Magnetic vector potential

The magnetostatic Maxwell equations are:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

A valid form of an expression for the vector potential is found using the source-free Maxwell equation. Since the divergence of a curl is always zero, then the magnetic field can be written as the curl of a vector function, (call it the magnetic vector potential).

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Longrightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

The additional constraint, $\vec{\nabla} \cdot \vec{A} = 0$, leads to an expression relating $\vec{A}$ to $\vec{J}$, the source of the magnetic field. This can be seen by substituting $\vec{nabla} \times \vec{A}$ for $\vec{B}$ in the other Maxwell equation.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} \tag{1}$$

The following identity will be used:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \tag{2}$$

Substitute Eq. 2 into Eq. 1, and for convenience set $(\vec{\nabla} \cdot \vec{A}) = 0$. Then:

$$\begin{aligned} \vec{\nabla}(\cancelto{0}{\vec{\nabla} \cdot \vec{A}}\,) - \nabla^2 \vec{A} &= \mu_0 \vec{J} \\ \nabla^2 \vec{A} &= -\mu_0 \vec{J} \end{aligned}$$

This has the same form as Poisson's equation. Since the solution to Poisson's equation is known, the solution for the vector potential is also known:

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad \longrightarrow \qquad V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho d\tau'}{\mathscr{r}}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \qquad \longrightarrow \qquad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\, d\tau'}{\mathscr{r}}$$

Or, for surface or line currents the magnetic vector potential $\vec{A}$ is given by:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\, da'}{\mathscr{r}}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\, dl'}{\mathscr{r}}$$

The choice $\vec{\nabla} \cdot \vec{A} = 0$ is called the 'Coulomb gauge'. If the given vector potential $\vec{A}_0$ is such that $\vec{\nabla} \cdot \vec{A}_0 \neq 0$, it is possible to choose another vector potential $\vec{A}$ with $\vec{\nabla} \cdot \vec{A} = 0$. This must be done without changing the physical magnetic field, $\vec{B} = \vec{\nabla} \times \vec{A}_0$. Since the curl of a gradient is zero, $\vec{A}_0$ can be shifted by the gradient of a scalar function $(\vec{\nabla}\lambda)$ without changing the field.

$$\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda$$

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\vec{A}_0 + \vec{\nabla}\lambda) = \vec{\nabla} \times \vec{A}_0 + \vec{\nabla} \times \vec{\nabla}\lambda$$

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}_0 + 0 = \vec{B} \qquad \text{As required}$$

OK, but what λ do we need to achieve $\vec{\nabla} \cdot \vec{A} = 0$? We can find out what λ must be in terms of our original vector potential $\vec{A}_0$:

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2\lambda$$

Then, in order to get $\vec{\nabla} \cdot \vec{A} = 0$, it must be that

$$\nabla^2\lambda = -\vec{\nabla} \cdot \vec{A}_0 \tag{3}$$

Note that Eq. 3 also has the form of Poisson's equation, so there is a known form of solution for λ:

$$\lambda = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}_0 \, d\tau'}{\mathscr{r}}$$

The coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$) is a convenient choice for magnetostatic problems, since there is an associated solution for $\vec{A}$ using the current distribution. However, you will find out that a different choice of gauge is more suitable for dynamical situations.

Magnetic vector potential in the far field region

In the far field limit, where the scale of the current distribution is much less than the distance to the observation point ($r' \ll r$), a multipole expansion of $\vec{A}$ is a convenient approximation for the magnetic vector potential. The expression for the magnetic vector potential, $\vec{A}$, is similar to that of the scaler potential, V, so the basic technique of doing the multipole expansion is similar.

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho d\tau'}{\mathscr{r}} \qquad \text{Scalar potential}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{I}\, dl'}{\mathscr{r}} \qquad \text{Magnetic vector potential}$$

In both cases, a binomial expansion can be performed on $1/\mathscr{r}$, allowing the potential to be expressed as a series of terms in increasing powers of (r'/r). In the far field limit, these terms then diminish in magnitude as their order increases. Then, the potential (or the field) may be approximated by keeping the first non-zero term of the expansion.

The expansion of $1/\mathscr{r}$ followed by collecting terms (r'/r) in order of their powers was done previously for the electrostatic potential. This resulted in the following:

$$\frac{1}{\mathscr{r}} = \frac{1}{r} \sum_n \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

For electrostatics, the monopole term is,

$$V(r)_{\text{monopole}} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int \rho(r') d\tau' = \frac{1}{4\pi\varepsilon_0} \left(\frac{Q_{\text{net}}}{r}\right)$$

where Q_{net} is the net charge.

For magnetostatics, the monopole term is,

$$\vec{A}(\vec{r})_{\text{monopole}} = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\vec{l'} = 0$$

since the integral of the displacement around a closed loop is zero.

There is never a magnetic monopole term. The dipole term is,

$$\begin{aligned}
\vec{A}(\vec{r})_{\text{dipole}} &= \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\theta' d\vec{l'} \\
&= \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot r')d\vec{l'} \qquad (4)\\
&= \frac{\mu_0 I}{4\pi r^2}\left(-\hat{r}\times\int d\vec{a'}\right) \qquad (5)\\
&= \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \qquad (6)
\end{aligned}$$

Reminder, the magnetic dipole moment is defined as,

$$\vec{m} = I\int d\vec{a'}$$

where the direction of an area element is normal to the surface of that element, $\hat{n}$. In simpler days, we had $\vec{m}$ of a flat coil of area A and N turns as: $\vec{m} = NIA\,\hat{n}$.

Aside (If you want a guide on how to get from Eq. 4 to Eq. 5):

That is, show that the following holds:

$$\oint (\hat{r}\cdot r')d\vec{l'} = -\hat{r}\times\int d\vec{a'}$$

First, show that,

$$\int \vec{\nabla}'T\times d\vec{a'} = -\oint T d\vec{l'}$$

Use Stokes theorem on $\vec{c}T$, where $\vec{c}$ is a constant vector,

$$\int \vec{\nabla}' \times (\vec{c}T) \cdot d\vec{a}' = \oint \vec{c}T \cdot d\vec{l}' \qquad\qquad \text{Stokes theorem} \tag{7}$$

By product rule 5 (front cover of Griffiths):

$$\vec{\nabla}' \times (\vec{c}T) = T(\cancelto{0}{\vec{\nabla}' \times \vec{c}}) - \vec{c} \times \vec{\nabla}' T \tag{8}$$

The second term on the right of Eq. 8 goes to zero because the curl of a constant vector is zero. (Or, a constant vector has no curl!)

While by the triple product rule,

$$(\vec{c} \times \vec{\nabla}' T) \cdot d\vec{a}' = \vec{c} \cdot (\vec{\nabla}' T \times d\vec{a}') \tag{9}$$

Combining equations 7, 8, and 9 we have:

$$\int \vec{\nabla}' \times (\vec{c}T) \cdot d\vec{a}' = -\int \vec{c} \cdot (\vec{\nabla}' T \times d\vec{a}') = \oint \vec{c}T \cdot d\vec{l}'$$

Pulling the constant vector outside the integrals,

$$\vec{c} \cdot \int \vec{\nabla}' T \times d\vec{a}' = -\vec{c} \cdot \oint T d\vec{l}'$$

So, it must be that,

$$\int \vec{\nabla}' T \times d\vec{a}' = -\oint T d\vec{l}' \tag{10}$$

Now that we have Eq. 10, for the second part of this derivation, let our heretofore arbitrary scalar T be defined as $T = \hat{r} \cdot r'$. Equation 10 is then written,

$$\int \vec{\nabla}'(\hat{r} \cdot r') \times d\vec{a}' = -\oint (\hat{r} \cdot r') d\vec{l}' \tag{11}$$

By product rule 4 (front cover of Griffiths):

$$\vec{\nabla}'(\hat{r} \cdot r') = \hat{r} \times \vec{\nabla}' \times \vec{r}' + \vec{r}' \times \vec{\nabla}' \times \hat{r} + (\hat{r} \cdot \vec{\nabla}')\vec{r}' + (\vec{r}' \cdot \vec{\nabla}')\hat{r} \tag{12}$$

Three of the terms on the right side of Eq. 12 are zero.

$$\begin{aligned} \hat{r} \times \vec{\nabla}' \times \vec{r}' &\qquad \longrightarrow \qquad \text{The curl of the radial vector is zero} \\ \vec{r}' \times \vec{\nabla}' \times \hat{r} &\qquad \longrightarrow \qquad \hat{r} \text{ has no primed variable dependence} \\ (\vec{r}' \cdot \vec{\nabla}')\hat{r} &\qquad \longrightarrow \qquad \hat{r} \text{ has no primed variable dependence} \end{aligned}$$

This leaves $(\hat{r} \cdot \vec{\nabla}')\vec{r'}$ as the only non-zero term.

In general for any constant vector, $\vec{b}$,

$$(\vec{b} \cdot \vec{\nabla})\vec{r} = \vec{b} \qquad \text{Where } \vec{b} \text{ is a constant vector} \tag{13}$$

Let's show Eq. 13 explicitly (since we are dragging through details in this section anyway!).

$$\left[(b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) \cdot \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\right]\vec{r}$$

$$= \left(b_x\frac{\partial}{\partial x} + b_y\frac{\partial}{\partial y} + b_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$$

$$= b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$$

$$= \vec{b}$$

Therefore,

$$\vec{\nabla}'(\hat{r} \cdot r') = (\hat{r} \cdot \vec{\nabla}')\vec{r'} = \hat{r}$$

so that,

$$\int \vec{\nabla}'(\hat{r} \cdot r') \times d\vec{a'} = \hat{r} \times \int d\vec{a'}$$

Then, Eq. 11 can be written,

$$\hat{r} \times \int d\vec{a'} = -\oint (\hat{r} \cdot r')d\vec{l'}$$

This is what we had set out to prove.

End of aside

Magnetic dipole field

Here, starting with $\vec{A}(\vec{r})_{\text{dipole}}$, the magnetic dipole potential (Eq. 6 above) it will be shown that the magnetic field of a dipole is given by:

$$\vec{B}(\vec{r})_{\text{dipole}} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}\right] \tag{14}$$

The magnetic field is given by the curl of the vector potential,

$$\vec{B}(\vec{r})_{\text{dipole}} = \vec{\nabla}\times\vec{A}(\vec{r})_{\text{dipole}} = \vec{\nabla}\times\frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\hat{r}}{r^2}$$

Since $(\vec{m}\times\hat{r})$ is a vector, and $\frac{1}{r^3}$ is a scalar, the following vector identity is useful here (with f a scalar, and $\vec{v}$ a vector):

$$\vec{\nabla}\times(f\vec{v}) = \vec{\nabla}f\times\vec{v} + f\vec{\nabla}\times\vec{v}$$

In this case,

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\vec{\nabla}\left(\frac{1}{r^3}\right)\times(\vec{m}\times\vec{r}) + \left(\frac{1}{r^3}\right)\vec{\nabla}\times(\vec{m}\times\vec{r})\right] \tag{15}$$

Use the following vector identity on the first term of the RHS of Eq. 15,

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

with $\vec{a} = \vec{\nabla}\frac{1}{r^3}$, $\vec{b} = \vec{m}$, and $\vec{c} = \vec{r}$.

Then, the first term on the RHS of Eq. 15 can be written,

$$\vec{\nabla}\left(\frac{1}{r^3}\right)\times(\vec{m}\times\vec{r}) = \left(\vec{\nabla}\left(\frac{1}{r^3}\right)\cdot\vec{r}\right)\vec{m} - \left(\vec{\nabla}\left(\frac{1}{r^3}\right)\cdot\vec{m}\right)\vec{r} \tag{16}$$

The gradient of r^{-3} is the following,

$$\vec{\nabla}\left(\frac{1}{r^3}\right) = \frac{\partial r^{-3}}{\partial r}\,\hat{r} = -3r^{-4}\hat{r} = -\frac{3}{r^4}\,\hat{r}$$

Then, the first term on the RHS of Eq. 16 is,

$$\left(\vec{\nabla}\left(\frac{1}{r^3}\right)\cdot\vec{r}\right)\vec{m} = -\frac{3\vec{m}}{r^3}$$

and the second term on the RHS of Eq. 16 is,

$$-\left(\vec{\nabla}\left(\frac{1}{r^3}\right)\cdot\vec{m}\right)\vec{r} = \frac{3(\hat{r}\cdot\vec{m})\hat{r}}{r^3}$$

It remains to deal with the second term on the RHS of of Eq. 15, starting with $\vec{\nabla}\times(\vec{m}\times\vec{r})$. There is another vector identity to help with this task,

$$\vec{\nabla}\times(\vec{a}\times\vec{b}) = \vec{a}(\vec{\nabla}\cdot\vec{b}) - \vec{b}(\vec{\nabla}\cdot\vec{a}) + (\vec{b}\cdot\vec{\nabla})\vec{a} - (\vec{a}\cdot\vec{\nabla})\vec{b}$$

With $\vec{a}=\vec{m}$, and $\vec{b}=\vec{r}$, the vector identity reads,

$$\vec{\nabla}\times(\vec{m}\times\vec{r}) = \vec{m}(\vec{\nabla}\cdot\vec{r}) - \vec{r}(\vec{\nabla}\cdot\vec{m}) + (\vec{r}\cdot\vec{\nabla})\vec{m} - (\vec{m}\cdot\vec{\nabla})\vec{r} \tag{17}$$

The second and third terms on the RHS of Eq. 17 are zero, because $\vec{m}$ is a constant vector (operating on a constant vector with $\vec{\nabla}$ yields zero). Examining $\vec{\nabla}\cdot\vec{r}$ (in the first term on the RHS of Eq. 17):

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

While examining the last term on the RHS of Eq. 17 we find:

$$(\vec{m}\cdot\vec{\nabla})\vec{r} = \hat{x}m_x\frac{\partial x}{\partial x} + \hat{y}m_y\frac{\partial y}{\partial y} + \hat{z}m_z\frac{\partial z}{\partial z} = \vec{m}$$

So, overall the second term in brackets in Eq. 15 can be written as follows,

$$\left(\frac{1}{r^3}\right)\vec{\nabla}\times(\vec{m}\times\vec{r}) = \left(\frac{1}{r^3}\right)(3\vec{m}-\vec{m}) = \frac{2\vec{m}}{r^3}$$

Putting all of this together, Eq. 15 simplifies:

$$\vec{B} = \frac{\mu_0}{4\pi}\left[-\frac{3\vec{m}}{r^3} + \frac{3(\hat{r}\cdot\vec{m})\hat{r}}{r^3} + \frac{2\vec{m}}{r^3}\right]$$

Finally simplifying to Eq. 14:

$$\boxed{\vec{B}(\vec{r})_{\text{dipole}} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{r})\hat{r}-\vec{m}}{r^3}\right]}$$