Solutions to Homework 8

Solutions to Multiple Choice Questions

1. **E** – A prediction interval based on a simple linear regression is a confidence interval for an anticipated value of Y given a specific value of X. The regression line allows us to make predictions about the response variable Y given a considered value of X. A prediction interval tells us the range of possible values for Y in which, with high confidence, we can locate the actual value of Y when it is observed.
2. **B** – In the simple regression model, the two variables involved have asymmetric roles and one variable is called an independent variable and the other is called a response variable. The regression model is about two variables observed in pairs, with one called an independent variable (also known as the explanatory variable or predictor) and the other is called a response variable (also known as the dependent or predicted variable). Their role is not interchangeable because one is used to explain variability of the other.
3. **B** – In the standard reporting of the results of a simple linear regression analysis, the *t*-ratios, which are the ratios of the regression coefficients to the corresponding standard errors, are used in tests of the statistical significance of the regression coefficients. The *t*-ratios are defined below. The first ratio is for determining if the intercept α is non-zero, and the second ratio is for determining if the slope β is non-zero.

$$\frac{\hat{\alpha}}{SE(\hat{\alpha})} \ and \ \frac{\hat{\beta}}{SE(\hat{\beta})}$$

4. **E** – The R^2 coefficient reports what proportion of variation in the response variable has been explained by its relation to the explanatory variable. If the total variation of the response variable Y is measured by the sum of squares of its deviations from its mean, then R^2 is proportional to this variation.
5. **A** – The correlation coefficient is closely related to the slope coefficient in simple linear regression; if one is 0, then so is the other. The correlation coefficient is another way of measuring the linear relationship between variables and thus is closely related to the slope coefficient. Namely, the relationship has the form:

$$\hat{\beta} = r\frac{S_Y}{S_X}$$

r	=	Correlation Coefficient
$\hat{\beta}$	=	Estimation of the slope

6. **B** – The correlation coefficient is positive when the values of the response variable tend to increase as the values of the explanatory variable increase. The correlation coefficient is always a number between -1 and 1, and its value indicates the strength of the linear relationship between two variables. A positive value indicates a proportional relationship i.e. increasing the explanatory variable will result in a proportional increase in the response variable. A negative value indicates an inverse proportional relationship i.e. increasing the explanatory variable will result in a proportional decrease in the response variable.

7. **D** – The residuals represent deviations of Y from the fitted line so they are no dependent on whether or not the slope is significantly non-zero. All residuals are equal to zero only if the data lie perfectly on a straight line.

Solutions to Problems

Answer to Question 1

- The fitted model we are using as a two sigma error band and is given by:

  $Y = 50 + 3.3X \pm 20$

  Y = Man Hours (Thousands)
  X = Pieces of Mail Handled (Millions)

  We are asked to predict the man hours for period 7, fiscal year 1962. The volume of mail associated with this year and period is 268. Therefore the predicted man hours will be given by:

  $Y_{1962,P7} = 50 + 3.3(268) \pm 20$

  $Y_{1962,P7} = 50 + 884.4 \pm 20$

  $Y_{1962,P7} = 934.4 \pm 20$

  We are asked to predict the man hours for period 6, fiscal year 1963. The volume of mail associated with this year and period is 180. Therefore the predicted man hours will be given by:

  $Y_{1963,P6} = 50 + 3.3(180) \pm 20$

  $Y_{1963,P6} = 50 + 594 \pm 20$

  $Y_{1963,P6} = 644 \pm 20$

  We are asked to predict the man hours for period 7, fiscal year 1963. The volume of mail associated with this year and period is 270. Therefore the predicted man hours will be given by:

  $Y_{1963,P7} = 50 + 3.3(270) \pm 20$

  $Y_{1963,P7} = 50 + 891 \pm 20$

  $Y_{1963,P7} = 941 \pm 20$

- The results are presented below in table form. The actual man hours are also given.

Year	Period	Pieces of Mail (Millions)	Predicted Man Hours (Thousands)	Actual Man Hours (Thousands)
1962	7	268	934.4 ± 20	1053
1963	6	180	644 ± 20	765
1963	7	270	941 ± 20	1070

For all three cases it can be seen that the prediction model underestimated the actual man hours.

- In the third part of this question we are asked to make predictions for three more periods. We are first asked to predict the man hours for period 6, fiscal year 1962. The volume of mail associated with this year and period is 184. Therefore the predicted man hours will be given by:

  $$Y_{1962,P6} = 50 + 3.3(184) \pm 20$$

  $$Y_{1962,P6} = 50 + 884.4 \pm 20$$

  $$Y_{1962,P6} = 657.2 \pm 20$$

  We are asked to predict the man hours for period 1, fiscal year 1963. The volume of mail associated with this year and period is 154. Therefore the predicted man hours will be given by:

  $$Y_{1963,P1} = 50 + 3.3(154) \pm 20$$

  $$Y_{1963,P1} = 50 + 508.2 \pm 20$$

  $$Y_{1963,P1} = 558.2 \pm 20$$

  We are asked to predict the man hours for period 5, fiscal year 1963. The volume of mail associated with this year and period is 191. Therefore the predicted man hours will be given by:

  $$Y_{1963,P7} = 50 + 3.3(191) \pm 20$$

  $$Y_{1963,P7} = 50 + 630.3 \pm 20$$

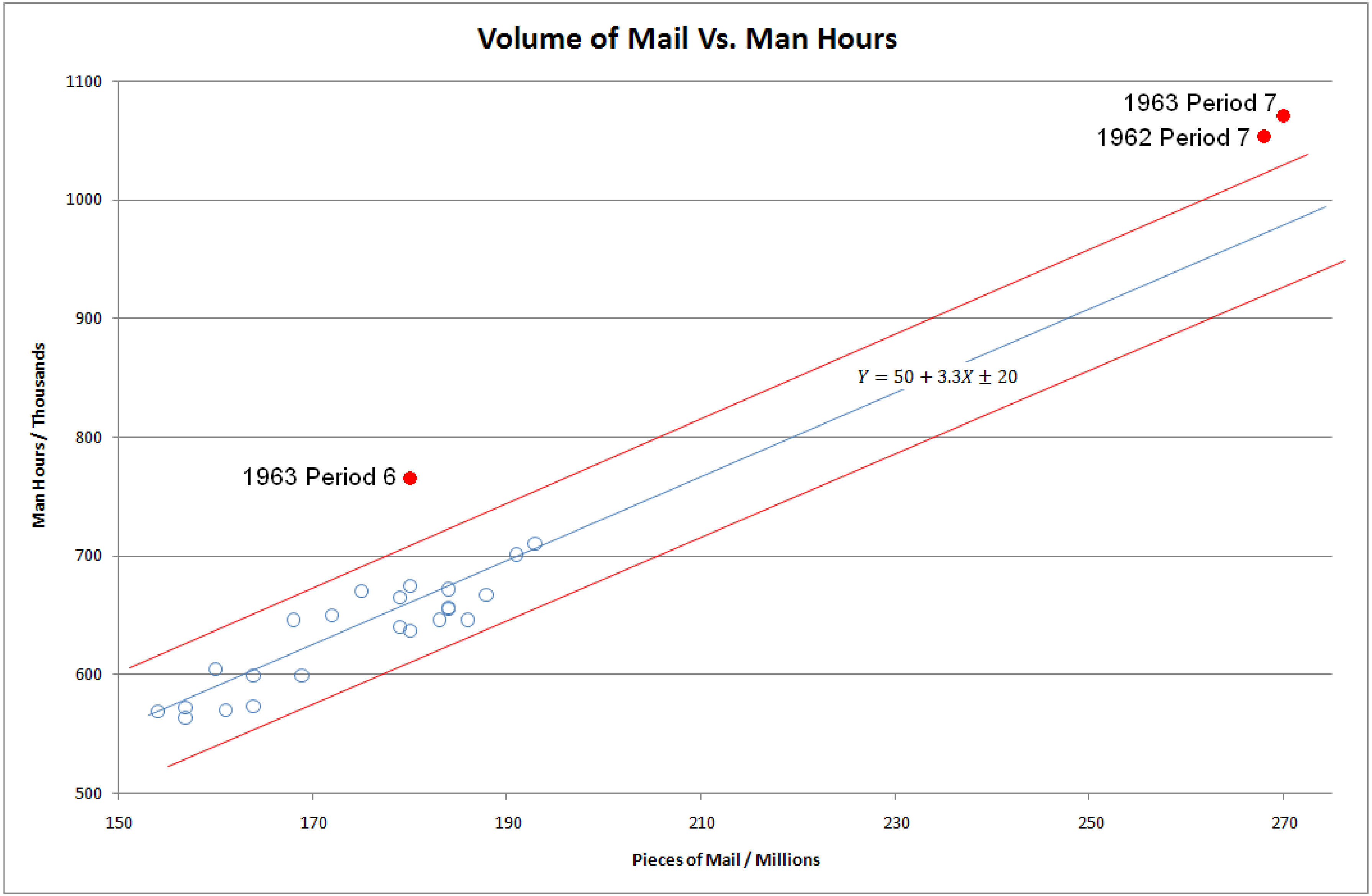
  $$Y_{1963,P7} = 680.3 \pm 20$$

- The results are presented below in table form. The actual man hours are also given.

Year	Period	Pieces of Mail (Millions)	Predicted Man Hours (Thousands)	Actual Man Hours (Thousands)
1962	6	184	657.2 ± 20	671
1963	1	154	558.2 ± 20	569
1963	5	191	680.3 ± 20	700

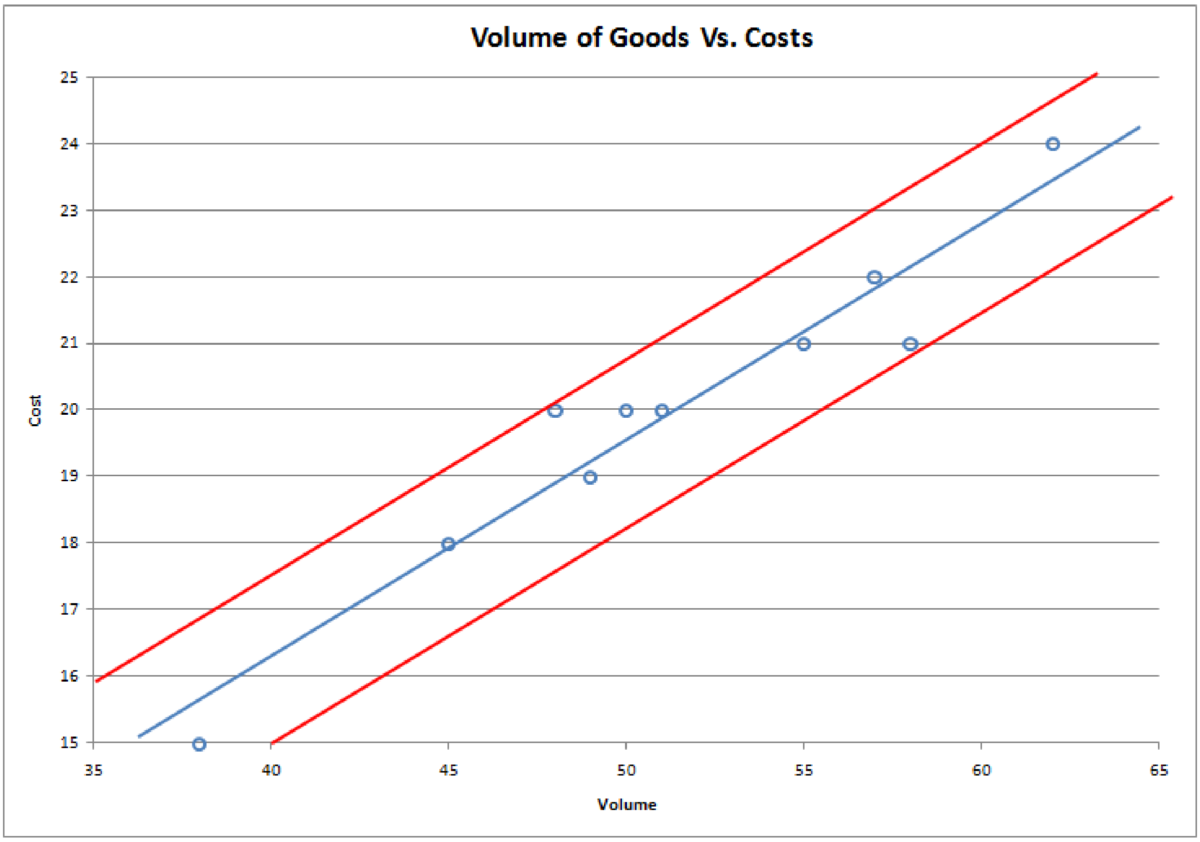
  For these three cases it can be seen that the actual (observed) values are within the two sigma error bands around the predicted values

- We are told in the problem that the three values in the first part of this question were not used when making the fitted model. We observe that the model underperforms for the first set of three values, and is quite accurate for the second set. One could suggest another model for the Christmas period, if more data were available for other Christmas periods.


Volume of Mail Vs. Man Hours
1963 Period 7
1962 Period 7
1963 Period 6
Y = 50 + 3.3X ± 20
Man Hours / Thousands
Pieces of Mail / Millions
1100
1000
900
800
700
600
500
150
170
190
210
230
250
270

Answer to Question 2

- The points are plotted on a graph shown below. The data does appear to follow a simple linear regression model, and there does not appear to be any suspicious points that should be excluded before fitting the model i.e. all the points appear to indicate a strong linear relationship between volume and cost.



- The simple linear regression model will have the form:

$\hat{Y} = \hat{\alpha} + \hat{\beta}X$

Where:

$\hat{\beta} = \dfrac{\sum_{i=1}^{n} X_i Y_i - n\overline{X}\overline{Y}}{\sum_{i=1}^{n} {X_i}^2 - n\overline{X}^2}$ $\qquad \hat{\alpha} = \overline{Y} - \hat{\beta}\overline{X}$

In the question we are told that:

$$\sum_{i=1}^{n} X_i = 513$$

$$\sum_{i=1}^{n} Y_i = 200$$

$$\sum_{i=1}^{n} {X_i}^2 = 26757$$

$$\sum_{i=1}^{n} {Y_i}^2 = 4052$$

$$\sum_{i=1}^{n} X_i Y_i = 10406$$

Firstly we must work out beta-hat using the equation:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{XY}}{\sum_{i=1}^{n} {X_i}^2 - n\bar{X}^2}$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i - n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)}{\sum_{i=1}^{n} {X_i}^2 - n\bar{X}^2} \quad Since \ldots\ldots\ldots\ldots..\overline{XY} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i - \frac{1}{n}(\sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} Y_i)}{\sum_{i=1}^{n} {X_i}^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2} \quad Since \ldots\ldots\ldots\ldots..\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

For our data we get:

$$\hat{\beta} = \frac{10406 - \frac{1}{10}(513)(200)}{26757 - 10\left(\frac{513}{10}\right)^2}$$

$$\hat{\beta} = \frac{10406 - 10260}{26757 - 10(51.3)^2}$$

$$\hat{\beta} = \frac{146}{26757 - 26316.9}$$

$$\hat{\beta} = \frac{146}{440.1}$$

$\hat{\beta} \approx 0.33$

Next we must work out alpha-hat using the equation:

$$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$$

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} Y_i - \hat{\beta}\bar{X} \quad Since \dots \dots \dots \dots . . \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} Y_i - \hat{\beta}\frac{1}{n}\sum_{i=1}^{n} X_i \quad Since \dots \dots \dots \dots . . \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

For our data we get:

$$\hat{\alpha} = \frac{1}{10}(200) - (0.33)\frac{1}{10}(513)$$

$$\hat{\alpha} = 20 - 0.33(51.3)$$

$$\hat{\alpha} = 20 - 16.929$$

$$\hat{\alpha} = 3.071$$

Therefore the fitted line is given by the below equation and is shown in the graph on page 6:

$$\hat{Y} = 3.071 + 0.33X$$

- The residuals for each of the ten points are the difference between the actual value, and the predicted value using the equation for the line.

Volume	Actual Cost	Predicted Cost	Residuals (e)
48	20	18.9	1.1
57	22	21.9	0.1
49	19	19.2	-0.2
45	18	17.9	0.1
50	20	19.6	0.4
62	24	23.5	0.5
58	21	22.2	-1.2
55	21	21.2	-0.2
38	15	15.6	-0.6
51	20	19.9	0.1

The estimator of the error term variance σ^2 is given by the formula:

$$S^2 = \frac{\sum_{i=1}^{n} {e_i}^2}{n-2}$$

Where:

e_i = Residual for sample i

For our data we get:

$$S^2 = \frac{([1.1]^2 + [0.1]^2 + [-0.2]^2 + [0.1]^2 + [0.4]^2 + [0.5]^2 + [-1.2]^2 + [-0.2]^2 + [-0.6]^2 + [0.1]^2)}{10-2}$$

$$S^2 \approx 0.44$$

- The estimated value of σ is given by:

$$\sigma = \sqrt{0.44}$$

$$\sigma = 0.66$$

Thus the two-sigma control limits are given by:

$$\hat{Y} = 3.071 + 0.33X \pm 2(0.66)$$

$$\hat{Y} = 3.071 + 0.33X \pm 1.33$$

The control limits are shown as red lines on the graph on page 6. All the points lie within the two-sigma control limits.

- The R^2 coefficient is given by the formula:

$$R^2 = 1 - \frac{\sum_{i=1}^{n} {e_i}^2}{\sum_{i=1}^{n} {Y_i}^2 - n\bar{Y}^2}$$

We can alter it as follows:

$$R^2 = 1 - \frac{\sum_{i=1}^{n} {e_i}^2}{\sum_{i=1}^{n} {Y_i}^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2} \quad Since \ldots\ldots\ldots\ldots.. \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$R^2 = 1 - \frac{\sum_{i=1}^{n} {e_i}^2}{4052 - 10\left(\frac{200}{10}\right)^2}$$

$$R^2 = 1 - \frac{\sum_{i=1}^{n} {e_i}^2}{4052 - 10(20)^2}$$

$$R^2 = 1 - \frac{\sum_{i=1}^{n} e_i^2}{4052 - 4000}$$

$$R^2 = 1 - \frac{\sum_{i=1}^{n} e_i^2}{52}$$

The sum of squared residuals is 3.57 therefore:

$$R^2 = 1 - \frac{3.57}{52}$$

$$R^2 = 1 - \frac{3.57}{52}$$

$$R^2 = 0.9313$$

We conclude therefore that 93% of the variation in costs has been explained by their relation to volume.

- The correlation coefficient is given by:

$$r = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{XY}}{\sqrt{\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)\left(\sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2\right)}}$$

We can alter it as follows:

$$r = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{XY}}{\sqrt{\left(\sum_{i=1}^{n} X_i^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right)\left(\sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2\right)}} \; Since \ldots\ldots\ldots\ldots.. \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$r = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{XY}}{\sqrt{\left(\sum_{i=1}^{n} X_i^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right)\left(\sum_{i=1}^{n} Y_i^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2\right)}} \; Since \ldots\ldots\ldots\ldots.. \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$r = \frac{\sum_{i=1}^{n} X_i Y_i - n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)}{\sqrt{\left(\sum_{i=1}^{n} X_i^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right)\left(\sum_{i=1}^{n} Y_i^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2\right)}} \; Since \ldots\ldots\ldots\ldots.. \overline{XY} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)$$

$$r = \frac{\sum_{i=1}^{n} X_i Y_i - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{\sqrt{\left(\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2\right)\left(\sum_{i=1}^{n} Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2\right)}}$$

For our data we get:

$$r = \frac{10406 - \frac{1}{10}(513)(200)}{\sqrt{\left(26757 - \frac{1}{10}(513)^2\right)\left(4052 - \frac{1}{10}(200)^2\right)}}$$

$$r = \frac{10406 - (513)(20)}{\sqrt{\left(26757 - \frac{1}{10}(513)^2\right)\left(4052 - \frac{1}{10}(200)^2\right)}}$$

$$r = \frac{10406 - 10260}{\sqrt{\left(26757 - \frac{1}{10}(513)^2\right)\left(4052 - \frac{1}{10}(200)^2\right)}}$$

$$r = \frac{146}{\sqrt{(26757 - 26316.9)(4052 - 4000)}}$$

$$r = \frac{146}{\sqrt{(440.1)(52)}}$$

$$r = \frac{146}{\sqrt{22885.2}}$$

$$r = \frac{146}{151.279}$$

$$r \approx 0.965$$

Since this value is close to one, we conclude that there is a strong linear relationship between volume and cost. In particular the reduction in the prediction error of Y given some value of X is approximately by factor:

$$\sqrt{\frac{(1 - r^2)(n - 1)}{n - 2}}$$

For our data we get:

$$\sqrt{\frac{(1 - 0.965^2)(10 - 1)}{10 - 2}}$$

$$\sqrt{\frac{(1 - 0.965^2)(9)}{8}}$$

$$0.278$$

This means that if we predict Y just using $\bar{Y}$, the prediction error based on the two-sigma rule is:

$$\pm 2S_Y$$

$$\pm 2\sqrt{\frac{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}{n-1}}$$

We can alter it as follows:

$$\pm 2\sqrt{\frac{\sum_{i=1}^{n} {Y_i}^2 - n\bar{Y}^2}{n-1}}$$

$$\pm 2\sqrt{\frac{\sum_{i=1}^{n} {Y_i}^2 - n\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2}{n-1}} \qquad Since \ldots\ldots\ldots\ldots.. \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

For our data:

$$\pm 2\sqrt{\frac{4052 - 10\left(\frac{200}{10}\right)^2}{10-1}}$$

$$\pm 2\sqrt{\frac{4052 - 10(20)^2}{9}}$$

$$\pm 2\sqrt{\frac{4052 - 10(400)}{9}}$$

$$\pm 2\sqrt{\frac{4052 - 4000}{9}}$$

$$\pm 2\sqrt{\frac{52}{9}}$$

$$\pm 2(2.40)$$

$$\pm 4.8$$

While if we predict Y using the regression model, then the approximated prediction error is:

$\pm 2S$

$$\pm 2\sqrt{\frac{\sum_{i=1}^{n}\left(Y_i - \hat{Y}_i\right)^2}{n-2}}$$

$$\pm 2\sqrt{\frac{\sum_{i=1}^{n} e_i{}^2}{n-2}}$$

For our data:

$$\pm 2\sqrt{\frac{3.57}{10-2}}$$

$$\pm 2\sqrt{\frac{3.57}{8}}$$

$\pm 2\sqrt{0.44625}$

± 1.336

We note quite a dramatic improvement in accuracy. Note: the ratio of 1.336 to 4.8 is approximately 0.278.

Answer to Question 3

The data for this problem is given below:

Year	Stamp Sales	Meter Sales
1964	234.8	121.3
1965	228.8	149.0
1966	230.1	153.7
1967	234.3	162.8
1968	238.6	169.3
1969	242.7	186.5
1970	226.4	197.5
1971	199.4	172.2
1972	205.4	192.8
1973	201.6	195.9
1974	191.1	199.6
1975	181.0	213.3
1976	174.9	240.9
1977	181.0	258.4

- The simple linear regression model will have the form:

  $\hat{Y} = \hat{\alpha} + \hat{\beta}X$

  $\hat{\alpha}$ = Intercept
  $\hat{\beta}$ = Slope

  From the computer readout we see that the intercept is 435.9786 and slope is -1.1752. Therefore, the equation for the least square fit to the data is:

  $\hat{Y} = 435.9786 - 1.1752X$

  The equation of the visually plotted line obtained from Homework 1, Problem 4 is:

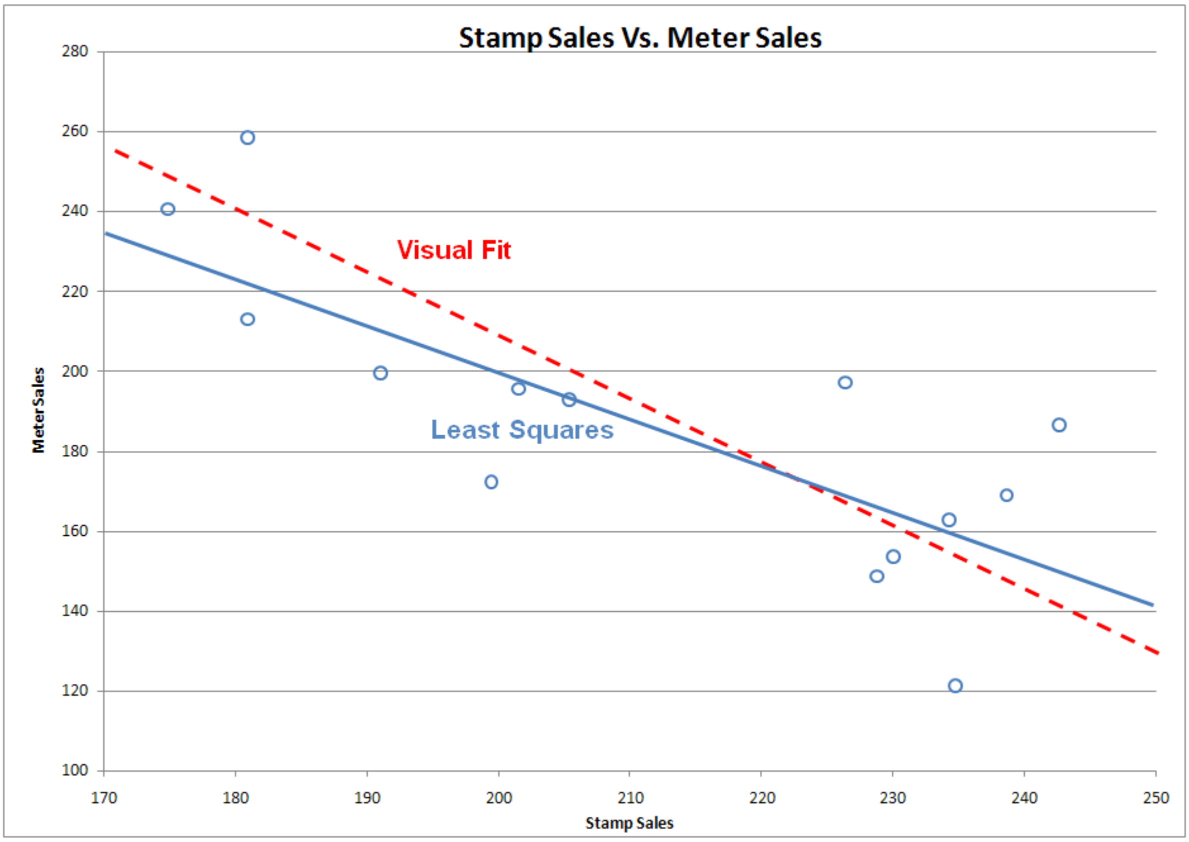
  $\hat{Y} = 520 - 1.5625X$

- The two lines are shown on the graph on the following page. We observe that the lines differ significantly, although they both capture the negative slope and the general inverse proportional relationship between stamp sales and meter sales.

- From the computer readout we see that the standard deviation is 23.5639. Therefore the two-sigma control limits for the fitted model are:

  $\hat{Y} = 435.9786 - 1.1752X \pm 2(23.5639)$

  $\hat{Y} = 435.9786 - 1.1752X \pm 47.1278$



- The null hypothesis is that the slope of the line is zero i.e. the line is horizontal. We will use a significance level of 5%. The critical value for this test we obtain from the Student's t-distribution table with *n - 2 = 12* degrees-of-freedom for a single tail of 2.5% which is 2.179. The standardised statistic for the slope coefficient is given by:

$$T = \frac{\hat{\beta}}{\widehat{SE}(\hat{\beta})}$$

$\hat{\beta}$	=	Beta-hat (-1.1752 from computer readout)
$\widehat{SE}(\hat{\beta})$	=	Standard Error of beta-hat (0.2716 from computer readout)
T	=	Test statistic

$$T = \frac{-1.1752}{0.2716}$$

$$T \approx -4.3277$$

Since the test statistic of -4.3277 lies outside the range of -2.179 and 2.179 we must reject the null hypothesis. There is strong statistical evidence to suggest that the slope should be non-zero. In fact the *p*-value as reported in the computer readout is 0.001 (.1%). As expected the *p*-

value is less than 5% since we already determined that the null hypothesis should be rejected. The value of 0.001 corresponds to the area under the Student's t-distribution curve with 12 degrees-of-freedom outside the range of [-4.3277, 4.3277].

- R-squared is a measure of variability that is explained by the linear regression between the two variables. From the computer readout it is shown to be 0.6095 or around 61%.

- The correlation between variables can be computed from the formula:

$$r = \hat{\beta}\frac{S_X}{S_Y}$$

$$r = -1.1752\frac{24.1}{36.2}$$

$$r = -0.78$$

- We use the below prediction formula with stamp sales (X) = 188.2

$$\hat{Y} = 435.9786 - 1.1752X \pm 47.1278$$

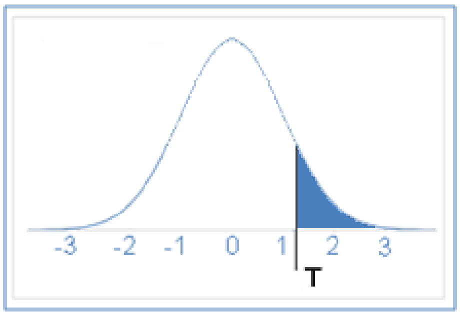
$$\hat{Y} = 435.9786 - 1.1752(188.2) \pm 47.1278$$

$$\hat{Y} = 435.9786 - 221.17264 \ \pm 47.1278$$

$$\hat{Y} \approx 215 \ \pm 47.13$$

$$\hat{Y} \ is \ in \ the \ range \ of \ [167.68, 261.92]$$

- The actual value of 240.8 is within the predicted two-sigma range of [167.68, 261.92]



Student’s t-Distribution Table[1]

Area	10%	5%	4%	2.5%	2.0%	1%	0.5%	0.25%	0.1%	0.05%
DoF										
1	3.078	6.314	7.916	12.706	15.894	31.821	63.656	127.321	318.289	636.578
2	1.886	2.920	3.320	4.303	4.849	6.965	9.925	14.089	22.328	31.600
3	1.638	2.353	2.605	3.182	3.482	4.541	5.841	7.453	10.214	12.924
4	1.533	2.132	2.333	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	1.476	2.015	2.191	2.571	2.757	3.365	4.032	4.773	5.894	6.869
6	1.440	1.943	2.104	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	1.415	1.895	2.046	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	1.397	1.860	2.004	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	1.383	1.833	1.973	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	1.372	1.812	1.948	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	1.363	1.796	1.928	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	1.356	1.782	1.912	$\boxed{2.179}$	2.303	2.681	3.055	3.428	3.930	4.318
13	1.350	1.771	1.899	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	1.345	1.761	1.887	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	1.341	1.753	1.878	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	1.337	1.746	1.869	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	1.333	1.740	1.862	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	1.330	1.734	1.855	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	1.328	1.729	1.850	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	1.325	1.725	1.844	2.086	2.197	2.528	2.845	3.153	3.552	3.85
21	1.323	1.721	1.840	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	1.321	1.717	1.835	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	1.319	1.714	1.832	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	1.318	1.711	1.828	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	1.316	1.708	1.825	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	1.315	1.706	1.822	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	1.314	1.703	1.819	2.052	2.158	2.473	2.771	3.057	3.421	3.689
28	1.313	1.701	1.817	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	1.311	1.699	1.814	2.045	2.150	2.462	2.756	3.038	3.396	3.660
30	1.310	1.697	1.812	2.042	2.147	2.457	2.75	3.030	3.385	3.646
inf	1.282	1.654	1.751	1.960	2.054	2.326	2.576	2.807	3.090	3.290

[1] Values used in the problems are boxed in red.