Chapter 2:

Functions of Several Variables and Partial Differentiation

Prerequisites: Derivatives of standard functions; Product, Quotient and Chain Rules; derivatives of exponential and logarithmic functions.

Functions of Several Variables (revisited)

The functions we have considered so far have one independent variable.

We now want to look at functions of two or more variables (called *several variables*).

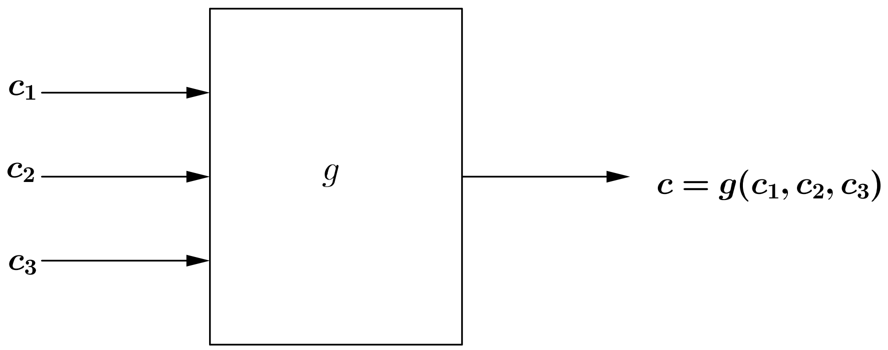
For example, the function $z = f(x, y) = x^2 - 2xy + 4y^2$ has two independent variables x and y.

For example in a gas, pressure depends on volume and temperature, so we could write $P = f(V, T)$.

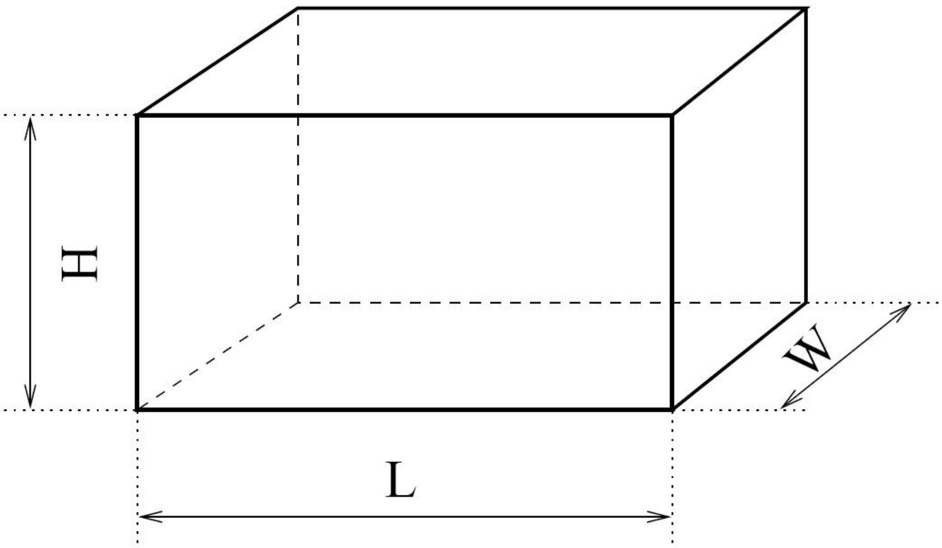
Both V and T can vary independently.

The concentration c of a pollutant in a river may be the result of emissions from three different sources, so we could write $c = g(c_1, c_2, c_3)$.

OUTPUT c IS A FUNCTION OF THREE INDEPENDENT VARIABLES c_1, c_2, c_3.



Example: volume of a box



Volume of box = Length $\times$ Width $\times$ Height

$V(L, W, H) = L \times W \times H$

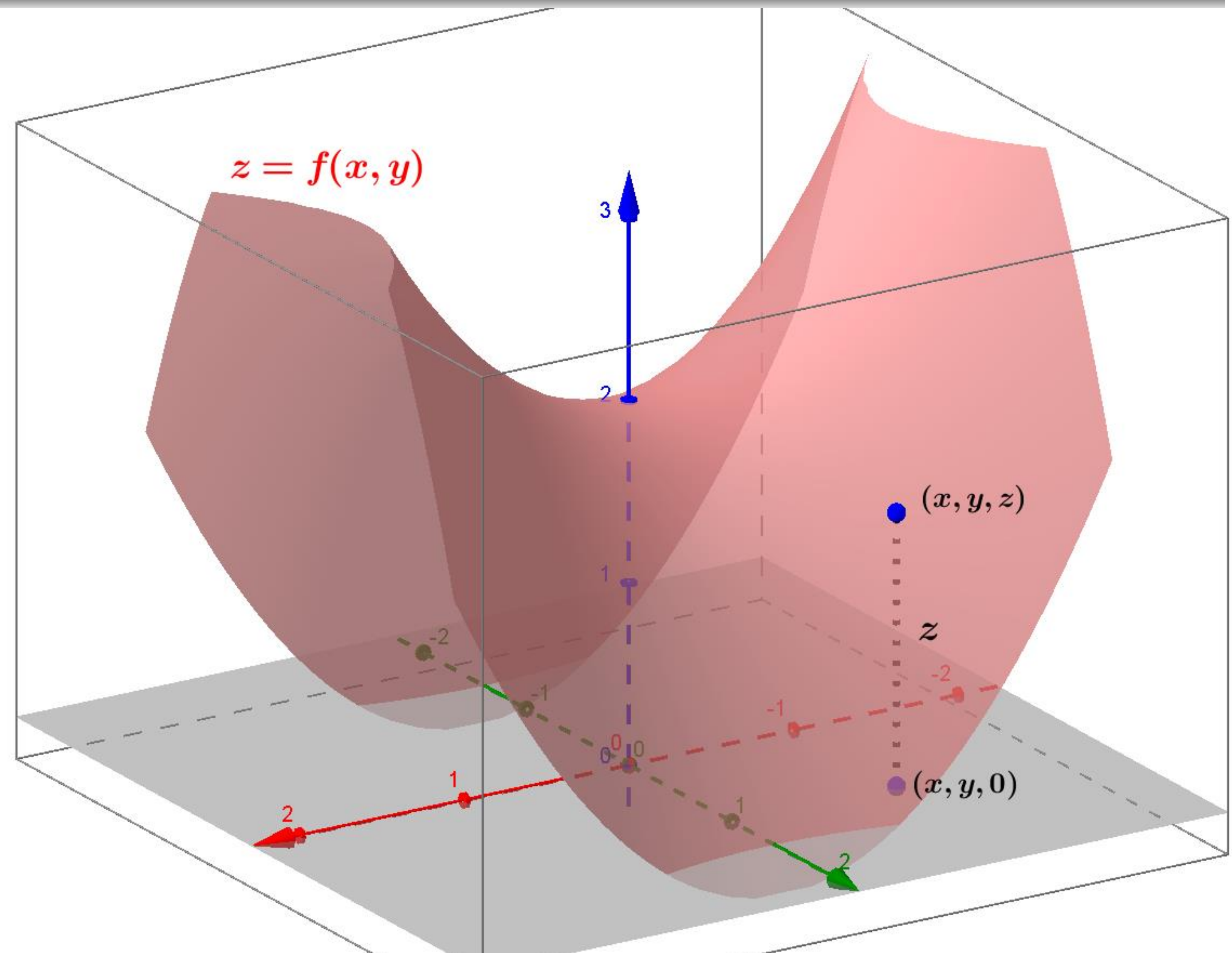
Graphing Functions of Two Variables

A function of two variables $z = f(x, y)$ can be visualised as a *surface* in three dimensions as follows:

- Think of (x, y) as the point $(x, y, 0)$ in the xy-plane ('sea-level').
- Use the function to calculate $z = f(x, y)$, which we think of as the 'height' above (or below) (x, y).
- Plot the point (x, y, z) in three dimensions.
- Repeat for every x and y.

For example, at the point $(2, 1)$ at 'sea-level', the function $z = f(x, y) = x^2 - y^2$ gives $z = 2^2 - 1^2 = 3$.

So this gives the point $(2, 1, 3)$ in (x, y, z) space.

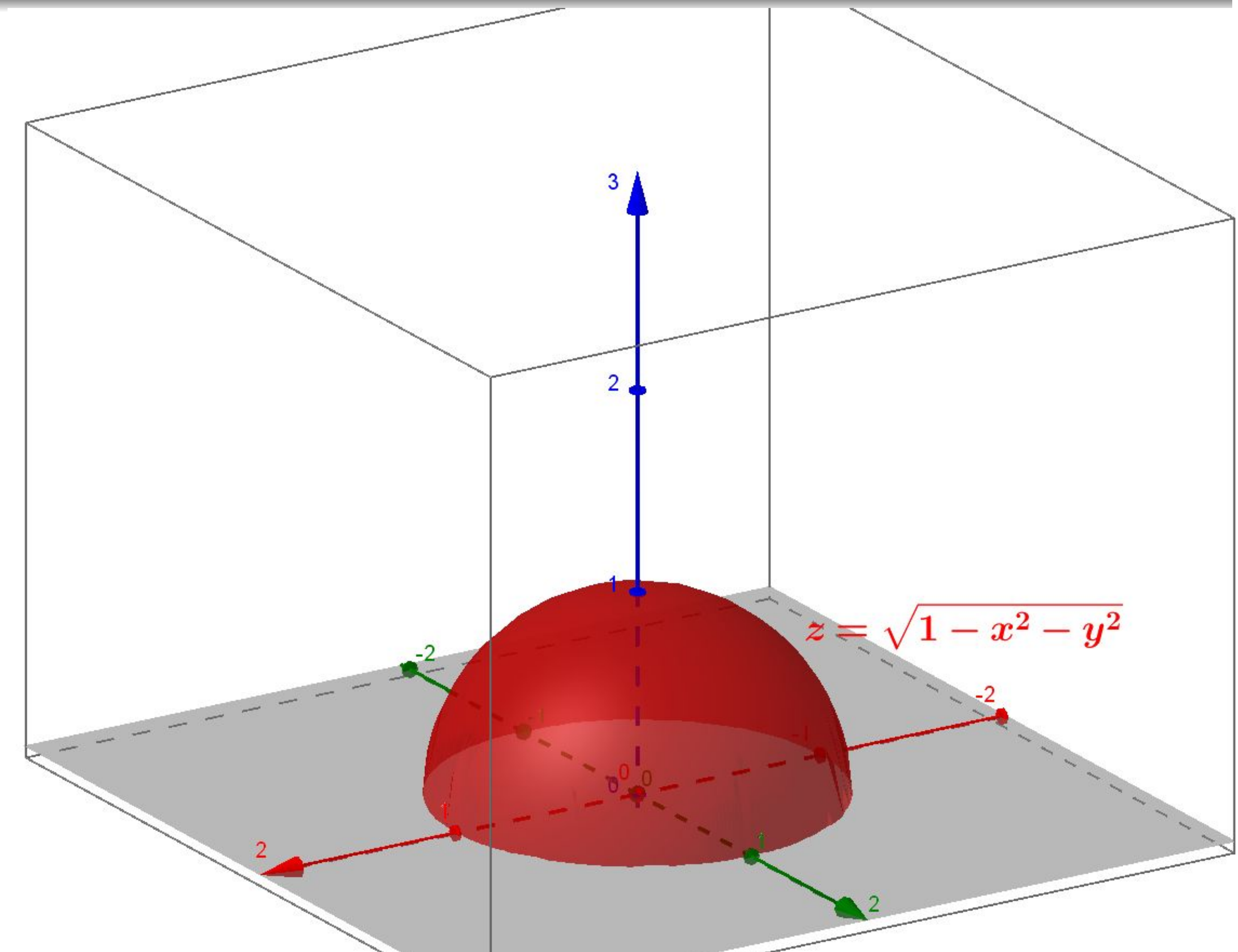


Example

The equation $x^2 + y^2 + z^2 = 1$ represents all points (x, y, z) which are distance one away from the origin; that is points on the surface of the sphere of radius 1, centred at the origin.

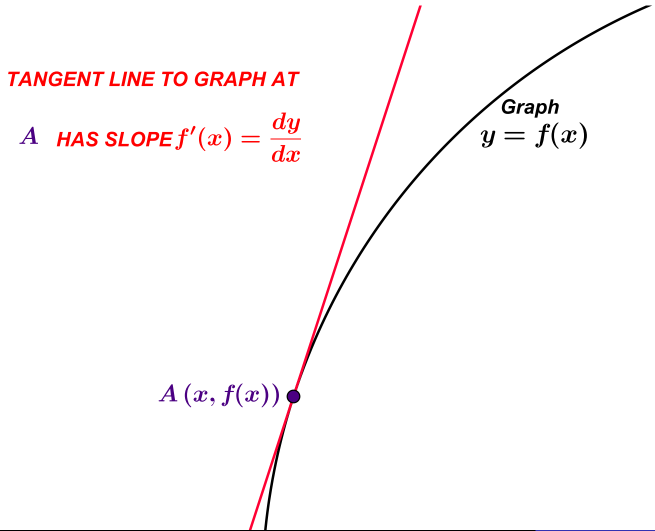
We can write this as $z^2 = 1 - x^2 - y^2$, which gives 'solutions' $z = \pm\sqrt{1 - x^2 - y^2}$.

The function $z = \sqrt{1 - x^2 - y^2}$ represents the 'northern hemisphere' of the sphere whereas the function $z = -\sqrt{1 - x^2 - y^2}$ represents the 'southern hemisphere'.



Partial Differentiation

For a function of one variable $y = f(x)$, we have $\dfrac{dy}{dx} = \dfrac{df(x)}{dx} = f'(x)$, which represents the slope of the (tangent line to the) graph of the function at the point $(x, f(x))$. It is the rate of change of y with respect to x.



A function of two variables $z = f(x, y)$ varies independently with both x and y.

Suppose we keep y fixed and vary x.

This gives the **first partial derivative with respect to x:**

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}.$$

It is the rate of change of z with x, keeping y fixed.

To calculate it, we differentiate z with respect to x,

treating y as if it were a constant!

Now suppose we keep x fixed and vary y.

This gives the **first partial derivative with respect to y:**

$$\frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}.$$

It is the rate of change of z with y, keeping x fixed.

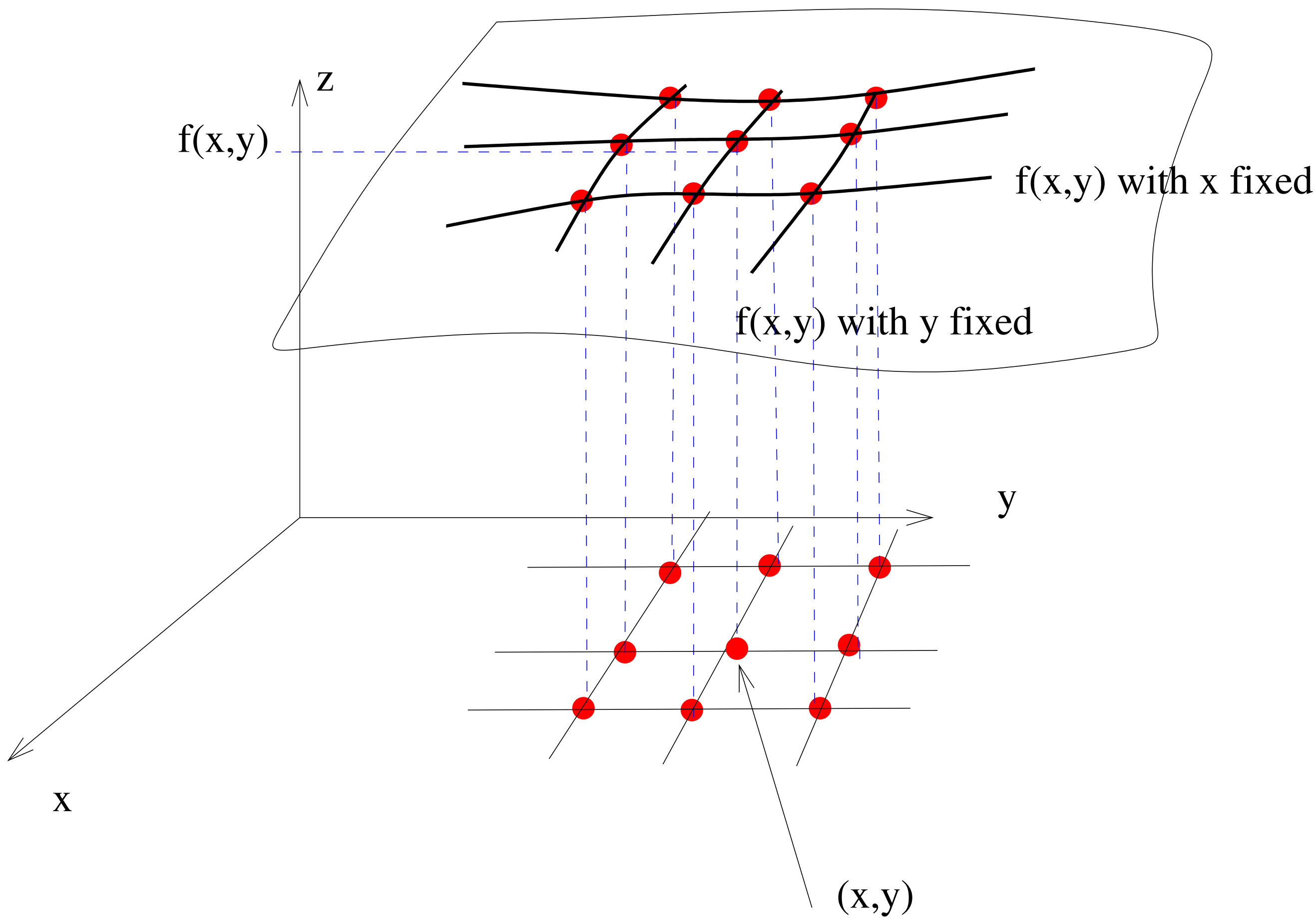
To calculate it, we differentiate z with respect to y,

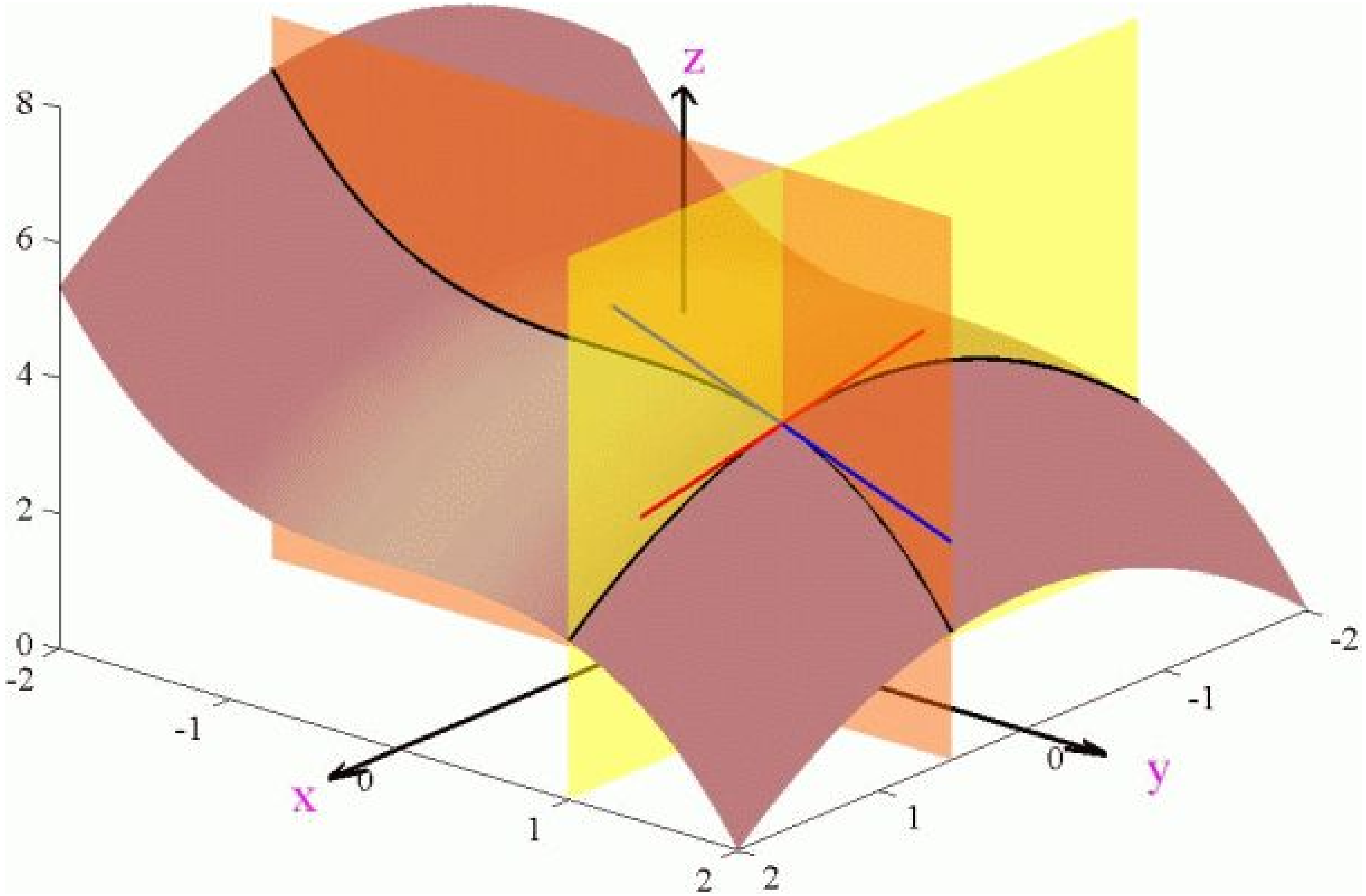
treating x as if it were a constant!

The technical definitions are (if the limits exist):

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

and $$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}.$$





Example

Find the first partial derivatives of

$z = f(x, y) = x^2y^2 + x^3 + y^3$.

We use colour to help us see what's happening:

$z = f(x, y) = {\color{red}x^2}{\color{blue}y^2} + {\color{red}x^3} + {\color{blue}y^3}$.

The first partial derivative with respect to ${\color{red}x}$:

$\frac{\partial z}{\partial {\color{red}x}} = {\color{red}(2x)}{\color{blue}y^2} + {\color{red}3x^2} + {\color{red}0} = 2xy^2 + 3x^2$.

Here ${\color{red}x}$ is active and ${\color{blue}y}$ is like a constant.

Observe how differentiating the term ${\color{blue}y^3}$ with respect to ${\color{red}x}$ gives zero.

Next the first partial derivative with respect to y:

$\frac{\partial z}{\partial y} = (x^2)(2y) + 0 + 3y^2 = 2x^2y + 3y^2$.

Here y is active and x is like a constant.

Observe how differentiating the term x^3 with respect to y gives zero.

More Examples

1. If $z = x\,e^y + y\ln x$, then the first partial derivatives are

$\frac{\partial z}{\partial x} = 1e^y + y\frac{1}{x} = e^y + \frac{y}{x}$,

$\frac{\partial z}{\partial y} = x\,e^y + 1\ln x = xe^y + \ln x$.

Note that even though there were products here, the Product Rule was not needed, since in each term one factor was treated like a constant.

2. Find the first partial derivatives of

$$z = f(x,y) = y^2\cos(3x) + \frac{x}{y}.$$

$$\frac{\partial z}{\partial x} = y^2\,(-3\sin(3x)) + \frac{1}{y} = -3y^2\sin(3x) + \frac{1}{y}.$$

$$\frac{\partial z}{\partial y} = (2y)\cos(3x) + x\frac{\partial}{\partial y}\left(\frac{1}{y}\right)$$

$$= 2y\cos(3x) + x\left(\frac{-1}{y^2}\right) = 2y\cos(3x) - \frac{x}{y^2}.$$

Note that we used the Power Rule to get

$$\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \frac{\partial(y^{-1})}{\partial y} = (-1)y^{-2} = \frac{-1}{y^2}.$$

3. Find the first partial derivatives of

$$c(c_1,c_2,c_3) = c_1^3c_2 - 2c_1c_2c_3 + c_2^2c_3 - 3c_1c_3^2.$$

First we colour it: $c = c_1^3c_2 - 2c_1c_2c_3 + c_2^2c_3 - 3c_1c_3^2$.

$$\frac{\partial c}{\partial c_1} = (3c_1^2)c_2 - 2(1)c_2c_3 + 0 - 3(1)c_3^2$$

$$= 3c_1^2c_2 - 2c_2c_3 - 3c_3^2,$$

$$\frac{\partial c}{\partial c_2} = c_1^3(1) - 2c_1(1)c_3 + (2c_2)c_3 - 0$$

$$= c_1^3 - 2c_1c_3 + 2c_2c_3,$$

$$\frac{\partial c}{\partial c_3} = 0 - 2c_1c_2(1) + c_2^2(1) - 3c_1(2c_3)$$

$$= -2c_1c_2 + c_2^2 - 6c_1c_3.$$

4. Find the first partial derivatives of $z = x^4 e^{2x+3y}$.

To do the x-derivative, we need the Product Rule.

Write $z = uv$, where $u = x^4$ and $v = e^{2x+3y}$.

Then $\dfrac{\partial u}{\partial x} = 4x^3$ and $\dfrac{\partial v}{\partial x} = 2e^{2x+3y}$.

(Note: when only one variable appears, the partial and ordinary derivatives are the same thing).

$$\text{So } \frac{\partial z}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = (x^4)(2e^{2x+3y}) + (e^{2x+3y})(4x^3)$$
$$= e^{2x+3y}(2x^4 + 4x^3).$$

$$\text{Next } \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^4 e^{2x+3y}) = 3x^4 e^{2x+3y}.$$

(Note: only the e^{3y} gets differentiated.)

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5. Let $x = \dfrac{st}{s^2+t^2}$. Find $\dfrac{\partial x}{\partial s}$ and $\dfrac{\partial x}{\partial t}$.

We use the Quotient Rule: $\dfrac{\partial}{\partial s}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{\partial u}{\partial s} - u\frac{\partial v}{\partial s}}{v^2}$.

Here $u = st$ and $v = s^2 + t^2$.

So $\dfrac{\partial u}{\partial s} = t$ and $\dfrac{\partial v}{\partial s} = 2s$.

$$\text{So } \frac{\partial x}{\partial s} = \frac{(s^2+t^2)(t) - (st)(2s)}{(s^2+t^2)^2} = \frac{t^3 - s^2 t}{(s^2+t^2)^2}.$$

The expression for x is symmetric under interchange $s \leftrightarrow t$.

So by interchanging s and t, we get $\dfrac{\partial x}{\partial t} = \dfrac{s^3 - t^2 s}{(t^2+s^2)^2}$.

You should check this directly by doing the t derivative!

6. Find the first partial derivatives of $z = \ln(x^3 + y^4)$.

Recall from the ordinary derivative that $\dfrac{d \ln u}{dx} = \dfrac{\left(\frac{du}{dx}\right)}{u}$ (essentially the Chain Rule).

So $\dfrac{\partial z}{\partial x} = \dfrac{\frac{\partial\left(x^3+y^4\right)}{\partial x}}{x^3 + y^4} = \dfrac{3x^2}{x^3 + y^4}$.

Similarly $\dfrac{\partial z}{\partial y} = \dfrac{\frac{\partial\left(x^3+y^4\right)}{\partial y}}{x^3 + y^4} = \dfrac{4y^3}{x^3 + y^4}$.

Higher partial derivatives

Given a function $z = f(x, y)$ we can calculate its first partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

We can continue to find the x and y partial derivatives of each of these. This gives us four *second partial derivatives.*

The first two are 'pure':

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right).$$

There are also two **mixed partial derivatives**

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

For 'nice' functions, these last two are the same, that is

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x},$$ so it doesn't matter whether we differentiate with respect to x first and then y or *vice versa*.

Example *Find the first and second partial derivatives of* $z = x^2 y + y^2$.

We colour it: $z = x^2 y + y^2$.

First partial derivatives:

$$\frac{\partial z}{\partial x} = (2x)y + 0 = 2xy, \qquad \frac{\partial z}{\partial y} = x^2(1) + 2y = x^2 + 2y.$$

Second partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2xy) = 2y \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(x^2 + 2y) = 0 + 2 = 2.$$

Mixed derivatives:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(x^2 + 2y) = 2x + 0 = 2x \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(2xy) = 2x.$$

We observe that, as expected, $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$.

Shorthand notation for partial derivatives

For a function $z = f(x, y)$,

$\dfrac{\partial z}{\partial x}$ is denoted by z_x or f_x, $\dfrac{\partial z}{\partial y}$ is denoted by z_y or f_y,

$\dfrac{\partial^2 z}{\partial x^2}$ is denoted by z_{xx} or f_{xx}, $\dfrac{\partial^2 z}{\partial y^2}$ is denoted by z_{yy} or f_{yy},

$\dfrac{\partial^2 z}{\partial x \partial y}$ is denoted by z_{yx} (note reverse order),

$\dfrac{\partial^2 z}{\partial y \partial x}$ is denoted by z_{xy} (note reverse order).

Since $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$, then also $z_{yx} = z_{xy}$.

Examples

1. If $z = x^2y^3 + x^4y$, then

$z_x = 2xy^3 + 4x^3y$

$z_y = x^2(3y^2) + x^4(1) = 3x^2y^2 + x^4$

$z_{xx} = 2y^3 + 4(3x^2)y = 2y^3 + 12x^2y$

$z_{yy} = 3x^2(2y) + 0 = 6x^2y$

$z_{xy} = 2x(3y^2) + 4x^3(1) = 6xy^2 + 4x^3$

$z_{yx} = 3(2x)y^2 + 4x^3 = 6xy^2 + 4x^3$

Observe that $z_{xy} = z_{yx}$, as expected.

2. If $z = x^2 \sin y + y^2 e^{3x}$, then

The first partial derivatives are

$$z_x = 2x \sin y + 3y^2 e^{3x} \text{ and}$$

$$z_y = x^2 \cos y + 2ye^{3x}.$$

The second partial derivatives are

$$z_{xx} = 2 \sin y + 9y^2 e^{3x},$$

$$z_{yy} = -x^2 \sin y + 2e^{3x} \text{ and}$$

$$z_{xy} = 2x \cos y + 6ye^{3x} = z_{yx}.$$

We can also consider third order derivatives:

for example $\frac{\partial^3 z}{\partial x^3} = z_{xxx} = \frac{\partial z_{xx}}{\partial x}$. Similarly $\frac{\partial^3 z}{\partial y^3} = z_{yyy}$.

There are also two mixed partial third derivatives

$$\frac{\partial^3 z}{\partial y \partial x^2} = z_{yxx} \quad \text{and} \quad \frac{\partial^3 z}{\partial x \partial y^2} = z_{xyy}.$$

In the above example, we have

$$z_{xxx} = 27y^2 e^{3x},$$

$$z_{xxy} = 2 \cos y + 18ye^{3x},$$

$$z_{xyy} = -2x \sin y + 6e^{3x},$$

$$z_{yyy} = -x^2 \cos y.$$

The fourth partial derivatives are $\frac{\partial^4 z}{\partial x^4} = z_{xxxx}$, etc.

3. The equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ is a *partial differential equation*, called the **Heat Equation** (also known as the **Diffusion Equation**).

Show that $u(x,t) = e^{-\alpha t} \sin x$ is a solution of the Heat Equation.

We can write the Heat Equation as $u_t - \alpha u_{xx} = 0$.

First $u_t = -\alpha e^{-\alpha t} \sin x$.

Next $u_x = e^{-\alpha t} \cos x$

and hence $u_{xx} = -e^{-\alpha t} \sin x$.

So $u_t - \alpha u_{xx} = -\alpha e^{-\alpha t} \sin x + \alpha e^{-\alpha t} \sin x = 0 \quad \checkmark$.

Partial Derivative Notation

Sometimes (especially in Physical Chemistry), $\frac{\partial z}{\partial x}$ is written as $\left(\frac{\partial z}{\partial x}\right)_y$, emphasising the fact that y is held constant.

For example the *Ideal Gas Law* states that $PV = kT$.

We can think of any one of P, V, T as a function of the other two, for example $P = \frac{kT}{V}$.

So we get $\left(\frac{\partial P}{\partial T}\right)_V = \frac{k}{V}$ and $\left(\frac{\partial P}{\partial V}\right)_T = -\frac{kT}{V^2}$.

We could also have written $V = \frac{kT}{P}$ which gives

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{k}{P} \quad \text{and} \quad \left(\frac{\partial V}{\partial P}\right)_T = -\frac{kT}{P^2}.$$

Finally, writing $T = \frac{PV}{k}$ we get

$$\left(\frac{\partial T}{\partial V}\right)_P = \frac{P}{k} \quad \text{and} \quad \left(\frac{\partial T}{\partial P}\right)_V = \frac{V}{k}.$$