

## Lecture 16 – Degenerate Fermi gas I

PREVIOUSLY: Quantum density  $n_Q(T) = \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2}$

$n_Q$  is the particle density at which the wavefunctions begin to overlap, and quantum effects become important. When  $n \approx n_Q$ , multiple occupancy of energy levels (orbitals) is possible and quantum statistics must be taken into account.

Degenerate Fermi gases are gases of fermions under the condition that  $n \sim n_Q$  (Note: here, meaning of “degenerate” is different than usual meaning in QM where states have same energy)

Examples of degenerate Fermi gases:

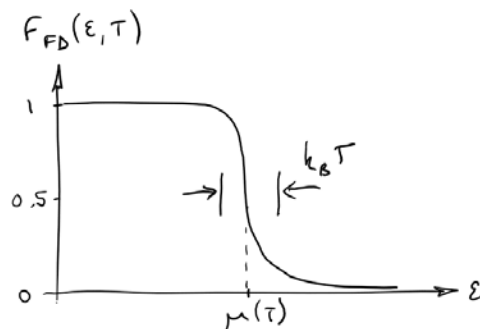
- conduction  $e^-$  in metals – outer  $e^-$  free from ions
- white dwarf – atoms are ionized,  $e^-$  in gas
- neutron star – gas of neutrons
- nuclear matter – can be approximated as gas of protons & neutrons
- liquid  $^3\text{He}$  – fermions too, though ideal gas is not a good approximation

TODAY: conduction  $e^-$  in metals

Define degeneracy temperature  $T_d$  as temperature when  $n = n_Q(T_d)$ :  $T_d = \frac{2\pi\hbar^2}{mk_B} n^{2/3}$

$$\frac{T_d}{T} = \left( \frac{n}{n_Q} \right)^{2/3} \quad \text{when} \quad \begin{cases} T < T_d, & n > n_Q \\ T > T_d, & n < n_Q \end{cases}$$

For typical metals,  $n = 1 - 5 \times 10^{22}$  conduction  $e^-$  per  $\text{cm}^3$  ( $1 e^-$  in a volume of  $(3-4 \text{ \AA})^3$ ) and  $T_d \sim 50,000 - 100,000 \text{ K}$ , so  $n \gg n_Q$  at room temperature,  $T = 300 \text{ K}$ . Metals are strongly degenerate.



This will mean that even at room  $T$ , the behavior of a metal is close to that at  $T = 0$

For example, the occupancy  $f_{FD}(\epsilon, T)$  is very close to a step function at  $T = 0$ . We will make use of this fact to make approximations.

KEY CONCEPT: The Sommerfeld theory of metals

How do we treat conduction  $e^-$  in metals?

- We'll make the approximation that they are a gas of free electrons, subject to no forces.
- In reality,  $e^-$  feel attractive forces from nearby ions, though long-range electrostatic forces can be neglected because metal is electrically neutral.

Recall that the chemical potential  $\mu$  must satisfy the constraint that

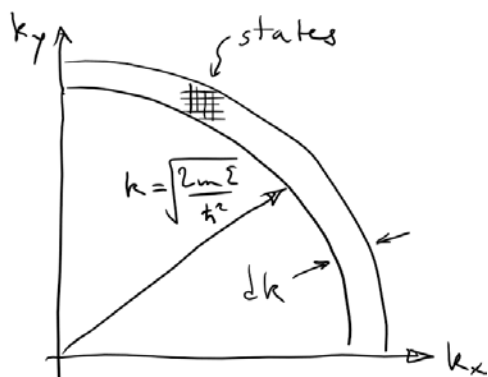
$$N = \langle N \rangle = \sum_n \langle N(\varepsilon_n, T) \rangle = \sum_n f_{FD}(\varepsilon_n, T) = \sum_n \frac{1}{e^{\beta(\varepsilon_n - \mu)} + 1}$$

Also, the average energy is given by

$$U(T) = \sum_n \langle N(\varepsilon_n, T) \rangle \varepsilon_n = \sum_n f_{FD}(\varepsilon_n, T) \varepsilon_n = \sum_n \frac{\varepsilon_n}{e^{\beta(\varepsilon_n - \mu)} + 1}$$

So, we can determine  $\mu(T)$  from the first equation, then use it to evaluate  $U(T)$  in the second

The energy levels  $\varepsilon_n$  are determined using the same "particle in a box" approach as before:



$$\varepsilon_n = \frac{\hbar^2 k_n^2}{2m} \text{ with } k_n^2 = k_x^2 + k_y^2 + k_z^2 \text{ and}$$

$$k_x = \frac{n_x \pi}{L_x} \text{ with } n_x = 1, 2, 3, \dots$$

same for  $y, z$

Putting in some numbers: the spacing between energy levels for a  $1 \text{ cm}^3$  sample of metal is

$$\Delta \varepsilon \sim \frac{\hbar^2 \pi^2}{2mL^2} = \frac{\hbar^2 \pi^2}{2mV^{2/3}} \sim 10^{-15} \text{ eV, i.e. } \Delta \varepsilon / k_B \sim 10^{-11} \text{ K}$$

So, for all reasonable temperatures we can consider the spectrum of energy levels (or orbitals) to be continuous, and we are justified in replacing the sums with integrals:

$$\sum_n \rightarrow \int_0^\infty d\varepsilon D(\varepsilon)$$

**Question 1: Write down an expression for  $D(\varepsilon)$  for the Fermi gas (in 3D)**

$$\begin{aligned} \text{\# of states with energy between } \varepsilon \text{ and } \varepsilon + d\varepsilon &\approx \text{\# spins} \times \frac{\text{vol. of shell in } k\text{-space with radius } k = \sqrt{2m\varepsilon} / \hbar}{\text{vol. in } k\text{-space per state}} \end{aligned}$$

Accounts for 2 spin states of each  $e^-$

↓

$$D(\varepsilon)d\varepsilon \approx 2 \frac{4\pi k^2 dk / 8}{\pi^3 / V} = \frac{V}{\pi^2} k^2 dk = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \varepsilon^{1/2} d\varepsilon$$

In the last step we used  $k = \sqrt{\frac{2m}{\hbar^2}} \varepsilon^{1/2}$ ,  $dk = \sqrt{\frac{2m}{\hbar^2}} \frac{d\varepsilon}{2\varepsilon^{1/2}}$  to express  $D$  in terms of  $\varepsilon$

(Note the difference in  $D(\varepsilon)$  for the photon gas, where  $\varepsilon = \hbar\omega = \hbar ck$ )

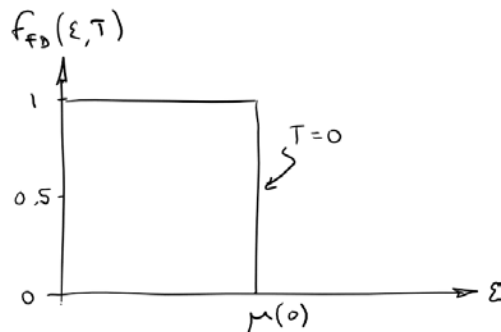
So, we have

$$N = \langle N \rangle = \int_0^\infty d\varepsilon D(\varepsilon) f_{FD}(\varepsilon, T) = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty d\varepsilon \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon - \mu)} + 1}$$

$$U = \int_0^\infty d\varepsilon \varepsilon D(\varepsilon) f_{FD}(\varepsilon, T) = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty d\varepsilon \frac{\varepsilon^{3/2}}{e^{\beta(\varepsilon - \mu)} + 1}$$

In general, we cannot solve these integrals analytically.

However, at  $T = 0$ ,  $f_{FD}(\varepsilon, T = 0)$  is a step function, and integrals are trivial.



KEY CONCEPTS: Fermi energy and temperature

First look at  $T = 0$

$$f_{FD}(\varepsilon, 0) = \begin{cases} 0 & \text{for } \varepsilon > \mu(0) \\ 1 & \text{for } \varepsilon < \mu(0) \end{cases}$$

It follows that

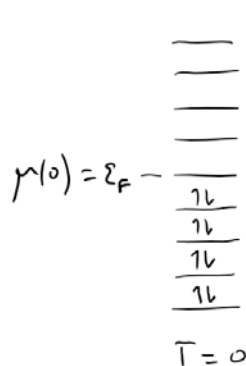
$$N = \int_0^\infty d\varepsilon D(\varepsilon) f_{FD}(\varepsilon, 0) = \int_0^{\mu(0)} d\varepsilon D(\varepsilon)$$

$$= \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\mu(0)} d\varepsilon \varepsilon^{1/2} = \frac{V}{3\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \mu(0)^{3/2}$$

Solving for the chemical potential at  $T = 0$ :

$$\varepsilon_f \equiv \mu(0) = \frac{\hbar^2}{2m} \left( 3\pi^2 \frac{N}{V} \right)^{2/3}$$

which we define as the Fermi energy



$\varepsilon_F$  is the highest energy level occupied with  $e^-$  at  $T = 0$

At  $T = 0$ ,  $e^-$  fill up energy levels from ground state up (like orbitals in an atom). Occupancy goes to 0 as soon as the  $N$   $e^-$  have been used up

Notice that  $\mu(0) = \varepsilon_F$  depends on  $N$ , but also on  $V$ , because the size of the box affects the spacing between energy levels  $\Delta\varepsilon \sim \hbar^2 \pi^2 / 2mV^{2/3}$ .

$\varepsilon_F \sim n^{2/3}$  is an intensive property of the metal

Next we calculate the energy at  $T = 0$

### Question 2: Write down an expression for $U(T = 0)$ for the Fermi gas

Now for the average energy at  $T = 0$ :

$$\begin{aligned}
 U(0) &= \int_0^\infty d\varepsilon \varepsilon D(\varepsilon) \langle n(\varepsilon, 0) \rangle_{FD} = \int_0^{\mu(0)} d\varepsilon \varepsilon D(\varepsilon) \\
 &= \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\mu(0)} d\varepsilon \varepsilon^{3/2} = \frac{2}{5} \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \varepsilon_F^{5/2} = \frac{3}{5} N \varepsilon_F
 \end{aligned}$$

Makes sense. The highest possible energy for an  $e^-$  at  $T = 0$  is  $\varepsilon_F$ . On average, an electron will have  $\sim 1/2$  that energy (3/5 to be precise, because the density of states is not uniform).

Compare to classical ideal gas with MB statistics:

$$U_{MB} = \frac{3}{2} N k_B T \quad \text{vs.} \quad U_{FD} = \frac{3}{5} N \varepsilon_F$$

Even at  $T = 0$ ,  $U_{FD} \gg 0$  because the Pauli exclusion principle requires that  $e^-$  keep filling up energy levels

Define Fermi temperature  $T_F = \varepsilon_F / k_B$

$T_F$  is almost the same as the degeneracy temperature  $T_d$ :

$$k_B T_d = \frac{2\pi\hbar^2}{m} n^{2/3} \approx (3\pi^2)^{2/3} \frac{\hbar^2}{2m} n^{2/3} = k_B T_F$$

$T_F \sim 50,000 - 100,000$  K in metals. This is just a reference point for temperatures. Metals liquefy long before this temperature can be reached experimentally. Practically  $k_B T \ll \varepsilon_F$ .

KEY CONCEPT: Degeneracy pressure

**Question 3: Write down an expression for the pressure  $p$  generated by the Fermi gas at  $T = 0$**

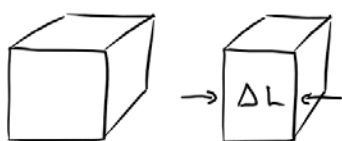
Pressure is defined as  $p(T) = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$ . At  $T = 0$ ,  $F = U$  so  $p(0) = -\left(\frac{\partial U}{\partial V}\right)_{T,N}$

$U(0) = \frac{3}{5} N \varepsilon_F$  and  $\varepsilon_F = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V}\right)^{2/3}$ , so

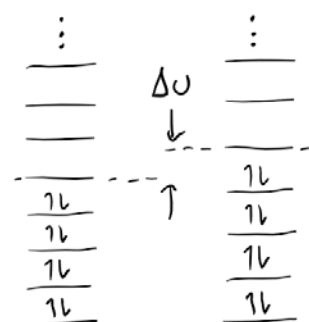
$$U(0) = \frac{3}{5} N \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V}\right)^{2/3} = \underbrace{\frac{3}{5} (3\pi^2)^{2/3} \frac{\hbar^2}{2m}}_{\alpha} \frac{N^{5/3}}{V^{2/3}}$$

It follows that

$$p(0) = -\frac{\partial}{\partial V} \frac{\alpha N^{5/3}}{V^{2/3}} = \frac{2}{3} \frac{\alpha N^{5/3}}{V^{5/3}} = \frac{2}{3} \frac{U(0)}{V}$$



Compare to a classical ideal gas – a Fermi gas exerts a pressure even at  $T = 0$ ! Why?



$e^-$  fill up energy levels due to the Pauli exclusion principle. The spacing between energy levels depends on  $V^{-2/3}$ , so if the metal is compressed, the energy level spacing increases, increasing the total energy.

This is called the degeneracy pressure. It's the reason why matter does not collapse under electrostatic forces pulling protons and electrons together.

This pressure is very large:  $p \sim 10^9 \text{ N/m}^2$