UNIVERSITY of LIMERICK

OLLSCOIL LUIMNIGH

Time Series Analysis

MS 4218

joseph.lynch@ul.ie

Outline

- Invertibility
- ARMA processes
- Backshift operator

Invertibility

For an MA(1) process, MA(θ) and MA($\frac{1}{\theta}$) have the same ρ value. Likewise for MA(2).

An MA process can also be expressed as an AR process.

$$
\begin{aligned}
Y_t &= e_t - \theta e_{t-1} \\
e_t &= Y_t + \theta e_{t-1} \;=\; Y_t + \theta(Y_{t-1} + \theta e_{t-2}) \\
&= Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \ldots \\
Y_t &= (-\theta Y_{t-1} - \theta^2 Y_{t-2} - \ldots) + e_t
\end{aligned}
$$

$-\theta^1 \cdots - \theta^q$ are termed the π_j weights for $j = 1 \ldots q$ and the substitutions can be infinite if $|\pi_j| < 1$.

1. MA invertibility

The MA(1) is said to be invertible if it can be converted to an infinite order AR model.

An MA(q) process has a characteristic polynomial given by:

$$\theta(x) \quad = \quad 1 - \theta_1 x - \theta_2 x^2 -, \ldots, -\theta_q x^q.$$

and a characteristic equation given by:

$$1 - \theta_1 x - \theta_2 x^2 -, \ldots, -\theta_q x^q \quad = \quad 0.$$

MA(q) is invertible if the roots of the characteristic equation are $> |1|$.

Invertible

There is only 1 set of parameter values that yield an invertible MA(q) process with a given auto-correlation function.

$$Y_t = e_t + 2e_{t-1}.$$

$$-\theta = 2; \qquad \theta = -2$$

$$\rho_1 = \frac{-\theta}{1+\theta^2} = \frac{2}{5}.$$

The characteristic equation

$$\begin{aligned}\theta(x) &= 1 - \theta x \\ &= 1 + 2x = 0\end{aligned}$$

has root $x = -\frac{1}{2}$ and is not invertible as $|x| < 1$.

Invertible MA(1)

$$\begin{aligned} Y_t &= e_t + \frac{1}{2} e_{t-1}. \\ -\theta &= \frac{1}{2}; \qquad \theta = -\frac{1}{2} \\ \rho_1 &= \frac{-\theta}{1+\theta^2} = \frac{1}{2} \div \frac{5}{4} = \frac{2}{5.} \end{aligned}$$

The characteristic equation

$$\begin{aligned} \theta(x) &= 1 - \theta x \\ &= 1 + \frac{1}{2} x = 0 \end{aligned}$$

has root $x = 2$ and is invertible as $|x| > 1$.

Stationary models cont.: ARMA model

A mixed auto-regressive and moving average model.

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\
&\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}.
\end{aligned}
$$

The model is stationary so we expect a constant mean and a variance that depends only on the lag k and not on the actual times involved in the lag.

ARMA(1,1)

There is 1 ϕ term in the AR component and 1 θ term in the MA component.

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}.$$

To derive Yule-Walker equations, we multiply both sides by Y_{t-k} and take expectations.

$$\begin{aligned} E(Y_t Y_{t-k}) &= \phi E(Y_{t-1} Y_{t-k}) + E(e_t Y_{t-k}) - \theta E(e_{t-1} Y_{t-k}) \\ \gamma_k &= \phi \gamma_{t-1-(t-k)} + E(e_t Y_{t-k}) - \theta E(e_{t-1} Y_{t-k}) \\ &= \phi \gamma_{k-1} + E(e_t Y_{t-k}) - \theta E(e_{t-1} Y_{t-k}). \end{aligned}$$

ARMA(1,1)

If $k = 0$

$$\begin{aligned}
\gamma_0 &= \phi\gamma_{(-1)} + E(e_t Y_t) - \theta E(e_{t-1} Y_t) \\
&= \phi\gamma_1 + E(e_t Y_t) - \theta E(e_{t-1} Y_t). \\
E(e_t Y_t) &= E\{e_t(\phi Y_{t-1} + e_t - \theta e_{t-1})\} \\
&= E(e_t^2) \\
&= \sigma_e^2.
\end{aligned}$$

ARMA(1,1)γ_0 cont.

$$
\begin{aligned}
E(e_{t-1}Y_t) &= E\{e_{t-1}(\phi Y_{t-1} + e_t - \theta e_{t-1})\} \\
&= E[e_{t-1}\{\phi(\phi Y_{t-2} + e_{t-1} - \theta e_{t-2}) + e_t - \theta e_{t-1}\}] \\
&= E(\phi e_{t-1}^2 - \theta e_{t-1}^2) \\
&= (\phi - \theta)\sigma_e^2.
\end{aligned}
$$

ARMA(1,1) cont.

$$
\begin{aligned}
\gamma_0 &= \phi\gamma_1 + E(e_t Y_t) - \theta E(e_{t-1} Y_t) \\
&= \phi\gamma_1 + \sigma_e^2 - \theta(\phi - \theta)\sigma_e^2 \\
&= \phi\gamma_1 + \sigma_e^2\{1 - \theta(\phi - \theta)\}.
\end{aligned}
$$

ARMA(1,1) cont.

$$
\begin{aligned}
\gamma_k &= \phi\gamma_{k-1} + E(e_t Y_{t-k}) - \theta E(e_{t-1} Y_{t-k}) \\
\gamma_1 &= \phi\gamma_0 + E(e_t Y_{t-1}) - E(\theta e_{t-1} Y_{t-1}) \\
&= \phi\gamma_0 + 0 - \theta E\{e_{t-1}(\phi Y_{t-2} + e_{t-1} - \theta e_{t-2})\} \\
&= \phi\gamma_0 - \theta E(e_{t-1}^2) \;=\; \phi\gamma_0 - \theta\sigma_e^2.
\end{aligned}
$$

ARMA(1,1) acvf summary

$$\gamma_0 = \phi\gamma_1 + \sigma_e^2\{1 - \theta(\phi - \theta)\}.$$

$$\gamma_1 = \phi\gamma_0 - \theta\sigma_e^2.$$

$$\gamma_k = \phi\gamma_{k-1} + E(e_t Y_{t-k}) - \theta E(e_{t-1} Y_{t-k}).$$

$$E(e_t Y_{t-k}) - \theta E(e_{t-1} Y_{t-k}) = 0 \quad \forall \quad k \geq 2.$$

$$\therefore \gamma_k = \phi\gamma_{k-1} \text{ for } k \geq 2.$$

ARMA(1,1) γ_0 in terms of parameters

$$\begin{aligned}
\gamma_0 &= \phi\gamma_1 + \sigma_e^2\{1 - \theta(\phi - \theta)\} \\
&= \phi(\phi\gamma_0 - \theta\sigma_e^2) + \sigma_e^2\{1 - \theta(\phi - \theta)\} \\
\gamma_0 - \phi^2\gamma_0 &= -\phi\theta\sigma_e^2 + \sigma_e^2 - \theta\phi\sigma_e^2 + \theta^2\sigma_e^2 \\
\gamma_0(1 - \phi^2) &= -2\phi\theta\sigma_e^2 + \sigma_e^2 + \theta^2\sigma_e^2 \\
\gamma_0 &= \frac{1 - 2\phi\theta + \theta^2}{1 - \phi^2}\sigma_e^2.
\end{aligned}$$

ARMA(1,1) acf

$$
\begin{aligned}
\rho_k &= \frac{\gamma_k}{\gamma_0} \\
&= \frac{\phi\gamma_{(k-1)}}{\frac{1-2\phi\theta+\theta^2}{1-\phi^2}\sigma_e^2} \\
&= \frac{\phi^2\gamma_{(k-2)}(1-\phi^2)}{(1-2\phi\theta+\theta^2)\sigma_e^2} \\
&= \frac{\phi^{k-1}\gamma_1(1-\phi^2)}{(1-2\phi\theta+\theta^2)\sigma_e^2}.
\end{aligned}
$$

We need to express γ_1 in terms of the parameters.

ARMA(1,1) γ_1 in terms of parameters

$$
\begin{aligned}
\gamma_1 &= \phi\gamma_0 - \theta\sigma_e^2 \\
&= \phi\frac{1-2\phi\theta+\theta^2}{1-\phi^2}\sigma_e^2 - \theta\sigma_e^2 \\
&= \frac{\phi-2\phi^2\theta+\theta^2\phi-\theta(1-\phi^2)}{1-\phi^2}\sigma_e^2 \\
&= \frac{\phi-\phi^2\theta+\theta^2\phi-\theta}{1-\phi^2}\sigma_e^2 \\
&= \frac{(\phi-\theta)(1-\phi\theta)}{1-\phi^2}\sigma_e^2.
\end{aligned}
$$

ARMA(1,1) acf

$$
\begin{aligned}
\rho_k &= \frac{\phi^{k-1}(1-\phi^2)\gamma_1}{(1-2\phi\theta+\theta^2)\sigma_e^2}. \\
&= \frac{\phi^{k-1}(1-\phi^2)}{(1-2\phi\theta+\theta^2)\sigma_e^2}\frac{(\phi-\theta)(1-\phi\theta)}{1-\phi^2}\sigma_e^2 \\
&= \frac{(\phi-\theta)(1-\phi\theta)}{1-2\phi\theta+\theta^2}\phi^{k-1}, \text{ for } k \geq 1.
\end{aligned}
$$

As $k\uparrow$, $\rho_k \downarrow$ exponentially when $\phi < 1$.

Many different shapes for ρ_k depending on sign of θ and of ϕ.

ARMA(p, q) stationarity

Given e_t is independent of $Y_{t-1}, Y_{t-2}, \dots$, then

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\ &\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \end{aligned}$$

has a stationary solution $\iff$ all of the roots of the AR characteristic equation $\phi(x) = 0$ are $> |1|$.

If stationary, the acf is given by:

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p} \qquad \text{for } k > q.$$

Backshift operator: B

The backshift operator shifts a time series back as follows:

$$
\begin{aligned}
BY_t &= Y_{t-1}. \\
BY_{t-1} &= Y_{t-2}. \\
B^2 Y_t &= Y_{t-2}. \\
B^m Y_t &= Y_{t-m}.
\end{aligned}
$$

Linearity property of B

$$
\begin{aligned}
B(aY_t + bX_t + c) &= B(aY_t) + B(bX_t) + B(c) \\
&= aB(Y_t) + bB(X_t) + c.
\end{aligned}
$$

MA(1) and B

$$\begin{aligned} Y_t &= e_t - \theta e_{t-1} \\ &= e_t - \theta B e_t \\ &= (1 - \theta B) e_t \\ &= \theta(B), \end{aligned}$$

where $\theta(B)$ is the characteristic polynomial “evaluated" at B.

MA(q) and B

$$
\begin{aligned}
Y_t &= e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \\
&= e_t - \theta_1 B e_t - \theta_2 B^2 e_t - \cdots - \theta_q B^q e_t \\
&= (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q) e_t. \\
&= \theta(B) e_t,
\end{aligned}
$$

where $\theta(B)$ is the MA characteristic polynomial “evaluated" at B.

AR(p) and B

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\
e_t &= Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \cdots - \phi_p Y_{t-p} \\
&= Y_t - \phi_1 B Y_t - \phi_2 B^2 Y_t - \cdots - \phi_p B^p Y_t \\
&= (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) Y_t \\
&= \phi(B) Y_t \;=\; e_t,
\end{aligned}
$$

where $\phi(B)$ is the AR characteristic polynomial “evaluated" at B.

General ARMA(p, q) and B

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\
&\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}. \\
\phi(B) &= Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \cdots - \phi_p Y_{t-p}. \\
\theta(B) &= e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}.
\end{aligned}
$$

$$\phi(B) Y_t = \theta(B) e_t.$$

Differencing operator, ∇, and B

$$
\begin{aligned}
\nabla Y_t &= Y_t - Y_{t-1}. \\
&= Y_t - BY_t. \\
&= (1 - B)Y_t.
\end{aligned}
$$

Differencing operator, ∇, and B cont.

$$
\begin{aligned}
\nabla^2 Y_t &= (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}) \\
&= Y_t - 2Y_{t-1} + Y_{t-2} \\
&= (1 - 2B + B^2) Y_t \\
&= (1 - B)^2 Y_t. \\
\nabla^m &= (1 - B)^m Y_t.
\end{aligned}
$$

Next

Model for non-stationary time series

- Stationarity through differencing
- ARIMA models
  - IMA(1,1) model
  - IMA(2,2) model
  - ARI(1,1) model
- Constant terms in ARIMA model
- Other transformations