Where were we? Find $\frac{\partial \vec{p}_{mech}}{\partial t}$, write in form of continuity equation, identify $\frac{\partial \vec{p}_{em}}{\partial t}$, $\vec{\nabla} \cdot \overleftrightarrow{T}$.

EM force on charges in a volume, V

$$\vec{F} = \int_V (\rho \vec{E} + \vec{J} \times \vec{B})\, d\tau$$

<u>Force per volume</u>, $\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$

<u>Write in terms of fields only</u>:

$$\left.\begin{aligned} \rho &= \epsilon_0 \vec{\nabla} \cdot \vec{E} \\ \vec{J} &= \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}\right\} \text{Maxwell equations}$$

$$\vec{f} = \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} + \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$$

Use every trick in the book to get this into a hideous form, and then simplify with the magic of notation. We're not going to let Griffiths do all the sweating, though.

Let's look at $\vec{f}$, and with 20-20 hindsight (we know the answer) figure out how to attack it. We know that the continuity equation has a $\frac{\partial \vec{p}_{em}}{\partial t}$ term, and the last term of $\vec{f}$ looks close, but it would be better if we had a time derivative of $\vec{E}$ <u>and</u> $\vec{B}$.

Like so:

$$\frac{\partial(\vec{E}\times\vec{B})}{\partial t} = \left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) + \left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \underbrace{\frac{\partial(\vec{E}\times\vec{B})}{\partial t}}_{\uparrow\ \text{Good}} - \underbrace{\vec{E}\times\frac{\partial\vec{B}}{\partial t}}$$

Let's try not to have any terms besides $\frac{\partial \vec{p}_{em}}{\partial t}$ with both $\vec{E}$ + $\vec{B}$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial(\vec{E}\times\vec{B})}{\partial t} + \underbrace{\vec{E}\times(\vec{\nabla}\times\vec{E})}_{\text{using } \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

Now we have for $\vec{f}$:

$$\vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] - \frac{1}{\mu_0}\left[\vec{B}\times(\vec{\nabla}\times\vec{B})\right]$$

$$-\epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

① Add in a $\underbrace{(\vec{\nabla}\cdot\vec{B})}_{0}\vec{B}$ to the terms with $\vec{B}$ for symmetry

② Use a product rule to get rid of triple cross-products.

Its not that we hate triple cross-products, but rather that we want all terms to have the form of a time derivative $\frac{\partial \wp_{em}}{\partial t}$, or a divergence $\vec{\nabla} \cdot \overleftrightarrow{T}$ when we're done.

So, for example

$$\vec{E} \times \vec{\nabla} \times \vec{E} = \frac{1}{2} \nabla (E^2) - (\vec{E} \cdot \vec{\nabla}) \vec{E}$$

Now we have:

$$\vec{f} = \epsilon_0 [(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E}] + \frac{1}{\mu_0} [(\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B}]$$

$$- \frac{1}{2} \vec{\nabla} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \epsilon_0 \frac{\partial (\vec{E} \times \vec{B})}{\partial t}$$

The circled term is the only time derivative, so it must be that

$$\frac{\partial \wp_{em}}{\partial t} = \epsilon_0 \frac{\partial (\vec{E} \times \vec{B})}{\partial t} = \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

The rest of $\vec{f}$ must be $\vec{\nabla} \cdot \overleftrightarrow{T}$, and now we have to figure out how to write $\overleftrightarrow{T}$!

Since we have so nicely separated E & B, we can work with only the E terms (for example) and extend the results to include B at the end.

We begin by comparing one component of the general expression $\vec{\nabla} \cdot \overleftrightarrow{T}$ (which is a vector) to one component of our result. Let's pick the x-component. A tensor dotted with a vector yields a vector:

$$(\nabla \cdot \overleftrightarrow{T})_x = \sum_{i=x,y,z} \nabla_i T_{ix}$$

$$= \nabla_x T_{xx} + \nabla_y T_{yx} + \nabla_z T_{zx}$$

Now pick out the x-component of the $\vec{E}$ terms of our vector:

$$\epsilon_0 \left[(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} \right] - \tfrac{1}{2} \vec{\nabla} \epsilon_0 E^2$$

$$\Downarrow$$

$$\epsilon_0 \left[(\vec{\nabla} \cdot \vec{E}) E_x + (\vec{E} \cdot \vec{\nabla}) E_x \right] - \tfrac{1}{2} \nabla_x \epsilon_0 E^2$$

To proceed further, we'll write this out term by term, and then group the terms according to derivative, i.e. $\frac{\partial}{\partial x}(\text{stuff}) + \frac{\partial}{\partial y}(\text{stuff}) + \frac{\partial (\text{stuff})}{\partial z}$ Finally we can equate these to

$\nabla_x T_{xx}$, $\nabla_y T_{yx}$, $\nabla_z T_{zx}$ thereby identifying the electric field portion of T_{xx}, T_{yx}, T_{zx}. Add the magnetic field part, and we'll be done. OK, here goes:

$$\epsilon_0 \Big[\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) E_x$$

$$+ \left(E_x \frac{\partial}{\partial x} + E_y \frac{\partial}{\partial y} + E_z \frac{\partial}{\partial z} \right) E_x$$

$$- \tfrac{1}{2} \frac{\partial}{\partial x} \left(E^2 \right) \Big]$$

This is, grouping by derivative type,

$$\epsilon_0 \Big[\left(E_x \frac{\partial E_x}{\partial x} + E_x \frac{\partial E_x}{\partial x} - \tfrac{1}{2} \frac{\partial}{\partial x} E^2 \right)$$

$$+ \left(E_x \frac{\partial E_y}{\partial y} + E_y \frac{\partial E_x}{\partial y} \right)$$

$$+ \left(E_x \frac{\partial E_z}{\partial z} + E_z \frac{\partial E_x}{\partial z} \right) \Big]$$

This is:

$$\epsilon_0 \left[\frac{\partial}{\partial x}\left(E_x E_x - \tfrac{1}{2} E^2 \right) + \frac{\partial}{\partial y}\left(E_x E_y \right) + \frac{\partial}{\partial z}\left(E_x E_z \right) \right]$$

similarly for B:

$$\frac{1}{\mu_0} \left[\frac{\partial}{\partial x}\left(B_x B_x - \tfrac{1}{2} B^2 \right) + \frac{\partial}{\partial y}\left(B_x B_y \right) + \frac{\partial}{\partial z}\left(B_x B_z \right) \right]$$

Since the sum should be $\nabla_x T_{xx} + \nabla_y T_{yx} + \nabla_z T_{zx}$

we can now see that

$$T_{xx} = \epsilon_0 \left(E_{xx} - \tfrac{1}{2} E^2 \right) + \frac{1}{\mu_0}\left(B_{xx} - \tfrac{1}{2} B^2 \right)$$

$$T_{yx} = \epsilon_0 \left(E_y E_x \right) + \frac{1}{\mu_0}\left(B_y B_x \right)$$

$$T_{zx} = \epsilon_0 \left(E_z E_x \right) + \frac{1}{\mu_0}\left(B_z B_x \right)$$

If we swap the indices in T_{yx}, we get the same result since $E_y E_x = E_x E_y$ and so on. So, $\overleftrightarrow{T}$ is a <u>symmetric</u> tensor, meaning that $T_{ij} = T_{ji}$. So, writing the result in general form:

$$T_{ij} = \epsilon_0 \left(E_i E_j - \tfrac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \tfrac{1}{2} \delta_{ij} B^2 \right)$$

Here, δ_{ij} is the Kronecker delta, as opposed to the Dirac delta function introduced in chapter 1. The Kronecker delta is defined as $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.
δ_{ij} is a tensor;

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, we're done. We have obtained a continuity equation representing conservation of momentum density:

$$\frac{\partial(p_{mech} + p_{em})}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T}$$

with $p_{em} = \mu_0 \epsilon_0 \vec{S} = \epsilon_0 (\vec{E} \times \vec{B})$

and $T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$