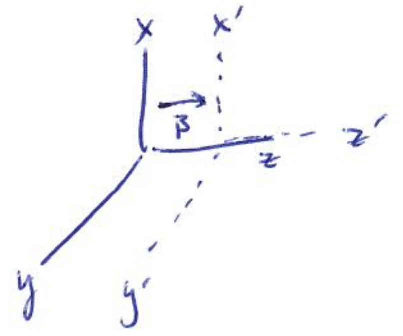
Covariance of the Electromagnetic Field

We've assumed some familiarity with special relativity, especially that particle energies & momenta transform like a Lorentz 4-vector: $cp^\mu = (E, \vec{p}c) = (\gamma mc^2, \gamma mc\vec{\beta})$.

How does the electromagnetic field "transform", or appear differently for different observers in relative motion?

First some notation. Recall: if one observer uses coordinates $x^\mu \equiv (ct, \vec{x})$ then the observer moving with constant velocity βc along the $\hat{z}$ direction would label the same events by coordinates:



$$t' = \gamma(t - \beta z/c)$$
$$z' = \gamma(z - \beta ct)$$
$$x' = x$$
$$y' = y$$

In matrix notation, $x^{\mu'} = \Lambda^\mu{}_\nu x^\nu$

$$\Lambda^\mu{}_\nu \equiv \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & 1 \end{pmatrix}$$

implements a "z-boost"

A more general form of the boost matrix is given in Jackson 11.98. All 4-vectors transform this way.
(definition)

We can define the proper time along a given trajectory

as $\tau = \int_{\text{trajectory}} d\tau = \int_{\text{trajectory}} \sqrt{dt^2 - \frac{1}{c^2} dx^2}$, ie $d\tau = dt/\gamma$

where γ is associated with the trajectory.

This is Lorentz invariant.

Then the "4-velocity" $U^\mu \equiv \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{d\vec{x}}{dt}\frac{dt}{d\tau} \right)$

$$= \gamma(c, \vec{v})$$

is a 4-vector, since x^μ is a 4-vec and τ is invariant.

If we multiply this by m, we get $(\gamma mc, \gamma mc\vec{\beta})$

which is just p^μ! So p^μ is also a 4-vector

$$p^\mu = (E/c, \vec{p})$$

For the z boost,
$$E' = \gamma(E - \beta c p_z)$$
$$p_z' = \gamma(p_z - \beta E/c)$$
$$p_x' = p_x$$
$$p_y' = p_y$$

or $\Lambda^\mu{}_\nu p^\nu = p^{\mu\prime}$

So $\frac{dp^\mu}{d\tau}$ is a 4-vector and $\frac{d\vec{p}}{d\tau}$ is its spatial 3-vector part. Now look at the Lorentz force law:

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{V}}{c}\times\vec{B}\right)$$

Jackson normalizes $\vec{B}$ with a $1/c$ here. Sometimes I absorb this in $\vec{B}$, so that $F = q(E + v\times B)$

We take this equation as given by experiment and use it to try to work out the necessary transformation properties of E and B, so that the equation is Lorentz invariant.

$$\Rightarrow \quad \gamma\frac{d\vec{p}}{dt} = \frac{d\vec{p}}{d\tau} = q\left(\gamma\vec{E} + \frac{\gamma\vec{V}}{c}\times\vec{B}\right)$$

$$= \frac{q}{c}\left(u^0 E + \vec{u}\times\vec{B}\right)$$

This tells us the RHS transforms like a 3-vector. The corresponding "time" is

$$\frac{dp^0}{d\tau} = \frac{d}{d\tau}\sqrt{m^2c^2+\vec{p}^{\,2}} = \frac{1}{p^0}\,\vec{p}\cdot\frac{d\vec{p}}{d\tau}$$

$$= \frac{q}{cp^0}\, m\vec{u}\cdot\left(u^0\vec{E} + \vec{u}\times\vec{B}\right)$$

$$= \frac{q}{c}\,\vec{u}\cdot\vec{E}$$

Therefore $\frac{q}{c}\left(\vec{U}\cdot\vec{E},\ U^0\vec{E}+\vec{U}\times\vec{B}\right)$ must be a 4-vector.

$= \frac{dp^\mu}{d\tau}$

this must determine the transformation properties of E and B...

The way to do it is to arrange $\vec{E}$ and $\vec{B}$ in a rank 2 antisymmetric tensor, the "Maxwell field strength tensor":

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & +B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Then $F^{\mu\nu} U_\nu = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} -\gamma c \\ +\gamma v_x \\ +\gamma v_y \\ +\gamma v_z \end{pmatrix}$

$$= -\begin{pmatrix} \vec{U}\cdot\vec{E} \\ U^0 E_x + (\vec{U}\times\vec{B})_x \\ U^0 E_y + (\vec{U}\times\vec{B})_y \\ U^0 E_z + (\vec{U}\times\vec{B})_z \end{pmatrix}$$

If we chose the "West Coast" metric convention, eta = diag(1, -1, -1, -1), the minus sign would not be here.

$(F^{\mu\nu})' \equiv \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$ is the transformation

law, so that $F^{\mu\nu} U_\nu$ is manifestly a 4-vector:

$$(F^{\mu\nu} U_\nu)' = F^{\mu\nu\,\prime} g_{\nu\beta} U^{\beta\,\prime}$$

$$= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma} g_{\nu\beta} \Lambda^\beta{}_\gamma U^\gamma$$

and since $\Lambda^\nu{}_\sigma g_{\nu\beta} \Lambda^\beta{}_\gamma = g_{\sigma\gamma}$

(Minkowski space is Lorentz invariant!)

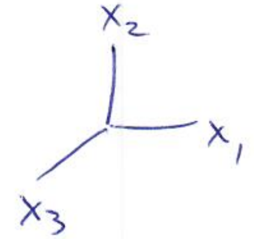
we get $\Lambda^\mu{}_\rho (F^{\rho\sigma} U_\sigma)$

$\Rightarrow$ 4-vector.

So $\frac{dp^\mu}{d\tau} = -\frac{q}{c} F^{\mu\nu} U_\nu$ is the "covariant form"

of the force law, and we can read off the

transformation properties of $\vec{E}$ and $\vec{B}$:



$$E_1' = E_1$$

$$E_2' = \gamma(E_2 - \beta B_3)$$

$$E_3' = \gamma(E_3 + \beta B_2)$$

$$B_1' = B_1$$

$$B_2' = \gamma(B_2 + \beta E_3)$$

$$B_3' = \gamma(B_3 - \beta E_2)$$

from boosts along the x_1 axis:

$$\underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & +B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}}_{F} \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^T}$$

More generally, $\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \dfrac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{E})$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{B})$$

if $\vec{B} = 0$, then $\vec{B}' = -\gamma \vec{\beta} \times \vec{E}$

Maxwell equations:

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0 \qquad \epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & 0123, \text{ even perms} \\ 0 & \text{else} \\ -1 & \text{odd perms} \end{cases}$$

$$\Box A^\mu = \mu_0 J^\mu \text{ (lorentz gauge)}$$

$$J = (c\rho, \vec{j})$$

$$(\star F)^{\alpha\beta} \equiv \tfrac{1}{2}\left(\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}\right)$$

eg $\nu = 0$

$$\underbrace{\partial_0 \cancel{F^{00}}}_{0} + \underbrace{\partial_i F^{i0}}_{\nabla\cdot\vec{E}} = \mu_0 c \rho$$

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \text{ gauss law}$$

$$\partial_0 F^{0i} + \partial_j F^{ji} = \mu_0 \vec{j}$$

$$\frac{1}{c}\dot{\vec{E}} \mp \nabla\times\vec{B} = \mu_0 \vec{j}$$

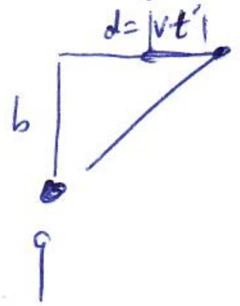
signs, c's.

Finally, let's use this to recover (in a simpler way!) the fields of a moving point charge, as used in the Weizsäcker-Williams analysis (later)

rest frame of observer (x^μ)

rest frame of charge ($x^{\mu'}$)

In the $x^{\mu'}$ coordinate system,



$$E_1 = \frac{vt'}{\sqrt{b^2+(vt')^2}} \frac{q}{4\pi\epsilon_0} \frac{1}{b^2+(vt')^2}$$

$$E_2 = \frac{b}{\sqrt{b^2+(vt')^2}} \frac{q}{4\pi\epsilon_0} \frac{1}{b^2+(vt')^2}$$

$$E_3 = \vec{B} = 0.$$

In the x^μ coordinate system, $E_1' = E_1$, $E_2' = \gamma E_2$, $B_3' = -\gamma\beta E_2$
all others $= 0$.

In terms of $t = t'/\gamma$,

$$E_1' = \frac{\gamma v t}{[b^2+(\gamma v t)^2]^{3/2}} \frac{q}{4\pi\epsilon_0}$$

$$E_2' = \frac{b\gamma}{[b^2+(\gamma v t)^2]^{3/2}} \frac{q}{4\pi\epsilon_0}$$

$$B_3' = -\beta E_2'$$