Crash course in contour integration

Holomorphic function : <u>complex differentiable</u>

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

If f' so defined is independent of how the limit is taken (the sequence of z's tending to z_0) the function is said to be complex differentiable or holomorphic.

write $f(x+iy) = u(x,y) + i\,v(x,y)$ with real x, y, u, v.

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0+h)-f(z_0)}{h} = \left.\frac{\partial f}{\partial x}\right|_{z_0}.$$

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0+ih)-f(z_0)}{ih} = \frac{1}{i}\left.\frac{\partial f}{\partial y}\right|_{z_0}$$

$$\Rightarrow \quad \left.\frac{\partial f}{\partial x}\right|_{z_0} = \frac{1}{i}\left.\frac{\partial f}{\partial y}\right|_{z_0}$$

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy riemann eq.

can show that CR → complex differentiable.
*(+ real differentiable)

- defining $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$\text{CR} \iff \left.\frac{\partial f}{\partial \bar{z}}\right|_z = 0.$$

- $\partial_x(\text{CR}) \Rightarrow \partial_x^2 u = \partial_{xy} v, \; \partial_{xy} u = -\partial_x^2 v$
  $\partial_y(\text{CR}) \Rightarrow \partial_{xy} u = \partial_y^2 v, \; \partial_y^2 u = -\partial_{xy} v$

  add & subtract to get $\nabla^2 u = 0 = \nabla^2 v$

  and
$$\begin{aligned} \nabla u \cdot \nabla v &= \partial_x u \, \partial_x v + \partial_y u \, \partial_y v \\ &= \partial_x u \, \partial_x v - \partial_x v \, \partial_x u \\ &= 0 \end{aligned}$$

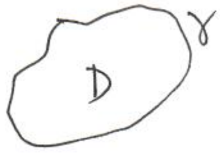
  u & v harmonic & orthogonal gradients if f is holomorphic.

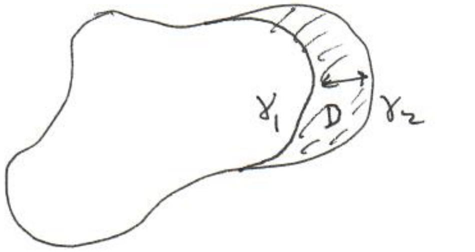
Cauchy integral theorem

let f be holomorphic. Then

closed contour

$$\begin{aligned} \oint_\gamma f \, dz &= \oint_\gamma (u+iv)(dx+idy) \\ &= \oint_\gamma (u\,dx - v\,dy) + i\oint_\gamma (u\,dy + v\,dx) \\ &= \int_D \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dx\,dy + i\int_D \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\,dy \\ &= 0 \text{ by CR.} \end{aligned}$$





This means

$$\oint_{\gamma_1} f = \oint_{\gamma_2} f$$

if f is holomorphic in the shaded region D — "contour deformation"

Cauchy integral formula

Suppose f is holomorphic in D.

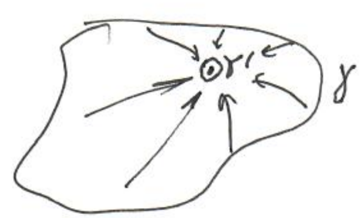
Then

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{f(z)}{z-a}\,dz = f(a)$$

The only point where the integrand is *not* complex differentiable is a, which is called a simple pole.

We can deform:

$$\frac{1}{2\pi i}\oint_{\gamma} \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\oint_{\gamma'} \frac{f(z)}{z-a}\,dz$$



let $z = a + \epsilon e^{i\theta}$ with $\epsilon \ll a$. Change vars $z \to \theta$.

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\, d\theta = \frac{1}{2\pi i} f(a) \int_0^{2\pi} i\, d\theta$$

$$= f(a).$$

$$f^{\overbrace{\prime\prime\cdots\prime}^{(n)}}(a) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

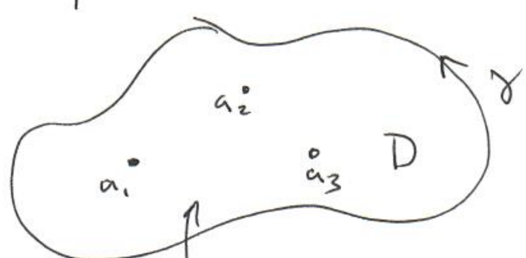
follows from previous by writing $\frac{d^n}{dz^n}\frac{1}{z-a} = (-1)^n n! \frac{1}{(z-a)^{n+1}}$ and then IBP n times (no surf term because γ closed)

(this can be used to prove that holomorphic $\Leftrightarrow$ analytic.)

Generalization: the residue theorem

Suppose $\tilde{f}$ is holomorphic in D except at a finite set of points $\{a_i\}$.

eg



$\tilde{f}$ is holomorphic in $D - \{a_i\}$

Then $\oint_\gamma \tilde{f}\,dz = 2\pi i \sum_i \mathrm{Res}(\tilde{f}, a_i)$

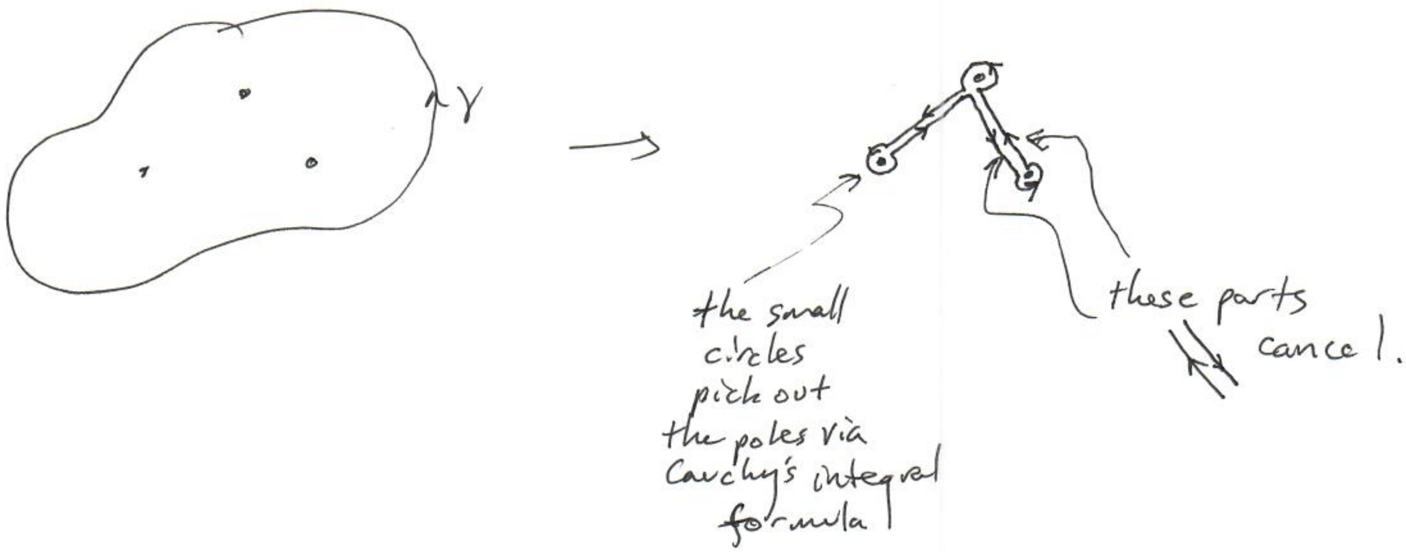
The residues of $\tilde{f}$ at a_i are the coefficients of the simple pole terms in an expansion of $\tilde{f}$ around a_i (called a Laurent expansion)

$$\tilde{f} = \cdots + \frac{c_{-3}}{(z-a_i)^3} + \frac{c_{-2}}{(z-a_i)^2} + \frac{\boxed{c_{-1}}}{z-a_i} + c_0 + c_1(z-a_i) + \cdots$$

simple pole term

$\mathrm{Res}(\tilde{f}, a_i) \equiv c_{-1}$

Intuitively, this is because we can deform contours:



(note that the higher degree poles don't contribute because the c_n are constant coeffs in the Laurent expansion.)

Physics applications/examples

- Retarded Green function for the wave operator

$$\left(-\partial_t^2 + \vec{\nabla}^2\right) G(t,\vec{x};t',\vec{x}') = \delta^{(4)}(x^\mu - x^{\mu'})$$

Fourier transform both sides: $\int_{-\infty}^{\infty} dt\, d^3x\, e^{i\omega(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')}$

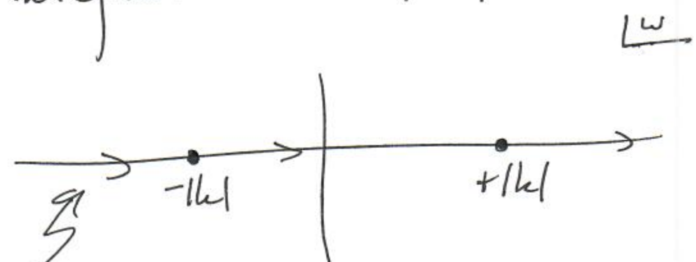
$$\Rightarrow \quad (\omega^2 - k^2)\,\tilde{G}(\omega,\vec{k}) = 1$$

$$\text{or} \quad \tilde{G} = \frac{1}{\omega^2 - k^2}$$

Invert: $$G(t,x;t',x') = \int_{-\infty}^{\infty} \frac{d\omega\, d^3k}{(2\pi)^4} \frac{e^{-i\omega(t-t') - i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\omega^2 - k^2}$$

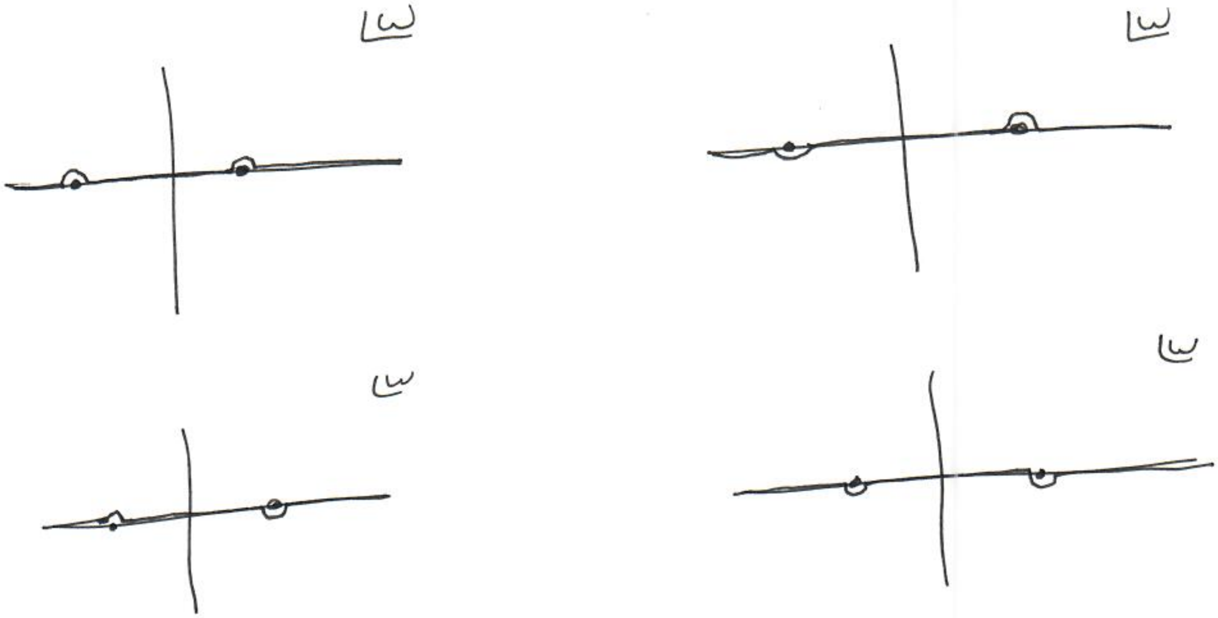
Do ω integral first. $\omega^2 - k^2 = (\omega - |k|)(\omega + |k|)$

So the integrand has simple poles at $\pm |k|$ in cplx ω plane



integration contour is the real ω axis

To define the green function we need to avoid the poles. There are 4 prescriptions possible!



The four prescriptions correspond to different boundary conditions in time. It turns out that for the <u>retarded</u> propagator, which vanishes if $t'<t$, we want the first prescription above. (This will be apparent after the fact.)

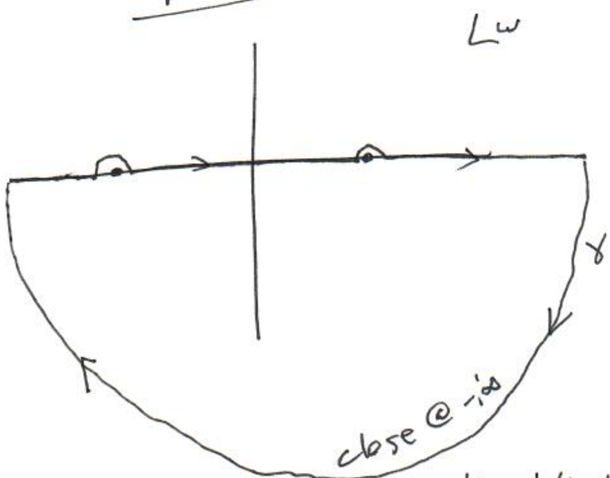
The contours are not closed, so we can't use residue calculus yet. But, it turns out we can close the contours for free "at $\pm i\infty$" depending on the sign of $t-t'$. (This is an example of the "Jordan lemma".)

$$e^{-i\omega(t-t')} = e^{\mathrm{Im}(\omega)(t-t')}\, e^{-i\,\mathrm{Re}(\omega)(t-t')}$$

If $\mathrm{sign}(\mathrm{Im}\,\omega) = -\,\mathrm{sign}(t-t')$ then the first factor exponentially suppresses the integrand as $|\mathrm{Im}\,\omega| \to \infty$

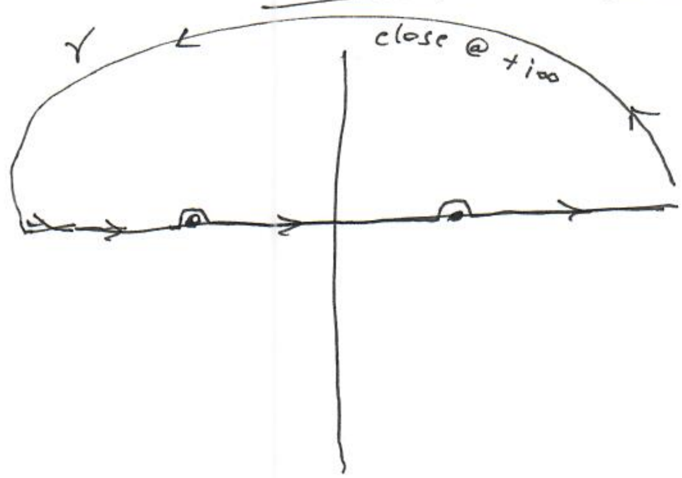
One can show that if we close the contour on a semicircle at "complex ∞", the semicircle part contributes nothing because of this exponential suppression. Two cases:

$\boxed{t > t'}$



integrand $\sim e^{-|\text{Im}\,\omega|(t-t')} \to 0$ on the semicircle

$\boxed{t < t'}$



integrand $\sim e^{-|\text{Im}\,\omega|(t'-t)} \to 0$ on the semicircle

This trick closes the contour in a way that doesn't change the integral, and allows us to use the residue theorem:

- for $t-t' < 0$, the integrand is holomorphic inside γ and the integral vanishes.
- for $t-t' > 0$, we pick up the simple poles at $\omega = \pm k$.

Residue @ $\omega = -|k|$ is $\dfrac{e^{-i(-|k|)(t-t')-i\bar{k}\cdot(\bar{x}-\bar{x}')}}{(-|k|-|k|)}$

Residue @ $\omega = +|k|$ is $\dfrac{e^{-i(|k|)(t-t')-i\bar{k}\cdot(\bar{x}-\bar{x}')}}{(+|k|+|k|)}$

And the contour was clockwise $\rightarrow$ extra -1

So the ω integral gives

$$-\int \frac{d^3k}{(2\pi)^4}\, 2\pi i \left(\frac{e^{-i\bar{k}\cdot(\bar{x}-\bar{x}')}}{2|k|}\right)\left(-e^{i|k|(t-t')} + e^{-i|k|(t-t')}\right)\Theta(t-t')$$

$$= -\int \frac{d^3k}{(2\pi)^3}\, \frac{e^{-i\bar{k}\cdot(\bar{x}-\bar{x}')}}{|k|}\, \sin(|k|(t-t'))\, \Theta(t-t')$$

(The rest doesn't use contour methods)

$$= -\int \frac{2\pi \sin\theta\, |k|^2\, d|k|\, d\theta}{(2\pi)^3\, |k|}\, e^{-i|k||x-x'|\cos\theta}\, \sin|k|(t-t')\, \Theta(t-t')$$

The $d\theta$ integral is simple and gives $\dfrac{2\sin(|k||x-x'|)}{|k|\,|x-x'|}$

$$\rightarrow -\int_0^\infty \frac{2\cancel{|k|^2}\, d|k|}{(2\pi)^2\cancel{|k|^2}\,|x-x'|}\, \sin(|k||x-x'|)\sin(|k|(t-t'))\,\Theta(t-t')$$

$$= \frac{-1}{2\pi^2|\bar{x}-\bar{x}'|}\,\Theta(t-t')\int_0^\infty d|k|\, \underbrace{\sin(|k||\bar{x}-\bar{x}'|)\sin(|k|(t-t'))}_{-\frac{1}{2}\left(\cos[|k|(t-t'+|x-x'|)] - \cos[|k|(t-t'-|x-x'|)]\right)}$$

The cosines are symmetric in $|k|$, so we can extend the range of the $|k|$ integration to $-\infty$. We get $2\pi\delta(t-t'\pm|x-x'|)$ but only $(t-t')-|x-x'|$ contributes since $(t-t'), |x-x'| \geq 0$

So we recover

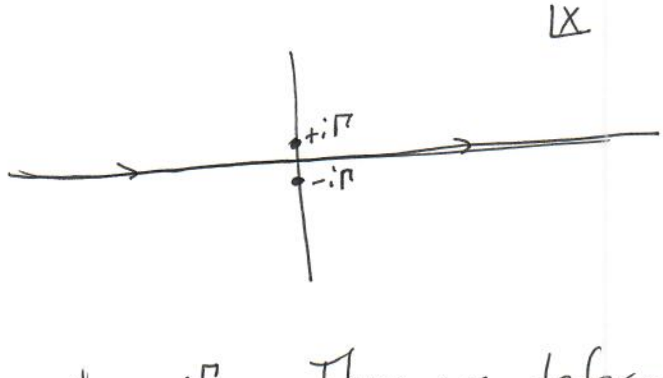
$$G(t,\bar{x},t',\bar{x}') = -\frac{1}{4\pi|x-x'|}\,\theta(t-t')\,\delta\big((t-t')-|x-x'|\big)$$

Example: Plemelj formula

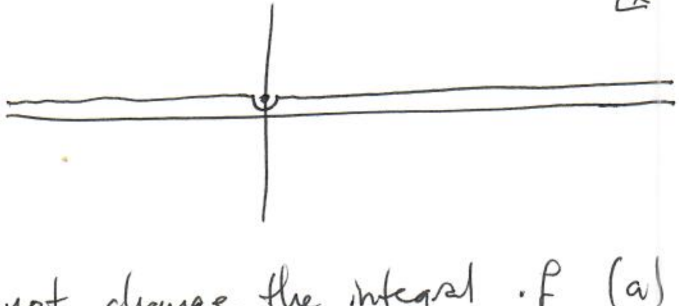
Closely related to this, we have used several times the prescription $\dfrac{1}{x \pm i\Gamma} = \mp i\pi\delta(x) + P\left(\dfrac{1}{x}\right)$

for small Γ. This means $\displaystyle\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x \pm i\Gamma} \to \mp i\pi f(0) + P\int_{-\infty}^{\infty} dx\, \frac{f(x)}{x}$ for small Γ.

We can derive it by considering a contour deformation.



suppose we have the $-i\Gamma$. Then we deform



which does not change the integral if (a) Γ is infinitesimal and (b) f has no other singularities near the real axis.

(Then 

would be a closed contour containing no poles, and its integral vanishes.)

The contour C  has two parts:

 and .

C_1 $\quad$ C_2

The former is the principal value:

$$\int_{C_1} dx \, \frac{f(x)}{x - i\Gamma} = P \int_{-\infty + i\Gamma}^{+\infty + i\Gamma} dx \, \frac{f(x)}{x - i\Gamma}$$

change vars to $y = x - i\Gamma$

$$= P \int_{-\infty}^{\infty} dy \, \frac{f(y + i\Gamma)}{y}$$

taylor expand to $O(\Gamma^0)$

$$\underset{O(\Gamma^0)}{\Rightarrow} P \int_{-\infty}^{\infty} dy \, \frac{f(y)}{y}$$

redefine $y \to x$

$$= P \int_{-\infty}^{\infty} dx \, \frac{f(x)}{x}$$

The C_2 contour is just "half the contour in cauchy's formula" so it picks up half of the residue

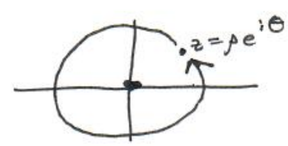
$$\int_{C_2} dx \, \frac{f(x)}{x - i\Gamma} = i\pi f(i\Gamma) \xrightarrow[O(\Gamma^0)]{} i\pi f(0)$$

so $$\int_{-\infty}^{\infty} dx \, \frac{f(x)}{x - i\Gamma} = i\pi f(0) + P \int_{-\infty}^{\infty} dx \, \frac{f(x)}{x} + O(\Gamma)$$

For completeness of the crash course, although we won't use it in this course, we conclude with Branch cuts & Branch cut integrals.

Because $e^{2\pi i} = 1$, some functions are multivalued in the complex plane.

<u>example</u>: $\sqrt{z}$. Let $z = \rho e^{i\theta}$. as $\theta \to \theta + 2\pi$, $z \to z$ but $\sqrt{z}$ goes from $\sqrt{\rho}\, e^{i\theta/2}$ to $-\sqrt{\rho}\, e^{i\theta/2}$



we say that $\sqrt{z}$ has a "<u>branch point</u>" at the origin. To keep $\sqrt{z}$ single valued, we remove a line from the plane

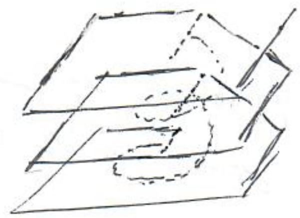
$\mathbb{C} - \mathbb{R}^+ =$

$0 < \theta < 2\pi$, and $\sqrt{z}$ is discontinuous across the cut.

- we can put the cut at any angle we want,

is just as good.

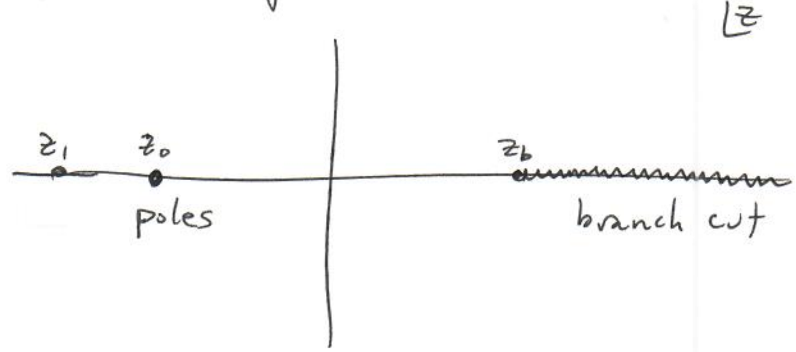
- $0 < \theta < 2\pi$ is called the principal branch. Another alternative treatment of "multibranched" functions is to introduce "Riemann sheets"



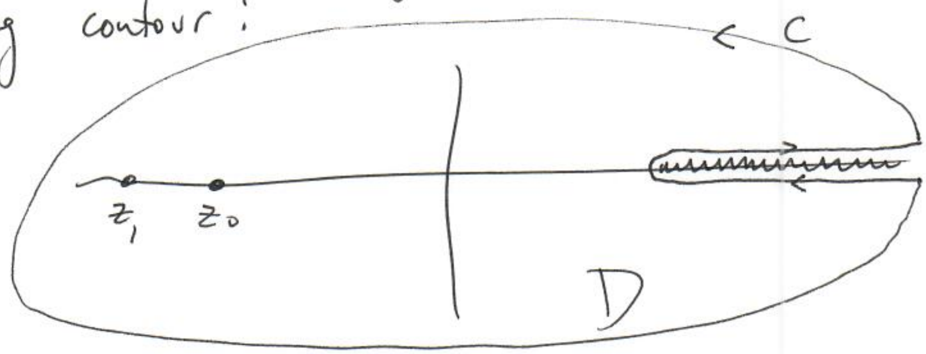
effectively extend the range of θ as much as needed so that the function is single-valued. for $\sqrt{z}$, there are only two sheets needed: $0 \leq \theta < 4\pi$

We'll continue with the branch cut formalism & the principal branch.

A common scenario in physics: we have a function f which is holomorphic apart from a branch cut and some poles, eg:



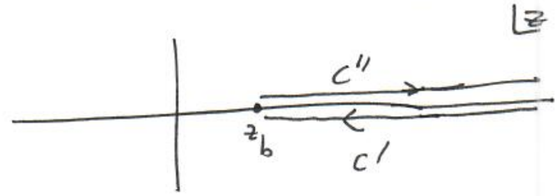
(For simplicity assume no pole @ z_b.)
We obtain some interesting integral relations by drawing the following contour:



Then $\int_C f\, dz = \sum_i 2\pi i \, \mathrm{Res}(f, z_i)$

Now suppose that f decays rapidly enough with $|z|$ so that the whole contour at infinity contributes nothing (eg $|z|^{-3/2}$ as $|z| \to \infty$.) Also suppose that f is real for real $z < z_b$.

Then $\int_C f\,dz$ also $= \left(\int_{C'} + \int_{C''}\right) f\,dz$



$$= \left(\int_{\infty - i\epsilon}^{z_b - i\epsilon} + \int_{z_b + i\epsilon}^{\infty + i\epsilon}\right) f\,dz$$

$$= \left(\int_{z_b + i\epsilon}^{\infty + i\epsilon} - \int_{z_b - i\epsilon}^{\infty - i\epsilon}\right) f\,dz$$

Because f is holomorphic away from the singularities, $f = f(z)$ (and not $f = f(z, z^*)$.) So for real $z < z_b$,

$$(f(z))^* = f(z) \quad \text{(since we assumed } f \text{ real here)}$$

Since holomorphic functions are analytic, $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in $D - \{z_i\}$. $(f(z))^* = f(z)$ for real $z < z_b$ means all the c_n are real, so for real $\hat{z} \geqq z_b$,

$$\text{Disc } f(\hat{z}) \equiv f(\hat{z} + i\epsilon) - f(\hat{z} - i\epsilon)$$

Discontinuity across cut

$$= \sum_n c_n (\hat{z} + i\epsilon)^n - \sum_n c_n (\hat{z} - i\epsilon)^n$$

$$= f(\hat{z} + i\epsilon) - (f(\hat{z} + i\epsilon))^*$$

$$= 2i \,\mathrm{Im}\, f(\hat{z} + i\epsilon)$$

So $\int_C f\,dz = 2i\,\mathrm{Im} \int_{z_b + i\epsilon}^{\infty + i\epsilon} f(z)\,dz = 2\pi i \sum_i \mathrm{Res}(f, z_i)$

or

(with the understanding that the contour is just above the cut) $\boxed{\mathrm{Im} \int_{z_1}^{\infty} f(z)\,dz = \pi \sum_i \mathrm{Res}(f, z_i)}$