Tutorial Sheet 2 Solutions

2.1
Suppose $E(X) = 2$, $Var(X) = 9$,
$E(Y) = 0$, $Var(Y) = 4$, and
$Corr(X, Y) = 0.25$.

Find:
(a) $Var(X + Y)$.

$$\begin{aligned}
Var(X+Y) &= Var(X) + Var(Y) + 2Cov(X,Y) \\
Corr(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \\
0.25 &= \frac{Cov(X,Y)}{\sqrt{9 \times 4}} \\
Cov(X,Y) &= 1.5 \\
\Rightarrow Var(X+Y) &= 9 + 4 + 2(1.5) \\
&= 16.
\end{aligned}$$

(b) $Cov(X, X + Y)$.

$$\begin{aligned}
Cov(X, X+Y) &= Cov(X,X) + Cov(X,Y) \\
&= Var(X) + Cov(X,Y) \\
&= 9 + 1.5 \ = \ 10.5.
\end{aligned}$$

(c) $Corr(X+Y, X-Y)$.

$$\begin{aligned}
Corr(X+Y, X-Y) &= \frac{Cov(X+Y, X-Y)}{\sqrt{Var(X+Y)Var(X-Y)}}. \\
Cov(X+Y, X-Y) &= Cov(X,X) - Cov(X,Y) + Cov(Y,X) - Cov(Y,Y) \\
&= Var(X) - Cov(X,Y) + Cov(X,Y) - Var(Y) \\
&= 9 - 4 \\
&= 5. \\
Var(X-Y) &= Var(X) + Var(Y) - 2Cov(X,Y) \\
&= 9 + 4 - 2(1.5) \\
&= 10. \\
Corr(X+Y, X-Y) &= \frac{5}{\sqrt{16 \times 10}} \\
&= 0.395.
\end{aligned}$$

2.2
If X and Y are dependent but $Var(X) = Var(Y)$, find $Cov(X+Y, X-Y)$.

$$\begin{aligned} Cov(X+Y, X-Y) &= Cov(X,X) - Cov(X,Y) + Cov(Y,X) - Cov(Y,Y). \\ &= Var(X) - Cov(X,Y) + Cov(X,Y) - Var(Y) \\ &= Var(X) - Var(X) \\ &= 0. \end{aligned}$$

2.3
Let X have a distribution with mean μ and variance σ^2.
Let $Y_t = X$ for all t.

(a) Show that $\{Y_t\}$ is strictly and weakly stationary.

Let $t_1, t_2, \ldots, t_n$ be a set of time points, and let k be any lag, then

$$\begin{aligned} Pr(Y_{t_1} \le y_{t_1}, \ldots, Y_{t_n} \le y_{t_n}) &= Pr(X \le y_{t_1}, \ldots, X \le y_{t_n}) \\ &= Pr(Y_{t_1-k} \le y_{t_1}, \ldots, Y_{t_n-k} \le y_{t_n}). \end{aligned}$$

Thus the process is strictly stationary.

As $Y_t = X$, and the process is strictly stationary, then for any k, $E(Y_{t-k})$ $= E(X) = \mu$, independent of t.

For weakly stationary, we also require auto-covariance function for $\{Y_t\}$ which is part(b).

(b) Find the auto-covariance function for $\{Y_t\}$.

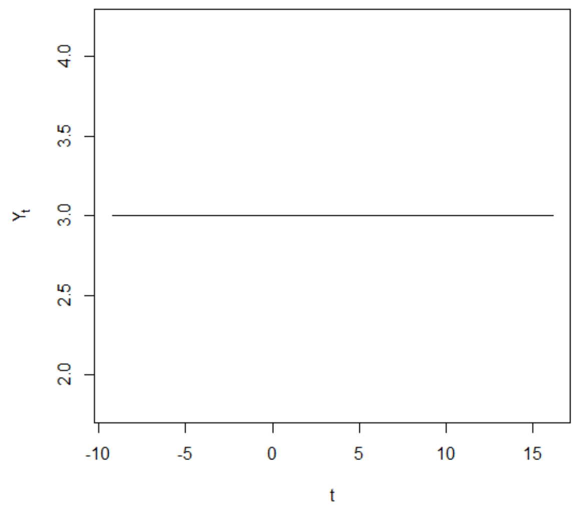
$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(X, X) \\ &= Var(X) \\ &= \sigma^2. \end{aligned}$$

(c) Sketch a "typical" time plot of Y_t.

Let $X \sim N(\mu = 3, \sigma^2 = 6)$.

$E(Y_t) = \mu = 3$.

```
x<-rnorm(100,3,6)
y<-rep(3,length(x))
plot(x,y,type="l",xlab="t",ylab=expression(Y[t]))
```



2.4
Suppose $Y_t = 5 + 2t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with auto-covariance function γ_k.

(a) Find the mean function for $\{Y_t\}$.

$$\begin{aligned} E(Y_t) &= E(5 + 2t + X_t) \\ &= E(5) + E(2t) + E(X_t) \\ &= 5 + 2t. \end{aligned}$$

Note here that, while X_t is a random variable, t is not, and thus its expectation is t itself.

(b) Find the auto-covariance function for $\{Y_t\}$.

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(5 + 2t + X_t, 5 + 2(t-k) + X_{t-k}) \\ &= Cov(X_t, X_{t-k}) \qquad \text{See slide 8 of TSLecture 2.} \\ &= \gamma_{t-(t-k)} \\ &= \gamma_k. \end{aligned}$$

We are told $\{X_t\}$ is a zero-mean stationary series. The mean is zero, independent of t, and the auto-covariance is only a function of the lag, and not of the actual times t and $t-k$.

(c) Is $\{Y_t\}$ stationary? Why or why not?

$E(Y_t)$ is a function of t, so $\{Y_t\}$ is not stationary, eventhough its auto-covariance points towards stationarity.

2.5
Suppose $Cov(X_t, X_{t-k}) = \gamma_k$ is free of t but that $E(X_t) = 3t$.

(a) Is $\{X_t\}$ stationary?

No, because $E(X_t) = 3t$ is a function of t.

(b) Let $Y_t = 7 - 3t + X_t$. Is $\{Y_t\}$ stationary?

$$\begin{aligned} E(Y_t) &= E(7 - 3t + X_t) \\ &= E(7) - E(3t) + E(X_t) \\ &= 7 - 3t + 3t \\ &= 7, \qquad \text{independent of } t. \end{aligned}$$

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov\{7 - 3t + X_t, 7 - 3(t-k) + X_{t-k}\} \\ &= Cov(X_t, X_{t-k}) \\ &= \gamma_k, \text{ given, and, independent of } t. \end{aligned}$$

So $\{Y_t\}$ is stationary.

2.6
Let $Y_1 = \theta_0 + e_1$, and then for $t > 1$ define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here θ_0 is a constant. The process $\{Y_t\}$ is called a random walk with drift.

(a) Show that Y_t may be rewritten as $Y_t = t\theta_0 + e_t + e_{t-1} + \ldots + e_1$.

$$\begin{aligned}
Y_1 &= \theta_0 + e_1 \\
Y_2 &= \theta_0 + Y_1 + e_2 \\
&= \theta_0 + \theta_0 + e_1 + e_2 \\
&= 2\theta_0 + e_1 + e_2. \\
Y_3 &= \theta_0 + Y_2 + e_3 \\
&= \theta_0 + 2\theta_0 + e_1 + e_2 + e_3 \\
&= 3\theta_0 + e_1 + e_2 + e_3. \\
Y_4 &= 4\theta_0 + e_1 + e_2 + e_3 + e_4. \\
Y_t &= t\theta_0 + e_1 + e_2 + e_3 + e_4 + \ldots + e_t.
\end{aligned}$$

(b) Find the mean function for Y_t.

$$\begin{aligned}
E(Y_t) &= E(t\theta_0 + e_1 + e_2 + e_3 + e_4 + \ldots + e_t) \\
&= E(t\theta_0) + E(e_1) + E(e_2) + E(e_3) + E(e_4) + \ldots + E(e_t) \\
&= t\theta_0.
\end{aligned}$$

(c) Find the auto-covariance function for Y_t.

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov\{t\theta_0 + e_t + e_{t-1} + \ldots + e_1, (t-k)\theta_0 + e_{t-k} + e_{t-k-1} \ldots + e_1\} \\ &= Cov(e_t + e_{t-1} + \ldots + e_{t-k} + e_{t-k-1} \ldots + e_1, e_{t-k} + e_{t-k-1} \ldots + e_1) \text{ for } t \geq k \\ &= Cov(e_{t-k} + e_{t-k-1} \ldots + e_1, e_{t-k} + e_{t-k-1} \ldots + e_1) \\ &= (t-k)\sigma_e^2. \end{aligned}$$

2.7
Let $\{X_t\}$ be a time series in which we are interested. However, because the measurement process itself is not perfect, we actually observe $Y_t = X_t + e_t$.

We assume that $\{X_t\}$ and $\{e_t\}$ are independent processes. We call X_t the signal and e_t the measurement noise or error process.

If $\{X_t\}$ is stationary with auto-correlation function ρ_k, show that $\{Y_t\}$ is also stationary, and for $k \geq 1$, that

$$Corr(Y_t, Y_{t-k}) = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_X^2}}.$$

We call $\frac{\sigma_X^2}{\sigma_e^2}$ the signal-to-noise ratio, or SNR. Note that the larger the SNR, the closer the autocorrelation function of the observed process $\{Y_t\}$ is to the autocorrelation function of the desired signal $\{X_t\}$.

As $\{e_t\}$ is a noise process, we assume it has zero mean and variance of σ_e^2.

$$\begin{aligned} E(Y_t) &= E(X_t + e_t) \\ &= E(X_t) + E(e_t) \\ &= \mu_X + 0. \end{aligned}$$

$\{X_t\}$ is stationary, so μ_X is independent of t and thus, $\{Y_t\}$ is stationary in the mean.

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(X_t + e_t, X_{t-k} + e_{t-k}) \\ &= Cov(X_t, X_{t-k}) + Cov(X_t, e_{t-k}) + Cov(e_t, X_{t-k}) + Cov(e_t, e_{t-k}). \end{aligned}$$

$\{X_t\}$ and $\{e_t\}$ are independent processes, so they do not co-vary.

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(X_t, X_{t-k}) + Cov(e_t, e_{t-k}) \\ &= Cov(X_t, X_{t-k}) + 0 \text{ when } k \geq 1 \end{aligned}$$

As $\{X_t\}$ is stationary in terms of auto-covariance, then $\{Y_t\}$ is stationary also in terms of auto-covariance.

$$
\begin{aligned}
\rho_k &= Corr(X_t, X_{t-k}) \\
&= \frac{Cov(X_t, X_{t-k})}{\sqrt{Var(X_t)Var(X_{t-k})}} \\
&= \frac{Cov(X_t, X_{t-k})}{\sqrt{Var(X_t)Var(X_t)}} \qquad \{X_t\} \text{ is stationary} \\
&= \frac{Cov(X_t, X_{t-k})}{Var(X_t)} \\
\Rightarrow Cov(X_t, X_{t-k}) &= \rho_k Var(X_t).
\end{aligned}
$$

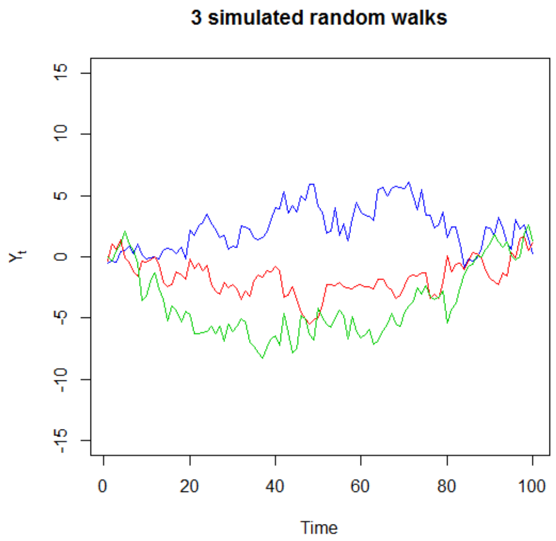
$$
\begin{aligned}
Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)Var(Y_{t-k})}} \\
&= \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)Var(Y_t)}} \qquad \{Y_t\} \text{ is stationary} \\
&= \frac{Cov(Y_t, Y_{t-k})}{Var(Y_t)} \\
&= \frac{Cov(X_t, X_{t-k})}{Var(Y_t)} \\
&= \frac{\rho_k Var(X_t)}{Var(X_t + e_t)} \\
&= \frac{\rho_k Var(X_t)}{Var(X_t) + Var(e_t) + 2Cov(X_t, e_t)} \\
&= \frac{\rho_k Var(X_t)}{Var(X_t) + Var(e_t)} \\
&= \frac{\rho_k \sigma_X^2}{\sigma_X^2 + \sigma_e^2} \\
&= \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_X^2}}.
\end{aligned}
$$

2.8

A random walk has equation: $Y_t = Y_{t-1} + e_t$, with $Y_1 = e_1$.

If $e_i \sim N(\mu = 0, \sigma^2 = 1)$, use R to produce 3 simultaneous random walks over 100 time points in 3 different colours.

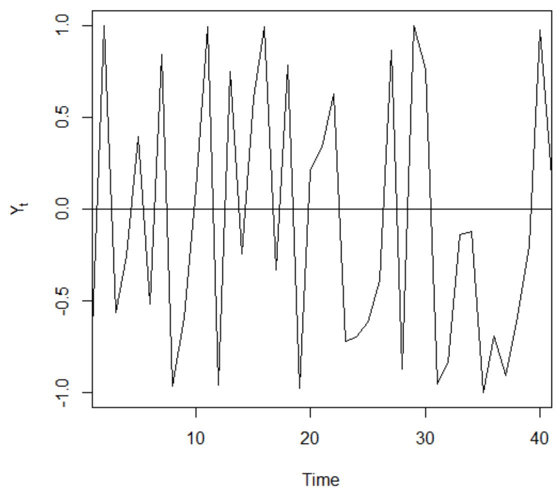
```
set.seed(100)
y<-e<-rnorm(100)
for(t in 2:100)
y[t]<-y[t-1]+e[t]
plot(y,type="l",col=4,ylim=c(-15,15),
main="3 simulated random walks",
xlab="Time",ylab=expression(Y[t]))
x<-e<-rnorm(100)
for(t in 2:100)
x[t]<-x[t-1]+e[t]
z<-e<-rnorm(100)
for(t in 2:100)
z[t]<-z[t-1]+e[t]
lines(x,col=2)
lines(z,col=3)
```



2.9
A random cosine wave has equation: $Y_t = cos\left\{2\pi\left(\frac{t}{12}+\Phi\right)\right\}$.

If $\Phi \sim U(a=0, b=1)$, use R to produce random cosine wave with 40 time points.

```
t<-seq(-20,20,by=1)
phi<-runif(length(t),0,1)
plot(cos(2*pi*(t/(12)+phi)),type="l",
xlab="Time",
ylab=expression(Y[t]),xaxs="i")
abline(h=0)
```



2.10
Suppose that $\{Y_t\}$ is stationary with auto-covariance function γ_k.

(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary by finding the mean and auto-covariance function for $\{W_t\}$.

$$
\begin{aligned}
E(W_t) &= E(Y_t - Y_{t-1}) \\
&= E(Y_t) - E(Y_{t-1}) \\
&= 0, \qquad \text{because } \{Y_t\} \text{ is stationary.} \\
Cov(W_t, W_{t-k}) &= Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\
&= Cov(Y_t, Y_{t-k}) - Cov(Y_t, Y_{t-k-1}) - Cov(Y_{t-1}, Y_{t-k}) + Cov(Y_{t-1}, Y_{t-k-1}) \\
&= \gamma_{t-(t-k)} - \gamma_{t-(t-k-1)} - \gamma_{t-1-(t-k)} + \gamma_{t-1-(t-k-1)} \\
&= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\
&= 2\gamma_k - \gamma_{k+1} - \gamma_{k-1}.
\end{aligned}
$$

All of these covariances are dependent only on the lag k, and, because of time-independent zero mean, $\{W_t\}$ is stationary.

(b) Show that $U_t = \nabla^2 Y_t = \nabla(Y_t - Y_{t-1}) = Y_t - 2Y_{t-1} + Y_{t-2}$ is stationary. (You need not find the mean and auto-covariance function for $\{U_t\}$.)

By part(a), $\{\nabla Y_t\}$ is stationary.

$\{U_t\}$ is the difference of a stationary process, and again by part(a), is itself a stationary process.

(c) Suppose $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with auto-covariance function γ_k, and β_0 and β_1 are constants.

Show that $\{Y_t\}$ is not stationary but that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary.

$$\begin{aligned}
E(Y_t) &= E(\beta_0 + \beta_1 t + X_t) \\
&= E(\beta_0) + E(\beta_1 t) + E(X_t) \\
&= \beta_0 + \beta_1 t,
\end{aligned}$$

which is not independent of t, and thus is not statonary.

$$\begin{aligned}
E(W_t) &= E(Y_t - Y_{t-1}) \\
&= E[\beta_0 + \beta_1 t + X_t - \{\beta_0 + \beta_1(t-1) + X_{t-1}\}] \\
&= E(\beta_0 - \beta_0 + \beta_1 t - \beta_1 t + \beta_1 + X_t - X_{t-1}) \\
&= E(\beta_1) + E(X_t) - E(X_{t-1}) \\
&= \beta_1,
\end{aligned}$$

which is independent of t.

By using part(a), a differenced process, $\{W_t\}$, is stationary.