Time Series – Tutorial 3 – Solutions

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Question 6

a)
$$\begin{aligned} W_t &= Y_t - Y_{t-12} \\ &= \beta_0 + \beta_1 t + s_t + e_t \\ &\quad - \beta_0 - \beta_1(t-12) - s_{t-12} - e_{t-12} \\ &= 12\beta_1 + e_t - e_{t-12} \qquad \text{(since } s_t = s_{t-12}) \end{aligned}$$

The mean function is:

$$\begin{aligned} E(W_t) &= 12\beta_1 + E(e_t) - E(e_{t-12}) \\ &= 12\beta_1 + 0 - 0 = 12\beta_1. \end{aligned}$$

The autocovariance is given by

$$\begin{aligned} &\mathrm{Cov}(W_t, W_{t-k}) \\ &= \mathrm{Cov}(12\beta_1 + e_t - e_{t-12},\ 12\beta_1 + e_{t-k} - e_{t-k-12}) \\ &= \mathrm{Cov}(e_t - e_{t-12},\ e_{t-k} - e_{t-k-12}) \\ &= \mathrm{Cov}(e_t, e_{t-k}) - \mathrm{Cov}(e_t, e_{t-k-12}) \\ &\quad - \mathrm{Cov}(e_{t-12}, e_{t-k}) + \mathrm{Cov}(e_{t-12}, e_{t-k-12}) \end{aligned}$$

- $\mathrm{Cov}(e_t, e_{t-k}) = \sigma_e^2$ when $k = 0$.
- $\mathrm{Cov}(e_t, e_{t-k-12}) = 0$ always.
- $\mathrm{Cov}(e_{t-12}, e_{t-k}) = \sigma_e^2$ when $k = 12$.
- $\mathrm{Cov}(e_{t-12}, e_{t-k-12} = \sigma_e^2$ when $k = 0$.

$$\mathrm{Cov}(W_t, W_{t-k}) = \begin{cases} 2\sigma_e^2 & \text{when } k = 0 \\ -\sigma_e^2 & \text{when } k = 12 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the mean is constant and the autocovariance depends only on k, i.e., we can write γ_k.
The autocorrelation is given by

$$\begin{aligned} \rho_{t,t-k} &= \frac{\gamma_{t,t-k}}{\sqrt{\gamma_{t,t}\,\gamma_{t-k,t-k}}} \\ &= \frac{\gamma_k}{\sqrt{\gamma_0\,\gamma_0}} \qquad \text{(series is stationary)} \\ &= \frac{\gamma_k}{\gamma_0} \\ &= \begin{cases} \frac{2\sigma_e^2}{2\sigma_e^2} = 1 & \text{when } k = 0 \\ \frac{-\sigma_e^2}{2\sigma_e^2} = -0.5 & \text{when } k = 12 \\ \frac{0}{2\sigma_e^2} = 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus, applying ∇_{12}, i.e., seasonal differencing, eliminates linear and seasonal trend.

b) The mean function would be the same since $E(X_t) = 0$ just like for e_t.

However, unlike e_t, covariance may be non-zero for various lags, k (for e_t, it is only non-zero at lag $k = 0$).

Thus, using the fact that the autocovariance for X_t is $\mathrm{Cov}(X_t, X_t - k) = \gamma_k^*$ (i.e., it only depends on the lag), the autocovariance for Y_t is

$$\begin{aligned} &\mathrm{Cov}(W_t, W_{t-k}) \\ &= \mathrm{Cov}(X_t, X_{t-k}) - \mathrm{Cov}(X_t, X_{t-k-12}) \\ &\quad - \mathrm{Cov}(X_{t-12}, X_{t-k}) + \mathrm{Cov}(X_{t-12}, X_{t-k-12}) \\ &= \gamma^*_{t-(t-k)} - \gamma^*_{t-(t-k-12)} \\ &\quad - \gamma^*_{t-12-(t-k)} + \gamma^*_{t-12-(t-k-12)} \\ &= \gamma_k^* - \gamma_{k+12}^* - \gamma_{k-12}^* + \gamma_k^* \\ &= 2\gamma_k^* - \gamma_{k+12}^* - \gamma_{k-12}^*. \end{aligned}$$

This still depends only on k (i.e., stationary) is more complicated than in part (a). Note that the ACF is:

$$\begin{aligned} \rho_k = \frac{\gamma_k}{\gamma_0} &= \frac{2\gamma_k^* - \gamma_{k+12}^* - \gamma_{k-12}^*}{\gamma_0} \\ &= \frac{2\gamma_k^* - \gamma_{k+12}^* - \gamma_{k-12}^*}{2\gamma_0^* - \gamma_{12}^* - \gamma_{-12}^*} \\ &= \frac{2\gamma_k^* - \gamma_{k+12}^* - \gamma_{k-12}^*}{2\gamma_0^* - 2\gamma_{12}^*} \\ &\qquad \text{(Since } \gamma_k = \gamma_{-k} \text{ by definition)} \end{aligned}$$

c) Here we have $Y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + s_t + e_t$ and therefore:

$$\begin{aligned} W_t &= Y_t - Y_{t-12} \\ &= \beta_0 + \beta_1 t + \beta_2 t^2 + s_t + e_t \\ &\quad - \beta_0 - \beta_1(t-12) - \beta_2(t-12)^2 - s_{t-12} - e_{t-12} \\ &= 12\beta_1 + 24\beta_2 t - 144\beta_2 + e_t - e_{t-12} \\ &= 12\beta_1 - 144\beta_2 + 24\beta_2 t + e_t - e_{t-12} \end{aligned}$$

This is clearly non-stationary since

$$E(W_t) = 12\beta_1 - 144\beta_2 + 24\beta_2 t.$$

However, the autocovariance is still the same as in part (a) since $12\beta_1 - 144\beta_2 + 24\beta_2 t$ is non-random and, hence, disappears from covariance.

d) Here we have

$$\begin{aligned} \nabla W_t &= 12\beta_1 - 144\beta_2 + 24\beta_2 t + e_t - e_{t-12} \\ &\quad - 12\beta_1 + 144\beta_2 - 24\beta_2(t-1) - e_{t-1} + e_{t-13} \\ &= 24\beta_2 + \underbrace{e_t - e_{t-12} - e_{t-1} + e_{t-13}}_{\text{sum of stationary series}}. \end{aligned}$$

This is clearly stationary as it is the sum of stationary series plus a constant.

This question shows that applying $\nabla\nabla_{12}$, i.e., seasonal differencing and then differencing, eliminates quadratic and seasonal trend.