Tutorial Sheet 4c

Part 4c

4.10
Sketch the auto-correlation function for each of the following ARMA models.

(a) $\phi = 0.7$ and $\theta = 0.4$.

An ARMA(1,1) process has an exponentially decaying auto-correlation starting from lag one, but not from lag zero.

```
ACF=ARMAacf(ar=0.7, ma = -0.4 ,lag.max=20)
# Remember that R uses the negative of our theta values.
ACF[-1]
#           1            2            3            4            5            6
#0.3600000000 0.2520000000 0.1764000000 0.1234800000 0.0864360000 0.0605052000
#           7            8            9           10           11           12
#0.0423536400 0.0296475480 0.0207532836 0.0145272985 0.0101691090 0.0071183763
#          13           14           15           16           17           18
#0.0049828634 0.0034880044 0.0024416031 0.0017091221 0.0011963855 0.0008374699
#          19           20
#0.0005862289 0.0004103602
```

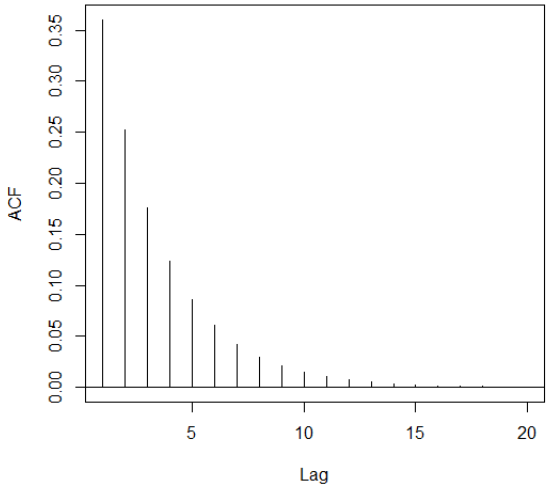
Alternatively, we can write our own function using the formula for ρ_k given in Lecture 4c page 17, i.e.,

$$\rho_k = \frac{(\phi-\theta)(1-\phi\theta)}{1-2\phi\theta+\theta^2}\phi^{k-1}, \text{ for } k \geq 1.$$

Because the function is not inbuilt, we just use the parameters as given.

```
rhok<-function(phi,theta)
{
rhok<-rep(NA,15)
for(k in 1:length(rhok))
rhok[k]<-((phi-theta)*(1-theta*phi))/(1-2*theta*phi+theta^2)*phi^(k-1)
return( rhok)
}
rhok(0.7,0.4)
# [1] 0.360000000 0.252000000 0.176400000 0.123480000 0.086436000 0.060505200
# [7] 0.042353640 0.029647548 0.020753284 0.014527299 0.010169109 0.007118376
#[13] 0.004982863 0.003488004 0.002441603
```

```
plot(y=ACF[-1],x=1:20,xlab='Lag',ylab='ACF',type='h')
abline(h=0)
```



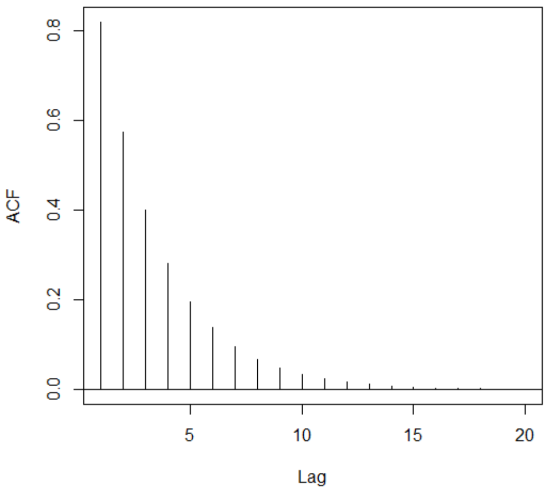
(b) $\phi = 0.7$ and $\theta = -0.4$.

```
ACF=ARMAacf(ar=0.7, ma = +0.4 ,lag.max=20)
ACF[-1]
#           1            2            3            4            5            6
#0.8186046512 0.5730232558 0.4011162791 0.2807813953 0.1965469767 0.1375828837
#           7            8            9           10           11           12
#0.0963080186 0.0674156130 0.0471909291 0.0330336504 0.0231235553 0.0161864887
#          13           14           15           16           17           18
#0.0113305421 0.0079313795 0.0055519656 0.0038863759 0.0027204632 0.0019043242
#          19           20
#0.0013330269 0.0009331189
```

Alternatively, using our own function

```
rhok(0.7,-0.4)
# [1] 0.818604651 0.573023256 0.401116279 0.280781395 0.196546977 0.137582884
# [7] 0.096308019 0.067415613 0.047190929 0.033033650 0.023123555 0.016186489
#[13] 0.011330542 0.007931379 0.005551966
```

```
plot(y=ACF[-1],x=1:20,xlab='Lag',ylab='ACF',type='h')
abline(h=0)
```



4.11
For the ARMA(1,2) model, $Y_t = 0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2}$, show that

(a) $\rho_k = 0.8\rho_{k-1}$ for $k > 2$.

Assume, with no loss of generality, that the mean is zero.

$$\begin{aligned}
Cov(Y_t, Y_{t-k}) &= E(Y_t Y_{t-k}) - E(Y_t)E(Y_{t-k}) \\
&= E(Y_t Y_{t-k}) \\
&= E\{(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})Y_{t-k}\} \\
&= 0.8E(Y_{t-1}Y_{t-k}) + E(e_t Y_{t-k}) + 0.7E(e_{t-1}Y_{t-k}) + 0.6E(e_{t-2}Y_{t-k}) \\
&= 0.8E(Y_{t-1}Y_{t-k}) + 0 + 0 + 0 \qquad \text{because } k > 2 \\
&= 0.8\gamma_{\{(t-1)-(t-k)\}} \\
\gamma_k &= 0.8\gamma_{k-1}. \\
\rho_k &= \frac{\gamma_k}{\gamma_0} \\
\Rightarrow \rho_k &= 0.8\rho_{k-1}, \qquad \text{for } k > 2.
\end{aligned}$$

(b) $\rho_2 = 0.8\rho_1 + 0.6\sigma_e^2/\gamma_0$.

$$\begin{aligned}
Cov(Y_t, Y_{t-2}) &= E(Y_t Y_{t-2}) \\
&= E\{(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2})Y_{t-2}\} \\
&= E\{(0.8Y_{t-1} + 0.6e_{t-2})Y_{t-2}\} \\
&= 0.8E(Y_{t-1}Y_{t-2}) + 0.6E(e_{t-2}Y_{t-2}) \\
&= 0.8Cov(Y_{t-1}, Y_{t-2}) + 0.6E\{e_{t-2}(0.8Y_{t-3} + e_{t-2} + 0.7e_{t-3} + 0.6e_{t-4})\}
\end{aligned}$$

$$
\begin{aligned}
&= 0.8\gamma_1 + 0.6E(e_{t-2}^2) \\
\gamma_2 &= 0.8\gamma_1 + 0.6\sigma_e^2. \\
\rho_2 &= \frac{\gamma_2}{\gamma_0} \\
&= 0.8\frac{\gamma_1}{\gamma_0} + 0.6\frac{\sigma_e^2}{\gamma_0} \\
&= 0.8\rho_1 + 0.6\frac{\sigma_e^2}{\gamma_0}.
\end{aligned}
$$

4.12
Consider two MA(2) processes, one with $\theta_1 = \theta_2 = \frac{1}{6}$, and another with $\theta_1 = -1$ and $\theta_2 = 6$.

(a) Show that these processes have the same auto-correlation function.

(i) $\theta_1 = \theta_2 = \frac{1}{6}$.

As in Question 4.4, we have:

```
ACF<-ARMAacf(ma=list(-1/6,-1/6))
ACF
#         0          1          2          3
# 1.0000000 -0.1315789 -0.1578947  0.0000000
```

We can write our own function as before, and this time, we use the sign of θ as given in the question because this is not an inbuilt R function.

```
MA2rho<-function(theta1,theta2)
{
rho0<-1
rho1<-(-theta1+theta1*theta2)/(1+theta1^2+theta2^2)
rho2<-(-theta2)/(1+theta1^2+theta2^2)
rho3<-0
return(list=c(rho0,rho1,rho2,rho3))
}
MA2rho(1/6,1/6)
#[1]  1.0000000 -0.1315789 -0.1578947  0.0000000
```

(ii) $\theta_1 = -1$ and $\theta_2 = 6$.

```
ARMAacf(ma=c(1,-6))
#         0          1          2          3
# 1.0000000 -0.1315789 -0.1578947  0.0000000

MA2rho(-1,6)
#[1]  1.0000000 -0.1315789 -0.1578947  0.0000000
```

(b) How do the roots of the corresponding characteristic polynomials compare?

The characteristic equation $1 - \frac{1}{6}x - \frac{1}{6}x^2$ has roots given by:

```
polyroot(c(1,-1/6,-1/6))
#[1]  2-0i -3+0i
```

The characteristic equation $1 + x - 6x^2$ has roots given by:

```
polyroot(c(1,1,-6))
#[1]  0.5000000-0i -0.3333333+0i
```

The roots of the two equations are reciprocals of one another.

Only the MA(2) model with $\theta_1 = \theta_2 = \frac{1}{6}$ is invertible because both of its roots are > 1.

4.13
Consider the AR(1) model $Y_t = \phi Y_{t-1} + e_t$. Show by taking variances of both sides that, if $|\phi| = 1$, the process cannot be stationary.

$$
\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
Var(Y_t) &= \phi^2 Var(Y_{t-1}) + Var(e_t).
\end{aligned}
$$

If we assume stationary, $Var(Y_t) = Var(Y_{t-1})$.

$$
\begin{aligned}
Var(Y_t) &= \phi^2 Var(Y_t) + \sigma_e^2 \\
Var(Y_t) &= \frac{\sigma_e^2}{1-\phi^2}.
\end{aligned}
$$

If $|\phi| = 1$, then $1 - \phi^2 = 0$, and thus $Var(Y_t)$ is not defined.

4.14
Consider an MA(6) model with $\theta_1 = 0.5, \theta_2 = -0.25, \theta_3 = 0.125, \theta_4 = -0.0625, \theta_5 = 0.03125$ and $\theta_6 = -0.015625$.

Find a much simpler model that has nearly the same ψ weights.

Notice that these coefficients are decreasing exponentially at a rate of 0.5, while alternating in sign.

Furthermore, the coefficients have nearly died out by θ_6.

Thus an AR(1) model with $\phi = -0.5$ would be nearly the same process.