Seasonal ARIMA Models

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1 Seasonality

Seasonality arises when the series repeats itself at regular intervals. Often this occurs for monthly data where the period is 12, i.e., the series repeats itself every 12 time units.

In Lecture 3 (Section 3.2) we modelled seasonal trend directly using seasonal effects or harmonic terms. However, it has generally been our approach to eliminate trend rather than developing a trend model (for more information see Example 3.1 and Section 3.4 in Lecture 3).

We have also shown in Tutorial 3 (Question 6) that seasonal differencing can eliminate seasonal trend. Recall that a seasonal difference is given by

$$\nabla_{12} = Y_t - Y_{t-12} = (1 - B^{12})Y_t.$$

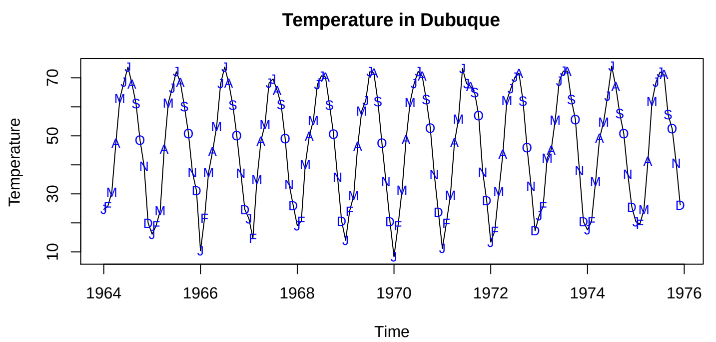
Note that we are assuming here that this is monthly data (period = 12). For quarterly data we would have $(1 - B^4)Y_t$ or, more generally, $(1 - B^s)Y_t$ where s is the period.

Aside from seasonal trend, we may also observe **seasonal autocorrelation** which must also be handled. This feature is the focus of this lecture.

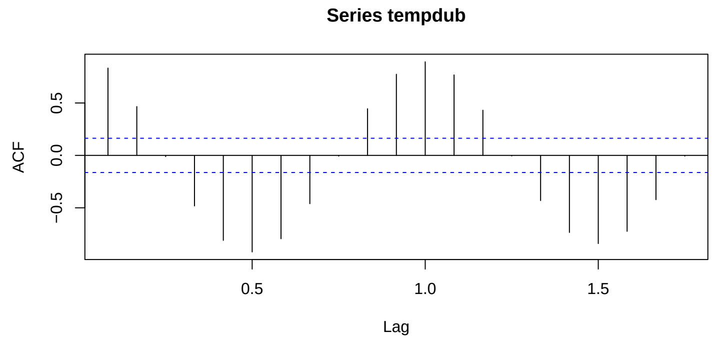
Example 1.1. *Temperature in Dubuque*
Consider the `tempdub` series which contains monthly temperatures in Dubuque.

```
dev.new(width=8, height=4)
plot(tempdub)
points(x=as.vector(time(tempdub)), y=as.vector(tempdub),
    pch=as.vector(season(tempdub)), cex=0.8, col=4)

acf(tempdub)
```

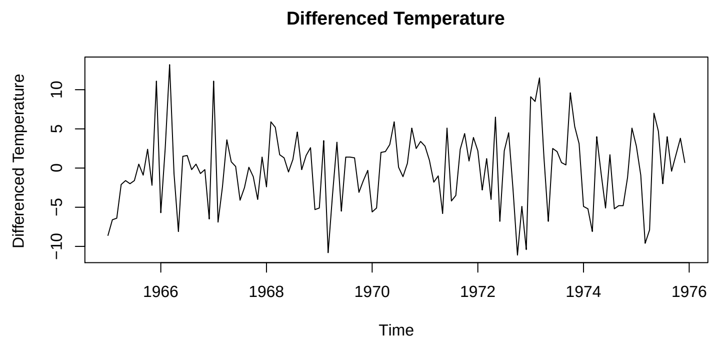


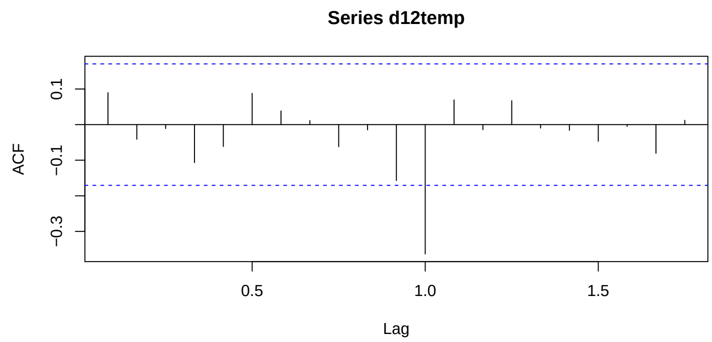
The regular seasonal trend is very clear from the time plot and is also displayed in the sample ACF which has significant autocorrelation at lag 12 (and lags 6 and 18). The fact that the ACF fails to decay is evidence of non-stationarity, i.e., the mean is not constant - it varies with the season.



We can eliminate the seasonal trend by seasonal differencing.

```
d12temp <- diff(tempdub, lag=12)
plot(d12temp)
acf(d12temp)
```





This looks stationary. Note however there is a significant lag-12 autocorrelation which must be captured.

An MA(12) model allows the first 12 autocorrelations to be non-zero. Thus, we might think of using an MA(12) model in order to capture this lag-12 autocorrelation.

The code required is as follows:

```
tempmod1 <- arima(d12temp, order=c(0,0,12) )
tempmod1
tsdiag(tempmod1)
```

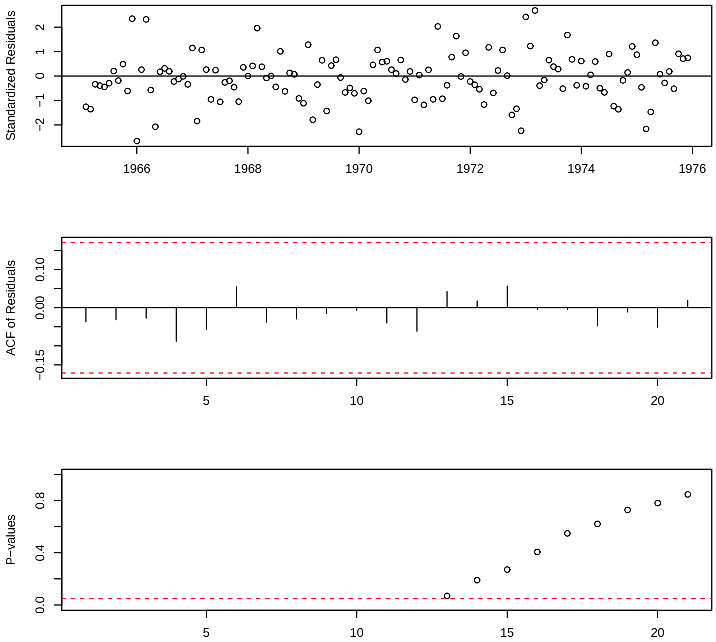
Although the diagnostics below look reasonable (apart from one Ljung-Box p-value being close to 0.05), there is a clear sense that this model is more complicated than necessary since only the $\hat{\theta}_{12}$ value is significantly different to zero.

```
Coefficients:
         ma1      ma2      ma3      ma4      ma5
      0.1151  -0.1221  -0.0079  -0.0897  -0.0084
s.e.  0.0904   0.0837   0.0910   0.0941   0.0869

         ma6     ma7      ma8      ma9     ma10
      0.2010  0.0513  -0.1111  -0.0936  -0.0781
s.e.  0.1156  0.0883   0.0840   0.1057   0.0935

        ma11     ma12  intercept
     -0.0566  -0.7999    -0.0002
s.e.  0.0819   0.1102     0.0822

sigma^2 estimated as 12.08:
log likelihood = -362.41,  aic = 750.81
```



2 Seasonal ARIMA

Note: capital letters will be used to distinguish these seasonal models from the standard models.

2.1 SMA$(1)_{12}$

The previous example suggests that we should consider models such as

$$Y_t = e_t - \Theta\, e_{t-12}$$

which is a special case of an MA(12) model where only the 12th coefficient is non-zero. We may refer to this as a seasonal MA(1) model, i.e., SMA$(1)_{12}$ where the **subscript 12 indicates that the period is 12**, i.e., monthly data. Obviously we can have other periods.

It is easy to show that

$$\gamma_k = \mathrm{Cov}(Y_t, Y_{t-k}) = \mathrm{Cov}(e_t - \Theta\, e_{t-12}, e_{t-k} - \Theta\, e_{t-k-12})$$
$$= \begin{cases} (1+\Theta^2)\,\sigma_e^2 & k=0 \\ -\Theta\,\sigma_e^2 & k=12 \\ 0 & \text{otherwise} \end{cases}$$

and, hence,

$$\rho_k = \begin{cases} 1 & k=0 \\ \dfrac{-\Theta}{1+\Theta^2} & k=12 \\ 0 & \text{otherwise} \end{cases}$$

which behaves exactly like a standard MA(1) except that the non-zero autocorrelation appears at lag-12.

This seems to be just the type of model that we require for the `tempdub` series.

2.2 SMA$(Q)_{12}$

More generally, the seasonal MA(Q) process is given by

$$\begin{aligned} Y_t &= e_t - \Theta_1\, e_{t-12} - \Theta_2\, e_{t-24} - \cdots - \Theta_Q\, e_{t-12Q} \\ &= (1 - \Theta_1 B^{12} - \Theta_2 B^{24} - \cdots - -\Theta_Q B^{12Q})\, e_t \\ &= \Theta(B)\, e_t \end{aligned}$$

where $\Theta(B) = 1 - \Theta_1 B^{12} - \Theta_2 B^{24} - \cdots - \Theta_Q B^{12Q}$ is the **seasonal MA polynomial**. As before, the roots of this polynomial must have magnitude greater than one for invertibility.

2.3 SAR$(1)_{12}$

We may also consider the seasonal AR(1) model:

$$Y_t = \Phi\, Y_{t-12} + e_t$$

which is a special case of an AR(12) model where only the 12th coefficient is non-zero.

By assuming stationarity, we find that for $k > 0$:

$$\begin{aligned} \mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(\Phi\, Y_{t-12} + e_t, Y_{t-k}) \\ \gamma_k &= \Phi \gamma_{k-12} \end{aligned}$$

$$\Rightarrow \rho_k = \Phi \rho_{k-12}.$$

Considering $k = 12, 24, 36, \ldots$ gives

$$\begin{aligned}
\rho_{12} &= \Phi\rho_0 = \Phi \\
\rho_{24} &= \Phi\rho_{12} = \Phi^2 \\
\rho_{36} &= \Phi\rho_{24} = \Phi^3 \\
&\vdots \quad \vdots \\
\rho_{12i} &= \Phi\rho_{12i-12} = \Phi^i
\end{aligned}$$

Considering $k = 1$ gives

$$\rho_1 = \Phi\rho_{-11} = \Phi\rho_{11}$$

and then, $k = 11$ gives

$$\begin{aligned}
\rho_{11} &= \Phi\rho_1 \\
&= \Phi^2\rho_{11} \qquad \text{(using the result for } k = 1\text{)} \\
(1 - \Phi^2)\rho_{11} &= 0 \\
\Rightarrow \rho_{11} &= 0 \\
\Rightarrow \rho_1 &= \Phi\rho_{11} = 0.
\end{aligned}$$

This same argument can be applied for other values of k to show that the autocorrelation function is always zero if k is not some multiple of 12.

Thus, the autocorrelation function is given by

$$\rho_k = \begin{cases} 1 & k = 0 \\ \Phi^i & k = 12i \\ 0 & \text{otherwise} \end{cases}$$

As with the standard AR(1) this decays over time (oscillating if $\Phi < 0$) since, for stationarity, we require $|\Phi| < 1$. The difference is that the autocorrelation is only non-zero for k values which are multiples of 12.

2.4 SAR$(P)_{12}$

More generally, the seasonal AR(P) process is given by

$$Y_t = \Phi_1 Y_{t-12} + \Phi_2 Y_{t-24} - \cdots - \Phi_P Y_{t-12P} + e_t$$

$$\begin{aligned}
\Rightarrow (1 - \Phi_1 B^{12} + \Phi_2 B^{24} - \cdots - \Phi_P B^{12P})\, Y_t &= e_t \\
\Phi(B)\, Y_t &= e_t
\end{aligned}$$

where $\Phi(B) = 1 - \Phi_1 B^{12} + \Phi_2 B^{24} - \cdots - \Phi_P B^{12P}$ is the **seasonal AR polynomial**. As before, the roots of this polynomial must have magnitude greater than one for stationarity.

2.5 SARMA$(P, Q)_{12}$

The seasonal ARMA(P, Q) process is obviously given by

$$\Phi(B)\, Y_t = \Theta(B)\, e_t.$$

The behaviour of the autocorrelation function for this process will be the same as the standard ARMA(p, q) but the autocorrelation structure repeats every 12 units.

2.6 SARIMA$(P, D, Q)_{12}$

In Example 1.1 the data was seasonal differenced, i.e., $W_t = (1 - B^{12})\, Y_t$. We saw that the $SMA(1)_{12}$ model looked appropriate for W_t. Thus, we have that

$$\begin{aligned}
W_t &= e_t - \Theta e_{t-12} \\
\Rightarrow (1 - B^{12})\, Y_t &= (1 - \Theta B^{12})\, e_t.
\end{aligned}$$

More generally we can define the seasonal ARIMA(P, D, Q) model

$$\Phi(B)\, (1 - B^{12})^D\, Y_t = \Theta(B)\, e_t$$

which is analogous to the standard ARIMA(p, d, q) model.

Example 2.1. *Temperature in Dubuque*

We can fit the SARIMA$(0, 1, 1)_{12}$ model to the `tempdub` series as follows:

```
tempmod2 <- arima(tempdub, order=c(0,0,0),
                  seasonal=list(order=c(0,1,1), period=12) )
tempmod2
```

The seasonal orders are clearly indicated and also the period.

Note that the usual `order` term (for standard ARIMA(p, d, q) models) is set to `c(0,0,0)`, i.e., there is not standard ARIMA component. However, the fact that this is here suggests we can also have (p, d, q) orders. This is the subject of Section 3.

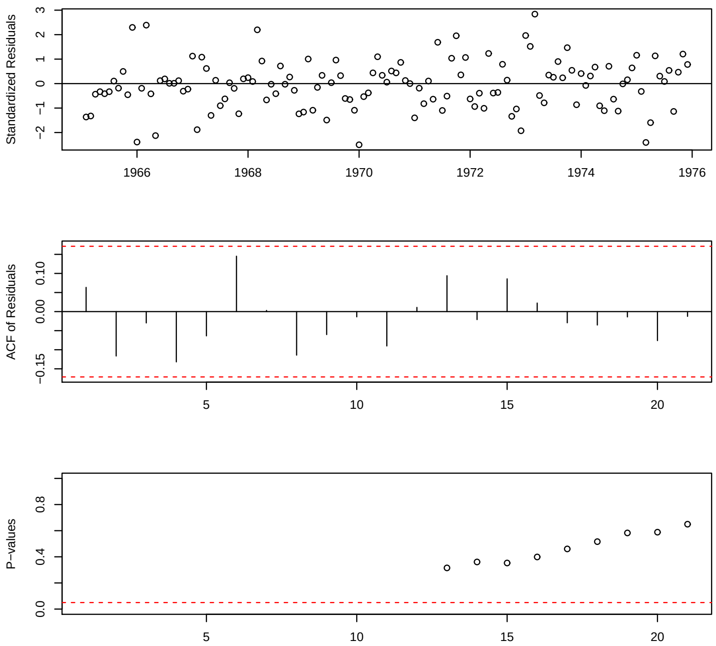
The output of the above code is:

```
Coefficients:
          sma1
       -1.0000
s.e.    0.0967

sigma^2 estimated as 11.69:
log likelihood = -364.48,  aic = 730.96
```

This is a lot simpler than the MA(12) model attempted in Example 1.1 which is reflected by a much lower AIC.

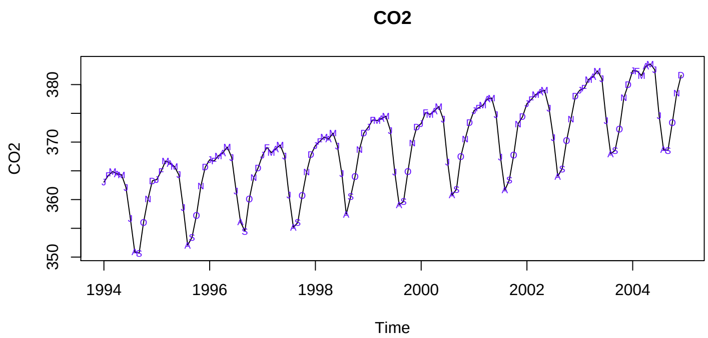
The diagnostics for this model also look better also (in terms of the Ljung-Box p-values).



Example 2.2. *CO2 Data*

We consider now the `co2` series which contains monthly CO2 levels.

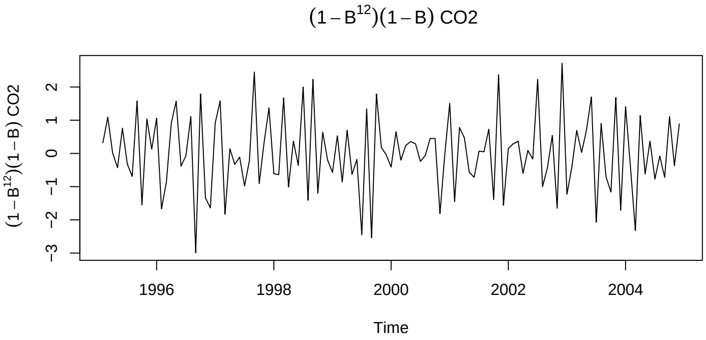
```
dev.new(width=8, height=4)
plot(co2, ylab="CO2", main="CO2")
points(x=as.vector(time(co2)), y=as.vector(co2),
    pch=as.vector(season(co2)), cex=0.6, col=4)
```



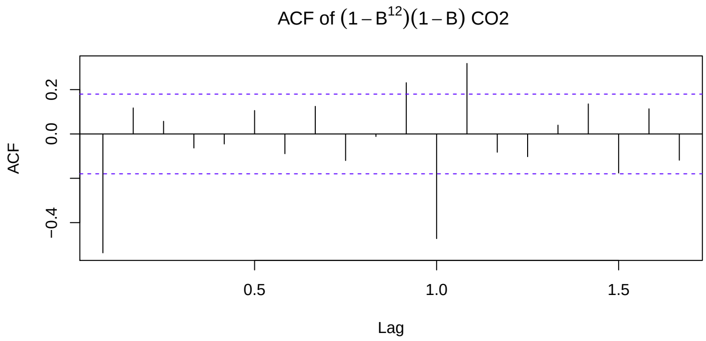
This series has quite clear evidence of both seasonal and overall trend (we could also confirm this using `decompose` - see Lecture 3). We can eliminate the overall trend through differencing and the seasonal trend through seasonal differencing, i.e., by considering

$$W_t = (1 - B^{12})(1 - B)Y_t$$

```
d12dco2 <- diff(diff(co2,lag=1),lag=12)
plot(d12dco2)
acf(d12dco2)
```



This series appears to be relatively stationary now. Note that the ACF shows significant autocorrelation at lag 1, lag 12 (either side of the lag 12, i.e., lags 11 and 13). This suggests that not only is there seasonal autocorrelation (correlation between values 12 months apart) but also short term autocorrelation (correlation between neighbouring values, i.e., one month apart).



This type of structure cannot be handled by the SARIMA models described above since these only incorporate the long-term seasonal autocorrelation but not the short-term autocorrelation.

For example, we could try to fit an SMA$(1)_{12}$ model to the differenced series, W_t, as follows:

```
co2mod1 <- arima(co2, order=c(0,1,0),
              seasonal=list(order=c(0,1,1), period=12) )
co2mod1
```

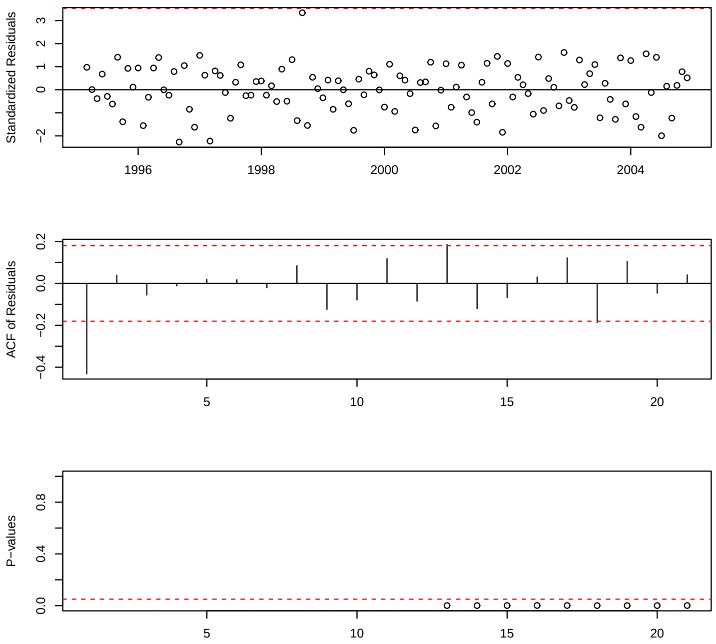
Note that differencing appears under `order` and seasonal differencing appears under `order` within `seasonal`.

The fitted model is:

```
Coefficients:
         sma1
      -0.9197
s.e.   0.2083

sigma^2 estimated as 0.6792:
log likelihood = -156,  aic = 314
```

When we examine the residuals, we observe that there is still a strong lag-1 autocorrelation in the residuals and the Ljung-Box p-values are all less than 0.05. However, we expected this model would not work.



What we really require is a combination of MA and SMA terms, i.e., combining standard and seasonal ARIMA models. This is the subject of the next section.

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3 ARIMA$(p,d,q)\times(P,D,Q)_{12}$

In the previous example we combined differencing with seasonal differencing and then fitted a SMA$(1)_{12}$ model to this differenced series, W_t, i.e.,

$$\begin{aligned} W_t &= (1-\Theta B^{12})\, e_t \\ \Rightarrow (1-B)(1-B^{12})\, Y_t &= (1-\Theta B^{12})\, e_t \end{aligned}$$

Thus, we had in fact already taken a step in the direction of combining ARIMA and SARIMA models in the sense that we mixed these different types of differencing.

It is then a small step to further combine ARMA and seasonal ARMA terms to arrive at the general ARIMA$(p,d,q)\times(P,D,Q)_{12}$ model:

$$\phi(B)\,\Phi(B)\ (1-B)^d\,(1-B^{12})^D\,Y_t = \theta(B)\,\Theta(B)\,e_t$$

where

$$\phi(B) = 1 - \phi_1\,B + \phi_2\,B^2 - \cdots - \phi_p B^p$$

$$\Phi(B) = 1 - \Phi_1\,B^{12} + \Phi_2\,B^{24} - \cdots - \Phi_P B^{12P}$$

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$$

$$\Theta(B) = 1 - \Theta_1 B^{12} - \Theta_2 B^{24} - \cdots - \Theta_Q B^{12Q}$$

Note that we have assumed the period is 12 but we can also define ARIMA$(p,d,q)\times(P,D,Q)_s$ models more generally where s is the period.

This is a very flexible class of models as it handles all of the following:

- Overall trend via differencing $(1-B)^d$
- Seasonal trend via seasonal differencing $(1-B^{12})^D$
- Short-term autocorrelation (i.e., neighbouring series values) via the ARMA components $\phi(B)$ and $\theta(B)$
- Long-term autocorrelation (i.e., series values one season apart) via the SARMA components $\Phi(B)$ and $\Theta(B)$

Important: We previously noted that differencing once or twice is usually enough to produce a stationary series in practice. This guideline also holds when mixing differencing, i.e., the total number of differences $d + D$ is usually one or two. For example, one difference ($d = 1$) and one seasonal difference ($D = 1$) is two differences in total.

Example 3.1. *CO2 Data*
For the `co2` series it looked like we needed an MA(1) term in addition to the SMA$(1)_{12}$ term where this model would be applied to $W_t = (1-B)(1-B^{12})\,Y_t$.

Thus, the model is given by

$$W_t = (1-\theta B)\,(1-\Theta B^{12})\,e_t$$

or, in terms of Y_t,

$$(1-B)\,(1-B^{12})\,Y_t = (1-\theta B)\,(1-\Theta B^{12})\,e_t.$$

This is an ARIMA$(0,1,1)\times(0,1,1)_{12}$ model.

Note that multiplying out the MA and SMA polynomials gives

$$W_t = (1-\theta B - \Theta B^{12} + \theta\Theta B^{13})\,e_t$$

which shows that this is a special case of an MA(13) model for W_t. However, as highlighted in Example 1.1, fitting these high order ARMA models to capture seasonal autocorrelation is much less satisfactory than fitting models with SARMA terms.
This model can be fitted as follows:

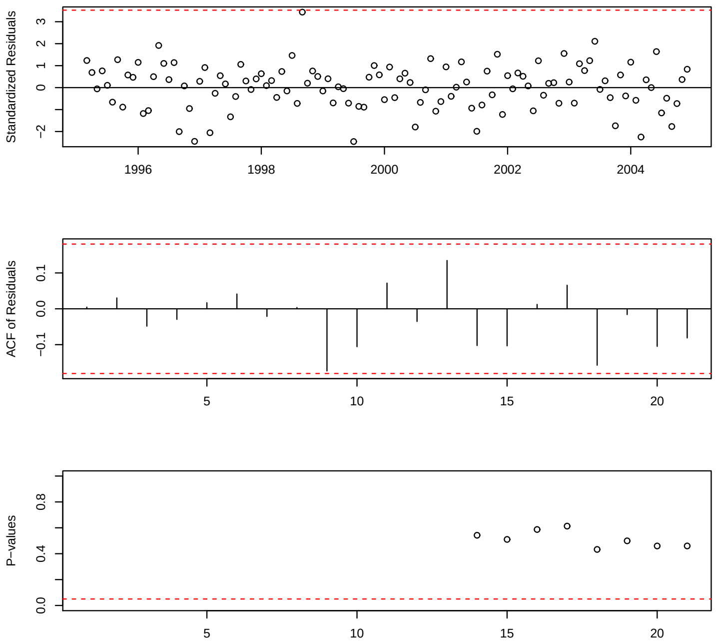
```
co2mod2 <- arima(co2, order=c(0,1,1),
                 seasonal=list(order=c(0,1,1), period=12) )
co2mod2
```

The fitted model is:

```
Coefficients:
          ma1     sma1
      -0.5792  -0.8206
s.e.   0.0791   0.1137

sigma^2 estimated as 0.5446:
log likelihood = -139.54,  aic = 283.08
```

Note that the AIC is much lower than the model fitted in Example 2.2 and, furthermore, the diagnostics look good as shown below.



We do note the presence of one large outlier (standardised residual ≈ 3) but this does not seem to be cause for much concern.

We could further test the model through overfitting (discussed in Lecture 7) to investigate if there is any room for further improvement. For example we might try some more general models such as:

- ARIMA$(0,1,1)\times(\mathbf{1},1,1)_{12}$
- ARIMA$(\mathbf{1},1,1)\times(0,1,1)_{12}$
- ARIMA$(\mathbf{1},1,1)\times(\mathbf{1},1,1)_{12}$
- ARIMA$(0,1,1)\times(0,1,\mathbf{2})_{12}$
- ARIMA$(0,1,\mathbf{2})\times(0,1,1)_{12}$
- ARIMA$(0,1,\mathbf{2})\times(0,1,\mathbf{2})_{12}$