UNIVERSITY of LIMERICK

OLLSCOIL LUIMNIGH

Time Series Analysis

MS 4218

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Outline

Models for stationary time series cont.

Auto-Regressive (AR) processes

- AR(1)
- AR(2)
- AR(p)

AR(p): Auto-regressive process of order p

A linear combination of the p most recent values of Y_t and a white noise term.

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t.$$

e_t incorporates everything new in the series at time t and is independent of all Y_t's.

Assumed to be zero mean after process mean has been subtracted.

$$(Y_t - \mu) = \phi(Y_{t-1} - \mu) + \cdots + \phi_p(Y_{t-p} - \mu) + e_t \rightarrow$$

$$Y_t = \phi Y_{t-1} + \cdots + \phi_p Y_{t-p} + e_t.$$

$$E(Y_t) = 0.$$

AR(1) process

$$Y_t = \phi Y_{t-1} + e_t.$$

$$Var(Y_t) = \phi^2 Var(Y_{t-1}) + Var(e_t).$$

Assumed to be stationary $\Rightarrow Var(Y_t) = Var(Y_{t-1})$.

$$Var(Y_t) = Cov(Y_t, Y_t) = \gamma_0$$

$$\Rightarrow \gamma_0 - \phi^2\gamma_0 = \sigma_e^2$$

$$\therefore \gamma_0 = \frac{\sigma_e^2}{1-\phi^2}.$$

As $Var(Y_t) > 0$, then $1 - \phi^2 > 0$, and so $|\phi| < 1$.

Auto-covariance for AR(1) process

Multiply both sides of $Y_t = \phi Y_{t-1} + e_t$ by Y_{t-k} and take expectation.

$$E(Y_t Y_{t-k}) \;=\; \phi E(Y_{t-1} Y_{t-k}) + E(e_t Y_{t-k})$$

If $E(Y_t) \;=\; 0 \;=\; E(Y_{t-k})$, then

$E(Y_t Y_{t-k}) = Cov(Y_t, Y_{t-k}) = \gamma_{t-(t-k)} = \gamma_k.$

$$\gamma_k \;=\; \phi(\gamma_{t-1-(t-k)} = \gamma_{k-1}) + 0$$

$$\gamma_k \;=\; \phi\gamma_{k-1}, \text{ for } k = 1, 2, 3, \ldots$$

Auto-covariance for AR(1) process

$$
\begin{aligned}
\gamma_1 &= \phi\gamma_0 \\
&= \phi\frac{\sigma_e^2}{1-\phi^2} \\
\gamma_2 &= \phi\gamma_1 = \phi^2\gamma_0 \\
&= \phi^2\frac{\sigma_e^2}{1-\phi^2}. \\
\gamma_k &= \phi^k\gamma_0 \\
&= \phi^k\frac{\sigma_e^2}{1-\phi^2}.
\end{aligned}
$$

Auto-correlation for AR(1) process

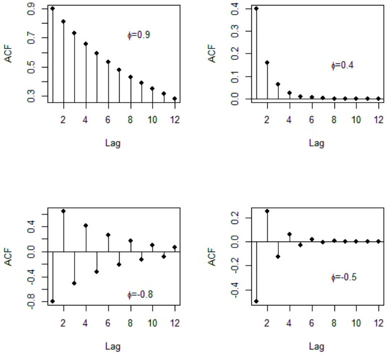
$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\phi^k \gamma_0}{\gamma_0} = \phi^k \text{ for } k = 1, 2, 3, \ldots$$

$|\phi| < 1 \Rightarrow \rho \downarrow$ exponentially as $k \uparrow$.

$\phi > 0 \Rightarrow \rho > 0$.

$\phi < 0 \Rightarrow \rho > 0$ when k is even, and $\rho < 0$ when k is odd.

Auto-correlation for several AR(1) models

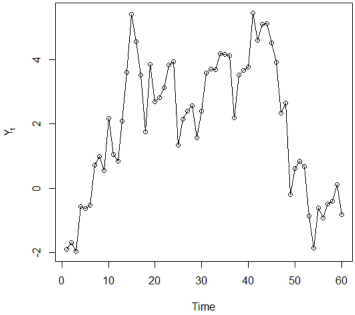


Slow decline in ρ_k for large ϕ.
ρ_k has alternate declining $+$ and $-$ values when $\phi < 0$.

R code for first of Exhibit 4.12

```
par(mfrow=c(2,2))
ACF=ARMAacf(ar=0.9,lag.max=12)
plot(y=ACF[-1],x=1:12,xlab='Lag',
ylab='ACF',type='h')
points(ACF[-1],pch=19)
legend(6,0.8,legend=~phi*"=0.9",bty="n")
abline(h=0)
```

Time plot of an AR(1) series



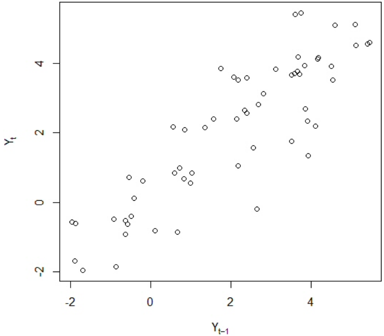
$\phi = 0.9$

```
data(ar1.s)
plot(ar1.s,ylab=expression(Y[t]),type='o')
```

Impression of trend due to high auto-correlation in data.

Plot of Y_t v Y_{t-1} for AR(1)

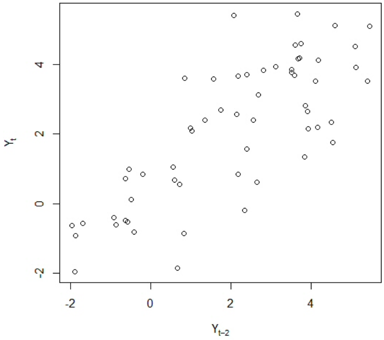
$$\phi = 0.9;\ \rho_1 = \phi^1 = 0.9^1.$$




```
r1=cor(ar1.s[-1],ar1.s[-n])
r1 #0.869
```

Plot of Y_t v Y_{t-2} for AR(1)

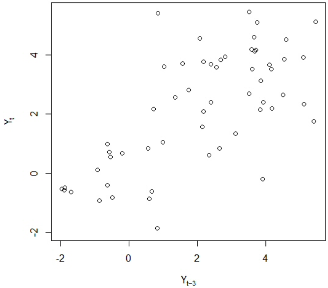
$$\phi = 0.9;\ \rho_2 = \phi^2 = 0.9^2 = 0.81.$$




```
r2=cor(ar1.s[-(1:2)],ar1.s[-((n-1):n)]);
r2 #0.779
```

Plot of Y_t v Y_{t-3} for AR(1)

$$\phi = 0.9;\ \rho_3 = \phi^3 = 0.9^3 = 0.729.$$




```
r3=cor(ar1.s[-(1:3)],ar1.s[-((n-2):n)])
r3 #0.678
```

General linear process version of AR(1) model

We wish to express AR(1) as an infinite sum of white noise terms.

$$
\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
&= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\
&= e_t + \phi e_{t-1} + \phi^2 Y_{t-2} \\
&= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^k Y_{t-k} \\
&= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots
\end{aligned}
$$

if $|\phi| < 1$ and $k \to \infty$. $\quad (\psi_j = \phi^j)$

Stationarity of AR(1) process

$$Y_t = \phi Y_{t-1} + e_t$$

$\iff |\phi| < 1.$

This is the AR(1) stationarity condition.

AR(2) process

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t \\
\Rightarrow e_t &= Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2}.
\end{aligned}
$$

The AR(2) process characteristic polynomial and equation are:

$$
\phi(x) = 1 - \phi_1 x - \phi_2 x^2.
$$

$$
1 - \phi_1 x - \phi_2 x^2 = 0.
$$

The two quadratic roots may or may not be complex.

Stationarity conditions of AR(2) process

$\exists$ a stationary solution to $1 - \phi_1 x - \phi_2 x^2 = 0 \iff$ **both of the roots are $> |1|$.**

The roots are given by:

$$x = \frac{\phi_1 \pm \sqrt{(-\phi_1)^2 + 4\phi_2}}{-2\phi_2}.$$

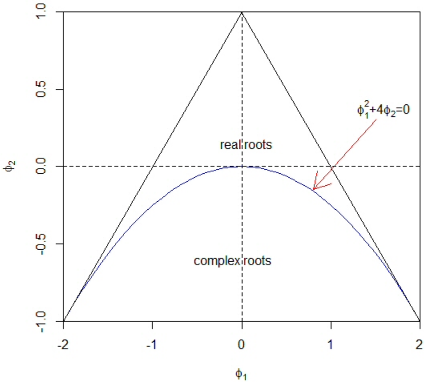
Stationarity exists if:

$$\phi_1 + \phi_2 < 1.$$

$$\phi_2 - \phi_1 < 1.$$

$$|\phi_2| < 1.$$

Stationarity parameter region for AR(2)



For complex roots, $\sqrt{\phi_1^2 + 4\phi_2} < 0$, thus ϕ_2 must be < 0.

Acvf of AR(2) process

Multiply both sides of $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$ by Y_{t-k} and take expectation.

$$E(Y_t Y_{t-k}) \;=\; \phi_1 E(Y_{t-1} Y_{t-k}) + \phi_2 E(Y_{t-2} Y_{t-k}) + E(e_t Y_{t-k})$$

If $E(Y_t) \;=\; 0 \;=\; E(Y_{t-k})$, then

$$E(Y_t Y_{t-k}) = Cov(Y_t, Y_{t-k}) = \gamma_{t-(t-k)} = \gamma_k.$$

$$\gamma_k \;=\; \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \text{ for } k = 1, 2, 3, \ldots$$

We will do $Var(Y_t) = \gamma_{(k=0)}$ shortly.

Acf of AR(2) process in terms of parameters

$$
\begin{aligned}
\frac{\gamma_k}{\gamma_0} &= \phi_1 \frac{\gamma_{k-1}}{\gamma_0} + \phi_2 \frac{\gamma_{k-2}}{\gamma_0} \\
\rho_k &= \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}. \\
\rho_1 &= \phi_1 \rho_0 + \phi_2 \rho_{-1} \; = \; \phi_1 + \phi_2 \rho_1. \\
\rho_1 - \phi_2 \rho_1 &= \phi_1 \\
\Rightarrow \rho_1 &= \frac{\phi_1}{1 - \phi_2}.
\end{aligned}
$$

Acf of AR(2) process in terms of parameters cont.

$$\rho_1 = \frac{\phi_1}{1-\phi_2}.$$

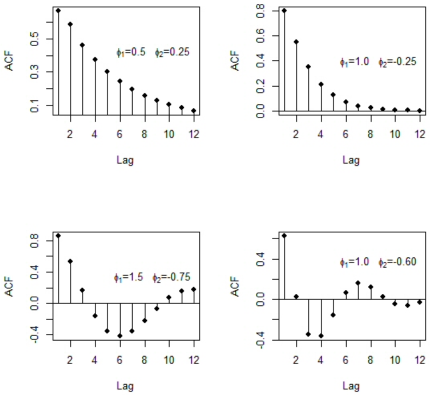
$$\rho_2 = \phi_1\rho_1 + \phi_2\rho_0 = \phi_1\frac{\phi_1}{1-\phi_2} + \phi_2$$

$$\rho_2 = \frac{\phi_1^2 + \phi_2(1-\phi_2)}{1-\phi_2}.$$

Once we have ρ_1 and ρ_2, we can calculate any ρ_k via $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$.

Auto-correlation functions for several AR(2) models

Acf can have many different shapes.



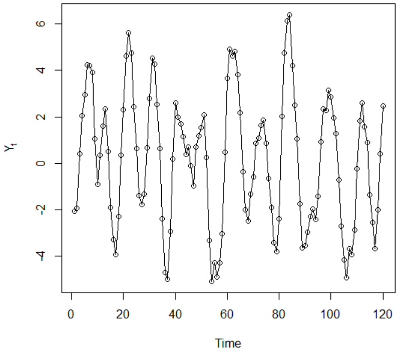
Large negative ϕ_2 may have damped cosine wave where $R = \sqrt{-\phi_2}$ is the damping factor.

R code for first graph of preceding figure

```
par(mfrow=c(2,2))
ACF=ARMAacf(ar=c(0.5,0.25),lag.max=12)
plot(y=ACF[-1],x=1:12,
xlab='Lag',ylab='ACF',type='h')
points(ACF[-1],pch=19)
#legend(4,0.5,legend=expression(paste(
#phi[1],"=0.5","   ",phi[2],"=0.25")),
#bty="n")
legend(4,0.5,
legend=~phi[1]*"=0.5"*"   "*phi[2]*"=0.25",bty="n")
abline(h=0)
```

Time plot of AR(2) model: $\phi_1 = 1.5$ and $\phi_2 = -0.75$

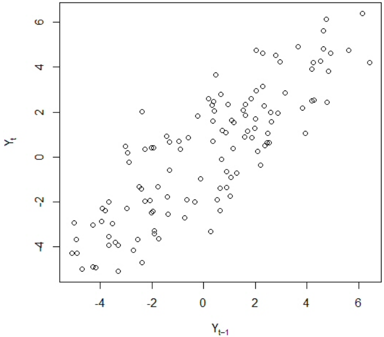
```
data(ar2.s)
plot(ar2.s,ylab=expression(Y[t]),type='o')
```



Periodic every 12 time units.

Plot of Y_t v Y_{t-1} for AR(2)

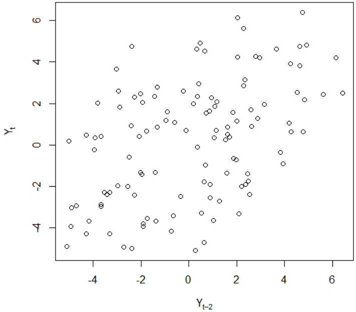
$$\phi_1 = 1.5,\ \phi_2 = -0.75;\ \rho_1 = \frac{\phi_1}{1-\phi_2} = 0.857.$$




```
n<-length(ar2.s)
r1=cor(ar2.s[-1],ar2.s[-n]) ;
r1 #0.837
```

Plot of Y_t v Y_{t-2} for AR(2)

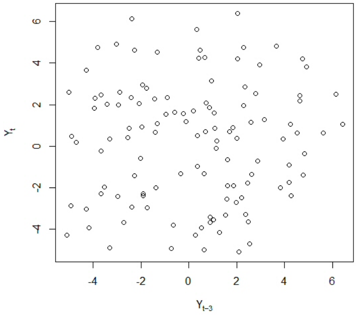
$$\phi_1 = 1.5,\ \phi_2 = -0.75;\ \rho_2 = \frac{\phi_1^2 + \phi_2(1-\phi_2)}{1-\phi_2} \ = \ 0.536.$$




```
r2=cor(ar2.s[-(1:2)],ar2.s[-((n-1):n)])
r2 #0.463.
```

Plot of Y_t v Y_{t-3} for AR(2)

$$\phi_1 = 1.5, \phi_2 = -0.75$$
$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \Rightarrow \rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1 = 0.161.$$




```
r3=cor(ar2.s[-(1:3)],ar3.s[-((n-2):n)]); r3 #0.033
```

Variance for AR(2) model in terms of parameters

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t. \\
Var(Y_t) &= \phi_1^2 Var(Y_{t-1}) + \phi_2^2 Var(Y_{t-2}) + 2\phi_1\phi_2 Cov(Y_{t-1}, Y_{t-2}) + \sigma_e^2. \\
\gamma_0 &= (\phi_1^2 + \phi_2^2)\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2. \\
\gamma_1 &= \phi_1\gamma_0 + \phi_2\gamma_1 \Rightarrow \gamma_1 - \phi_2\gamma_1 \ = \ \phi_1\gamma_0 \\
\therefore \gamma_1 &= \frac{\phi_1}{1-\phi_2}\gamma_0. \\
\gamma_0(1-\phi_1^2-\phi_2^2) &= 2\phi_1\phi_2\frac{\phi_1}{1-\phi_2}\gamma_0 + \sigma_e^2 \\
\sigma_e^2 &= \gamma_0\left(1-\phi_1^2-\phi_2^2-2\frac{\phi_1^2\phi_2}{1-\phi_2}\right) \\
\gamma_0 &= \frac{\sigma_e^2(1-\phi_2)}{(1-\phi_1^2-\phi_2^2)(1-\phi_2)-2\phi_1^2\phi_2} = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_e^2}{(1-\phi_2)^2-\phi_1^2}.
\end{aligned}
$$

ψ coefficients for AR(2) model

For the general linear process, let $\psi_j = \phi_j$.

$$
\begin{aligned}
Y_t &= \psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \dots \\
Y_{t-1} &= \psi_0 e_{t-1} + \psi_1 e_{t-2} + \psi_2 e_{t-3} + \dots \\
Y_{t-2} &= \psi_0 e_{t-2} + \psi_1 e_{t-3} + \psi_2 e_{t-4} + \dots \\
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t \\
&= e_t + \phi_1(\psi_0 e_{t-1} + \psi_1 e_{t-2} + \psi_2 e_{t-3} + \dots) \\
&\quad + \phi_2(\psi_0 e_{t-2} + \psi_1 e_{t-3} + \psi_2 e_{t-4} + \dots).
\end{aligned}
$$

ψ coefficients for AR(2) model cont.

Equating coefficients:

$$
\begin{aligned}
e_t : \qquad \psi_0 &= 1. \\
e_{t-1} : \qquad \psi_1 &= \phi_1 \psi_0 = \phi_1. \\
e_{t-2} : \qquad \psi_2 &= \phi_1 \psi_1 + \phi_2 \psi_0 \\
&= \phi_1^2 + \phi_2. \\
e_{t-k} : \qquad \psi_k &= \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2}.
\end{aligned}
$$

General AR(p) process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

$$\Rightarrow e_t = Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \cdots - \phi_p Y_{t-p} + e_t.$$

The characteristic polynomial and equation are:

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p.$$

$$1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0.$$

$\exists$ a stationary solution $\iff$ **the p roots are all $> |1|$.**

$\phi_1 + \phi_2 + \cdots + \phi_p < 1$ and $|\phi_p| < 1$ are necessary but not sufficient conditions.

Deriving Yule-Walker equations

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t.$$

Multiply the p^{th} order equation by Y_{t-k} and take expectations.

$$\begin{aligned} E(Y_t Y_{t-k}) &= \phi_1 E(Y_{t-1} Y_{t-k}) + \phi_2 E(Y_{t-2} Y_{t-k}) + \ldots \\ &\quad + \phi_p E(Y_{t-p} Y_{t-k}) + E(e_t Y_{t-k}). \end{aligned}$$

Assuming zero means and stationarity, we can get the acvf:

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \cdots + \phi_p \gamma_{k-p}.$$

On $\div$ each term by γ_0, we can get the acf:

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p}.$$

Deriving Yule-Walker equations cont.

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p}.$$

Using $\rho_0 = 1$ and $\rho_{-1} = \rho_1$ and then setting $k = 1, 2, \ldots, p$,

$$\rho_1 = \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 + \cdots + \phi_p\rho_{p-1}.$$

$$\rho_2 = \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 + \cdots + \phi_p\rho_{p-2}.$$

$$\vdots$$

$$\rho_p = \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \phi_3\rho_{p-3} + \cdots + \phi_p.$$

Given $\phi_1, \ldots, \phi_p$, $\rho_1, \ldots, \rho_p$ can be solved numerically.

Variance of AR(p) in terms of parameters

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t.$$

Multiply both sides by Y_t and take expectations.

$$\begin{aligned} E(Y_t Y_t) &= \phi_1 E(Y_t Y_{t-1}) + \phi_2 E(Y_t Y_{t-2}) + \cdots + \phi_p E(Y_t Y_{t-p}) \\ &\quad + E(Y_t e_t). \end{aligned}$$

Assuming $E(Y_t) = 0$, we have

$$\begin{aligned} \gamma_0 &= \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + E\{(\cdots + e_t)e_t\} \\ &= \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + \sigma_e^2. \end{aligned}$$

Variance of AR(p) in terms of parameters

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + \sigma_e^2.$$

Divide across by γ_0.

$$1 = \phi_1\rho_1 + \phi_2\rho_2 + \cdots + \phi_p\rho_p + \frac{\sigma_e^2}{\gamma_0}$$

$$\Rightarrow \gamma_0 = \frac{\sigma_e^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p}$$

The $\rho_1 \ldots \rho_p$ values are known from the Yule-Walker equations and thus the $Var(Y_t)$ can be evaluated.

Next

- Invertibility
- ARMA processes
- Backshift operator