UNIVERSITY of LIMERICK

OLLSCOIL LUIMNIGH

Time Series Analysis

MS 4218

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Outline

Forecasting

- Deterministic trends
- ARIMA forecasting
- Prediction limits
- Examples
- Forecasting transformed series

Mean Square Error Forecasting

If we have an observed time series, $Y_1, Y_2, \ldots, Y_{t-1}, Y_t$, we often wish to predict a future value, Y_{t+l}, which is l time units ahead.

Time t is the time origin, and l is the lead time for the forecast.

The forecast time, $\hat{Y}_t(l)$, is called the mean square error forecast and is conditional on the times up to time t.

$$\hat{Y}_t(l) = E(Y_{t+l}|Y_1, Y_2, \ldots, Y_t).$$

Deterministic trends

$$Y_t = \mu_t + X_t.$$

Assume X_t is white noise with variance of γ_0.

$$\begin{aligned}
\hat{Y}_t(l) &= E\{(\mu_{t+l} + X_{t+l}) | Y_1, Y_2, \ldots, Y_t\} \\
&= E(\mu_{t+l} | Y_1, Y_2, \ldots, Y_t) + E(X_{t+l} | Y_1, Y_2, \ldots, Y_t) \\
&= \mu_{t+l} + E(X_{t+l}) = \mu_{t+l}.
\end{aligned}$$

Forecasting is just extrapolating the deterministic time trend into the future, but is trend constant?

Seasonal Models

If $\mu_t = \mu_{t+12}$, then

$$\begin{aligned}\hat{Y}_t(l) &= \mu_{t+12+l} \\ &= \hat{Y}_t(12+l).\end{aligned}$$

The forecast is also periodic, which is desirable.

Forecast Error

The forecast error is unbiased:

$$
\begin{aligned}
e_t(l) &= Y_{t+l} - \hat{Y}_t(l) \\
&= (\mu_{t+l} + X_{t+l}) - \mu_{t+l} \\
&= X_{t+l}. \\
\therefore E\{e_t(l)\} &= E(X_{t+l}) = 0. \\
Var\{e_t(l)\} &= Var(X_{t+l}) \\
&= \gamma_0.
\end{aligned}
$$

Cosine Trend

In TSLecture3, the cosine trend model for the average monthly temperature was estimated as follows:

```
data(tempdub)
har.=harmonic(tempdub,m=1)

modelCos=lm(tempdub~har.)
summary(modelCos)

#Coefficients:
#                 Estimate Std. Error t value Pr(>|t|)
#(Intercept)       46.2660     0.3088 149.816  < 2e-16 ***
#har.cos(2*pi*t) -26.7079     0.4367 -61.154  < 2e-16 ***
#har.sin(2*pi*t)  -2.1697     0.4367  -4.968 1.93e-06 ***

#Residual standard error: 3.706 on 141 degrees of freedom
#Multiple R-squared: 0.9639,     Adjusted R-squared: 0.9634
#F-statistic:  1882 on 2 and 141 DF,  p-value: < 2.2e-16
```

Cosine Trend cont.

Thus the regression equation is given by:

$$\hat{\mu}_t = 46.266 + (-26.7079)Cos(2\pi t) + (-2.1697)Sin(2\pi t)$$

Time origin is January 1964 and time end is December 1975.

To forecast June (end of May) 1976 T°, we use $t = 1976.41667$.

$$\begin{aligned}\hat{\mu}_t &= 46.266 + (-26.7079)Cos(2\pi 1976.41667) \\ &\quad +(-2.1697)Sin(2\pi 1976.41667) \\ &= 68.3^\circ\text{F}.\end{aligned}$$

AR(1) Forecasting one step ahead

An AR(1) process with non-zero mean is given by:

$$Y_t - \mu \quad = \quad \phi(Y_{t-1} - \mu) + e_t.$$

To forecast one time unit ahead, i.e., at Y_{t+1}, we replace t with $t+1$.

$$Y_{t+1} - \mu \quad = \quad \phi(Y_t - \mu) + e_{t+1}.$$

Given $Y_1, Y_2, \ldots, Y_t$, we take the expectation of both sides

$$\begin{aligned} E\{(Y_{t+1} - \mu)|Y_1, Y_2, \ldots, Y_t\} &= E[\{\phi(Y_t - \mu) + e_{t+1}\}|Y_1, \ldots, Y_t] \\ &= E\{\phi(Y_t - \mu)|Y_1, Y_2, \ldots, Y_t\} \\ &\quad + E(e_{t+1}|Y_1, Y_2, \ldots, Y_t). \end{aligned}$$

AR(1) Forecasting 1 step ahead

$$
\begin{aligned}
E\{(Y_{t+1} - \mu) | Y_1, Y_2, \ldots, Y_t\} &= E(Y_{t+1} | Y_1, Y_2, \ldots, Y_t) - \mu \\
&= \hat{Y}_t(1) - \mu. \\
E\{\phi(Y_t - \mu) | Y_1, Y_2, \ldots, Y_t\} &= \phi\{E(Y_t | Y_1, Y_2, \ldots, Y_t) - \mu\} \\
&= \phi(Y_t - \mu). \\
E(e_{t+1} | Y_1, Y_2, \ldots, Y_t) &= 0 \because \text{ of independence} \\
\therefore \hat{Y}_t(1) &= \mu + \phi(Y_t - \mu).
\end{aligned}
$$

AR(1) Forecasting / steps ahead

$$E\{(Y_{t+l}-\mu)|Y_1,Y_2,\ldots,Y_t\} = E[\{\phi(Y_{t+l-1}-\mu)+e_{t+l}\}|Y_1,\ldots,Y_t]$$

$$E(Y_{t+l}|Y_1,Y_2,\ldots,Y_t) = \hat{Y}_t(l).$$

$$E\{(Y_{t+l-1})|Y_1,Y_2,\ldots,Y_t\} = \hat{Y}_t(l-1).$$

$$E(e_{t+l}|Y_1,Y_2,\ldots,Y_t) = 0 \because \text{ of independence.}$$

$$\hat{Y}_t(l) = \mu+\phi\{\hat{Y}_t(l-1)-\mu\},\ \text{for } l>1.$$

AR(1) Forecasting cont.

$$
\begin{aligned}
\hat{Y}_t(l) &= \mu + \phi\{\hat{Y}_t(l-1) - \mu\}, \text{ for } l > 1 \\
&= \mu + \phi[\mu + \phi\{\hat{Y}_t(l-2) - \mu\} - \mu] \\
&= \mu + \phi^2\{\hat{Y}_t(l-2) - \mu\} \\
&= \mu + \phi^{l-1}\{\hat{Y}_t(1) - \mu\} \\
&= \mu + \phi^l(Y_t - \mu).
\end{aligned}
$$

For $\phi < 1$, as $l \uparrow$, $\phi^l \downarrow$, and thus, in the long term, the forecast approaches the mean.

MLE for AR(1) model for colour data

```
data(color)
m1.color=arima(color,order=c(1,0,0))
m1.color

#Call:
#arima(x = color, order = c(1, 0, 0))
#Coefficients:
#          ar1  intercept
#       0.5705    74.3293
#s.e.   0.1435     1.9151
#sigma^2 estimated as 24.83:
#log likelihood = -106.07,  aic = 216.15
```

If stationary, the intercept, θ_0, is the process mean, μ.

Forecasting colour series one step ahead

$n = 35$ and color[35] $= 67$.

$$\begin{aligned}\hat{Y}_t(1) &= \mu + \phi(Y_t - \mu) \\ &= 74.3293 + (0.5705)(67 - 74.3293) \\ &= 70.14793.\end{aligned}$$

Forecasting colour series two steps ahead

$$
\begin{aligned}
\hat{Y}_t(2) &= \mu + \phi\{\hat{Y}_t(1) - \mu\} \\
&= 74.3293 + (0.5705)(70.14793 - 74.3293) \\
&= 71.94383 \\
&= \mu + \phi^2(Y_t - \mu) \\
&= 74.3293 + (0.5705)^2(67 - 74.3293).
\end{aligned}
$$

Forecasting error for AR(1) one step ahead

See pages 6, 9 and 10.

$$\begin{aligned} e_t(1) &= Y_{t+1} - \hat{Y}_t(1) \\ &= \{\phi(Y_t - \mu) + \mu + e_{t+1}\} - \{\phi(Y_t - \mu) + \mu\} \\ &= e_{t+1}. \end{aligned}$$

Thus the white noise is a sequence of one-step ahead forecast errors and is independent of the history of the process.

$$\begin{aligned} E\{e_t(1)\} &= 0. \text{ Unbiased forecast error.} \\ Var\{e_t(1)\} &= \sigma_e^2. \end{aligned}$$

MA(1) Forecasting one step ahead

An MA(1) process with non-zero mean is given by:

$$Y_t = \mu + e_t - \theta e_{t-1}.$$

To forecast one time unit ahead, i.e., at Y_{t+1}, we replace t with $t+1$, and given $Y_1, Y_2, \ldots, Y_t$, we take the expectation of both sides.

$$\begin{aligned} E(Y_{t+1}|Y_1, Y_2, \ldots, Y_t) &= \hat{Y}_t(1) \\ &= \mu + E\{e_{t+1}|Y_1, Y_2, \ldots, Y_t\} \\ &\quad -\theta E(e_t|Y_1, Y_2, \ldots, Y_t) \\ &= \mu - \theta E(e_t|Y_1, Y_2, \ldots, Y_t). \end{aligned}$$

MA(1) Forecasting one step ahead

In TSLecture4c page 3, we saw that an invertible MA process can be represented as an AR(∞) process by substitution:

$$Y_t = -\theta Y_{t-1} - \theta^2 Y_{t-2} - \cdots + e_t.$$

Thus e_t is a function of $Y_1, Y_2, \ldots, Y_t$.

$$E(e_t | Y_1, Y_2, \ldots, Y_t) = e_t.$$

$$\hat{Y}_t(1) = \mu - \theta e_t.$$

Forecasting error for MA(1) one step ahead

$$\begin{aligned} e_t(1) &= Y_{t+1} - \hat{Y}_t(1) \\ &= (\mu + e_{t+1} - \theta e_t) - (\mu - \theta e_t) \\ &= e_{t+1}. \end{aligned}$$

For larger lags, the process estimate is the mean because all the conditional expectations are 0.

$$\hat{Y}_t(l) = \mu \text{ for } l > 1$$

ARMA(1,1) Forecasting 1 step ahead

$$Y_t = \theta_0 + \phi Y_{t-1} + e_t - \theta e_{t-1}.$$

Again, replacing t with $t+1$ and taking conditional expectations, we end up with

$$\hat{Y}_t(1) = \theta_0 + \phi Y_t - \theta e_t.$$

$$\hat{Y}_t(2) = \theta_0 + \phi \hat{Y}_t(1).$$

Forecasting for Non-stationary ARIMA series

To illustrate, consider the random walk with drift defined by

$$Y_t = \theta_0 + Y_{t-1} + e_t.$$

$$\begin{aligned}\hat{Y}_t(1) &= \theta_0 + E\{(Y_t + e_{t+1})|Y_1 \ldots, Y_t\} \\ &= \theta_0 + Y_t.\end{aligned}$$

Similarly, the difference equation form for the lead l forecast is

$$\hat{Y}_t(l) = \theta_0 + \hat{Y}_t(l-1) \qquad l \geq 1,$$

and iterating backward on l yields the explicit expression

$$\hat{Y}_t(l) = \theta_0 l + Y_t \qquad l \geq 1.$$

Forecasting for Non-stationary ARIMA series

In contrast to the AR forecast, the forecast does not converge for long leads but rather follows a straight line with slope $l\theta_0 \; \forall \; l$.

The presence or absence of the constant term $l\theta_0$ significantly alters the nature of the forecast.

For this reason, constant terms should not be included in nonstationary ARIMA models unless the evidence is clear that the mean of the differenced series is significantly different from zero.

Error Forecasts for Non-stationary ARIMA series

The one and l step-ahead forecast errors are:

$$
\begin{aligned}
e_t(1) &= Y_{t+1} - \hat{Y}_t(1) \;=\; e_{t+1}. \\
e_t(l) &= Y_{t+l} - \hat{Y}_t(l) \\
&= \theta_0 l + Y_t + e_{t+1} +, \ldots, e_{t+l} - (\theta_0 l + Y_t) \\
&= e_{t+1} +, \ldots, e_{t+l}. \\
Var\{e_t(l)\} &= l\sigma_e^2.
\end{aligned}
$$

This property is characteristic of the forecast error variance for all nonstationary ARIMA processes.

Prediction Limits for Deterministic Trends

Limits are required for assessing the precision of estimates.

For the trend with white noise X_t, we have seen:

$$\hat{Y}_t(l) = \mu_{t+l}.$$

$$Var\{e_t(l)\} = \gamma_0 \text{ Slide 6.}$$

If X_t follows a Normal distribution, then the forecast error $e_t(l) = Y_{t+l} - \hat{Y}_t(l) = X_{t+l}$ is also Normally distributed.

Prediction Limits for Deterministic Trends cont.

The confidence interval is then the usual form, i.e., the prediction $\pm$ a number of standard errors of the estimate, e.g.,

we can be 95% confident that the estimate at lead l is given by:

$$\hat{Y}_t(l) \pm z_{0.025} \times \sqrt{Var\{e_t(l)\}} \quad =$$

$$\hat{Y}_t(l) \pm 1.96 \times \sqrt{\sigma_e^2}.$$

Prediction limits Example

The output for the Cosine model gave a Residual standard error: 3.706.

This is σ_e, i.e., $\sqrt{\sigma_e^2} = \sqrt{\gamma_0}$.

The 95% confidence interval for the average June 1976 temperature is given by:

$$68.3 \pm (1.96 \times 3.706) \quad \approx \quad [61.5^\circ \rightarrow 75.55^\circ]\mathrm{F}.$$

Prediction Limits for Arima Models

The limits are more complicated here, e.g.,

for an AR(1) model,

$$Var\{e_t(l)\} = \sigma_e^2 \left(\frac{1 - \phi^{2l}}{1 - \phi^2} \right).$$

This is what R calculates. Let R do all the work for the given model!

In an exam situation, just use $\pm 1.96\sqrt{\sigma_e^2}$.

Deterministic Cosine Trend Forecasts

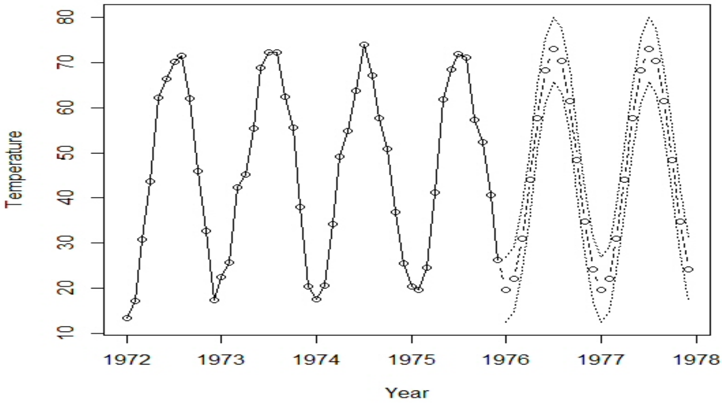
```
data(tempdub)

har.=harmonic(tempdub,1)

m5.tempdub=arima(tempdub,order=c(0,0,0),xreg=har.)
newhar.=harmonic(ts(rep(1,24), start=c(1976,1),freq=12),1)

win.graph(width=4.875, height=2.5,pointsize=8)
plot(m5.tempdub,n.ahead=24,n1=c(1972,1),newxreg=newhar.,
type="b",ylab="Temperature",xlab="Year")
```

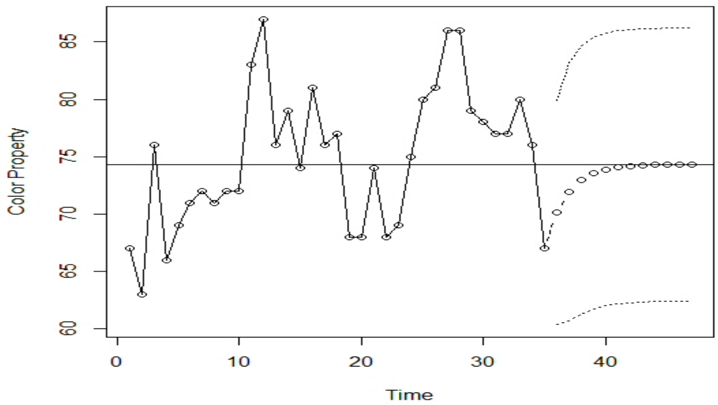
Forecasts and Limits for Temperature Cosine Trend



Arima Forecast and Limits for Colour series

```
data(color);m1.color=arima(color,order=c(1,0,0))
plot(m1.color,n.ahead=12,type="b",
xlab="Time", ylab="Color Property")
abline(h=coef(m1.color)[names(coef(m1.color))=='intercept'])
```

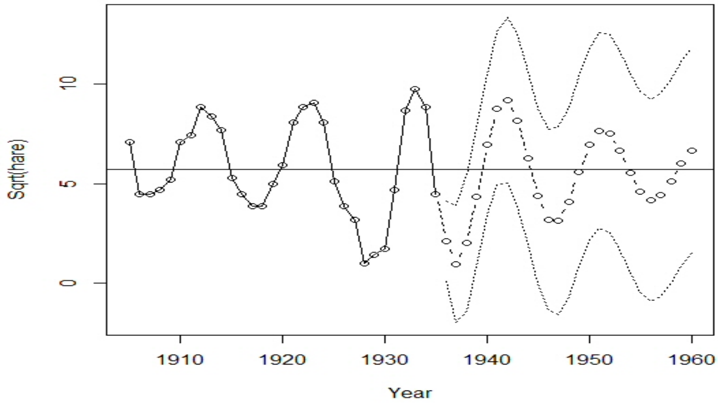
Forecast and Limits for Colour series



Forecast and Limits for Hare abundance series

```
data(hare)
m1.hare=arima(sqrt(hare),order=c(3,0,0))
plot(m1.hare, n.ahead=25,type='b',
xlab="Year",ylab="Sqrt(hare)")
abline(h=coef(m1.hare)[names(coef(m1.hare))=='intercept'])
```

Forecast and Limits for Hare abundance series



Forecasting Transformed Series

Forecasting series made stationary by differencing.

Either

- 1. forecast original series or
- 2. forecast the stationary differenced series and then undoing the difference by summing to obtain the forecast in original terms.

1. Forecasting IMA(1,1) model

$$
\begin{aligned}
Y_t &= Y_{t-1} + e_t - \theta e_{t-1}. \\
E(Y_{t+1} | Y_1, Y_2, \ldots, Y_t) &= \hat{Y}_t(1) \\
&= Y_t - \theta e_t. \\
\hat{Y}_t(l) &= \hat{Y}_t(l-1) \text{ for } l > 1.
\end{aligned}
$$

2. Forecasting differenced series

$$
\begin{aligned}
W_t &= Y_t - Y_{t-1} \\
E(W_t | Y_1, Y_2, \ldots, Y_t) &= \hat{W}_t(1) \\
&= -\theta e_t \;=\; \hat{Y}_t(1) - Y_t. \\
\hat{W}_t(l) &= \hat{Y}_t(l) - \hat{Y}_t(l-1) \\
&= 0 \text{ for } l > 1.
\end{aligned}
$$

Forecasting Log Transformed Series

Mean square error forecasting is appropriate for Normally distributed random variables.

Taking exponent to undo the transformation converts a random variable to an asymmetrical log-Normal random variable.

$Z_t \sim N(\mu, \sigma)$ and $Y_t = \exp(Z_t)$ and $\log(Y_t) \sim N(\mu, \sigma)$.

For this distribution, the optimal forecast is the median of the distribution of Z_{t+l} conditional on $Z_t, Z_{t-1}, \ldots, Z_2, Z_1$.

The median is unaffected by the skew.

Next

- Seasonal Arima models
- Time Series in the Frequency domain