Time Series – Tutorial 8 – Solutions

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Question 1

We have that $Y_t = 31.92 + 0.57Y_{t-1} + e_t$ where $\hat{\sigma}_e^2 = 24.83$.

a)
$$\begin{aligned}\hat{Y}_{t+1} &= 31.92 + 0.57E(Y_t \,|\, \mathcal{H}_t) + E(e_{t+1} \,|\, \mathcal{H}_t)\\ &= 31.92 + 0.57Y_t\\ &= 31.92 + 0.57(67) = 70.11.\end{aligned}$$

$$\begin{aligned}\hat{Y}_{t+2} &= 31.92 + 0.57E(Y_{t+1} \,|\, \mathcal{H}_t) + E(e_{t+2} \,|\, \mathcal{H}_t)\\ &= 31.92 + 0.57\hat{Y}_{t+1}\\ &= 31.92 + 0.57(70.11) = 71.8827.\end{aligned}$$

Note that $E(e_{t+1} \,|\, \mathcal{H}_t) = E(e_{t+1})$ since e_{t+1} is independent of the past and, furthermore $E(e_{t+1}) = 0$. Similarly, $E(e_{t+2} \,|\, \mathcal{H}_t) = 0$.

The prediction limits for Y_{t+1} are given by

$$\begin{gathered}\hat{Y}_{t+1} \pm 1.96\sigma_e\\ 70.11 \pm 1.96\sqrt{24.83}\\ [60.34,\ 79.88]\end{gathered}$$

The prediction limits for Y_{t+2} are given by

$$\hat{Y}_{t+2} \pm 1.96\sigma_e\sqrt{\psi_0^2 + \psi_1^2}$$

where $\psi_0 = 1$ and $\psi_1 = \phi$ since this is an AR(1) process (see Lecture 4, Section 4.4). Thus,

$$\begin{gathered}\hat{Y}_{t+2} \pm 1.96\sigma_e\sqrt{1 + \phi^2}\\ 71.8827 \pm 1.96\sqrt{24.83}\sqrt{1 + (0.57)^2}\\ [60.64,\ 83.12]\end{gathered}$$

Question 2

a) The fitted MA(1) model is given by

$$Y_t = e_t - \tfrac{1}{2}e_{t-1}$$

where $\hat{\sigma}_e^2 = \frac{1}{3}$.

Thus, the one-ahead prediction is:

$$\begin{aligned}\hat{Y}_{t+1} &= E(e_{t+1} \,|\, \mathcal{H}_t) - \tfrac{1}{2}E(e_t \,|\, \mathcal{H}_t)\\ &= -\tfrac{1}{2}e_t\\ &= -\tfrac{1}{2}(0)\\ &= 0\end{aligned}$$

since, from Tutorial 7 $e_3 = \frac{1}{2} - \theta = \frac{1}{2} - \frac{1}{2} = 0$. The two-step ahead prediction is

$$\begin{aligned}\hat{Y}_{t+2} &= E(e_{t+2} \,|\, \mathcal{H}_t) - \tfrac{1}{2}E(e_{t+1} \,|\, \mathcal{H}_t)\\ &= 0\end{aligned}$$

The prediction limits for Y_{t+1} are given by

$$\begin{gathered}\hat{Y}_{t+1} \pm 1.96\sigma_e\\ 0 \pm 1.96\sqrt{\tfrac{1}{3}}\\ [-1.132,\ 1.132]\end{gathered}$$

The prediction limits for Y_{t+2} are given by

$$\begin{gathered}\hat{Y}_{t+2} \pm 1.96\sigma_e\sqrt{\psi_0^2 + \psi_1^2}\\ 0 \pm 1.96\sqrt{\tfrac{1}{3}}\sqrt{1 + (-\tfrac{1}{2})^2}\\ 0 \pm 1.96\sqrt{\tfrac{1}{3}}\sqrt{\tfrac{5}{4}}\\ [-1.265,\ 1.265]\end{gathered}$$

Question 3

The model is

$$Y_t = 10.8 - 0.5(Y_{t-1} - 10.8) + e_t$$

where $\sigma_e^2 = 4$. Note that $Y_t = 12.2$.

a) Taking conditional expectations as in the previous questions (also see Lecture 9 for further details) we get

$$\begin{aligned}\hat{Y}_{t+1} &= 10.8 - 0.5(Y_t - 10.8)\\ &= 10.8 - 0.5(12.2 - 10.8)\\ &= 10.1\end{aligned}$$

$$\begin{aligned}\hat{Y}_{t+2} &= 10.8 - 0.5(\hat{Y}_{t+1} - 10.8)\\ &= 10.8 - 0.5(10.1 - 10.8)\\ &= 11.15\end{aligned}$$

The prediction limits for Y_{t+1} are given by

$$\begin{gathered}\hat{Y}_{t+1} \pm 1.96\sigma_e\\ 10.1 \pm 1.96\sqrt{4}\\ [6.18,\ 14.02]\end{gathered}$$

The prediction limits for Y_{t+2} are given by

$$\begin{gathered}\hat{Y}_{t+2} \pm 1.96\sigma_e\sqrt{\psi_0^2 + \psi_1^2}\\ 11.15 \pm 1.96\sqrt{4}\sqrt{1 + (0.5)^2}\\ [6.77,\ 15.53]\end{gathered}$$

b) Note that from Lecture 9 (Section 2.1) we derived a general formula for AR(1) forecasts:

$$\hat{Y}_{t+l} = \mu + \phi^l(\hat{Y}_t - \mu).$$

Thus,

$$\begin{aligned}\hat{Y}_{t+10} &= 10.8 + (-0.5)^{10}(12.2 - 10.8)\\ &= 10.80137\end{aligned}$$

For the variance of this prediction we will need to evaluate

$$\begin{aligned}\sum_{i=0}^{9}\psi_i^2 &= \sum_{i=0}^{9}(\phi^i)^2\\ &\text{($\psi_i = \phi^i$ for an AR(1) process - see Lecture 4)}\\ &= \sum_{i=0}^{9}(\phi^2)^i\\ &= \frac{1-(\phi^2)^9}{1-\phi^2}\\ &\text{(since this is a geometric series)}\\ &= \frac{1-(0.25)^9}{1-0.25}\\ &= 1.333328\end{aligned}$$

Thus, the prediction limits for Y_{t+10} are

$$\hat{Y}_{t+10} \pm 1.96\sigma_e\sqrt{\sum_{i=0}^{9}\psi_i^2}$$
$$10.80137 \pm 1.96\sqrt{4}\sqrt{1.333328}$$
$$[6.27,\ 15.33]$$

Question 4

$$Y_t = 5 + 1.1Y_{t-1} - 0.5Y_{t-2} + e_t$$

where $\sigma_e^2 = 2$.

a) Taking conditional expectations gives

$$\begin{aligned}\hat{Y}_{2008} = \hat{Y}_{t+1} &= 5 + 1.1Y_t - 0.5Y_{t-1}\\ &= 5 + 1.1(10) - 0.5(11)\\ &= 10.5\end{aligned}$$

$$\begin{aligned}\hat{Y}_{2009} = \hat{Y}_{t+2} &= 5 + 1.1\hat{Y}_{t+1} - 0.5Y_t\\ &= 5 + 1.1(10.5) - 0.5(10)\\ &= 11.55\end{aligned}$$

b) The prediction limits for Y_{t+1} are given by

$$\hat{Y}_{t+1} \pm 1.96\sigma_e$$
$$10.5 \pm 1.96\sqrt{2}$$
$$[7.73,\ 13.27]$$

The prediction limits for Y_{t+2} are given by

$$\hat{Y}_{t+2} \pm 1.96\sigma_e\sqrt{\psi_0^2+\psi_1^2}$$
$$\hat{Y}_{t+2} \pm 1.96\sigma_e\sqrt{1+\phi_1^2}$$
(for an AR(2) model $\psi_1 = \phi_1$ - see below)
$$11.55 \pm 1.96\sqrt{2}\sqrt{1+(1.1)^2}$$
$$[9.45,\ 13.65]$$

Note that above we used the fact that $\psi_1 = \phi_1$ for an AR(2) process. We can find this by writing the AR(2) process, $\phi(B)\,Y_t = e_t$, as a general linear process, $Y_t = \psi(B)\,e_t$. Substituting the second expression into the first gives:

$$\begin{aligned}\phi(B)\,\psi(B)\,e_t &= e_t\\ \phi(B)\,\psi(B) &= 1\end{aligned}$$

$$\begin{aligned}\Rightarrow (1-\phi_1B-\phi_2B^2)&(1+\psi_1B+\psi_2B^2+\cdots)\\ &= 1 + 0B + 0B^2 + \cdots\end{aligned}$$

Hence, we need to multiply out the AR and linear process polynomials and match the coefficients with the right hand side:

$$\begin{aligned}\phi(B)\,\psi(B) &= (1-\phi_1B-\phi_2B^2)(1+\psi_1B+\psi_2B^2+\cdots)\\ &= 1+\psi_1B-\phi_1B+\cdots\\ &\text{(we are only interested in the B coefficient here)}\\ &= 1+(\psi_1-\phi_1)B+\cdots\end{aligned}$$

Since the coefficient of B must be zero, we have that $\psi_1 = \phi_1$.

An alternative approach is to use the formula $Y_t = 5 + 1.1Y_{t-1} - 0.5Y_{t-2} + e_t$ and carry out back-substitution (as was done for an AR(1) process in Lecture 4, Section 4.4) to find the coefficient of e_{t-1}.