PHYS 427: Discussion 1

Jan. 28, 2025

1. ***Binomial distribution.*** You are given a glass box containing a gas of N particles. The particles almost never collide with each other (we say they are approximately *non-interacting*). The particles are moving quickly around the box and bouncing off the walls. When you look closely, you notice that each particle is labeled with a number from 1 to N that distinguishes it from the other particles. You imagine dividing the box into M imaginary parts of equal volume, which we'll call *zones*.

   (a) With a camera, you take a picture of the box. Looking at the picture, you carefully mark down which zone each particle is in. For example, you might find:

particle	1	2	3	⋯	N
found in zone	3	8	1	⋯	3

   This table defines a *microstate* of the system. How many distinct microstates are there? In how many microstates is particle 3 located in the first zone?

   Each microstate is equally probable (make sure this sounds reasonable to you). From your previous answers, calculate the probability that particle 3 will be in the first zone when you take a picture.

   (b) You take many more pictures of the box. In each picture, you count how many particles are in the first zone, without caring about the labels on the particles. Then you take the average of these counts. Without doing any calculation, what do *expect* the average count to be? (We will calculate this in part (f).)

   (c) How many microstates have exactly n particles in the first zone (without caring about the labels on the particles)?

   (d) Using your answer to part (c), show that the probability of finding exactly n particles in the first zone is equal to

   $$\binom{N}{n} p^n (1-p)^{N-n},$$

   where

   $$\binom{N}{n} \equiv \frac{N!}{n!(N-n)!}$$

   is the **binomial coefficient** and $p = 1/M$ is the probability that you found in part (a).

   (e) Explain why the expected (i.e. average) number of particles in the first zone is given by

   $$\langle n \rangle = \sum_{n=0}^{N} n \binom{N}{n} p^n (1-p)^{N-n}.$$

   (The first zone isn't special, so this is the expected number of particles in any one zone.)

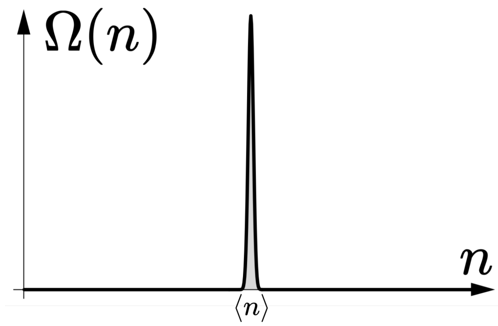
   (f) Compute the sum in part (e). Does it agree with your expectations from part (b)? *Hint: There are multiple ways to do this. A slick way involves taking a derivative of both sides of the* ***binomial identity****,*

   $$(p+q)^N = \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n},$$

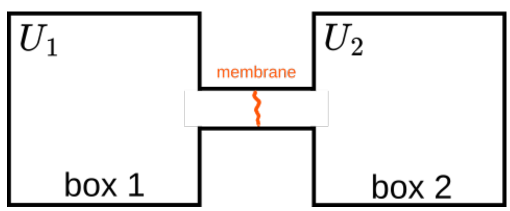
   which holds for any p and q. [If it's unfamiliar, prove it by multiplying out the left-hand side.]

2. ***Definition of temperature.*** In this problem, we will motivate the definition of temperature by considering two systems in thermal contact. It's the same thing you saw in lecture today, so if you're feeling comfortable with that, you can skip this problem.

Think back to problem 1(c). If we know the number n of particles in the first zone, we will say that we know the **macrostate**. What we calculated in problem 1(d) is called the **multiplicity** $\Omega(n)$ of the macrostate. Below is a sketch of $\Omega(n)$ when the total number of particles N is very large. From the plot, we see that the *vast majority* of microstates belong to a very small range of macrostates, namely those for which n is very close to the expected value $\langle n \rangle$. The **fundamental assumption of statistical mechanics** states that, for a closed system in thermal equilibrium, all microstates are equally likely. Therefore, the most likely *macrostate* by far is $n \approx \langle n \rangle$. ***Lesson: in equilibrium, the system settles into the macrostate with the highest multiplicity*** Ω***!!!***



Ok. Now forget about the zones, and focus your attention on the total energy of all the particles in the box. Someone gives you another box containing another gas. You connect the two boxes with an *energy-permeable barrier*. This barrier is simply a rubber membrane (a drumhead). Gas particles from box 1 can strike the membrane, transferring vibrational energy into the membrane. The vibrating membrane can then strike gas particles from box 2, transferring kinetic energy into box 2. In this way, energy can be exchanged between the boxes, but it cannot be created or destroyed. The particles themselves cannot pass through the membrane.



If we know the energy U_1 of the gas in box 1, we will say we know the macrostate of box 1. Similarly, U_2 specifies the macrostate of box 2. The multiplicity functions for the two boxes are denoted $\Omega_1(U_1)$ and $\Omega_2(U_2)$. (They are not necessarily the same function, because the boxes may be a different size and may contain different amounts of gas.)

(a) Show that, when the entire system (box 1 + box 2) settles into thermal equilibrium, we will have

$$\frac{\partial \ln \Omega_1}{\partial U_1} = \frac{\partial \ln \Omega_2}{\partial U_2}. \tag{1}$$

You have found a quantity which is the same for two systems in *thermal contact* when they reach thermal equilibrium. That's exactly the sort of behavior we expect of the quantity called "temperature". The **temperature** T_1 of box 1 is *defined* as

$$\frac{1}{T_1} \equiv k_B \frac{\partial \ln \Omega_1}{\partial U_1}, \tag{2}$$

where $k_B \approx 1.38 \times 10^{-23} J/K$ is called Boltzmann's constant, introduced for historical reasons. (In a perfect world, we would set k_B to 1 and measure temperature in the same units as energy.) We define the **entropy** of box 1 as

$$S_1 \equiv k_B \ln \Omega_1, \tag{3}$$

so that

$$\frac{1}{T_1} = \frac{\partial S_1}{\partial U_1}. \tag{4}$$

(b) Suppose that initially $T_1 > T_2 > 0$. Show that box 1 will lose energy (and box 2 will gain energy) as the combined system approaches equilibrium. We say heat flows from hot to cold.

3. ***Paramagnet.*** Consider a system of N identical non-interacting spin-1/2 particles that are are fixed in place, e.g. on the sites of a crystal lattice. Each particle can be in one of two states: spin pointed along $+\hat{\boldsymbol{z}}$ ("spin up", denoted $\uparrow$) or spin pointed along $-\hat{\boldsymbol{z}}$ ("spin down", denoted $\downarrow$). In a uniform external magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, a spin can lower its energy by aligning with the field. The energy of a particle is $-\mu B$ if it's spin-up and $+\mu B$ if it's spin-down. Here μ is the magnetic moment, which characterizes the strength of the interaction between the spin and the magnetic field.

   Treat the collection of spins as a closed system in thermal equilibrium.

   (a) Let $N_\uparrow$ denote the total number of particles that are spin-up. Write the total energy U of the system in terms of $N_\uparrow$ and the other variables defined above.

   (b) Compute the entropy S of the system as a function of U, N and B in the limit where N, $N_\uparrow$ and $N_\downarrow \equiv N - N_\uparrow$ are each much greater than 1. *Hint: First find the multiplicity Ω in terms of N, $N_\uparrow$ and $N_\downarrow$, then compute S using Stirling's approximation to simplify:* $\ln(q!) \approx q \ln q - q$ *when* $q \gg 1$. *Finally, use your answer to part (a) to express $N_\uparrow$ and $N_\downarrow$ in terms of U, N and B. The answer will not be pretty.*

   (c) Compute the energy of the system as a function of its temperature T defined by $T^{-1} \equiv (\partial S/\partial U)_{N,B}$. *Hint: to simplify your work, recall the definition of the hyperbolic tangent,* $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

   (d) Sketch a graph of $U(T)$. Conclude that at very low temperature, the vast majority of spins are aligned parallel to the applied magnetic field – this feature of the model is known as **paramagnetism**. It's also seen in more realistic models of spin systems and in real magnetic materials.

   (e) A typical spin magnetic moment is about 10^{-4} electronvolts per tesla. Estimate the ratio $N_\uparrow/N$ in a field of 1 tesla at room temperature. *Hint: No calculator needed! Room temperature is about* 300 *kelvin, and room temperature times k_B is about* $1/40 = 0.025$ *electronvolts. These are useful to memorize. Also note* $\tanh x \approx x$ *when* $x \ll 1$.