Today's outline - January 31, 2023

- Dense Coding
- Quantum teleportation
- Phase shift and rotation operators
- Operator decomposition

Reading Assignment: Reiffel: 5.5-5.6 Wong: 4.6.1-4.6.4

Homework Assignment #03:
due Thursday, February 02, 2023

Homework Assignment #04:
due Tuesday, February 09, 2023

Dense coding

One application of simple gates is dense coding, where a single qubit and a shared EPR pair is used to transmit two classical bits

Make an entangled pair of qubits $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ then send the first to Alice and the second to Bob

Alice wishes to transmit the state of two classical bits (values 0, 1, 2, 3) and applies one of four Pauli transformations to her qubit according to the table

$$
\begin{aligned}
0 &\longrightarrow |\psi_0\rangle = (I \otimes I)|\psi_0\rangle \\
1 &\longrightarrow |\psi_1\rangle = (X \otimes I)|\psi_0\rangle \\
2 &\longrightarrow |\psi_2\rangle = (Z \otimes I)|\psi_0\rangle \\
3 &\longrightarrow |\psi_3\rangle = (Y \otimes I)|\psi_0\rangle
\end{aligned}
$$

$$
\begin{aligned}
I \otimes I|\psi_0\rangle &= (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes I \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
X \otimes I|\psi_1\rangle &= (|1\rangle\langle 0| + |0\rangle\langle 1|) \otimes I \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\
Z \otimes I|\psi_2\rangle &= (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes I \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
Y \otimes I|\psi_3\rangle &= (|0\rangle\langle 1| - |1\rangle\langle 0|) \otimes I \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle)
\end{aligned}
$$

Alice then sends the transformed qubit to Bob who now has both qubits together

Dense coding (cont.)

Bob decodes the information by applying a controlled-NOT to the two qubits of the entangled pair to separate them followed by a Hadamard transformation to Alice's qubit

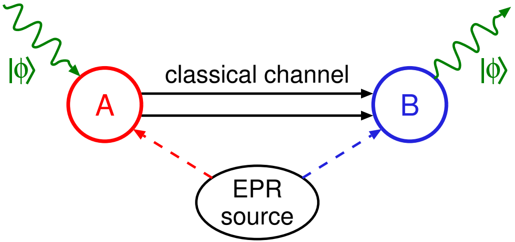
$$\left.\begin{aligned} |\psi_0\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\psi_1\rangle &= \tfrac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\ |\psi_2\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\psi_3\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}\right\} \xrightarrow{C_{not}} \left\{\begin{aligned} &\tfrac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\ &\tfrac{1}{\sqrt{2}}(|11\rangle + |01\rangle) \\ &\tfrac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \\ &\tfrac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \end{aligned}\right\} = \left\{\begin{aligned} &\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle \\ &\tfrac{1}{\sqrt{2}}(|1\rangle + |0\rangle)|1\rangle \\ &\tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle \\ &\tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle \end{aligned}\right.$$

$$= \left.\begin{aligned} &\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle \\ &\tfrac{1}{\sqrt{2}}(|1\rangle + |0\rangle)|1\rangle \\ &\tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle \\ &\tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle \end{aligned}\right\} \xrightarrow{H\otimes I} \left\{\begin{aligned} &|0\rangle|0\rangle \\ &|0\rangle|1\rangle \\ &|1\rangle|0\rangle \\ &|1\rangle|1\rangle \end{aligned}\right\} = \left\{\begin{aligned} &|\psi_0\rangle \\ &|\psi_1\rangle \\ &|\psi_2\rangle \\ &|\psi_3\rangle \end{aligned}\right.$$

and Bob recovers the two qubits that Alice started with

Quantum teleportation

Another common application is quantum teleportation, where Alice wants to transmit an unknown qubit, $|\phi\rangle = a|0\rangle + b|1\rangle$, to Bob by means of two classical bits



Start with an EPR pair of qubits and send one to Alice and the other to Bob

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice also has $|\phi\rangle$, making a three qubit system with Bob controlling the last one and Alice controlling the first two: $|\phi\rangle|\psi_0\rangle = \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)$

Alice applies C_{not} and then $H \otimes I$ to the two bits she controls

$$\begin{aligned}
&(H \otimes I \otimes I)(C_{not} \otimes I)\frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle) \\
&= (H \otimes I \otimes I)\frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle) \\
&= \frac{1}{2}\left[a(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + b(|010\rangle + |001\rangle - |110\rangle - |101\rangle\right]
\end{aligned}$$

Quantum teleportation

$$
\begin{aligned}
|\phi\rangle|\psi_0\rangle &= \tfrac{1}{2}\left[a(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + b(|010\rangle + |001\rangle - |110\rangle - |101\rangle\right] \\
&= \tfrac{1}{2}\left[|00\rangle(a|0\rangle + b|1\rangle) + |01\rangle(a|1\rangle + b|0\rangle) + |10\rangle(a|0\rangle - b|1\rangle) + |11\rangle(a|1\rangle - b|0\rangle)\right]
\end{aligned}
$$

Alice now measures her two qubits and gets one of four states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ with equal probability and sends the 2 classical bit result to Bob

depending on Alice's result, Bob's qubit is projected into one of four states

$$
a|0\rangle + b|1\rangle, \quad a|1\rangle + b|0\rangle, \quad a|0\rangle - b|1\rangle, \quad a|1\rangle - b|0\rangle
$$

Bob can now reconstruct the original state of the unknown $|\phi\rangle$ by applying the Pauli gate corresponding to the classical bits he receives from Alice

$$
\begin{aligned}
00 &\longrightarrow \ I(a|0\rangle + b|1\rangle) = a|0\rangle + b|1\rangle = |\phi\rangle \\
01 &\longrightarrow X(a|1\rangle + b|0\rangle) = a|0\rangle + b|1\rangle = |\phi\rangle \\
10 &\longrightarrow Z(a|0\rangle - b|1\rangle) = a|0\rangle + b|1\rangle = |\phi\rangle \\
11 &\longrightarrow Y(a|1\rangle - b|0\rangle) = a|0\rangle + b|1\rangle = |\phi\rangle
\end{aligned}
$$

Quantum teleportation experiment

Experimental quantum teleportation

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Quantum teleportation—the transmission and reconstruction over arbitrary distances of the state of a quantum system—is demonstrated experimentally. During teleportation, an initial photon which carries the polarization that is to be transferred and one of a pair of entangled photons are subjected to a measurement such that the second photon of the entangled pair acquires the polarization of the initial photon. This latter photon can be arbitrarily far away from the initial one. Quantum teleportation will be a critical ingredient for quantum computation networks.

The dream of teleportation is to be able to travel by simply reappearing at some distant location. An object to be teleported can be fully characterized by its properties, which in classical physics can be determined by measurement. To make a copy of that object at a distant location one does not need the original parts and pieces—all that is needed is to send the scanned information so that it can be used for reconstructing the object. But how precisely can this be a true copy of the original? What if these parts and pieces are electrons, atoms and molecules? What happens to their individual quantum properties, which according to the Heisenberg's uncertainty principle cannot be measured with arbitrary precision?

Bennett *et al.*[1] have suggested that it is possible to transfer the quantum state of a particle onto another particle—the process of quantum teleportation—provided one does not get any information about the state in the course of this transformation. This requirement can be fulfilled by using entanglement, the essential feature of quantum mechanics[2]. It describes correlations between quantum systems much stronger than any classical correlation could be.

The possibility of transferring quantum information is one of the cornerstones of the emerging field of quantum communication and quantum computation[3]. Although there is fast progress in the theoretical description of quantum information processing, the difficulties in handling quantum systems have not allowed an equal advance in the experimental realization of the new proposals. Besides the promising developments of quantum cryptography[4] (the first provably secure way to send secret messages), we have only recently succeeded in demonstrating the possibility of quantum dense coding[5], a way to quantum mechanically enhance data compression. The main reason for this slow experimental progress is that, although there exist methods to produce pairs of entangled photons[6], entanglement has been demonstrated for atoms only very recently[7] and it has not been possible thus far to produce entangled states of more than two quanta.

Here we report the first experimental verification of quantum teleportation. By producing pairs of entangled photons by the process of parametric down-conversion and using two-photon interferometry for analysing entanglement, we could transfer a quantum property (in our case the polarization state) from one photon to another. The methods developed for this experiment will be of great importance both for exploring the field of quantum communication and for future experiments on the foundations of quantum mechanics.

The problem

To make the problem of transferring quantum information clearer, suppose that Alice has some particle in a certain quantum state $|\psi\rangle$ and she wants Bob, at a distant location, to have a particle in that state. There is certainly the possibility of sending Bob the particle directly. But suppose that the communication channel between Alice and Bob is not good enough to preserve the necessary quantum coherence or suppose that this would take too much time, which could easily be the case if $|\psi\rangle$ is the state of a more complicated or massive object. Then, what strategy can Alice and Bob pursue?

As mentioned above, no measurement that Alice can perform on $|\psi\rangle$ will be sufficient for Bob to reconstruct the state because the state of a quantum system cannot be fully determined by measurements. Quantum systems are so evasive because they can be in a superposition of several states at the same time. A measurement on the quantum system will force it into only one of these states—this is often referred to as the projection postulate. We can illustrate this important quantum feature by taking a single photon, which can be horizontally or vertically polarized, indicated by the states $|\leftrightarrow\rangle$ and $|\updownarrow\rangle$. It can even be polarized in the general superposition of these two states

$$|\psi\rangle = \alpha|\leftrightarrow\rangle + \beta|\updownarrow\rangle \tag{1}$$

where α and β are two complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. To place this example in a more general setting we can replace the states $|\leftrightarrow\rangle$ and $|\updownarrow\rangle$ in equation (1) by $|0\rangle$ and $|1\rangle$, which refer to the states of any two-state quantum system. Superpositions of $|0\rangle$ and $|1\rangle$ are called qubits to signify the new possibilities introduced by quantum physics into information science[8].

If a photon in state $|\psi\rangle$ passes through a polarizing beamsplitter—a device that reflects (transmits) horizontally (vertically) polarized photons—it will be found in the reflected (transmitted) beam with probability $|\alpha|^2$ ($|\beta|^2$). Then the general state $|\psi\rangle$ has been projected either onto $|\leftrightarrow\rangle$ or onto $|\updownarrow\rangle$ by the action of the measurement. We conclude that the rules of quantum mechanics, in particular the projection postulate, make it impossible for Alice to perform a measurement on $|\psi\rangle$ by which she would obtain all the information necessary to reconstruct the state.

The concept of quantum teleportation

Although the projection postulate in quantum mechanics seems to bring Alice's attempts to provide Bob with the state $|\psi\rangle$ to a halt, it was realised by Bennett *et al.*[1] that precisely this projection postulate enables teleportation of $|\psi\rangle$ from Alice to Bob. During teleportation Alice will destroy the quantum state at hand while Bob receives the quantum state, with neither Alice nor Bob obtaining information about the state $|\psi\rangle$. A key role in the teleportation scheme is played by an entangled ancillary pair of particles which will be initially shared by Alice and Bob.

Suppose particle 1 which Alice wants to teleport is in the initial state $|\psi\rangle_1 = \alpha|\leftrightarrow\rangle_1 + \beta|\updownarrow\rangle_1$ (Fig. 1a), and the entangled pair of particles 2 and 3 shared by Alice and Bob is in the state:

$$|\Psi^-\rangle_{23} = \frac{1}{\sqrt{2}}\left(|\leftrightarrow\rangle_2|\updownarrow\rangle_3 - |\updownarrow\rangle_2|\leftrightarrow\rangle_3\right) \tag{2}$$

That entangled pair is a single quantum system in an equal superposition of the states $|\leftrightarrow\rangle_2|\updownarrow\rangle_3$ and $|\updownarrow\rangle_2|\leftrightarrow\rangle_3$. The entangled state contains no information on the individual particles; it only indicates that the two particles will be in opposite states. The important property of an entangled pair is that as soon as a measurement on one of the particles projects it, say, onto $|\leftrightarrow\rangle$ the state of the other one is determined to be $|\updownarrow\rangle$, and vice versa. How could a measurement on one of the particles instantaneously influence the state of the other particle, which can be arbitrarily far away? Einstein, among many other distinguished physicists, could simply not accept this "spooky action at a distance". But this property of entangled states has now been demonstrated by numerous experiments (for reviews, see refs 9, 10).

The teleportation scheme works as follows. Alice has the particle 1 in the initial state $|\psi\rangle_1$ and particle 2. Particle 2 is entangled with particle 3 in the hands of Bob. The essential point is to perform a specific measurement on particles 1 and 2 which projects them onto the entangled state:

$$|\Psi^-\rangle_{12} = \frac{1}{\sqrt{2}}\left(|\leftrightarrow\rangle_1|\updownarrow\rangle_2 - |\updownarrow\rangle_1|\leftrightarrow\rangle_2\right) \tag{3}$$

This is only one of four possible maximally entangled states into which any state of two particles can be decomposed. The projection of an arbitrary state of two particles onto the basis of the four states is called a Bell-state measurement. The state given in equation (3) distinguishes itself from the three other maximally entangled states by the fact that it changes sign upon interchanging particle 1 and particle 2. This unique antisymmetric feature of $|\Psi^-\rangle_{12}$ will play an important role in the experimental identification, that is, in measurements of this state.

Quantum physics predicts[1] that once particles 1 and 2 are projected into $|\Psi^-\rangle_{12}$, particle 3 is instantaneously projected into the initial state of particle 1. The reason for this is as follows. Because we observe particles 1 and 2 in the state $|\Psi^-\rangle_{12}$ we know that whatever the state of particle 1 is, particle 2 must be in the opposite state, that is, in the state orthogonal to the state of particle 1. But we had initially prepared particle 2 and 3 in the state $|\Psi^-\rangle_{23}$, which means that particle 2 is also orthogonal to particle 3. This is only possible if particle 3 is in the same state as particle 1 was initially. The final state of particle 3 is therefore:

$$|\psi\rangle_3 = \alpha|\leftrightarrow\rangle_3 + \beta|\updownarrow\rangle_3 \tag{4}$$

We note that during the Bell-state measurement particle 1 loses its identity because it becomes entangled with particle 2. Therefore the state $|\psi\rangle_1$ is destroyed on Alice's side during teleportation.

This result (equation (4)) deserves some further comments. The transfer of quantum information from particle 1 to particle 3 can happen over arbitrary distances, hence the name teleportation. Experimentally, quantum entanglement has been shown[11] to survive over distances of the order of 10 km. We note that in the teleportation scheme it is not necessary for Alice to know where Bob is. Furthermore, the initial state of particle 1 can be completely unknown not only to Alice but to anyone. It could even be quantum mechanically completely undefined at the time the Bell-state measurement takes place. This is the case when, as already remarked by Bennett *et al.*[1], particle 1 itself is a member of an entangled pair and therefore has no well-defined properties on its own. This ultimately leads to entanglement swapping[12,13].

It is also important to notice that the Bell-state measurement does not reveal any information on the properties of any of the particles. This is the very reason why quantum teleportation using coherent two-particle superpositions works, while any measurement on one-particle superpositions would fail. The fact that no information whatsoever is gained on either particle is also the reason why quantum teleportation escapes the verdict of the no-cloning theorem[14]. After successful teleportation particle 1 is not available in its original state any more, and therefore particle 3 is not a clone but is really the result of teleportation.

A complete Bell-state measurement can not only give the result that the two particles 1 and 2 are in the antisymmetric state, but with equal probabilities of 25% we could find them in any one of the three other entangled states. When this happens, particle 3 is left in one of three different states. It can then be brought by Bob into the original state of particle 1 by an accordingly chosen transformation, independent of the state of particle 1, after receiving via a classical communication channel the information on which of the Bell-state

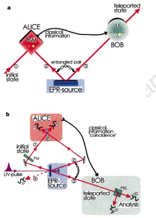
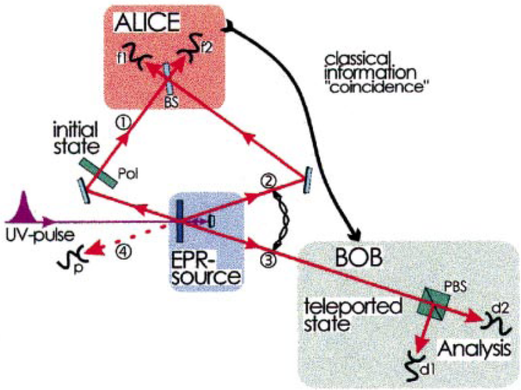


Figure 1 Scheme showing principles involved in quantum teleportation (**a**) and the experimental set-up (**b**). **a**, Alice has a quantum system, particle 1, in an initial state which she wants to teleport to Bob. Alice and Bob also share an ancillary entangled pair of particles 2 and 3 emitted by an Einstein–Podolsky–Rosen (EPR) source. Alice then performs a joint Bell-state measurement (BSM) on the initial particle and one of the ancillaries, projecting them also onto an entangled state. After she has sent the result of her measurement as classical information to Bob, he can perform a unitary transformation (U) on the other ancillary particle resulting in it being in the state of the original particle. **b**, A pulse of ultraviolet radiation passing through a nonlinear crystal creates the ancillary pair of photons 2 and 3. After retroflection during its second passage through the crystal the ultraviolet pulse creates another pair of photons, one of which will be prepared in the initial state of photon 1 to be teleported, the other one serving as a trigger indicating that a photon to be teleported is under way. Alice then looks for coincidences after a beam splitter BS where the initial photon and one of the ancillaries are superposed. Bob, after receiving the classical information that Alice obtained a coincidence count in detectors f1 and f2 identifying the $|\Psi^-\rangle_{12}$ Bell state, knows that his photon 3 is in the initial state of photon 1 which he then can check using polarization analysis with the polarizing beam splitter PBS and the detectors d1 and d2. The detector p provides the information that photon 1 is under way.

"Experimental quantum teleportation," D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zellinger, *Nature* **390**, 575 (1997).

Quantum teleportation experiment

Experimental single photon teleportation using 3 and 4 coincidence measurements



Parametric down-conversion produces an EPR pair 2 & 3 in state
$|\Psi^-\rangle_{23} = \frac{1}{\sqrt{2}}(|\rightarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\rightarrow\rangle)$

The reflected beam produces photons 1 & 4

1 & 2 are mixed in a beam splitter and a coincidence is detected by detectors f_1 and f_2 if Bell state $|\Psi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|\rightarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\rightarrow\rangle)$ is present

Bob measures photon 3 with a polarizing beam splitter and two detectors d_1 and d_2 when he knows that Alice has the Bell state $|\Psi^-\rangle_{12}$

Quantum teleportation experiment

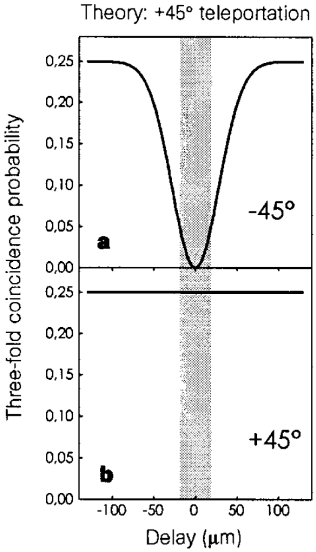
Initial experiment with photon 1 polarized at 45°

Coincidence between f_1 and f_2 will occur 25% of the time

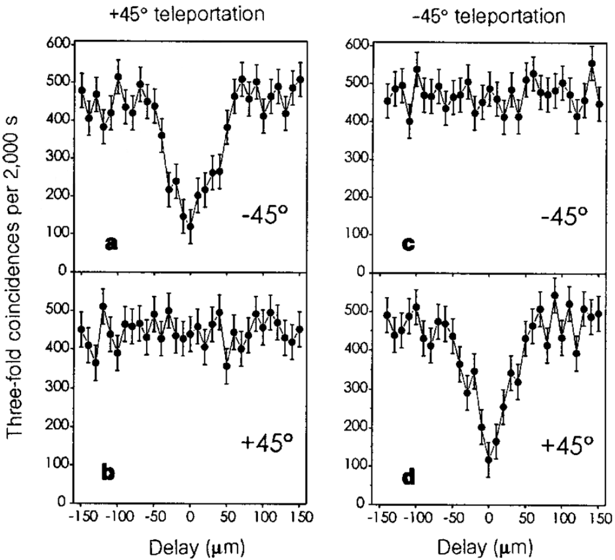
Bob's polarizer is also set to 45° and detector d_2 should give a coincident pulse with f_1 and f_2 to demonstrate teleportation

A variable delay is applied to photon 2 to obtain the temporal overlap needed for the Bell-state measurement

Coincidence between d_1, f_1, and f_2 should drop to zero when teleportation occurs



Quantum teleportation: three photon coincidence



Quantum teleportation experiment

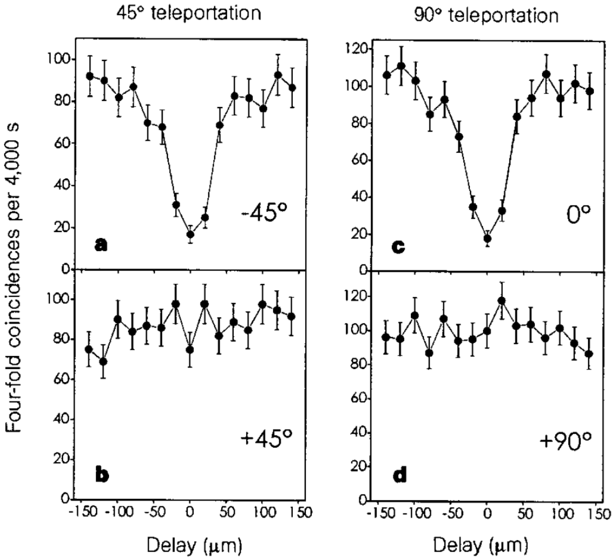
These results are confirmed by measuring a number of different polarizations

Polarization	*Visibility*
$+45^\circ$	0.63 ± 0.02
-45°	0.64 ± 0.02
0°	0.66 ± 0.02
90°	0.61 ± 0.02
Circular	0.57 ± 0.02

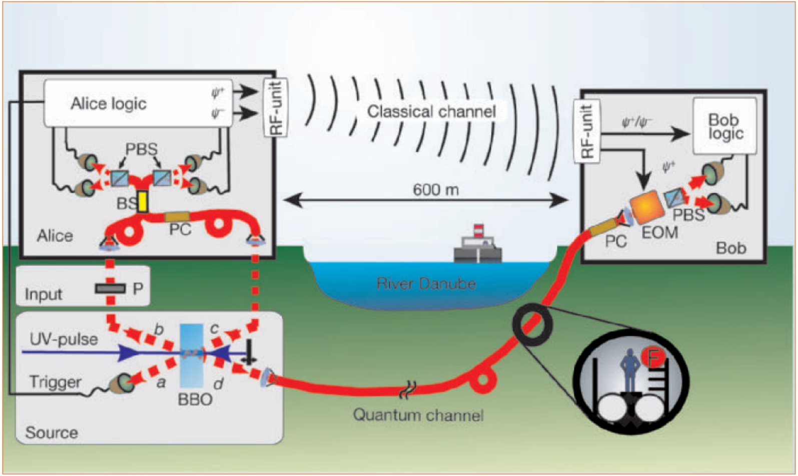
Visibility is a measure of the dip

The background in the three-photon coincidence can be eliminated at the cost of forcing photon 1 into a single particle state by measuring the coincidence with photon 4 in detector p

Quantum teleportation: four photon coincidence



Quantum teleportation over long distance



"Quantum teleportation across the Danube," R. Ursin, T. Jennewein, M. Aspelmeyer, R. Kaltenbaek, M. Lindenthal, P. Walther, and A. Zeilinger, *Nature* **430**, 849 (2004).

Phase shift and rotation operators

All single-qubit transformations can be written as a combination of three types of transformations, phase shifts $K(\delta)$, rotations $R(\beta)$, and phase rotations $T(\alpha)$

$$K(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \qquad R(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \qquad T(\alpha) = \begin{pmatrix} e^{+i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

with the properties that

$$K(\delta_1 + \delta_2) = K(\delta_1)K(\delta_2) \qquad R(\beta_1 + \beta_2) = R(\beta_1)R(\beta_2) \qquad T(\alpha_1 + \alpha_2) = T(\alpha_1)T(\alpha_2)$$

furthermore, the phase shift operator $K(\delta)$ commutes with both $R(\beta)$ and $T(\alpha)$

$$[K(\delta), R(\beta)] = K(\delta)R(\beta) - R(\beta)K(\delta) = 0 \qquad [K(\delta), T(\alpha)] = K(\delta)T(\alpha) - T(\alpha)K(\delta) = 0$$

K applies a global phase shift and can be written just as the phase factor alone, $e^{i\delta}$, while $R(\alpha)$ and $T(\alpha)$ rotate the qubit by 2α about the y- and z- axes respectively

Operator decomposition

If Q is a single-qubit unitary transformation, it can be represented by a sequence of rotations and phase shifts such that $Q = K(\delta)T(\alpha)R(\beta)T(\gamma)$

The general form of the transformation Q is given by

$$Q = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$$

Because the transformation must be unitary we can write

$$QQ^\dagger = I = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} \begin{pmatrix} \overline{u_{00}} & \overline{u_{10}} \\ \overline{u_{01}} & \overline{u_{11}} \end{pmatrix}$$

The off-diagonal elements must equal 0 and the diagonal elements must equal 1

$$= \begin{pmatrix} |u_{00}|^2 + |u_{01}|^2 & u_{00}\overline{u_{10}} + u_{01}\overline{u_{11}} \\ u_{10}\overline{u_{00}} + u_{11}\overline{u_{01}} & |u_{10}|^2 + |u_{11}|^2 \end{pmatrix}$$

Rearrange the off-diagonal terms and multiply them together

$$u_{00}\overline{u_{10}} = -u_{11}\overline{u_{01}}, \qquad \overline{u_{00}}u_{10} = -\overline{u_{11}}u_{01} \quad \longrightarrow \quad |u_{00}|^2|u_{10}|^2 = |u_{11}|^2|u_{01}|^2$$

Solve for $|u_{01}|^2$ and $|u_{10}|^2$ and use these in combination with the two equations from the diagonal terms

$$|u_{00}|^2 + |u_{01}|^2 = 1, \qquad |u_{10}|^2 + |u_{11}|^2 = 1$$

Operator decomposition (cont.)

$$|u_{01}|^2 = |u_{10}|^2 \frac{|u_{00}|^2}{|u_{11}|^2} \qquad |u_{10}|^2 = |u_{01}|^2 \frac{|u_{11}|^2}{|u_{00}|^2}$$

$$1 = |u_{00}|^2 + |u_{01}|^2 \qquad 1 = |u_{10}|^2 + |u_{11}|^2$$

$$|u_{00}|^2 + |u_{10}|^2 \frac{|u_{00}|^2}{|u_{11}|^2} = 1 \;\longrightarrow\; |u_{00}|^2|u_{11}|^2 + |u_{10}|^2|u_{00}|^2 = |u_{11}|^2 = |u_{00}|^2 \left(|u_{11}|^2 + |u_{10}|^2\right)$$

Thus we find that $|u_{00}|^2 = |u_{11}|^2$ and by consequence $|u_{01}|^2 = |u_{10}|^2$ and

$$|u_{00}|^2 + |u_{01}|^2 = 1 \quad \longrightarrow \quad |u_{00}| = \cos\beta, \;\; |u_{01}| = \sin\beta$$
$$|u_{10}|^2 + |u_{11}|^2 = 1 \quad \longrightarrow \quad |u_{11}| = \cos\beta, \;\; |u_{10}| = \sin\beta$$

The absolute values imply that there is an arbitrary phase factor associated with each element in the matrix

$$Q = \begin{pmatrix} e^{i\theta_{00}}\cos\beta & e^{i\theta_{01}}\sin\beta \\ -e^{i\theta_{10}}\sin\beta & e^{i\theta_{11}}\cos\beta \end{pmatrix}$$

The phase factors are constrained by the relation $u_{10}\overline{u_{00}} + u_{11}\overline{u_{01}} = 0$

$$\theta_{10} - \theta_{00} = \theta_{11} - \theta_{01}$$

Operator decomposition (cont.)

Since we assert that Q can be decomposed into the combination of $K(\delta)T(\alpha)R(\beta)T(\gamma)$ we write the matrix as

$$Q = \begin{pmatrix} e^{i\theta_{00}}\cos\beta & e^{i\theta_{01}}\sin\beta \\ -e^{i\theta_{10}}\sin\beta & e^{i\theta_{11}}\cos\beta \end{pmatrix} = K(\delta)T(\alpha)R(\beta)T(\gamma) = \begin{pmatrix} e^{i(\delta+\alpha+\gamma)}\cos\beta & e^{i(\delta+\alpha-\gamma)}\sin\beta \\ -e^{i(\delta-\alpha+\gamma)}\sin\beta & e^{i(\delta-\alpha-\gamma)}\cos\beta \end{pmatrix}$$

This selection can be shown to satisfy $\theta_{10} - \theta_{00} = \theta_{11} - \theta_{01}$

$$\theta_{00} = \delta + \alpha + \gamma$$
$$\theta_{01} = \delta + \alpha - \gamma$$
$$\theta_{10} = \delta - \alpha + \gamma$$

$$\begin{aligned} \theta_{11} &= \theta_{10} - \theta_{00} + \theta_{01} \\ &= \cancel{\delta} - \cancel{\alpha} + \cancel{\gamma} - \cancel{\delta} - \alpha - \cancel{\gamma} + \delta + \cancel{\alpha} - \gamma \\ &= \delta - \alpha - \gamma \end{aligned}$$

This is another form for the general unitary transformation which forms the building blocks, along with the C_{not} operator for all arbitrary n-qubit operators

Singly controlled transformations

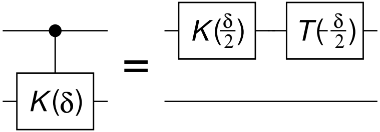
We wish to implement a controlled operator $\bigwedge Q$ where $Q = K(\delta)T(\alpha)R(\beta)T(\delta)$ and

$$K(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \qquad R(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \qquad T(\alpha) = \begin{pmatrix} e^{+i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

Because the $K(\delta)$ operator is a global phase shift it is possible to write that
$\bigwedge Q = \bigwedge K(\delta) \bigwedge (T(\alpha)R(\beta)T(\gamma)) = (\bigwedge K(\delta))(\bigwedge Q')$

The conditional phase shift, $\bigwedge K_\delta$ can be implemented using

$$\begin{aligned} \bigwedge K_\delta &= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes K(\delta) \\ &= |0\rangle\langle 0| \otimes I + e^{i\delta}|1\rangle\langle 1| \otimes I \\ &= (K(\tfrac{\delta}{2})T(-\tfrac{\delta}{2})) \otimes I \end{aligned}$$



Note that the conditional phase shift is realized by acting on the first qubit only since a phase shift changes the entire state