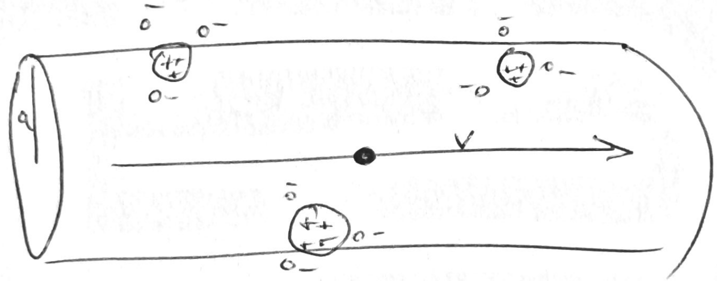
Cherenkov Radiation

We worked out the electromagnetic fields of a charged particle moving with <u>constant velocity</u> through a <u>polarizable</u>, <u>absorptive</u> medium which we modeled as a <u>constant density</u> of <u>harmonically bound electrons</u>



$$\tilde{\vec{E}}(\vec{k},\omega) = -i\left(\vec{k} - \omega\vec{v}/\bar{c}^2\right)\cdot \frac{2\pi z e}{\epsilon(\omega)\left(k^2-\omega^2/\bar{c}^2\right)} \cdot \delta(\vec{k}\cdot\vec{v}-\omega)$$

$$\tilde{\vec{B}}(\vec{k},\omega) = \frac{i\,\vec{k}\times\vec{v}}{\bar{c}^2} \cdot \frac{2\pi z e}{\epsilon(\omega)\left(k^2-\omega^2/\bar{c}^2\right)} \cdot \delta(\vec{k}\cdot\vec{v}-\omega)$$

$$\epsilon(\omega) = \epsilon_0\left(1 + \frac{N Z_t e^2}{m_e}\,\frac{1}{(\omega_0^2-\omega^2) - i\omega\Gamma}\right)$$

frequency dependent dielectric constant $\tilde{\vec{P}}(\omega)/\tilde{\vec{E}}(\omega) = \epsilon(\omega) - \epsilon_0$

$N Z_t =$ e^- number density

typical atomic frequency

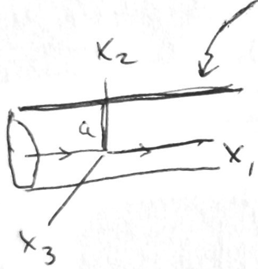
absorptive effects

$$\bar{c} = 1/\sqrt{\mu_0\,\epsilon(\omega)}$$

To calculate the energy loss we used the Poynting vector:

$$\left.\frac{dE}{dx}\right)_{b>a} = \int_{(\text{cylinder})} (\text{Power flux}) \times \left(\frac{dt}{dx}\right)$$

$$= \frac{1}{V} \int_{(\text{cylinder})} \vec{S} \cdot \hat{n} \, dA$$



$$= \frac{-1}{\mu_0 V} \int_{-\infty}^{\infty} (2\pi a)\, B_3(x_1, a, 0, t) E_1(x_1, a, 0, t)\, dx_1$$

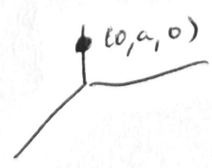
using cylindrical symmetry and the fact that only B_3 and E_1 contribute to S_2 along the line $(x_1, x_2 = a, x_3 = 0)$

The integral over all x_1 at a fixed t is v times the integral over all t at fixed x_1 (mathematically, since we assume constant v, we change variables $x_1 \to x_1 - vt$)

So

$$\left.\frac{dE}{dx}\right)_{b>a} = \frac{-2\pi a}{\mu_0} \int_{-\infty}^{\infty} dt\, B_3(0, a, 0, t) E_1(0, a, 0, t)$$

setting the fixed x_1 to zero



We need the fields at a fixed $\vec{x}$ and all t.

We have the fields for $\tilde{k}, \omega$. We invert the spatial part of the fourier transform and use a variant of Parseval's theorem for the frequency dependence to write

$$\left.\frac{dE}{dx}\right)_{b>a} = \frac{-2\pi a}{\mu_0}\int_0^\infty 2\,\mathrm{Re}\left[B_3^*(\vec{x}=(0,a,0),\omega)\,E_1(\vec{x}=(0,a,0),\omega)\right]\frac{d\omega}{2\pi}$$

frequency domain

from explicitly inverting the spatial fourier transform

$$B_3(\vec{x}=(0,a,0),\omega) = \frac{v\,E_2(0,a,0,\omega)}{c^2}$$

$$E_2(0,a,0,\omega) = \frac{ze}{2\pi^2\epsilon v}\lambda K_1(\lambda a)$$

$$E_1(0,a,0,\omega) = \frac{-ize\omega}{2\pi^2\epsilon v^2}\left(1-v^2/c^2\right)K_0(\lambda a)$$

$$\lambda = \frac{\omega}{v}\sqrt{1-v^2/c^2}$$

The "i" in E_1, and the "Re" in front of $B_3^* E_1$, means we need something else to be complex to get a nonzero answer.

There are two possibilities.

(1) Last time, we used the "i" in $\epsilon(\omega)$, sitting in front of the absorptive term Γ, and we assumed $v < \bar{c}$.

[Comment: before we considered density effects, we modeled energy loss as scattering off a free electron, with no Γ or ω_0. Our new analysis now looks very different. But in fact there is a limit where the earlier result is recovered: For $\Gamma \ll \omega_0$,

$$\mathrm{Im} \frac{1}{(\omega_0^2 - \omega^2) - i\Gamma\omega} = i\pi \underbrace{\delta(\omega_0^2 - \omega^2)}_{\text{independent of } \Gamma!} \text{ using}$$

the Plemelj formula. Then in the nonrelativistic limit and for $\lambda a \ll 1$ it is straightforward to do the ω integral and recover the old result.]

This gave us the energy loss into atomic e^-.

Looking at E and B, there's another way we might get an "i": if $v > \bar{c}$, λ is complex even if we ignore the "$i\Gamma$" in ϵ (which we will do now for simplicity.)

It turns out this is connected with radiation, so with malice aforethought let us change our attention from atomic a ($|\lambda| a \ll 1$ for optical $|\lambda|$) to distant a ($|\lambda| a \gg 1$, as we do in radiation problems where energy is carried off to ∞.)

$$\text{Then } K_0(\lambda a) \longrightarrow \sqrt{\frac{\pi}{2\lambda a}}\, e^{-\lambda a}$$

$$K_1(\lambda a) \longrightarrow (\text{Same})$$

$$\text{So } \frac{-4\pi a}{\mu_0} B_3^{*} E_1 \longrightarrow \frac{i(Ze)^2 \omega}{2\pi^2 \epsilon v^2}\left(1 - \frac{v^2}{\bar{c}^2}\right)\sqrt{\frac{\lambda^*}{\lambda}}\, e^{-(\lambda+\lambda^*)a}$$

for $|\lambda| a \gg 1$

If λ has a positive real part then this $\to 0$ exponentially as $a \to \infty$.

But if λ is imaginary, no suppression!

$(\lambda + \lambda^* = 0,)$

$\sqrt{\lambda^*/\lambda} = i$

As mentioned above we are assuming ϵ is real now (no absorption) - Jackson notes that this is not strictly necessary, and small absorption can be included, but it complicates the analysis needlessly with subtle orders of limits. So let's agree to set $\Gamma \to 0$.

Then ϵ and $\bar{c}$ are real and $\delta = \frac{\omega}{v}\sqrt{1 - v^2/\bar{c}^2}$ is imaginary <u>iff $v > \bar{c}$</u>, or $v > \sqrt{\frac{\epsilon_0}{\epsilon}}\, c$.

$\therefore$ If the beam particle speed exceeds the <u>phase velocity</u> of EM waves in the medium, there is <u>radiation</u>! (Nonzero power/unit solid angle @ ∞)

This is called Cherenkov radiation, observed by Cherenkov in 1934.

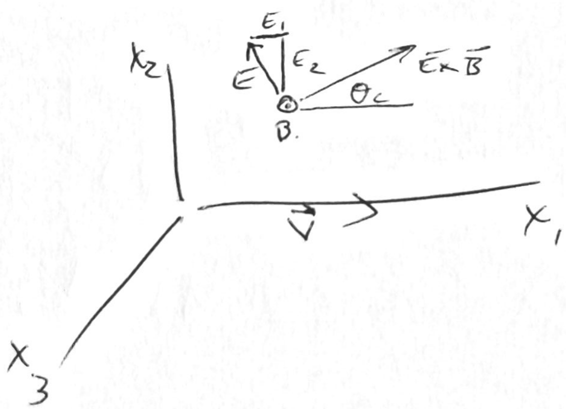
$$\left.\frac{dE}{dx}\right)_{rad} = \frac{-4\pi a}{\mu_0} \int_\Sigma \frac{d\omega}{2\pi} \, \mathrm{Re}\left(B_3^*(\omega) E_1(\omega)\right)$$

where $\Sigma = \{\omega : \bar{c} < v\}$

$$= \frac{(Ze)^2}{2\pi^2 v^2} \int_{\Sigma} \frac{d\omega}{2\pi}\, \omega \left(\frac{v^2}{\bar{c}^2} - 1\right) \frac{1}{\epsilon}$$

Evidently a somewhat complicated frequency dependence determined by the detailed behavior of $\epsilon(\omega)$.
For relativistic particles we just need $\epsilon(\omega) \gtrsim 1$.

The direction of emission is given by the direction of $\bar{S} \sim \bar{E} \times \vec{B}$. We had only B_3 in the x_1-x_2 plane



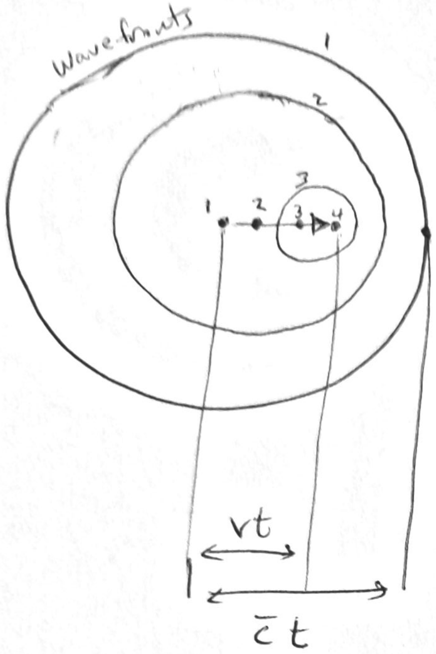
So the angle θ_c relative to $\vec{v}$ is $\tan \theta_c = \left|\frac{E_1}{E_2}\right|$

In the far field $|\lambda a| \gg 1$ the ratio is

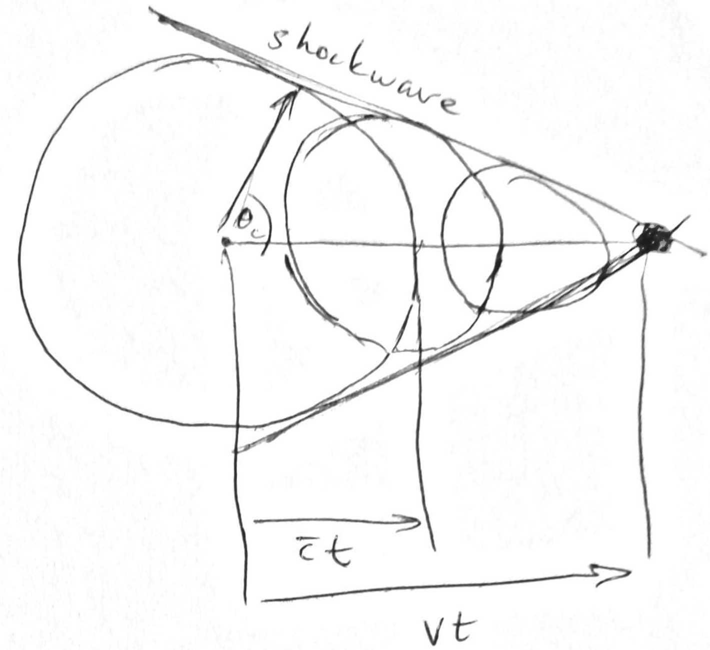
$$\left|\frac{E_1}{E_2}\right| = \left|\frac{\omega/v\,(1 - v^2/\bar{c}^2)}{\omega/v\,\sqrt{1 - v^2/\bar{c}^2}}\right| = \left|\sqrt{1 - \bar{v}^2/\bar{c}^2}\right|$$

$$= \sqrt{v^2/\bar{c}^2 - 1}$$ in the relevant regime

The angle is larger the farther above $\bar{c}$ we get.



subluminal



superluminal

cherenkov radiation is analogous to a wake behind a boat.

The sensitivity of θ_c to v allows cherenkov radiation to be a useful velocity measurement tool, if the medium dielectric function is known.