Time Series – Tutorial 6 – Solutions

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Question 5

For all of the below we need to calculate $1.96\frac{1}{\sqrt{n}}$. Any autocorrelations whose magnitude is larger than this are statistically significant.

a) $1.96\frac{1}{\sqrt{100}} = 0.196$.
The first three autocorrelations are greater than 0.196 suggesting an MA(3). However, $r_3 = -0.21$ is only slightly bigger than this and, therefore, we might consider an MA(2).

b) $1.96\frac{1}{\sqrt{121}} = 0.178$.
The first two partial autocorrelations are greater than 0.178 suggesting an AR(2).

c) $1.96\frac{1}{\sqrt{121}} = 0.151$.
All of these autocorrelations are above this limit and are decaying. This may suggest an AR(1) model. However, recall that $\rho_k = \phi^k$, i.e., $\rho_1 = \phi$ and this is also the rate of decay. However, $r_5 \neq (0.41)^5 \approx 0.012$ which suggests the decay is slower than we would expect for an AR(1). Perhaps an AR(2) is appropriate or maybe an ARMA(1,1) model. The PACF and EACF would help us further but these are unavailable here.

d) For the ACF of Y_t the cut-off is $1.96\frac{1}{\sqrt{100}} = 0.196$. However, when you difference the data, you lose one data point so the cut-off for the ACF of ∇Y_t is $1.96\frac{1}{\sqrt{99}} = 0.197$. Of course this is a minor technical point here with a sample size of 100.
For Y_t, the ACF values fail to decay quickly (and all are statistically significant) which may indicate non-stationarity. On the other hand, for ∇Y_t the first two are significant - however the second one is only just significant. We would suggest an MA(1) model for ∇Y_t, i.e., an IMA(1,1) for Y_t.

Question 6

a)

$$\begin{aligned} Y_t &= \alpha Y_{t-1} + e_t \\ Y_t - Y_{t-1} &= \alpha Y_{t-1} - Y_{t-1} + e_t \\ \nabla Y_t &= (\alpha - 1)Y_{t-1} + e_t \\ &= \beta Y_{t-1} + e_t \end{aligned}$$

as required.

Question 7

a) $Y_t = \alpha Y_{t-1} + X_t$

$$\begin{aligned} \Rightarrow X_t &= Y_t - \alpha Y_{t-1} \\ &= Y_t - \alpha Y_{t-1} + \alpha Y_t - \alpha Y_t \\ &= (1-\alpha)Y_t + \alpha(Y_t - Y_{t-1}) \\ &= (1-\alpha)Y_t + \alpha \nabla Y_t. \end{aligned}$$

b)

$$\begin{aligned} Y_t &= \alpha Y_{t-1} + X_t \\ Y_t - Y_{t-1} &= \alpha Y_{t-1} - Y_{t-1} + X_t \\ \nabla Y_t &= (\alpha - 1)Y_{t-1} + X_t \\ &= (\alpha - 1)Y_{t-1} + \phi X_{t-1} + e_t \\ &= (\alpha - 1)Y_{t-1} + \phi[(1-\alpha)Y_{t-1} + \alpha\nabla Y_{t-1}] + e_t \\ &= (\alpha - 1)(1-\phi)Y_{t-1} + \alpha\phi\nabla Y_{t-1} + e_t \\ &= \beta Y_{t-1} + \alpha\phi\nabla Y_{t-1} + e_t \end{aligned}$$

c) When $\beta = 0$ we have

$$\begin{aligned} (\alpha - 1)(1 - \phi) &= 0 \\ \Rightarrow \alpha &= 1 \\ \text{or } \phi &= 1 \end{aligned}$$

However for an AR(1) process, $|\phi| < 1$, i.e., we cannot have $\phi = 1$. Hence, $\beta = 0 \Rightarrow \alpha = 1$.

Question 8

a) An AR(p) process is defined by

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} \cdots + \phi_p Y_{t-p} + e_t$$

For $k > p$ we have

$$\begin{aligned} \hat{Y}_t &= E(Y_t \,|\, Y_{t-1}, Y_{t-2}, \ldots Y_{t-k+1}) \\ &= E(\phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + e_t \,|\, Y_{t-1}, \ldots Y_{t-k+1}). \end{aligned}$$

Note that $k > p$. We *know* the values of $Y_{t-1}, \ldots, Y_{t-k+1}$ (this is what "given" means above).

Consider what this means:

$$\begin{aligned} k = p+1 &\Rightarrow Y_{t-1}, \ldots, Y_{t-p} \\ k = p+2 &\Rightarrow Y_{t-1}, \ldots, Y_{t-p}, Y_{t-p-1} \\ &\vdots \\ k > p &\Rightarrow Y_{t-1}, \ldots, Y_{t-p}, \ldots, Y_{t-k+1} \end{aligned}$$

i.e., we know *all* of the Y variables in $\phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p}$ and hence the above conditional expectation becomes

$$\hat{Y}_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + E(e_t)$$

where $E(e_t \,|\, Y_{t-1}, \ldots Y_{t-k+1}) = E(e_t)$ since e_t is independent of the past.

$$\begin{aligned} \Rightarrow \hat{Y}_t &= \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + 0 \\ &= \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p}. \end{aligned}$$

Thus, the residual is

$$\begin{aligned} Y_t - \hat{Y}_t &= \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + e_t \\ &\quad - (\phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p}) \\ &= e_t. \end{aligned}$$

Thus the the partial autocorrelation function (for $k > p$) is given by

$$\begin{aligned} \tau_k &= \mathrm{Corr}(Y_t - \hat{Y}_t, Y_{t-k} - \hat{Y}_{t-k}) \\ &= \mathrm{Corr}(e_t, Y_{t-k} - \hat{Y}_{t-k}) \\ &= 0 \end{aligned}$$

since e_t is independent of the past and $Y_{t-k} - \hat{Y}_{t-k}$ is a function of the past values in the series $Y_{t-1}, Y_{t-2}, \ldots Y_{t-k+1}, Y_{t-k}$, i.e.,

$$Y_{t-k} - \hat{Y}_{t-k} = Y_{t-k} - E(Y_{t-k} \,|\, Y_{t-1}, Y_{t-2}, \ldots Y_{t-k+1}).$$

Question 9

a)
$$\begin{aligned} \mathrm{Var}(Y_t - \hat{Y}_t) &= \mathrm{Var}(Y_t - \beta Y_{t-1}) \\ &= \mathrm{Var}(Y_t) + \beta^2 \mathrm{Var}(Y_{t-1}) \\ &\quad - 2\beta \mathrm{Cov}(Y_t, Y_{t-1}) \\ &= \gamma_0 + \beta^2 \gamma_0 - 2\beta\gamma_1 \\ &\qquad \text{(by stationarity of } Y_t\text{)} \end{aligned}$$

b)
$$\tfrac{d}{d\beta} \mathrm{Var}(Y_t - \hat{Y}_t) = 2\beta\gamma_0 - 2\gamma_1.$$

Set this equal to zero to minimize:

$$\begin{aligned} 2\beta\gamma_0 - 2\gamma_1 &= 0 \\ 2\beta\gamma_0 &= 2\gamma_1 \\ \beta &= \frac{\gamma_1}{\gamma_0} = \rho_1. \end{aligned}$$

c)
$$\begin{aligned} \mathrm{Var}(Y_{t-2} - \hat{Y}_{t-2}) &= \mathrm{Var}(Y_{t-2} - \beta Y_{t-1}) \\ &= \mathrm{Var}(Y_{t-2}) + \beta^2 \mathrm{Var}(Y_{t-1}) \\ &\quad - 2\beta \mathrm{Cov}(Y_{t-2}, Y_{t-1}) \\ &= \gamma_0 + \beta^2\gamma_0 - 2\beta\gamma_{-1} \\ &\qquad \text{(by stationarity of } Y_t\text{)} \\ &= \gamma_0 + \beta^2\gamma_0 - 2\beta\gamma_1. \end{aligned}$$

This is exactly the same function as in part (a) and, hence, is also maximised at $\beta = \rho_1$.

d)
$$\begin{aligned} &\mathrm{Cov}(Y_t - \hat{Y}_t, Y_{t-2} - \hat{Y}_{t-2}) \\ &= \mathrm{Cov}(Y_t - \rho_1 Y_{t-1}, Y_{t-2} - \rho_1 Y_{t-1}) \\ &= \mathrm{Cov}(Y_t, Y_{t-2}) - \rho_1 \mathrm{Cov}(Y_t, Y_{t-1}) \\ &\qquad - \rho_1 \mathrm{Cov}(Y_{t-1}, Y_{t-2}) + \rho_1^2 \mathrm{Cov}(Y_{t-1}, Y_{t-1}) \\ &= \gamma_2 - \rho_1\gamma_1 - \rho_1\gamma_1 + \rho_1^2\gamma_0 \\ &= \gamma_2 - 2\rho_1\gamma_1 + \rho_1^2\gamma_0 \\ &= \gamma_0(\rho_2 - 2\rho_1\rho_1 + \rho_1^2) \\ &= \gamma_0(\rho_2 - 2\rho_1^2 + \rho_1^2) \\ &= \gamma_0(\rho_2 - \rho_1^2) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(Y_t - \hat{Y}_t) = \mathrm{Var}(Y_{t-2} - \hat{Y}_{t-2}) &= \gamma_0 + \beta^2\gamma_0 - 2\beta\gamma_1 \\ &\qquad \text{(from parts (a) and (c))} \\ &= \gamma_0 + \rho_1^2\gamma_0 - 2\rho_1\gamma_1 \\ &\qquad (\beta = \rho_1) \\ &= \gamma_0(1 + \rho_1^2 - 2\rho_1\rho_1) \\ &= \gamma_0(1 - \rho_1^2) \end{aligned}$$

$$\begin{aligned} \Rightarrow \tau_2 &= \mathrm{Corr}(Y_t - \hat{Y}_t, Y_{t-2} - \hat{Y}_{t-2}) \\ &= \frac{\gamma_0(\rho_2 - \rho_1^2)}{\sqrt{\gamma_0(1 - \rho_1^2)\gamma_0(1 - \rho_1^2)}} \\ &= \frac{\gamma_0(\rho_2 - \rho_1^2)}{\gamma_0(1 - \rho_1^2)} \\ &= \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}. \end{aligned}$$

e) For an AR(1) process, $\rho_1 = \phi$ and $\rho_2 = \phi^2$. Thus,

$$\begin{aligned} \tau_2 &= \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} \\ &= \frac{\phi^2 - \phi^2}{1 - \phi^2} = \frac{0}{1 - \phi^2} = 0. \end{aligned}$$

For an AR(2) process, $\rho_1 = \frac{\phi_1}{1-\phi_2}$ and $\rho_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1-\phi_2}$.

$$\begin{aligned} \rho_2 - \rho_1^2 &= \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1 - \phi_2} - \frac{\phi_1^2}{(1-\phi_2)^2} \\ &= \frac{\phi_1^2 + \phi_2 - \phi_2^2 - \phi_2\phi_1^2 - \phi_2^2 + \phi_2^3 - \phi_1^2}{(1-\phi_2)^2} \\ &= \frac{\phi_2(1 - 2\phi_2 + \phi_2^2 - \phi_1^2)}{(1-\phi_2)^2} \end{aligned}$$

and

$$\begin{aligned} 1 - \rho_1^2 &= 1 - \frac{\phi_1^2}{(1-\phi_2)^2} \\ &= \frac{1 - 2\phi_2 + \phi_2^2 - \phi_1^2}{(1-\phi_2)^2} \end{aligned}$$

$$\Rightarrow \tau_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \phi_2.$$