General Homogeneous Cases

We now return to the more general equation

$$y'' + ay' + by = 0 \qquad (\star)$$

where a and b are constants, with $a \neq 0$ and $b \neq 0$.

This type of equation arises, for example, in a mass-spring system with *damping*, that is an extra resistance, such as a friction, air resistance or surface contact.

To solve it, we look for an exponential type solution: $y(t) = e^{mt}$. Then $y' = me^{mt}$, and $y'' = m^2e^{mt}$.

Substituting these into $(\star)$ gives $(m^2 + am + b)e^{mt} = 0$.

Since the exponential function is never zero, we must have $m^2 + am + b = 0$.

This is called the **auxiliary equation** for $(\star)$.

So $y(t) = e^{mt}$ satisfies the ODE, provided that $m^2 + am + b = 0$.

The solutions of the quadratic equation are

$$m = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

Three situations arise:

(I) $a^2 - 4b > 0$; (II) $a^2 - 4b = 0$; (III) $a^2 - 4b < 0$.

Case (I) $\quad a^2 - 4b > 0$.

Here the quadratic has two real roots, which we call

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \quad \text{and} \quad m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

So $e^{m_1 t}$ and $e^{m_2 t}$ satisfy $(\star)$. The general solution is

$$y(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}.$$

So the solutions are a combination of exponential growth/decay-type solutions.

If we exclude exponential growth-type solutions, then the parameters must satisfy $a > 0$ and $b > 0$, (so that $m_1 < 0$ and $m_2 < 0$).

In most applications this is the case.

Example 1

Find the general solution of $y'' + 5y' + 4y = 0$.

Find the solution with initial conditions
$y(0) = 0, \quad y'(0) = 6$.

The auxiliary equation is $m^2 + 5m + 4 = 0$.

This factors into $(m+1)(m+4) = 0$.

So the roots are $m_1 = -1 \quad$ and $\quad m_2 = -4$.

So the general solution is $y = C_1 e^{-t} + C_2 e^{-4t}$.

Before we apply the initial conditions we get
$y' = -C_1 e^{-t} - 4C_2 e^{-4t}$.

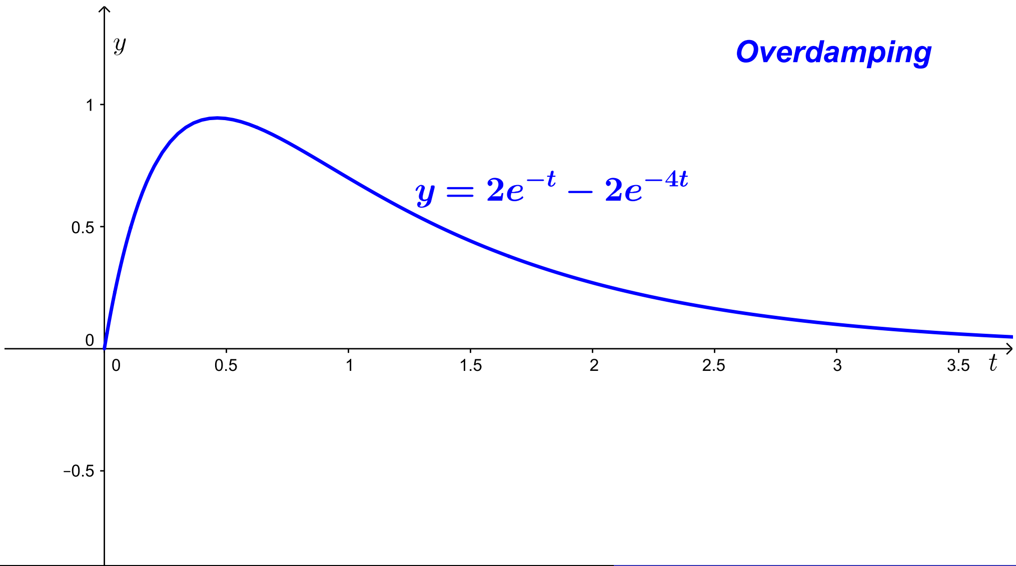
So $y(0) = 0$ gives $C_1 + C_2 = 0$ and

$y'(0) = 6$ gives $-C_1 - 4C_2 = 6$.

Solving these gives $C_1 = 2$ and $C_2 = -2$.

So the solution is $y = 2e^{-t} - 2e^{-4t}$.

This case is called *overdamping* and corresponds to a 'stiff spring'.



Case (II) $a^2 - 4b = 0$.

Here $b = \dfrac{a^2}{4}$ so the auxiliary equation becomes

$$m^2 + am + \frac{a^2}{4} = 0.$$

That is $(m + \dfrac{a}{2})^2 = 0$, so we get a double root at $m_1 = m_2 = -\dfrac{a}{2}$.

So we get one solution $y(t) = C_1 e^{-\frac{1}{2}at}$.

But we need a second solution, independent of this.

It turns out that $te^{-\frac{1}{2}at}$ is also a solution, which you can verify by substituting it into the ODE.

So the general solution is $y(t) = C_1 e^{-\frac{at}{2}} + C_2 t e^{-\frac{at}{2}}$.

The second term is like exponential decay, but slower.

Example 2

Find the general solution of $y'' + 4y' + 4y = 0$.

Find the solution with initial conditions

$y(0) = 0, \quad y'(0) = 6$.

The auxiliary equation is $m^2 + 4m + 4 = 0$.

This factors into $(m + 2)(m + 2) = 0$.

So there is a double root $m_1 = -2$.

So the general solution is $y = C_1 e^{-2t} + C_2 t e^{-2t}$.

Before we apply the initial conditions, we need

$y' = -2C_1 e^{-2t} + C_2(e^{-2t} - 2te^{-2t})$,

(using the Product Rule on the second term).

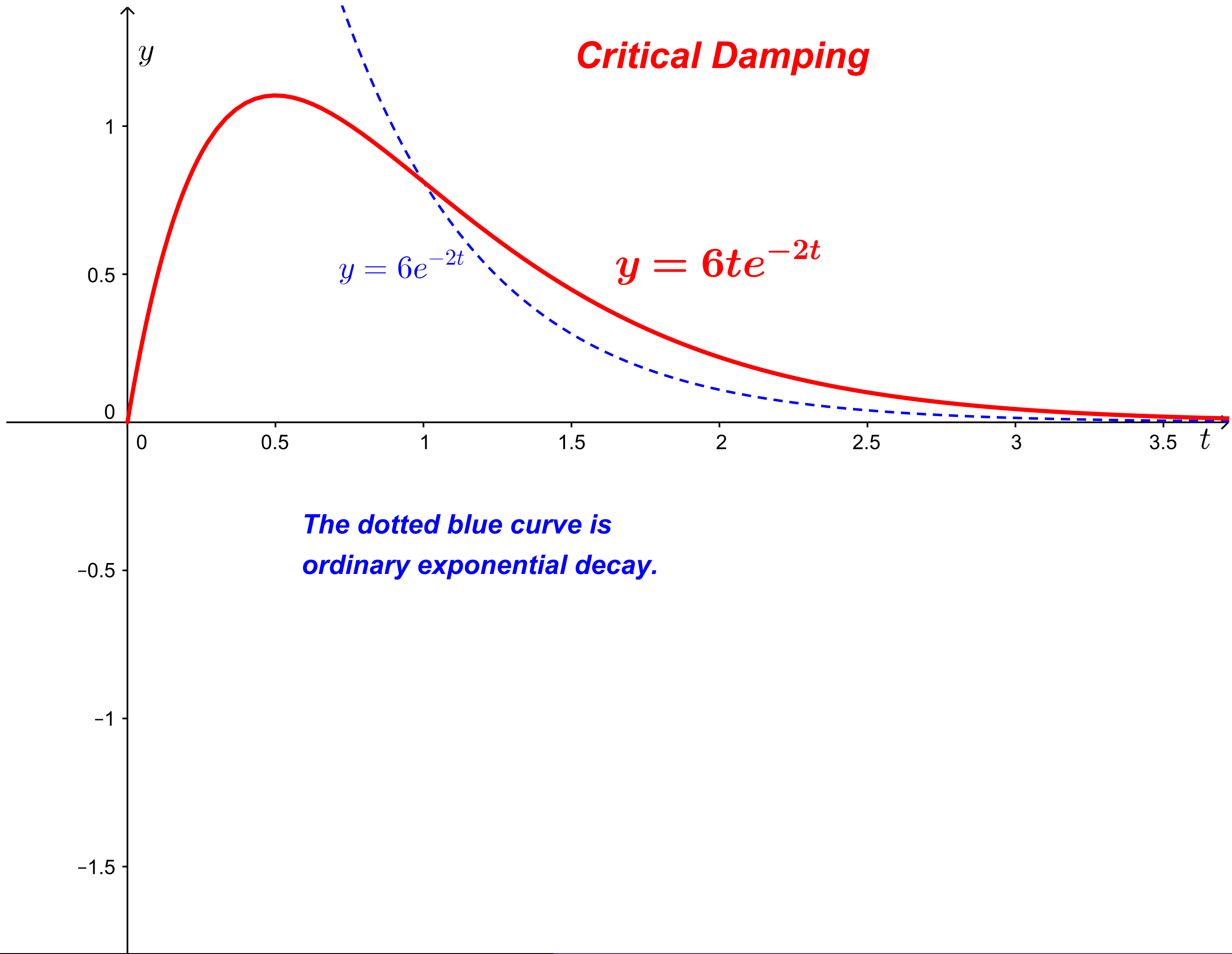
So $y(0) = 0$ gives $C_1 = 0$

and $y'(0) = 6$ gives $-2C_1 + C_2 = 6$.

Solving these gives $C_1 = 0 \quad \text{and} \quad C_2 = 6$.

So the solution is $y = 6te^{-2t}$.

This case is called *critical damping* and is the 'in-between' case.



Case (III) $\qquad a^2 - 4b < 0.$

Here the quadratic has complex roots!

For convenience, we define $\omega = \sqrt{b - \dfrac{a^2}{4}} > 0.$

So $a^2 - 4b = -4\omega^2 < 0.$

The roots of the auxiliary equation are

$$\begin{aligned} m_1 &= \frac{-a + \sqrt{a^2 - 4b}}{2} = \frac{-a + \sqrt{-4\omega^2}}{2} \\ &= \frac{-a + 2i\omega}{2} = -\frac{a}{2} + i\omega \quad \text{and} \\ m_2 &= \frac{-a - \sqrt{a^2 - 4b}}{2} = \frac{-a - \sqrt{-4\omega^2}}{2} \\ &= \frac{-a - 2i\omega}{2} = -\frac{a}{2} - i\omega. \end{aligned}$$

Consequently, we have two solutions of the ODE in case (III), which are of the form

$$e^{(-\frac{a}{2}+i\omega)t} = e^{-\frac{at}{2}}\, e^{i\omega t} = e^{-\frac{at}{2}}(\cos(\omega t) + i\sin(\omega t)) \text{ and}$$

$$e^{(-\frac{a}{2}-i\omega)t} = e^{-\frac{at}{2}}\, e^{-i\omega t} = e^{-\frac{at}{2}}(\cos(\omega t) - i\sin(\omega t)).$$

Here we used the definition of complex exponential, that $e^{ix} = \cos x + i\sin x.$ Also $e^{-ix} = \cos x - i\sin x$.

Adding these solutions and dividing by 2 gives the solution $e^{-\frac{at}{2}}\cos(\omega t)$.

Subtracting the two solutions and dividing by $2i$ gives the solution $e^{-\frac{at}{2}}\sin(\omega t)$.

These are two independent *real* solutions of the ODE.

Therefore the general solution is

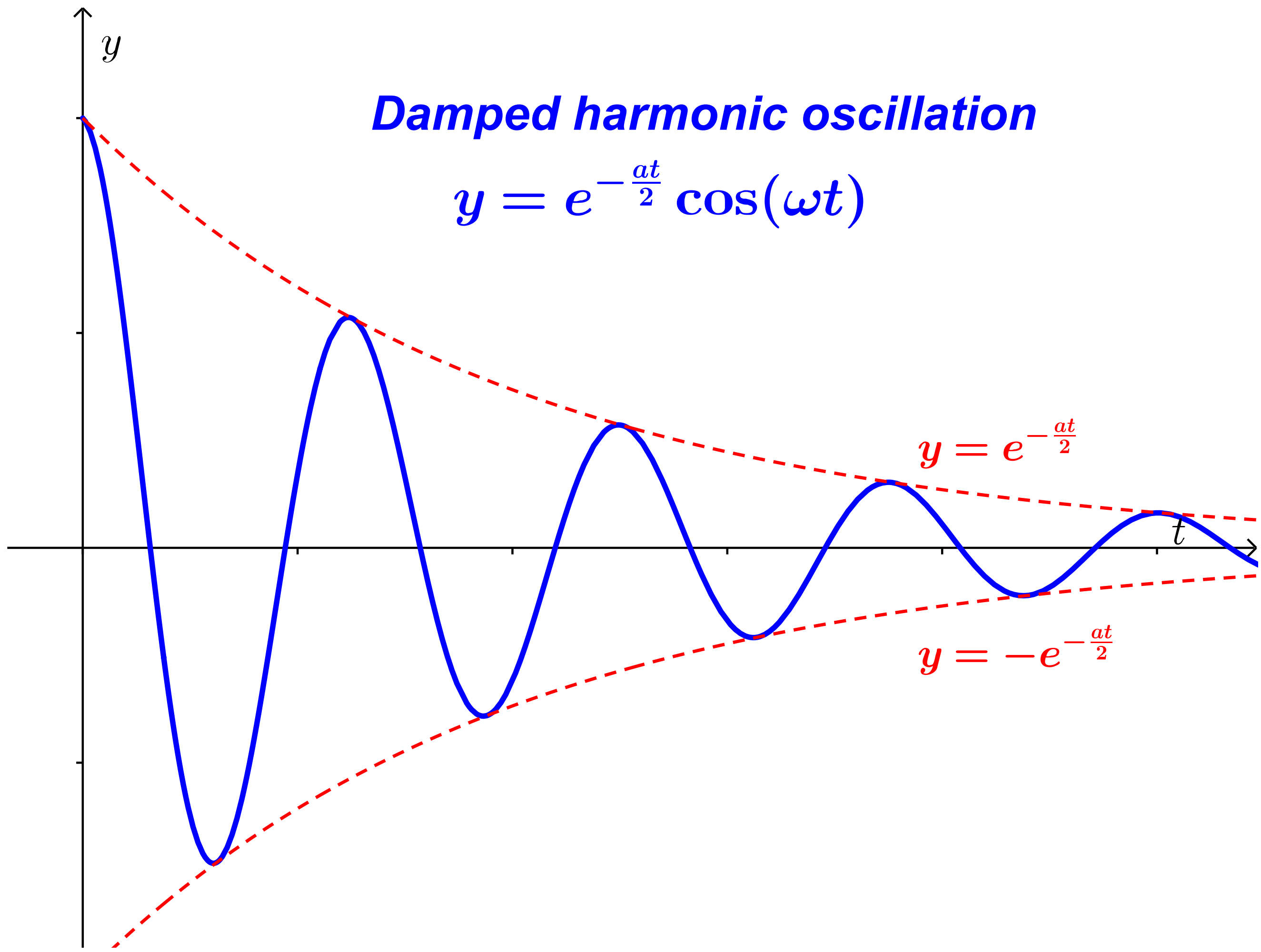
$$y(t) = e^{-\frac{at}{2}}(C_1\cos(\omega t) + C_2\sin(\omega t)).$$

This is a simple harmonic oscillator, multiplied with an exponentially decaying amplitude $e^{-\frac{a}{2}t}$.

It is called *underdamping*, which corresponds to a lightly damped spring (for example, by air resistance).

It is called a *damped harmonic oscillator*.

(Technical aside: above we used the property of $(\star)$ that adding two solutions gives another solution, and multiply a solution by a constant gives another solution.)



Example 3

Solve $y'' + 2y' + 5y = 0$*.*

Find the solution with initial conditions

$y(0) = 4, \quad y'(0) = -2$*.*

The auxiliary equation is $m^2 + 2m + 5 = 0$.

The roots of this equation are

$$m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

So the roots are $m_1 = -1 + 2i$ and $m_2 = -1 - 2i$.

The "-1" contributes to the exponential decay, that is e^{-t}, and the "$2i$" to the oscillatory terms, that is $\cos(2t)$ and $\sin(2t)$.

So the general solution is

$$y(t) = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t).$$

Before we apply the initial conditions, we need $y'(t)$.

This in turn requires two applications of the Product Rule:

$$\frac{d}{dt}(e^{-t}\cos(2t)) = e^{-t}\frac{d}{dt}(\cos(2t)) + \frac{d}{dt}(e^{-t})(\cos(2t))$$

$$= e^{-t}(-2\sin(2t)) - e^{-t}\cos(2t) = -e^{-t}(2\sin(2t) + \cos(2t)).$$

$$\frac{d}{dt}(e^{-t}\sin(2t)) = e^{-t}\frac{d}{dt}(\sin(2t)) + \frac{d}{dt}(e^{-t})(\sin(2t))$$

$$= e^{-t}(2\cos(2t)) - e^{-t}\sin(2t) = e^{-t}(2\cos(2t) - \sin(2t)).$$

So

$$y'(t) = -C_1 e^{-t}(2\sin(2t)+\cos(2t)) + C_2 e^{-t}(2\cos(2t) - \sin(2t)).$$

So $y(0) = 4$ gives $C_1(1) + C_2(0) = 4,$ that is $C_1 = 4$ and $y'(0) = -2$ gives $-C_1(0+1) + C_2(2-0) = -2,$ that is $-C_1 + 2C_2 = -2$. Using $C_1 = 4$. this gives $C_2 = 1$.

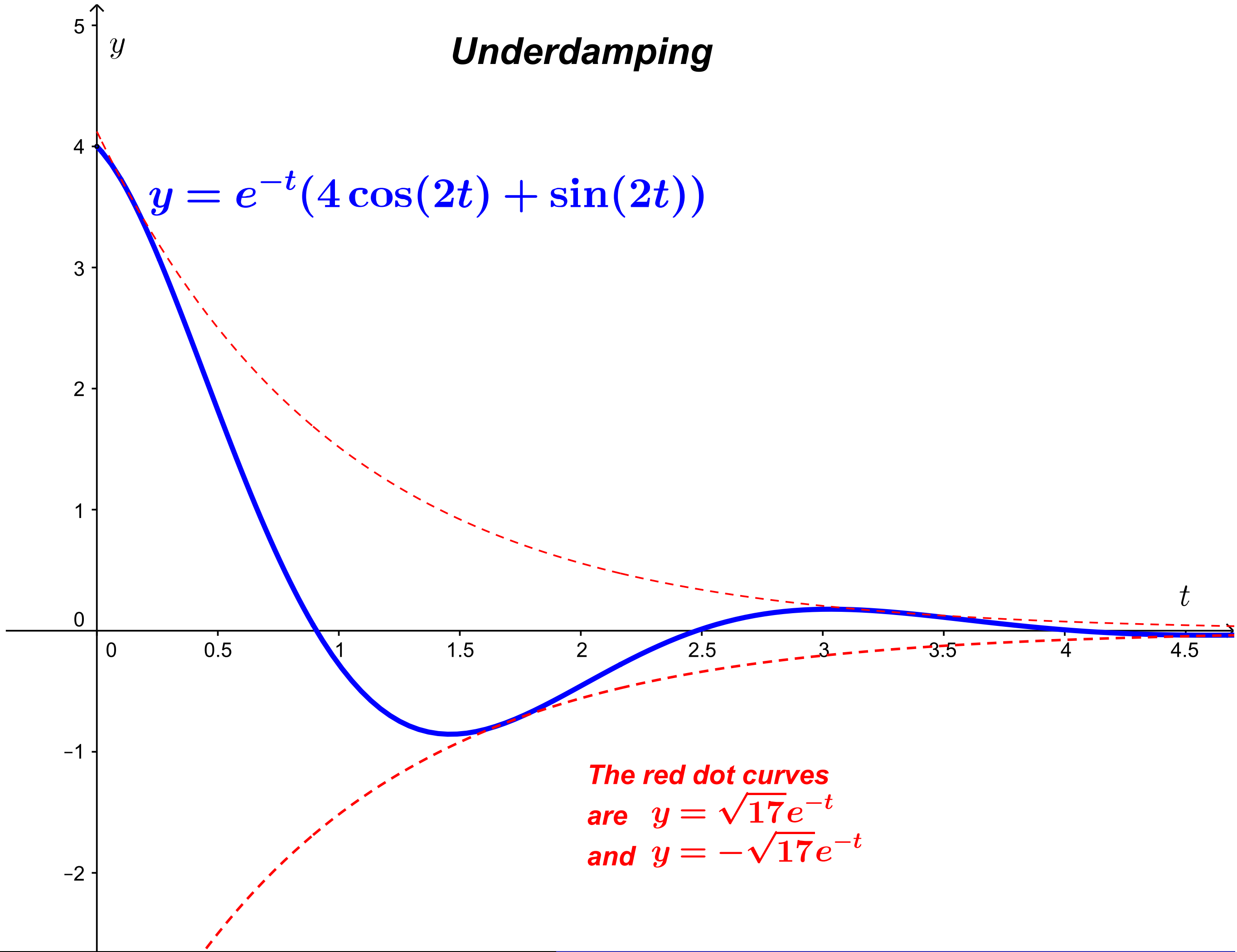
So the solution is

$$y(t) = 4e^{-t}\cos(2t) + e^{-t}\sin(2t) = e^{-t}(4\cos(2t) + \sin(2t)).$$

Note the amplitude of the sinusoidal part is $\sqrt{4^2 + 1^2} = \sqrt{17},$ so we can write

$y(t) = \sqrt{17}\, e^{-t}\sin(2t + \phi),$ for some angle ϕ.

This says the solution graph is bounded between $\sqrt{17}\, e^{-t}$ and $-\sqrt{17}\, e^{-t}$.



Example 4

Find the general solution of $y'' + 0.4y' + 4y = 0.$

Find the solution with initial conditions

$y(0) = 0, \quad y'(0) = 6.$

The auxiliary equation is $m^2 + 0.4m + 4 = 0.$

The roots of this equation are

$$m = \frac{-0.4 \pm \sqrt{(0.4)^2 - (4)4}}{2} = \frac{-0.4 \pm \sqrt{-15.84}}{2}$$

$$\approx \tfrac{-0.4 \pm 3.98i}{2} = -0.2 \pm 1.99i.$$

We round these and take the roots as

$m_1 = -0.2 + 2i$ and $m_2 = -0.2 - 2i.$

The "-0.2" contributes to the exponential decay, that is $e^{-0.2t}$, and the "$2i$" to the oscillatory terms, that is $\cos(2t)$ and $\sin(2t)$.

So the general solution is approximately

$$y(t) = C_1 e^{-0.2t}\cos(2t) + C_2 e^{-0.2t}\sin(2t).$$

Before we apply the initial conditions, we need $y'(t)$.

In a similar manner to above, we use the Product Rule:

$$\begin{aligned} y'(t) &= -C_1 e^{-0.2t}(2\sin(2t) + 0.2\cos(2t)) \\ &\quad + C_2 e^{-0.2t}(2\cos(2t) - 0.2\sin(2t)). \end{aligned}$$

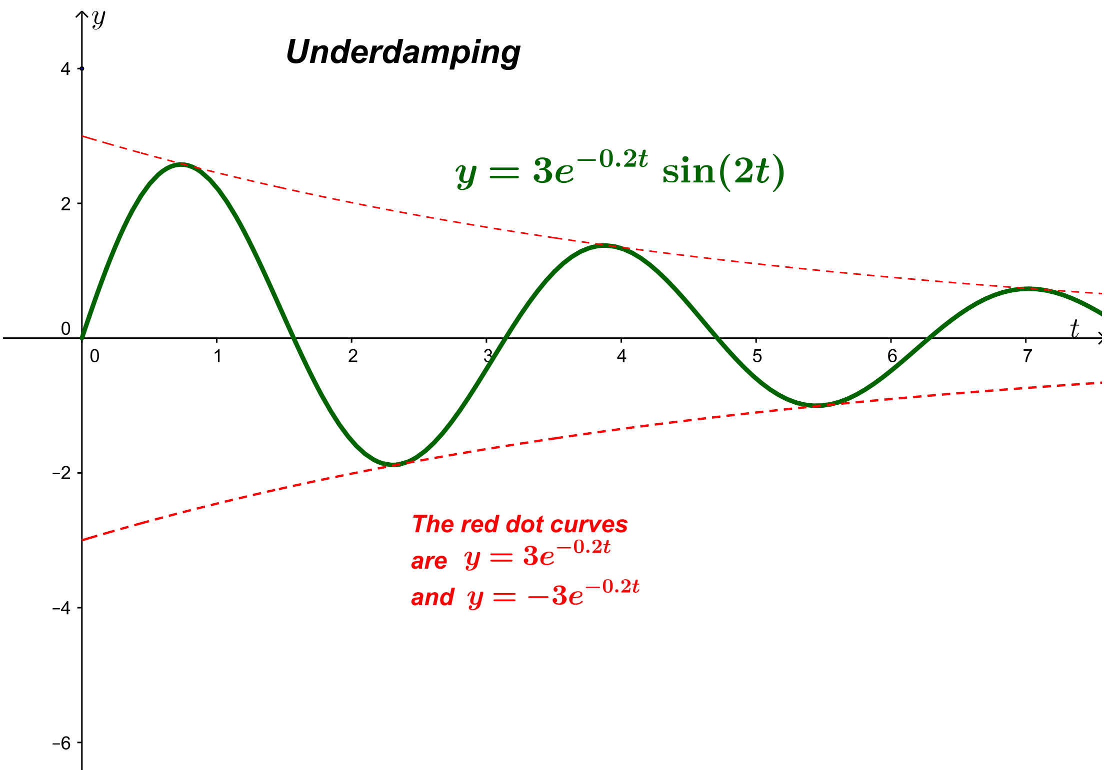
So $y(0) = 0$ gives $C_1(1) + C_2(0) = 0$, that is $C_1 = 0$

and $y'(0) = 6$ gives $-C_1(0.2) + C_2(2-0) = 6$,

that is $-0.2C_1 + 2C_2 = 6$.

So $C_1 = 0$ and $C_2 = 3$.

So the solution is $y(t) = 3e^{-0.2t}\sin(2t)$.



Effect of Damping

Examples 1, 2, and 4 are all the following problem:

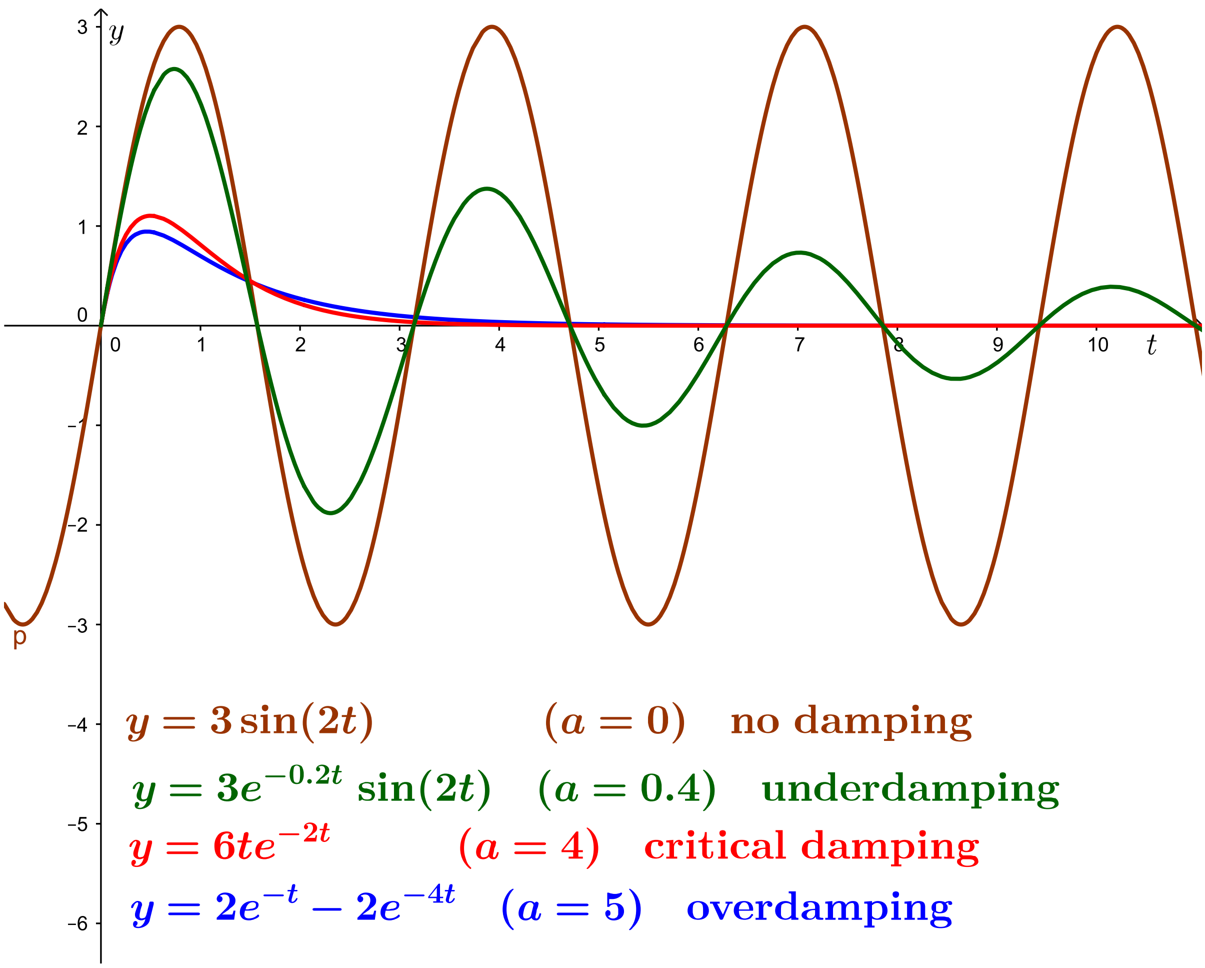
Find the solution of $y'' + ay' + 4y = 0$,

with initial conditions $y(0) = 0, \quad y'(0) = 6$.

In the three examples, a takes three different values: $a = 5, 4$ and 0.4. The spring constant and the initial conditions are the same in all three.

We also add in the solution of the undamped SHO problem ($a = 0$), namely $y(t) = 3\sin(2t)$.

We draw all four graphs in the one picture.



Lots of Examples

Example 5 *Solve* $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 0.$

The auxiliary equation is $m^2 + 3m + 2 = 0,$ which gives $(m+1)(m+2) = 0.$ So the roots are $m_1 = -1,\ m_2 = -2,$ (two real roots, overdamping).

So the (general) solution is $y(t) = C_1 e^{-t} + C_2 e^{-2t}$.

Example 6 *Solve* $y'' + 6y' + 8y = 0,$ *subject to* $y(0) = 4,\ y'(0) = -10.$

The auxiliary equation is

$m^2 + 6m + 8 = (m+2)(m+4) = 0.$

So there are two real roots: $m_1 = -2$ and $m_2 = -4.$

So the general solution is $y = C_1 e^{-2t} + C_2 e^{-4t}.$

Imposing the first initial condition, we get
$4 = y(0) = C_1 + C_2.$

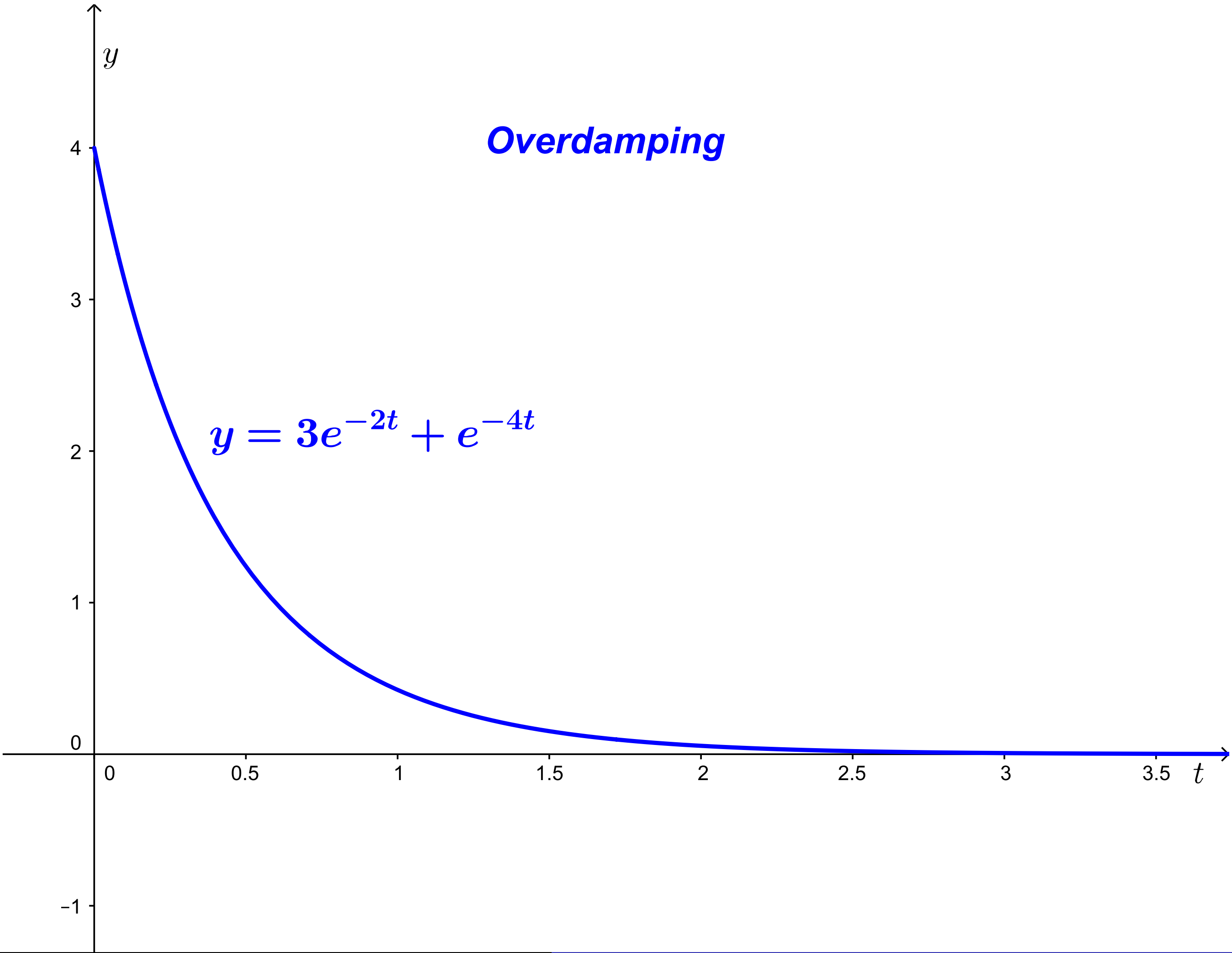
For the second initial condition we need to compute

$y' = -2C_1 e^{-2t} - 4C_2 e^{-4t},$ which gives $-10 = -2C_1 - 4C_2.$

We then solve $C_1 + C_2 = 4$ and $C_1 + 2C_2 = 5$

to get $C_2 = 1$ and $C_1 = 3.$

So the required solution is $y = 3e^{-2t} + e^{-4t}.$



Example 7 *Solve* $y'' + 6y' + 9y = 0$.

Find the solution with $y(0) = 2$ *and* $y'(0) = -1$.

This has auxiliary equation
$m^2 + 6m + 9 = (m + 3)(m + 3) = 0$.

So there is a double root $m = -3$.

This says the solution is like Case (II), critical damping.

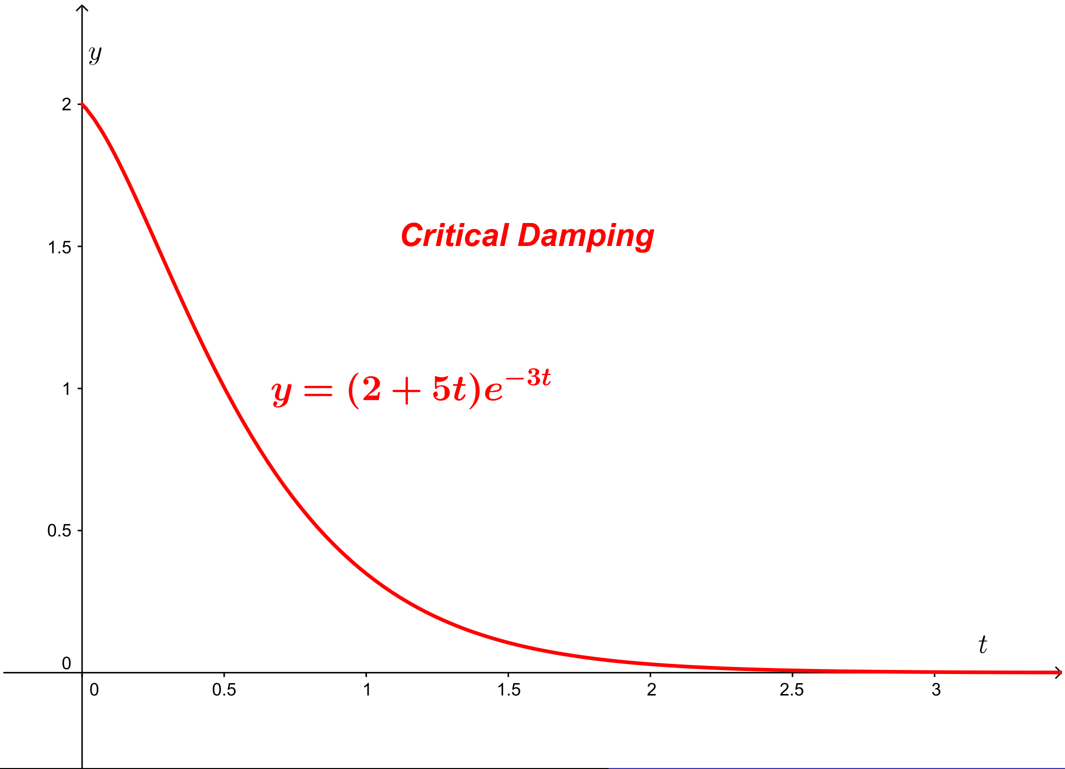
So the general solution is $y = C_1 e^{-3t} + C_2 t e^{-3t}$.

To apply the initial conditions, we first need
$y' = -3C_1 e^{-3t} + C_2(e^{-3t} - 3te^{-3t})$,
(using the Product Rule on the second term).

So $y(0) = 2$ gives $C_1 = 2$

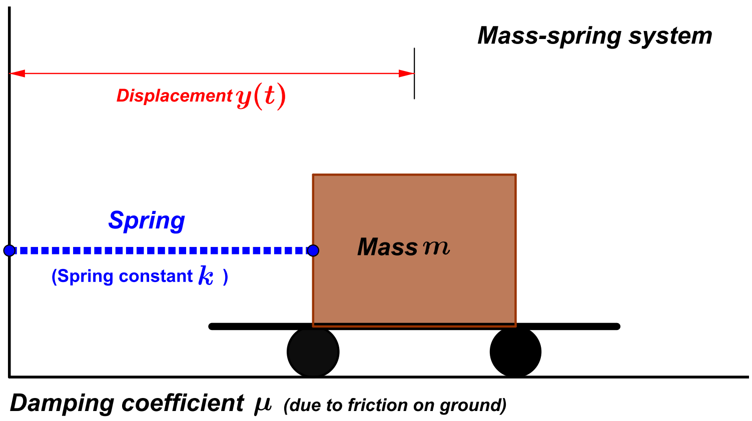
and $y'(0) = -1$ gives $-3C_1 + C_2 = -1$, so $C_2 = 5$.

The solution is $y = 2e^{-3t} + 5te^{-3t} = (2+5t)e^{-3t}$.



Example 8 *In the mass-spring system with damping, the displacement $y(t)$ at time t satisfies $my'' + \mu y' + ky = 0$. Here m is the mass, k is the spring constant (in Hooke's law) and μ is the coefficient of damping.*

If we take $m = 1, \quad \mu = 6, \quad k = 13$, find the displacement $y(t)$ and the velocity $y'(t)$, if the mass is released from rest with an initial displacement of 4.



Putting in the numbers, we want to solve the ODE

$y'' + 6y' + 13y = 0,$ subject to the initial conditions

$y(0) = 4, \quad y'(0) = 0.$

The ODE has auxiliary equation $m^2 + 6m + 13 = 0.$

Solving this gives $m = \dfrac{-6 \pm \sqrt{36 - 52}}{2}$

$$= \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i.$$

So the roots are $m_1 = -3 + 2i$ and $m_2 = -3 - 2i.$

The "-3" gives e^{-3t}, and the "$\pm 2i$" give $\cos(2t)$ and $\sin(2t)$.

So the general solution is

$y(t) = C_1 e^{-3t} \cos(2t) + C_2 e^{-3t} \sin(2t)$.

Applying the first initial condition gives $y(0) = 4 = C_1.$

To apply the second initial condition, we first need to calculate

$$y'(t) = C_1 \frac{d}{dt}(e^{-3t}\cos(2t)) + C_2 \frac{d}{dt}(e^{-3t}\sin(2t))$$
$$= C_1 \left(e^{-3t}(-2\sin(2t)) - 3e^{-3t}\cos(2t)\right)$$
$$+ C_2 \left(e^{-3t}(2\cos(2t)) - 3e^{-3t}\sin(2t)\right).$$

This is actually the velocity, which we will return to later.

Applying the second initial condition then gives

$0 = y'(0) = C_1(-3) + C_2(2)$.

Since $C_1 = 4,$ this then gives $C_2 = 6.$

So the solution is $y(t) = 4e^{-3t}\cos(2t) + 6e^{-3t}\sin(2t)$.

To calculate the velocity, we put in the values of C_1 and C_2 into $y'(t)$.

$$
\begin{aligned}
y'(t) &= 4\left(-2e^{-3t}\sin(2t) - 3e^{-3t}\cos(2t)\right) \\
&\quad +6\left(2e^{-3t}\cos(2t) - 3e^{-3t}\sin(2t)\right). \\
&= e^{-3t}\cos(2t)(-12+12) + e^{-3t}\sin(2t)(-8-18) \\
&= -26\,e^{-3t}\sin(2t).
\end{aligned}
$$

So the velocity is $y'(t) = -26\,e^{-3t}\sin(2t)$.