Relativistic Point Charges

We want to generalize our power flux $\frac{dP}{d\Omega}$ and our Energy flux/freq interval $\frac{dE}{d\Omega\, d\omega}$ to fast charges, $|\vec{\beta}|^2 \sim \mathcal{O}(1)$. There are two approaches. In the direct approach, we put $\vec{E}_{rad}$ & $\vec{B}_{rad}$ into $\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$ <u>without</u> taking $\beta \to 0$. This is fine but it does require a lot of vector identities to massage into a simple form. Alternatively, we use the fact that the total power is Lorentz invariant. Takes a little work to show this. One way is to go into the (instantaneous) rest frame of the charge where the nonrelativistic formulas apply, and see that the total momentum carried off by the radiation fields is zero (in an infinitesimal time Δt):

$$(\Delta\vec{P})_{tot} = \int_{\text{sphere},\, R=c(t-t')} \frac{\vec{S}}{c}\, d\Omega \qquad \vec{S} \sim \vec{a}_\perp \times (\hat{n}\times\vec{a}_\perp) \sim |a|^2 \sin^2\theta\, \hat{n}$$

$$= 0$$

Cancels in opposite directions

Therefore the Lorentz transformation of the total energy carried off is

$$(\Delta E)' = \gamma(\Delta E - \vec{v} \cdot \Delta\vec{p})$$

$$= \gamma \Delta E \quad \text{since } \Delta\vec{p} = 0.$$

$\gamma = \frac{1}{\sqrt{1-\vec{v}^2}}$ is the boost parameter.

But the time interval dilates:

$$(\Delta t)' = \gamma \Delta t.$$

So $\frac{(\Delta E)'}{(\Delta t)'} = \frac{\Delta E}{\Delta t}$ = total power.

So, we can look for a Lorentz invariant generalization of Larmor's formula. It turns out to be unique if it depends only on the 4-velocity u^μ and the 4-accel $a^\mu = \frac{du^\mu}{d\tau}$ where τ is proper time.

Re Write Larmor using $m\vec{a} = \frac{d\vec{p}}{dt}$ $\vec{p}$ = 3 momentum.

$$P = \frac{e^2 a^2}{6\pi\epsilon_0 c^3} = \frac{e^2}{6\pi\epsilon_0 c^3 m^2}\left(\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}\right)$$

Try replacing $\frac{d\vec{p}}{dt} \rightarrow \frac{dp^\mu}{d\tau}$ $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

a Lorentz 4-vector. $= \gamma(cm, \vec{p})$

$$P = -\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\underbrace{\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}}_{\text{Lorentz invariant product,}}$$ is the correct generalization.

$$= \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 - \left(\frac{d\vec{p}}{d\tau}\right)^2$$

Now $E = \gamma mc^2$ and $|\vec{p}| = \gamma m v$
$= \sqrt{|\vec{p}|^2c^2 + m^2c^4}$

So $$\frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 = \frac{1}{c^2}\left(\frac{c^2}{2E}\frac{d|\vec{p}|^2}{d\tau}\right)^2 = \frac{1}{\cancel{c^2}}\cancel{c^2}\left(\frac{|\vec{p}|c}{E}\frac{d|\vec{p}|}{d\tau}\right)^2$$
$$= \beta^2\left(\frac{d|\vec{p}|}{d\tau}\right)^2$$

$$\Rightarrow \quad P = -\frac{e^2}{6\pi\epsilon_0 c^3 m^2}\left[\beta^2\left(\frac{d|\vec{p}|}{d\tau}\right)^2 - \left(\frac{d\vec{p}}{d\tau}\right)\cdot\left(\frac{d\vec{p}}{d\tau}\right)\right]$$

(In the nonrelativistic limit, $\beta \to 0$ and $\tau \to t$ and we recover previous results.)

For a linear accelerator, motion is 1D

So $$\left|\frac{d\vec{p}}{d\tau}\right| = \left|\frac{dp_z}{d\tau}\right| = \frac{d|p_z|}{d\tau} = \frac{d|p|}{d\tau}$$

so $$-\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = \frac{1}{\gamma^2}\left(\frac{d|p|}{d\tau}\right)^2 = \left(\frac{d|p|}{dt}\right)^2$$

$d\tau = dt/\gamma$

$\frac{dp}{dt} = \frac{dE}{dx}$, so

$$P = \frac{e^2}{6\pi\epsilon_0 c^3 m^2}\left(\frac{dE}{dx}\right)^2 \quad \text{(linear accel)}$$

Some proposed e^+e^- linear colliders like CLIC would have $\mathcal{O}(100)$ MeV/m accelerations, and lengths of $\mathcal{O}(10-100)$ km. So the run down the linac takes, say, $\frac{30\text{km}}{3\times10^8\text{m/s}} = 10^{-4}$ s (since the particles are relativistic after a very short distance)

The total energy radiated is

$$E \approx \underbrace{\left(\frac{e^2}{6\pi\epsilon_0}\right)}_{\approx 1\,\text{MeV}\cdot\text{fm}} \underbrace{\left(\frac{100\text{ MeV}}{\text{m}}\right)^2}_{dE/dx} \underbrace{\left(\frac{1}{m^2c^4}\right)}_{\approx \frac{4}{\text{MeV}^2}} \cdot (10^{-4}\text{s}) \cdot c \approx 10^{-6}\,\text{MeV}$$

Which is totally negligible. For the radiated power to come close to the input power, we would need

$$1 \simeq \frac{P}{dE/dt} = \frac{P}{dE/dx \; dx/dt} \simeq \frac{P}{c\,\frac{dE}{dx}}$$

$$= \left(\frac{e^2}{6\pi\epsilon_0}\right)\left(\frac{1}{m^2c^4}\right)\frac{dE}{dx}$$

$$\rightarrow \frac{1/4 \text{ MeV}^2}{1 \text{ MeV}\cdot\text{fm}} = \frac{dE}{dx} \sim \underline{\underline{10^{14}}} \text{ MeV/m}.$$

a trillion times CLIC...

The story is quite different for circular colliders (LEP, Fermilab Tevatron, LHC, $\underbrace{\text{FCC, CEPC, ...}}_{\text{future}}$)

Here $\frac{d|\vec{p}|}{d\tau} \ll \left|\frac{d\vec{p}}{d\tau}\right|$ because <u>direction</u> changes a lot in one loop, while

magnitude only grows over many loops.

So $P \simeq \frac{e^2}{6\pi\epsilon_0 c^3 m^2} \left|\frac{d\vec{p}}{d\tau}\right|^2$

$\vec{p} = \gamma m\vec{v}$ and $\tau = t/\gamma$ so

$$P = \left(\frac{e^2}{6\pi\epsilon_0}\right)\left(\frac{1}{c^3}\right)\gamma^4 \left|\frac{d\vec{v}}{dt}\right|^2$$

But in circular motion $\frac{d\vec{v}}{dt} = \frac{v^2}{r}$, so

(for $v \simeq c$) we get

$$P = \left(\frac{e^2}{6\pi\epsilon_0}\right)\frac{c\gamma^4}{r^2}$$

In one revolution the energy loss is

$$P \cdot \frac{2\pi r}{c} = (\text{MeV fm})\left(\frac{E}{mc^2}\right)^4 \frac{2\pi}{r}$$

$$= \left(\frac{E}{\text{GeV}}\right)^4 \underbrace{\left(\frac{\text{GeV}}{mc^2}\right)^4}_{16 \cdot 10^{12} \text{ for } e^-} \underbrace{\left(\frac{\text{fm}}{\text{m}}\right)}_{10^{-15}}\left(\frac{\text{m}}{r}\right) 2\pi \text{ MeV}$$

$$\simeq \left(0.1\ E^4/r\right) \text{MeV} \ (E \text{ in GeV}, r \text{ in m})$$

If you tried to put a Higgs factory ($E \sim 100$ GeV) in a football-field-sized building ($r \sim 100$m), the radiative losses in 1 turn would be $\mathcal{O}(100\text{ GeV})$! 100%...

$((10^{-1}\, 10^{8}\, 10^{-2})$ MeV $= 10^{5}$ MeV $= 100$ GeV$)$

You do much better w/ big colliders & heavy particles.

Fermilab was $\mathcal{O}(1\,\text{TeV})$ in $\mathcal{O}(1\,\text{km})$, but p^+ instead of e^-, lowering the γ^4 factor by

$$\left(\frac{M_e}{M_p}\right)^4 \sim \left(\frac{10^{-3}\,\text{GeV}}{1\,\text{GeV}}\right)^4 \sim 10^{-12}$$

$$\text{loss/turn} \approx 0.1 \cdot \frac{(1000)^4}{10^3} \cdot 10^{-12}\ \text{MeV} \approx 10^{-4}\ \text{MeV/turn}.$$

negligible

Angular distribution of Radiation.

Recall that the electric radiation field was

$$\vec{E}_{rad}(\vec{x},t) = \frac{e}{4\pi\epsilon_0 c}\left[\frac{\hat{n}\times([\hat{n}-\vec{\beta}]\times\dot{\vec{\beta}})}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right]_{ret}$$

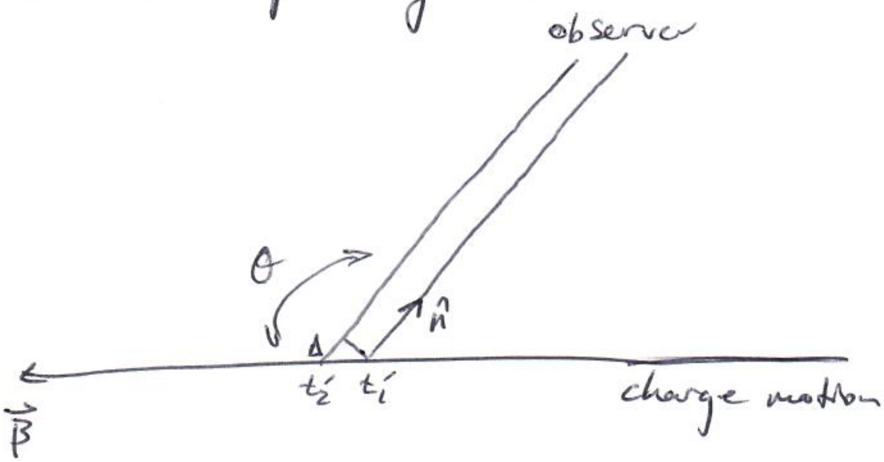
and $\vec{B}_{rad}(x,t) = \frac{1}{c}[\hat{n}\times\vec{E}_{rad}]_{ret}$

$$\text{or}\quad \vec{E}_{rad} = \frac{-e}{4\pi\epsilon_0 c}\left(\frac{\vec{a}_\perp/c - \hat{n}\times(\vec{a}\times\vec{\beta})}{R(1-\vec{\beta}\cdot\hat{n})^3}\right)_{ret}$$

$$\text{with}\quad \vec{a}_\perp = \vec{a}-(\vec{a}\cdot\hat{n})\hat{n}$$

Things are simple for the nonrelativistic limit $|\vec{\beta}|\to 0$.
Nonzero $|\vec{\beta}|$ makes for some interesting complications.

If we observe power $\frac{dP}{d\Omega}$ at some time t, that is not exactly the same as the power emitted at the corresponding retarded time t'_{ret}.



In a small time $\Delta t' = t_2' - t_1'$, the charge moves $|\vec{\beta}\,\Delta t'\,c|$. As a result, radiation emitted at t_2' has to travel a little further – an extra distance $\Delta = |\vec{\beta}\,\Delta t'\,c|\,|\cos\theta|$ – and so it arrives at the observer at $t_2' + R/c + \Delta/c$. Radiation emitted at t_1' arrives at $t_1' + R/c$. So

$$\frac{\text{power radiated}}{\quad} = \frac{\text{time to observe fixed amt of } E}{[\text{illegible}] \text{ same amt of } E}$$

$$= \quad 1 + |\vec{\beta}||\cos\theta| \quad = \quad 1 - \vec{\beta}\cdot\hat{n}$$

Think of this as a factor $\frac{dt}{dt'}$ converting emission time to obs time.

The Poynting vector tells us the Power/unit area detected at t by the observer. We evaluate $\vec{\beta}$, R, $\hat{n}$ in $(\vec{S}\cdot\hat{n})R^2$ at $t'_{ret} = t - R(t'_{ret})/c$. The power was emitted at t'_{ret}. So the emitted power was actually $\frac{dP(t')}{d\Omega} = R^2(\vec{S}\cdot\hat{n})\frac{dt}{dt'}$ in terms of the charge's time t'.

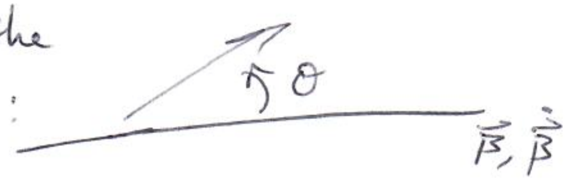
Plugging in $\vec{E}_{rad}$ & $\vec{B}_{rad}$,

$$R^2(\vec{S}\cdot\hat{n})\frac{dt}{dt'} = \frac{e^2}{(4\pi\epsilon_0)^2 c^5 \mu_0} \frac{|\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}|^2}{(1-\vec{\beta}\cdot\hat{n})^5}$$

This is the angular distribution of power radiated in a small time window around t'.

In linear motion, $\vec{\beta}$ and $\dot{\vec{\beta}}$ are $\parallel$, so

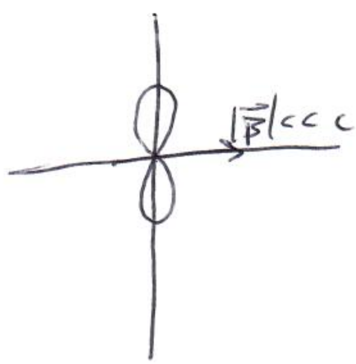
defining the angle :

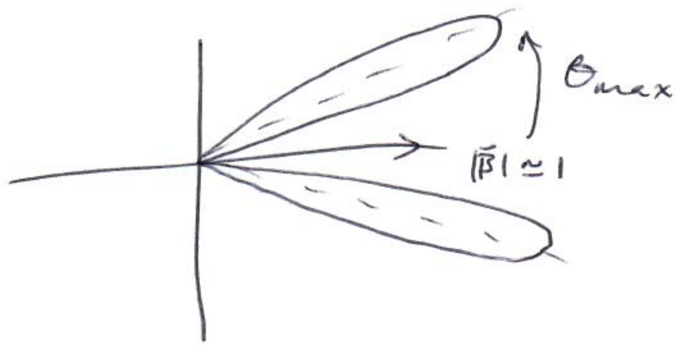


the distribution simplifies :

$$\frac{|n \times ((\hat{n} - \cancel{\vec{\beta}}) \times \dot{\vec{\beta}})|^2}{(1 - \hat{n} \cdot \vec{\beta})^5} \to \frac{\sin^2\theta \, |\dot{\vec{\beta}}|^2}{(1 - \beta\cos\theta)^5} .$$

as $\beta \to 0$, recover Larmor. But as $\beta \to 1$, there is a strange peak toward $\theta \to 0$!





$$\frac{d}{d\theta}\left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right) = 0 \Rightarrow \theta_{max} \simeq \frac{1}{2\gamma} \quad \text{as } \beta \to 1.$$

$$\text{and } \left.\frac{dP}{d\Omega}\right|_{\theta_{max}} \propto \gamma^8 \leftarrow !!!$$

(In these plots, the distance to a point on the lobes at angle θ is proportional to the power radiated in that direction.)

For relativistic accelerated ptcl, radiation pattern is confined to a narrow cone close to the direction of motion, if $\dot{\vec{\beta}} \parallel \vec{\beta}$. If they are not $\parallel$, the patterns change. Explicit formulas for circular motion, for example, are given in Jackson sec 14.3.