Chapter 8

Now we would like to describe the energy in a dynamic situation, where the sources of fields may change (charges move). To find the energy of an inductor with a steady current, the principle of conservation of energy was used. The energy required to overcome the back emf was equated to the energy stored in the inductor. Conservation of energy can be used to get an expression for energy flow relevant to the dynamic case.

Since in the static case we were able to represent the energy of the system in terms of the fields, we might guess that the fields themselves carry energy. Moving charges radiate, that is, produce fields that move through space. Indeed, electromagnetic radiation, like any wave, transports energy.

An important idea in chapter 8 is that a local conservation law can be expressed with a continuity equation.

Local Conservation Law $\Longleftrightarrow$ Continuity Equation

We've come across this already, local charge conservation being expressed as $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$

What does 'local' conservation law mean? Take, for example, local charge conservation:

If charge decreases in a certain volume of space, it does so by flowing out of the volume. Charge doesn't simultaneously disappear out of the volume and reappear somewhere else far away without flowing through the space in between.

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$$

If charge decreases at some place...

There must be a flow of charge away from that place

If local conservation of energy holds for electromagnetism, we will be able to write a continuity equation:

$$\frac{\partial}{\partial t}(u_{total}) = -\vec{\nabla} \cdot \vec{S}$$

A change in energy density at some place...

must be accompanied by a flow of energy density away from that place

Consider a volume enclosing some matter in the form of charged particles. There will be an energy density in the volume due to the energy of these particles, which we'll call u_{mech} as per Griffiths. This can change with time if the electromagnetic fields do work on the particles (or vice vs.). There may be energy stored within the fields themselves, call this energy density u_{em}. Here is an energy balance expression ($\sim$ continuity equation) of sorts:

$$\Delta E_{\substack{\text{mech}\\ \text{in volume}}} + \Delta E_{\substack{\text{fields}\\ \text{in volume}}} + \Delta E_{\substack{\text{flowing}\\ \text{across volume}\\ \text{boundaries}}} = 0$$

Or;

$$\underbrace{\frac{\partial u_{mech}}{\partial t}}_{\text{we can calculate this}} + \underbrace{\frac{\partial u_{em}}{\partial t} + \left(\vec{\nabla} \cdot \vec{S}, \ \substack{\text{energy density}\\ \text{flow across boundaries}} \right)}_{\text{we don't know what these are}} = 0$$

Once we calculate $\frac{\partial u_{mech}}{\partial t}$, we can try to identify the result as the contributions $\frac{\partial u_{em}}{\partial t}$, and $\vec{\nabla} \cdot \vec{S}$ (the flow of energy density across the volume boundary).

If we succeed, we have demonstrated local conservation of energy density holds, because the equation has the form of a continuity equation. If we fail, I'm wasting your time even more than usual.

$$\frac{dW}{dt} = \frac{d}{dt}\int_V u_{mech}\, d\tau$$

work done by
em forces on charges in
volume V in time interval dt

Work done on a single charge:

$$w = \vec{F}\cdot d\vec{l} = q(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{v}\, dt = q\vec{E}\cdot\vec{v}\,dt$$

$$\frac{dw}{dt} = q\vec{E}\cdot\vec{v}$$

For a distribution of charges $q = \int_V \rho\, d\tau$ and $\rho v = J$

$$\frac{dW}{dt} = \int_V \vec{E}\cdot\rho\vec{v}\, d\tau = \int_V (\vec{E}\cdot\vec{J})\, d\tau$$

Eliminate $\vec{J}$ using the Maxwell eq:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{J} = \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \underbrace{\vec{E} \cdot (\vec{\nabla} \times \vec{B})}_{\text{Re-write using product rule}} \underbrace{- \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}}_{-\epsilon_0 \left(\frac{1}{2} \frac{\partial E^2}{\partial t} \right)}$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$= -\vec{B} \cdot \left(\frac{\partial \vec{B}}{\partial t} \right) - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$= -\frac{1}{2} \frac{\partial B^2}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$\vec{E} \cdot \vec{J} = -\frac{1}{2} \frac{\partial}{\partial t} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$\frac{dW}{dt} = -\frac{d}{dt} \int_V \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau - \frac{1}{\mu_0} \int \nabla \cdot (\vec{E} \times \vec{B}) \, d\tau$$

Poynting's theorem

This is:

$$\int \frac{\partial u_{mech}}{\partial t}\, d\tau = -\int \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\right] d\tau - \frac{1}{\mu_0}\int \nabla \cdot (\vec{E} \times \vec{B})\, d\tau$$

Equating the integrands:

$$\frac{\partial u_{mech}}{\partial t} + \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\right] + \vec{\nabla} \cdot \left[\frac{1}{\mu_0}(\vec{E} \times \vec{B})\right] = 0$$

Now compare to our original guess at a continuity equation (page 3 of these notes):

$$\frac{\partial u_{mech}}{\partial t} + \frac{\partial u_{em}}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0$$

We do have this! Apparently,

$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$ is the energy per volume stored in the electromagnetic fields. u_{em} has the same form that was obtained for statics, when the energy was written in terms of the fields. (Happy, warm feelings.)

Also,

$$\vec{S} \equiv \frac{1}{\mu_0} (\vec{E} \times \vec{B})$$ The Poynting vector

$\vec{S}$ is the energy per unit time, per unit area transported by the fields. $\vec{S}$ desribes the flow of energy density across the volume boundaries, and is the "<u>energy flux density</u>."