

### 0.0.1 The LME model as a general linear model

Henderson's equations in (??) can be rewritten  $(T'W^{-1}T)\delta = T'W^{-1}y_a$  using

$$\delta = \begin{pmatrix} \beta \\ b \end{pmatrix}, y_a = \begin{pmatrix} y \\ \psi \end{pmatrix}, T = \begin{pmatrix} X & Z \\ 0 & I \end{pmatrix}, \text{ and } W = \begin{pmatrix} \Sigma & 0 \\ 0 & D \end{pmatrix},$$

where ? describe  $\psi = 0$  as quasi-data with mean  $E(\psi) = b$ . Their formulation suggests that the joint estimation of the coefficients  $\beta$  and  $b$  of the linear mixed effects model can be derived via a classical augmented general linear model  $y_a = T\delta + \varepsilon$  where  $E(\varepsilon) = 0$  and  $\text{var}(\varepsilon) = W$ , with *both*  $\beta$  and  $b$  appearing as fixed parameters. The usefulness of this reformulation of an LME as a general linear model will be revisited.

# 1 Hierarchical likelihood

Inferential method was developed for the mixed linear model via Lee and Nelder's (1996) hierarchical-likelihood (h-likelihood).

## 2 Importance-Weighted Least-Squares (IWLS)

### 3 Augmented GLMs

With the use of h-likelihood, a random effected model of the form can be viewed as an 'augmented GLM' with the response variables  $(y^t, \phi_m^t)^t$ , (with  $\mu = E(y), u = E(\phi)$ ,  $var(y) = \theta V(\mu)$ ). The augmented linear predictor is

$$\eta_{ma} = (\eta^t, \eta_m^t)^t = T\omega.$$

The subscript  $M$  is a label referring to the mean model.

$$\begin{pmatrix} Y \\ \psi_M \end{pmatrix} = \begin{pmatrix} X & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} \beta \\ \nu \end{pmatrix} + e^* \quad (1)$$

The error term  $e^*$  is normal with mean zero. The variance matrix of the error term is given by

$$\Sigma_a = \begin{pmatrix} \Sigma & 0 \\ 0 & D \end{pmatrix}. \quad (2)$$

$$y_a = T\delta + e^* \quad (3)$$

Weighted least squares equation

#### 3.1 The Augmented Model Matrix

$$X = \begin{pmatrix} T & Z \\ 0 & I \end{pmatrix} \delta = \begin{pmatrix} \beta \\ \nu \end{pmatrix} \quad (4)$$

Generalized linear models are a generalization of classical linear models.

### 4 Augmented GLMs

With the use of h-likelihood, a random effected model of the form can be viewed as an 'augmented GLM' with the response variables  $(y^t, \phi_m^t)^t$ , (with  $\mu = E(y), u = E(\phi)$ ,  $var(y) = \theta V(\mu)$ ). The augmented linear predictor is

$$\eta_{ma} = (\eta^t, \eta_m^t)^t = T\omega.$$

The subscript  $M$  is a label referring to the mean model.

$$\begin{pmatrix} Y \\ \psi_M \end{pmatrix} = \begin{pmatrix} X & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} \beta \\ \nu \end{pmatrix} + e^* \quad (5)$$

The error term  $e^*$  is normal with mean zero. The variance matrix of the error term is given by

$$\Sigma_a = \begin{pmatrix} \Sigma & 0 \\ 0 & D \end{pmatrix}. \quad (6)$$

$$y_a = T\delta + e^*$$

Weighted least squares equation

#### 4.1 The Augmented Model Matrix

$$X = \begin{pmatrix} T & Z \\ 0 & I \end{pmatrix} \delta = \begin{pmatrix} \beta \\ \nu \end{pmatrix} \quad (7)$$

### 5 The LME model as a general linear model

Henderson's equations in can be rewritten  $(T'W^{-1}T)\delta = T'W^{-1}y_a$  using

```
\[
\delta = \pmatrix{\beta \cr b},
\ y_{\{a\}} = \pmatrix{
y \cr \psi
},
\ T = \pmatrix{
X \& Z \cr
0 \& I
},
\ \text{and} \ W = \pmatrix{
\Sigma \& 0 \cr
0 \& D },
\]
```

where **cite[Lee:Neld:Pawi:2006]** describe  $\psi = 0$  as quasi-data with mean  $E(\psi) = b$ . Their formulation suggests that the joint estimation of the coefficients  $\beta$  and  $b$  of the linear mixed effects model can be derived via a classical augmented general linear model  $y_a = T\delta + \varepsilon$  where  $E(\varepsilon) = 0$  and  $\text{var}(\varepsilon) = W$ , with *both*  $\beta$  and  $b$  appearing as fixed parameters. The usefulness of this reformulation of an LME as a general linear model will be revisited.

Generalized linear models are a generalization of classical linear models.

## 6 Augmented GLMs

With the use of h-likelihood, a random effected model of the form can be viewed as an ‘augmented GLM’ with the response variables  $(y^t, \phi_m^t)^t$ , (with  $\mu = E(y), u = E(\phi)$ ,  $var(y) = \theta V(\mu)$ ). The augmented linear predictor is

$$\eta_{ma} = (\eta^t, \eta_m^t)^t = T\omega.$$

The subscript  $M$  is a label referring to the mean model.

$$\begin{pmatrix} Y \\ \psi_M \end{pmatrix} = \begin{pmatrix} X & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} \beta \\ \nu \end{pmatrix} + e^* \quad (8)$$

The error term  $e^*$  is normal with mean zero. The variance matrix of the error term is given by

$$\Sigma_a = \begin{pmatrix} \Sigma & 0 \\ 0 & D \end{pmatrix}. \quad (9)$$

Weighted least squares equation

### 6.1 The Augmented Model Matrix

$$X = \begin{pmatrix} T & Z \\ 0 & I \end{pmatrix} \delta = \begin{pmatrix} \beta \\ \nu \end{pmatrix} \quad (10)$$

### 6.2 Importance-Weighted Least-Squares (IWLS)

### 6.3 H-Likelihood

## 7 Application to MCS

Let  $\hat{\beta}$  denote the least square estimate of  $\beta$  based upon the full set of observations, and let  $\hat{\beta}^{(k)}$  denoted the estimate with the  $k^{th}$  case excluded.

## 8 Grubbs’ Data

For the Grubbs data the  $\hat{\beta}$  estimated are  $\hat{\beta}_0$  and  $\hat{\beta}_1$  respectively. Leaving the fourth case out, i.e.  $k = 4$  the corresponding estimates are  $\hat{\beta}_0^{-4}$  and  $\hat{\beta}_1^{-4}$

$$Y^{-Q} = \hat{\beta}^{-Q} X^{-Q} \quad (11)$$

When considering the regression of case-wise differences and averages, we write  $D^{-Q} = \hat{\beta}^{-Q} A^{-Q}$

	F	C	D	A
1	793.80	794.60	-0.80	794.20
2	793.10	793.90	-0.80	793.50
3	792.40	793.20	-0.80	792.80
4	794.00	794.00	0.00	794.00
5	791.40	792.20	-0.80	791.80
6	792.40	793.10	-0.70	792.75
7	791.70	792.40	-0.70	792.05
8	792.30	792.80	-0.50	792.55
9	789.60	790.20	-0.60	789.90
10	794.40	795.00	-0.60	794.70
11	790.90	791.60	-0.70	791.25
12	793.50	793.80	-0.30	793.65

$$Y^{(k)} = \hat{\beta}^{(k)} X^{(k)} \quad (12)$$

Consider two sets of measurements , in this case F and C , with the vectors of case-wise averages  $A$  and case-wise differences  $D$  respectively. A regression model of differences on averages can be fitted with the view to exploring some characteristics of the data.

When considering the regression of case-wise differences and averages, we write

$$D^{-Q} = \hat{\beta}^{-Q} A^{-Q} \quad (13)$$

Let  $\hat{\beta}$  denote the least square estimate of  $\beta$  based upon the full set of observations, and let  $\hat{\beta}^{(k)}$  denoted the estimate with the  $k^{th}$  case excluded.

For the Grubbs data the  $\hat{\beta}$  estimated are  $\hat{\beta}_0$  and  $\hat{\beta}_1$  respectively. Leaving the fourth case out, i.e.  $k = 4$  the corresponding estimates are  $\hat{\beta}_0^{-4}$  and  $\hat{\beta}_1^{-4}$

$$Y^{(k)} = \hat{\beta}^{(k)} X^{(k)} \quad (14)$$

Consider two sets of measurements , in this case F and C , with the vectors of case-wise averages  $A$  and case-wise differences  $D$  respectively. A regression model of differences on averages can be fitted with the view to exploring some characteristics of the data.

```
Call: lm(formula = D ~ A)
```

```
Coefficients: (Intercept)          A
-37.51896      0.04656
```

When considering the regression of case-wise differences and averages, we write

$$D^{-Q} = \hat{\beta}^{-Q} A^{-Q} \quad (15)$$

## 8.1 Influence measures using R

R provides the following influence measures of each observation.

	dfb.1_	dfb.A	dffit	cov.r	cook.d	hat
1	0.42	-0.42	-0.56	1.13	0.15	0.18
2	0.17	-0.17	-0.34	1.14	0.06	0.11
3	0.01	-0.01	-0.24	1.17	0.03	0.08
4	-1.08	1.08	1.57	0.24	0.56	0.16
5	-0.14	0.14	-0.24	1.30	0.03	0.13
6	-0.00	0.00	-0.11	1.31	0.01	0.08
7	-0.04	0.04	-0.08	1.37	0.00	0.11
8	0.02	-0.02	0.15	1.28	0.01	0.09
9	0.69	-0.68	0.75	2.08	0.29	0.48
10	0.18	-0.18	-0.22	1.63	0.03	0.27
11	-0.03	0.03	-0.04	1.53	0.00	0.19
12	-0.25	0.25	0.44	1.05	0.09	0.12

Augmented GLMs

Generalized linear models are a generalization of classical linear models.

## 9 Augmented GLMs

With the use of h-likelihood, a random effected model of the form can be viewed as an ‘augmented GLM’ with the response varaibkes  $(y^t, \phi_m^t)^t$ , (with  $\mu = E(y), u = E(\phi)$ ,  $var(y) = \theta V(\mu)$ ). The augmented linear predictor is

$$\eta_{ma} = (\eta^t, \eta_m^t)^t = T\omega.$$

The subscript  $M$  is a label referring to the mean model.

$$\begin{pmatrix} Y \\ \psi_M \end{pmatrix} = \begin{pmatrix} X & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} \beta \\ \nu \end{pmatrix} + e^* \quad (16)$$

The error term  $e^*$  is normal with mean zero. The variance matrix of the error term is given by

$$\Sigma_a = \begin{pmatrix} \Sigma & 0 \\ 0 & D \end{pmatrix}. \quad (17)$$

Weighted least squares equation

### 9.1 The Augmented Model Matrix

$$X = \begin{pmatrix} T & Z \\ 0 & I \end{pmatrix} \delta = \begin{pmatrix} \beta \\ \nu \end{pmatrix} \quad (18)$$

## 9.2 Importance-Weighted Least-Squares (IWLS)

## 9.3 H-Likelihood

Application to Method Comparison Studies

## 10 Application to MCS

Let  $\hat{\beta}$  denote the least square estimate of  $\beta$  based upon the full set of observations, and let  $\hat{\beta}^{(k)}$  denoted the estimate with the  $k^{th}$  case excluded.

## 11 Grubbs' Data

For the Grubbs data the  $\hat{\beta}$  estimated are  $\hat{\beta}_0$  and  $\hat{\beta}_1$  respectively. Leaving the fourth case out, i.e.  $k = 4$  the corresponding estimates are  $\hat{\beta}_0^{-4}$  and  $\hat{\beta}_1^{-4}$

$$Y^{-Q} = \hat{\beta}^{-Q} X^{-Q} \tag{19}$$

When considering the regression of case-wise differences and averages, we write  $D^{-Q} = \hat{\beta}^{-Q} A^{-Q}$



	F	C	D	A
1	793.80	794.60	-0.80	794.20
2	793.10	793.90	-0.80	793.50
3	792.40	793.20	-0.80	792.80
4	794.00	794.00	0.00	794.00
5	791.40	792.20	-0.80	791.80
6	792.40	793.10	-0.70	792.75
7	791.70	792.40	-0.70	792.05
8	792.30	792.80	-0.50	792.55
9	789.60	790.20	-0.60	789.90
10	794.40	795.00	-0.60	794.70
11	790.90	791.60	-0.70	791.25
12	793.50	793.80	-0.30	793.65

$$Y^{(k)} = \hat{\beta}^{(k)} X^{(k)} \quad (20)$$

Consider two sets of measurements , in this case F and C , with the vectors of case-wise averages  $A$  and case-wise differences  $D$  respectively. A regression model of differences on averages can be fitted with the view to exploring some characteristics of the data.

When considering the regression of case-wise differences and averages, we write

$$D^{-Q} = \hat{\beta}^{-Q} A^{-Q} \quad (21)$$

Let  $\hat{\beta}$  denote the least square estimate of  $\beta$  based upon the full set of observations, and let  $\hat{\beta}^{(k)}$  denoted the estimate with the  $k^{th}$  case excluded.

For the Grubbs data the  $\hat{\beta}$  estimated are  $\hat{\beta}_0$  and  $\hat{\beta}_1$  respectively. Leaving the fourth case out, i.e.  $k = 4$  the corresponding estimates are  $\hat{\beta}_0^{-4}$  and  $\hat{\beta}_1^{-4}$

$$Y^{(k)} = \hat{\beta}^{(k)} X^{(k)} \quad (22)$$

Consider two sets of measurements , in this case F and C , with the vectors of case-wise averages  $A$  and case-wise differences  $D$  respectively. A regression model of differences on averages can be fitted with the view to exploring some characteristics of the data.

```
Call: lm(formula = D ~ A)
```

```
Coefficients: (Intercept)          A
-37.51896      0.04656
```

When considering the regression of case-wise differences and averages, we write

$$D^{-Q} = \hat{\beta}^{-Q} A^{-Q} \quad (23)$$

## 11.1 Influence measures using R

R provides the following influence measures of each observation.

	dfb.1_	dfb.A	dffit	cov.r	cook.d	hat
1	0.42	-0.42	-0.56	1.13	0.15	0.18
2	0.17	-0.17	-0.34	1.14	0.06	0.11
3	0.01	-0.01	-0.24	1.17	0.03	0.08
4	-1.08	1.08	1.57	0.24	0.56	0.16
5	-0.14	0.14	-0.24	1.30	0.03	0.13
6	-0.00	0.00	-0.11	1.31	0.01	0.08
7	-0.04	0.04	-0.08	1.37	0.00	0.11
8	0.02	-0.02	0.15	1.28	0.01	0.09
9	0.69	-0.68	0.75	2.08	0.29	0.48
10	0.18	-0.18	-0.22	1.63	0.03	0.27
11	-0.03	0.03	-0.04	1.53	0.00	0.19
12	-0.25	0.25	0.44	1.05	0.09	0.12

Augmented GLMs

Generalized linear models are a generalization of classical linear models.

## 12 Augmented GLMs

With the use of h-likelihood, a random effected model of the form can be viewed as an ‘augmented GLM’ with the response variables  $(y^t, \phi_m^t)^t$ , (with  $\mu = E(y), u = E(\phi)$ ,  $var(y) = \theta V(\mu)$ ). The augmented linear predictor is

$$\eta_{ma} = (\eta^t, \eta_m^t)^t = T\omega.$$

The subscript  $M$  is a label referring to the mean model.

$$\begin{pmatrix} Y \\ \psi_M \end{pmatrix} = \begin{pmatrix} X & Z \\ 0 & I \end{pmatrix} \begin{pmatrix} \beta \\ \nu \end{pmatrix} + e^* \quad (24)$$

The error term  $e^*$  is normal with mean zero. The variance matrix of the error term is given by

$$\Sigma_a = \begin{pmatrix} \Sigma & 0 \\ 0 & D \end{pmatrix}. \quad (25)$$

$$y_a = T\delta + e^* \quad (26)$$

Weighted least squares equation

### 12.1 The Augmented Model Matrix

$$X = \begin{pmatrix} T & Z \\ 0 & I \end{pmatrix} \delta = \begin{pmatrix} \beta \\ \nu \end{pmatrix} \quad (27)$$

## 12.2 Importance-Weighted Least-Squares (IWLS)

## 12.3 H-Likelihood

Application to Method Comparison Studies

## 13 Application to MCS

Let  $\hat{\beta}$  denote the least square estimate of  $\beta$  based upon the full set of observations, and let  $\hat{\beta}^{(k)}$  denoted the estimate with the  $k^{th}$  case excluded.

## 14 Grubbs' Data

For the Grubbs data the  $\hat{\beta}$  estimated are  $\hat{\beta}_0$  and  $\hat{\beta}_1$  respectively. Leaving the fourth case out, i.e.  $k = 4$  the corresponding estimates are  $\hat{\beta}_0^{-4}$  and  $\hat{\beta}_1^{-4}$

$$Y^{-Q} = \hat{\beta}^{-Q} X^{-Q} \quad (28)$$

When considering the regression of case-wise differences and averages, we write  $D^{-Q} = \hat{\beta}^{-Q} A^{-Q}$

	F	C	D	A
1	793.80	794.60	-0.80	794.20
2	793.10	793.90	-0.80	793.50
3	792.40	793.20	-0.80	792.80
4	794.00	794.00	0.00	794.00
5	791.40	792.20	-0.80	791.80
6	792.40	793.10	-0.70	792.75
7	791.70	792.40	-0.70	792.05
8	792.30	792.80	-0.50	792.55
9	789.60	790.20	-0.60	789.90
10	794.40	795.00	-0.60	794.70
11	790.90	791.60	-0.70	791.25
12	793.50	793.80	-0.30	793.65

$$Y^{(k)} = \hat{\beta}^{(k)} X^{(k)} \quad (29)$$

Consider two sets of measurements , in this case F and C , with the vectors of case-wise averages  $A$  and case-wise differences  $D$  respectively. A regression model of differences on averages can be fitted with the view to exploring some characteristics of the data.

When considering the regression of case-wise differences and averages, we write

$$D^{-Q} = \hat{\beta}^{-Q} A^{-Q} \quad (30)$$

Let  $\hat{\beta}$  denote the least square estimate of  $\beta$  based upon the full set of observations, and let  $\hat{\beta}^{(k)}$  denoted the estimate with the  $k^{th}$  case excluded.

For the Grubbs data the  $\hat{\beta}$  estimated are  $\hat{\beta}_0$  and  $\hat{\beta}_1$  respectively. Leaving the fourth case out, i.e.  $k = 4$  the corresponding estimates are  $\hat{\beta}_0^{-4}$  and  $\hat{\beta}_1^{-4}$

$$Y^{(k)} = \hat{\beta}^{(k)} X^{(k)} \quad (31)$$

Consider two sets of measurements , in this case F and C , with the vectors of case-wise averages  $A$  and case-wise differences  $D$  respectively. A regression model of differences on averages can be fitted with the view to exploring some characteristics of the data.

```
Call: lm(formula = D ~ A)
```

```
Coefficients: (Intercept)          A
-37.51896      0.04656
```

When considering the regression of case-wise differences and averages, we write

$$D^{-Q} = \hat{\beta}^{-Q} A^{-Q} \quad (32)$$

## 14.1 Influence measures using R

R provides the following influence measures of each observation.

	dfb.1_	dfb.A	dffit	cov.r	cook.d	hat
1	0.42	-0.42	-0.56	1.13	0.15	0.18
2	0.17	-0.17	-0.34	1.14	0.06	0.11
3	0.01	-0.01	-0.24	1.17	0.03	0.08
4	-1.08	1.08	1.57	0.24	0.56	0.16
5	-0.14	0.14	-0.24	1.30	0.03	0.13
6	-0.00	0.00	-0.11	1.31	0.01	0.08
7	-0.04	0.04	-0.08	1.37	0.00	0.11
8	0.02	-0.02	0.15	1.28	0.01	0.09
9	0.69	-0.68	0.75	2.08	0.29	0.48
10	0.18	-0.18	-0.22	1.63	0.03	0.27
11	-0.03	0.03	-0.04	1.53	0.00	0.19
12	-0.25	0.25	0.44	1.05	0.09	0.12