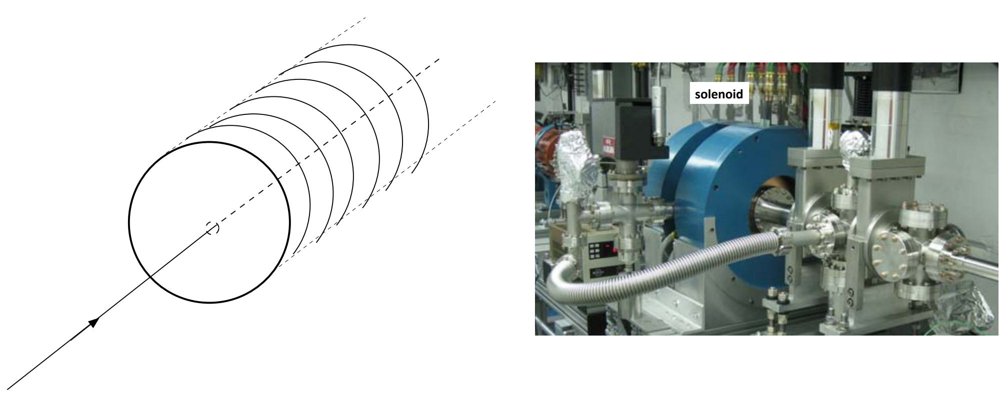
1 Electromagnets

The advantage of an electromagnet is that the magnetic field can be increased by increasing the current through the magnet coils. This enables the same particle trajectory to be maintained as even though the particle energy changes. A beam passes through the aperture of a magnet, a field region under vacuum containing no material. The magnetic field that results from powering the coils is present whether the beam is there or not, so magnets can be designed assuming no source of current within the aperture.

To contain a diverging beam near a target or particle source, solenoid magnets are often used. A solenoid is a conducting coil, usually wound cylindrically so that the beam and perhaps other beamline elements may be located inside in the field region. An ideal solenoid has a uniform longitudinal magnetic field inside and zero field outside. Ideally, there is no transverse magnetic field component. As a particle moves through a solenoid, any component of its motion perpendicular to the intended trajectory will cause the particle to move helically, corkscrewing around the uniform magnetic field direction as it moves downstream. In the absence of the magnetic field, an initial transverse particle motion would cause it to move further and further from the longitudinal design trajectory.



(a) cartoon of solenoid (b) actual solenoid

Figure 1: The left figure shows a ridiculous cartoon of a solenoid. The right figure shows a solenoid (blue) surrounding an RF gun at the Argonne Wakefield Accelerator.

In contrast, accelerators and beamlines (away from targets or sources) typically utililize magnets that have fields in the transverse direction only. There are usually separate magnets to do different jobs; dipole magnets bend the entire beam, quadrupole magnets focus a beam, and sextupole magnets are used to control chromaticity. A picture of these three types of magnets is shown in Fig. 2.

Figure 2: The electromagnets from left to right are a quadrupole magnet, a dipole magnet, and a sextupole magnet. Courtesy Fermilab visual media services.

When drawing magnetic field lines to represent a field, these must leave (diverge from) North poles, and enter (converge on) South Poles of a magnet. The force on a particle from a magnetic field is given by the Lorentz force law $\vec{F} = q(\vec{v} \times \vec{B})$, where q is the charge of the particle, $\vec{v}$ is the particle velocity, and $\vec{B}$ is the magnetic field. Since $\vec{B}$ is a vector, to determine the action of a magnet, it must be known which are the north poles, and which the south.

1.1 Dipole magnets

A uniform, vertically oriented magnetic field is needed to bend a beam in the horizontal plane. A typical conventional dipole electromagnet achieves a uniform magnetic field through the use of a pair of conducting coils (multi-turn current loops) wrapped around a steel core.

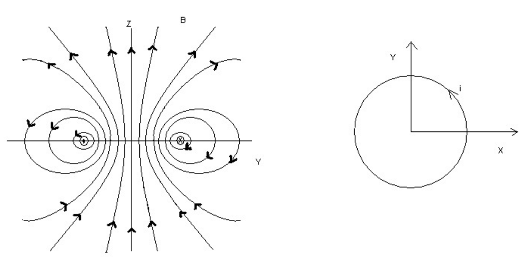


Figure 3: The axis of symmetry is the z-axis, the loop lies in the x-y plane. Current flows counter-clockwise when looking down from the positive z-axis. The direction of the field is given by the right-hand-rule; that is, if the fingers of your right hand curl around the loop in the dirction of current flow, your thumb will point in the direction of the magnetic field along the symmetry axis of the loop.

To see why this is, consider a single current carrying loop of wire. It generates a magnetic field that passes through the loop perpendicular to the plane of the loop, as sketched in

Fig. 3. The orientation of the magnetic field depends on whether the current flows clockwise or counterclockwise around the loop. If two coils are placed around the same axis of symmetry, one below the other then the current between the two coils is more uniform than it would be if the second coil were missing. Dipole electromagnets are

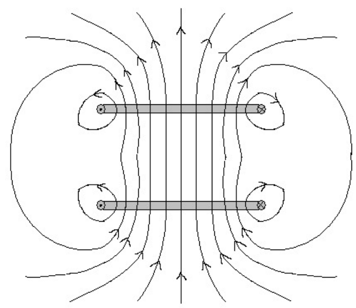


Figure 4: Magnetic field sketch for two parallel current coils.

constructed with a pair of current loops. There is a reasonably uniform magnetic field in the region between the loops.

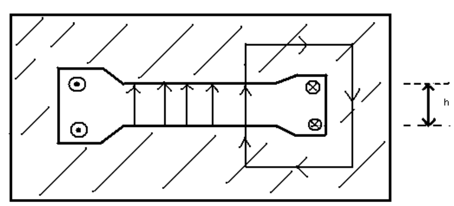
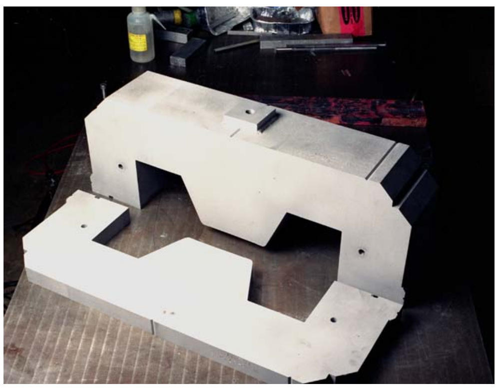


Figure 5: Two coils with surrounded by steel, with vacuum gap of height h. The steel concentrates the strength of the magnetic field within the vacuum gap. It can be shown that the field is related to the current in the coils as $B_0 = \frac{2\mu_0 NI}{h}$, where N is the number of turns per coil, I is the current in each turn, and h is the gap height of the magnet.

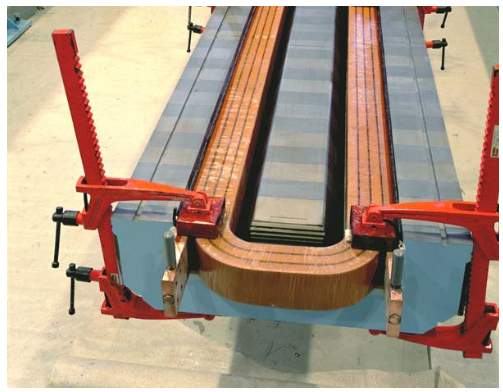
The field can be made stronger in the region between the coils if they are wound around blocks of steel as shown in Fig. 5, because the permeability of the steel is much larger than the vacuum permeability.

Material	Permeability
vacuum (μ_0)	$4\pi \times 10^{-7}$
FNAL Main Injector dipole magnet steel	2.3×10^{-4}
Iron	6.3×10^{-3}

Stages of the Fermilab Main Injector dipole construction are shown in Fig. 6. Steel laminations are punched out in a two-step process (see Fig. 6(a)). The magnets are built in two halves; the steel part of each half is called a half-core. A half-core is a stack of laminations coated with epoxy, sandwiched together between two endpacks and held together by welded tie plates on each side and the top. The current generating the magnetic field flows through two sets of copper coils. Each coil individually produces a magnetic field, but the combination of two coils gives a more uniform field in the region between the coils. The Main Injector magnets are cooled using deionized water flowing directly through the copper coils.



(a) steel laminations



(b) half core with coil

(c) minus one coil

(d) finished

Figure 6: Fermilab Main Injector dipole magnet in various stages of construction. Courtesy Fermilab visual media services.

1.2 Quadrupole magnets

Quadrupole magnets have four coils. The pole tips around which the coils are wrapped around are alternating north and south poles. Figure 7 shows a quadrupole magnet. The

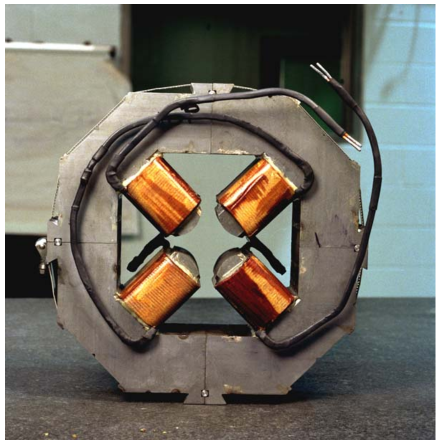


Figure 7: Quadrupole electromagnet. Courtesy Fermilab visual media services.

magnetic field in the aperture of a quadrupole is not uniform, it changes in both transverse directions. Along either transverse axis (x or y) the magnitude of the field increases linearly with distance from the quadrupole center. The proportionality relating the magnitude of the field to the distance from the origin is called the magnetic field gradient $B^{'}$. A quadrupole can only focus a beam in one transverse direction, it will defocus in the other transverse direction. A quadrupole is named according to its effect in the horizontal plane. So, if it focuses the beam horizontally and defocuses it vertically, it is called a focusing quadrupole. A defocusing quadrupole defocuses horizontally, and focuses vertically.

Example: Calculate quadrupole field gradient dependence using Ampere's law

Ampere's law can be used to calculate the dependence of the magnetic field gradient, $\vec{B}^{'}$, on the current in the coils and the size of the vacuum gap. Ampere's law relates the magnetic field integrated around a closed loop to the current passing through the plane enclosed by the loop;

$$\oint \vec{H} \cdot \vec{dl} = \int \vec{J} \cdot \vec{da}$$

where J is the current density (current/area), integrated over the area of the loop, and H is related to the magnetic field, B, by the permeability of the material, $\vec{H} = \frac{\vec{B}}{\mu}$

A convenient loop of integration crosses the gap from pole tip to pole tip in the $\hat{y}$ direction, with the remainder of the loop going through the iron of the magnet, as shown in Fig. 8. The distance from the center of the magnet to any pole tip is R. The current carrying coils shown in the figure are passing through the loop perpendicular to the plane of the loop. The loop shown has current from two coils penetrating the loop, each coil has

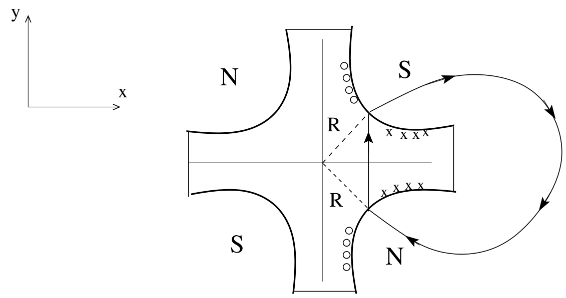


Figure 8: Sketch of an Amperian loop for a quadrupole magnet.

N turns carrying current I. The total enclosed current is then,

$$\int \vec{J} \cdot d\vec{a} = \int J da = 2NI$$

The length of the path through the gap is $\sqrt{2}R$ (by Pythagorean theorem). In the gap

$$\vec{H} = \frac{1}{\mu_0}\vec{B}$$

where μ_0 is the permeability of free space. The permeability of the iron is much larger than the permeability of vacuum ($\mu_0 << \mu_{iron}$), which results in the contribution of the portion of the loop in the iron to be negligible compared to the contribution from the portion of the loop in the gap.

$$\begin{aligned}
\oint \vec{H} \cdot d\vec{l} &= \int \vec{H}_{gap} \cdot d\vec{l}_{gap} + \int \vec{H}_{iron} \cdot d\vec{l}_{iron} \\
&= \frac{1}{\mu_0}\int_0^{\sqrt{2}R} \vec{B} \cdot \hat{y} dy + \frac{1}{\mu_{iron}} \int \vec{B}_{iron} \cdot d\vec{l}_{iron} \\
&\approx \frac{1}{\mu_0}\int_0^{\sqrt{2}R} \vec{B} \cdot \hat{y} dy
\end{aligned}$$

The magnetic field in the quadrupole is $B^{'}y\hat{x} + B^{'}x\hat{y}$, where $B^{'}$ is a constant characteristic of the particular quadrupole magnet. Derivation of this field will follow later in the notes. Substitute the field of a focusing quadrupole in for $\vec{B}$:

$$\oint \vec{H} \cdot d\vec{l} = \frac{1}{\mu_0}\int_0^{\sqrt{2}R} (B^{'}y\hat{x} + B^{'}x\hat{y}) \cdot \hat{y} dy$$

The x position is constant along the path of integration in the magnet gap, $x = \frac{R}{\sqrt{2}}$ by geometry. Then,

$$B^{'}\left(\frac{R}{\sqrt{2}}\right)(\sqrt{2}R) = B^{'}R^2 = 2\mu_0 NI$$

This can be solved for B', the magnetic field gradient of the quadrupole.

$$B' = \frac{2\mu_0 NI}{R^2} \qquad \text{T/m}$$

2 Calculating magnetic fields

The following table gives the magnetic fields of some of typically used magnets.

Magnet type	field $\vec{B}$
Horizontally bending dipole	$B_0\,\hat{y}$
Focusing quadrupole	$B'y\,\hat{x} + B'x\,\hat{y}$
Focusing sextupole	$B''xy\,\hat{x} + \frac{B''}{2}(x^2 - y^2)\,\hat{y}$
Skew quadrupole	$B'x\,\hat{x} + B'y\,\hat{y}$
Skew sextupole	$\frac{B''}{2}(x^2 - y^2)\,\hat{x} - B''xy\,\hat{y}$

This remainder of these notes can be skipped if preferred.

Finding fields in electromagnets is a magnetostatics problem. Ultimately, since the magnetic potential satisfies Laplaces equation for source free statics problems, the powerful method of separation of variables can be used in a circular geometry to solve for the magnetic potential. The magnet pole faces are shaped to insure that only one term in the series solution is needed to completely describe a specific magnet. The first term of the series is the dipole term, the second is the quadrupole term, and so on. Once the potential is known, the field can be calculated. The two Maxwell's equations for the magnetic field are:

$$\nabla \cdot \vec{B} = 0 \tag{1}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{2}$$

Electromagnets are made to produce a field inside the beam chamber, but the coils of electromagnets are outside the beam chamber region. If the coils carry a steady current, the problem of finding the field in the chamber reduces to a nneveu@hawk.iit.edumagnetostatics problem. So, both of the source terms on the right side of Eq. 2 for this problem are zero. Since the curl of a gradient is zero, then $\vec{B}$ can be written as the gradient of a scalar function (say, Φ_m). Putting $\vec{B} = -\vec{\nabla}\Phi_m$ into Eq. 1 results in Laplace's equation,

$$\nabla^2\Phi_m = 0$$

Laplace's equation may be solved using the separation of variables technique, resulting in solutions that are sums of harmonics appropriate to the geometry of the problem. In the lecture on accelerating structures, it was mentioned that expansions in a Cartesian coordinate system were sine and cosine functions, while in cylindrical coordinates the radial harmonics were Bessel functions. Here, it is assumed that the magnetic fields in the dipoles, quadrupoles, etc. are uniform in the longitudinal direction. Then, the field varies only in the transverse cross-section, and Laplace's equation is two dimensional. Circular coordinates are appropriate, so the general solution to Laplace's equation is given as

$$\Phi_m = A_0 + B_0 \ln(r) + \sum_{k=1}^{\infty} \left(A_k r^k + B_k r^{-k}\right)\left(C_k \cos(k\theta) + D_k \sin(k\theta)\right) \tag{3}$$

Since there is no current inside the vacuum chamber, B_0 and B_k in Eq. 3 must be zero, or else Φ_m would become infinite as $r \to 0$, which is not physical. Further, A_0 may also be set to zero since the derivative of a constant is zero and will not change $\vec{B}$.

Then,

$$\Phi_m = \sum_{k=1}^{\infty} r^k \left(a_k \cos(k\theta) + b_k \sin(k\theta)\right) \tag{4}$$

where the as yet undetermined coefficient A_k has been absorbed into the new (still undetermined) coefficients a_k and b_k. In addition, only one of the two terms in the expansion ($a_k \cos(k\theta)$ or $b_k \sin(k\theta)$) should be non-zero, the choice depending on whether even or odd symmetry is required. Normally oriented magnets are associated with the sine terms, while skew elements (rotated 90° with respect to a normal element) are associated with the cosine terms. Separated function magnets ideally are described by only one term in the harmonic expansion for the magnetic potential.

The only term desired for a horizontally bending dipole (with a vertically oriented magnetic field) is the $k = 1$ sine term, $b_1 r \sin(\theta)$. Note that in Cartesian coordinates $\Phi_m = b_1 r \sin(\theta) = b_1 y$. The magnetic field of the dipole is given by

$$\begin{aligned} \vec{B} &= -\vec{\nabla}\Phi_m = \left(-\hat{x}\frac{\partial}{\partial x} - \hat{y}\frac{\partial}{\partial y}\right) b_1 y \\ &= -b_1 \hat{y} \end{aligned}$$

The magnetic field is constant in the y direction. Also note that an equipotential will be a line at some constant value of y. Figure 9 shows a sketch of a dipole magnet

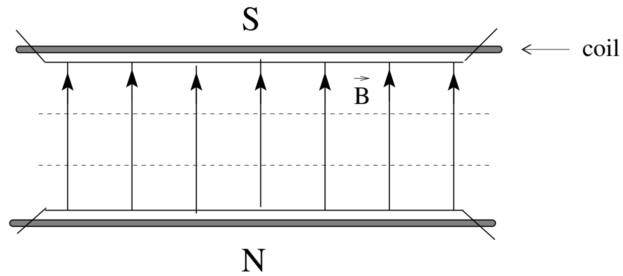


Figure 9: Cross-sectional sketch of a dipole magnet. A beam would go into or out of the plane of the paper. The force on the beam from the field would be $\vec{F} = q\vec{v} \times \vec{B}$, so for a positively charged particle going into the paper, the force would be to the right.

cross-section. The magnetic field lines are in the $\hat{y}$ direction, and some equipotentials are shown as dashed lines at constant y.

Now consider a quadrupole magnet. The $k = 2$ term in the expansion for Φ_m is the quadrupole term,

$$\Phi_{quad} = b_2 r^2 \sin(2\theta)$$

Using the following trigonometric identity,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

gives the following

$$\begin{aligned}\Phi_{quad} &= 2b_2(r\sin(\theta))(r\cos(\theta)) \\ &= 2b_2 xy\end{aligned}$$

The resulting magnetic field is then,

$$\begin{aligned}\vec{B} &= -\vec{\nabla}\Phi_{quad} = -\hat{x}\,2b_2\frac{\partial(xy)}{\partial x} - \hat{y}\,2b_2\frac{\partial(xy)}{\partial y} \\ &= -2b_2 y\,\hat{x} - 2b_2 x\,\hat{y} \\ &= B^{'} y\,\hat{x} + B^{'} x\,\hat{y}\end{aligned}$$

where the constant $-2b_2$ has been renamed $B^{'}$. It must have units of T/m, to have the field end up in units of Tesla. Lets check out the force this field would exert on a particle of charge q traveling with speed v in the $\hat{z}$ direction. The force is given by $\vec{F} = q\vec{v} \times \vec{B}$.

$$\vec{F} = q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & v \\ B^{'}y & B^{'}x & 0 \end{vmatrix} = q\hat{x}(-vB^{'}x) + q\hat{y}(vB^{'}y)$$

Notice that there is focusing in the $\hat{x}$ direction; the force is negative (restoring) and proportional to the offset x of the particle from the design orbit. In the $\hat{y}$ direction, there is a defocusing force, also linearly proportional to the particle offset y from the design orbit. A 'focusing' quadrupole focuses the beam horizontally but defocuses the beam in the vertical direction. A 'defocusing' quadrupole focuses vertically and defocuses horizontally. Net focusing in both planes in achievable, it requires the proper pattern of focusing and defocusing quadrupoles.

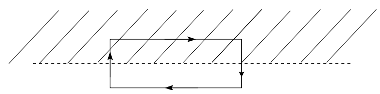


Figure 10: A loop across a boundary between vacuum and a ferromagnet.

How can a magnet be made to produce a field that is represented be a single term in the expansion of Φ_m? Shaping the pole faces correctly can achieve this goal. Ferromagnetic materials used for electromagnets have a very large permeability, μ. The curl equation $\vec{\nabla} \times \vec{H} = 0$ requires that the tangential component of $\vec{H}$ be continuous across a boundary. Place a rectangular loop straddling the boundary of two materials, in this case, the vacuum in the beam chamber and the ferromagnetic material the coil is wrapped around. This is depicted in Fig. 10. Since $\vec{\nabla} \times \vec{H} = 0$, the integral of the curl over the area of the loop must be zero;

$$\begin{aligned} \int \vec{\nabla} \times \vec{H} \cdot d\vec{a} &= 0 && (5) \\ \oint \vec{H} \cdot d\vec{l} &= 0 && (6) \\ \int_{top} \vec{H} \cdot d\vec{l} + \int_{bottom} \vec{H} \cdot d\vec{l} &= 0 \\ H^{||}_{top} l + H^{||}_{bottom} l &= 0 \\ \frac{1}{\mu} B^{||}_{top} + \frac{1}{\mu_0} B^{||}_{bottom} &= 0 && (7) \end{aligned}$$

Applying the curl theorem to Eq. 5 results in Eq. 6. The sides of the loop perpendicular to the interface can be made arbitrarily short and contribute nothing to the loop integral. Evaluating the contributions from the top and bottom sides of the loop results in Eq. 7. Since the top side of the boundary has a very large permeability, $\frac{1}{\mu} B^{||}_{top} \approx 0$. Then, Eq. 7, implies that there can be no component of $\vec{B}$ parallel to the magnetic surface. All magnetic field lines must be perpendicular to the ferromagnetic surface. Ferromagnetic surfaces are equipotential surfaces, just as the surfaces of perfect conductors are equipotential surfaces for electrostatic fields. (Perfect conductors cannot support a tangential component of the electrostatic field, or else current would flow along the surface.)

Then, to build a dipole, the pole faces should be flat at constant y with respect to the center plane of the magnet, just as depicted in Fig. 9. The expression for constant potential for a quadrupole is $Constant = \Phi_m = B^{'}xy$. This is the equation for a rotated hyperbola (such as the rough sketch of the pole faces in Fig. 8).

3 Magnetic field expansion

The magnetic fields present in particle accelerators, storage rings and transport lines can be represented with a multipole expansion, an expression where all the lower harmonics of Eq. 4 are kept, rather than a single term. In Cartesian coordinates the normal and skew components of the magnetic field takes the form:

$$B_y + iB_x = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x + iy)^n \tag{8}$$

where n gives the order of the pole. For example, the $n = 0$ term corresponds to a dipole field, the $n = 1$ term to a quadrupole field, the $n = 2$ term to a sextupole field, and so on. The b_n coefficients go with the normal magnetic fields and the a_n coefficients with the skew magnetic fields.

Examining the dipole term, we have: $B_y + iB_x = B_0(b_0 + ia_0)$, or $B_y = B_0b_0$ and $B_x = B_0a_0$. If the dipole is an ideal horizontal dipole, with a constant field $B_y = B_0$, then $b_0 = 1$ and $a_0 = 0$.

Now check out the quadrupole term ($n = 1$) in Eq. 8. Equating the real part of the left side of the equation to the real part of the right side of the equation (and similarly for the imaginary part):

$$B_x = B_0(b_1y + a_1x) \tag{9}$$

$$B_y = B_0(b_1x - a_1y) \tag{10}$$

To solve Eq. 10 for b_1, take the derivative of both sides with respect to x:

$$b_1 = \frac{1}{B_0}\frac{\partial B_y}{\partial x}$$

This tells us that B_y in a quadrupole magnet is $B_y = \frac{\partial B_y}{\partial x}x = B^{'}x$, and that the field gradient in a quadrupole is a constant (since b_1 is a constant).

The general magnetic field expansion may also be expressed in polar coordinates, the radial component of the field in polar coordinates has the form

$$B_r = B_0 \sum_{n=0}^{\infty} r^n[b_n \sin((n + 1)\theta) + a_n \cos((n + 1)\theta)]$$

The angular component of the field in polar coordinates has the form

$$B_\theta = B_0 \sum_{n=0}^{\infty} r^n [b_n \cos((n+1)\theta) - a_n \sin((n+1)\theta)]$$

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