# Chapter 10, lecture 3

The retarded potentials are solutions of the inhomogeneous wave equations. Knowing the retarded potentials, the fields may be calculated.

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

There are a bunch of derivatives to handle, but they are similar to those we have already done. Then;

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int \left[ \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} - \frac{\dot{\rho}(\vec{r}\,',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c^2\mathscr{r}} \right] d\tau'$$

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \left[ \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c\mathscr{r}} \right] \times \hat{\mathscr{r}}\, d\tau'$$

Now it is time to be more specific, by calculating the potentials and fields due to a moving point charge.

The retarded potentials for a moving point charge are called the Liénard-Wiechert potentials:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\imath c - \vec{\imath}\cdot\vec{v})}$$

$$A(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{(\imath c - \vec{\imath}\cdot\vec{v})} = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

Check out the scalar potential:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\imath c - \vec{\imath}\cdot\vec{v})}$$

This piece makes it different than the static potential

Once we understand the nature of the difference between the static potential of a point charge and the Liénard-Wiechert potential, we'll look at the special case of a point charge moving at constant velocity.

Examining this case, we see a form which later will be recognizable in terms of a Lorentz transformation. The Maxwell equations lead naturally to the Lorentz transformation, which in fact Lorentz himself discovered while studying these equations of electricity and magnetism. The theory of electromagnetism is already relativistically correct.

The first issue is where does the extra $\vec{\mathscr{r}}\cdot\vec{v}$ in the potential for a point charge come from? Here lets follow Griffiths exactly.

$$V_{\substack{pt\\ charge}}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$$

A single point charge has a definite trajectory, its position is a function of time, $\vec{w}(t) \equiv$ position of $q$ at time $t$.

$\vec{r}'$ at retarded time $\equiv \vec{w}(t_r)$

$$\vec{\mathscr{r}}_{\substack{ret\\ time}} = \vec{r} - \vec{r}'_{\substack{ret\\ time}} = \vec{r} - \vec{w}(t_r)$$

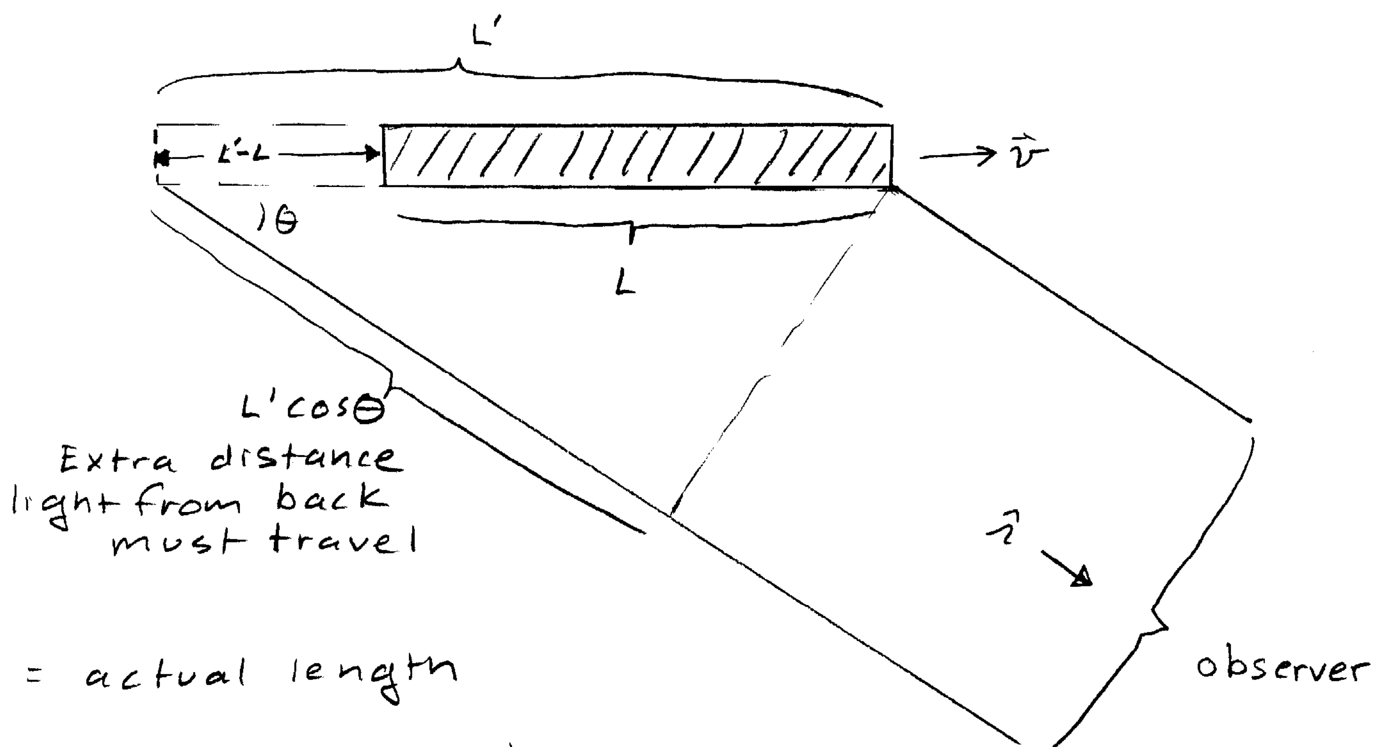
The integral $V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}} d\tau'$

is evaluated at a specific retarded time. Then, for a point charge, $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$ may be taken out of the integral.

$$V_{\text{pt charge}}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{1}{\mathscr{r}} \int \rho(\vec{r}\,', t_r) d\tau'$$

For a moving charge, $\int \rho(\vec{r}\,', t_r) d\tau'$ is not equal to $q$! A point charge is the limit of an extended charge when the size goes to zero. For a charge distribution the motion of the charge affects the time it takes for a signal to propagate from the "back" of the distribution as compared to the time it takes to propagate from the "front" of the distribution.

Lets check out Griffith's moving train analogy. What we want to know is how long the train looks to an observer far away. This is not the actual length of the train since there it takes longer for light from the back of the train to reach the observer than it will take light from the front of the train to reach the observer. This wouldn't matter if the train were not moving.




L = actual length

L' = apparent length

The apparent length of the train may be found by realizing that the length the train <u>seems</u> to be is governed by

| | | |
|---|---|---|
| Time train takes to move to new position | = | Time difference for signal to travel from front of train compared to back of train |

which is

$$\frac{\text{Distance train has moved}}{\text{velocity of train}} = \frac{\text{Extra distance light travels}}{c}$$

which is

$$\frac{L' - L}{v} = \frac{L' \cos\theta}{c}$$

solving for $L'$

$$L' = \frac{c}{c - v\cos\theta} L$$

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may also be written:

$$L' = \frac{c}{c - \hat{\imath}\cdot\vec{v}} L$$

The apparent volume of the train is

$$\tau' = \frac{\tau}{1 - \frac{\hat{\imath}\cdot\vec{v}}{c}}$$

Since the dimensions of the train perpendicular to the motion are not distorted. Thus, a volume integral taken at the retarded time is equal to the volume integral at $t$ (un-retarded time) adjusted by the factor $\dfrac{1}{1 - \frac{\hat{\imath}\cdot\vec{v}}{c}}$

$$\Rightarrow \int_{\substack{p^+ \\ \text{charge}}} \rho(\vec{r}', t_r) = \frac{q}{1 - \frac{\hat{\imath}\cdot\vec{v}}{c}}$$

So, we have

$$V = \frac{1}{4\pi\epsilon_0} \frac{1}{\mathscr{r}} \int \rho(\vec{r}', t_r)\, d\tau' = \frac{1}{4\pi\epsilon_0} \frac{qc}{\mathscr{r}c - \mathscr{r}\hat{\mathscr{r}} \cdot \vec{v}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{qc}{\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v}}$$

For a point charge $\vec{J} = \rho \vec{v}$ =>

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{\mathscr{r}c - \vec{\mathscr{r}} \cdot \vec{v}} = \frac{\vec{v}}{c^2} V(\vec{r}, t)$$

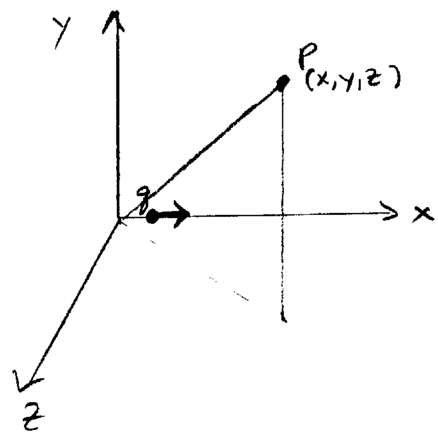
Now, lets repeat Griffith's Example 10.3, but take the special case of a point charge moving with constant velocity in the $\hat{x}$ direction. (From Feynman) We will:

1. Write expressions for $\mathscr{r}$ and $t_r$
2. Combine to solve for $\mathscr{r}$ and $t_r$
3. Substitute into the expression for the potential

$$t_r = t - \mathscr{r}/c \longrightarrow \mathscr{r} = c(t - t_r)$$

$\mathscr{r}$ is the distance between the source and the observer at the retarded time,

$$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r) \Rightarrow \quad \mathscr{r} = \left[(x - vt_r)^2 + y^2 + z^2\right]^{1/2}$$




coordinates of P:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

coordinates of $q$ :

$$\vec{r}\,' = vt\,\hat{x}$$

Combine and solve for $t_r$ :

$$c^2(t^2 + t_r^2 - 2tt_r) = (x^2 + v^2 t_r^2 - 2xvt_r) + y^2 + z^2$$

$$(c^2 - v^2)t_r^2 + (-2tc^2 + 2xv)t_r + c^2t^2 - x^2 - y^2 - z^2 = 0$$

$$(1 - v^2/c^2)t_r^2 + 2\left(-t + \frac{xv}{c^2}\right)t_r + t^2 - \frac{1}{c^2}(x^2 + y^2 + z^2) = 0$$

Now apply the quadratic formula to get $t_r$

$$a t_r = -\tfrac{1}{2} b \pm \sqrt{(b/2)^2 - ac}$$

$$(1 - v^2/c^2)\, t_r = t - \frac{vx}{c^2} - \frac{1}{c}\sqrt{(x - vt)^2 + (1 - v^2/c^2)(y^2 + z^2)}$$

The negative root was selected to insure $t > t_r$ for all $t$.

OK - we are ready!

$$V(r,t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{ᴧc - \vec{ᴧ}\cdot\vec{v}} = \frac{1}{4\pi\epsilon_0} \frac{q}{ᴧ - \frac{\vec{ᴧ}\cdot\vec{v}}{c}}$$

$$ᴧ - \frac{\vec{ᴧ}\cdot\vec{v}}{c} = c(t - t_r) - [(x - vt_r)\hat{x} + y\hat{y} + z\hat{z}] \cdot \frac{v}{c}\hat{x}$$

$$= ct - ct_r - (x - vt_r)\frac{v}{c} = \frac{v^2}{c} t_r - c\, t_r + c\, t - \frac{xv}{c}$$

$$= -c\,(1 - v^2/c^2)\, t_r + ct - \frac{xv}{c}$$

*Now substitute the result at the top of the page:*

$$-c\left(\cancel{t - \frac{vx}{c^2}}\right) + \sqrt{(x - vt)^2 + (1 - v^2/c^2)(y^2 + z^2)} + \cancel{ct - \frac{xv}{c}}$$

$$= \sqrt{1 - v^2/c^2}\,\sqrt{\frac{(x - vt)^2}{1 - v^2/c^2} + (y^2 + z^2)}$$

$$V(r,t) = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{1-v^2/c^2}} \frac{1}{\sqrt{\frac{(x-vt)^2}{\sqrt{1-v^2/c^2}} + y^2 + z^2}}$$

$$\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}$$

$$V(r,t) = \gamma \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}}$$