Chapter 10, Lecture 2

Now for the second difficulty, the delay from what the source does to when you actually know about it. Or, to say it another way, what you 'see' actually happened at an earlier time. That earlier time is called 'the retarded time', and must be $t_r = t - \mathscr{r}/c$, $\mathscr{r}$ as always being the distance between the observer and the source, $\vec{r} - \vec{r}'$, and $c \equiv$ speed of light.

1.) We are looking for a particular solution to the potential equations, for example, $\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$

2.) The time dependence of the solution will be on the retarded time, $t_r = t - \mathscr{r}/c$

3.) If the time dependance goes away, the solution must reduce to the static solution

$$\nabla^2 V - \cancel{\mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2}} = -\rho/\epsilon_0$$

$$\longrightarrow \qquad V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')\, d\tau'}{\mathscr{r}}$$

Further, since it is the charge distribution which is now varying in time, it is reasonable to expect that the time dependence of V is due to the time dependence of ρ

4.) It is also interesting to consider the equation for the potential without the source term:

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = 0 \quad \Big\} \text{ wave equation!}$$

Suppose we had one little charge buzzing around. It is going to generate fields (emit radiation). Symmetry leads us to expect it to be the same in all directions, or some kind of spherical wave. The wave equation in spherical coordinates with no angular dependence is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} =$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r V\right) - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} =$$

$$\frac{\partial^2}{\partial r^2}(rV) - \mu_0 \epsilon_0 \frac{\partial^2 (rV)}{\partial t^2}$$

The solution to the wave equation (outgoing waves) has the form:

$$(rV) = f(r - ct) = f(t - r/c)$$

$$V = \frac{f(t - r/c)}{r}$$

The outgoing spherical wave decreases in amplitude as it moves out radially. Note that this solution is not valid at $r = 0$ (V blows up to ∞). This is because it is a point charge generating the wave, and the potential of a point charge is infinite at the location of the charge.

Also note if there is a distribution of charges, we must sum their contributions to the potential.

Considering the region near the charge distribution, $r \ll c$, and put the source term back into the equation to account for the presence of the charge.

Static point charge:

$$V = \frac{q/4\pi\epsilon_0}{r}$$

static distribution:

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')\, d\tau'}{\mathscr{r}}$$

<u>Near</u> a non-static charge distribution:

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t)}{\mathscr{r}}\, d\tau$$

Spherical wave:

$$V = \frac{f(t - \mathscr{r}/c)}{\mathscr{r}}$$

Total solution:

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}}\, d\tau'$$

What have we done? Basically checked limiting cases as far as possible and guessed at a solution consistent with all the physics we know.

Educated guess:

$$V_{ret}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{\text{\textscriptr}} \, d\tau'$$

$$A_{ret}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,', t_r)}{\text{\textscriptr}} \, d\tau'$$

Now we need to verify that these expressions for the potentials of non-static sources are indeed valid particular solutions to the equations for the potentials.

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

Is the retarded potential a solution to the inhomogeneous wave equation above? To check, Griffiths calculates $\nabla^2 V_{ret}$ and shows this is the same as $-\rho/\epsilon_0 + \mu_0 \epsilon_0 \frac{\partial^2 V_{ret}}{\partial t^2}$

The calculation is longish & painfulish. We won't flinch in the face of danger.

$$V_{ret}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{\eta} \, d\tau' \, , \qquad t_r = t - \eta/c$$

First find $\nabla V_{ret}(\vec{r},t)$

Then find $\nabla^2 V_{ret}$

∇ differentiates with respect to unprimed variables

For now, let $k \equiv \frac{1}{4\pi\epsilon_0}$

$$\nabla V_r = k \int \frac{1}{\eta} \vec{\nabla}\rho + \rho \vec{\nabla}(1/\eta) \, d\tau'$$

① $\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) \overset{?}{=}$

$$\vec{\nabla}\left(\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}\right)$$

$$= -\frac{1}{2}\frac{1}{(\mathscr{r}^2)^{3/2}}\left\{\frac{\partial(x-x')^2}{\partial x}\hat{x} + \frac{\partial(y-y')^2}{\partial y}\hat{y} + \frac{\partial(z-z')^2}{\partial z}\hat{z}\right\}$$

$$= -\frac{1}{2}\frac{1}{\mathscr{r}^3}\left\{2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z}\right\}$$

$$= -\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

② $\vec{\nabla}\rho(\vec{r}\,',t_r) \overset{?}{=}$

$$\vec{\nabla}\rho = \hat{x}\frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial x} + \hat{y}\frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial y} + \hat{z}\frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial z}$$

$$= \frac{\partial\rho}{\partial t_r}\left(\hat{x}\frac{\partial t_r}{\partial x} + \hat{y}\frac{\partial t_r}{\partial y} + \hat{z}\frac{\partial t_r}{\partial z}\right)$$

$$= \frac{\partial t}{\partial t_r}\frac{\partial\rho}{\partial t}\vec{\nabla}t_r = \dot{\rho}\,\vec{\nabla}t_r$$

ρ does not depend on $\vec{r}$ directly (only $\vec{r}\,'$)
The only dependence on $\vec{r}$ is through $t_r = t - \mathscr{r}/c$

since

$$\frac{\partial t_r}{\partial t} = \frac{\partial(t - ᴫ/c)}{\partial t} = 1 \implies \frac{\partial t}{\partial t_r} = 1$$

Note: $ᴫ$ is independent of t <u>in this case</u> since $\vec{ᴫ} = \vec{r} - \vec{r}\,'$

$$\vec{\nabla}\rho = \dot{\rho}\,\vec{\nabla} t_r = -\frac{\dot{\rho}}{c}\,\vec{\nabla} ᴫ$$

$$\vec{\nabla}(ᴫ) = \frac{1/2}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}\left\{\hat{x}\,\frac{\partial(x-x')^2}{\partial x} + \ldots\right\}$$

$$= \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}} = \frac{\vec{ᴫ}}{ᴫ} = \hat{ᴫ}$$

$$\vec{\nabla}\rho = -\frac{\dot{\rho}}{c}\,\hat{ᴫ}$$

<u>Round 2:</u>

$$\nabla^2 V_{ret} = K\int \vec{\nabla}\cdot\left(\frac{\hat{ᴫ}}{ᴫ}\left(-\frac{\dot{\rho}}{c}\right) + \rho\left(-\frac{\hat{ᴫ}}{ᴫ^2}\right)\right) d\tau'$$

$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot\vec{\nabla}f$$

$$\nabla^2 V_{ret} = -K\int\left\{\frac{\dot{\rho}}{c}\left[\vec{\nabla}\cdot\left(\frac{\hat{ᴫ}}{ᴫ}\right)\right] + \frac{\hat{ᴫ}}{ᴫc}\cdot\nabla\dot{\rho} + \rho\left(\vec{\nabla}\cdot\frac{\hat{ᴫ}}{ᴫ^2}\right) + \frac{\hat{ᴫ}}{ᴫ^2}\cdot\nabla\rho\right\} d\tau'$$

① $\vec{\nabla}\dot{\rho} \stackrel{?}{=}$ Use the same procedure as before

$$= \hat{x}\,\frac{\partial\dot{\rho}}{\partial t_r}\frac{\partial t_r}{\partial x} + \hat{y}\,\frac{\partial\dot{\rho}}{\partial t_r}\frac{\partial t_r}{\partial y} + \hat{z}\,\frac{\partial\dot{\rho}}{\partial t_r}\frac{\partial t_r}{\partial z}$$

$$= \frac{\partial t}{\partial t_r}\frac{\partial\dot{\rho}}{\partial t}\vec{\nabla} t_r = \ddot{\rho}\,\vec{\nabla} t_r = -\frac{\ddot{\rho}}{c}\vec{\nabla}\mathscr{r}$$

$$= -\frac{\ddot{\rho}}{c}\hat{\mathscr{r}}$$

② $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right) = \vec{\nabla}\cdot\frac{\vec{\mathscr{r}}}{\mathscr{r}^2} = \frac{1}{\mathscr{r}^2}\left(\vec{\nabla}\cdot\vec{\mathscr{r}}\right) + \vec{\mathscr{r}}\cdot\vec{\nabla}\frac{1}{\mathscr{r}^2}$

$$\vec{\nabla}\cdot\vec{\mathscr{r}} = \frac{\partial(x-x')}{\partial x} + \frac{\partial(y-y')}{\partial y} + \frac{\partial(z-z')}{\partial z} = 3$$

$$\vec{\nabla}\left(\frac{1}{\mathscr{r}^2}\right) = -\frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^2}\left\{2(x-x')\hat{x} + 2(y-y')\hat{y} + \ldots\right\}$$

$$\vec{\mathscr{r}}\cdot\vec{\nabla}\left(1/\mathscr{r}^2\right) = -2\,\frac{(x-x')^2+(y-y')^2+(z-z')^2}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^2}$$

$$= -\frac{2}{\mathscr{r}^2}$$

$$\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right) = 3/\mathscr{r}^2 - \frac{2}{\mathscr{r}^2} = \frac{1}{\mathscr{r}^2}$$

③ $\vec{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})$

since (see Griffiths page 45 and page 50)

$\vec{\nabla} \cdot \left(\hat{\mathscr{r}}/\mathscr{r}^2\right)$ is zero everywhere but $\mathscr{r} = 0$ where it is undefined

$\oint \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) \cdot d\vec{a} = 4\pi$

$\vec{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})$ by definition

Now put it all together:

$$\nabla^2 V_{ret} = -k \int \Big\{ \cancel{\frac{\dot{\rho}}{c}\frac{1}{\mathscr{r}^2}} + \frac{\hat{\mathscr{r}}}{c\mathscr{r}} \cdot \left(-\frac{\ddot{\rho}}{c}\right)\hat{\mathscr{r}} + \rho\, 4\pi\,\delta^3(\vec{\mathscr{r}})$$

$$+ \cancel{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \left(-\frac{\dot{\rho}}{c}\, \hat{\mathscr{r}}\right)} \Big\}\, d\tau'$$

$$= -\frac{\rho(\vec{r},t)}{\epsilon_0} + \frac{1}{c^2 4\pi\epsilon_0}\frac{\partial^2}{\partial t^2}\int \frac{\rho(r',t_r)}{\mathscr{r}}\, d\tau'$$

$$= \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{\rho(\vec{r},t)}{\epsilon_0}$$

The retarded potential is a solution of the inhomogeneous wave equation