The method of Green functions is a powerful approach to such problems. Given an operator L_x, construct its inverse or Green function satisfying

$$L_x \, G(\vec{x}, \vec{x}') = \delta^{(n)}(\vec{x} - \vec{x}')$$

Then $\int d^n x' \, G(\vec{x}, \vec{x}') \, f(\vec{x}') = \varphi(\vec{x})$ is the solution

check: $\int d^n x' \left(L_x \, G(\vec{x}, \vec{x}') \right) f(\vec{x}') = L_x \varphi$

$$= \int d^n x' \left(\delta^{(n)}(\vec{x} - \vec{x}') \right) f(\vec{x}') = f(\vec{x}) \checkmark$$

Notes:

(1) We have to specify BCs. Different BCs $\Rightarrow$ Different Green fn for same L_x

(2) If there are zero modes $L_x \varphi_0 = 0$ consistent with the BC's, then we have to do a little more work, because L_x is only invertible modulo addition of these zero modes

(3) If L is self-adjoint wrt some inner product, a very general powerful formula for G is

$$G = \text{"}\sum_k\text{"} \frac{\psi_k(x) \psi_k^*(x')}{\lambda_k}$$

eigenfunctions

eigenvalues

We will construct a Green function for the wave operator $\Box$. But first it is useful to recall an example you first saw in high school physics:

Say there is a unit point charge at the origin. 
Then the electrostatic potential is $\varphi = \frac{1}{4\pi\epsilon_0 r}$

But we also know $-\nabla^2\varphi = \rho/\epsilon_0$

The charge density is $\rho = \delta(x)\delta(y)\delta(z) = \delta^3(x)$

(so that $\int_V \rho\, d^3x = Q = 1$ for any V enclosing the origin.)

$$\therefore \quad \nabla^2\left(\frac{-1}{4\pi r}\right) = \delta^3(x)$$

and $\frac{-1}{4\pi r}$ is a Green function for the Laplace operator on $\mathbb{R}^3$ with vanishing BCs as $r \to \infty$.

$$\text{check:} \quad \underbrace{-\frac{1}{4\pi}\int_{R} d^3x}_{\substack{\text{sphere}\\ \text{of radius } R}} \underbrace{\nabla^2\left(\frac{1}{r}\right)}_{\substack{\text{derivs}\\ \text{undef}\\ \text{@ origin}}} \overset{\substack{\text{divergence}\\ \text{theorem}}}{=} -\frac{1}{4\pi}\int_{r=R} R^2 d\Omega\, \hat{r}\cdot\underbrace{\vec{\nabla}\left(\frac{1}{r}\right)}_{\substack{\text{well def}\\ \text{@ } r=R}}$$

$$= 1$$

indep of R ✓

We'll use this momentarily. We seek G s.th.

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) G(t,t',\vec{x},\vec{x}') = \delta(t-t')\,\delta^3(x-x').$$

Let's look for $G = G(t-t', \widetilde{|x-x'|})$ since the delta function source is rotationally invariant and time-translation invariant.

Fourier in time:

$$\int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')} \text{ [both sides]}$$

$$\Rightarrow \left(-\frac{\omega^2}{c^2} - \nabla^2\right)\tilde{G}(\omega,r) = \delta^3(x-x')$$

$$\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r \qquad r = |x-x'|$$

Let's try $\tilde{G} = u(\omega,r) \times \left(G_{\text{-Laplace}} = \frac{1}{4\pi r}\right)$ ★ this trick only works in 4D

$$\Rightarrow -\frac{\omega^2 u}{4\pi r c^2} - \left(\partial_r^2 + \frac{2}{r}\partial_r\right)\left(\frac{u(\omega,r)}{4\pi r}\right) = \delta^3(x-x')$$

$$\Rightarrow -\frac{\omega^2 u}{4\pi r c^2} - \cancel{\frac{(\partial_r^2 + \frac{2}{r}\partial_r)u}{4\pi r}} - \frac{2}{4\pi}\cancel{(\partial_r u)\left(\partial_r \frac{1}{r}\right)} = \delta^3(x-x') + u\left(\nabla^2 \frac{1}{4\pi r}\right)$$

The terms on the RHS cancel if $u(r=0) = 1$.

On the LHS we have a familiar operator... 1+1D waves!

$$\frac{1}{4\pi r}\left(\frac{-\omega^2}{c^2} - \partial_r^2\right) u = 0$$

$$\rightarrow \quad \partial_r^2 u = -\frac{\omega^2}{c^2} u$$

$$\Rightarrow \quad u = A e^{i\omega r/c} + B e^{-i\omega r/c}$$

@ $r=0$, $u = A + B$, so we need $A + B = 1$.

$$\Rightarrow \quad \tilde{G}(\omega, r) = \frac{1}{4\pi r}\left(A e^{i\omega r/c} + B e^{-i\omega r/c}\right)$$

and $G(t-t', r) = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{G}(\omega, r)$

The integral contains things like $\int \frac{d\omega}{2\pi} e^{-i\omega[(t-t')-r/c]}$

$$= \delta(t-t'-r/c)$$

All together,

$$G(t-t', r) = \frac{1}{4\pi r}\left[A\,\delta(t-t'-r/c) + B\,\delta(t-t'+r/c)\right]$$

$$A + B = 1$$

A & B can be fixed by choosing boundary conditions in time.

Say there is no field for $t < t'$, and at t' a charge or current is introduced @ position $\vec{x}'$.

(x', t')

It takes time $\frac{|x - x'|}{c}$ for the signal to propagate to another point $\bar{x}$. So we want $G = 0$ for $t < (t' + r/c)$. This picks out the "retarded" Green function with $A = 1$, $B = 0$

$$\boxed{G(t - t', r) = \frac{\delta(t - t' - r/c)}{4\pi r}}$$

Now we can write a very general class of solutions to $\Box A^\mu = \mu_0 J^\mu$!

$$A^\mu = \mu_0 \int d^3x' dt' \, J^\mu(t', x') \, G(t - t', r = |\bar{x} - \bar{x}'|)$$

$$= \mu_0 \int d^3x' \, dt' \; J^\mu(t', x') \frac{\delta(t-t'-r/c)}{4\pi r}$$

$$= \boxed{\mu_0 \int d^3x' \; \frac{J^\mu(t-r/c, x')}{4\pi r} = A^\mu(t, \vec{x})}$$

$$J^\mu = (c\rho, \vec{j}) \quad A^\mu = (\varphi/c, \vec{A})$$

We can compute expressions for $\vec{E}$ & $\vec{B}$ from A^μ, or recall that $\vec{E}$ & $\vec{B}$ also satisfied wave equations,

$$\Box \vec{E} = -\vec{\nabla}(\rho/\epsilon_0) - \partial_t(\mu_0 \vec{j})$$

$$\Box \vec{B} = \mu_0(\vec{\nabla} \times \vec{j})$$

Again these are inhomogeneous wave eq that we can solve with our Green function

As before let's define $\vec{r} =$ "$\overrightarrow{\text{obs} - \text{source}}$" $= \vec{x} - \vec{x}'$ points from source to obs @ $\vec{x}$

$$\vec{E}(t, \vec{x}) = \frac{1}{4\pi\epsilon_0} \int dt' \int d^3x' \, \frac{1}{r} \left[-\vec{\nabla}'\rho - \frac{1}{c^2} \partial_{t'} \vec{j} \right] \delta(t-t'-r/c)$$

$$\equiv \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{1}{r} \left[-\vec{\nabla}'\rho - \frac{1}{c^2} \partial_{t'} \vec{j} \right]_{ret}$$

$[\vec{\nabla}'\rho]_{ret}$ means compute $\partial_{x',y',z'}\rho$ at fixed t', THEN evaluate @ $t' = t - r/c$; since r depends on $\vec{x}'$ this is important:

If we did it in the opposite order...

$$\vec{\nabla}'[\rho_{ret}] = \vec{\nabla}'[\rho(t' = t - r/c, \vec{x}')]$$

$$= [\vec{\nabla}'\rho]_{ret} + \left.\frac{\partial\rho}{\partial t'}\right|_{t'=t-r/c} \vec{\nabla}'(t - r/c)$$

$$= [\vec{\nabla}'\rho]_{ret} + \left[\frac{\partial\rho}{\partial t'}\right]_{ret}\left(-\frac{\hat{r}}{c}\right)$$

$$\vec{\nabla}' r = \hat{r} = \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|}$$

We can now trade ~~[illegible]~~ $[\vec{\nabla}'\rho]_{ret}$ for $\vec{\nabla}'[\rho_{ret}] + \ldots$ and (I)ntegrate (B)y (P)arts the $\vec{\nabla}'$, using $\vec{\nabla}'\left(\frac{1}{r}\right) = \frac{-\hat{r}}{r^2}$

We get

$$\vec{E}(t,x) = \frac{1}{4\pi\epsilon_0}\int d^3x' \Big\{ \underbrace{\frac{\hat{r}}{r^2}[\rho(x',t')]_{ret}}_{\text{here we IBP}} + \underbrace{\frac{\hat{r}}{cr}[\partial_{t'}\rho(x't')]_{ret}}_{\text{from replacing } (\nabla'\rho)_{ret} \text{ by } \nabla'(\rho_{ret})}$$

$$- \frac{1}{c^2 r}[\partial_{t'}\vec{j}(t'x')]_{ret}\Big\}$$

If j and ρ are time-indep, just recover Coulomb's Law from the 1st term.

Similarly, we can use

$$(\vec{\nabla}'\times\vec{j})_{ret} = \vec{\nabla}'\times(\vec{j}_{ret}) - \underbrace{\left[\frac{\partial\vec{j}}{\partial t'}\right]_{ret}\times\vec{\nabla}'(t - r/c)}_{-\frac{1}{c}\left(\frac{\partial\vec{j}}{\partial t'}\right)_{ret}\times\hat{r}}$$

to write the solution for $\vec{B}$:

$$\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \left\{ \vec{j}(t',x') \times \frac{\hat{r}}{r^2} + \left[\frac{\partial \vec{j}}{\partial t'}\right]_{ret} \times \frac{\hat{r}}{cr} \right\}$$

In the integrands, we can also write

$$\left[\frac{\partial \vec{j}(t',x')}{\partial t'}\right]_{ret} = \frac{\partial \vec{j}_{ret}}{\partial t}$$

$$\left[\frac{\partial \rho(t',x')}{\partial t'}\right]_{ret} = \frac{\partial \rho_{ret}}{\partial t}$$

because ret time $t' = t - r/c$ is linear in t w/ coeff 1

"Jefimenko's Equations"

Huge amount of Electrodynamics in these 2 eq.
(Still need generalizations to media and diff BCs,
but otherwise pretty complete.)

We'll plug in some j's later when we discuss antennae.

For now, we'll spend awhile on point charges.

The simplest way to think about a point charge:

It has some trajectory $\vec{s} = \vec{s}(t)$

The charge density is $\rho(\vec{x}, t) = e\,\delta^{(3)}(\vec{x} - \vec{s}(t))$

The current is $\vec{j}(x,t) = e \underbrace{\frac{d\vec{s}}{dt}}_{\vec{v}} \delta^{(3)}(\vec{x} - \vec{s}(t))$.

Aside: you'll often see this in a more covariant form.

let $s^\mu = (cs^0, \vec{s}(s^0))$ where s^0 is some "time" parameter for the trajectory.

then $(c\rho, \vec{j}) = J^\mu = e \begin{pmatrix} c \\ \left.\frac{d\vec{s}}{ds^0}\right|_{s^0=t} \end{pmatrix} \times \delta^{(3)}(\vec{x} - \vec{s}(s^0=t))$

$$= e\int ds^0 \frac{\partial s^\mu}{\partial s^0} \underbrace{\delta(t-s^0)\,\delta^{(3)}(\vec{x}-\vec{s}(s^0))}_{\delta^{(4)}(\underbrace{x^\mu}_{(ct,\vec{x})} - \underbrace{s^\mu}_{(cs^0,\vec{s})})}$$

Note: $a\,\delta(ax) = \delta(x)$

$$= e\int d\tau \frac{\partial s^\mu}{\partial \tau} \delta^{(4)}(x^\mu - s^\mu)$$

τ = proper time

$c^2 d\tau^2 = c^2 ds^{0\,2} - d\vec{s}^{\,2}$

$\Rightarrow d\tau = ds^0 \sqrt{1 - \frac{d\vec{s}^{\,2}}{c^2 ds^{0\,2}}}$

$= ds^0\sqrt{1-\beta^2}$

$$\Rightarrow \boxed{J^\mu_{\text{point charge}} = e\int d\tau\, u^\mu\, \delta^4(x^\mu - s^\mu)}$$