~~For larger frequencies such that $\epsilon(\omega) > 0$, k is real again and waves can propagate. "ultraviolet transparency" of metals. Determined by ω_p, via the number density of free e^-.~~

Causality & Kramers-Kronig.

Recall that $\vec{D}_\omega(\vec{x}) \equiv \epsilon(\omega) \vec{E}_\omega(\vec{x})$

$$\left[\begin{array}{l} \epsilon = \epsilon_0 (1 + \chi_e), \ \epsilon_0 \chi_e = P/E_{net} \text{ for linear media;} \\ \text{then } \vec{\nabla} \cdot \vec{D}_\omega = \rho_\omega \text{ and } \vec{\nabla} \times \vec{H}_\omega = -i\omega \vec{D}_\omega + \vec{J}_\omega \end{array} \right]$$

Let's Fourier transform back to the time domain.

$$\vec{D}(t,x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \epsilon(\omega) E_\omega(x)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \epsilon(\omega) \int_{-\infty}^{\infty} dt' e^{i\omega t'} \vec{E}(x,t')$$

$$= \epsilon_0 \vec{E}(x,t) + \int_{-\infty}^{\infty} G(\tau) \vec{E}(x, t-\tau) \, d\tau$$

$$\text{with } G(\tau) = \int_{-\infty}^{\infty} d\omega/_{2\pi} \, e^{-i\omega\tau} \underbrace{\chi_e(\omega)}_{\frac{\epsilon}{\epsilon_0} - 1}$$

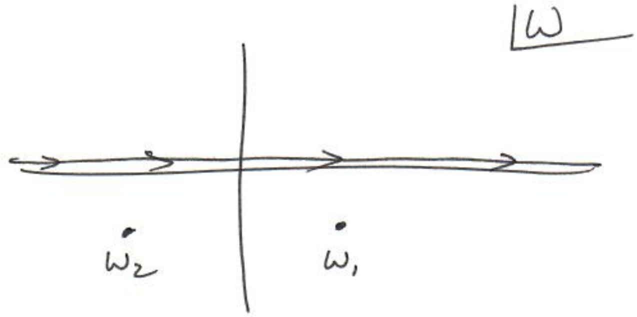
So $\vec{D}(t)$ depends on $\vec{E}$ at other times!

(this is why, for time dep fields in media, it is convenient to work with $\vec{E}$ [illegible] $\vec{D}$ [illegible])

Consider the one-resonance model,

$$\epsilon(\omega)/\epsilon_0 - 1 = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

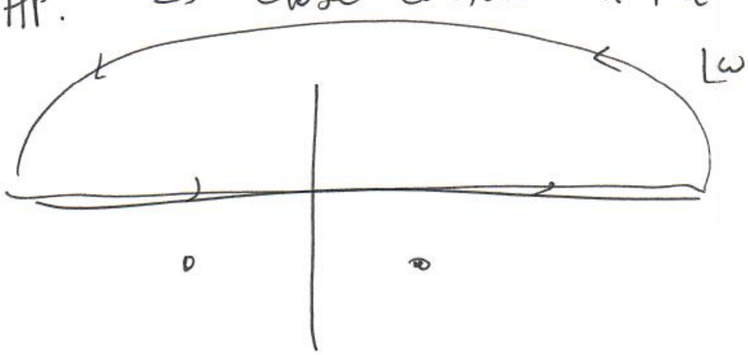
Then $$G(\tau) = \omega_p^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$



poles in the complex ω-plane at $\omega = \omega_{1,2} = \frac{-i\gamma}{2} \pm \nu_0$,

$$\nu_0^2 = \omega_0^2 - \gamma^2/4$$

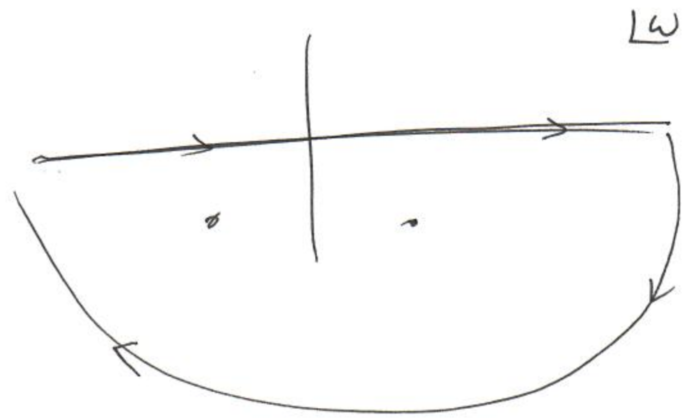
For $\tau < 0$, $e^{-i\omega\tau} \to 0$ exponentially fast with $\mathrm{Im}\,\omega$ in the UHP. $\Rightarrow$ close contour in the UHP



closed contour contains no poles & vanishes by the Cauchy theorem.

For $\tau > 0$, $e^{-i\omega\tau} \to 0$ exponentially fast in the LHP.

Close contour in LHP, pick up both poles:



Since the contour is clockwise we get $-2\pi i \times$ (sum of residues)

$$G(\tau) = \omega_p^2 \left(\frac{e^{-i\omega_1\tau}}{\omega - \omega_2} + \frac{e^{-i\omega_2\tau}}{\omega - \omega_1} \right)$$

$$= \omega_p^2 \, e^{-\gamma\tau/2} \frac{\sin \nu_0 \tau}{\nu_0} \, \theta(\tau)$$

So $\vec{D}(\vec{x}, t)$ depends on $\vec{E}(\vec{x}, t')$ "averaged" over t'

in a time window of order $1/\gamma$ $\quad \underset{t - 1/\gamma}{|} \longleftrightarrow \underset{t}{|}$

Because of $\theta(\tau)$, $\vec{D}$ does not depend on $\vec{E}(t' > t)$.

<u>Causality</u>. So

$$\vec{D}(x,t) = \epsilon_0 \left\{ \vec{E}(x,t) + \int_{\underline{\underline{0}}}^{\infty} G(\tau) \, \vec{E}(x, t - \tau) \, d\tau \right\}$$

This is quite general and applies beyond the simple resonance model for G, since it is a basic causality property.

We can rearrange and write

$$\epsilon(\omega)/\epsilon_0 = 1 + \int_0^\infty G(\tau) e^{i\omega\tau} d\tau$$

G is real because $\vec{E}(t)$ and $\vec{D}(t)$ are real. So for complex ω, $\epsilon^*(\omega^*)/\epsilon_0 = \epsilon(-\omega)/\epsilon_0$.

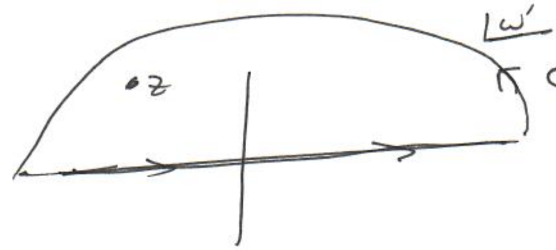
If $G(\tau)$ is finite for all τ, then $\epsilon(\omega)$ is analytic in the UHP. On the real axis there might be a pole at $\omega = 0$ (conductors) but otherwise ϵ is analytic.

$$\int_0^\infty \longleftrightarrow \text{causality} \longleftrightarrow e^{i\omega\tau} \longleftrightarrow \text{analyticity in UHP.}$$

Since $\epsilon(\omega)$ is analytic in the UHP,

$$\epsilon(z)/\epsilon_0 - 1 = \frac{1}{2\pi i} \oint_C \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - z} d\omega'$$

any pt in UHP

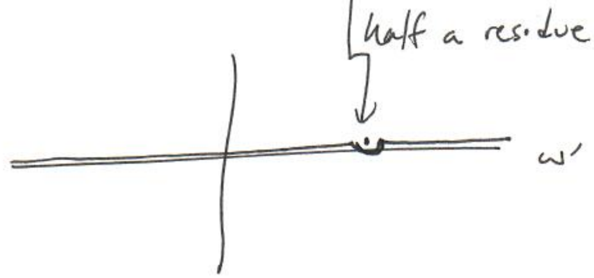


The contour at ∞ does not contribute because $\epsilon(\omega)/\epsilon_0 - 1$ falls sufficiently fast at infinity (see Jackson pg 333 for a proof.)

So
$$\epsilon(z)/\epsilon_0 - 1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - z}$$

Now let $z = \omega + i\delta$ with small δ and use Plemelj:

$$\frac{1}{\omega'-\omega-i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + i\pi\,\delta(\omega'-\omega)$$



We can rearrange this into

$$\epsilon(\omega)/\epsilon_0 = 1 + \frac{1}{i\pi} P\int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - \omega}$$

or

$$\mathrm{Re}\left(\epsilon(\omega)/\epsilon_0\right) = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Im}\left(\epsilon(\omega')/\epsilon_0\right)}{\omega'-\omega}\, d\omega'$$

$$\mathrm{Im}\left(\epsilon(\omega)/\epsilon_0\right) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathrm{Re}\left(\epsilon(\omega')/\epsilon_0\right) - 1}{\omega'-\omega}\, d\omega'$$

Since $\epsilon^*(\omega^*) = \mathrm{Re}\,\epsilon(\omega^*) - i\,\mathrm{Im}\,\epsilon(\omega^*)$
$\qquad\qquad\qquad = \mathrm{Re}\,\epsilon(-\omega) + i\,\mathrm{Im}\,\epsilon(-\omega)$

and ω is real $\Rightarrow$ $\mathrm{Re}\,\epsilon$ is symmetric in ω
and $\mathrm{Im}\,\epsilon$ is antisymmetric.

So we can rewrite the integrals as

$$\mathrm{Re}\left(\epsilon(\omega)/\epsilon_0\right) = 1 + \frac{2}{\pi} P\int_0^{\infty} \frac{\omega'\,\mathrm{Im}\left(\epsilon(\omega')/\epsilon_0\right)}{\omega'^2-\omega^2}\, d\omega'$$

$\left[\dfrac{2\omega'}{\omega'^2-\omega^2} = \dfrac{1}{\omega'+\omega} + \dfrac{1}{\omega'-\omega}\right]$

$$\mathrm{Im}\left(\epsilon(\omega)/\epsilon_0\right) = -\frac{2\omega}{\pi} P\int_0^{\infty} \frac{\mathrm{Re}\left(\epsilon(\omega')/\epsilon_0\right) - 1}{\omega'^2-\omega^2}\, d\omega'$$

"Kramers - Kronig Relations" (small modification required for conductors & pole in ϵ at $\omega = 0$.)

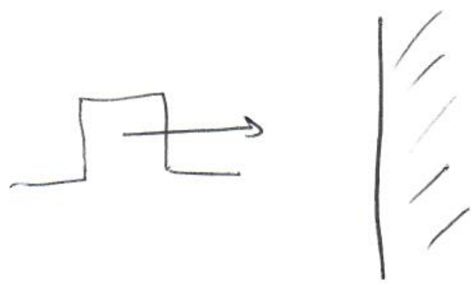
You can measure $\mathrm{Im}\,\epsilon(\omega)$ with absorption studies and then <u>calculate</u> $\mathrm{Re}\,\epsilon(\omega)$ with the 1st eq.

"Dispersion Relations" – very little assumed in their derivation.

More on causality

Suppose a wave pulse is incident on a medium at $x=0$ at $t=0$

$t<0$



$x=0$

$t=0$

$x=0$

We investigate the electric field in the region $x>0$.

Each fourier component behaves as:

$$E_\omega(x,t) = \left(A(\omega)\,\frac{2}{1+n(\omega)}\right) e^{i(k(\omega)x-\omega t)}$$

for $x>0$, where $n(\omega) = c/\bar{c}(\omega)$, $\omega = \bar{c}(\omega)k \Rightarrow k(\omega) = \omega/\bar{c}(\omega)$,

The factor $\frac{2}{1+n}$ is a refraction coefficient, and

$$A(\omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, E(x=0^-,t)\,e^{i\omega t}$$ is the ampl just outside.

- Since $E(x=0^-, t<0)=0$, $A(\omega)$ is analytic in the upper-half complex ω-plane. (The integral $\int_{-\infty}^{\infty} dt\, E(x=0^-, t) e^{i\omega t}$ can be restricted to $\int_0^{\infty} dt\, E(x=0^-, t) e^{i\omega t}$ and assuming E is finite for all t the integral converges in the UHP. Since it is holomorphic in ω it is also analytic.)
- We already saw that $\epsilon(\omega)$ is analytic in the UHP. So $n \sim 1/\sqrt{\epsilon}$ can also be taken analytic in the UHP (it may have branch cuts in the LHP.)
- $n(\omega) \to 1$ as $|\omega| \to \infty$ $\left(\epsilon(\omega)/\epsilon_0 \sim 1 - \omega_p^2/\omega^2 \right.$ so $\left. \epsilon \to \epsilon_0 \right)$

Therefore $i(k(\omega)x - \omega t) \to \dfrac{i\omega(x-ct)}{c}$ as $|\omega| \to \infty$

so $e^{-i(k(\omega)x-\omega t)} \to e^{i\omega(x-ct)/c}$ and

if $x - ct > 0$ we can close the contour in the UHP. Then cauchy tells us that, since the whole integrand is analytic (A, n, $e^{-i(\omega t - kx)}$ all analytic in the UHP), the integral vanishes.

- No signal propagates faster than the speed of light in vacuum, regardless of medium -