Tutorial Sheet 4b

Part 4.b

4.5
Describe the important characteristics of the auto-correlation function for AR(1) and AR(2) models.

An AR(1) process has exponentially decaying auto-correlations starting from lag 0.

If $\phi > 0$, then all auto-correlations are positive.

If $\phi < 0$, then auto-correlations are alternatively negative and positive.

An AR(2) process has has several different patterns.

If the roots of the characteristic equation, $1 - \phi_1 x - \phi_2 x^2 = 0$, are complex, then the auto-correlation pattern will be a damped cosine wave with a decaying magnitude.

The roots are given by:

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2},$$

and thus the roots are complex if

$$\phi_1^2 + 4\phi_2 < 0.$$

The damped cosine wave from these complex roots has damping factor $R = \sqrt{-\phi_2}$.

4.6
Calculate and sketch the auto-correlation function for each of the following AR(1) models. Plot for sufficient lags that the auto-correlation function has nearly died out.

(a)$\phi = 0.6$.

(b)$\phi = -0.6$.

(c)$\phi = 0.95$. Do out to 20 lags.

(d)$\phi = 0.3$.

(a) $\phi = 0.6$

We can write our own function and use the ϕ value as specified.

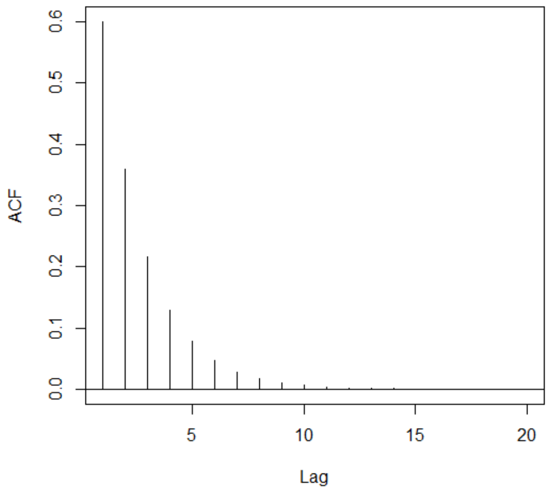
```
AR1rho<-function(phi)
{
rho<-rep(NA,20)
for(k in 1:length(rho))
rho[k]<-phi^k
return(round(rho,4))
}
```

```
AR1rho(0.6)
# [1] 0.6000 0.3600 0.2160 0.1296 0.0778 0.0467 0.0280 0.0168 0.0101 0.0060
#[11] 0.0036 0.0022 0.0013 0.0008 0.0005 0.0003 0.0002 0.0001 0.0001 0.0000
```

Unlike in the case of the θ value in the MA process, the inbuilt ARMAacf() function in R uses the same signed value as our ϕ value in the AR process.

```
ACF=ARMAacf(ar=0.6,lag.max=8)
ACF
#          0          1          2          3          4          5          6
#1.00000000 0.60000000 0.36000000 0.21600000 0.12960000 0.07776000 0.04665600
#          7          8
#0.02799360 0.01679616

plot(y=ACF[-1],x=1:8,xlab='Lag',ylab='ACF',type='h')
abline(h=0)
```

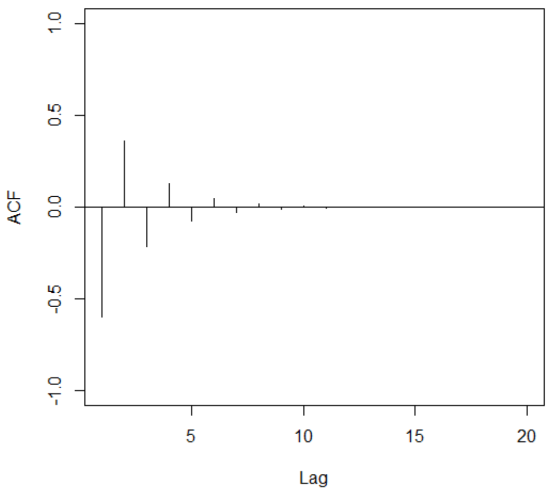

ACF
0.0
0.1
0.2
0.3
0.4
0.5
0.6
5
10
15
20
Lag

(b) $\phi = -0.6$.

```
AR1rho(-0.6)
# [1] -0.6000  0.3600 -0.2160  0.1296 -0.0778  0.0467 -0.0280  0.0168 -0.0101
#[10]  0.0060 -0.0036  0.0022 -0.0013  0.0008 -0.0005  0.0003 -0.0002  0.0001
#[19] -0.0001  0.0000

ACF=ARMAacf(ar=-0.6,lag.max=8)
ACF
#          0           1           2           3           4           5
# 1.00000000 -0.60000000  0.36000000 -0.21600000  0.12960000 -0.07776000
#          6           7           8
# 0.04665600 -0.02799360  0.01679616

plot(y=ACF[-1],x=1:8,xlab='Lag',ylab='ACF',type='h',ylim=c(-1,1))
abline(h=0)
```

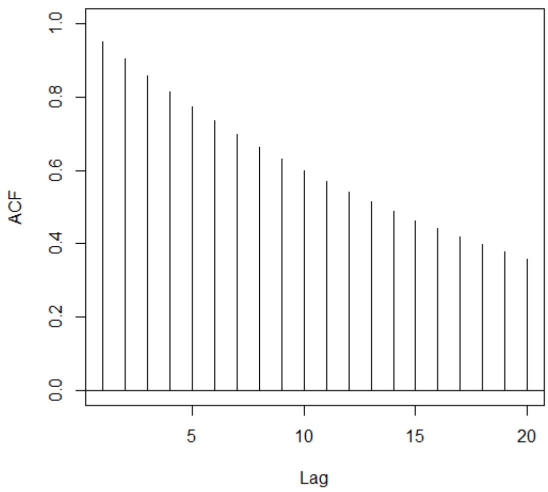


(c) $\phi = 0.95$

```
AR1rho(0.95)
# [1] 0.9500 0.9025 0.8574 0.8145 0.7738 0.7351 0.6983 0.6634 0.6302 0.5987
#[11] 0.5688 0.5404 0.5133 0.4877 0.4633 0.4401 0.4181 0.3972 0.3774 0.3585

ACF=ARMAacf(ar=0.95,lag.max=20)
ACF
#        0         1         2         3         4         5         6         7
#1.0000000 0.9500000 0.9025000 0.8573750 0.8145062 0.7737809 0.7350919 0.6983373
#        8         9        10        11        12        13        14        15
#0.6634204 0.6302494 0.5987369 0.5688001 0.5403601 0.5133421 0.4876750 0.4632912
#       16        17        18        19        20
#0.4401267 0.4181203 0.3972143 0.3773536 0.3584859

plot(y=ACF[-1],x=1:20,xlab='Lag',ylab='ACF',type='h',ylim=c(0,1))
abline(h=0)
```

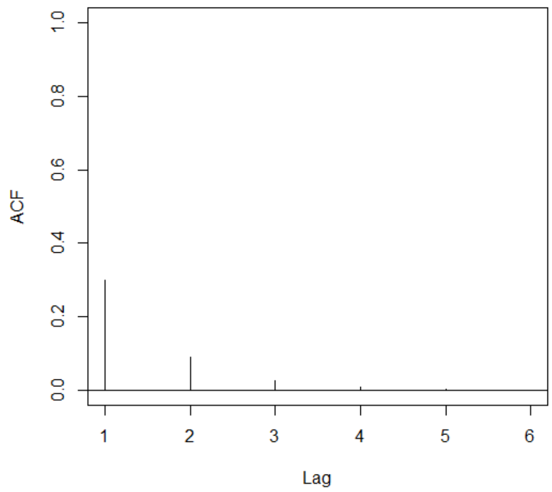


(d) $\phi = 0.3$

```
AR1rho(0.3)
# [1] 0.3000 0.0900 0.0270 0.0081 0.0024 0.0007 0.0002 0.0001 0.0000 0.0000
#[11] 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

ACF=ARMAacf(ar=0.3,lag.max=6)
ACF
#       0        1        2        3        4        5        6
#1.000000 0.300000 0.090000 0.027000 0.008100 0.002430 0.000729

plot(y=ACF[-1],x=1:6,xlab='Lag',ylab='ACF',type='h',ylim=c(0,1))
abline(h=0)
```



4.7
Let $\{Y_t\}$ be an AR(1) process with $-1 < \phi < +1$.

(a) Find the auto-covariance function for $W_t = \nabla Y_t = Y_t - Y_{t-1}$ in terms of ϕ and σ_e^2.

In class, we have seen that, for a stationary AR(1) process:

$$
\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t. \\
Var(Y_t) &= Var(\phi Y_{t-1} + e_t) \\
&= Var(\phi Y_{t-1}) + Var(e_t) \\
&= \phi^2 Var(Y_{t-1}) + Var(e_t) \\
&= \phi^2 Var(Y_t) + \sigma_e^2 \\
Var(Y_t)(1-\phi^2) &= \sigma_e^2 \\
\therefore Var(Y_t) &= \frac{\sigma_e^2}{1-\phi^2} \\
&= \gamma_0. \\
\gamma_1 &= \phi\gamma_0. \qquad \text{Page 6 Lecture 4b} \\
\gamma_k &= \phi^k \gamma_0 \\
&= \phi^k \frac{\sigma_e^2}{1-\phi^2}.
\end{aligned}
$$

For $k > 0$, we have:

$$
\begin{aligned}
Cov(W_t, W_{t-k}) &= Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\
&= Cov(Y_t, Y_{t-k}) + Cov(Y_t, -Y_{t-k-1}) + Cov(-Y_{t-1}, Y_{t-k}) \\
&\quad + Cov(-Y_{t-1}, -Y_{t-k-1}) \\
&= \gamma_{t-(t-k)} - \gamma_{t-(t-k-1)} - \gamma_{(t-1)-(t-k)} + \gamma_{(t-1)-(t-k-1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\
&= (\phi^k - \phi^{k+1} - \phi^{k-1} + \phi^k)\frac{\sigma_e^2}{1-\phi^2} \\
&= (2\phi^k - \phi^{k+1} - \phi^{k-1})\frac{\sigma_e^2}{1-\phi^2} \\
&= (2\phi - \phi^2 - 1)\phi^{k-1}\frac{\sigma_e^2}{1-\phi^2} \\
&= -(1-\phi)^2\phi^{k-1}\frac{\sigma_e^2}{(1-\phi)(1+\phi)} \\
&= -\frac{1-\phi}{1+\phi}\phi^{k-1}\sigma_e^2.
\end{aligned}
$$

(b) In particular, show that $Var(W_t) = \frac{2\sigma_e^2}{1+\phi}$.

$$
\begin{aligned}
Var(W_t) &= Var(Y_t - Y_{t-1}) \\
&= Var(Y_t) + (-1)^2 Var(Y_{t-1}) - 2Cov(Y_t, Y_{t-1}) \\
&= \frac{\sigma_e^2}{1-\phi^2} + \frac{\sigma_e^2}{1-\phi^2} - 2\left(\phi^1\frac{\sigma_e^2}{1-\phi^2}\right) \\
&= 2(1-\phi)\frac{\sigma_e^2}{1-\phi^2} \\
&= 2(1-\phi)\frac{\sigma_e^2}{(1-\phi)(1+\phi)} \\
&= 2\frac{\sigma_e^2}{1+\phi}.
\end{aligned}
$$

4.8
Let $\{Y_t\}$ be an AR(2) process of the special form $Y_t = \phi_2 Y_{t-2} + e_t$.

Use first principles to find the range of values for which the process is stationary.

$$\begin{aligned}
Y_t &= \phi_2 Y_{t-2} + e_t. \\
Var(Y_t) &= \phi_2^2 Var(Y_{t-2}) + Var(e_t) \\
Var(Y_t) &= \phi_2^2 Var(Y_t) + \sigma_e^2 \qquad \text{(stationary)} \\
Var(Y_t)(1-\phi_2^2) &= \sigma_e^2 \\
\therefore Var(Y_t) &= \frac{\sigma_e^2}{1-\phi_2^2}.
\end{aligned}$$

Variance must be ≥ 0, and as $\sigma_e^2 \geq 0$, we must have $1 - \phi_2^2 > 0$, i.e.,

$$-1 < \phi_2 < 1.$$

4.9
Use the recursive Yule-Walker equations:

$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$

for $k = 1, 2, 3, \ldots$ to calculate and sketch the auto-correlation functions for the following AR(2) models with parameters as specified.

In each case, specify whether the roots are real or complex, and, if complex, find the damping factor, R.

(a) $\phi_1 = 0.6$ and $\phi_2 = 0.3$.

(b) $\phi_1 = -0.4$ and $\phi_2 = 0.5$.

(c) $\phi_1 = 1.2$ and $\phi_2 = -0.7$.

(d) $\phi_1 = -1$ and $\phi_2 = -0.6$.

(e) $\phi_1 = 0.5$ and $\phi_2 = -0.9$.

(f) $\phi_1 = -0.5$ and $\phi_2 = -0.6$.

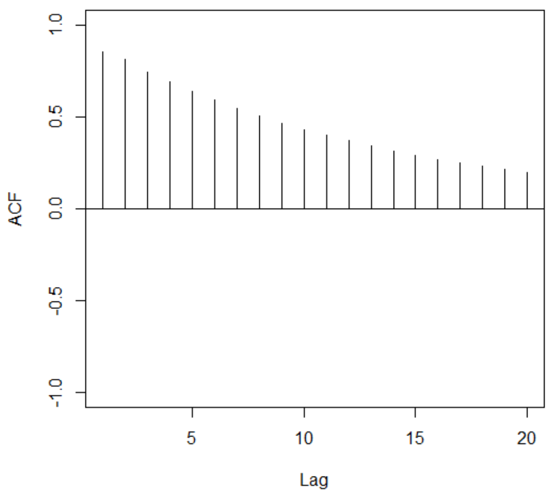
(a) $\phi_1 = 0.6$ and $\phi_2 = 0.3$.

```
rho=NULL
phi1=0.6
phi2=0.3
max.lag=20

#page 20 Lect4b
rho1=phi1/(1-phi2)
#page 21 Lect4b
rho2=(phi2*(1-phi2)+phi1^2)/(1-phi2)
rho[1]=rho1
rho[2]=rho2

#Yule-Walker equations
for (k in 3:max.lag)
rho[k]=phi1*rho[k-1]+phi2*rho[k-2]
rho # to display the values

plot(y=rho,x=1:max.lag,
type='h',ylim=c(-1,+1),
ylab='ACF',xlab='Lag')
abline(h=0)
```



We can get the same output using the inbuilt ARMAacf function.

```
ACF<-ARMAacf(ar=c(0.6,0.3),lag.max=10)
ACF
plot(y=ACF,x=1:length(ACF),
type='h',ylim=c(-1,+1),
ylab='ACF',xlab='Lag')
abline(h=0)
```

However, we were asked to use the recursive Yule-Walker equations, so we will continue using them.

The roots of the characteristic equation $1 - \phi_1 x - \phi_2 x^2 = 0$ can be confirmed in R with the function

```
polyroot(c(1,-phi1,-phi2))
```

```
#[1]  1.081666+0i -3.081666+0i
```

As $\phi_2 > 0$, the roots are not complex.

For the stationarity conditions of an AR(2) process, see page 17 Lecture 4b.

This process is stationary because both of the roots are $> |1|$.

In addition, the three conditions:

$\phi_1 + \phi_2 < 1$,

$\phi_2 - \phi_1 < 1$, and

$|\phi_2| < 1$ all hold.

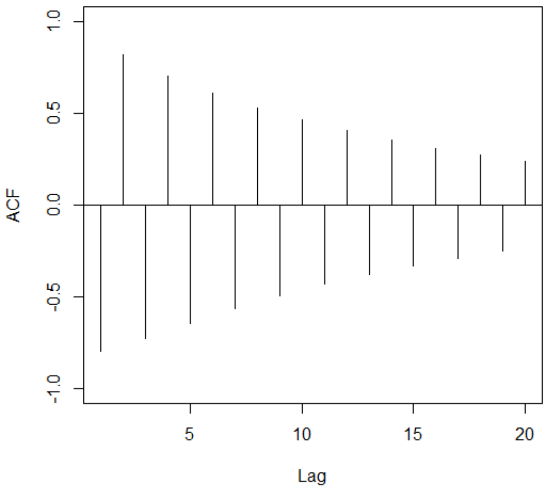
(b) $\phi_1 = -0.4$ and $\phi_2 = 0.5$.

```
rho=NULL
phi1=-0.4
phi2=0.5
max.lag=20

rho1=phi1/(1-phi2)
rho2=(phi2*(1-phi2)+phi1^2)/(1-phi2)
rho[1]=rho1
rho[2]=rho2

for (k in 3:max.lag)
rho[k]=phi1*rho[k-1]+phi2*rho[k-2]
rho # to display the values

plot(y=rho,x=1:max.lag,
type='h',ylim=c(-1,+1),
ylab='ACF',xlab='Lag')
abline(h=0)
```



The roots of the characteristic equation are

```
polyroot(c(1,-phi1,-phi2))

#[1] -1.069694+0i  1.869694-0i
```

This process is stationary because both of the roots are $> |1|$.

In addition, the three conditions:

$\phi_1 + \phi_2 < 1$

$\phi_2 - \phi_1 < 1$ and

$|\phi_2| < 1$ all hold.

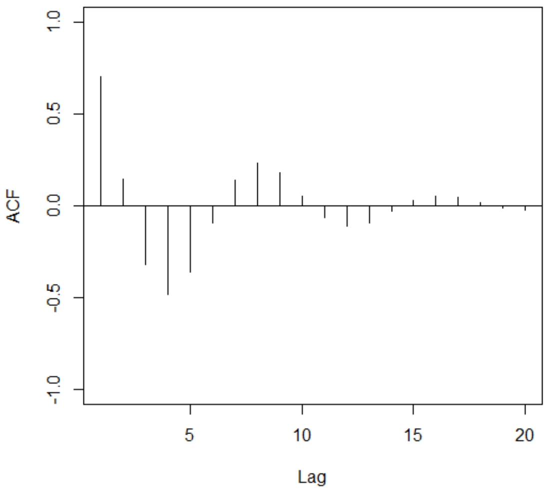
(c) $\phi_1 = 1.2$ and $\phi_2 = -0.7$.

```
rho=NULL
phi1=1.2
phi2=-0.7
max.lag=20

rho1=phi1/(1-phi2)
rho2=(phi2*(1-phi2)+phi1^2)/(1-phi2)
rho[1]=rho1
rho[2]=rho2

for (k in 3:max.lag)
rho[k]=phi1*rho[k-1]+phi2*rho[k-2]
rho # to display the values

plot(y=rho,x=1:max.lag,
type='h',ylim=c(-1,+1),
ylab='ACF',xlab='Lag')
abline(h=0)
```



Because $\phi_2 < 0$, the roots of the characteristic equation are complex:

```
polyroot(c(1,-phi1,-phi2))

#[1] 0.8571429+0.8329931i 0.8571429-0.8329931i
```

```
#Damping factor

R=sqrt(-phi2)
R

#[1] 0.83666
```

This process is stationary because both of the roots are $> |1|$.

$$|x + iy| = \sqrt{x^2 + y^2}.$$

$$\sqrt{0.86^2 + 0.83^2} = 1.195.$$

$$\sqrt{0.86^2 + (-0.83)^2} = 1.195.$$

In addition, the three conditions:

$\phi_1 + \phi_2 < 1$,

$\phi_2 - \phi_1 < 1$, and

$|\phi_2| < 1$ all hold.

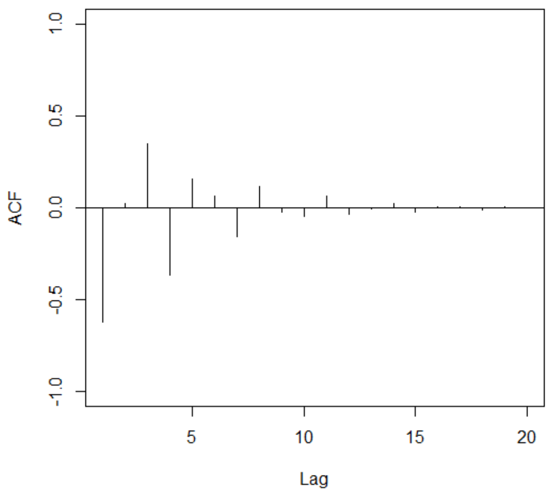
(d) $\phi_1 = -1$ and $\phi_2 = -0.6$.

```
rho=NULL
phi1=-1
phi2=-0.6
max.lag=20

rho1=phi1/(1-phi2)
rho2=(phi2*(1-phi2)+phi1^2)/(1-phi2)
rho[1]=rho1
rho[2]=rho2

for (k in 3:max.lag)
rho[k]=phi1*rho[k-1]+phi2*rho[k-2]
rho # to display the values

plot(y=rho,x=1:max.lag,
type='h',ylim=c(-1,+1),
ylab='ACF',xlab='Lag')
abline(h=0)
```



The roots of the characteristic equation are

```
polyroot(c(1,-phi1,-phi2))

#[1] -0.8333333+0.9860133i -0.8333333-0.9860133i
```

```
#Damping factor

R=sqrt(-phi2)
R

#[1] 0.7745967
```

This process is stationary because both of the roots are $> |1|$.

$$\sqrt{(-0.83)^2 + 0.98^2} = 1.28.$$
$$\sqrt{(-0.83)^2 + (-0.98)^2} = 1.28.$$

In addition, the three conditions:

$\phi_1 + \phi_2 < 1$,

$\phi_2 - \phi_1 < 1$, and

$|\phi_2| < 1$ all hold.

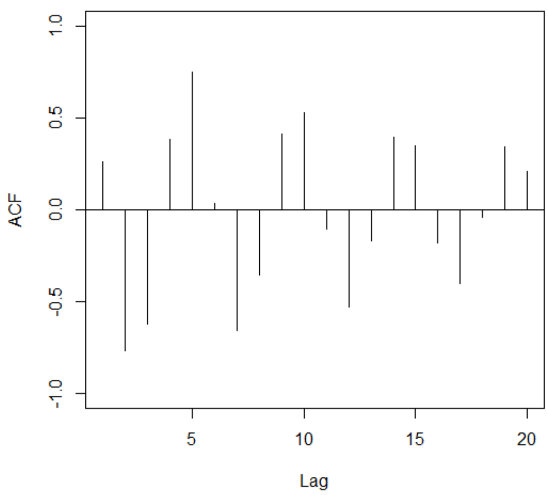
(e) $\phi_1 = 0.5$ and $\phi_2 = -0.9$.

```
rho=NULL
phi1=0.5
phi2=-0.9
max.lag=20

rho1=phi1/(1-phi2)
rho2=(phi2*(1-phi2)+phi1^2)/(1-phi2)
rho[1]=rho1
rho[2]=rho2

for (k in 3:max.lag)
rho[k]=phi1*rho[k-1]+phi2*rho[k-2]
rho # to display the values

plot(y=rho,x=1:max.lag,
type='h',ylim=c(-1,+1),
ylab='ACF',xlab='Lag')
abline(h=0)
```



The roots of the characteristic equation are

```
polyroot(c(1,-phi1,-phi2))

#[1] 0.277778+1.016834i 0.277778-1.016834i
```

This process is stationary because both of the roots are $> |1|$.

$$\sqrt{0.27^2 + 1.02^2} = 1.055.$$
$$\sqrt{0.27^2 + (-1.02)^2} = 1.055.$$

In addition, the three conditions:

$\phi_1 + \phi_2 < 1$,

$\phi_2 - \phi_1 < 1$, and

$|\phi_2| < 1$ all hold.

```
#Damping factor

R=sqrt(-phi2)
R

#[1] 0.9486833
```

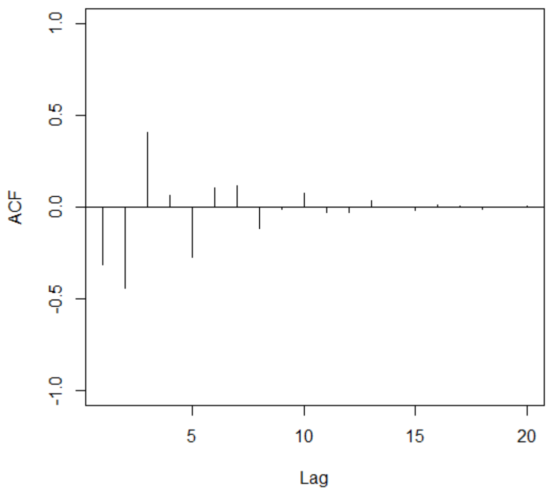
(f) $\phi_1 = -0.5$ and $\phi_2 = -0.6$.

```
rho=NULL
phi1=-0.5
phi2=-0.6
max.lag=20

rho1=phi1/(1-phi2)
rho2=(phi2*(1-phi2)+phi1^2)/(1-phi2)
rho[1]=rho1
rho[2]=rho2

for (k in 3:max.lag)
rho[k]=phi1*rho[k-1]+phi2*rho[k-2]
rho # to display the values

plot(y=rho,x=1:max.lag,
type='h',ylim=c(-1,+1),
ylab='ACF',xlab='Lag')
abline(h=0)
```



The roots of the characteristic equation are

```
polyroot(c(1,-phi1,-phi2))

#[1] -0.416667+1.221907i -0.416667-1.221907i
```

```
#Damping factor
R=sqrt(-phi2)
R

#[1] 0.7745967
```

This process is stationary because both of the roots are $> |1|$.

$$\sqrt{(-0.42)^2 + 1.22^2} = 1.29.$$
$$\sqrt{(-0.42)^2 + (-1.22)^2} = 1.29.$$

In addition, the three conditions:

$\phi_1 + \phi_2 < 1$,

$\phi_2 - \phi_1 < 1$, and

$|\phi_2| < 1$ all hold.