UNIVERSITY of LIMERICK

OLLSCOIL LUIMNIGH

Time Series Analysis

MS 4218

joseph.lynch@ul.ie

Outline

Time Series in the Frequency domain

- Spectral Density functions
- Applications to AR and MA processes

Spectral Analysis

Time series can be described in terms of an average frequency composition.

Spectral analysis distributes the variance of a time series over frequency.

Used a lot in acoustics, communication engineering, geophysical and biomedical sciences.

Fourier sums or series

Well-behaved functions can be approximated over a finite interval by a weighted combination of Sin and Cos functions:

$$Y_t = \sum_{j=1}^{m} \left\{ A_j Cos(2\pi f_j t) + B_j Sin(2\pi f_j t) \right\},$$

$0 < f_1 < f_2 < \cdots < f_m < \frac{1}{2}$ are fixed, and

A_j and B_j are independent Normal(0,σ_j^2) random variables.

$\{Y_t\}$ is stationary with zero mean.

$\gamma_k = \sum_{j=1}^{m} \sigma_j^2 Cos(2\pi k f_j)$, and $\gamma_0 = \sum_{j=1}^{m} \sigma_j^2$.

Spectral density function

A spectral density function has all the properties of a probability density function on the interval $(-\frac{1}{2}, \frac{1}{2}]$, except that the total area under the curve is the process variance rather than 1.

For $-\frac{1}{2} < f \leq \frac{1}{2}$, the spectral density function is given by:

$$S_f \ = \ \gamma_0 + 2\sum_{k=1}^{\infty} \gamma_k Cos(2\pi fk).$$

$S(f)$ cont.

$S(f)$ is the discrete Fourier transform of the sequence,
$\cdots, \gamma_{-2}, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2, \cdots$

γ_k is the inverse Fourier transform of $S(f)$, for $-\frac{1}{2} < f \leq \frac{1}{2}$.

$$\gamma_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} S(f) Cos(2\pi f k) \mathrm{d}f.$$

$S(f)$ and γ_k

It can be shown that

$$S(f) = \sum_{-\infty}^{\infty} \gamma_k e^{-2\pi i k f},$$

which looks more like a standard discrete Fourier transform.

Also

$$\gamma_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} S(f) e^{-2\pi i k f} \mathrm{d}f.$$

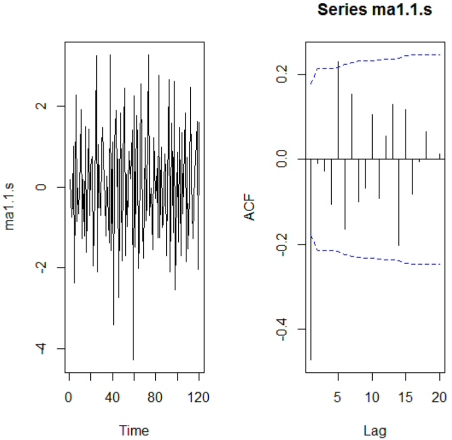
Spectral Density for White Noise

$$S(f) = \sigma_e^2 \; \forall \; -\tfrac{1}{2} < f \leq \tfrac{1}{2}.$$

All frequencies receive equal weight in the spectral representation of white noise.

For any series, $S(f)$ are symmetric about zero frequency, so they are generally only plotted for positive frequencies.

MA(1) $\theta > 0$




```
data(ma1.1.s); plot(ma1.1.s)
acf(ma1.1.s,ci.type="ma")
```

MA(1) Spectral Density

Time series plots shows high frequency oscillation when $\theta > 0$.

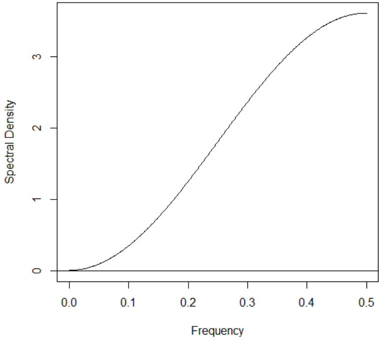
Auto-correlation function shows a strong negative correlation at lag 1 and then tends $\rightarrow 0$ thereafter.

$S(f) = \{1 + \theta^2 - 2\theta Cos(2\pi f)\}\sigma_e^2$.

When $\theta > 0$, $\frac{\mathrm{d}}{\mathrm{d}f} S(f) = 4\pi\theta\sigma_e^2 Sin(2\pi f) > 0$,

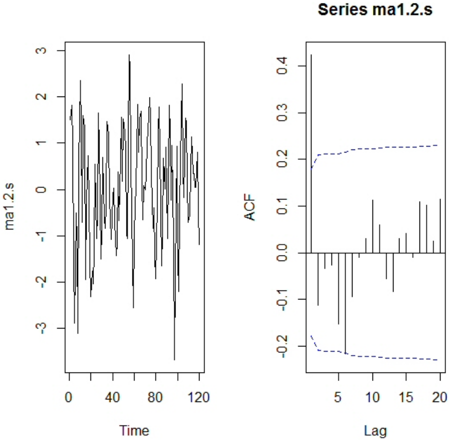
$S(f)$ is an increasing function of non-negative frequency.

MA(1) Spectral Density with $\theta = 0.9$




```
theta=0.9
ARMAspec(model=list(ma=-theta))
```

MA(1) $\theta < 0$: plot graphs separately to see effect!




```
data(ma1.2.s)
plot(ma1.2.s);acf(ma1.2.s,ci.type="ma")
```

MA(1) Spectral Density

Time series plots shows lower frequency oscillation when $\theta < 0$.

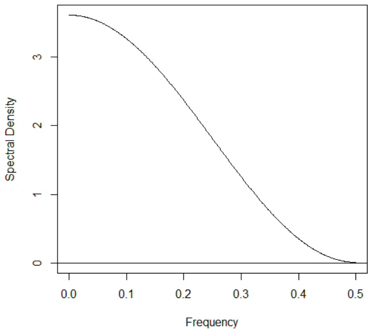
Auto-correlation function shows a strong positive correlation at lag 1 and then tends $\rightarrow 0$ thereafter.

$S(f) = \{1 + \theta^2 - 2\theta Cos(2\pi f)\}\sigma_e^2$.

When $\theta < 0$, $\frac{d}{df}S(f) = 4\pi\theta\sigma_e^2 Sin(2\pi f) < 0$,

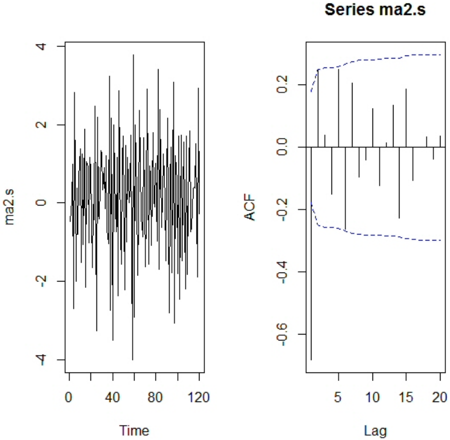
$S(f)$ is a decreasing function of non-negative frequency.

MA(1) Spectral Density with $\theta < 0$




```
theta=-0.9
ARMAspec(model=list(ma=-theta))
```

MA(2) Process




```
data(ma2.s);plot(ma2.s)
acf(ma2.s,ci.type="ma")
```

MA(2) Spectral Density

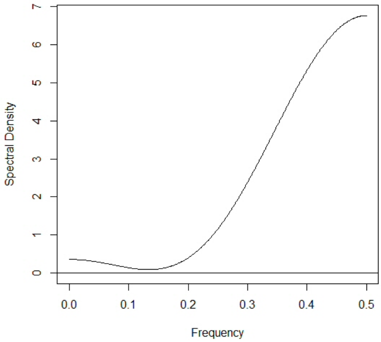
Time series plots shows high frequency oscillation.

Auto-correlation function shows a strong negative correlation at lag 1, and positive value at lag 2, and then tends $\rightarrow$ 0 thereafter.

$$S(f) = \{1 + \theta_1^2 + \theta_2^2 - 2\theta_1(1-\theta_2)Cos(2\pi f) - 2\theta_2 Cos(4\pi f)\}\sigma_e^2.$$

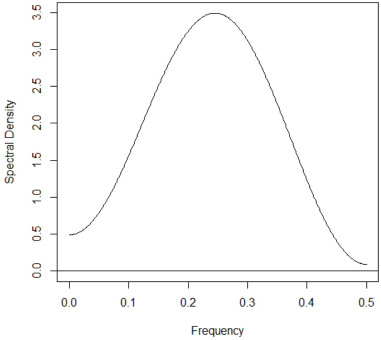
The sign of the derivative $\frac{d}{df}S(f)$ depends on the values of θ_1 and θ_2.

MA(2) Spectral Density with $\theta_1 = 1$ and $\theta_2 = -0.6$



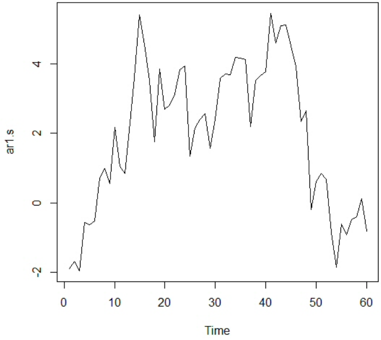

```
theta1=1; theta2=-0.6
ARMAspec(model=list(ma=c(-theta1,-theta2))
```

MA(2) Spectral Density with $\theta_1 = -0.5$ and $\theta_2 = 0.8$




```
theta1=-0.5; theta2=0.8
ARMAspec(model=list(ma=c(-theta1,-theta2))
```

AR(1) with $\phi > 0$




```
data(ar1.s)
plot(ar1.s)
```

AR(1) Spectral Density

Time series plots shows low frequency oscillation when $\phi > 0$.

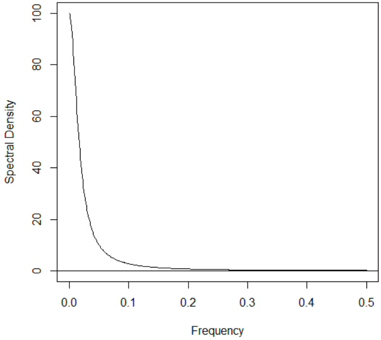
$S(f) = \frac{\sigma_e^2}{1+\phi^2-2\phi Cos(2\pi f)}$.

When $\phi > 0$, the first derivative of the denominator

$\frac{\mathrm{d}}{\mathrm{d}f}\{1+\phi^2-2\phi Cos(2\pi f)\} = 4\pi\phi\sigma_e^2 Sin(2\pi f) > 0$,

so $S(f)$ is a **decreasing** function of non-negative frequency.

AR(1) Spectral Density with $\phi > 0$




```
phi=0.9
ARMAspec(model=list(ar=phi))
```

AR(1) Spectral Density with $\phi < 0$

Time series plots, relatively speaking, shows higher frequency oscillation when $\phi < 0$.

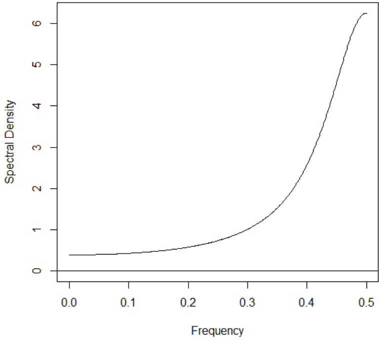
$S(f) = \frac{\sigma_e^2}{1+\phi^2-2\phi Cos(2\pi f)}$.

When $\phi < 0$, the first derivative of the denominator

$\frac{d}{df}\{1+\phi^2-2\phi Cos(2\pi f)\} = 4\pi\phi\sigma_e^2 Sin(2\pi f) < 0$,

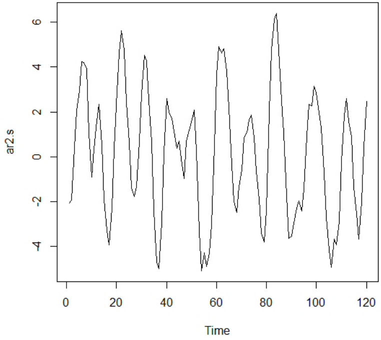
so $S(f)$ is a **increasing** function of non-negative frequency.

AR(1) Spectral Density with $\phi < 0$




```
phi=-0.9
ARMAspec(model=list(ar=phi))
```

AR(2)process




```
data(ar2.s)
plot(ar2.s)
```

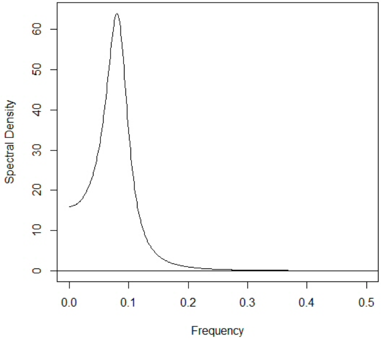
AR(2) Spectral Density

The shape of the time series plot depends on the values of ϕ_1 and of ϕ_2.

$$S(f) = \frac{\sigma_e^2}{1+\phi_1^2+\phi_2^2-2\phi_1(1-\phi_2)Cos(2\pi f)-2\phi_2 Cos(4\pi f)}.$$

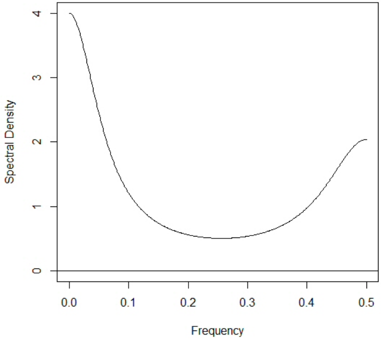
The sign of the derivative $\frac{d}{df}S(f)$ depends on the values of ϕ_1 and ϕ_2.

AR(2) Spectral Density $\phi_1 = 1.5$; $\phi_2 = -0.75$




```
phi1=1.5; phi2=-0.75
ARMAspec(model=list(ar=c(phi1,phi2)))
```

AR(2) Spectral Density $\phi_1 = 0.1; \phi_2 = 0.4$




```
phi1=0.1; phi2=0.4
ARMAspec(model=list(ar=c(phi1,phi2)))
```

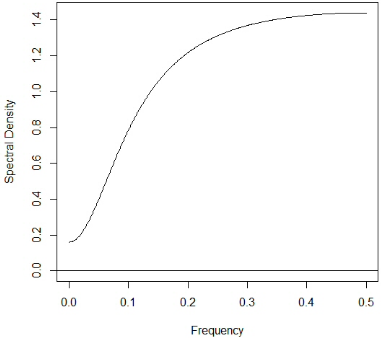
ARMA(1,1) Spectral Density

The spectral density is a combination of MA(1) and AR(1) spectral densities.

$$S(f) = \frac{1+\theta^2-2\theta Cos(2\pi f)}{1+\phi^2-2\phi Cos(2\pi f)}\sigma_e^2.$$

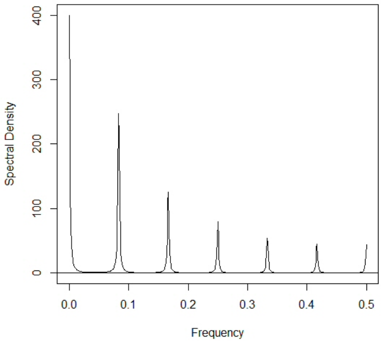
The sign of the derivative $\frac{d}{df}S(f)$ depends on the values of ϕ and θ.

ARMA(1,1) Spectral Density cont.



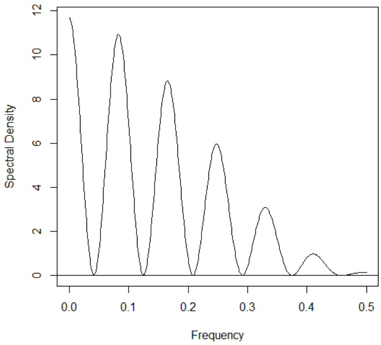

```
phi=0.5; theta=0.8
ARMAspec(model=list(ar=phi, ma=-theta))
```

Seasonal AR Spectral Density




```
phi=0.5; PHI=0.9
ARMAspec(model=list(ar=phi,
seasonal=list(sar=PHI,period=12)))
```

Seasonal MA Spectral Density




```
theta=0.9; THETA=0.9
ARMAspec(model=list(ma=theta,
seasonal=list(sma=THETA,period=12)))
```

Next

The End.