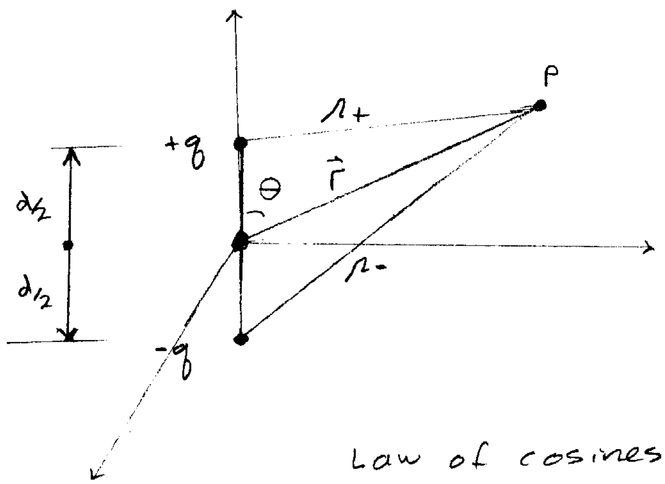
Multipole expansion

Far away from a distribution of charges, the field due to the charges is not sensitive to details in the structure of the distribution. For example, far enough away from a group of charges with net charge Q, the group of charges looks approximately like a point charge Q. What if the overall set of charges is neutral? Still, there is something there, the next possibility is that the group of charges looks like a dipole. If the dipole moment vanishes, the quadrupole... and so on.

Following Griffiths, we'll show the dominance of the dipole term for a physical dipole, and then develop the multipole expansion for a general distribution of charge. These involve:

1) Writing an explicit expression for $\mathscr{r}$
2) Being at a large distance from the charge enables a binomial expansion of the expression for $\mathscr{r}$.
3) Once an expansion has been done, the potential is a sum of terms which go up in order of powers of $1/r$; i.e. $1/r$, $1/r^2$, $1/r^3$ ....

The dipole:



Law of cosines:

$$r_+^2 = r^2 + (d/2)^2 - 2(d/2) r\cos\theta$$

$$r_-^2 = r^2 + (d/2)^2 - 2(d/2) r\cos(\pi - \theta)$$

$$\cos(\pi-\theta) = \cos\pi\cos\theta + \cancelto{0}{\sin\pi}\sin\theta = -\cos\theta$$

binomial expansion $(x^2 < 1)$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$\frac{1}{r_+} = \left(r^2 + (d/2)^2 - rd\cos\theta\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 + \left(\frac{d}{2r}\right)^2 - \frac{d}{r}\cos\theta\right)^{-1/2}$$

$$\approx \frac{1}{r}\left(1 - d/r\cos\theta\right)^{-1/2} \qquad \text{since } \frac{d}{r}\cos\theta \gg \frac{d^2}{2r^2}$$

$$\approx \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right) \qquad \text{using binomial expansion}$$

Similarly

$$\frac{1}{r^-} \approx \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$$

Then, $V(r) = \frac{1}{4\pi\epsilon_0}\left(q/r^+ - \frac{q}{r^-}\right)$ is approximately (at large r):

$$V(r) \approx \frac{1}{4\pi\epsilon_0}\frac{q}{r}\left(\cancel{1} + \frac{d}{2r}\cos\theta - \cancel{1} + \frac{d}{2r}\cos\theta\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

note that $p \equiv qd$ is the magnitude of the dipole moment

Notice that in this calculation of the potential due to a dipole, the lower order term (monopole) vanishes due to having equal & opposite charges; and higher order terms are neglected as the decrease in successive powers of $(1/r)$.

We handle a general charge distribution the same way, but all terms are present until such time as a particular distribution is specified.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \rho(\vec{r}\,') \, d\tau'$$

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\theta'$$

$$= r^2 \left(1 + (r'/r)^2 - 2(r'/r)\cos\theta' \right)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r} \left(1 + (r'/r)(r'/r - 2\cos\theta') \right)^{-1/2}$$

$$= \frac{1}{r} \Big\{ 1 - \tfrac{1}{2} (r'/r)(r'/r - 2\cos\theta')$$

$$+ \tfrac{3}{8} (r'/r)^2 (r'/r - 2\cos\theta')^2 - \ldots \Big\}$$

collect terms of the same order in (r'/r):

$$\frac{1}{\mathscr{r}} = \frac{1}{r} \left\{ 1 + (r'/r)(\cos\theta') + (r'/r)^2 \left(\tfrac{3}{2}\cos^2\theta' - \tfrac{1}{2} \right) + \ldots \right\}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos\theta')$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta') \rho(\vec{r}\,') \, d\tau'$$

problem 3.26

$$\rho(r',\theta') = k\frac{R}{r'^2}(R-2r')\sin\theta'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\vec{r}\,')\,d\tau'$$

since $r = z$, θ' is measured from z-axis and θ' in $\rho(r',\theta')$ is the same as θ' in $\mathscr{r}$. Also, $r = z$

substituting expansion for $1/\mathscr{r}$:

$$\int \frac{1}{\mathscr{r}}\rho(\vec{r}\,')\,d\tau' = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \int (r')^n P_n(\cos\theta')\frac{kR}{r'^2}(R-2r')\sin\theta'\,d\tau'$$

$$= \frac{2\pi kR}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \int_0^R (r')^n \frac{1}{(\cancel{r'})^2}(R-2r')\cancel{r'^2}\,dr' \int_0^\pi P_n(\cos\theta')\sin^2\theta'\,d\theta'$$

$$= \frac{2\pi kR}{z}\sum_n \left(\frac{1}{z}\right)^n \int_0^R \left[R(r')^n - 2(r')^{n+1}\right]dr' \int_0^\pi P_n(\cos\theta')\sin^2\theta'\,d\theta'$$

$$= \frac{2\pi kR}{z}\sum_n \left(\frac{1}{z}\right)^n \left\{R\frac{(R)^{n+1}}{n+1} - 2\frac{R^{n+2}}{n+2}\right\} \int_0^\pi P_n(\cos\theta')\sin^2\theta'\,d\theta'$$

To get any further (since $\sin\theta'$ is not obviously Legendre polynomial of some order) we must evaluate this term by term.

problem 3.26

$n = 0$ (monopole term):

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{2\pi K R}{z} \cancelto{0}{\left(R^2 - 2\frac{R^2}{2} \right)} \int_0^\pi \sin^2\theta' d\theta' = 0$$

$n = 1$ (dipole term):

$$V(\vec{r}) = \frac{KR}{2\epsilon_0 z^2} \left(\frac{R^3}{2} - 2\frac{R^3}{3} \right) \int_0^\pi \cos\theta' \sin^2\theta' d\theta'$$

$$= -\frac{KR^4}{12\epsilon_0 z^2} \int_0^0 u^2 du = 0$$

$n = 2$ (quadrupole term):

$$V(\vec{r}) = \frac{KR}{2\epsilon_0 z^3} \left(\frac{R^4}{3} - 2\frac{R^4}{4} \right) \frac{1}{2} \int_0^\pi \left(3\cos^2\theta' - 1 \right) \sin^2\theta' d\theta'$$

$$= -\frac{KR^5}{24\epsilon_0 z^3} \left\{ 3\int_0^\pi (\sin\theta' \cos\theta')^2 d\theta' - \int_0^\pi \sin^2\theta' d\theta' \right\}$$

$$= -\frac{KR^5}{24\epsilon_0 z^3} \left\{ \tfrac{3}{2}\left(\tfrac{1}{2}\right) \int_0^\pi \sin^2 2\theta' d2\theta' - \int_0^\pi \sin^2\theta' d\theta' \right\}$$

$$\int \sin^2\theta \, d\theta = \frac{1}{2}\int (1-\cos 2\theta)\, d\theta \quad , \text{ then}$$

$$\frac{3}{8}\int_0^{2\pi} \sin^2 u \, du - \int_0^{\pi} \sin^2 u \, du = \frac{3}{16}\int_0^{2\pi} (1-\cos 2u)\, du - \frac{1}{2}\int_0^{\pi} (1-\cos 2u)\, du$$

$$= \frac{3}{16}\left\{ 2\pi - \frac{1}{2}\cancelto{0}{\sin v}\Big|_0^{4\pi} \right\} - \frac{\pi}{2} + \frac{1}{2}\cancelto{0}{\sin v}\Big|_0^{2\pi} \Big\} = -\frac{\pi}{8}$$

<u>$n = 2$</u> :

$$V(r) = + \frac{k R^5 \pi}{192\, \epsilon_0 Z^3}$$

Since terms decrease in magnitude with increasing order, n (when not equal to zero) the first non-zero term is an approximate value for $V(\vec{r})$.