Time Series – Tutorial 4 – Solutions

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Question 1

$Y_t = 5 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}$

a) This is an MA(2) process.

b)
$$\begin{aligned} E(Y_t) &= 5 + E(e_t) - \tfrac{1}{2}E(e_{t-1}) + \tfrac{1}{4}E(e_{t-2}) \\ &= 5 + 0 - \tfrac{1}{2}(0) + \tfrac{1}{4}(0) = 5. \end{aligned}$$

$$\begin{aligned} \gamma_k &= \text{Cov}(Y_t, Y_{t-k}) \\ &= \text{Cov}(5 + e_t - \tfrac{1}{2}e_{t-1} + \tfrac{1}{4}e_{t-2}, \\ &\qquad 5 + e_{t-k} - \tfrac{1}{2}e_{t-k-1} + \tfrac{1}{4}e_{t-k-2}) \\ &= \text{Cov}(e_t - \tfrac{1}{2}e_{t-1} + \tfrac{1}{4}e_{t-2}, \\ &\qquad e_{t-k} - \tfrac{1}{2}e_{t-k-1} + \tfrac{1}{4}e_{t-k-2}) \\ &= \begin{cases} [(1)^2 + (\frac{1}{2})^2 + (\frac{1}{4})^2]\sigma_e^2 = \frac{21}{16}\sigma_e^2 & \text{for } k = 0 \\ [-\frac{1}{2} + \frac{1}{4}(-\frac{1}{2})]\sigma_e^2 = -\frac{5}{8}\sigma_e^2 & \text{for } k = 1 \\ [\frac{1}{4}]\sigma_e^2 = \frac{1}{4}\sigma_e^2 & \text{for } k = 2 \\ 0 & \text{for } k \geq 3 \end{cases} \end{aligned}$$

c) This process is a general linear process with $\psi_1 = -\frac{1}{2}$ and $\psi_2 = \frac{1}{4}$. Therefore:

$$\gamma_k = \sigma_e^2 \sum_{j=0}^{\infty} \psi_j\, \psi_{j+k}.$$

$$= \begin{cases} [\psi_0^2 + \psi_1^2 + \psi_2^2 + \psi_3^2 + \ldots]\sigma_e^2 & k = 0 \\ [\psi_0\psi_1 + \psi_1\psi_2 + \psi_2\psi_3 + \ldots]\sigma_e^2 & k = 1 \\ [\psi_0\psi_2 + \psi_1\psi_3 + \ldots]\sigma_e^2 = \frac{1}{4}\sigma_e^2 & k = 2 \\ 0 & k \geq 3 \end{cases}$$

$$= \begin{cases} [(1)^2 + (\frac{1}{2})^2 + (\frac{1}{4})^2 + (0)^2 + \ldots]\sigma_e^2 = \frac{21}{16}\sigma_e^2 & k = 0 \\ [(1)(-\frac{1}{2}) + (-\frac{1}{2})(\frac{1}{4}) + (\frac{1}{4})(0) + \ldots]\sigma_e^2 = -\frac{5}{8}\sigma_e^2 & k = 1 \\ [(1)(\frac{1}{4}) + (\frac{1}{4})(0) + \ldots]\sigma_e^2 = \frac{1}{4}\sigma_e^2 & k = 2 \\ 0 & k \geq 3 \end{cases}$$

d)
$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \frac{21/16\ \sigma_e^2}{21/16\ \sigma_e^2} = 1 & \text{for } k = 0 \\ \frac{-5/8\ \sigma_e^2}{21/16\ \sigma_e^2} = -\frac{10}{21} = -0.476 & \text{for } k = 1 \\ \frac{1/4\ \sigma_e^2}{21/16\ \sigma_e^2} = \frac{4}{21} = 0.190 & \text{for } k = 2 \\ 0 & \text{for } k \geq 3 \end{cases}$$

Question 2

The formulae are:

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$
$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

a) $\theta_1 = 0.5,\ \theta_2 = 0.4.$

$$\rho_1 = \frac{-0.5 + 0.5(0.4)}{1 + (0.5)^2 + (0.4)^2} = -0.2128$$
$$\rho_2 = \frac{-0.4}{1 + (0.5)^2 + (0.4)^2} = -0.2837$$

b) $\theta_1 = 1.2,\ \theta_2 = 0.4.$

$$\rho_1 = \frac{-1.2 + 1.2(0.4)}{1 + (1.2)^2 + (0.4)^2} = -0.2769$$
$$\rho_2 = \frac{-0.4}{1 + (1.2)^2 + (0.4)^2} = -0.1538$$

c) $\theta_1 = -1,\ \theta_2 = 0.6.$

$$\rho_1 = \frac{-(-1) + (-1)(0.6)}{1 + (-1)^2 + (0.6)^2} = 0.1695$$
$$\rho_2 = \frac{-0.6}{1 + (-1)^2 + (0.6)^2} = -0.2542$$

d)
```
> ARMAacf(ma=c(-0.5,-0.4),lag.max=2)
        0          1          2
1.0000000 -0.2127660 -0.2836879

> ARMAacf(ma=c(-1.2,-0.4),lag.max=2)
        0          1          2
1.0000000 -0.2769231 -0.1538462

> ARMAacf(ma=c(1,-0.6),lag.max=2)
        0          1          2
1.0000000  0.1694915 -0.2542373
```

Question 3

The two MA(1) processes are:

$$Y_t = e_t - \theta e_{t-1}$$
$$Y_t = e_t - \frac{1}{\theta}e_{t-1}$$

and assume that $|\theta| < 1$.

a) For the first process we have

$$\rho_1 = \frac{-\theta}{1 + \theta^2}$$

and for the second process we have

$$\begin{aligned} \rho_1 = \frac{-\frac{1}{\theta}}{1 + (\frac{1}{\theta})^2} &= \frac{-\frac{1}{\theta}}{1 + \frac{1}{\theta^2}} \\ &= \frac{-\frac{1}{\theta}}{1 + \frac{1}{\theta^2}} \times \frac{\theta^2}{\theta^2} \\ &= \frac{-\theta}{\theta^2 + 1} \end{aligned}$$

which is the same as the first case.

b) In the first case the MA equation is

$$1 - \theta B = 0$$

which yields the root $B = \frac{1}{\theta}$. Since it is stated in the question that $|\theta| < 1$, we have that $|\frac{1}{\theta}| > 1$ which is required for invertibility.

In the second case the MA equation is

$$1 - \frac{1}{\theta} B = 0$$

which yields the root $B = \theta$. Since $|\theta| < 1$, this process is *not* invertible.

Therefore, even though both processes have the same correlation structure only the first is invertible. Thus we prefer the first since invertibility is needed for prediction.

Question 4

The two MA(2) processes are: one with $\theta_1 = \theta_2 = \frac{1}{6}$ and the other with $\theta_1 = -1$ and $\theta_2 = 6$.

a) For the first process we have

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{-\frac{1}{6} + \frac{1}{6}(\frac{1}{6})}{1 + (\frac{1}{6})^2 + (\frac{1}{6})^2} = -0.1315789$$
$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{-\frac{1}{6}}{1 + (\frac{1}{6})^2 + (\frac{1}{6})^2} = -0.1578947$$

For the second process we have

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{-(-1) + (-1)(6)}{1 + (-1)^2 + (6)^2} = -0.1315789$$
$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{-6}{1 + (-1)^2 + (6)^2} = -0.1578947$$

b) For the first process the MA equation is

$$\begin{aligned} 1 - \theta_1 B - \theta_2 B^2 &= 0 \\ 1 - \tfrac{1}{6}B - \tfrac{1}{6}B^2 &= 0 \end{aligned}$$

$$\begin{aligned} B^2 + B - 6 &= 0 \qquad (\times -6) \\ B^2 + 3B - 2B - 6 &= 0 \\ B(B+3) - 2(B+3) &= 0 \\ (B-2)(B+3) &= 0 \end{aligned}$$

$\Rightarrow$ The roots are $B = 2$ and $B = -3$. These are greater than 1 in magnitude. Thus, this process is invertible. For the first process the MA equation is

$$\begin{aligned} 1 - \theta_1 B - \theta_2 B^2 &= 0 \\ 1 - (-1)B - 6B^2 &= 0 \end{aligned}$$

$$\begin{aligned} -6B^2 + B + 1 &= 0 \\ -6B^2 + 3B - 2B + 1 &= 0 \\ -6B^2 + 3B - 2B + 1 &= 0 \\ 3B(-2B+1) + 1(-2B+1) &= 0 \\ (3B+1)(-2B+1) &= 0 \end{aligned}$$

$\Rightarrow$ The roots are $B = -\frac{1}{3}$ and $B = \frac{1}{2}$. These are less than 1 in magnitude. Thus, this process is *not* invertible.

Question 5

$\{Y_t\}$ is an AR(1) process with with $\gamma_k = \phi^k \frac{\sigma_e^2}{1-\phi^2}$ for $k \geq 0$. $W_t = \nabla Y_t = Y_t - Y_{t-1}$.

a)

$$\begin{aligned} &\mathrm{Cov}(W_t, W_{t-k}) \\ &= \mathrm{Cov}(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\ &= \mathrm{Cov}(Y_t, Y_{t-k}) - \mathrm{Cov}(Y_t, Y_{t-k-1}) \\ &\qquad - \mathrm{Cov}(Y_{t-1}, Y_{t-k}) + \mathrm{Cov}(Y_{t-1}, Y_{t-k-1}) \\ &= \gamma_{t-(t-k)} - \gamma_{t-(t-k-1)} - \gamma_{t-1-(t-k)} + \gamma_{t-1-(t-k-1)} \\ &= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\ &= (\phi^k - \phi^{k+1} - \phi^{k-1} + \phi^k)\frac{\sigma_e^2}{1-\phi^2} \\ &= (2\phi^k - \phi^{k+1} - \phi^{k-1})\frac{\sigma_e^2}{1-\phi^2} \\ &= (2\phi - \phi^2 - 1)\phi^{k-1}\frac{\sigma_e^2}{1-\phi^2} \\ &= -(\phi^2 - 2\phi + 1)\phi^{k-1}\frac{\sigma_e^2}{(1-\phi)(1+\phi)} \\ &= -(1-\phi)^2\phi^{k-1}\frac{\sigma_e^2}{(1-\phi)(1+\phi)} \\ &= -\left(\frac{1-\phi}{1+\phi}\right)\phi^{k-1}\sigma_e^2. \end{aligned}$$

b) In part (a) we found that

$$\mathrm{Cov}(W_t, W_{t-k}) = \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k$$

and developed this for $k \geq 1$. Now consider the case when $k = 0$:

$$\begin{aligned}
\mathrm{Var}(W_t) &= \mathrm{Cov}(W_t, W_t) \\
&= \gamma_0 - \gamma_1 - \gamma_{-1} + \gamma_0 \\
&= \gamma_0 - \gamma_1 - \gamma_1 + \gamma_0 \qquad\qquad (\gamma_{-1} = \gamma_1) \\
&= 2(\gamma_0 - \gamma_1) \\
&= 2(\phi^0 - \phi)\,\frac{\sigma_e^2}{1-\phi^2} \\
&= 2(1-\phi)\,\frac{\sigma_e^2}{(1-\phi)(1+\phi)} \\
&= \frac{2\sigma_e^2}{1+\phi}
\end{aligned}$$

Question 6

Let $\{Y_t\}$ be an AR(2) process of the special form

$$Y_t = \phi Y_{t-2} + e_t.$$

a) Note that the process can be written as

$$\begin{aligned}
Y_t - \phi Y_{t-2} &= e_t \\
(1 - \phi B^2)\, Y_t &= e_t \\
\phi(B)\, Y_t &= e_t.
\end{aligned}$$

If it can be expressed as a general linear process then it can be written as

$$Y_t = \psi(B) e_t$$

where $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots$.

Inserting $Y_t = \psi(B)e_t$ into $\phi(B)\, Y_t = e_t$ we get

$$\phi(B)\psi(B)\, e_t = e_t$$

$$\Rightarrow \phi(B)\psi(B) = 1$$

Thus,

$$\begin{aligned}
&(1-\phi B^2)(1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) \\
&= 1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots \\
&\qquad - \phi B^2 - \phi\psi_1 B^3 - \phi\psi_2 B^4 - \phi\psi_3 B^5 - \cdots \\
&= 1 + \psi_1 B + (\psi_2 - \phi) B^2 + (\psi_3 - \phi\psi_1) B^3 \\
&\qquad + (\psi_4 - \phi\psi_2) B^4 + (\psi_5 - \phi\psi_3) B^5 \cdots
\end{aligned}$$

Since $\phi(B)\psi(B) = 1$ the coefficients of the B^i terms $(i \geq 1)$ must be zero.

$$\begin{array}{lll}
B: & & \psi_1 = 0 \\
B^2: & \psi_2 - \phi = 0 \Rightarrow & \psi_2 = \phi \\
B^3: & \psi_3 - \phi\psi_1 = 0 \Rightarrow & \psi_3 = \phi\psi_1 = \phi(0) = 0 \\
B^4: & \psi_4 - \phi\psi_2 = 0 \Rightarrow & \psi_4 = \phi\psi_2 = \phi\phi = \phi^2 \\
B^5: & \psi_5 - \phi\psi_3 = 0 \Rightarrow & \psi_5 = \phi\psi_3 = \phi(0) = 0 \\
B^6: & \psi_6 - \phi\psi_4 = 0 \Rightarrow & \psi_6 = \phi\psi_4 = \phi\phi^2 = \phi^3 \\
 & \vdots &
\end{array}$$

Thus we can see that

$$\psi_i = \begin{cases} \phi^{i/2} & i = 0, 2, 4, 6, \ldots = i \text{ even} \\ 0 & i = 1, 3, 5, 7, \ldots = i \text{ odd} \end{cases}$$

Thus, $Y_t = \phi Y_{t-2} + e_t$ can be written as a general linear process as follows:

$$Y_t = e_t + \phi e_{t-2} + \phi^2 e_{t-4} + \phi^3 e_{t-6} + \cdots$$

and, for this sum to be meaningful way, we require $|\phi| < 1$.

Note: we could also have written this as a general linear process by successively back-substituting (as was done in Section 4.4 of Lecture 4 for an AR(1) process).

b) The AR equation is

$$\begin{aligned}
1 - \phi B^2 &= 0 \\
\Rightarrow B^2 &= \frac{1}{\phi} \\
B &= \pm\sqrt{\frac{1}{\phi}} = \pm\frac{1}{\sqrt{\phi}}
\end{aligned}$$

We require the roots to be greater than 1 in absolute value

$$\begin{aligned}
\left|\pm\frac{1}{\sqrt{\phi}}\right| &> 1 \\
\left|\frac{1}{\sqrt{\phi}}\right| &> 1 \\
\left|\sqrt{\phi}\right| &< 1 \\
|\phi| &< 1
\end{aligned}$$

which is what we found by looking at the general linear process.

c) We can apply the formula $\gamma_k = \sigma_e^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$.

$$\begin{aligned}
\gamma_k &= \sigma_e^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \\
&= \sigma_e^2 \sum_{i \text{ even}}^{\infty} \psi_i \psi_{i+k}
\end{aligned}$$

because the terms where i is odd are zero since $\psi_i = 0$ when i is odd.

Now notice that if k is odd then $\psi_{i+k} = 0$ since $i+k =$ even + odd = odd. Thus, $\gamma_k = 0$ if k is odd.

Let's now consider the case where k is even:

$$\begin{aligned}
\Rightarrow \gamma_k &= \sigma_e^2 \sum_{i \text{ even}}^{\infty} \psi_i \psi_{i+k} \\
&= \sigma_e^2 \sum_{i \text{ even}}^{\infty} \phi^{i/2} \phi^{(i+k)/2} \\
&= \sigma_e^2 \phi^{k/2} \sum_{i \text{ even}}^{\infty} \phi^{i/2+i/2} \\
&= \sigma_e^2 \phi^{k/2} \sum_{i \text{ even}}^{\infty} \phi^{i} \\
&= \sigma_e^2 \phi^{k/2} \underbrace{(1 + \phi^2 + \phi^4 + \cdots)}_{\text{geometric sum}} \\
&= \sigma_e^2 \phi^{k/2} \frac{1}{1-\phi^2} \\
&= \phi^{k/2} \frac{\sigma_e^2}{1-\phi^2}.
\end{aligned}$$

Therefore, the autocovariance function is

$$\gamma_k = \begin{cases} \phi^{k/2} \frac{\sigma_e^2}{1-\phi^2} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

d) We will use the usual tricks for AR processes: 1. assume the process is stationary and 2. leave Y_{t-k} in the covariance calculation.

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(\phi Y_{t-2} + e_t, Y_{t-k}) \\
&= \phi \mathrm{Cov}(Y_{t-2}, Y_{t-k}) + \mathrm{Cov}(e_t, Y_{t-k})
\end{aligned}$$

$$\Rightarrow \gamma_k = \phi \gamma_{k-2} + \mathrm{Cov}(e_t, Y_{t-k})$$

By definition of e_t being independent of the past $\mathrm{Cov}(e_t, Y_{t-k}) = 0$ for $k \geq 1$. Then, for $k = 0$, we have

$$\begin{aligned}
\mathrm{Cov}(e_t, Y_{t-k}) &= \mathrm{Cov}(e_t, Y_t) \\
&= \mathrm{Cov}(e_t, \phi Y_{t-2} + e_t) \\
&= \phi \mathrm{Cov}(e_t, Y_{t-2}) + \mathrm{Cov}(e_t, e_t) \\
&= 0 + \sigma_e^2
\end{aligned}$$

Therefore, $\gamma_k = \phi \gamma_{k-2} + \mathrm{Cov}(e_t, Y_{t-k})$ becomes

$$\gamma_k = \begin{cases} \phi \gamma_2 + \sigma_e^2 & k = 0 \\ \phi \gamma_{k-2} & k \geq 1 \end{cases}$$

From this we get

$$\begin{aligned}
\gamma_0 &= \phi \gamma_2 + \sigma_e^2 \\
&= \phi(\phi \gamma_0) + \sigma_e^2 \\
(1-\phi^2)\gamma_0 &= \sigma_e^2 \\
\gamma_0 &= \frac{\sigma_e^2}{1-\phi^2}
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= \phi \gamma_1 \\
(1-\phi)\gamma_1 &= 0 \\
\Rightarrow \gamma_1 &= 0
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= \phi \gamma_0 = \phi \frac{\sigma_e^2}{1-\phi^2} \\
\gamma_3 &= \phi \gamma_1 = 0 \\
\gamma_4 &= \phi \gamma_2 = \phi^2 \frac{\sigma_e^2}{1-\phi^2} \\
\gamma_5 &= \phi \gamma_3 = 0 \\
&\vdots
\end{aligned}$$

The pattern is clearly

$$\gamma_k = \begin{cases} \phi^{k/2} \frac{\sigma_e^2}{1-\phi^2} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

which is what we found in part (c).

Question 7

$Y_t = 0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2}$

a) Here we use the same tricks as for the AR process.

$$\begin{aligned}
&\mathrm{Cov}(Y_t, Y_{t-k}) \\
&= \mathrm{Cov}(0.8Y_{t-1} + e_t + 0.7e_{t-1} + 0.6e_{t-2},\ Y_{t-k}) \\
&= 0.8\mathrm{Cov}(Y_{t-1}, Y_{t-k}) + \mathrm{Cov}(e_t, Y_{t-k}) + \\
&\qquad 0.7\mathrm{Cov}(e_{t-1}, Y_{t-k}) + 0.6\mathrm{Cov}(e_{t-2}, Y_{t-k})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \gamma_k &= 0.8\gamma_{k-1} + \mathrm{Cov}(e_t, Y_{t-k}) + \\
&\qquad 0.7\mathrm{Cov}(e_{t-1}, Y_{t-k}) + 0.6\mathrm{Cov}(e_{t-2}, Y_{t-k}) \\
&= 0.8\gamma_{k-1} \qquad \text{for } k > 2
\end{aligned}$$

and dividing across by γ_0 gives

$$\rho_k = 0.8\rho_{k-1} \qquad \text{for } k > 2.$$

b) For $k = 2$, $\mathrm{Cov}(e_t, Y_{t-k}) = \mathrm{Cov}(e_t, Y_{t-2}) = 0$ and $\mathrm{Cov}(e_{t-1}, Y_{t-k}) = \mathrm{Cov}(e_{t-1}, Y_{t-2}) = 0$ whereas

$$\begin{aligned}
&\mathrm{Cov}(e_{t-2}, Y_{t-k}) \\
&= \mathrm{Cov}(e_{t-2}, Y_{t-2}) \\
&= \mathrm{Cov}(e_{t-2}, 0.8Y_{t-3} + e_{t-2} + 0.7e_{t-3} + 0.6e_{t-4}) \\
&= \mathrm{Cov}(e_{t-2}, e_{t-2}) = \sigma_e^2.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \gamma_2 &= 0.8\gamma_1 + 0.6\sigma_e^2 \\
\tfrac{\gamma_2}{\gamma_0} &= \tfrac{0.8\gamma_1 + 0.6\sigma_e^2}{\gamma_0} \\
\rho_2 &= 0.8\rho_1 + 0.6\tfrac{\sigma_e^2}{\gamma_0}
\end{aligned}$$

Question 8

a) The model is:

$$\begin{aligned}Y_t &= e_t - 0.5e_{t-1} + 0.25e_{t-2} - 0.125e_{t-3}\\ &\quad + 0.0625e_{t-4} - 0.03125e_{t-5} + 0.015625e_{t-6}\end{aligned}$$

which is a general linear process with $\psi_1 = -0.5$, $\psi_2 = 0.25$, $\psi_3 = -0.125$, $\ldots$, $\psi_6 = 0.015625$.

Recall in Lecture 4 we found that the AR(1) process is a general linear process with $\psi^i = \phi^i$, i.e., the weights start at $\psi_i = \phi$ and decay by the factor ϕ thereafter.

Note that these weights for the above MA(6) model start at -0.5 and decay by the factor -0.5 until $\psi_6 = 0.015625$ (and are zero thereafter). This is therefore similar to an AR(1) model with $\phi = -0.5$, i.e.,

$$\begin{aligned}Y_t &= \phi Y_{t-1} + e_t\\ &= -0.5Y_{t-1} + e_t.\end{aligned}$$

The models are not exactly the same as $\psi_7 = 0$ for the MA(6) process but for the AR(1) process $\psi_7 = \phi^7 = (-0.5)^7 = -0.0078125$. However, this is quite close to zero and the next ψ weight will be even closer to zero.

b) The model is:

$$\begin{aligned}Y_t &= e_t - 1e_{t-1} + 0.5e_{t-2} - 0.25e_{t-3}\\ &\quad + 0.125_{t-4} - 0.0625e_{t-5} + 0.03125e_{t-6}\\ &\quad - 0.015625e_{t-7}.\end{aligned}$$

which is a general linear process with $\psi_1 = -1$, $\psi_2 = 0.5$, $\psi_3 = 0.125$, $\ldots$, $\psi_7 = -0.015625$.

This is similar to a model where the weights start at $\psi_1 = -1$ and decay by the factor -0.5 thereafter. This behaviour cannot be captured by the AR(1) process since the first weight and the decay factor must be equal in this case.

However this is like an ARMA(1,1) process where, from Lecture 4, we have found that $\psi_i = \phi^{i-1}(\phi - \theta)$ for $i \geq 1$. Thus, for the ARMA(1,1) process $\psi_1 = \phi - \theta$ and the weights then decay by the factor ϕ, e.g., $\psi_2 = \phi(\phi - \theta)$, $\psi_3 = \phi^2(\phi - \theta)$ and so on.

Thus we have

$$\text{Decay factor:} \qquad \phi = -0.5$$

$$\begin{aligned}\text{First term:} \qquad \phi - \theta &= -1\\ -0.5 - \theta &= -1\\ \Rightarrow \theta &= 0.5.\end{aligned}$$

So the above MA(7) model will be similar to the ARMA(1,1) model:

$$\begin{aligned}Y_t &= \phi Y_{t-1} + e_t - \theta e_{t-1}\\ &= -0.5Y_{t-1} + e_t - 0.5e_{t-1}.\end{aligned}$$

Question 9

a) $Y_t = 0.4Y_{t-1} + 0.45Y_{t-2} + e_t + e_{t-1} + 0.25e_{t-2}$

$$\begin{aligned}\Rightarrow Y_t - 0.4Y_{t-1} - 0.45Y_{t-2} &= +e_t + e_{t-1} + 0.25e_{t-2}\\ (1 - 0.4B - 0.45B^2)\, Y_t &= (1 + B + 0.25B^2)\, e_t\\ \phi(B)\, Y_t &= \theta(B)\, e_t\end{aligned}$$

b) The AR equation is:

$$-0.45B^2 - 0.4B + 1 = 0$$

$$\begin{aligned}\Rightarrow B &= \frac{-(-0.4) \pm \sqrt{(-0.4)^2 - 4(-0.45)(1)}}{2(-0.45)}\\ &= \frac{0.4 \pm 1.4}{-0.9}\end{aligned}$$

Thus the roots are $B = -2$ and $B = \frac{10}{9}$ and the AR polynomial can be factorised as:

$$1 - 0.4B - 0.45B^2 = (1 + \tfrac{1}{2}B)(1 - \tfrac{9}{10}B)$$

The MA equation is:

$$0.25B^2 + B + 1 = 0$$

$$\begin{aligned}\Rightarrow B &= \frac{-1 \pm \sqrt{(1)^2 - 4(0.25)(1)}}{2(0.25)}\\ &= \frac{-1 \pm 0}{0.5}\end{aligned}$$

Thus we have repeated roots: $B = -2$ and the MA polynomial can be factorised as:

$$1 - 0.4B - 0.45B^2 = (1 + \tfrac{1}{2}B)^2$$

So the "ARMA(2,2)" model is

$$\begin{aligned}(1 + \tfrac{1}{2}B)(1 - \tfrac{9}{10}B)\, Y_t &= (1 + \tfrac{1}{2}B)^2\, e_t\\ (1 - \tfrac{9}{10}B)\, Y_t &= (1 + \tfrac{1}{2}B)\, e_t\end{aligned}$$

which is actually an ARMA(1,1) model. Note that we could re-express the model as

$$\begin{aligned}Y_t &= \tfrac{9}{10}Y_{t-1} + e_t + \tfrac{1}{2}e_{t-1}\\ &= 0.9Y_{t-1} + e_t + 0.5e_{t-1}\end{aligned}$$

c) The AR root is $\frac{10}{9}$ which is greater than one in absolute value $\Rightarrow$ the model is stationary.

The MA root is -2 which is greater than one in absolute value $\Rightarrow$ the model is invertible.