Dr. Kevin Burke

Question 7

a) $E(aX+b) = aE(X)+b$

$$\begin{aligned} E(aX+b) &= \int (ax+b)\, f(x)\, dx \\ &= \int ax\, f(x)\, dx + \int b\, f(x)\, dx \\ &= a\int x\, f(x)\, dx + b\int f(x)\, dx \\ &= aE(X) + b(1) \end{aligned}$$

(by definition of $E(X)$ and fact that $\int f(x)\, dx = 1$)

$$= aE(X)+b$$

b) $E(aX+bY) = aE(X)+bE(Y)$
Let $f(x,y)$ be the joint density function for X and Y. Now we have

$$\begin{aligned} &E(aX+bY) \\ &= \int_y\int_x (ax+by)\, f(x,y)\, dx\, dy \\ &= \int_y\int_x ax\, f(x,y)\, dx\, dy + \int_y\int_x by\, f(x,y)\, dx\, dy \\ &= a\int_x x \underbrace{\int_y f(x,y)\, dy}_{f(x)}\, dx + b\int_y y \underbrace{\int_x f(x,y)\, dx}_{f(y)}\, dy \end{aligned}$$

(marginal probability)

$$\begin{aligned} &= a\int_x x\, f(x)\, dx + b\int_y y\, f(y)\, dy \\ &= aE(X)+bE(Y) \end{aligned}$$

c) $\mathrm{Var}(X) = E(X^2) - (EX)^2$

$$\begin{aligned} \mathrm{Var}(X) &= E(X-EX)^2 \\ &= E(X^2 - 2XEX + (EX)^2) \\ &= E(X^2) - 2EXEX + (EX)^2 \\ &= E(X^2) - 2(EX)^2 + (EX)^2 \\ &= E(X^2) - (EX)^2 \end{aligned}$$

d) $\mathrm{Var}(aX+b) = a\mathrm{Var}(X)$

$$\begin{aligned} \mathrm{Var}(aX+b) &= E[aX+b-E(aX+b)]^2 \\ &= E[aX+b-aE(X)-b]^2 \\ &= E[aX-aE(X)]^2 \\ &= E[a\{X-E(X)\}]^2 \\ &= E[a^2\{X-E(X)\}^2] \\ &= a^2E[\{X-E(X)\}^2] \\ &= a^2\mathrm{Var}(X) \end{aligned}$$

e) $E(XY) = (EX)(EY)$ if X, Y independent.
Let $f(x,y)$ be the joint density function for X and Y and, by independence, $f(x,y) = f(x)f(y)$. Now we have

$$\begin{aligned} E(XY) &= \int_y\int_x xy\, f(x,y)\, dx\, dy \\ &= \int_y\int_x xy\, f(x)f(y)\, dx\, dy \\ &= \int_y y\, f(y)\int_x x\, f(x)\, dx\, dy \\ &= \int_y y\, f(y)E(X)\, dy \\ &= E(X)\int_y y\, f(y)\, dy \\ &= E(X)E(Y) \end{aligned}$$

Note that $\mathrm{Cov} = E(XY) - E(X)E(Y)$ so, if X and Y are independent, $\mathrm{Cov} = E(X)E(Y) - E(X)E(Y) = 0$. Note that it can also happen that X and Y are *dependent* but it just so happens that $E(XY) = E(X)E(Y)$.
In other words independence implies $\mathrm{Cov}(X,Y) = 0$ but $\mathrm{Cov}(X,Y) = 0$ does not imply independence.

f) $\mathrm{Cov}(aX+b, cY+d) = a\,c\,\mathrm{Cov}(X,Y)$

$$\begin{aligned} &\mathrm{Cov}(aX+b, cY+d) \\ &= E[aX+b-E(aX+b)][cY+d-E(cY+d)] \\ &= E[aX+b-aE(X)-b][cY+d-cE(Y)-d] \\ &= E[aX-aE(X)][cY-cE(Y)] \\ &= E[a\{X-E(X)\}][c\{Y-E(Y)\}] \\ &= acE[X-E(X)][Y-E(Y)] \\ &= ac\mathrm{Cov}(X,Y) \end{aligned}$$

g) $\mathrm{Cov}(X_1+X_2, Y_1) = \mathrm{Cov}(X_1,Y_1) + \mathrm{Cov}(X_2,Y_1)$

$$\begin{aligned} &\mathrm{Cov}(X_1+X_2, Y_1) \\ &= E[(X_1+X_2)Y_1] - [E(X_1+X_2)]E(Y_1) \\ &= E[X_1Y_1 + X_2Y_1] \\ &\qquad - [E(X_1)+E(X_2)]E(Y_1) \\ &= E(X_1Y_1) + E(X_2Y_1) \\ &\qquad - E(X_1)E(Y_1) - E(X_2)E(Y_1) \\ &= E(X_1Y_1) - E(X_1)E(Y_1) \\ &\qquad + E(X_2Y_1) - E(X_2)E(Y_1) \\ &= \mathrm{Cov}(X_1,Y_1) + \mathrm{Cov}(X_2,Y_1) \end{aligned}$$

h) $\mathrm{Corr}(X, aX+b) = \mathrm{sign}(a)$
First assume $a ne 0$.

$$\begin{aligned}
\mathrm{Corr}(X, aX+b) &= \frac{\mathrm{Cov}(X, aX+b)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(aX+b)}} \\
&= \frac{a\mathrm{Cov}(X, X)}{\sqrt{\mathrm{Var}(X)\,a^2\mathrm{Var}(X)}} \\
&= \frac{a\mathrm{Var}(X)}{\sqrt{a^2}\sqrt{\mathrm{Var}(X)^2}} \\
&= \frac{a\mathrm{Var}(X)}{|a|\,|\mathrm{Var}(X)|} \\
&= \frac{a\mathrm{Var}(X)}{|a|\,\mathrm{Var}(X)} \quad (\text{since } \mathrm{Var}(X) > 0) \\
&= \frac{a}{|a|}
\end{aligned}$$

This equals 1 if a is positive and -1 if a is negative. Note that when $a = 0$, $\mathrm{Corr}(X, aX+b) = \mathrm{Corr}(X, b) = 0$ since X cannot be correlated with the constant b.

Thus, $\mathrm{Corr}(X, aX+b) = \mathrm{sign}(a)$ where the "sign function" is defined as

$$\mathrm{sign}(a) = \begin{cases} -1 = & a < 0 \\ 0 = & a = 0 \\ 1 = & a > 0 \end{cases}$$