PHYS 535 Homework 1 Solutions

February 7, 2024

Problem 1

(a) The conditions on our matrix L are $L_{jk} = 0$ for $j < k$ and $L_jj > 0$. By definition, we have

$$M_{ij} = \sum_{k=1}^{n} L_{ik}(L^t)_{kj} = \sum_{k=1}^{n} L_{ik}L_{jk} \tag{1}$$

Considering the case $i = j = 1$ yields

$$M_{11} = \sum_{k=1}^{n} L_{1k}^2 = L_{11}^2 \tag{2}$$

Since all the principal leading minors D_k of M are positive, we have $D_1 = M_{11} > 0$, and so we can take the square root to obtain

$$L_{11} = \sqrt{M_{11}} \tag{3}$$

where we have chosen the positive root to ensure $L_11 > 0$.

(b) Returning to equation (1) with $j = 1$ and i arbitrary, we have

$$M_{i1} = \sum_{k=1}^{n} L_{ik}L_{1k} = L_{i1}L_{11} \tag{4}$$

and so

$$L_{i1} = \frac{M_{i1}}{L_{11}} = \frac{M_{i1}}{\sqrt{M_{11}}} \tag{5}$$

(c) Once again using (1),

$$M_{kk} = \sum_{l=1}^{n} L_{kl}^2 = L_{kk} + \sum_{l=1}^{k-1} L_{kl}^2 \tag{6}$$

which is simply rearranged to yield

$$L_{kk} = \sqrt{M_{kk} - \sum_{l=1}^{k-1} L_{kl}^2} \tag{7}$$

(d) For a matrix A, let A_k denote the $k \times k$ diagonal submatrix obtained by deleting the last $n-k$ rows and columns. That is, A_k is the matrix such that $D_k = \det(A_k)$. Using the fact that L is lower triangular once again, we have

$$(M_k)_{ij} = \sum_{l=1}^{n} L_{il}L_{jl} = \sum_{l=1}^{k} L_{il}L_{jl} \implies M_k = L_kL_k^t \tag{8}$$

That is, unlike for an arbitrary product of matrices, to compute M_k we only need the $k \times k$ submatrix L_k of L. So,

$$D_k = \det(M_k) = \det(L_k)^2 = \prod_{i=1}^{k} L_{ii}^2 \tag{9}$$

since the determinant of a triangular matrix is given by the product of its diagonal elements. Then we clearly also have $D_{k-1} = \prod_{i=1}^{k-1} L_{ii}^2$, from which we conclude

$$L_{kk} = \sqrt{\frac{D_k}{D_{k-1}}} \tag{10}$$

(e) From (1) once again,

$$M_{ik} = \sum_{l=1}^{n} L_{il}L_{kl} = \sum_{l=1}^{k} L_{il}L_{kl} = L_{ik}L_{kk} + \sum_{l=1}^{k-1} L_{il}L_{kl} \tag{11}$$

and so

$$L_{ik} = \frac{1}{L_{kk}}\left(M_{ik} - \sum_{l=1}^{k-1} L_{il}L_{kl}\right) \qquad k \le i \le n \tag{12}$$

Problem 2

Recall that a linear operator T is bounded if there exists some $M > 0$ such that for all $x \in V$,

$$\|Tx\| \le M\|x\|. \tag{13}$$

For an inner product space V with inner product $\langle \cdot, \cdot \rangle$, we have an induced norm

$$\|\phi\|_V = \sqrt{\langle \phi, \phi \rangle} \tag{14}$$

So our linear operators $L : V \to W$ are bounded if and only if for all $\chi \in V$ there exists $M > 0$ such that

$$\|L\chi\|_W \le M\|\chi\|_V \tag{15}$$

First we consider $L_\phi : V \to \mathbb{C}$ given by $L_\phi = \langle \phi, \cdot \rangle$. Using the Cauchy Schwartz inequality, one has

$$\|L_\phi \chi\|_{\mathbb{C}} = |\langle \phi, \chi \rangle| \le \|\phi\|_V \|\chi\|_V \tag{16}$$

So the choice $M = \|\phi\|_V$ shows that L_ϕ is indeed bounded.

Next we have $L_{\psi,\phi} : V \to V$ with $L_{\psi,\phi} = \phi\langle \psi, \cdot \rangle$. Again using Cauchy Schwartz as well as homogeneity of the norm, we obtain

$$\|L_{\psi,\phi}\chi\|_V = \|\phi\langle \psi, \chi \rangle\|_V = |\langle \psi, \chi \rangle| \|\phi\|_V \le \|\phi\|_V \|\psi\|_V \|\chi\|_V \tag{17}$$

and so $M = \|\phi\|_V \|\psi\|_V$ yields the result that $L_{\psi,\phi}$ is bounded.

Problem 3

We need to calculate:

$$\min_W \left(\sum_{i=1}^{m} \left\| x^i - P_W(x^i) \right\|_2^2 \right) = \min_W \|X - P_W X\|_F^2 \tag{18}$$

where X is the $n \times m$ matrix given by $X_{ij} = x_i^j$. Using the singular value decomposition (SVD) to write $X = U\Sigma V^t$, and performing the unitary transformation $P_W \to UP_W U^t$, we can take advantage of the unitary invariance of $\|\cdot\|_F$ to write

$$\min_W \|X - P_W X\|_F^2 = \min_W \|\Sigma - P_W \Sigma\|_F^2 = \min_W \sum_{i=1}^{n} \sum_{j=1}^{m} |\Sigma_{ij} - \sum_{k=1}^{n} (P_W)_{ik}\Sigma_{kj}|^2 \tag{19}$$

Now, let us use the formula for the Frobenius norm $\|A\|_F = \sqrt{\mathrm{Tr}(AA^t)}$ to compute

$$\|\Sigma - P_W\Sigma\|_F^2 = \mathrm{Tr}((\Sigma - P_W\Sigma)(\Sigma^t - \Sigma^t P_W)) = \mathrm{Tr}(\Sigma\Sigma^t - P_W\Sigma\Sigma^t). \tag{20}$$

In performing these manipulations, we have used two standard facts about orthogonal projections, namely that real projections are symmetric and idempotent. As equations, these read $P_W^t = P_W$ and $P_W^2 = P_W$. Using these facts along with cyclicity of the trace give us the final expression in (20).

Now, returning to the minimization, we have

$$\min_W \|\Sigma - P_W\Sigma\|_F^2 = \min_W \mathrm{Tr}(\Sigma\Sigma^t - P_W\Sigma\Sigma^t) = \min_W \left(\sum_i \sigma_i^2 - \mathrm{Tr}(P_W\Sigma\Sigma^t)\right) \tag{21}$$

where σ_i are the singular values of X. From this expression, we can see that our original minimization is equivalent to maximizing the quanitity

$$\mathrm{Tr}(P_W\Sigma\Sigma^t) \tag{22}$$

Intuitively, it is clear that this can be achieved if P_W simply projects onto the first k largest singular values. More rigorously, we can solve the maximization directly using Lagrange multipliers. Recall that an orthogonal projection can be completely described by an orthonormal set of vectors $\{n^{(i)}\}$, such that the action of the corresponding projector on a vector v is

$$Pv = \sum_i (n^{(i)}, v) n^{(i)} \tag{23}$$

The trace of PA for any matrix A is then given by

$$\mathrm{Tr}(PA) = \sum_i (n^{(i)}, An^{(i)}) = \sum_i (n^{(i)})^t A n^{(i)} \tag{24}$$

So, we can replace the optimization over subspaces W with an optimization over sets of orthonormal vectors $\{n^{(i)}\}_{i=1}^k$ which reads

$$\max_W \mathrm{Tr}(P_W\Sigma\Sigma^t) = \max_{\{n^{(i)}\}_{i=1}^k} \sum_i (n^{(i)})^t \Sigma\Sigma^t n^{(i)} \tag{25}$$

We can find such a maximum using a Lagrangian

$$\mathcal{L} = \sum_i (n^{(i)})^t \Sigma\Sigma^t n^{(i)} + \sum_{ij} \lambda_{ij} \left((n^{(i)})^t n^{(j)} - \delta_{ij}\right) \tag{26}$$

where the second term consists of a symmetric matrix of Lagrange multipliers λ_{ij} which enforce orthonormality of the $n^{(i)}$. Now, taking a derivative with respect to a component $n_l^{(i)}$, and using the fact that $\Sigma\Sigma^t$ is a diagonal matrix with the singular values σ_l^2 along the diagonal, we have

$$\frac{\partial \mathcal{L}}{\partial n_l^{(i)}} = 2n_l^{(i)}\sigma_l^2 + 2\sum_j \lambda_{ji} n_l^{(j)} = 0 \tag{27}$$

This condition requires that *either* λ_{ij}, viewed as a $l-$independent $k \times k$ matrix, have eigenvalues σ_l, *or* $n_l^{(i)} = 0, i = 1, \ldots, k$. The solution which satisfies these conditions is $n_l^{(i)} = \delta_{il}$ and $\lambda_{ij} = \sigma_i^2\delta_{ij}$. Plugging this back into the Lagrangian, the problem reduces to maximizing a choice of k singular values, for which the solution is obviously to pick the $n^{(i)}$ to align with the k singular vectors with corresponding largest k singular values.

Problem 4

(a) Writing the SVD $L = U\Sigma V^t$ and $x = x_1 - x_2$, we have

$$\|L(x_1) - L(x_2)\|_2 = \left\|U\Sigma V^t x\right\| = \left\|\Sigma V^t x\right\| \tag{28}$$

where we have used the fact that the ℓ_2 norm is invariant under unitary transformations. Also writing $x' = V^t x$, we then have

$$\|\Sigma x'\|_2^2 = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \Sigma_{ij} x'_j \right)^2 = \sum_{i=1}^{\min(n,m)} |\sigma_i x'_i|^2 \leq \max_i \sigma_i^2 \|x\|_2^2 \tag{29}$$

where we have used $\|x'\|_2 = \|x\|_2$. To show that the choice $C = \max_i \sigma_i$ is the minimal such C, we just need to find a particular vector x' saturating the above bound. Clearly (for singular values ordered from greatest to lowest), the choice $x' = (1, 0, \ldots, 0)^t$ achieves the said saturation, and so $C = \sigma_1$ $(= \max_i \sigma_i)$.

(b) Since we are using the ℓ_2 norm, we have

$$\max_{x \in V, x \neq 0} \frac{\|Lx\|_2^2}{\|x\|_2^2} = \max_{x \in V, \|x\|_2 = 1} \|Lx\|_2^2 = \max_{x \in V, \|x\|_2 = 1} x^t L^t L x \tag{30}$$

So we would like to extremize $x^t L^t L x$ with the constraint $x^t x = 1$, which can be stated as the extremization of the function

$$\mathcal{L}(x, \lambda) = x^t L^t L x - \lambda(x^t x - 1). \tag{31}$$

Recall the following matrix calculus identities for the function $f = y^t M x$:

$$\frac{\partial f}{\partial x} = y^t M, \quad \frac{\partial f}{\partial y} = x^t M^t. \tag{32}$$

Taking derivatives of $\mathcal{L}$, we then obtain

$$\frac{\partial \mathcal{L}}{\partial x^t} = 0 = L^t L x - \lambda x \tag{33}$$

which implies that a solution x_i is an eigenvector of $L^t L$ with eigenvalue λ_i. The set of such eigenvalues λ_i are simply σ_i^2. So for an extremal vector x_* we have

$$\|Lx\|_2^2 = \sigma_i^2 \tag{34}$$

which is clearly maximized if we pick the x_* to be the eigenvector of $L^t L$ with largest eigenvalue. Thus we once again have $C = \max_i \sigma_i = \sigma_1$.

Problem 5

(a) Let M be the Pauli matrix $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We have

$$M^k v = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^k \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ (-1)^k v_2 \end{pmatrix} \tag{35}$$

So the vector does not converge to an eigenvector (or converge at all!) but instead oscillates between v and $v' = Mv$.

(b) Notice that for the nilpotent matrix N given, we have $N^n = 0$. So for $k > n$, we have

$$M^k = (\lambda I + N)^k = \sum_{p=1}^{n} \binom{k}{p} \lambda^{k-p} N^p = \sum_{p=1}^{n-1} \binom{k}{p} \lambda^{k-p} N^p \tag{36}$$

Now if we were to apply the power iteration method to M and take k large, the leading order behavior of the binomial coefficient for large k and fixed p is

$$\binom{k}{p} \approx \frac{k^p}{p!} + \mathcal{O}(k^{p-1}) \tag{37}$$

and so intuitively the term which dominates the sum at large k is then the one with largest p, i.e. $p = n-1$:

$$M^k \approx \frac{\lambda^{k-n-1} k^{n-1}}{(n-1)!} N^{n-1}. \tag{38}$$

A bit more carefully, note that N^k and $N^{k'}$ have no shared non-zero matrix elements when $k \neq k'$. So, if we compare coefficients of the N^p in the sum to the value $p = n-1$, we find

$$\frac{\binom{k}{p}\lambda^{k-p}}{\binom{k}{n-1}\lambda^{k-(n-1)}} = \frac{(n-1)!(k-n-1)!}{p!(k-p)!}\lambda^{n-1-p} \overset{k\to\infty}{\longrightarrow} 0 \tag{39}$$

So, focusing on the term $k = n-1$ in the sum, the matrix N^{n-1} is given by $(N^{n-1})_{ij} = \delta_{1i}\delta_{nj}$, and so when acting on a vector,

$$(M^k v)_i \propto v_n \delta_{i1}. \tag{40}$$

That is, at large k, M^k projects onto the direction $\hat{e}_1 = (1, 0, 0, \cdots, 0)$. We can check that this is indeed an eigenvector:

$$M\hat{e}_1 = \lambda\hat{e}_1 + N\hat{e}_1 = \lambda\hat{e}_1 \tag{41}$$

since $N\hat{e}_1 = 0$. So the power iteration method converges on the eigenvector $\hat{e}_1$ with corresponding eigenvalue λ.

Problem 6

Let v_m be a vector of length m and v_n of length n, and write the $n+m$ length vector v as

$$v = \begin{pmatrix} v_m \\ v_n \end{pmatrix} \tag{42}$$

Then

$$\mathrm{Sym}(M)v = \begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix} \begin{pmatrix} v_m \\ v_n \end{pmatrix} = \begin{pmatrix} Mv_n \\ M^t v_m \end{pmatrix} \tag{43}$$

The eigenequation can then be written as the system of equations

$$Mv_n = \lambda v_m \tag{44}$$
$$M^t v_m = \lambda v_n \tag{45}$$

Acting on the first equation by M^t and the second by M we have

$$M^t M v_n = \lambda^2 v_n \tag{46}$$
$$MM^t v_m = \lambda^2 v_m \tag{47}$$

So v_n and v_m must be eigenvectors of M^tM and MM^t, respectively and with the same eigenvalue. Thus, $v_n^i = \alpha_i v^i$ and $v_m^i = \beta_i u^i$ where v^i and u^i are the right and left singular vectors with corresponding singular value σ_i and α_i, β_i are constants. WLOG, we may choose $\alpha_i = 1$. The eigenvalues λ_i are related to the singular values by

$$\lambda_i^2 = \sigma_i^2 \implies \lambda_i = \pm\sigma_i \tag{48}$$

Plugging these values of lambda back into the eigenvalue equation and using the SVD of $M = \sum_i \sigma_i u^i (v^i)^t$ we have

$$\sigma_i u^i = \pm\lambda\beta_i u^i \tag{49}$$
$$\beta_i \sigma_i v^i = \pm\lambda v^i \tag{50}$$

So, when we take the positive root $\lambda_i = \sigma_i$, $\beta_i = 1$ and when we take the negative root $\lambda_i = -\sigma_i$, we obtain $\beta_i = -1$. We conclude that $\mathrm{Sym}(M)$ has $2r$ eigenvalues $\pm\sigma_i$ with corresponding eigenvectors

$$v_i = \begin{pmatrix} \pm u^i \\ v^i \end{pmatrix}, \tag{51}$$

where r is the rank of the matrix $M^t M$. Finally, notice there are $m - r$ vectors z_i which give $Mz_i = 0$ and $n - r$ vectors y_i which give $M^t y_i = 0$, and so there are $m + n - 2r$ vectors, of the form $(0, z_i)$ and $(y_i, 0)$ with eigenvalue zero.

Problem 7

For convenience, let S_p be the unit sphere in V with respect to the ℓ_p norm.

(a) The induced norm is

$$\|M\|_{1,2} = \max_{x \in S_1} \|Mx\|_2 = \max_{x \in S_1} \sqrt{\sum_{i=1}^{m} \left(\sum_{j=1}^{n} M_{ij} x_j \right)^2} \tag{52}$$

Because $\sum_i |x_i| = 1$, and the square of a function is convex, we may use Jensen's inequality to state

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} M_{ij} x_j \right)^2 \leq \sum_{i=1}^{m} |M_{ij}|^2 |x_j| \leq \max_j \sum_{i=1}^{m} |M_{ij}|^2 \sum_j |x_j| = \max_j \|M_{:j}\|_2 \tag{53}$$

Now we can saturate this inequality with the choice $x_i = \delta_{ki}$ with $k = \arg\max_{k'} \|M_{:k'}\|_2$, and so

$$\|M\|_{1,2} = \max_k \|M_{:k}\|_2 \tag{54}$$

(b) Now we have

$$\|M\|_{1,\infty} = \max_{x \in S_1} \|Mx\|_\infty = \max_{x \in S_1} \max_i \left| \sum_{j=1}^{n} M_{ij} x_j \right| \tag{55}$$

Let $i = \arg\max_{i'} \left| \sum_{j=1}^{n} M_{i'j} x_j \right|$. We have

$$\left| \sum_{j=1}^{n} M_{ij} x_j \right| \leq \sum_{j=1}^{n} |M_{ij}||x_j| \leq \max_j |M_{ij}| \leq \max_{k,j} |M_{kj}| \tag{56}$$

where in the last inequality we are simply using the fact that $\max_{k,j} |M_{kj}|$ is greater than or equal to *any* element of M. However, we can indeed saturate this bound by choosing $x_i = \delta_{ji}$ with $j = \arg\max_{j,i} |M_{ij}|$. Then

$$\|Mx\|_\infty = \max_i \left| \sum_{j=1'}^{n} M_{ij'} \delta_{jj'} \right| = \max_i |M_{ij}| = \max_{i,j} |M_{ij}| \tag{57}$$

and so,

$$\|M\|_{1,\infty} = \max_{ij} |M_{ij}| \tag{58}$$

(c) Finally, we have

$$\|M\|_{2,\infty} = \max_{x \in S_2} \|Mx\|_\infty = \max_{x \in S_2} \max_i \left| \sum_{j=1}^{n} M_{ij} x_j \right| \tag{59}$$

Again let $i = \arg\max_{i'} \left|\sum_{j=1}^n M_{i'j} x_j\right|$. Also, let $\langle \cdot, \cdot \rangle$ be the standard Euclidean inner product $\langle v, w \rangle = \sum_i v_i w_i$. Then we can use Cauchy Schwartz to write

$$\left|\sum_{j=1}^n M_{ij} x_j\right| = |\langle M_{i:}, x \rangle| \leq \|M_{i:}\|_2 \|x\|_2 = \|M_{i:}\|_2 \leq \max_i \|M_{i:}\|_2 \tag{60}$$

We can saturate this bound with the choice $x = \frac{1}{\|M_{i:}\|_2}(M_{i:})^t$ where $i = \arg\max_i' \|M_{i:}\|_2$. For this choice of x, we have

$$\|Mx\|_\infty = \max_k \left|\sum_{j=1}^n M_{kj} x_j\right| = \frac{1}{\|M_{i:}\|_2} \max_k |M_{kj} M_{ji}| = \frac{1}{\|M_{i:}\|_2} \left(\|M_{i:}\|_2^2\right) = \|M_{i:}\|_2 \tag{61}$$

and so we conclude

$$\|M\|_{2,\infty} = \max_i \|M_{i:}\|_2 \tag{62}$$