Tutorial Sheet 2

2.1
Suppose $E(X) = 2$, $Var(X) = 9$,
$E(Y) = 0$, $Var(Y) = 4$, and
$Corr(X, Y) = 0.25$.

Find:
(a) $Var(X + Y)$.

(b) $Cov(X, X + Y)$.

(c) $Corr(X + Y, X - Y)$.

2.2
If X and Y are dependent but $Var(X) = Var(Y)$, find $Cov(X + Y, X - Y)$.

2.3
Let X have a distribution with mean μ and variance σ^2.
Let $Y_t = X$ for all t.

(a) Show that $\{Y_t\}$ is strictly and weakly stationary.

(b) Find the auto-covariance function for $\{Y_t\}$.

(c) Sketch a "typical" time plot of Y_t.

2.4
Suppose $Y_t = 5 + 2t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with auto-covariance function γ_k.

(a) Find the mean function for $\{Y_t\}$.

(b) Find the auto-covariance function for $\{Y_t\}$.

(c) Is $\{Y_t\}$ stationary? Why or why not?

2.5
Suppose $Cov(X_t, X_{t-k}) = \gamma_k$ is free of t but that $E(X_t) = 3t$.

(a) Is $\{X_t\}$ stationary?

(b) Let $Y_t = 7 - 3t + X_t$. Is $\{Y_t\}$ stationary?

2.6
Let $Y_1 = \theta_0 + e_1$, and then for $t > 1$ define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here θ_0 is a constant. The process $\{Y_t\}$ is called a random walk with drift.

(a) Show that Y_t may be rewritten as $Y_t = t\theta_0 + e_t + e_{t-1} + \ldots + e_1$.

(b) Find the mean function for Y_t.

(c) Find the auto-covariance function for Y_t.

2.7
Let $\{X_t\}$ be a time series in which we are interested. However, because the measurement process itself is not perfect, we actually observe $Y_t = X_t + e_t$.

We assume that $\{X_t\}$ and $\{e_t\}$ are independent processes. We call X_t the signal and e_t the measurement noise or error process.

If $\{X_t\}$ is stationary with autocorrelation function ρ_k, show that $\{Y_t\}$ is also stationary, and for $k \geq 1$, that

$$Corr(Y_t, Y_{t-k}) = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_X^2}}.$$

We call $\frac{\sigma_X^2}{\sigma_e^2}$ the signal-to-noise ratio, or SNR. Note that the larger the SNR, the closer the autocorrelation function of the observed process $\{Y_t\}$ is to the autocorrelation function of the desired signal $\{X_t\}$.

2.8
A random walk has equation: $Y_t = Y_{t-1} + e_t$, with $Y_1 = e_1$.

If $e_i \sim N(\mu = 0, \sigma^2 = 1)$, use R to produce 3 simultaneous random walks over 100 time points in 3 different colours.

2.9
A random cosine wave has equation: $Y_t = cos\left\{2\pi\left(\frac{t}{12} + \Phi\right)\right\}$.

If $\Phi \sim U(a = 0, b = 1)$, use R to produce random cosine wave with 40 time points.

2.10

Suppose that $\{Y_t\}$ is stationary with auto-covariance function ρ_k.

(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary by finding the mean and auto-covariance function for $\{W_t\}$.

(b) Show that $U_t = \nabla^2 Y_t = \nabla(Y_t - Y_{t-1}) = Y_t - 2Y_{t-1} + Y_{t-2}$ is stationary. (You need not find the mean and auto-covariance function for $\{U_t\}$.)

(c) Suppose $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with auto-covariance function γ_k and β_0 and β_1 are constants.

Show that $\{Y_t\}$ is not stationary but that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary.