$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2} \vec{\nabla}(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})$$

$$\vec{\nabla}(\vec{\mathscr{r}}\cdot\vec{v}) = \vec{\mathscr{r}} \times \vec{\nabla} \times \vec{v} + \vec{v} \times \vec{\nabla} \times \vec{\mathscr{r}} + (\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{\mathscr{r}}$$

① $$(\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{v} = \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right)\vec{v}$$

$$\frac{\partial \vec{v}}{\partial x} = \frac{\partial \vec{v}}{\partial t_r}\frac{\partial t_r}{\partial x} \qquad \text{etc.}$$

$$\frac{\partial \vec{v}}{\partial t_r}\left(\mathscr{r}_x \frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{\partial t_r}{\partial z}\right)$$

$$\vec{a}\,(\vec{\mathscr{r}}\cdot\vec{\nabla} t_r)$$

② $$(\vec{v}\cdot\vec{\nabla})\vec{\mathscr{r}} = \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)\vec{\mathscr{r}}$$

$$\frac{\partial \vec{\mathscr{r}}}{\partial x} = \frac{\partial \vec{\mathscr{r}}}{\partial x} + \frac{\partial \vec{\mathscr{r}}}{\partial t_r}\frac{\partial t_r}{\partial x} = \frac{\partial \vec{r}}{\partial x} - \frac{\partial \vec{w}(t_r)}{\partial t_r}\frac{\partial t_r}{\partial x}$$

$$= \hat{x} - \frac{\partial \vec{w}(t_r)}{\partial t_r}\frac{\partial t_r}{\partial x} = \hat{x} - \vec{v}\frac{\partial t_r}{\partial x}$$

$$(\vec{v}\cdot\vec{\nabla})\vec{\imath} = v_x\hat{x} - v\,v_x\frac{\partial t_r}{\partial x} + v_y\hat{y} - v\,v_y\frac{\partial t_r}{\partial y}$$

$$+ v_z\hat{z} - v\,v_z\frac{\partial t_r}{\partial z}$$

$$= \vec{v} - \vec{v}\,(\vec{v}\cdot\vec{\nabla}t_r)$$

(3) $\vec{\imath} \times \vec{\nabla} \times \vec{v}(t_r)$

$$(\vec{\nabla}\times\vec{v}(t_r)) = \left(\frac{\partial v_z}{\partial t_r}\frac{\partial t_r}{\partial y} - \frac{\partial v_y}{\partial t_r}\frac{\partial t_r}{\partial z}\right)\hat{x}$$

$$+ \left(\frac{\partial v_x}{\partial t_r}\frac{\partial t_r}{\partial z} - \frac{\partial v_z}{\partial t_r}\frac{\partial t_r}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial t_r}\frac{\partial t_r}{\partial x} - \frac{\partial v_x}{\partial t_r}\frac{\partial t_r}{\partial y}\right)\hat{z}$$

$$= \left(a_z\frac{\partial t_r}{\partial y} - a_y\frac{\partial t_r}{\partial z}\right)\hat{x} + \left(a_x\frac{\partial t_r}{\partial z} - a_z\frac{\partial t_r}{\partial x}\right)\hat{y}$$

$$+ \left(a_y\frac{\partial t_r}{\partial x} - a_x\frac{\partial t_r}{\partial y}\right)\hat{z} = -\vec{a}\times\vec{\nabla}t_r$$

$$-\left(\vec{\imath}\times\vec{a}\times\vec{\nabla}t_r\right) = -\vec{a}\,(\vec{\imath}\cdot\vec{\nabla}t_r) + \vec{\nabla}t_r\,(\vec{\imath}\cdot\vec{a})$$

using triple product rule

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})$$

(4) $\vec{v} \times (\vec{\nabla} \times \vec{\mathscr{r}}) =$

$$\vec{\nabla} \times (\vec{r} - \vec{w}(t_r)) = \vec{\nabla} \times \vec{r} - \vec{\nabla} \times \vec{w}(t_r) = -\vec{\nabla} \times \vec{w}(t_r)$$

$$\vec{\nabla} \times \vec{w}(t_r) = \left(\frac{\partial w_z}{\partial t_r} \frac{\partial t_r}{\partial y} - \frac{\partial w_y}{\partial t_r} \frac{\partial t_r}{\partial z} \right) \hat{x}$$

$$\left(\frac{\partial w_x}{\partial t_r} \frac{\partial t_r}{\partial z} - \frac{\partial w_z}{\partial t_r} \frac{\partial t_r}{\partial x} \right) \hat{y} + \left(\frac{\partial w_y}{\partial t_r} \frac{\partial t_r}{\partial x} - \frac{\partial w_x}{\partial t_r} \frac{\partial t_r}{\partial y} \right) \hat{z}$$

$$= \left(v_z \frac{\partial t_r}{\partial y} - v_y \frac{\partial t_r}{\partial z} \right) \hat{x} + \left(v_x \frac{\partial t_r}{\partial z} - v_z \frac{\partial t_r}{\partial x} \right) \hat{y}$$

$$+ \left(v_y \frac{\partial t_r}{\partial x} - v_x \frac{\partial t_r}{\partial y} \right) \hat{z} = -\vec{v} \times \vec{\nabla} t_r$$

$$\vec{v} \times \vec{v} \times \vec{\nabla} t_r =$$

$$\vec{v}(\vec{v} \cdot \vec{\nabla} t_r) - \vec{\nabla} t_r (\vec{v} \cdot \vec{v}) = \vec{v}(\vec{v} \cdot \vec{\nabla} t_r) - v^2 \vec{\nabla} t_r$$

Add these results to get $\vec{\nabla}(\vec{\mathscr{r}} \cdot \vec{v})$

$$\cancel{\vec{a}(\vec{\mathscr{r}} \cdot \vec{\nabla} t_r)} + \vec{v} - \cancel{\vec{v}(\vec{v} \cdot \vec{\nabla} t_r)} - \cancel{\vec{a}(\vec{\mathscr{r}} \cdot \vec{\nabla} t_r)}$$

$$+ \vec{\nabla} t_r (\vec{\mathscr{r}} \cdot \vec{a}) + \cancel{\vec{v}(\vec{v} \cdot \vec{\nabla} t_r)} - v^2 \vec{\nabla} t_r$$

$$= \vec{v} + (\vec{\mathscr{r}} \cdot \vec{a} - v^2) \vec{\nabla} t_r$$

Putting it together:

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2} \left[\vec{\nabla}(\vec{\mathscr{r}}\cdot\vec{v}) + c^2 \vec{\nabla} t_r \right]$$

$$= \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2} \left[\vec{v} + (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a}) \vec{\nabla} t_r \right]$$

$$-c\vec{\nabla} t_r = \vec{\nabla} \mathscr{r}$$

Find $\vec{\nabla} t_r$ the tricky book way

$$\vec{\nabla} (\vec{\mathscr{r}}\cdot\vec{\mathscr{r}})^{1/2} = \frac{1}{2\mathscr{r}} \vec{\nabla}(\vec{\mathscr{r}}\cdot\vec{\mathscr{r}})$$

$$= \frac{1}{2\mathscr{r}}\, 2 \left[\vec{\mathscr{r}} \times \vec{\nabla} \times \vec{\mathscr{r}} + (\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{\mathscr{r}} \right]$$

$$(\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{\mathscr{r}} = \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z} \right) (\vec{r} - \vec{w}(t_r))$$

$$= \left(\mathscr{r}_x \frac{\partial x \hat{x}}{\partial x} + \mathscr{r}_y \hat{y} + \mathscr{r}_z \hat{z} \right) - \mathscr{r}_x \frac{\partial \vec{w}}{\partial t_r}\frac{\partial t_r}{\partial x} - \mathscr{r}_y \frac{\partial \vec{w}}{\partial t_r}\frac{\partial t_r}{\partial y}$$
$$- \mathscr{r}_z \frac{\partial \vec{w}}{\partial t_r}\frac{\partial t_r}{\partial z} = \vec{\mathscr{r}} - \vec{v}(\vec{\mathscr{r}}\cdot\vec{\nabla} t_r)$$

$$\vec{\nabla} \times \vec{\mathscr{r}} = \cancelto{0}{\vec{\nabla} \times \vec{r}} - \vec{\nabla} \times \vec{w}(t_r)$$

we did this one earlier,

$$\vec{\nabla} \times \vec{\mathscr{r}} = \vec{v} \times \vec{\nabla} t_r$$

$$\vec{\mathscr{r}} \times \vec{v} \times \vec{\nabla} t_r = \vec{v}(\vec{\mathscr{r}} \cdot \vec{\nabla} t_r) - \vec{\nabla} t_r (\vec{\mathscr{r}} \cdot \vec{v})$$

$$\vec{\nabla} \mathscr{r} = \frac{1}{\mathscr{r}} \left[\cancel{\vec{v}(\vec{\mathscr{r}} \cdot \vec{\nabla} t_r)} - \vec{\nabla} t_r (\vec{\mathscr{r}} \cdot \vec{v}) + \vec{\mathscr{r}} - \cancel{\vec{v}(\vec{\mathscr{r}} \cdot \vec{\nabla} t_r)} \right]$$

$$-c \vec{\nabla} t_r = \frac{1}{\mathscr{r}} \left[\vec{\mathscr{r}} - (\vec{\mathscr{r}} \cdot \vec{v}) \vec{\nabla} t_r \right]$$

$$\vec{\nabla} t_r = \frac{-\vec{\mathscr{r}}}{\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v}}$$

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v})^3} \left[(\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v}) \vec{v} - (c^2 - v^2 + \vec{\mathscr{r}} \cdot \vec{a}) \vec{\mathscr{r}} \right]$$

Alternative (longer) way of finding $\vec{\nabla} t_r$

$$\vec{\nabla} \eta = -c \vec{\nabla} t_r$$

Also

$$\vec{\nabla} \eta = \vec{\nabla} \left[r^2 - 2\vec{r} \cdot \vec{w}(t_r) + w^2(t_r) \right]^{1/2}$$

$$= \frac{1}{2} \frac{1}{\eta} \vec{\nabla} \left[\vec{r} \cdot \vec{r} - 2\vec{r} \cdot \vec{w}(t_r) + \vec{w} \cdot \vec{w} \right]$$

Look at $\frac{\partial}{\partial x}$ term of $\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$

$$\frac{\partial}{\partial x} \left[\vec{r} \cdot \vec{r} - 2\vec{r} \cdot \vec{w}(t_r) + \vec{w}(t_r) \cdot \vec{w}(t_r) \right] =$$

$$2\vec{r} \cdot \frac{\partial \vec{r}}{\partial x} - 2 \frac{\partial \vec{r}}{\partial x} \cdot \vec{w}(t_r) - 2\vec{r} \cdot \frac{\partial \vec{w}(t_r)}{\partial t_r} \frac{\partial t_r}{\partial x} + 2\vec{w} \cdot \frac{\partial \vec{w}}{\partial t_r} \frac{\partial t_r}{\partial x} =$$

$$2\vec{r} \cdot \hat{x} - 2\hat{x} \cdot \vec{w}(t_r) - 2\vec{r} \cdot \vec{v} \frac{\partial t_r}{\partial x} + 2\vec{w} \cdot \vec{v} \frac{\partial t_r}{\partial x} =$$

$$2x - 2w_x(t_r) - 2(\vec{r} \cdot \vec{v}) \frac{\partial t_r}{\partial x} + 2\vec{w} \cdot \vec{v} \frac{\partial t_r}{\partial x}$$

Then,

$$(\vec{\nabla} \eta)_x = \frac{1}{\eta} \left[\hat{x} x - \hat{x} w_x - [\vec{r} \cdot \vec{v} - \vec{w} \cdot \vec{v}] \hat{x} \frac{\partial t_r}{\partial x} \right]$$

Then $\vec{\nabla} \mathscr{r} = -c \vec{\nabla} t_r =$

$$\frac{1}{\mathscr{r}} \Big[\hat{x} x + \hat{y} y + \hat{z} z - (\hat{x} w_x + \hat{y} w_y + \hat{z} w_z)$$

$$- [\vec{r} \cdot \vec{v} - \vec{w} \cdot \vec{v}] \left(\hat{x} \frac{\partial t_r}{\partial x} + \hat{y} \frac{\partial t_r}{\partial y} + \hat{z} \frac{\partial t_r}{\partial z} \right) \Big]$$

$$= \frac{1}{\mathscr{r}} \left[\vec{r} - \vec{w}(t_r) - \left((\vec{r} - \vec{w}(t_r)) \cdot \vec{v} \right) \vec{\nabla} t_r \right]$$

Solving for $\vec{\nabla} t_r$

$$-\mathscr{r} c \vec{\nabla} t_r + \left[(\vec{r} - \vec{w}(t_r)) \cdot \vec{v} \right] \vec{\nabla} t_r = \vec{\mathscr{r}}$$

$$\vec{\nabla} t_r = \frac{-\vec{\mathscr{r}}}{\mathscr{r} c - \vec{\mathscr{r}} \cdot \vec{v}}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \qquad \left(\begin{array}{l} \text{use results} \\ \text{of problem 10.17} \end{array} \right)$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3} \Big\{ \cancel{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})\vec{v}}$$

$$- (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\vec{\mathscr{r}} + (\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})(\cancel{-\vec{v}} + \mathscr{r}\vec{a}/c)$$

$$+ \frac{\mathscr{r}}{c}(c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\vec{v} \Big\}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3} \Big\{ -(c^2 - v^2)\hat{\mathscr{r}}c\frac{\mathscr{r}}{c} + (c^2 - v^2)\frac{\mathscr{r}}{c}\vec{v}$$

$$- (\vec{\mathscr{r}}\cdot\vec{a})\hat{\mathscr{r}}c\frac{\mathscr{r}}{c} + (\vec{\mathscr{r}}\cdot\vec{a})\vec{v}\frac{\mathscr{r}}{c} + \mathscr{r}^2\vec{a} - (\hat{\mathscr{r}}\cdot\vec{v})\vec{a}\frac{\mathscr{r}}{c} \Big\}$$

$$= \frac{q\mathscr{r}}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3} \Big\{ (c^2 - v^2)(\hat{\mathscr{r}}c - \vec{v})$$

$$(\vec{\mathscr{r}}\cdot\vec{a})(\hat{\mathscr{r}}c - \vec{v}) - \mathscr{r}c\vec{a} + (\vec{\mathscr{r}}\cdot\vec{v})\vec{a} \Big\}$$

$$= \frac{q\mathscr{r}}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3} \Big\{ (c^2 - v^2)(\hat{\mathscr{r}}c - \vec{v}) + (\hat{\mathscr{r}}c - \vec{v})(\vec{\mathscr{r}}\cdot\vec{a})$$

$$- \vec{a}(\vec{\mathscr{r}}\cdot\hat{\mathscr{r}}c - \vec{\mathscr{r}}\cdot\vec{v}) \Big\}$$

Let $\vec{u} \equiv c\hat{r} - \vec{v}$, then this is:

$$\vec{E} = \frac{qr}{4\pi\epsilon_0} \frac{1}{[\vec{r}\cdot(\hat{r}c - \vec{v})]^3} \Big\{ (c^2 - v^2)\vec{u}$$

$$+ \vec{u}\,(\vec{r}\cdot\vec{a}) - \vec{a}\,(\vec{r}\cdot\vec{u}) \Big\}$$

$$\vec{E} = \frac{qr}{4\pi\epsilon_0 [\vec{r}\cdot\vec{u}]^3} \Big\{ (c^2 - v^2)\vec{u} + \vec{r} \times (\vec{u} \times \vec{a}) \Big\}$$

using $\vec{r} \times \vec{u} \times \vec{a} = \vec{u}(\vec{r}\cdot\vec{a}) - \vec{a}(\vec{r}\cdot\vec{u})$

Since $\vec{A} = \frac{\vec{v}}{c^2} V$, $\vec{B} = \frac{1}{c^2} \vec{\nabla} \times (\vec{v} V)$

$$\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times \vec{\nabla} f \quad \Big\} \text{Product Rule}$$

$$\vec{B} = \frac{1}{c^2} \left[V(\vec{\nabla} \times \vec{v}) - \vec{v} \times \vec{\nabla} V \right]$$

From before:

$$\vec{\nabla} V = \frac{1}{4\pi\epsilon_0} \frac{qc}{(rc - \vec{r}\cdot\vec{v})^3} \left[(rc - \vec{r}\cdot\vec{v})\vec{v} - (c^2 - v^2 + \vec{r}\cdot\vec{a})\vec{r} \right]$$

Also from before,

$$\vec{\nabla} \times \vec{v} = -\vec{a} \times \vec{\nabla} t_r$$

$$= -\vec{a} \times \left(\frac{-\vec{\mathscr{r}}}{\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v}} \right)$$

$$\vec{B} = \frac{1}{c^2} \left[\frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v})} \left(\vec{a} \times \frac{\vec{\mathscr{r}}}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v})} \right) - \vec{v} \times \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v})^3} \left((\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v})\vec{v} - (c^2 - v^2 + \vec{\mathscr{r}} \cdot \vec{a})\vec{\mathscr{r}} \right) \right]$$

$$= \frac{q}{c\, 4\pi\epsilon_0} \left[\frac{-1}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v})^2} (\vec{\mathscr{r}} \times \vec{a}) + \frac{(c^2 - v^2 + \vec{\mathscr{r}} \cdot \vec{a})}{(\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v})^3} (\vec{v} \times \vec{\mathscr{r}}) \right]$$

$$= \frac{-q}{c\, 4\pi\epsilon_0} \frac{1}{(\vec{u} \cdot \vec{\mathscr{r}})^3} \left[(\vec{u} \cdot \vec{\mathscr{r}})(\vec{\mathscr{r}} \times \vec{a}) - (c^2 - v^2 + \vec{\mathscr{r}} \cdot \vec{a})(\vec{v} \times \vec{\mathscr{r}}) \right]$$

$$= -\frac{q}{c\, 4\pi\epsilon_0} \frac{1}{(\vec{u} \cdot \vec{\mathscr{r}})^3} \vec{\mathscr{r}} \times \left[\vec{a}(\vec{u} \cdot \vec{\mathscr{r}}) + \vec{v}(c^2 - v^2 + \vec{\mathscr{r}} \cdot \vec{a}) \right]$$

$$= -\frac{q}{c\, 4\pi\epsilon_0} \frac{1}{(\vec{\mathscr{r}} \cdot \vec{u})^3} \vec{\mathscr{r}} \times \left[(c^2 - v^2)\vec{v} + (\vec{\mathscr{r}} \cdot \vec{a})\vec{v} + (\vec{\mathscr{r}} \cdot \vec{u})\vec{a} \right]$$