Today's outline - January 19, 2023

- Outer products
- Linear transformations
- Projection operators
- Qubit measurement revisited
- The EPR paradox

Reading Assignment: Reiffel: 4.3-4.4 Wong: 6.2.1-6.2.6

Homework Assignment #02:
due Thursday, January 26, 2023

Homework Assignment #03:
due Thursday, February 02, 2023

Outer products

Using Dirac bra-ket notation is a convenient way to represent linear transformations which operate on vectors

Given two vectors $|a\rangle$ and $|b\rangle$, their inner product, defined as $\langle a|b\rangle$ is a scalar quantity

Their outer product, $|a\rangle\langle b|$ however, is an operator which has the property

$$(|a\rangle\langle b|)\,|c\rangle = |a\rangle\,(\langle b|c\rangle) = (\langle b|c\rangle)\,|a\rangle$$

The outer product is a matrix operator which acts on a vector and transforms it into a new vector

One example is the projection operator, for a vector space V associated with a single qubit system, an example of a projection operator is $|0\rangle\langle 0|$ with respect to $\{|0\rangle, |1\rangle\}$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Linear transformations

Another example of a linear transformation on the same space is $|0\rangle\langle 1|$ which maps $|1\rangle$ to $|0\rangle$ and $|0\rangle$ to the null vector

$$|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 1|\;|1\rangle = |0\rangle\;\langle 1|1\rangle = |0\rangle 1 = |0\rangle \qquad |0\rangle\langle 1|\;|0\rangle = |0\rangle\;\langle 1|0\rangle = |0\rangle 0 = 0$$

The four simple transformations in this 2-dimensional space are thus

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A general transformation in this space can be easily written as

$$a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1| = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Examples of linear transformations

A linear transformation, X, that swaps $|0\rangle$ and $|1\rangle$ is with an alternative notation being

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X : \begin{cases} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{cases}$$

In a 2-qubit system, what is the transformation that exchanges $|00\rangle$ and $|10\rangle$ but does not disturb the rest?

This will be a 4×4 matrix and the corresponding outer products are

$$|00\rangle\langle 10| + |01\rangle\langle 01| + |10\rangle\langle 00| + |11\rangle\langle 11| = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operator formalism

It is evident that an operator in an n-qubit system which maps $|j\rangle \mapsto |i\rangle$ and leaves all the others the same in the standard basis is $O = |i\rangle\langle j|$

a general operator with entries a_{ij} is

$$O = \sum_i \sum_j a_{ij} |i\rangle\langle j|$$

taking the expectation value of the operator, will pick out a specific coefficient

$$\langle m|O|n\rangle = \langle m| \sum_i \sum_j a_{ij} |i\rangle\langle j|n\rangle = \langle m| \sum_i a_{in} |i\rangle = a_{mn}$$

the result of applying this operator to a vector $|\psi\rangle = \sum_k b_k |k\rangle$ can be worked out

$$O|\psi\rangle = \left(\sum_i \sum_j a_{ij} |i\rangle\langle j| \right) \left(\sum_k b_k |k\rangle \right) = \sum_i \sum_j \sum_k a_{ij} b_k |i\rangle\langle j|k\rangle = \sum_i \sum_j a_{ij} b_j |i\rangle$$

the operator can be written in the same way for any basis $\{|\beta_i\rangle\}$ as $O = \sum_i \sum_j b_{ij} |\beta_i\rangle\langle\beta_j|$

Measuring with projection operators

Previously used projection onto a detector to describe measurement, now generalize

Consider a subspace, S of V all of whose vectors are orthogonal to a subspace $S^\perp$ such that $V = S \oplus S^\perp$

Any vector $|v\rangle \in V$ can be written as $|v\rangle = \vec{s_1} + \vec{s_2}$ where $\vec{s_1} \in S$ and $\vec{s_2} \in S^\perp$

For any subspace S, the projection operator P_S is the linear operator $P_S : V \to S$ that sends $|v\rangle \mapsto \vec{s_1}$

To generalize, for any direct sum decomposition of $V = S_1 \oplus \cdots \oplus S_k$ into k orthogonal subspaces, there are k related projection operators $P_i : V \to S_i$ such that

$$P_i|v\rangle = \vec{s_i}, \qquad |v\rangle = \vec{s_1} + \cdots + \vec{s_k}, \quad s_i \in S_i$$

The state, $\vec{s_i}$, resulting from the projection operator P_i applied to a state $|\psi\rangle$ is not necessarily normalized so a detector, with associated decomposition $V = S \oplus S^\perp$ is applied to $|\psi\rangle$ must produce a normalized state $|\phi\rangle$

$$P_i|\psi\rangle = c_i|\phi\rangle \quad \longrightarrow \quad |\phi\rangle = P_i|\psi\rangle / |P_i|\psi\rangle|$$

Projector examples

Given a single qubit state

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

apply the projector $|0\rangle\langle 0|$

$$|0\rangle\langle 0|\psi\rangle = a\cancelto{1}{\langle 0|0\rangle}|0\rangle + b\cancelto{0}{\langle 0|1\rangle}|0\rangle = a|0\rangle$$

Given a 2-qubit state $|\phi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$, apply the projector $|10\rangle\langle 10|$

$$|10\rangle\langle 10|\phi\rangle = a_{00}|10\rangle\cancelto{0}{\langle 10|00\rangle} + a_{01}|10\rangle\cancelto{0}{\langle 10|01\rangle} + a_{10}|10\rangle\cancelto{1}{\langle 10|10\rangle} + a_{11}|10\rangle\cancelto{0}{\langle 10|11\rangle} = a_{10}|10\rangle$$

If P_S is a projector from an n-dimensional vector space V onto an k-dimensional subspace S with basis $\{|\alpha_0\rangle, \dots, |\alpha_{k-1}\rangle\}$ then

$$P_S = \sum_{i=0}^{k-1} |\alpha_i\rangle\langle\alpha_i| = |\alpha_0\rangle\langle\alpha_0| + \cdots + |\alpha_{k-1}\rangle\langle\alpha_{k-1}|$$

If $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ and S is a subspace spanned by $|00\rangle$, $|01\rangle$ then

$$P_S = |00\rangle\langle 00| + |01\rangle\langle 01| \quad \longrightarrow \quad P_S|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle, \qquad |P_S|\psi\rangle|^2 \neq 1$$

Adjoint operators

if operator O acts on spaces V and W as $$O : W \to V$$

its adjoint, $O^\dagger$ acts as and is defined by $$O^\dagger : V \to W$$

where $\vec{v} \in V$ and $\vec{w} \in W$ $$O^\dagger \vec{v} \cdot \vec{w} = \vec{v} \cdot O\vec{w}$$

In terms of matrices, $O^\dagger$ is the conjugate transpose of O

Recall that $\langle x|$ is the conjugate transpose of $|x\rangle$ so that given an operator A and its adjoint $A^\dagger$, we have $(\langle x|A^\dagger) = (A|x\rangle)^\dagger$

The inner product of $O^\dagger|x\rangle$ and $|w\rangle$ is thus equal to the inner product of $|x\rangle$ and $O|w\rangle$

$$(O^\dagger|x\rangle)^\dagger \equiv (\langle x|O) \quad \longrightarrow \quad (\langle x|O)|w\rangle = \langle x|O|w\rangle = \langle x|(O|w\rangle)$$

The projection operator is self-adjoint (or Hermitian) so that $P = P^\dagger$ and applying it multiple times is the same as applying it once. Take $P = |\alpha\rangle\langle\alpha|$

$$PP|v\rangle = P(P|v\rangle) = P(|\alpha\rangle\langle\alpha|v\rangle) = (P|\alpha\rangle)\langle\alpha|v\rangle = (|\alpha\rangle\cancelto{1}{\langle\alpha|\alpha\rangle})\langle\alpha|v\rangle = |\alpha\rangle\langle\alpha|v\rangle = P|v\rangle$$

Measurement of a single qubit

V is the vector space associated with a single-qubit system and the direct sum decomposition of V in the standard basis is $V = S \oplus S'$ where S is generated by $|0\rangle$ and S' is generated by $|1\rangle$

$$P = |0\rangle\langle 0|, \qquad P : V \to S$$
$$P' = |1\rangle\langle 1|, \qquad P' : V \to S'$$

Measurement of state $|\psi\rangle = a|0\rangle + b|1\rangle$ is done as

$$P|\psi\rangle = |0\rangle\langle 0|(a|0\rangle + b|1\rangle) = |0\rangle(a\cancelto{1}{\langle 0|0\rangle} + b\cancelto{0}{\langle 0|1\rangle}) = a|0\rangle, \qquad P|0\rangle \longrightarrow \frac{a}{|a|}|0\rangle$$

$$P'|\psi\rangle = |1\rangle\langle 1|(a|0\rangle + b|1\rangle) = |1\rangle(a\cancelto{0}{\langle 1|0\rangle} + b\cancelto{1}{\langle 1|1\rangle}) = b|1\rangle, \qquad P'|0\rangle \longrightarrow \frac{b}{|b|}|1\rangle$$

with probabilities given by

$$|P|\psi\rangle|^2 = \langle\psi|P^\dagger P|\psi\rangle = \langle\psi|PP|\psi\rangle = \langle\psi|P|\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle = \bar{a}a = |a|^2$$
$$|P'|\psi\rangle|^2 = \langle\psi|P'|\psi\rangle = \langle\psi|1\rangle\langle 1|\psi\rangle = |b|^2$$

Measuring a 2-qubit state

If V is a vector space in a 2-qubit system such that $V = S_{00} \oplus S_{01} \oplus S_{10} \oplus S_{11}$ is its decomposition for subspaces S_{ij} spanned by $|ij\rangle$ the projection operators are

$$P_{00} = |00\rangle\langle 00|, \qquad P_{01} = |01\rangle\langle 01|, \qquad P_{10} = |10\rangle\langle 10|, \qquad P_{11} = |11\rangle\langle 11|$$

Measuring a general state $|\phi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle = \sum_{m,n} a_{mn}|mn\rangle$ with a projection operator gives

$$P_{ij}|\phi\rangle = |ij\rangle\langle ij| \sum_{m,n} a_{mn}|mn\rangle = a_{ij}|ij\rangle, \qquad P_{ij}|\phi\rangle \quad \longrightarrow \quad \frac{a_{ij}}{|a_{ij}|}|ij\rangle$$

The state after measurement is in the normalized form which differs from $|ij\rangle$ only by a global phase and so are equal in the complex projective space

$$\frac{a_{ij}}{|a_{ij}|}|ij\rangle = e^{i\varphi}|ij\rangle \sim |ij\rangle$$

Measuring bits for equality

In a 2-qubit system, V is the vector space with associated decomposition $V = S_1 \oplus S_2$ where the two subspaces are spanned by $\{|00\rangle, |11\rangle\}$ and $\{|01\rangle, |10\rangle\}$ respectively

The projection operators are $P_1 = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $P_2 = |01\rangle\langle 01| + |10\rangle\langle 10|$

What is the result of measuring a general state $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$?

After measurement, we get one of two values with probabilities

$$P_1|\psi\rangle \longrightarrow |u\rangle = \frac{(a_{00}|00\rangle + a_{11}|11\rangle)}{\sqrt{|a_{00}|^2 + |a_{11}|^2}}$$

$$|P_1|\psi\rangle|^2 = |a_{00}|^2 + |a_{11}|^2$$

if this is the result, we know the two qubits are equal

$$P_2|\psi\rangle \longrightarrow |v\rangle = \frac{(a_{01}|01\rangle + a_{10}|10\rangle)}{\sqrt{|a_{01}|^2 + |a_{01}|^2}}$$

$$|P_2|\psi\rangle|^2 = |a_{01}|^2 + |a_{10}|^2$$

if this is the result, the qubits must be unequal

Note that we do not know the values of the qubits, just whether they are equal or not

Measurement in the Bell decomposition

Recall the four Bell states for a 2-qubit system

$$|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\Phi^-\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad |\Psi^-\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

If the vector space V has a decomposition $\qquad V = S_{\Phi^+} \oplus S_{\Phi^-} \oplus S_{\Psi^+} \oplus S_{\Psi^-}$

when we measure a qubit in state $|v\rangle = |00\rangle$ with this decomposition, what results do we get?

First realize that we can write $|00\rangle = \tfrac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)$ so that

$$P_{\Phi^+}|00\rangle \longrightarrow |u\rangle = |\Phi^+\rangle \qquad\qquad P_{\Phi^-}|00\rangle \longrightarrow |u\rangle = |\Phi^-\rangle$$

$$|P_{\Phi^+}|00\rangle|^2 = \frac{1}{2} \qquad\qquad |P_{\Phi^-}|00\rangle|^2 = \frac{1}{2}$$

Einstein Podolsky Rosen paradox

of lanthanum is 7/2, hence the nuclear magnetic moment as determined by this analysis is 2.5 nuclear magnetons. This is in fair agreement with the value 2.8 nuclear magnetons determined from La III hyperfine structures by the writer and N. S. Grace.[9]

[9] M. F. Crawford and N. S. Grace, Phys. Rev. **47**, 536 (1935).

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Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

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In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one

such way, whenever the conditions set down in it occur. Regarded not as a necessary, but merely as a sufficient, condition of reality, this criterion is in agreement with classical as well as quantum-mechanical ideas of reality.

To illustrate the ideas involved let us consider the quantum-mechanical description of the behavior of a particle having a single degree of freedom. The fundamental concept of the theory is the concept of *state*, which is supposed to be completely characterized by the wave function ψ, which is a function of the variables chosen to describe the particle's behavior. Corresponding to each physically observable quantity A there is an operator, which may be designated by the same letter.

If ψ is an eigenfunction of the operator A, that is, if

$$\psi' \equiv A\psi = a\psi, \tag{1}$$

where a is a number, then the physical quantity A has with certainty the value a whenever the particle is in the state given by ψ. In accordance with our criterion of reality, for a particle in the state given by ψ for which Eq. (1) holds, there is an element of physical reality corresponding to the physical quantity A. Let, for example,

$$\psi = e^{(2\pi i/h)p_0 x}, \tag{2}$$

where h is Planck's constant, p_0 is some constant number, and x the independent variable. Since the operator corresponding to the momentum of the particle is

$$p = (h/2\pi i)\partial/\partial x, \tag{3}$$

we obtain

$$\psi' = p\psi = (h/2\pi i)\partial\psi/\partial x = p_0\psi. \tag{4}$$

Thus, in the state given by Eq. (2), the momentum has certainly the value p_0. It thus has meaning to say that the momentum of the particle in the state given by Eq. (2) is real.

On the other hand if Eq. (1) does not hold, we can no longer speak of the physical quantity A having a particular value. This is the case, for example, with the coordinate of the particle. The operator corresponding to it, say q, is the operator of multiplication by the independent variable. Thus,

$$q\psi = x\psi \neq a\psi. \tag{5}$$

In accordance with quantum mechanics we can only say that the relative probability that a measurement of the coordinate will give a result lying between a and b is

$$P(a,b) = \int_a^b \bar{\psi}\psi dx = \int_a^b dx = b - a. \tag{6}$$

Since this probability is independent of a, but depends only upon the difference $b-a$, we see that all values of the coordinate are equally probable.

A definite value of the coordinate, for a particle in the state given by Eq. (2), is thus not predictable, but may be obtained only by a direct measurement. Such a measurement however disturbs the particle and thus alters its state. After the coordinate is determined, the particle will no longer be in the state given by Eq. (2). The usual conclusion from this in quantum mechanics is that *when the momentum of a particle is known, its coordinate has no physical reality*.

More generally, it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say A and B, do not commute, that is, if $AB \neq BA$, then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first.

From this follows that either (1) *the quantum-mechanical description of reality given by the wave function is not complete* or (2) *when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality*. For if both of them had simultaneous reality—and thus definite values—these values would enter into the complete description, according to the condition of completeness. If then the wave function provided such a complete description of reality, it would contain these values; these would then be predictable. This not being the case, we are left with the alternatives stated.

In quantum mechanics it is usually assumed that the wave function *does* contain a complete description of the physical reality of the system in the state to which it corresponds. At first

"Can quantum-mechanical description of physical reality be considered complete?," A. Einstein, B. Podolsky, and N. Rosen, *Physical Review* **47**, 777-779 (1935).

Bohm's thought experiment

Suppose a pair of photons are generated in the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

The first photon is sent to Alice and the second to Bob who are far apart

Alice and Bob can only measure the single photon they have received

Alice can measure only with an observable of the form $O \otimes I$

Bob can only measure with an observable of the form $I \otimes O'$

Now Alice measures her photon and sees that it is the the $|0\rangle$ state which forces the original state to collapse: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow |00\rangle$

When Bob now measures his photon he will get $|0\rangle$ with 100% certainty

Similarly, if Alice measures $|1\rangle$ so will Bob

This is true irrespective of who measures their photon "first" since because of special relativity, it is always possible to find a frame of reference where either Alice or Bob is measuring first

There is no causality, just correlated random behavior

Einstein Podolsky Rosen paradox

This so-called "spooky action at a distance" profoundly bothers many including Einstein, Podolsky, and Rosen

> *"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."*

This implies that when the two photons are created, there is some additional hidden state that is created along with the two photons which contains the information about how the result of Alice's and Bob's measurements will turn out

This local hidden variable is generated with a random value such that the measurements are random

If such a theory is correct, then the result of the measurements is determined before the photons are separated and no possible violations of causality can occur