UNIVERSITY of LIMERICK

OLLSCOIL LUIMNIGH

Time Series Analysis

MS 4218

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Outline

Models for stationary time series

- General linear processess
- Moving Average (MA) processes
  - MA(1)
  - MA(2)
  - MA(q)

General linear processes

Y_t is an observed time series, and e_t is unobserved white noise.

A weighted linear combination of e_t's where the sum of the squares of the weights is finite yields a general linear process.

$$Y_t = (\psi_0 = 1)e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \dots$$

If $\psi_j = \phi^j$ and $|\phi| < 1$, we have a convergent series:

$$Y_t \quad = \quad e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots$$

Expectation and Variance

$$
\begin{aligned}
E(Y_t) &= E(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots) = 0 \\
Var(Y_t) &= Var(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots) \\
&= Var(e_t) + Var(\phi e_{t-1}) + Var(\phi^2 e_{t-2}) + \dots \\
&= \sigma_e^2 (1 + \phi^2 + \phi^4 + \dots); \qquad (a = 1, r = \frac{u_2}{u_1} = \phi^2) \\
&= \sigma_e^2 \left(S_\infty = \frac{a}{1-r} = \frac{1}{1-\phi^2} \right) \\
&= \frac{\sigma_e^2}{1-\phi^2.}
\end{aligned}
$$

Auto-covariance

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots, \\ &\quad e_{t-1} + \phi e_{t-2} + \phi^2 e_{t-3} + \dots) \end{aligned}$$

Matching only the like terms:

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(\phi e_{t-1}, e_{t-1}) + Cov(\phi^2 e_{t-2}, \phi e_{t-2}) + \dots \\ &= \phi\sigma_e^2 + \phi^3\sigma_e^2 + \phi^5\sigma_e^2 + \dots \\ &= \phi\sigma_e^2(1 + \phi^2 + \phi^4 + \dots); \qquad (a = 1, r = \phi^2) \\ &= \frac{\phi\sigma_e^2}{1 - \phi^2.} \end{aligned}$$

Auto-correlation

$$
\begin{aligned}
Corr(Y_t, Y_{t-1}) = \rho_1 &= \frac{\gamma_1}{\gamma_0} \\
&= \frac{\phi\sigma_e^2}{1-\phi^2.} \div \frac{\sigma_e^2}{1-\phi^2} \\
&= \phi^1. \\
Corr(Y_t, Y_{t-k}) &= \phi^k.
\end{aligned}
$$

A constant mean and an autocovariance that depends only on the lag means the process is stationary.

Auto-covariance again

$$
\begin{aligned}
Cov(Y_t, Y_{t-1}) &= Cov(\psi_0 e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \dots, \\
&\quad \psi_0 e_{t-1} + \psi_1 e_{t-2} + \psi_2 e_{t-3} + \dots) \\
&= E(\psi_0\psi_1 e_{t-1}^2 + \psi_1\psi_2 e_{t-2}^2 + \psi_2\psi_3 e_{t-3}^2 + \dots) \\
&= \sigma_e^2(\psi_0\psi_1 + \psi_1\psi_2 + \psi_2\psi_3 + \dots) \\
&= \sigma_e^2 \sum_{i=0}^{\infty}(\psi_i\psi_{i+1}) \;=\; \gamma_1. \\
Cov(Y_t, Y_{t-k}) &= \sigma_e^2 \sum_{i=0}^{\infty}(\psi_i\psi_{i+k}) \;=\; \gamma_k.
\end{aligned}
$$

Moving average (MA) processes

A general linear process with a finite number (order $= q$) of non-zero ψ weights.

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} -, \ldots, -\theta_q e_{t-q}.$$

This MA process is written as MA(q).

An MA process of order 1 has 1 θ.

An MA(2) has a θ_1 and θ_2.

MA(1) process

A first order MA process is given by:

$$
\begin{aligned}
Y_t &= e_t - \theta e_{t-1}. \\
E(Y_t) &= E(e_t - \theta e_{t-1}) \\
&= E(e_t) - \theta E(e_{t-1}) \\
&= 0.
\end{aligned}
$$

Variance for MA(1) process

$$
\begin{aligned}
Var(Y_t) &= Var(e_t - \theta e_{t-1}) \\
&= Var(e_t) + (-\theta)^2 Var(e_{t-1}) \\
&= \sigma_e^2 + \theta^2 \sigma_e^2 \\
&= \sigma_e^2 (1 + \theta^2) \\
&= \gamma_0.
\end{aligned}
$$

Auto-covariance for MA(1) process

$$
\begin{aligned}
Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}). \\
&= Cov(e_t, e_{t-1}) + Cov(e_t, -\theta e_{t-2}) \\
&\quad + Cov(-\theta e_{t-1}, e_{t-1}) + Cov(-\theta e_{t-1}, -\theta e_{t-2}) \\
&= -\theta Cov(e_{t-1}, e_{t-1}) \\
&= -\theta \sigma_e^2.
\end{aligned}
$$

Auto-covariance for MA(1) process

$$
\begin{aligned}
Cov(Y_t, Y_{t-2}) &= Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) \\
&= Cov(e_t, e_{t-2}) + Cov(e_t, -\theta e_{t-3}) \\
&\quad + Cov(-\theta e_{t-1}, e_{t-2}) + Cov(-\theta e_{t-1}, -\theta e_{t-3}) \\
&= 0.
\end{aligned}
$$

Acvf MA(1)

$$Cov(Y_t, Y_{t-k}) = 0 \quad \forall \quad k > 1.$$

$$\gamma_k = \begin{cases} \sigma_e^2(1+\theta^2) & \text{if } k = 0. \\ -\theta\sigma_e^2 & \text{if } k = 1. \\ 0 & \text{if } k > 1. \end{cases}$$

Acf MA(1)

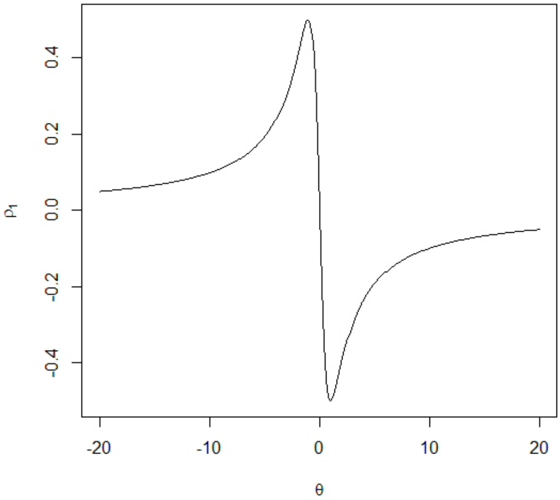
$$
\begin{aligned}
\rho_k &= \frac{\gamma_k}{\gamma_0} \\
&= \begin{cases} 1 & \text{if } k = 0. \\ -\frac{\theta\sigma_e^2}{\sigma_e^2(1+\theta^2)} = -\frac{\theta}{1+\theta^2} & \text{if } k = 1. \\ 0 & \text{if } k > 1. \end{cases}
\end{aligned}
$$

Maximum correlation for MA(1)

$$\begin{aligned}\rho_1 &= -\frac{\theta}{1+\theta^2}.\\ \frac{d}{d\theta}\rho_1 &= \frac{\theta^2-1}{(1+\theta^2)^2}\\ &= 0 \text{ when } \theta = \pm 1.\end{aligned}$$

$$\begin{aligned}\theta &= -1 \Rightarrow \rho = \frac{1}{2}.\\ \theta &= +1 \Rightarrow \rho = -\frac{1}{2}.\end{aligned}$$

Lag 1 auto-correlation of an MA(1) process for different θ values



Non-uniqueness (and later invertibility)

2 MA(1) processes with parameters a.$\theta \neq 0$ and b.$\frac{1}{\theta}$ have same value of ρ_1.

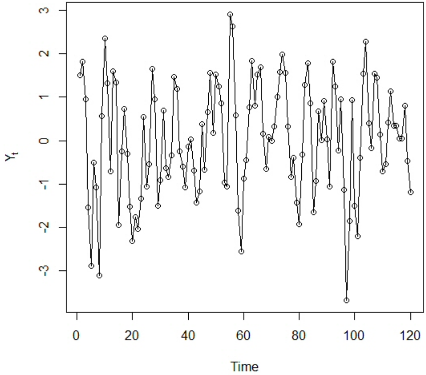
$$\begin{aligned} a.\rho_1 &= -\frac{\theta}{1+\theta^2}. \\ b.\rho_1 &= -\frac{1}{\theta} \div \left(1 + \frac{1}{\theta^2}\right) \\ &= -\frac{1}{\theta} \times \frac{\theta^2}{1+\theta^2} \\ &= -\frac{\theta}{1+\theta^2}. \end{aligned}$$

ma1.2.s data in library(TSA)

e_t's are white noise and $\theta = -0.9$.

```
data(ma1.2.s)
ma1.2.s
plot(ma1.2.s,
ylab=expression(Y[t]),type='o')
```

Timeplot of ma1.2.s



Relatively smooth.

Sample and theoretical correlation at lag 1 in ma1.2.s data

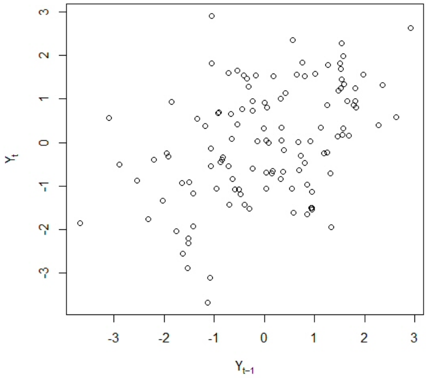
```
n<-length(ma1.2.s)
r1<-cor(ma1.2.s[-1],ma1.2.s[-n]); r1 = 0.428
```

$$
\begin{aligned}
\rho_1 &= -\frac{\theta}{1+\theta^2} \\
&= -\frac{-0.9}{1+(-0.9)^2} = 0.4972.
\end{aligned}
$$

```
plot(y=ma1.2.s,x=zlag(ma1.2.s),
ylab=expression(Y[t]),
xlab=expression(Y[t-1]),type='p')
```

Plot of Y_t v Y_{t-1} for MA(1) with $\theta = -0.9$

$$\rho_1 = 0.4972; r_1 = 0.428.$$



Moderately strong upward trend.

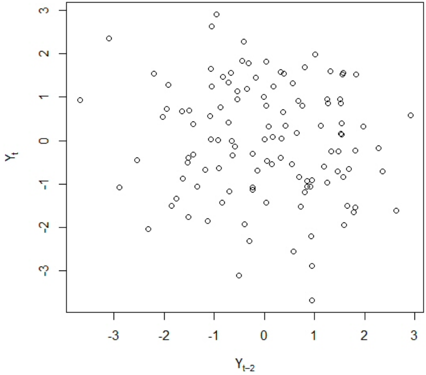
Sample and theoretical correlation at lag 2 in ma1.2.s data

```
n<-length(ma1.2.s)
r2<-cor(ma1.2.s[-(1:2)],ma1.2.s[-((n-1):n)])
r2=-0.1152471
```

$$\rho_2 = 0.$$

```
plot(y=ma1.2.s,x=zlag(ma1.2.s,2),
ylab=expression(Y[t]),
xlab=expression(Y[t-2]),type='p')
```

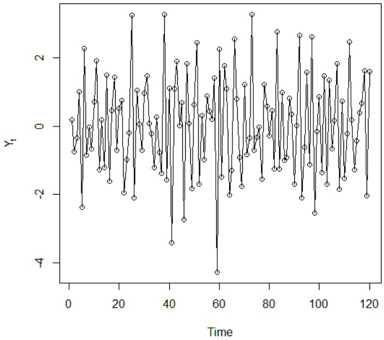
Plot of Y_t v Y_{t-2} for MA(1) with $\theta = -0.9$



Zero auto-correlation at lag 2: $\rho_2 = 0$; $r_2 = -0.1152471$.

Time plot of MA(1) process: ma1.1.s data in library(TSA)

```
data(ma1.1.s)
plot(ma1.1.s,ylab=expression(Y[t]),type='o')
```



Jagged plot, and here, $\theta = +0.9$

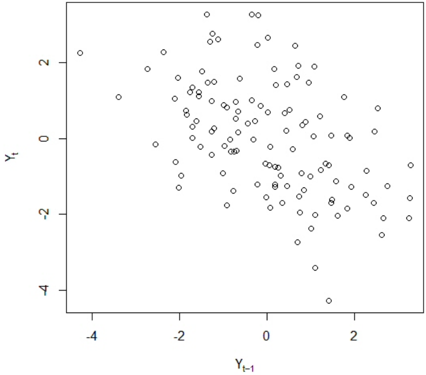
Sample and theoretical correlation at lag 1 in *ma1.1.s* data

```
n<-length(ma1.1.s)
r1<-cor(ma1.2.s[-1],ma1.2.s[-n]); r1 = -0.476
```

$$
\begin{aligned}
\rho_1 &= -\frac{\theta}{1+\theta^2} \\
&= -\frac{0.9}{1+0.9^2} = -0.4972.
\end{aligned}
$$

```
plot(y=ma1.2.s,x=zlag(ma1.2.s),
ylab=expression(Y[t]),
xlab=expression(Y[t-1]),type='p')
```

Plot of Y_t v Y_{t-1} for MA(1) with $\theta = +0.9$



$\rho_1 = -0.4972$ and $r_2 = -0.476$.
Moderately strong downward trend.

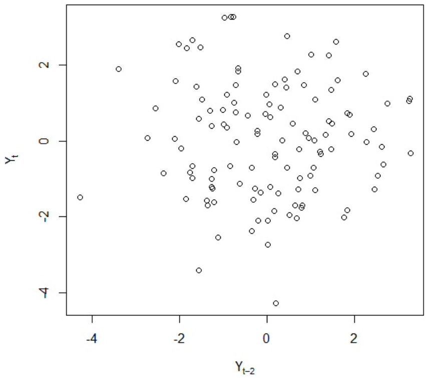
Sample and theoretical correlation at lag 2 in ma1.1.s data

```
n<-length(ma1.1.s)
r2<-cor(ma1.1.s[-(1:2)],ma1.1.s[-((n-1):n)])
r2=-0.009831165.
```

$\rho_2 = 0$.

```
plot(y=ma1.1.s,x=zlag(ma1.1.s,2),
ylab=expression(Y[t]),
xlab=expression(Y[t-2]),type='p')
```

Plot of Y_t v Y_{t-2} for MA(1) with $\theta = +0.9$



Zero auto-correlation at lag 2. $r_2 = -0.009831165$.

MA(2) process

$$
\begin{aligned}
Y_t &= e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}. \\
E(Y_t) &= E(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\
&= E(e_t) - \theta_1 E(e_{t-1}) - \theta_2 E(e_{t-2}) \\
&= 0.
\end{aligned}
$$

MA(2) Variance

$$
\begin{aligned}
\gamma_0 &= Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\
&= Var(e_t) + Var(-\theta_1 e_{t-1}) + Var(-\theta_2 e_{t-2}) \\
&= \sigma_e^2 + \theta_1^2 \sigma_e^2 + \theta_2^2 \sigma_e^2 \\
&= \sigma_e^2 (1 + \theta_1^2 + \theta_2^2).
\end{aligned}
$$

MA(2) Auto-covariance

$$
\begin{aligned}
\gamma_1 &= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\
&= Cov(-\theta_1 e_{t-1}, e_{t-1}) + Cov(-\theta_2 e_{t-2}, -\theta_1 e_{t-2}) \\
&= \sigma_e^2(-\theta_1 + \theta_1\theta_2).
\end{aligned}
$$

MA(2) Auto-covariance cont.

$$
\begin{aligned}
\gamma_2 &= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, \\
&\quad e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) \\
&= Cov(-\theta_2 e_{t-2}, e_{t-2}) \\
&= -\theta_2 \sigma_e^2. \\
Cov(Y_t, Y_{t-k}) &= 0 \quad \forall \quad k > 2.
\end{aligned}
$$

MA(2) Auto-covariance summary

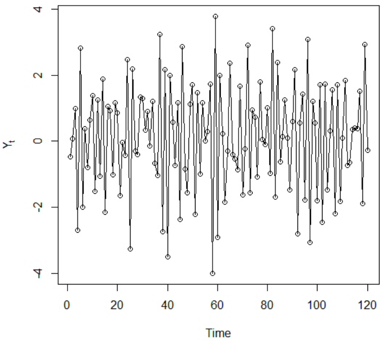
$$\gamma_k = \begin{cases} (1+\theta_1^2+\theta_2^2)\sigma_e^2 & \text{if } k=0. \\ (-\theta_1+\theta_1\theta_2)\sigma_e^2 & \text{if } k=1. \\ -\theta_2\sigma_e^2 & \text{if } k=2. \\ 0 & \text{if } k>2. \end{cases}$$

Acf MA(2)

$$
\begin{aligned}
\rho_k &= \frac{\gamma_k}{\gamma_0} \\
&= \begin{cases} 1 & \text{if } k = 0. \\ \frac{-\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if } k = 1. \\ \frac{-\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if } k = 2. \\ 0 & \text{if } k > 2. \end{cases}
\end{aligned}
$$

Time plot of MA(2) process: data ma2.s with $\theta_1 = 1$ & $\theta_2 = -0.6$

```
data(ma2.s)
plot(ma2.s,ylab=expression(Y[t]),type='o')
```



It oscillates across midline each time.

Sample and theoretical correlation at lag 1 in ma2.s data

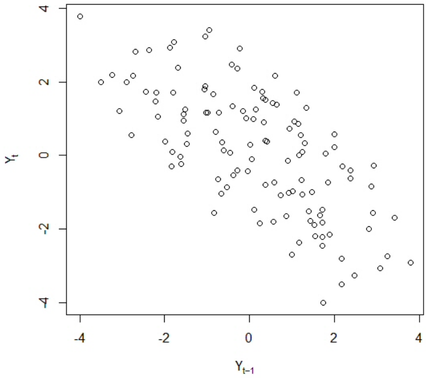
```
n<-length(ma2.s)
r1<-cor(ma2.s[-1],ma2.s[-n])
r1 = -0.684
```

$$\begin{aligned}\rho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \\ \rho_1 &= \frac{-1 + (1)(-0.6)}{1 + 1^2 + (-0.6)^2} \\ &= -0.678.\end{aligned}$$

```
plot(y=ma2.s,x=zlag(ma2.s),
ylab=expression(Y[t]),
xlab=expression(Y[t-1]),type='p')
```

Plot of Y_t v Y_{t-1} for MA(2)

$$\rho_1 = -0.678;\ r_1 = -0.684.$$



Sample and theoretical correlation at lag 2 in ma2.s data

```
n<-length(ma2.s)
r2<-cor(ma2.s[-(1:2)],ma2.s[-((n-1):n)])
r2 = 0.251
```

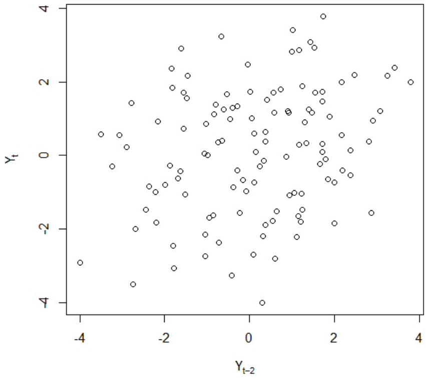
$$\rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2} = \frac{-(-0.6)}{1+1^2+(-0.6)^2}$$

$$\rho_2 = 0.254.$$

```
plot(y=ma2.s,x=zlag(ma2.s,2),
ylab=expression(Y[t]),
xlab=expression(Y[t-2]),type='p')
```

Plot of Y_t v Y_{t-2} for MA(2)

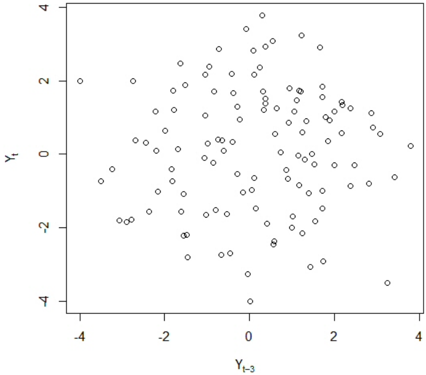
$$\rho_2 = 0.254;\ r_2 = 0.251.$$



Plot of Y_t v Y_{t-3} for MA(2)

```
r3<-cor(ma2.s[-(1:3)],ma2.s[-((n-2):n)])
r_3=0.0391.
```

$\rho_3 = 0.$



General MA(q) process: Acf 0 after lag q

$$
\begin{aligned}
Y_t &= e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}. \\
\gamma_0 &= (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_e^2. \\
\rho_k &= \begin{cases} \frac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & \text{if } k = 1, 2, \ldots, q \\ 0 & \text{if } k > q. \end{cases}
\end{aligned}
$$

Next

Models for stationary time series cont.

- Auto-Regressive (AR) processes
  - AR(1)
  - AR(2)
  - AR(p)
- Mixed ARMA processes
- Invertibility