Time Series – Tutorial 7 – Solutions

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Question 1

a)
$$
\begin{aligned}
Y_t - \mu &= \phi_1(Y_{t-1} - \mu) + \cdots + \phi_p(Y_{t-p} - \mu) \\
&\quad + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow E(Y_t) - \mu &= \phi_1(E(Y_{t-1}) - \mu) + \cdots + \phi_p(E(Y_{t-p}) - \mu) \\
&\quad + E(e_t) + \theta_1 E(e_{t-1}) + \cdots + \theta_q E(e_{t-q}) \\
&\qquad \text{(applying expectation)} \\
\mu^* - \mu &= \phi_1(\mu^* - \mu) + \cdots + \phi_p(\mu^* - \mu) \\
&\quad + 0 + \theta_1(0) + \cdots + \theta_q(0)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow (1 - \phi_1 - \cdots - \phi_p)\mu^* &= (1 - \phi_1 - \cdots - \phi_p)\mu \\
\mu^* &= \mu \\
E(Y_t) &= \mu
\end{aligned}
$$

b)
$$
\begin{aligned}
Y_t - \mu &= \phi_1(Y_{t-1} - \mu) + \cdots + \phi_p(Y_{t-p} - \mu) \\
&\quad + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}
\end{aligned}
$$

$$
\begin{aligned}
Y_t - \mu &= \phi_1(Y_{t-1} - \mu) + \cdots + \phi_p(Y_{t-p} - \mu) \\
&\quad + \theta(B) e_t
\end{aligned}
$$

$$
\begin{aligned}
Y_t - \phi_1 Y_{t-1} &- \cdots - \phi_p Y_{t-p} \\
&= \mu - \phi_1\mu - \cdots - \phi_p\mu + \theta(B) e_t
\end{aligned}
$$

$$
\begin{aligned}
\phi(B) Y_t &= (1 - \phi_1 - \cdots - \phi_p)\mu + \theta(B) e_t \\
\phi(B) Y_t &= \beta_0 + \theta(B) e_t
\end{aligned}
$$

Question 2

Consider the `color` dataset.

a) It is easy to find that $r_1 = 0.528$ and $\hat{\gamma}_0 = 37.1$. Note that we also need $\bar{y} = 74.885$ for part (b).

b) For the method of moments we have that

$$
\begin{aligned}
\hat{\phi} &= r_1 = 0.528 \\
\hat{\beta}_0 &= (1 - \hat{\phi})\, \bar{y} = (1 - 0.528)\,(74.885) = 35.33 \\
\hat{\sigma}_e^2 &= (1 - \hat{\phi}^2)\, \hat{\gamma}_0 = (1 - 0.528^2)\, 37.1 = 26.75.
\end{aligned}
$$

c) The fitted model (using method of moments) is

$$Y_t = 35.33 + 0.528\, Y_{t-1} + e_t$$

where $\hat{\sigma}_e^2 = 26.75$.

d) The fitted model using maximum likelihood (see `Tutorial7Solutions.R`) is

$$Y_t = 31.92 + 0.57\, Y_{t-1} + e_t$$

where $\hat{\sigma}_e^2 = 24.83$.

Clearly the two approaches yield very similar estimates here.

Question 3

have $r_1 = 0.736$, $r_2 = 0.304$, $\hat{\gamma}_0 = 5.88$ and $\bar{w} = 5.819$.

a) Using the results of Lecture 7 (Section 3.2) we have the following:

$$
\begin{aligned}
\hat{\phi}_1 &= \frac{r_1(1 - r_2)}{1 - r_1^2} = 1.118 \\
\hat{\phi}_2 &= \frac{r_2 - r_1^2}{1 - r_1^2} = -0.519 \\
\hat{\beta}_0 &= (1 - \hat{\phi}_1 - \hat{\phi}_2)\bar{y} = 2.334 \\
\hat{\sigma}_e^2 &= (1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2)\hat{\gamma}_0 = 1.969
\end{aligned}
$$

b) The fitted model (using method of moments) is

$$
\begin{aligned}
W_t &= 2.334 + 1.118\, W_{t-1} - 0.519\, W_{t-2} + e_t \\
\Rightarrow \sqrt{Y_t} &= 2.334 + 1.118\, \sqrt{Y_{t-1}} - 0.519\, \sqrt{Y_{t-2}} + e_t
\end{aligned}
$$

where $\hat{\sigma}_e^2 = 1.969$.

c) - e) See `Tutorial7Solutions.R`.

Question 4

a) Using `R` we find that

$$\bar{w} = 0.0044, \qquad \hat{\gamma}_0 = 0.0072, \qquad r_1 = 0.2117$$

b) We must solve

$$r_1 = \frac{-\theta}{1 + \theta^2}$$

$$
\begin{aligned}
&\Rightarrow r_1\theta^2 + \theta + r_1 = 0. \\
&\Rightarrow \hat{\theta} = \frac{-1 \pm \sqrt{1 - 4r_1^2}}{2r_1} \\
&\qquad = \frac{-1 \pm \sqrt{1 - 4(0.2117^2)}}{2(0.2117)}
\end{aligned}
$$

This yields two possible solutions: $\hat{\theta} = -4.5$ and $\hat{\theta} = -0.22$. Recall that we require $|\theta| < 1$ for invertibility and, hence, we use $\hat{\theta} = -0.22$.
For MA processes the intercept is the sample mean, i.e., $\hat{\beta}_0 = \bar{w} = 0.0044$.

Finally, the estimate of the error variance is given by

$$\hat{\sigma}_e^2 = \frac{\hat{\gamma}_0}{1+\hat{\theta}_1^2} = \frac{0.0072}{1+(-0.22)^2} = 0.0068.$$

c) The fitted model (using method of moments) is

$$\begin{aligned} W_t &= 0.0044 + e_t - (-0.22)\, e_{t-1} \\ W_t &= 0.0044 + e_t + 0.22\, e_{t-1} \\ \nabla \log Y_t &= 0.0044 + e_t + 0.22\, e_{t-1} \\ \log Y_t - \log Y_{t-1} &= 0.0044 + e_t + 0.22\, e_{t-1} \\ \log Y_t &= 0.0044 + \log Y_{t-1} + e_t + 0.22\, e_{t-1} \end{aligned}$$

where $\hat{\sigma}_e^2 = 1.969$.

It is worth noting however that, intercepts are typically omitted from models which incorporate differencing. This then gives:

$$\log Y_t = \log Y_{t-1} + e_t + 0.22\, e_{t-1}$$

Question 5

a) An MA(1) process (with no intercept) is given by

$$\begin{aligned} & Y_t = e_t - \theta e_{t-1} \\ \Rightarrow\ & e_t = Y_t + \theta e_{t-1} \\ \Rightarrow\ & e_1 = Y_1 + \theta e_0 = Y_1 \qquad \text{(setting } e_0 = 0) \\ & e_2 = Y_2 + \theta e_1 = Y_2 + \theta Y_1 \\ & e_3 = Y_3 + \theta e_2 = Y_3 + \theta Y_2 + \theta^2 Y_1 \end{aligned}$$

For the observed values $y_1 = 0$, $y_2 = -1$ and $y_3 = \frac{1}{2}$, this then becomes

$$\begin{aligned} e_1 &= Y_1 = 0 \\ e_2 &= Y_2 + \theta Y_1 = -1 \\ e_3 &= \tfrac{1}{2} - \theta \end{aligned}$$

Hence, the sum of squared errors (which we aim to minimise) is given by

$$\begin{aligned} S(\theta) = \sum e_i^2 &= (0)^2 + (-1)^2 + (\tfrac{1}{2} - \theta)^2 \\ &= 1 + (\tfrac{1}{2} - \theta)^2. \end{aligned}$$

It is easy to see that $\hat{\theta} = \frac{1}{2}$ minimised this function. Nonetheless, if we follow the usual procedure of differentiation

$$\tfrac{d}{d\theta} S(\theta) = 0 + 2(\tfrac{1}{2} - \theta)(-1)$$

and, solve $\frac{d}{d\theta} S(\hat{\theta}) = 0$, we clearly get $\hat{\theta} = \frac{1}{2}$.

b) The standard least squares estimate for the error variance is given by

$$\begin{aligned} \hat{\sigma}_e^2 &= \frac{1}{n} \sum \hat{e}_t \\ &= \frac{1}{n}[1 + (\tfrac{1}{2} - \hat{\theta})^2] \\ &= \frac{1}{3}[1 + (\tfrac{1}{2} - \tfrac{1}{2})^2] \\ &= \tfrac{1}{3} \end{aligned}$$

Question 6

In Lecture 7 we found in Section 4.1 that, for an AR(1) process with an intercept, the least squares estimates are $\hat{\phi} = r_1$ and $\hat{\beta}_0 = (1 - r_1)\bar{y}$ (when n is large) which are the same as the method of moments estimators.

The estimate of the error variance is given by

$$\begin{aligned} \hat{\sigma}_e^2 &= \frac{1}{n} \sum_{t=2}^{n} \hat{e}_t^2 \\ &= \frac{1}{n} \sum_{t=2}^{n} (y_t - \hat{\beta}_0 - \hat{\phi} y_{t-1})^2 \\ &= \frac{1}{n} \sum_{t=2}^{n} (y_t - (1 - r_1)\bar{y} - r_1 y_{t-1})^2 \\ &= \frac{1}{n} \sum_{t=2}^{n} (y_t - \bar{y} + r_1 \bar{y} - r_1 y_{t-1})^2 \\ &= \frac{1}{n} \sum_{t=2}^{n} [(y_t - \bar{y}) - r_1(y_{t-1} - \bar{y})]^2 \\ &= \frac{1}{n} \sum_{t=2}^{n} [(y_t - \bar{y})^2 - 2r_1(y_{t-1} - \bar{y})(y_t - \bar{y}) \\ &\qquad + r_1^2 (y_{t-1} - \bar{y})^2] \\ &= \frac{1}{n} \left[\sum_{t=2}^{n} (y_t - \bar{y})^2 - 2r_1 \sum_{t=2}^{n} (y_{t-1} - \bar{y})(y_t - \bar{y}) \right. \\ &\qquad \left. + r_1^2 \sum_{t=2}^{n} (y_{t-1} - \bar{y})^2 \right] \end{aligned}$$

Before we proceed, recall that

$$\begin{aligned} \hat{\gamma}_0 &= \frac{1}{n} \sum_{t=1}^{n} (y_t - \bar{y})^2 \\ \Rightarrow n\hat{\gamma}_0 &= \sum_{t=1}^{n} (y_t - \bar{y})^2 \end{aligned}$$

Also

$$
\begin{aligned}
r_1 &= \frac{\sum_{t=1}^{n-1}(y_t-\bar{y})(y_{t+1}-\bar{y})}{\sum_{t=1}^{n}(y_t-\bar{y})^2}. \\
&= \frac{\sum_{t=1}^{n-1}(y_t-\bar{y})(y_{t+1}-\bar{y})}{n\hat{\gamma}_0}. \\
\Rightarrow n\hat{\gamma}_0\, r_1 &= \sum_{t=1}^{n-1}(y_t-\bar{y})(y_{t+1}-\bar{y})
\end{aligned}
$$

Now, let's consider the terms in

$$
\hat{\sigma}_e^2 = \frac{1}{n}\left[\sum_{t=2}^{n}(y_t-\bar{y})^2 - 2r_1\sum_{t=2}^{n}(y_{t-1}-\bar{y})(y_t-\bar{y}) + r_1^2\sum_{t=2}^{n}(y_{t-1}-\bar{y})^2\right]
$$

Note that $\sum_{t=2}^{n}(y_t-\bar{y})^2 \approx \sum_{t=1}^{n}(y_t-\bar{y})^2 = n\hat{\gamma}_0$ when n is large, i.e., the missing $(y_1-\bar{y})^2$ term is negligible in this case - particularly for a stationary series.

Similarly, $\sum_{t=2}^{n}(y_{t-1}-\bar{y})^2 \approx n\hat{\gamma}_0$.

Finally, $\sum_{t=2}^{n}(y_{t-1}-\bar{y})(y_t-\bar{y}) = \sum_{t=1}^{n-1}(y_t-\bar{y})(y_{t+1}-\bar{y}) = n\hat{\gamma}_0 r_1$.

Thus,

$$
\begin{aligned}
\hat{\sigma}_e^2 &\approx \frac{1}{n}\left[n\hat{\gamma}_0 - r_1\, 2n\hat{\gamma}_0 r_1 + r_1^2\, n\hat{\gamma}_0\right] \\
&= \hat{\gamma}_0 - 2\hat{\gamma}_0 r_1^2 + \hat{\gamma}_0 r_1^2 \\
&= \hat{\gamma}_0 - \hat{\gamma}_0 r_1^2 \\
&= \hat{\gamma}_0\,(1-r_1^2)
\end{aligned}
$$

as required.

Question 7

b) From R we have that

```
arima(x = color, order = c(0, 0, 1))

Coefficients:
         ma1  intercept
      0.4443    74.7712
s.e.  0.1315     1.2752

sigma^2 estimated as 27.76:
log likelihood = -107.94,  aic = 219.88
```

Thus, the fitted model is

$$
Y_t = 74.77 + e_t + 0.44\, e_{t-1}
$$

where $\hat{\sigma}_e^2 = 27.76$.

Remember that MA coefficients in R are not defined with minus signs before them.

Question 8

b) • For the IMA(1,1) model we have

```
arima(x = log(oil.price), order = c(0, 1, 1))

Coefficients:
         ma1
      0.2956
s.e.  0.0693

sigma^2 estimated as 0.006689:
log likelihood = 260.29,  aic = -518.58
```

Thus, the fitted model is

$$
\begin{aligned}
\nabla \log Y_t &= e_t + 0.2956 e_{t-1} \\
\log Y_t - \log Y_{t-1} &= e_t + 0.2956 e_{t-1} \\
\log Y_t &= \log Y_{t-1} + e_t + 0.2956 e_{t-1}
\end{aligned}
$$

where $\hat{\sigma}_e^2 = 0.006689$.

• For the ARI(1,1) model we have

```
arima(x = log(oil.price), order = c(1, 1, 0))

Coefficients:
         ar1
      0.2364
s.e.  0.0660

sigma^2 estimated as 0.006787:
log likelihood = 258.55,  aic = -515.11
```

Thus, the fitted model is

$$
\begin{aligned}
\nabla \log Y_t &= 0.2364\, \nabla \log Y_{t-1} + e_t \\
\log Y_t - \log Y_{t-1} &= 0.2364(\log Y_{t-1} - \log Y_{t-2}) + e_t \\
\log Y_t &= 1.2364 \log Y_{t-1} - 0.2364 \log Y_{t-2} + e_t
\end{aligned}
$$

where $\hat{\sigma}_e^2 = 0.006787$.