Chapter 7 - lecture 2

Now that we are back in the circuit frame of mind, we can go after those coupling terms. It was discovered that a change of magnetic flux through a circuit loop generates an emf, driving a current in the loop.

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

From this you can see that the flux can change in various ways. The area of the circuit loop immersed in the field may change (by moving the loop, or moving the source of $\vec{B}$), or $\vec{B}$ may change with time. The emf must be in the form of an electric field to move the charges in the circuit! Here is where we begin to realize that a changing $\vec{B}$ field induces an $\vec{E}$ field.

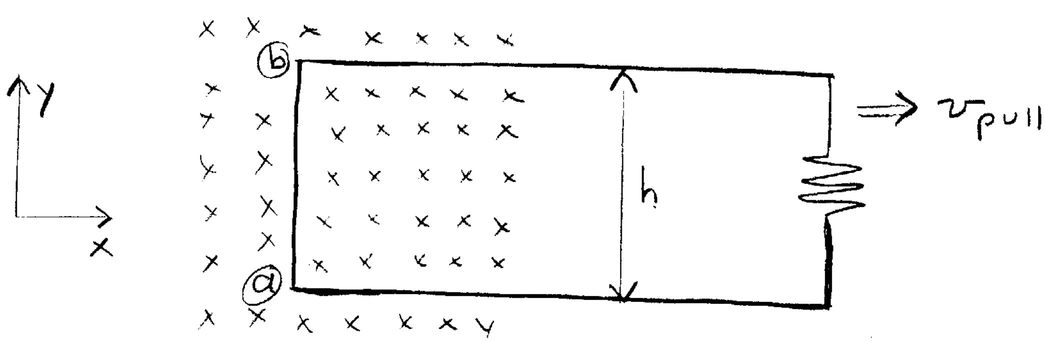
$$\mathcal{E} = \oint \vec{E}\cdot d\vec{\ell} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

Now invoke the curl theorem:

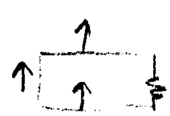
$$\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = - \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

$$\boxed{\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}}$$

Classic example, pull a current loop out of a field:

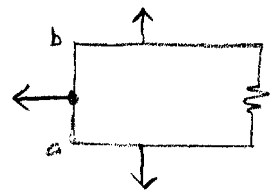


From v_{pull}, $\vec{F} = q \vec{v}_p \times \vec{B}$, the force is in the y direction

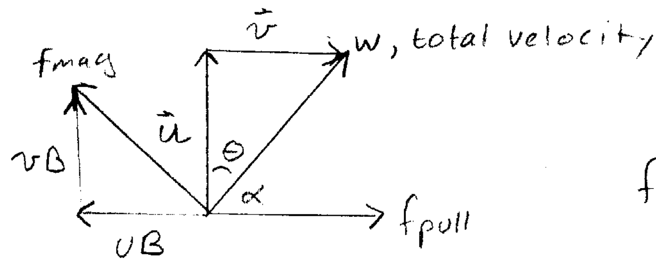


and causes current flow up wire segment ab (and then around circuit).

Once the charges begin moving (current) there is a resulting force due to the magnetic field:



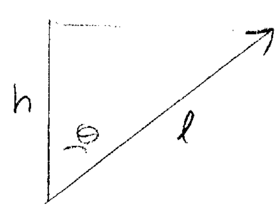
This force opposes the force due to the manual pulling, and the force of that pull is what does the work of the emf. Let's find that emf. Segment ab is where work is done, so here is the vector diagram for that wire segment:



$f_{pull} = uB$ for constant v to the right

$$\mathcal{E} = \int \vec{f}_{pull} \cdot d\vec{l} = uB\cos\alpha \int dl$$

$$= uB\sin\theta \int dl = uBh\left(\frac{\sin\theta}{\cos\theta}\right) = \cancel{u}Bh\left(\frac{v}{\cancel{u}}\right)$$

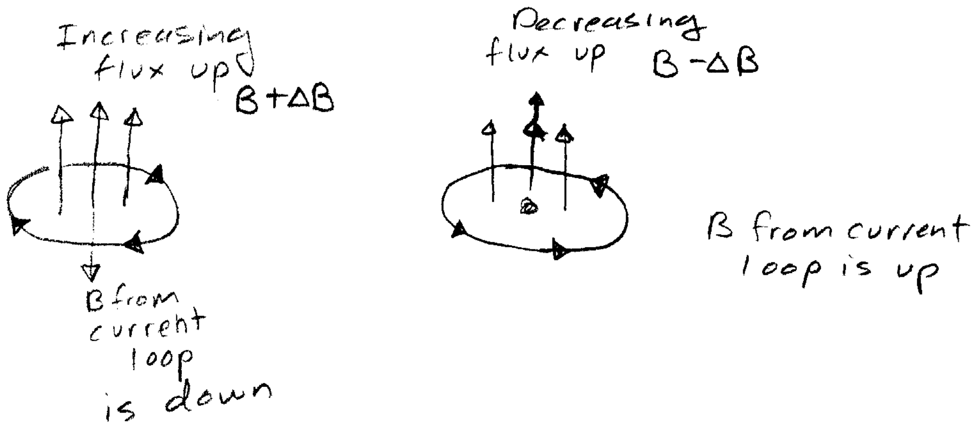


$$\frac{h}{l} = \cos\theta$$

$$l = \frac{h}{\cos\theta}$$

$$\mathcal{E} = Bhv = -Bh\frac{dx}{dt} = -\frac{d}{dt}(Bhx) = -\frac{d}{dt}\varphi$$

we have the dependence of emf on the changing flux, but what is the direction of the induced current flow? The current flows to produce a magnetic field which opposes the change in flux.



This is known as Lens' law. The induced field must oppose the change in flux, for if the changing field were enhanced, we'd be getting something for nothing. The energy of the field would increase at no cost elsewhere.

And speaking of energy.. Just as a pair of charged plates can support an electric field and store energy, so a coil with a steady current can support a magnetic field and store energy.

When a voltage is applied to a pair of electrodes, they will become charged with a charge that depends on their capacitance, which in turn depends largely on the geometry of the electrodes. Similarly, a current applied to a coil will result in a flux through the coil which depends largely on the geometry of the constructed coil.

$$\underbrace{C}_{\text{capacitance}} = \frac{\overbrace{Q}^{\text{get } Q}}{\underbrace{V}_{\text{apply } V}} \qquad \underbrace{L}_{\text{self-inductance}} = \frac{\overbrace{\Phi}^{\text{get flux in coil}}}{\underbrace{i}_{\text{apply } i}} \qquad \underbrace{M_{21}}_{\text{mutual inductance}} = \frac{\overbrace{\Phi_2}^{\text{get flux in nearby coil 2}}}{\underbrace{i_1}_{\text{apply } i \text{ in coil 1}}}$$

One method to find inductance, L:

1) Relate the flux Φ to B $\rightarrow$ $\Phi = \int \vec{B} \cdot d\vec{a}$

2) Relate B to i (Ampere's law?)

3) Plug into ratio Φ/i and cancel B

Now, for the remaining Maxwell's equation:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The second term was not as easy to observe in circuits as the Emf generated by a change in the magnetic flux. It was in fact added to Maxwell's equations before confirmed by observations. One argument in favor of the extra term was that it took care of an inconsistency if $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ was always correct:

$$\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}_{\text{always zero}} = \underbrace{\mu_0 (\vec{\nabla} \cdot \vec{J})}_{\text{not always zero}}$$

Hmm - smarty pants Maxwell figured out how to fix it up. One method is to add another cancelling term on the RHS. What cancels $\vec{\nabla} \cdot \vec{J}$? We know from the continuity equation that $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

Now use Guass' law; $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t} \epsilon_0 (\vec{\nabla} \cdot \vec{E}) = -\nabla \cdot \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

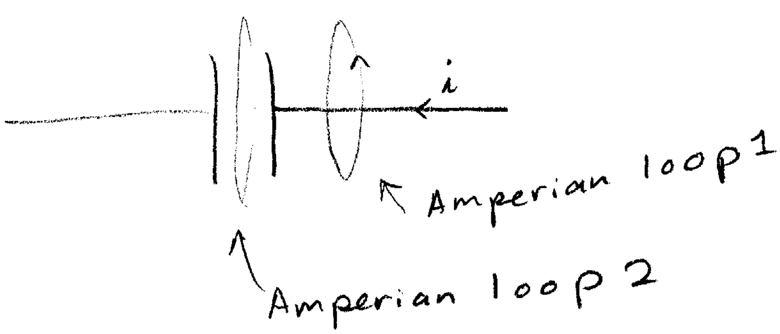
Then, $\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \cdot \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0$

So, now we have a consistent equation:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Does this work in practice? why, yes!

Consider a charging capacitor. This is really <u>not</u> a static situation, and so Ampere's law alone (remember Ampere's law comes from $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$) doesn't make it.



Amperian loop 1 gives the following:

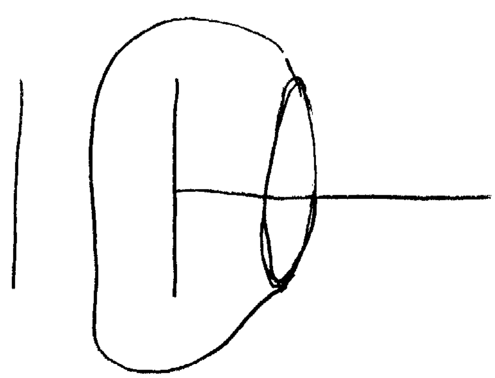
$$\int B \cdot dl = B(2\pi s) = \mu_0 i$$

$$\longrightarrow \quad B = \frac{\mu_0 i}{2\pi s}$$

Amperian loop 2 gives us:

$$\int B \cdot dl = \mu_0 i_{enc} = 0 \quad \longrightarrow \quad B = 0$$

Does this make sense? This says there is $\vec{B}$ field all around wire, but suddenly no field between plates. Take it one step further, keep the boundary of Amperian loop 1 but have a surface with no i_{enc}, like so (the balloon surface)



Same location in space, different answer. $(B=0)$

The answer (we know now) is that while current is flowing, the plates are charging (or discharging), and so the electric field between the plates is <u>changing</u>.

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{\partial q}{\partial t}$$

$$\int \vec{B}\cdot d\vec{l} = \mu_0 \cancelto{0}{i_{enc}} + \mu_0\epsilon_0 \int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a} = \mu_0\epsilon_0 \frac{\partial \Phi_E}{\partial t}$$

$$\int B\cdot dl = \mu_0 \cancel{\epsilon_0} \frac{1}{A\cancel{\epsilon_0}} \frac{\partial q}{\partial t} \cancel{\int da} = \mu_0 \frac{\partial q}{\partial t}$$

so, $\frac{\partial q}{\partial t}$ was called displacement current
(which everyone agrees is a dumb name)

and

$$\int \vec{B}\cdot d\vec{l} = \mu_0 i_{enc} + \mu_0 i_d$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\vec{J}_d$$

The important thing to recognize is that a charging capacitor has a $\frac{\partial E}{\partial t}$, and so in this quasi-static world of circuits it can be shown that a changing electric field induces a magnetic field. Maxwell's equations equations can now be written in their final dynamical form.

Example problem from Haliday & Resnick: Chapter 32, problem # 37:

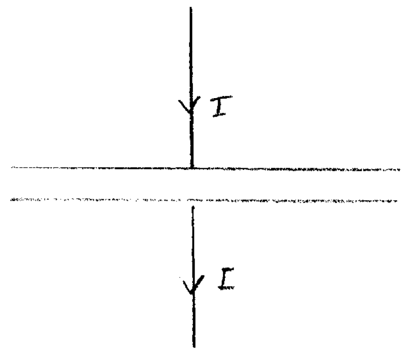
A parallel plate capacitor has square plates 1.0 m on a side. A current of 2A charges the capacitor, producing a uniform electric field between the plates, with $\vec{E}$ perpendicular to the plates.

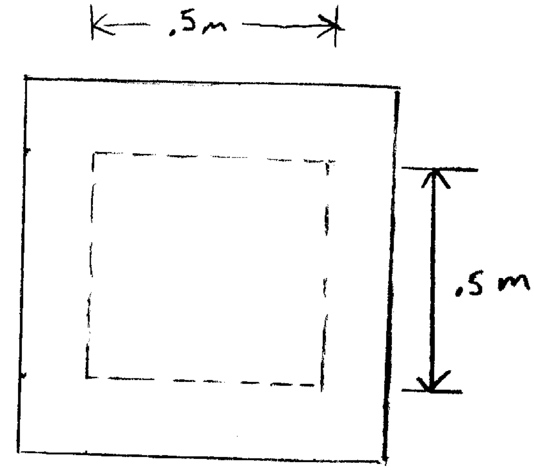
a) what is the displacement current i_d through the region between the plates?

b) What is $\frac{\partial E}{\partial t}$ in this region?

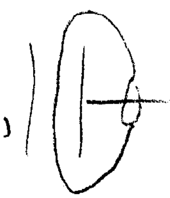
c) What is the displacement current through the square dashed path between the plates shown in the figure on the next page?

d) What is $\oint \vec{B} \cdot d\vec{\ell}$ around this square path?





a) Thinking of the balloon example we know that the 'displacement current' is equal to the actual current



$$i_d = 2A$$

b) $\mu_0 i_d = \mu_0 \epsilon_0 \int \frac{\partial E}{\partial t} \cdot da$, the area is constant and $\vec{E}$ is independent of spatial coordinate between the plates

$$i_d = \epsilon_0 \left(\frac{\partial E}{\partial t}\right) A$$

$$\frac{\partial E}{\partial t} = \frac{2A}{\epsilon_0 (1m)^2} = \frac{2}{8.85\times10^{-12}} \left[\frac{V}{ms}\right] = 2.26\times10^{11}\ V/ms$$

c) Since $\vec{E}$ is uniform between the plates, the ratio of areas will equal the ratio of displacement currents:

$$\frac{i_d'(.5m^2)}{i_d(1m^2)} = \frac{(.5)^2}{1^2} \Rightarrow i_d' = 2/4\ A = .5A$$

d) $\oint B \cdot ds = \mu_0 i_d = (1.26\times10^{-6})(\tfrac{1}{2}) = 6.3\times10^{-7}$ Tm

Lastly, let's be material people in a material world. Lets for convenience write Maxwell's equations in terms of free currents and free charges. There is one additional source of current in the dynamical case, $\vec{J}_p$ the density of current from a flow of bound charge when the polarization changes.

So, the total current is $\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p$

while $\rho = \rho_f + \rho_b$ does not change (changing $E + B$ can cause charge to flow, but are not sources of static charge)

What is J_p?

consider a small volume with area $\perp$ to P:

Then $\sigma_p = P$

$$\Delta \sigma_p = \Delta P$$

$$\frac{\partial P}{\partial t} = \frac{\partial \sigma_p}{\partial t} = J_p$$

The Maxwell equations that do not depend on $\vec{J}$ will not change.

Then only the $\vec{\nabla}\times\vec{B}$ equation changes

$$\vec{\nabla}\times\vec{B} = \mu_0 (J_f + J_b + J_p) + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\left(\vec{B}/\mu_0 - \vec{M}\right) = \vec{J}_f + \vec{J}_p + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \frac{\partial \epsilon_0 E}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \qquad (\vec{D} = \epsilon_0\vec{E} + P)$$

Then:

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho_f \\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{H} &= J_f + \frac{\partial \vec{D}}{\partial t} \end{aligned}}$$

The time derivatives don't change the boundary conditions, for the flux of the fields in the small loops used to get boundary conditions goes to zero.

$$\boxed{\begin{aligned} D_a^{\perp} - D_b^{\perp} &= \sigma_f & \qquad E_a^{\parallel} - E_b^{\parallel} &= 0 \\ B_a^{\perp} - B_b^{\perp} &= 0 & \qquad H_a^{\parallel} - H_b^{\parallel} &= \vec{K}_f \times \hat{n} \end{aligned}}$$