Magnetic Dipoles & Electric Quadrupoles

$$\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \left\{ \left(\vec{J}(x') \times \hat{r}\right) e^{ikr}\left(\frac{1}{r^2} - \frac{i\omega}{cr}\right)\right\}$$

keep $1/r^2$ now, for a page or so!

We'll use the same $e^{ikr} \to e^{ikr}(1 - ik\hat{n}\cdot\vec{x}' + \ldots)$ appx as before, where on the rhs r is now a const fiducial vector from some "source center" to the observer. Now we'll look at the 1st subleading terms:

$$\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \left\{ \left(\vec{J}(x') \times \left(\hat{n} - \hat{n}\,\frac{\hat{n}\cdot\vec{x}'}{r}\right)\right) e^{ikr}(1 - ik\hat{n}\cdot\vec{x}') \cdot \left(\frac{1}{r^2} - \frac{i\omega}{cr}\left(1 + \frac{\hat{n}\cdot\vec{x}}{r}\right)\right)\right\}$$

where we expanded $\hat{r}$, $\frac{1}{r}$, and e^{ikr} about the fixed fiducial r. The $1/r^2$ terms are

$$\left(\vec{B}\right)_{1/r^2} = \frac{\mu_0 e^{ikr}}{4\pi r^2}\int d^3x' \left\{ \vec{J} \times \hat{n}\left(1 + \cancel{\frac{i\omega\,\hat{n}\cdot\vec{x}'}{c}} - \cancel{\frac{i\omega\,\hat{n}\cdot\vec{x}'}{c}}\right)\right\}$$

and the $\frac{d}{r\lambda^2}$ terms are:

$$\left(\vec{B}\right)_{d/\lambda^2 r} = -\frac{\mu_0 e^{ikr}}{4\pi r}\underbrace{\left(\frac{k\omega}{c}\right)}_{= k^2}\int d^3x' \left(\vec{J} \times \hat{n}\right)\left(\hat{n}\cdot\vec{x}'\right)$$

We see explicitly that there are really two separate expansions. One is in powers of d/r, and the other is in powers of d/λ. It's consistent, therefore, to go into the radiation zone and drop $(\vec{B})_{1/r^2}$ while retaining $1/r$ terms that are suppressed by d/λ. Let's do this.

Now we get weird and write

$$(\hat{n}\cdot\vec{x}')\vec{J} = \frac{1}{2}\left[(\hat{n}\cdot\vec{x}')\vec{J} + (\hat{n}\cdot\vec{J})\vec{x}'\right] + \frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{n}$$

as you can verify by applying the triple product rule to the last term.

Note that the terms in [ ] are symmetric under exchanging the directions of $\vec{J}$ and $\vec{x}'$, while the last term is antisymmetric.

Focusing on the antisymmetric term, we get a contribution to $\vec{B}$:

$$\frac{-\mu_0 e^{ikr}}{4\pi r} k^2 \left(\left(\underbrace{\frac{1}{2}\int d^3x'\, \vec{x}'\times\vec{J}}_{\equiv \vec{m}\text{, the magnetic dipole moment.}}\right)\times\hat{n}\right)\times\hat{n}$$

So one term in $\vec{B}$ is

$$\vec{B}_{mag.dip.} = -\frac{\mu_0 e^{ikr}}{4\pi r} k^2 \hat{n} \times (\hat{n} \times \vec{m}).$$

The remaining stuff is

$$\frac{\mu_0 e^{ikr}}{4\pi r} \int d^3x' \Big[\underbrace{\cancel{\left(\vec{J} \times \frac{\hat{n}}{r} \right)}}_{\substack{\text{already agreed} \\ \text{to drop, } \mathcal{O}(1/r^2)}} - \Big(\underbrace{k^2 \frac{\vec{J}\,(\hat{n}\cdot\vec{x}')}{2} + \frac{(\hat{n}\cdot\vec{J})}{2} \vec{x}' k^2}_{\substack{\text{leftover from} \\ \text{symm/antisymm} \\ \text{decomposition.}}} \Big) \times \hat{n} \Big]$$

Using $\vec{\nabla}\cdot\vec{J} = i\omega\rho$ and some IBP trickery, this can be rewritten as

two $\vec{x}'$'s now

$$\frac{i\mu_0 e^{ikr} k^3}{8\pi r \; c} \left(\underbrace{\int d^3x' \, (\hat{n}\cdot\vec{x}') \, \vec{x}' \rho(x')}_{} \right) \times \hat{n}$$

It turns out this is proportional to the electric quadrupole moment tensor,

$$Q_{\alpha\beta} = \int d^3x \left(3 x_\alpha x_\beta - r^2 \delta_{\alpha\beta} \right) \rho(x)$$

Q is traceless $\left(Tr(\delta_{\alpha\beta}) = 3, \; Tr(3x_\alpha x_\beta) = 3r^2 \right)$

and symmetric $Q_{\alpha\beta} = Q_{\beta\alpha}$

If we define a vector $(\vec{Q})_\alpha \equiv \sum_\beta Q_{\alpha\beta} n_\beta$

(remember that homework??) then

one can show that

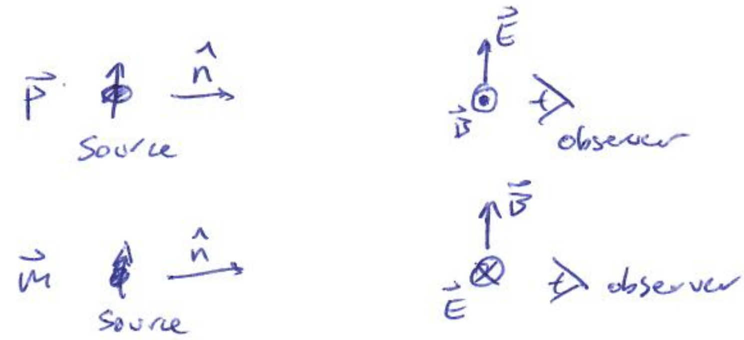
$$\hat{n} \times \int \vec{x}\,' (\hat{n}\cdot\vec{x}\,') \rho(x') d^3x' = \frac{1}{3} \hat{n} \times \vec{Q}$$

All together,

$$\vec{B}_{\text{subleading}} = -\frac{\mu_0 e^{ikr}}{4\pi r} k^2 \hat{n} \times \left(\hat{n} \times \vec{m} + \frac{ik}{6c} \vec{Q} \right)$$

We can compute $\vec{E}$ using $\frac{\vec{E}}{c} = \vec{B} \times \hat{n}$ in the radiation zone / far field.

The magnetic dipole term is very similar to the electric dipole term, but the polarizations are different.



electric dipole radiation

magnetic dipole radiation.

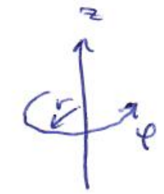
If we swap $\vec{p} \to \frac{\vec{m}}{c}$, $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$, the formulas for mag & elec dipole rad are the same. So we can easily get the power from prev.

Example: Loop Antenna

$I = I_0 e^{-i\omega t}$

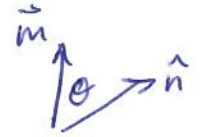
N coils, cross sectional area A

Then $\vec{m}_\omega = \frac{N}{2}\int (r\,d\varphi)\,(r\hat{r})\times(I_0\hat{\varphi})$



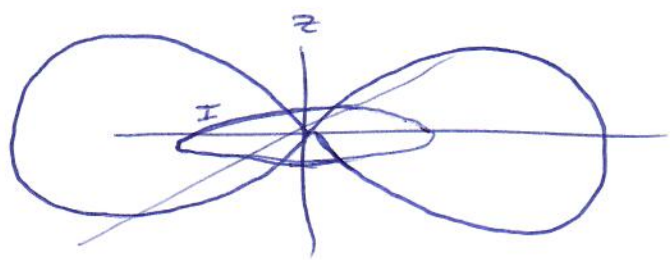
$$= N I_0 \pi r^2 \hat{z} = N I_0 A$$

same as elec dipole, & $\vec{p} \leftrightarrow \frac{\vec{m}}{c}$



so $\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{k^4}{8\pi c}\left|(\hat{n}\times\vec{m})\right|^2$

$$= \frac{1}{4\pi\epsilon_0}\frac{k^4}{8\pi c}|\vec{m}|^2\sin^2\theta$$



power diagram — maximum radiation in the plane of the loop ($\sin\theta \to 1$) and same for all φ.

$$P_{total} = \frac{1}{4\pi\epsilon_0}\frac{k^4}{3c}|\vec{m}|^2$$

$$= \frac{1}{4\pi\epsilon_0}\frac{k^4}{3c}(NAI_0)^2 \equiv \frac{1}{2}R_{eff}I_0^2$$

Again of "I^2R" form – the RF generator supplying I "sees" a non-ideal inductor of resistance R_{eff}.

An $N=10$ coil with $A = 1\,m^2$ and $\lambda = 20\,m$ ($\omega \sim 100$ MHz)

has $R_{eff} = \frac{1}{4\pi\epsilon_0} \frac{2(\omega/c)^4}{3c} (NA)^2 \approx 20\,\Omega$

So if $I_0 = 10$ Amp then $P \approx 1$ kW.

Now let's look at electric quadrupoles. Since $Q_{\alpha\beta}$ is hard to visualize, let's just do some examples.

$$Q_{\alpha\beta} = \int \left(3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}\right) \rho(x)\, d^3x$$

- for a spherically symmetric charge distribution, we expect only a monopole moment, so Q should vanish. Explicitly, $\int d\theta\, d\varphi\; x_\alpha x_\beta$

  must vanish if $\alpha \neq \beta$ ($\int_{angles} xy = 0$ because half the time $xy > 0$ and half the time $xy < 0$ on a sphere.)

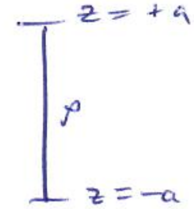
  while for $\alpha = \beta$ $\int d\theta d\varphi\, x^2 = \int d\theta d\varphi\, y^2 = \int d\theta d\varphi\, z^2$

$$= \frac{1}{3} \int d\theta d\varphi\, r^2$$

So $\int d\theta d\varphi dr\, r^2 \sin\theta \left(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}\right)\rho$

$$= \int d\theta\, d\varphi\, dr\; r^2 \sin\theta\, \underbrace{\left(3 \cdot \tfrac{1}{3} r^2 \delta_{\alpha\beta} - r^2 \delta_{\alpha\beta}\right)}_{=0} \rho$$

• $\rho(\vec{x}) = \begin{cases} \lambda\, \delta(x)\, \delta(y) & |z| < a \\ 0 & |z| > a \end{cases}$



line charge

$\alpha \neq \beta$: $Q_{\alpha\beta} = \int 3\, x_\alpha x_\beta\, \lambda\, \delta(x)\, \delta(y)\, d^3x$

But at least one of the x_α, x_β must be x or y (since $\alpha \neq \beta$ and there are only 3 choices!) so the δ-functions kill it. $\alpha \neq \beta \Rightarrow Q_{\alpha\beta} = 0$

$\alpha = \beta$! $Q_{\alpha\alpha} = \int (3\, x_\alpha^2 - r^2)\, \lambda\, \delta(x)\, \delta(y)\, d^3x$

$\alpha = 1$: $x_\alpha \to x$, $Q_{11} = -\lambda \int_{-a}^{a} dz\, z^2 = \frac{-2\lambda a^3}{3}$

$\alpha = 2$: Same, $Q_{22} = \frac{-2\lambda a^3}{3}$

$\alpha = 3$: $Q_{33} = \lambda \int_{-a}^{a} dz\, (3z^2 - z^2)$

$= \frac{4\lambda a^3}{3}$

$$Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \times \frac{2\lambda a^3}{3}$$

• $\rho(\vec{x}) = Q\,\delta^{(3)}(\vec{x}-\vec{a})$

$$\alpha \neq \beta: \quad Q_{\alpha\beta} = 3Q\int x_\alpha x_\beta\, \delta^{(3)}(\vec{x}-\vec{a})\, d^3x = 3a_\alpha a_\beta Q$$

$$\alpha = \beta: \quad Q_{\alpha\alpha} = Q\int (3x_\alpha^2 - r^2)\, \delta^{(3)}(x-a)\, d^3x$$

$$= 2\left(a_\alpha^2 - a_\beta^2 - a_\gamma^2\right) Q$$

where $\alpha\neq\beta$, $\alpha\neq\gamma$, $\beta\neq\gamma$.

$$\Rightarrow Q_{\alpha\beta} = Q\begin{bmatrix} 2a_x^2 - a_y^2 - a_z^2 & 3a_x a_y & 3a_x a_z \\ 3a_x a_y & 2a_y^2 - a_z^2 - a_x^2 & 3a_y a_z \\ 3a_x a_z & 3a_y a_z & 2a_z^2 - a_x^2 - a_y^2 \end{bmatrix}$$

if $\vec{a} = a\hat{z}$, reduces to $Q\,a^2\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

So, for example, a line charge with harmonically oscillating charge density will produce a quadrupole pattern.

Quadrupole radiation is also important in gravity. the "charge" is the mass, and $\dot{\vec{P}}$ is like the total momentum. It can't oscillate since momentum conservation $\Rightarrow \dddot{\vec{P}} = 0$. Similarly $\vec{m} \rightarrow$ angular momentum and can't oscillate. The lowest multipole that can oscillate & radiate (gravitational waves) is the quadrupole (of the mass distribution.)

The time-averaged quadrupole radiation power is

$$\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{ck^6}{288\pi}\left|(\hat{n}\times\vec{Q})\times\hat{n}\right|^2$$

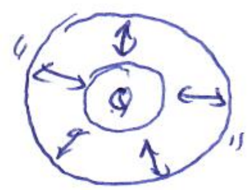
The angular dependence is complicated because $\vec{Q}$ also depends on $\hat{n}$. Jackson obtains

$$P = \frac{1}{4\pi\epsilon_0}\frac{ck^6}{360}\sum_{\alpha,\beta}\left|Q_{\alpha\beta}\right|^2$$

for a single Fourier component.

ASIDE

[By the way, we skipped "monopole radiation." There are no (yet observed) magnetic monopoles, but what about a sphere of charge with pulsing radius?



far zone: $E_\omega^{mono} = -\frac{ik e^{ikr}}{4\pi\epsilon_0 r}\int d^3x'\left(\hat{n}\rho_\omega - j_\omega/c\right)$

but $\int d^3x' \vec{j} = -i\omega\int d^3x' \vec{x}'\rho(x') = 0$ by symmetry of ρ

so $E^{mono} = -\frac{ik e^{ikr}}{4\pi\epsilon_0 r} Q_{total}\,\hat{n} + O(1/r^2)$

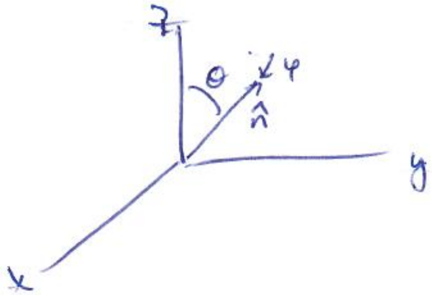
But $B_\omega^{mono} \propto \left(\int d^3x' \vec{j}\right)\times\hat{n} = 0$

so $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = 0$ in the far zone.

$\Rightarrow$ NO MONOPOLE RAD.]

if $Q_{\alpha\beta} \propto \begin{pmatrix} -1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$, like for our oscillating line,

$$\frac{dP}{d\Omega} \propto \left[\underbrace{\left(\epsilon_{ijk}\, n_j\, Q_{kl}\, n_l\right)}_{(\hat{n}\times\vec{Q})} \underbrace{\epsilon_{mip}\, n_p}_{\times\,\hat{n}}\right]\left[\epsilon_{i'j'k'}\, n_{j'}\, Q^{*}_{k'l'}\, n_{l'}\, \epsilon_{mi'p'}\, n_{p'}\right]$$



$\hat{n} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$

Plug in, and when the dust clears,

$$\frac{dP}{d\Omega} \propto \sin^2\theta\,\cos^2\theta$$

max @ $\theta = \pi/4,\ 3\pi/4$