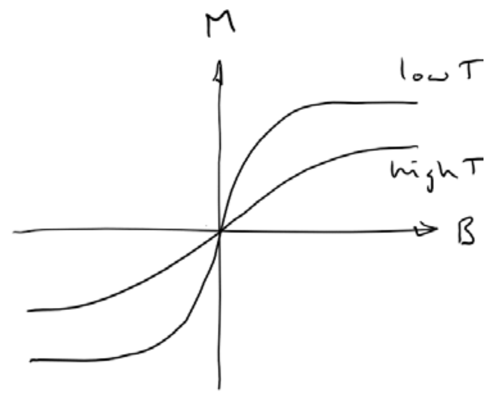
Lecture 21 – Weiss model of ferromagnetism

PREVIOUSLY: We investigated a paramagnet with independent, non-interacting spins.
Recall the simple 2-state paramagnet (e.g. spin ½ e^-): in a magnetic field B pointing up, each spin can either be aligned or anti-aligned, with energy $\varepsilon = -\mu_B B s$ and $s = \pm 1$ for spin up or spin down, respectively ($\mu_B \equiv e\hbar / 2m_e$ is the "Bohr magneton").

For a single spin:

$$Z_1 = e^{\mu_B B/k_B T} + e^{-\mu_B B/k_B T} = 2\cosh\frac{\mu B}{k_B T}$$

$$\langle s \rangle = \frac{1\cdot e^{\mu_B B/k_B T} - 1\cdot e^{-\mu_B B/k_B T}}{Z_1} = \tanh\frac{\mu_B B}{k_B T}$$



For a material with a density n of spins, the magnetization density is:

$$M(B,T) = n\mu_B \langle s \rangle = n\mu_B \tanh\frac{\mu_B B}{k_B T}$$

For small B, $\tanh x \approx x$ and $M(B,T) = n\mu_B^2 \dfrac{B}{k_B T}$

The <u>magnetic susceptibility</u> is defined as: $\chi(B,T) \equiv \left.\dfrac{\partial M}{\partial B}\right|_{B=0}$

$$\chi(B,T) = \frac{n\mu_B^2}{k_B T} \propto \frac{1}{T} \qquad \underline{\text{Curie law}}$$

TODAY: Ferromagnetism
Notice that for paramagnets, $M \to 0$ as $B \to 0$. Ferromagnets (e.g. iron) have a spontaneous magnetization $M \neq 0$, even when $B = 0$

The simple theory we used previously cannot explain this phenomenon. We need interactions between spins to get ferromagnetism. The "exchange" interaction leads to tendency for neighboring spins to align.

(Note: this interaction is purely quantum mechanical and comes from the symmetry properties of multi-particle wavefunctions under particle exchange).

For a pair of spins i and j, $\varepsilon_{exch} = -2J_{ij}s_i s_j$, so

$$\varepsilon_{tot} = -\sum_{i=1}^{N} \mu_B B s_i - \frac{1}{2}\sum_{\substack{i=1 \\ i\neq j}}^{N}\sum_{j=1}^{N} 2J_{ij}s_i s_j$$

(Again, the factor of ½ is there to avoid double counting, and cancels the factor of 2 in the exchange term).

The exchange interaction is important only when wavefunctions of two particles overlap. It is therefore short-range, and we usually assume only nearest-neighbor spins interact, with a constant $J_{ij} = J$:

$$\varepsilon_{tot} = -\mu_B B \sum_i s_i - J \sum_{i,j}{}' s_i s_j$$

where $\sum_{i,j}{}'$ signifies a sum over nearest neighbors only.

Note: this is called the Ising model, exactly solvable only under certain circumstances (1-D, 2-D with B = 0)

KEY CONCEPT: Weiss molecular field
As for the van der Waals model, we will treat this problem approximately using a <u>mean field model</u>, replacing the interaction term with an <u>average over all spins</u> $j \neq i$.

For the i-th spin:

$$\varepsilon_i = -\mu_B B s_i - J s_i \sum_{j \neq i}{}' s_j \approx -\mu_B B s_i - \underbrace{J s_i \sum_{j \neq i}{}' \langle s_j \rangle}_{J s_i \langle s \rangle N_{n.n.}}$$

$N_{n.n.}$ = number of nearest neighbors

Since $M = n\mu_B \langle s \rangle$, the averaged exchange term is $\propto M$ and

$$\varepsilon_i \approx -\mu_B B s_i - \mu_B \lambda M s_i = -\mu_B (B + B_{eff}) s_i$$

with $\lambda = J \;\vee\; / \mu_B^2$.

The average interaction term acts like an effective B-field $B_{eff} = \lambda M$ generated by neighboring spins, which tend to align i-th spin. This effective B-field is called the <u>Weiss molecular field</u>.

The rest is easy: take paramagnet solution and add B_{eff} to the external B field!

$$M(B,T) = n\mu_B \tanh \frac{\mu_B (B + B_{eff})}{k_B T}$$

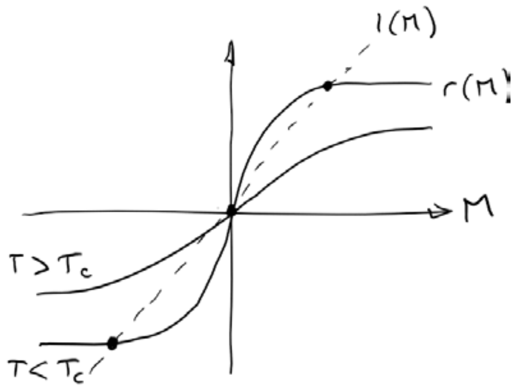
First consider case where external B field = 0:

$$M = n\mu_B \tanh \frac{\mu_B \lambda M}{k_B T}$$

a transcendental equation, since M is on both sides. Solve graphically

Define functions *r* and *l* for the right- and left-hand sides of the equation:

$$r(M) \equiv n\mu_B \tanh\frac{\mu_B \lambda M}{k_B T}, \qquad \text{and} \quad l(M) = M$$



Intersection(s) are the solutions

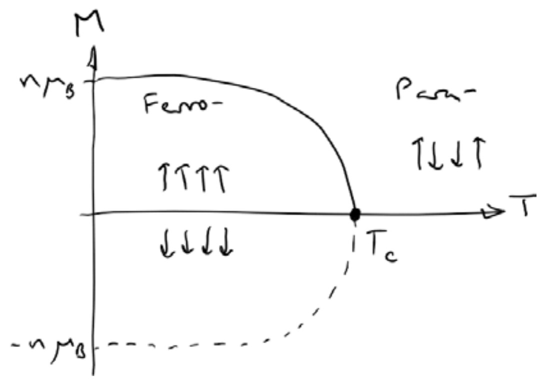
At high *T*, only one solution: *M* = 0

At low *T*, three solutions: $\underbrace{M > 0,\quad M < 0}_{\text{stable}},\quad \underbrace{M = 0}_{\text{unstable}}$

Below some temperature T_C, system has $M \neq 0$ despite *B* = 0: spontaneous magnetization

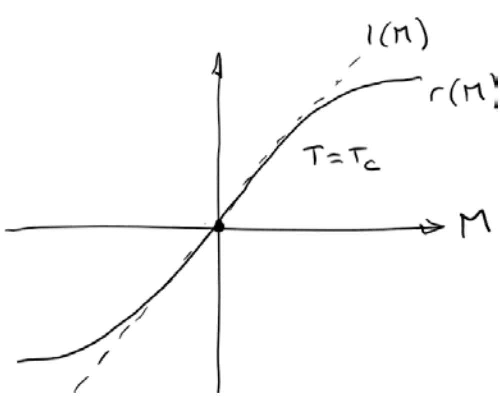
M > 0 = spins up, *M* < 0 = spins down

In the absence of an external B field, either state (↑↑↑ or ↓↓↓) is equally likely. We say symmetry is <u>spontaneously broken</u>.



Question 1: Derive an expression for T_C in terms of parameters of the system

This will occur when slopes of *r*(*M*) and *l*(*M*) at *M* = 0 are equal:



$$l'(M=0) = 1 = n\mu_B \frac{\mu_B \lambda}{k_B T_C} = r'(M=0)$$

(Since $\tanh x \approx x$ for small *x*)

Therefore, $T_C = \dfrac{n\mu_B^2 \lambda}{k_B}$

This is the <u>Curie temperature</u> – *T* at which system transitions from paramagnet to ferromagnet.

Some typical numbers: Ni has T_C = 631 K which gives λ ~ 2000

$$M(T=0) \approx 0.17\ \text{T}, \quad \text{so} \quad B_{eff} = \lambda M \sim 10^{2-3}\ \text{T}$$

Note: this is a very large field, much larger than B field acting on 1 spin due to neighboring spins:

$$B_{actual} \sim \frac{\mu_B}{a^3} \sim 0.1\ \text{T} \ll B_{eff} \quad (a = \text{lattice spacing})$$

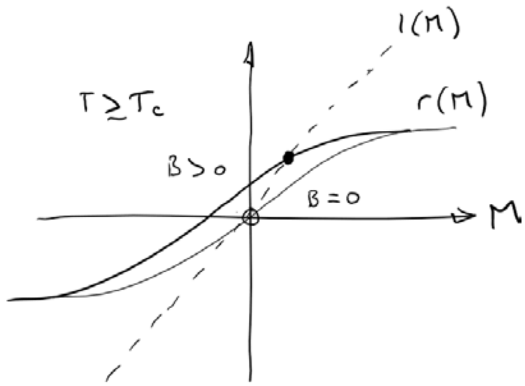
B_{eff} is not a classical dipole-dipole coupling interaction. Exchange interaction is purely quantum mechanical effect, much stronger than that expected classically.

Now consider case with external field B:

$$M(B,T) = n\mu_B \tanh\frac{\mu_B(B+\lambda M)}{k_B T}$$

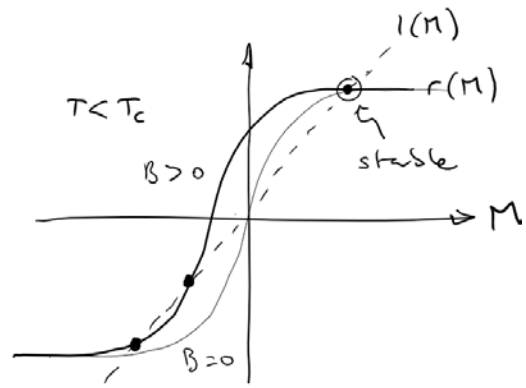
Question 2: Plot M vs. B for $T > T_C$ and $T < T_C$ by solving the above equation graphically

For $T \geq T_C$

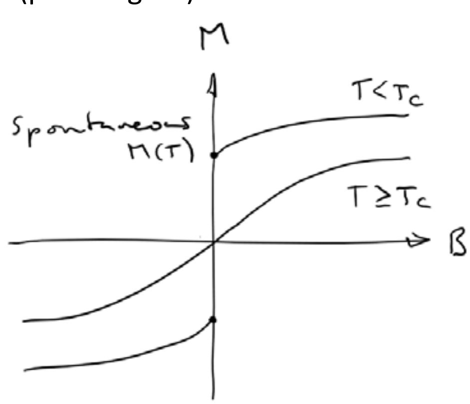


Magnetization dependent on B (paramagnet)

For $T < T_C$



Magnetization spontaneous, weakly dependent on B (ferromagnet)



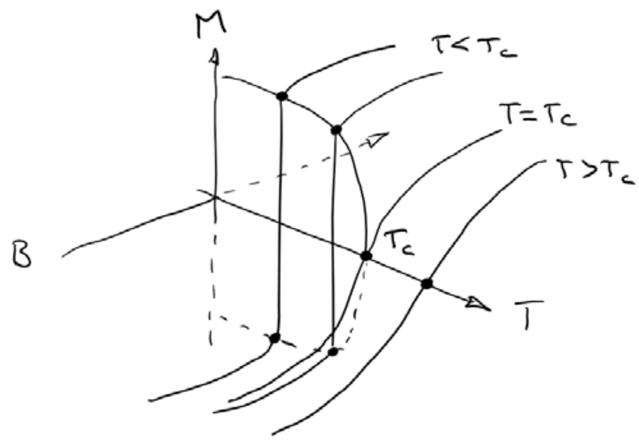
Plot M vs. B for different T

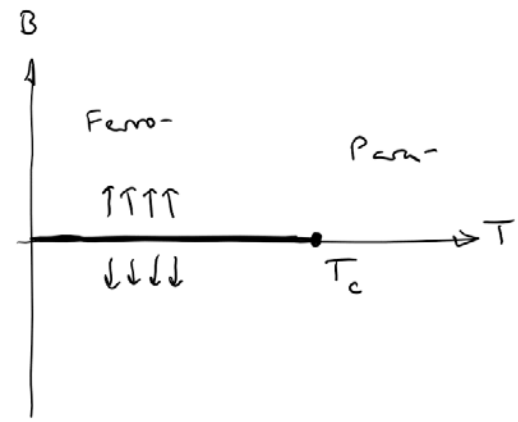
Note that the susceptibility

$$\chi(B,T) \equiv \left.\frac{\partial M}{\partial B}\right|_{B=0} = \text{slope of } M \text{ vs. } B \text{ at origin}$$

For $T \geq T_C$, $\chi > 0$; for $T < T_C$, $\chi \to \infty$

In 3-D, $M(T,B)$ vs. T and B:





Looking down along M axis, we get phase diagram
T_C is a critical point

This transition is second-order – no latent heat. Entropy changes continuously across transition, unlike gas-liquid transition (except at critical point).

KEY CONCEPT: Scaling laws near the critical point ($T \approx T_C$, $B \approx 0$)
Systems near a critical point exhibit <u>universal behavior</u>

Look at $M(T)$ and $\chi(T)$ near T_C:

$$M = n\mu_B \tanh\frac{\mu_B \lambda M}{k_B T}\,,\; T_C = \frac{n\mu_B^2\lambda}{k_B}$$

Let's define a normalized magnetization $m \equiv \frac{M}{n\mu_B}$, so we can write $m = \tanh\frac{mT_C}{T}$

Question 3: Show that near T_C, when m is small, the magnetization scales as $(T_C - T)^\beta$

When m is small, we can expand tanh to lowest order: $\tanh x \approx x - \frac{1}{3}x^3 + \cdots$ for $x << 1$

$$\cancel{m} \approx \cancel{m}\frac{T_C}{T} - \frac{1}{3}m^{\overset{2}{\cancel{3}}}\left(\frac{T_C}{T}\right)^3 + \cdots \quad \text{or} \quad 1 \approx \frac{T_C}{T}\left(1 - \frac{1}{3}\left(m\frac{T_C}{T}\right)^2\right)$$

Solving for m, we get: $m \approx \sqrt{3}\frac{T}{T_C}\left(1 - \frac{T}{T_C}\right)^{1/2}$

$M \sim (T_C - T)^\beta$ $\quad \beta$ = ½ is called a <u>critical exponent</u>

Now look at susceptibility χ for a small eternal field B

$$M = n\mu_B \tanh\left(\frac{\mu_B B}{k_B T} + \frac{\mu_B \lambda M}{k_B T}\right)$$

Define $b \equiv \frac{\mu_B B}{k_B}$, so that we can write $m = \tanh\left(\frac{b}{T} + m\frac{T_C}{T}\right)$

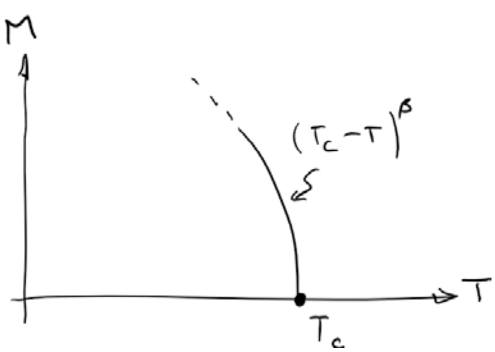
Here we only need to expand the tanh to linear order in m:

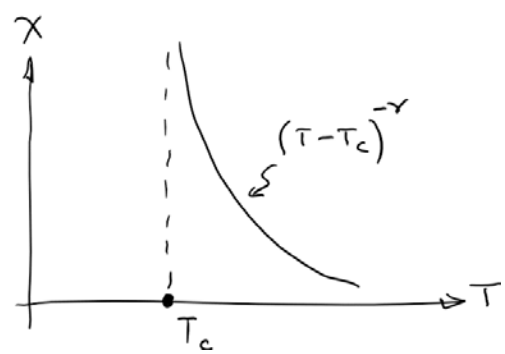
$$m = \frac{b}{T} + m\frac{T_C}{T} + \cdots$$

$$m\left(1 - \frac{T_C}{T}\right) = \frac{b}{T} \quad \text{and} \quad m = \frac{b}{T - T_C}$$

$$M \sim \frac{B}{T - T_C} \text{ and } \chi \sim \left(T - T_C\right)^{-\gamma} \ \gamma = 1$$

So χ diverges as $T \to T_C$. This is called the Curie-Weiss law





Amazingly, phase transitions for systems that appear unrelated (e.g. gas-liquid, ferromagnet-paramagnet) exhibit the same scaling laws near the critical point!

All mean field models can be shown to predict $\beta = \frac{1}{2}$ and $\gamma = 1$ (there are other critical exponents, named $\alpha, \delta, \eta, \nu$, related to other physical quantities). Actual measurements give $\beta \approx 0.33$ and $\gamma \approx 1.2$. More sophisticated field theoretical techniques are required to get better agreement with experiments.

Critical exponent	Uniaxial magnet (exp)	Liquid-vapor (exp)	Fluid mixture (exp)	Mean field theory	Perturbative field theory
β	0.325	0.324	0.327	½	0.326
γ	1.240	1.241	1.235	1	1.239