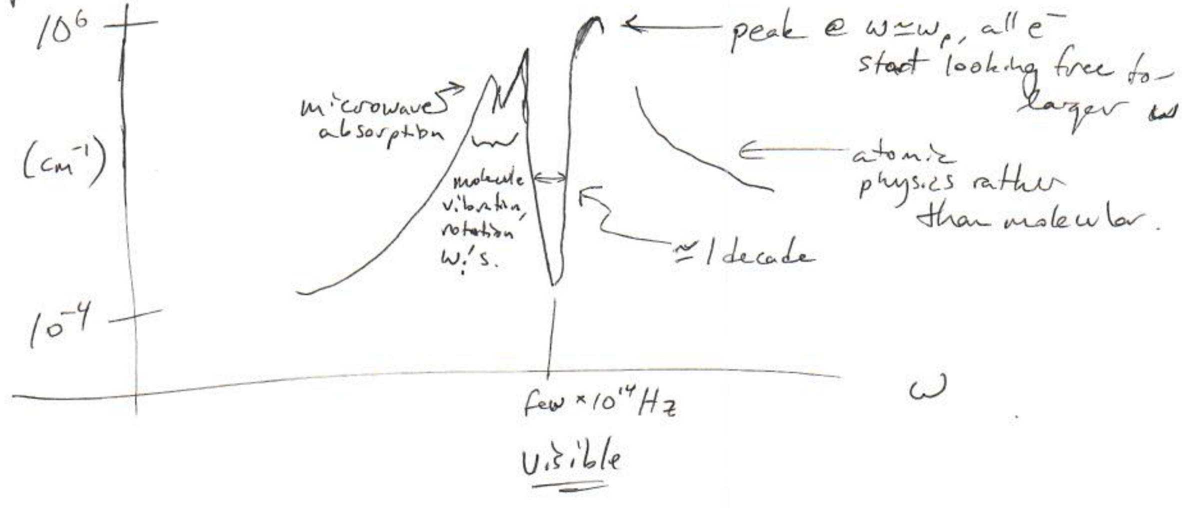
Then $$\omega = \bar{c}k = \frac{k}{\sqrt{\mu_0 \epsilon_0 (\epsilon/\epsilon_0)}} = \frac{ck}{\sqrt{1-\omega_p^2/\omega^2}}$$

or $\omega^2 = \omega_p^2 + c^2k^2$ massive relativistic dispersion relation.

- Only holds in ordinary dielectrics for $\omega \gg \omega_p$. ($\omega_i \sim \alpha^2 m_e \sim 10^{16}$ Hz $\gg \omega_p$)
- In actual plasmas, all e^- are free ($\omega_i = 0$) and this relation holds even for $\omega \ll \omega_p$. If $\omega < \omega_p$ k is imaginary ($\underline{\epsilon < 0}$) and the wave attenuates. Typical attenuation length $\sim \frac{c}{2\omega_p}$ "fields expelled by plasma!" For $NZ \sim 10^{18-22}\ e^-/m^3$, $\omega_p \sim 10^{11-13}$ Hz, $c/2\omega_p \sim 10\mu m - 1mm$.
- For metals, there is a population of free e^-. For $\omega \gg \gamma_0$,
$$\epsilon(\omega) \simeq \epsilon_b(\omega) - \frac{\omega_p^2}{\omega^2}\epsilon_0$$
For $\omega \ll \omega_p$ $\epsilon(\omega)$ is again < 0 and it behaves a lot like a plasma, expelling fields. ("Reflection")

For larger frequencies such that $\epsilon(\omega) > 0$, k is real and waves can propagate "ultraviolet transparency" of metals. [illegible]

There is a beautiful plot in Jackson fig 7.9 of the absorption coeff $\alpha(\omega)$ in water over 20 decades in ω.



Ionosphere

Now let's look at propagation in a plasma with a background B field. Assume uniform density electronic plasma, no collisions, small amplitude e^- motion, and a strong B (static)

$$m\ddot{\vec{x}} - e\vec{B}_0 \times \dot{\vec{x}} = -eE e^{-i\omega t}$$

Take incident waves to be transverse & circularly polarized, $\vec{E} = (\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)E$.

basis vecs of linear polarization. $\vec{\epsilon}_2$ 90° out of phase w/ $\vec{\epsilon}_1 \Rightarrow$ circ. pol.

let $\vec{B}$ be // to direction of propagation, for simplicity. Then we can take $\vec{x} = (\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)(x)(e^{-i\omega t})$

$$(B_0\hat{\epsilon}_3) \times (-i\omega x(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)) = -i\omega x B_0(\hat{\epsilon}_2 \mp i\hat{\epsilon}_1)$$
$$= \mp \omega x B_0(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)$$

So we get

$$-m\omega^2 x \pm eB_0\omega x = -eE$$

$$\Rightarrow \quad x = \frac{eE}{m\omega(\omega \mp \omega_B)}, \quad \omega_B \equiv \frac{eB_0}{m}$$

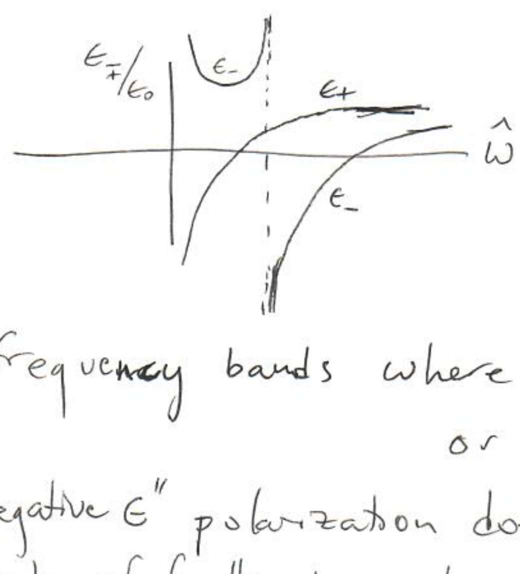
This amplitude of oscillation produces a frequency and helicity-dependent dipole moment and thus dielectric constant,

$$\frac{\epsilon_\mp}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)} \qquad \omega_p^2 = NZe^2/\epsilon_0 m$$

For the Earth's ionosphere, $\omega_p/\omega_B \sim 1$ (Both $\sim 10^7$ Hz)

Let $\hat{\omega} \equiv \omega/\omega_B$ and $\omega_p/\omega_B = 1$. Then

$$\frac{\epsilon_\mp}{\epsilon_0} = 1 - \frac{1}{\hat{\omega}(\hat{\omega} \mp 1)}$$



$$\epsilon_\mp = 0 \iff \hat{\omega} = \frac{1}{2}\left(\pm 1 + \sqrt{1 + 4\frac{\omega_p^2}{\omega_B^2}}\right)$$

There are frequency bands where $\epsilon_- > 0$ and $\epsilon_+ > 0$

or $\epsilon_+ > 0$ and $\epsilon_- < 0$

Here the "negative ϵ" polarization does not propagate: signals reflect off the ionosphere! Sensitive to $\# e^-/m^3$ via ω_p