Solutions to Homework 6

Solutions to Multiple Choice Questions

1. **E** – We use the Student's t-distribution for data that follows the normal model with unknown standard deviation, especially when sample sizes are small (usually under 30). It is used as a replacement for the normal distribution model. As the sample size increases, the Student's t-distribution moves towards corresponding with the normal distribution model.
2. **B** – The paired sample test is used to test the mean difference between elements of sampled pairs of values. It computes the difference between the two variables for each case, and tests to see if the average difference is significantly different from zero.
3. **B** – If we have two independent samples that are different and of small sample size, then we use the Student's t-distribution to test the difference of their means, provided their variances are equal.
4. **C** – The observed significance level (also called the *p*-value) can be used for testing by rejecting the null hypothesis if it is smaller than the significance level α, usually 5%. We can use this test in lieu of the test statistic and comparing that with the critical value for the significance level.
5. **A** – The significance level α is equal to the probability of falsely rejecting the null-hypothesis (Type I error). The significance level is generally set at 5%.

Solutions to Problems

Answer to Question 1

We are checking the null hypothesis that the means of the two samples are equal i.e.

H_O : $\mu_1 = \mu_2$

H_a : $\mu_1 \neq \mu_2$

Since the null hypothesis we are checking only uses the equality symbol (=) the test is two-sided.

- Because the sample sizes are small, we need to use the following test statistic:

  $$T = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$\bar{X}_1$	=	Sample mean of first dataset (78)
$\bar{X}_2$	=	Sample mean of second dataset (79)
n_1	=	Sample size of first dataset (9)
n_2	=	Sample size of second dataset (7)
S_p	=	Pooled standard deviation

  The pooled standard deviation is given by:

  $$S_p = \sqrt{\frac{{s_1}^2(n_1 - 1) + {s_2}^2(n_2 - 1)}{(n_1 - 1) + (n_2 - 1)}}$$

s_1	=	Standard deviation of first dataset (9.49)
s_2	=	Standard deviation of second dataset (6.88)

- This statistic follows the Student's t-distribution with ($n_1 + n_2 - 2$) degrees of freedom.

- We obtain the critical value for this test from the Student's t-distribution table. We know that the significance level is 5%, and since the test is two-sided when we consult the table we check for just one tail which will be 2.5%. We also know that the degrees-of-freedom is given by:

  $$Degrees\ of\ Freedom = (n_1 - 1) + (n_2 - 1)$$

  So the degrees-of-freedom for this test will be 14. Therefore the critical value for this test from the Student's t-distribution table is 2.145.

- To evaluate the test statistic we first need to determine the pooled standard deviation:

$$S_p = \sqrt{\frac{(9.49)^2(9-1) + (6.88)^2(7-1)}{9+7-2}}$$

$$S_p = \sqrt{\frac{8(9.49)^2 + 6(6.88)^2}{14}}$$

$$S_p = \sqrt{\frac{4(9.49)^2 + 3(6.88)^2}{7}}$$

$$S_p = \sqrt{\frac{4(90.061) + 3(47.3344)}{7}}$$

$$S_p = \sqrt{\frac{502.2472}{7}}$$

$$S_p = \sqrt{71.7496}$$

Now we work out the test statistic:

$$T = \frac{78-79}{\sqrt{71.7496}\sqrt{\frac{1}{9}+\frac{1}{7}}}$$

$$T = \frac{-1}{\sqrt{71.7496}\sqrt{\frac{7}{63}+\frac{9}{63}}}$$

$$T = \frac{-1}{\sqrt{71.7496}\sqrt{\frac{16}{63}}}$$

$$T = \frac{-1}{\sqrt{71.7496}}\sqrt{\frac{63}{16}}$$

$$T = -\sqrt{\frac{63}{1147.9936}}$$

$$T = -0.234$$

- Out test statistics is -0.234. In order to reject the null hypothesis the value should be either below the negative critical value of -2.145 or above the positive critical value of 2.145.

- Because our test statistic lies within the range of -2.145 and 2.145 we conclude that we do not reject the null hypothesis i.e. there is no significant difference in mean scores on the business statistics course between male and female students.

Answer to Question 2

- For the first part of this question using sample data, we want to test the null hypothesis that the mean tar content of cigarettes is less than or equal to 14mg per cigarette with a significance level of 5%.

H_O : $\mu \leq 14$

H_a : $\mu_1 > 14$

Since the null hypothesis we are checking includes the less-than symbol (<) the test is one-sided. Because the sample size is greater than 30, the test statistic is given by:

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$\bar{X}$	=	Sample mean (14.1)
μ	=	Theoretical mean (14)
σ	=	Standard deviation (0.3) (assume same as sample standard deviation *s*)
n	=	Sample size (100)
Z	=	Number of standard deviations from the mean

This statistic follows the normal distribution. The critical value for this test we obtain from the normal distribution table. We know that the significance level is 5%, and since the test is one-sided when we consult the table we check for just the right tail which will be 5%. When we check the table however, to find what value of Z_c will give us a value of 95% we find that 1.64 gives us a value of 94.95% and 1.65 gives us a value of 95.05%. Midway between these two percentages is 95%, therefore the Z-value will also be midway between the two corresponding values of 1.64 and 1.65. Therefore the critical value for this test from the normal distribution table is 1.645. Next, we calculate the test statistic.

$$Z = \frac{14.1 - 14}{\frac{0.3}{\sqrt{100}}}$$

$$Z = \frac{14.1 - 14}{\frac{0.3}{10}}$$

$$Z = \frac{0.1}{0.03}$$

$$Z = 3.33$$

Since 3.33 exceeds the critical value of 1.65, we reject the null hypothesis.

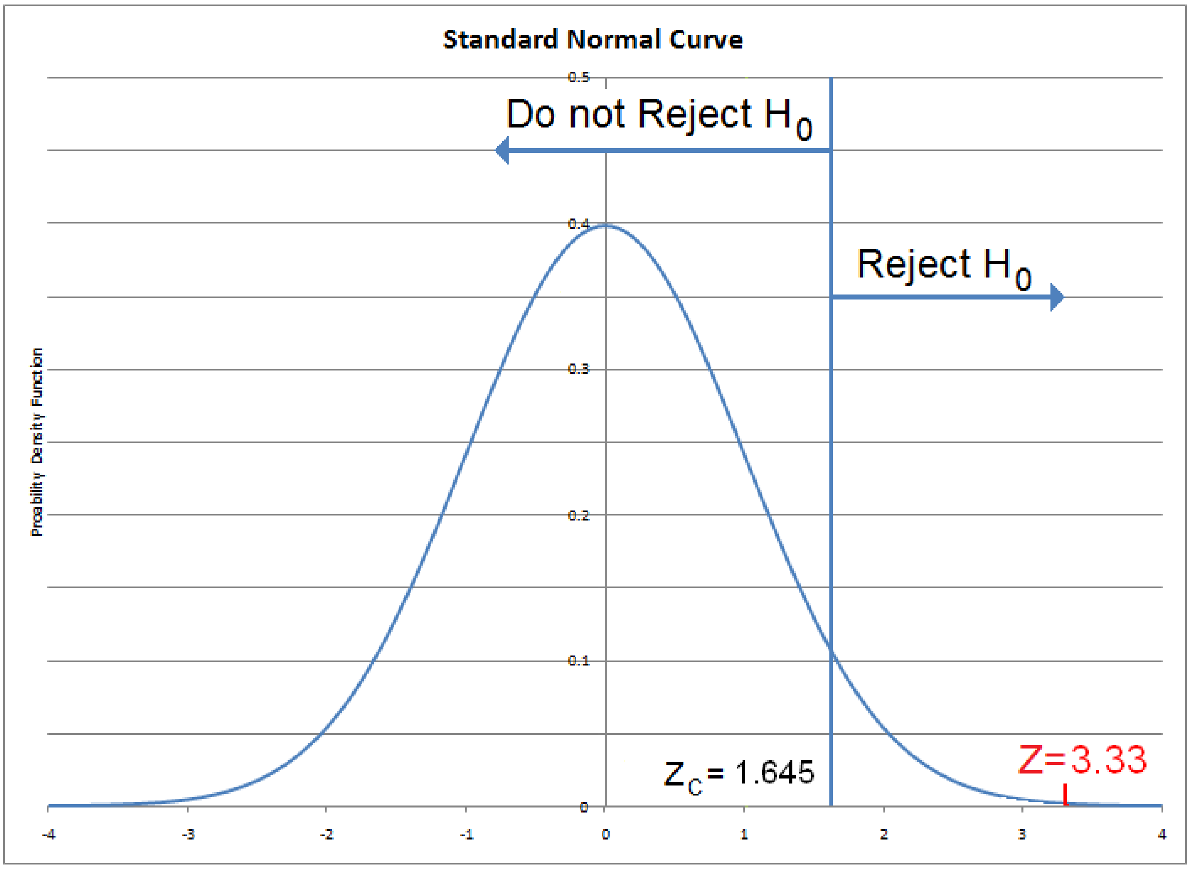
In the below graph, we see that the test statistic is 3.33 and is greater than the cutoff Z value of 1.645 (the area under the curve to the right of 1.645 is 5%). The *p*-value will be the area under the curve from 3.33 to plus infinity and will be given by:

$$Area = 1 - Z(3.33)$$

$$Area = 1 - 0.9995$$

$$Area = 0.0005$$

Therefore our *p*-value is 0.05%, which is very small. This means that if the null hypothesis is true, and the theoretical mean is less than or equal to 14 mg/cigarette, then we would only have a 0.05% chance of observing values at least as high as 14.1 mg/cigarette, the sample mean. Comparing the *p*-value with the significance level is an alternative way of testing the null hypothesis; since our *p*-value is significantly smaller than 5% we reject the null hypothesis.



- For the second part of this question using sample data from a different brand of cigarettes, we again want to test the null hypothesis that the mean tar content of these cigarettes is less than or equal to 14mg per cigarette with a significance level of 5%.

  H_O : $\mu \leq 14$

  H_a : $\mu_1 > 14$

  Since the null hypothesis we are checking includes the less-than symbol (<) the test is one-sided. Because the sample size is greater than 30, the test statistic is given by:

  $$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$\bar{X}$	=	Sample mean (14.2)
μ	=	Theoretical mean (14)
σ	=	Standard deviation (1.3) (assume same as sample standard deviation *s*)
n	=	Sample size (50)
Z	=	Number of standard deviations from the mean

  Again, like in the first part of this question the critical value for this test is 1.645.

  $$Z = \frac{14.2 - 14}{\frac{1.3}{\sqrt{50}}}$$

  $$Z = \frac{0.2}{\frac{1.3}{\sqrt{50}}}$$

  $$Z = 0.2\frac{\sqrt{50}}{1.3}$$

  $$Z = \frac{2\sqrt{50}}{13}$$

  $$Z = 1.09$$

  Since 1.09 does not exceed the critical value of 1.65, we do not reject the null hypothesis.

- The reason we do not reject the null hypothesis in the second brand is due the larger variability in the product (evidenced by a standard deviation of 1.3 compared with 0.3 in the first brand). In order to test this brand more effecivtly a larger sample size is required to account for the larger variability. This emphasises the fact that when testing the quality of a product it is very important to analyse variability as measured by standard deviation, together with the mean.

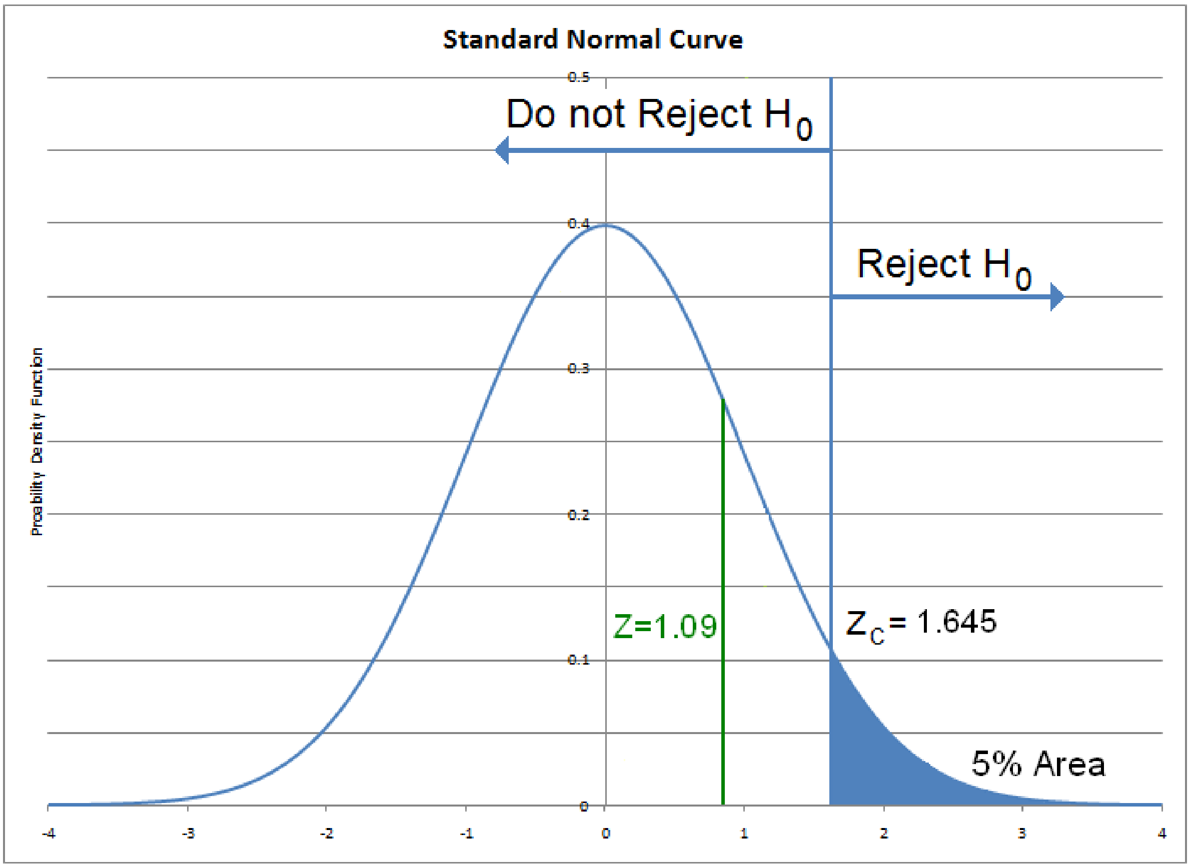
- In the below graph, we see that the test statistic is 1.09 and is less than the cutoff Z value of 1.09(the area under the curve to the right of 1.645 is 5%). The *p*-value will be the area under the curve from 1.09 to plus infinity and will be given by:

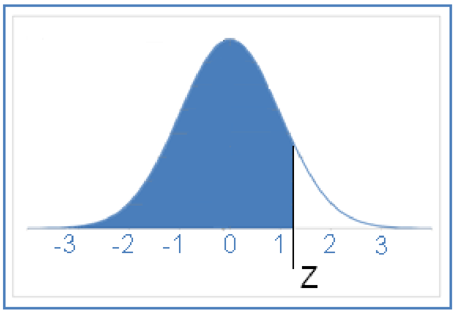
  $Area = 1 - Z(1.09)$

  $Area = 1 - 0.8621$

  $Area = 0.1379$

  Therefore our *p*-value is 13.79%. This means that if the null hypothesis is true, and the theoretical mean is less than or equal to 14 mg/cigarette, then we would only have a 13.79% chance of observing values at least as high as 13.79 mg/cigarette, the sample mean. Comparing the *p*-value with the significance level is an alternative way of testing the null hypothesis; since our *p*-value is greater than 5% we do not reject the null hypothesis.

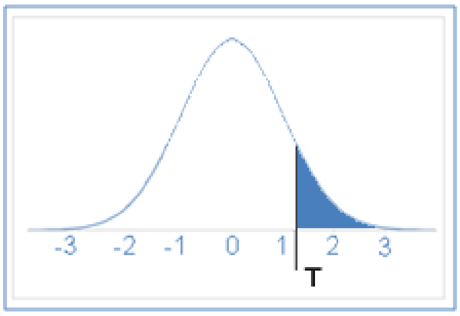




Z-Table (Area Under the Curve[1])

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997

[1] Values used in the problems are boxed in red.



Student’s t-Distribution Table[2]

Area	10%	5%	4%	2.5%	2.0%	1%	0.5%	0.25%	0.1%	0.05%
DoF										
1	3.078	6.314	7.916	12.706	15.894	31.821	63.656	127.321	318.289	636.578
2	1.886	2.920	3.320	4.303	4.849	6.965	9.925	14.089	22.328	31.600
3	1.638	2.353	2.605	3.182	3.482	4.541	5.841	7.453	10.214	12.924
4	1.533	2.132	2.333	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	1.476	2.015	2.191	2.571	2.757	3.365	4.032	4.773	5.894	6.869
6	1.440	1.943	2.104	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	1.415	1.895	2.046	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	1.397	1.860	2.004	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	1.383	1.833	1.973	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	1.372	1.812	1.948	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	1.363	1.796	1.928	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	1.356	1.782	1.912	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	1.350	1.771	1.899	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	1.345	1.761	1.887	$\boxed{2.145}$	2.264	2.624	2.977	3.326	3.787	4.140
15	1.341	1.753	1.878	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	1.337	1.746	1.869	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	1.333	1.740	1.862	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	1.330	1.734	1.855	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	1.328	1.729	1.850	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	1.325	1.725	1.844	2.086	2.197	2.528	2.845	3.153	3.552	3.85
21	1.323	1.721	1.840	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	1.321	1.717	1.835	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	1.319	1.714	1.832	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	1.318	1.711	1.828	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	1.316	1.708	1.825	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	1.315	1.706	1.822	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	1.314	1.703	1.819	2.052	2.158	2.473	2.771	3.057	3.421	3.689
28	1.313	1.701	1.817	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	1.311	1.699	1.814	2.045	2.150	2.462	2.756	3.038	3.396	3.660
30	1.310	1.697	1.812	2.042	2.147	2.457	2.75	3.030	3.385	3.646
inf	1.282	1.654	1.751	1.960	2.054	2.326	2.576	2.807	3.090	3.290

[2] Values used in the problems are boxed in red.