Time Series – Tutorial 2 – Solutions

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Question 1

First note that

$$\begin{aligned}\text{Corr}(X,Y) &= \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}\\ &= \frac{\text{Cov}(X,Y)}{\sqrt{(9)(4)}} = 0.25\end{aligned}$$

$$\Rightarrow \text{Cov}(X,Y) = 0.25\sqrt{(9)(4)} = 0.25(6) = 1.5.$$

a)
$$\begin{aligned}\text{Var}(X+Y) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)\\ &= 9 + 4 + 2(1.5) = 16.\end{aligned}$$

b)
$$\begin{aligned}\text{Cov}(X, X+Y) &= \text{Cov}(X,X) + \text{Cov}(X,Y)\\ &= \text{Var}(X) + \text{Cov}(X,Y)\\ &= 9 + 1.5 = 10.5.\end{aligned}$$

c)
$$\text{Corr}(X+Y, X-Y) = \frac{\text{Cov}(X+Y, X-Y)}{\sqrt{\text{Var}(X+Y)\,\text{Var}(X-Y)}}$$

$$\begin{aligned}\text{Cov}(X+Y, X-Y) &= \text{Cov}(X,X) + \text{Cov}(X,-Y)\\ &\quad + \text{Cov}(Y,X) + \text{Cov}(Y,-Y)\\ &= \text{Cov}(X,X) - \text{Cov}(X,Y)\\ &\quad + \text{Cov}(Y,X) - \text{Cov}(Y,Y)\\ &= \text{Var}(X) - \text{Var}(Y) = 9 - 4 = 5.\end{aligned}$$

$$\begin{aligned}\text{Var}(X-Y) &= \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X,Y)\\ &= 9 + 4 - 2(1.5) = 10.\end{aligned}$$

$$\text{Var}(X+Y) = 16 \qquad \text{(from (a))}$$

$$\Rightarrow \text{Corr}(X+Y, X-Y) = \frac{5}{\sqrt{(16)(10)}} = 0.395.$$

Question 2

a) From part (c) of Q1 we have

$$\text{Cov}(X+Y, X-Y) = \text{Var}(X) - \text{Var}(Y) = 0 \quad \text{(if } \text{Var}(X) = \text{Var}(Y))$$

Question 3

Suppose $Y_t = 5 + 2t + X_t$ where $\{X_t\}$ is a zero-mean stationary series with auto-covariance function γ_k.

a)
$$\begin{aligned}E(Y_t) = E(5 + 2t + X_t) &= 5 + 2t + E(X_t)\\ &= 5 + 2t + 0\\ &= 5 + 2t\end{aligned}$$

b)
$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(5 + 2t + X_t, 5 + 2(t-k) + X_{t-k})\\ &= \text{Cov}(X_t, X_{t-k})\\ &\qquad \text{(additive constants disappear from covariance)}\\ &= \gamma_k.\\ &\qquad \text{(since } \{X_t\} \text{ is a stationary series)}\end{aligned}$$

c) $\{Y_t\}$ non-stationary. Even though the autocovariance function depends only on the time-lag, k, the mean function depends on t.

Question 4

Suppose $\text{Cov}(X_t, X_{t-k}) = \gamma_k$ is free of t but that $E(X_t) = 3t$.

a) No – since the mean function depends on time.

b) The mean function is

$$\begin{aligned}E(Y_t) = E(7 + 3t + X_t) &= 7 - 3t + E(X_t)\\ &= 7 - 3t + 3t\\ &= 7\end{aligned}$$

and the autocovariance function is

$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(7 - 3t + X_t, 7 - 3(t-k) + X_{t-k})\\ &= \text{Cov}(X_t, X_{t-k})\\ &= \gamma_k.\end{aligned}$$

Thus, $\{Y_t\}$ is stationary.

Question 5

a) $Y_1 = \mu_0 + e_1$

(apply $Y_t = \mu_0 + Y_{t-1} + e_t$ repeatedly)

$$\begin{aligned}Y_2 &= \mu_0 + Y_1 + e_2 && = 2\mu_0 + e_1 + e_2\\ Y_3 &= \mu_0 + Y_2 + e_3 && = 3\mu_0 + e_1 + e_2 + e_3\\ &\vdots && \vdots\\ Y_t &= \mu_0 + Y_{t-1} + e_t && = t\mu_0 + e_1 + e_2 + e_3 + \ldots + e_t\\ & && = t\mu_0 + \sum_{i=1}^{t} e_i\end{aligned}$$

as required.

b)
$$E(Y_t) = E\left(t\mu_0 + \sum_{i=1}^{t} e_i\right)$$
$$= t\mu_0 + \sum_{i=1}^{t} E(e_i)$$
$$= t\mu_0 + \sum_{i=1}^{t} 0$$
$$= t\mu_0$$

$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}\left(t\mu_0 + \sum_{i=1}^{t} e_i, (t-k)\mu_0 + \sum_{j=1}^{t-k} e_j\right) \\ &= \text{Cov}\left(\sum_{i=1}^{t} e_i, \sum_{j=1}^{t-k} e_j\right) \\ &\qquad \text{(additive constants)} \\ &= \sum_{i=1}^{t}\sum_{j=1}^{t-k} \text{Cov}\left(e_i, e_j\right)\end{aligned}$$

Since $\text{Cov}(e_i, e_j) = 0$ when $i \neq j$, only when $i = j$ do we get non-zero terms in the above sum. Thus,

$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \sum_{i=1}^{t-k} \text{Cov}\left(e_i, e_i\right) \\ &= \sum_{i=1}^{t-k} \text{Var}\left(e_i, e_i\right) \\ &= \sum_{i=1}^{t-k} \sigma_e^2 \\ &= (t-k)\sigma_e^2\end{aligned}$$

Clearly $\{Y_t\}$ is non-stationary: both the mean function and acvf depend on t.

c)
$$\begin{aligned} Y_t &= \mu_0 + Y_{t-1} + e_t \\ \Rightarrow Y_t - Y_{t-1} &= \mu_0 + e_t \\ &\qquad \text{(subtracting } Y_t \text{ from both sides)}\end{aligned}$$

i.e., $X_t = \mu_0 + e_t$.

Thus, the mean function is

$$\begin{aligned}E(X_t) &= \mu_0 + (e_t) \\ &= \mu_0 + 0 \\ &= \mu_0\end{aligned}$$

and the acvf is

$$\begin{aligned}\text{Cov}(X_t, X_{t-k}) &= \text{Cov}(\mu_0 + e_t, \mu_0 + e_{t-k}) \\ &= \text{Cov}(e_t, e_{t-k}) \\ &= \begin{cases} \sigma_e^2 & \text{when } k = 0 \\ 0 & \text{when } k \neq 0 \end{cases}\end{aligned}$$

Therefore, $\{X_t\}$ is stationary since the mean function is constant and the acvf depends only on k.

d) See Tutorial 2 `R` code.

Question 6

a)
$$E(Y_t) = \theta_0 + tE(e_t) = \theta_0 + t(0) = \theta_0$$

$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(\theta_0 + te_t, \theta_0 + (t-k)e_{t-k}) \\ &= \text{Cov}(te_t, (t-k)e_{t-k}) \\ &= t(t-k)\text{Cov}(e_t, e_{t-k}) \\ &= \begin{cases} t^2\sigma_e^2 & \text{when } k = 0 \\ 0 & \text{when } k \neq 0 \end{cases}\end{aligned}$$

This is non-stationary since the acvf depends on t.

b) First we have that

$$\begin{aligned}W_t = \nabla Y_t &= Y_t - Y_{t-1} \\ &= \theta_0 + te_t - \theta_0 - (t-1)e_{t-1} \\ &= te_t - (t-1)e_{t-1}\end{aligned}$$

$$\begin{aligned}E(W_t) &= tE(e_t) - (t-1)E(e_{t-1}) \\ &= t(0) - (t-1)(0) = 0\end{aligned}$$

$$\begin{aligned}\text{Cov}(W_t, W_{t-k}) &= \text{Cov}(te_t - (t-1)e_{t-1}, \\ &\qquad te_{t-k} - (t-k-1)e_{t-k-1}) \\ &= \begin{cases} [t^2 + (t-1)^2]\sigma_e^2 & \text{when } k = 0 \\ -(t-1)t\sigma_e^2 & \text{when } k = 1 \\ 0 & \text{when } k \geq 2 \end{cases}\end{aligned}$$

W_t is non-stationary since the acvf depends on t.

c)
$$E(Y_t) = (-1)^t E(U_t) = (-1)^t(0) = 0$$

$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}[(-1)^t\, U_t, (-1)^{t-k}\, U_{t-k}] \\ &= (-1)^t\, (-1)^{t-k}\text{Cov}[U_t,\, U_{t-k}] \\ &= (-1)^{2t}(-1)^{-k}\text{Cov}[U_t,\, U_{t-k}] \\ &= (-1)^k\text{Cov}[U_t,\, U_{t-k}]\end{aligned}$$

where $\text{Cov}[U_t, U_{t-k}]$ depends only on k since $\{U_t\}$ is stationary. Thus, Y_t is stationary since its mean function is constant and the covariance depends only on k.

d)
$$E(S_t) = E(X_t) + E(Y_t) = \mu_X + \mu_Y$$

$$\begin{aligned}\text{Cov}(S_t, S_{t-k}) &= \text{Cov}(X_t + Y_t, X_{t-k} + Y_{t-k}) \\ &= \text{Cov}(X_t, X_{t-k}) + \text{Cov}(X_t, Y_{t-k}) \\ &\quad + \text{Cov}(Y_t, X_{t-k}) + \text{Cov}(Y_t, Y_{t-k}) \\ &= \text{Cov}(X_t, X_{t-k}) + \text{Cov}(Y_t, Y_{t-k}) \\ &\qquad \text{(since } \{X_t\} \text{ and } \{Y_t\} \text{ are independent)} \\ &= \gamma_{X,k} + \gamma_{Y,k} \\ &\qquad \text{(since } \{X_t\} \text{ and } \{Y_t\} \text{ are stationary)}\end{aligned}$$

Thus S_t is stationary.

e)
$$\begin{aligned} E(X_t) &= E(e_t\, e_{t-1}) \\ &= E(e_t)E(e_{t-1}) \\ &\text{(due to independence - see Q7(e) of Tutorial 1)} \\ &= 0(0) = 0. \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t-k}) &= E(X_t X_{t-k}) - E(X_t)E(X_{t-k}) \\ &= E(X_t X_{t-k}) - 0(0) \\ &= E(X_t X_{t-k}) \\ &= E(e_t\, e_{t-1} e_{t-k}\, e_{t-k-1}) \end{aligned}$$

When $k = 0$

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t-k}) &= E(e_t^2\, e_{t-1}^2) \\ &= E(e_t^2)E(e_{t-1}^2) \quad \text{(independence)} \\ &= \mathrm{Var}(e_t)\mathrm{Var}(e_{t-1}) \\ &\qquad \text{(since } E(e_t) = E(e_{t-1}) = 0) \\ &= \sigma_e^2 \sigma_e^2 \\ &= \sigma_e^4. \end{aligned}$$

When $k = 1$

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t-k}) &= E(e_t\, e_{t-1}^2\, e_{t-2}) \\ &= E(e_t)E(e_{t-1}^2)E(e_{t-2}) \\ &= 0(\sigma_e^2)(0) = 0 \end{aligned}$$

When $k = 2$

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t-k}) &= E(e_t\, e_{t-1} e_{t-2}\, e_{t-3}) \\ &= E(e_t)E(e_{t-1})E(e_{t-2})E(e_{t-3}) \\ &= 0. \end{aligned}$$

and this continues for $k > 2$. Summarising the above we have

$$\mathrm{Cov}(X_t, X_{t-k}) = \begin{cases} \sigma_e^4 & \text{when } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, X_t is stationary.

Question 7

a)
$$E(W_t) = E(Y_t) - E(Y_{t-1}) = \mu - \mu = 0$$
(where $E(Y_t) = \mu$ is a constant since $\{Y_t\}$ is stationary)

$$\begin{aligned} &\mathrm{Cov}(W_t, W_{t-k}) \\ &= \mathrm{Cov}(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\ &= \mathrm{Cov}(Y_t, Y_{t-k}) - \mathrm{Cov}(Y_t, Y_{t-k-1}) \\ &\quad - \mathrm{Cov}(Y_{t-1}, Y_{t-k}) + \mathrm{Cov}(Y_{t-1}, Y_{t-k-1}) \\ &= \gamma_{t-(t-k)} - \gamma_{t-(t-k-1)} - \gamma_{(t-1)-(t-k)} + \gamma_{(t-1)-(t-k-1)} \\ &= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\ &= 2\gamma_k - \gamma_{k+1} - \gamma_{k-1} \end{aligned}$$

Thus, W_t is stationary since the mean function is constant and the autocovariance depends only on k.

b) $U_t = \nabla(\nabla Y_t) = \nabla W_t$ where W_t is a stationary series as shown in part (a). Also in part (a) we have shown that differencing a stationary series leads to a stationary series and, therefore, U_t is a stationary series.

c)
$$\begin{aligned} E(Y_t) = \beta_0 + \beta_1 t + E(X_t) &= \beta_0 + \beta_1 t + E(X_t) \\ &= \beta_0 + \beta_1 t + 0 \\ &= \beta_0 + \beta_1 t \end{aligned}$$

Clearly, $\{Y_t\}$ is non-stationary since its mean function depends linearly on t, i.e., linear trend.

The differenced series is:

$$\begin{aligned} W_t = Y_t - Y_{t-1} &= \beta_0 + \beta_1 t + X_t - \beta_0 - \beta_1(t-1) - X_{t-1} \\ &= \beta_1 + X_t - X_{t-1} = \beta_1 + \nabla X_t. \end{aligned}$$

Thus,

$$E(W_t) = \beta_1 + E(X_t) - E(X_{t-1}) = \beta_1$$

i.e., the mean function is now constant, and

$$\begin{aligned} \mathrm{Cov}(Y_t, T_{t-k}) &= \mathrm{Cov}(\beta_1 + \nabla X_t, \beta_1 + \nabla X_{t-k}) \\ &= \mathrm{Cov}(\nabla X_t, \nabla X_{t-k}) \end{aligned}$$

We could do out the above or notice that this is the acvf of a differenced stationary series which we know, from part (a), depends only on k. Therefore, W_t is stationary - the linear trend has been eliminated.