

## 0.1 Case Deletion Diagnostics

Since the pioneering work of Cook in 1977, deletion measures have been applied to many statistical models for identifying influential observations. Case-deletion diagnostics provide a useful tool for identifying influential observations and outliers.

The key to making deletion diagnostics useable is the development of efficient computational formulas, allowing one to obtain the case deletion diagnostics by making use of basic building blocks, computed only once for the full model.

The computation of case deletion diagnostics in the classical model is made simple by the fact that estimates of  $\beta$  and  $\sigma^2$ , which exclude the  $i$ th observation, can be computed without re-fitting the model.

Preisser (1996) describes two type of diagnostics. When the set consists of only one observation, the type is called ‘*observation-diagnostics*’. For multiple observations, Preisser describes the diagnostics as ‘*cluster-deletion*’ diagnostics.

When applied to LME models, such update formulas are available only if one assumes that the covariance parameters are not affected by the removal of the observation in question. However, this is rarely a reasonable assumption.

Such update formulas are available in the mixed model only if you assume that the covariance parameters are not affected by the removal of the observation in question. This is rarely a reasonable assumption.

### 0.1.1 Deletion Diagnostics

Deletion diagnostics provide a means of assessing the influence of an observation (or groups of observations) on inference on the estimated parameters of LME models.

Data from single individuals, or a small group of subjects may influence non-linear mixed effects model selection. Diagnostics routinely applied in model building may

identify such individuals, but these methods are not specifically designed for that purpose and are, therefore, not optimal. We describe two likelihood-based diagnostics for identifying individuals that can influence the choice between two competing models.

## 0.2 Case Deletion Diagnostics

Christensen, Pearson and Johnson (1992) studied case deletion diagnostics, in particular the equivalent of Cook's distance, for diagnosing influential observations when estimating the fixed effect parameters and variance components.

## 0.3 Case Deletion Diagnostics for LME models

Haslett and Dillane (2004) remark that linear mixed effects models didn't experience a corresponding growth in the use of deletion diagnostics, adding that McCullough and Searle (2001) makes no mention of diagnostics whatsoever.

Christensen, Pearson and Johnson (1992) describes three propositions that are required for efficient case-deletion in LME models. The first proposition describes how to efficiently update  $V$  when the  $i$ th element is deleted.

$$V_{[i]}^{-1} = \Lambda_{[i]} - \frac{\lambda\lambda'}{\nu ii} \quad (1)$$

The second of Christensen's propositions is the following set of equations, which are variants of the Sherman Woodbury updating formula.

$$X'_{[i]} V_{[i]}^{-1} X_{[i]} = X' V^{-1} X - \frac{\hat{x}_i \hat{x}_i'}{s_i} \quad (2)$$

$$(X'_{[i]} V_{[i]}^{-1} X_{[i]})^{-1} = (X' V^{-1} X)^{-1} + \frac{(X' V^{-1} X)^{-1} \hat{x}_i \hat{x}_i' (X' V^{-1} X)^{-1}}{s_i - \bar{h}_i} \quad (3)$$

$$X'_{[i]} V_{[i]}^{-1} Y_{[i]} = X' V^{-1} Y - \frac{\hat{x}_i \hat{y}_i'}{s_i} \quad (4)$$

### 0.3.1 Case Deletion Diagnostics for Mixed Models

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## 0.4 Case Deletion Diagnostics for LME models

Schabenberger (2004) examines the use and implementation of influence measures in LME models.

Schabenberger (2004) describes a simple procedure for quantifying influence. Firstly a model should be fitted to the data, and estimates of the parameters should be obtained. The second step is that either single or multiple data points, specifically outliers, should be omitted from the analysis, with the original parameter estimates being updated. This is known as ‘leave one out’ or ‘leave k out’ analysis. The final step of the procedure is comparing the sets of estimates computed from the entire and reduced data sets to determine whether the absence of observations changed the analysis.

### **schabenberger**

Schabenberger (2004) notes that it is not always possible to derive influence statistics necessary for comparing full- and reduced-data parameter estimates. Haslett and Dil-

lane (2004) offers an procedure to assess the influences for the variance components within the linear model, complementing the existing methods for the fixed components. The essential problem is that there is no useful updating procedures for  $\hat{V}$ , or for  $\hat{V}^{-1}$ . Haslett and Dillane (2004) propose an alternative , and computationally inexpensive approach, making use of the ‘delete=replace’ identity.

Haslett (1999) considers the effect of ‘leave k out’ calculations on the parameters  $\beta$  and  $\sigma^2$ , using several key results from Haslett and Hayes (1998) on partitioned matrices.

### 0.4.1 Extension of Diagnostic Methods to LME models

Christensen et al. (1992) noted the case deletion diagnostics techniques had not been applied to linear mixed effects models and seeks to develop methodologies in that respect. Christensen et al. (1992) develops these techniques in the context of REML.

Christensen et al. (1992) develops case deletion diagnostics, in particular the equivalent of Cook’s distance, a well-known metric, for diagnosing influential observations when estimating the fixed effect parameters and variance components. Deletion diagnostics provide a means of assessing the influence of an observation (or groups of observations) on inference on the estimated parameters of LME models. We shall provide a fuller discussion of Cook’s distance in due course.

Demidenko (2004) extends several regression diagnostic techniques commonly used in linear regression, such as leverage, infinitesimal influence, case deletion diagnostics, Cook’s distance, and local influence to the linear mixed-effects model. In each case, the proposed new measure has a direct interpretation in terms of the effects on a parameter of interest, and reduces to the familiar linear regression measure when there are no random effects.

The new measures that are proposed by Demidenko (2004) are explicitly defined functions and do not require re-estimation of the model, especially for cluster deletion

diagnostics. The basis for both the cluster deletion diagnostics and Cook's distance is a generalization of Miller's simple update formula for case deletion for linear models. Furthermore Demidenko (2004) shows how Pregibon's infinitesimal case deletion diagnostics is adapted to the linear mixed-effects model.

Demidenko (2004) proposes two likelihood-based diagnostics for identifying individuals that can influence the choice between two competing models.

### 0.4.2 Extending deletion diagnostics to LMEs

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$$X = \begin{bmatrix} x'_i \\ X(i) \end{bmatrix}, Z = \begin{bmatrix} z'_{ij} \\ Z_{j(i)} \end{bmatrix}, Z = \begin{bmatrix} z'_{ij} \\ Z_{j(i)} \end{bmatrix},$$

$$y = \begin{bmatrix} y'_{ij} \\ y_{j(i)} \end{bmatrix} \text{ and } H = \begin{bmatrix} h_{ii} & h \\ h_{j(i)} & h \end{bmatrix}$$

For notational simplicity,  $\mathbf{A}_{(i)}$  denotes an  $n \times m$  matrix  $\mathbf{A}$  with the  $i$ -th row removed,  $\mathbf{a}_i$  denotes the  $i$ -th row of  $\mathbf{A}$ , and  $a_{ij}$  denotes the  $(i, j)$ -th element of  $\mathbf{A}$ .

$\mathbf{a}_{(i)}$  denotes a vector  $\mathbf{a}$  with the  $i$ -th element,  $a_i$ , removed.

$$\check{a}_i = \mathbf{a}_i - \mathbf{A}_{(i)} \mathbf{H}_{[i]} \mathbf{h}_i \quad (5)$$

## 0.5 Case Deletion Diagnostics for LME models

Christensen (19XX) describes three propositions that are required for efficient case-deletion in LME models. The first proposition describes how to efficiently update  $V$

when the  $i$ th element is deleted.

$$V_{[i]}^{-1} = \Lambda_{[i]} - \frac{\lambda\lambda'}{\nu ii} \quad (6)$$

The second of Christensen's propositions is the following set of equations, which are variants of the Sherman Wood bury updating formula.

$$X'_{[i]} V_{[i]}^{-1} X_{[i]} = X' V^{-1} X - \frac{\hat{x}_i \hat{x}'_i}{s_i} \quad (7)$$

$$(X'_{[i]} V_{[i]}^{-1} X_{[i]})^{-1} = (X' V^{-1} X)^{-1} + \frac{(X' V^{-1} X)^{-1} \hat{x}_i \hat{x}'_i (X' V^{-1} X)^{-1}}{s_i - \bar{h}_i} \quad (8)$$

$$X'_{[i]} V_{[i]}^{-1} Y_{[i]} = X' V^{-1} Y - \frac{\hat{x}_i \hat{y}'_i}{s_i} \quad (9)$$

## 0.6 Extension of technique to LME Models

Model diagnostic techniques, well established for classical models, have since been adapted for use with linear mixed effects models. Diagnostic techniques for LME models are inevitably more difficult to implement, due to the increased complexity.

Beckman, Nachtsheim and Cook (1987) applied the local influence method of Cook (1986) to the analysis of the linear mixed model.

While the concept of influence analysis is straightforward, implementation in mixed models is more complex. Update formulae for fixed effects models are available only when the covariance parameters are assumed to be known.

If the global measure suggests that the points in  $U$  are influential, the nature of that influence should be determined. In particular, the points in  $U$  can affect

- the estimates of fixed effects
- the estimates of the precision of the fixed effects
- the estimates of the covariance parameters
- the estimates of the precision of the covariance parameters
- fitted and predicted values

### 0.6.1 Methods and Measures

The key to making deletion diagnostics useable is the development of efficient computational formulas, allowing one to obtain the case deletion diagnostics by making use of basic building blocks, computed only once for the full model.

Zewotir and Galpin (2005) lists several established methods of analyzing influence in LME models. These methods include

- Cook’s distance for LME models,
- likelihood distance,
- the variance (information) ration,
- the Cook-Weisberg statistic,
- the Andrews-Prebigon statistic.

## 0.7 Notation for Case Deletion

### 0.7.1 Case deletion notation

For notational simplicity,  $\mathbf{A}(i)$  denotes an  $n \times m$  matrix  $\mathbf{A}$  with the  $i$ -th row removed,  $a_i$  denotes the  $i$ -th row of  $\mathbf{A}$ , and  $a_{ij}$  denotes the  $(i, j)$ -th element of  $\mathbf{A}$ .

### 0.7.2 Partitioning Matrices

Without loss of generality, matrices can be partitioned as if the  $i$ -th omitted observation is the first row; i.e.  $i = 1$ .

### 0.7.3 Effects on fitted and predicted values

$$\hat{e}_{i(U)} = y_i - x\hat{\beta}_{(U)} \tag{10}$$

### 0.7.4 Matrix Notation for Case Deletion

For notational simplicity,  $\mathbf{A}(i)$  denotes an  $n \times m$  matrix  $\mathbf{A}$  with the  $i$ -th row removed,  $a_i$  denotes the  $i$ -th row of  $\mathbf{A}$ , and  $a_{ij}$  denotes the  $(i, j)$ -th element of  $\mathbf{A}$ .



## 0.8 Terminology for Case Deletion diagnostics

Preisser (1996) describes two type of diagnostics. When the set consists of only one observation, the type is called 'observation-diagnostics'. For multiple observations, Preisser describes the diagnostics as 'cluster-deletion' diagnostics.

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