Time Series – Tutorial 5 – Solutions

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Question 1

a) The ARIMA(0,0,0) model is

$$\begin{aligned}(1)\,(1-B)^0\,Y_t &= (1)\,e_t \\ Y_t &= e_t.\end{aligned}$$

This is white noise.

b) The ARIMA(0,1,0) model is

$$\begin{aligned}(1)\,(1-B)^1\,Y_t &= (1)\,e_t \\ Y_t - Y_{t-1} &= e_t \\ Y_t &= Y_{t-1} + e_t.\end{aligned}$$

This is a random walk.

Question 2

a) $Y_t = Y_{t-1} - 0.25Y_{t-2} + e_t - 0.1e_{t-1}$.

$$\begin{aligned}\Rightarrow Y_t - Y_{t-1} + 0.25Y_{t-2} &= e_t - 0.1e_{t-1} \\ (1 - B + 0.25B^2)Y_t &= (1 - 0.1B)e_t\end{aligned}$$

The roots of the AR polynomial are

$$\frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.25)(1)}}{2(0.25)} = \frac{1}{0.5} = 2$$

i.e., repeated roots.

The root of the MA polynomial is 10.

Thus, the model is

$$(1 - 0.5\,B)^2\,Y_t = (1 - 0.1B)e_t$$

which is an ARIMA(2,0,1), i.e., an ARMA(2,1). The model is stationary and invertible since the roots of the AR and MA polynomials are greater than one.

The parameters are $\phi_1 = 1$, $\phi_2 = -0.25$, $\theta_1 = 0.1$.

b) $Y_t = 2Y_{t-1} - Y_{t-2} + e_t$.

$$\begin{aligned}\Rightarrow Y_t - 2Y_{t-1} + Y_{t-2} &= e_t \\ (1 - 2B + B^2)Y_t &= e_t\end{aligned}$$

The roots of the AR polynomial are

$$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)} = \frac{2}{2} = 1$$

i.e., repeated roots.

Thus, the model is

$$(1 - B)^2\,Y_t = e_t$$

which is an ARIMA(0,2,0).

c) $Y_t = 0.5Y_{t-1} - 0.5Y_{t-2} + e_t - 0.5e_{t-1} + 0.25e_{t-2}$.

$$\begin{aligned}\Rightarrow Y_t - 0.5Y_{t-1} + 0.5Y_{t-2} &= e_t - 0.5e_{t-1} + 0.25e_{t-2} \\ (1 - 0.5B + 0.5B^2)Y_t &= (1 - 0.5B + 0.25B^2)e_t\end{aligned}$$

The roots of the AR polynomial are

$$\frac{-(-0.5) \pm \sqrt{(-0.5)^2 - 4(0.5)(1)}}{2(0.5)} = 0.5 \pm \sqrt{1.75}i$$

Note that these roots have modulus greater than one: $\sqrt{(0.5)^2 + (\sqrt{1.75})^2} = \sqrt{2} \approx 1.414$.

The roots of the MA polynomial are

$$\begin{aligned}\frac{-(-0.5) \pm \sqrt{(-0.5)^2 - 4(0.25)(1)}}{2(0.25)} &= \frac{0.5 \pm \sqrt{0.75}i}{0.5} \\ &= 1 \pm \sqrt{3}i\end{aligned}$$

Note that these roots have modulus greater than one: $\sqrt{(1)^2 + (\sqrt{3})^2} = 2$.

This is an ARIMA(2,0,2), i.e., an ARMA(2,2). The model is stationary and invertible since the roots of the AR and MA polynomials have modulus greater than one.

The parameters are $\phi_1 = 0.5$, $\phi_2 = -0.5$, $\theta_1 = 0.5$, $\theta_2 = -0.25$.

d) $Y_t = 1.4Y_{t-1} - 0.35Y_{t-2} + 0.05Y_{t-3} + e_t + 0.1e_{t-1}$.

$$(1 - 1.4B + 0.35B^2 + 0.05B^3)Y_t = (1 + 0.1B)\,e_t$$

Confirm that $B = 1$ is a root of the AR polynomial:

$$1 - 1.4(1) + 0.35(1)^2 + 0.05(1)^3 = 0.$$

Thus, $B = 1$ is a root. Thus, $1 - B$ is a factor of the AR polynomial. To find the other roots, we first need to divide out the $1 - B$ factor. It is easy to see that

$$\begin{aligned}&(1 - B)(1 - 0.4B - 0.05B^2) \\ &\quad = (1 - 1.4B + 0.35B^2 + 0.05B^3)\end{aligned}$$

Thus, we need the roots of $1 - 0.4B - 0.05B^2$:

$$\begin{aligned}\frac{-(-0.4) \pm \sqrt{(-0.4)^2 - 4(1)(-0.05)}}{2(-0.05)} &= \frac{0.4 \pm 0.6}{-0.1} \\ &= 2 \text{ and } -10\end{aligned}$$

Thus, the model can be written as

$$\begin{aligned}(1 - 0.5B)(1 + 0.1B)(1 - B)Y_t &= (1 + 0.1B)\,e_t \\ (1 - 0.5B)(1 - B)Y_t &= e_t \\ &\text{(dividing across by } 1 + 0.1B)\end{aligned}$$

which is an ARIMA(1,1,0) or an ARI(1,1) with AR parameter $\phi = 0.5$.

Question 3

$$\begin{aligned}
\text{A:} \quad & Y_t = 0.9Y_{t-1} + 0.09Y_{t-2} + e_t \\
\text{B:} \quad & Y_t = Y_{t-1} + e_t - 0.1e_{t-1}
\end{aligned}$$

a) First consider model A: $Y_t = 0.9Y_{t-1} + 0.09Y_{t-2} + e_t$

$$(1 - 0.9B - 0.09B^2)\,Y_t = e_t$$

The roots of the AR polynomial are

$$\begin{aligned}
&\frac{-(-0.9) \pm \sqrt{(-0.9)^2 - 4(1)(-0.09)}}{2(-0.09)} \\
&= 1.009252 \text{ and } -11.009252
\end{aligned}$$

This is a stationary AR(2) model, i.e., an ARIMA(2,0,0). It is worth noting that one of the roots is very close to 1 which means it is close to being non-stationary.

model B: $Y_t = Y_{t-1} + e_t - 0.1e_{t-1}$

$$(1 - B)\,Y_t = (1 - 0.1B)\,e_t$$

This is an ARIMA(0,1,1), i.e., an IMA(1,1). This is a non-stationary model.

b) Model A is stationary but model B is non-stationary. It is worth reiterating that model A is close to being non-stationary.

c) Model A can be written as:

$$(1 - 0.9B - 0.09B^2)\,Y_t = e_t$$

For model B we have

$$\begin{aligned}
&(1 - B)\,Y_t = (1 - 0.1B)\,\pi(B)\,Y_t \\
&\Rightarrow (1 - B) = (1 - 0.1B)\,\pi(B)
\end{aligned}$$

$$\begin{aligned}
&(1 - 0.1B)\,\pi(B) \\
&= (1 - 0.1B)\,(1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots) \\
&= 1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots \\
&\qquad - 0.1B + 0.1\pi_1 B^2 + 0.1\pi_2 B^3 + 0.1\pi_3 B^4 - \cdots \\
&= 1 + (-\pi_1 - 0.1)B + (0.1\pi_1 - \pi_2)B^2 \\
&\qquad + (0.1\pi_2 - \pi_3)B^3 + (0.1\pi_3 - \pi_4)B^4 + \cdots
\end{aligned}$$

Since this equals $1 - B$ we have

$$\begin{aligned}
-\pi_1 - 0.1 = -1 &\Rightarrow \pi_1 = 0.9 \\
0.1\pi_1 - \pi_2 = 0 &\Rightarrow \pi_2 = 0.09 \\
0.1\pi_2 - \pi_3 = 0 &\Rightarrow \pi_3 = 0.009 \\
0.1\pi_3 - \pi_4 = 0 &\Rightarrow \pi_4 = 0.0009 \\
&\vdots
\end{aligned}$$

$\Rightarrow \pi_i = 0.9^i$. Note that we can also get the weights through back-substitution.

Thus, model B can be written

$$\begin{aligned}
&(1 - 0.9B - 0.09B^2 - 0.009B^3 \\
&\qquad - 0.0009B^4 - 0.00009B^5 - \cdots)\,Y_t = e_t
\end{aligned}$$

The first two π-weights are equal for both models and the remaining π-weights are similar (zero for model A and close to zero for model B). Thus, these models are very similar.