MA4604: Science Maths 4

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Chapter 1:

Complex Numbers

Prerequisites: Cosine and sine of standard angles (in radians). Properties of cosine and sine.

The number you have dialled is imaginary.

Please multiply by i and dial again.

The need for complex numbers

The real world is modelled well using real numbers (denoted by $\mathbb{R}$), but it also sometimes presents us with problems that cannot be solved within the real numbers.

By extending the number system to the *complex numbers* (denoted by $\mathbb{C}$), we are sometimes in a position to say more about real-world problems.

The quadratic equation $x^2 - 9 = 0$ can be factored into $(x - 3)(x + 3) = 0,$ thereby giving two *real* solutions $x = 3$ and $x = -3$.

However, an equation like $x^2 + 9 = 0$ has *no real solutions*, since $x^2 \geqslant 0$ so the left hand side (LHS) is always at least 9 and so can never be zero.

We can get around this by extending the real number system.

All we need to do is 'invent' one new number i (called the *imaginary unit*), that satisfies $x^2 + 1 = 0$.

So $i^2 + 1 = 0$ and hence $i^2 = -1$.

This is often written as

$$i = \sqrt{-1}$$

(In engineering j is used instead of i).

Now we can take the square root of **any** negative number.

For example, $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$.

$\sqrt{-2} = \sqrt{2}\sqrt{-1} = \sqrt{2}\,i = i\sqrt{2}$.

Notice that $-i$ is also a solution of $x^2 + 1 = 0$, since $(-i)^2 = (-i)(-i) = +i^2 = -1$.

So the two solutions of $x^2 + 1 = 0$ are $x = \pm i$.

Likewise the solutions of $x^2 + 16 = 0$ are got by writing it as $x^2 = -16$.

So $x = \pm\sqrt{-16} = \pm 4i$.

The Set of Complex Numbers $\mathbb{C}$

A *complex number* is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

A complex number essentially combines two real numbers, but we think of it as a single entity, often denoted by $z = a + bi$.

The set of all complex numbers is denoted by $\mathbb{C}$.

That is $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$.

Sometimes we denote $z = a + bi$ as $z = a + ib$ and even as $ib + a$ or $bi + a$.

Real and Imaginary Parts If $z = x + iy$, then x is called the *real part* of z (written $x = \mathfrak{Re}(z)$) and y is called the *imaginary part* of z (written $y = \mathfrak{Im}(z)$).

N.B. The imaginary part of z is a *real number*!

Complex numbers $x + i0$ are called **real** numbers and are written as just x.

Complex numbers $0 + iy$ are called **pure imaginary** and are written as iy or yi.

A complex number z is real if $\mathfrak{Im}(z) = 0$.

A complex number z is pure imaginary if $\mathfrak{Re}(z) = 0$.

The number $0 + 0i$ is called **zero** and is just written as 0.

Equality of Complex Numbers Two complex numbers are said *to be equal* if their real and imaginary parts are equal.

That is, if $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$, then $z = w$ if and only if $a = c$ and $b = d$.

Complex Numbers as Vectors

In what follows, we will see parallels between the algebra of complex numbers and the algebra of vectors in $\mathbb{R}^2$. It is often helpful to associate the complex number $z = a + bi \in \mathbb{C}$ with the vector $\mathbf{v} = \langle a, b \rangle \in \mathbb{R}^2$.

For example, the rules for addition and subtraction of vectors and complex numbers are essentially the same.

However, this association has its limitations, for example when we come to rules for multiplication; also for division (which isn't defined for vectors).

Algebra of Complex Numbers: Addition

We add two complex numbers $z = a + ib$ and $w = c + id$ according to the rule

$$z + w = (a + c) + i(b + d)$$

For example, $(3 + 4i) + (6 + 2i) = 9 + 6i$

The usual rules of addition apply:

Addition is commutative: $z + w = w + z$.

Addition is associative: $(z + w) + v = z + (w + v)$ for all complex numbers z, w, v.

So we can unambiguously omit brackets in a sum of three or more complex numbers.

Subtraction of Complex Numbers

We subtract two complex numbers $z = a + ib$ and $w = c + id$ according to the rule

$$z - w = (a - c) + i(b - d)$$

For example, $3 + 4i - (6 + 2i) = 3 - 6 + 4i - 2i = -3 + 2i$.

Multiplication of Complex Numbers

To multiply a complex number by a real number, we simply multiply both the real and imaginary parts by the real number

For example, $3(2 + 4i) = 6 + 12i$.

To multiply a complex number by another complex number, we treat i like a variable at first, and then use $i^2 = -1$ to simplify.

Specifically, if $z = a + ib$ and $w = c + id$, then

$zw = (a + ib)(c + id) = ac + aid + ibc + ibid$

$$= ac + iad + ibc + i^2bd = ac + i(ad + bc) + (-1)bd$$

$$= (ac - bd) + i(ad + bc).$$

That is $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$.

Example 1 $(2 + 3i)(4 + i) = 2(4) + (3i)4 + 2(i) + 3i(i)$

$$= 8 + 12i + 2i + 3i^2 = 8 + 14i - 3 = 5 + 14i.$$

Example 2 $i(4 - 6i) = 4i - 6i^2 = 4i + 6 = 6 + 4i$.

Example 3 $(5 - 2i)^2 = (5)^2 + 2(5)(-2i) + (-2i)^2$

$$= 25 - 20i + 4i^2 = 25 - 20i - 4 = 21 - 20i.$$

Properties of Multiplication

The usual rules of arithmetic apply (here z, w, v are any complex numbers):

- $zw = wz$ (multiplication is commutative)
- $z(wv) = (zw)v$ (multiplication is associative)
- $z(w + v) = zw + zv$ (distributive law)
- $0z = 0$ (multiplication by zero)
- $1z = z$ (multiplication by one)

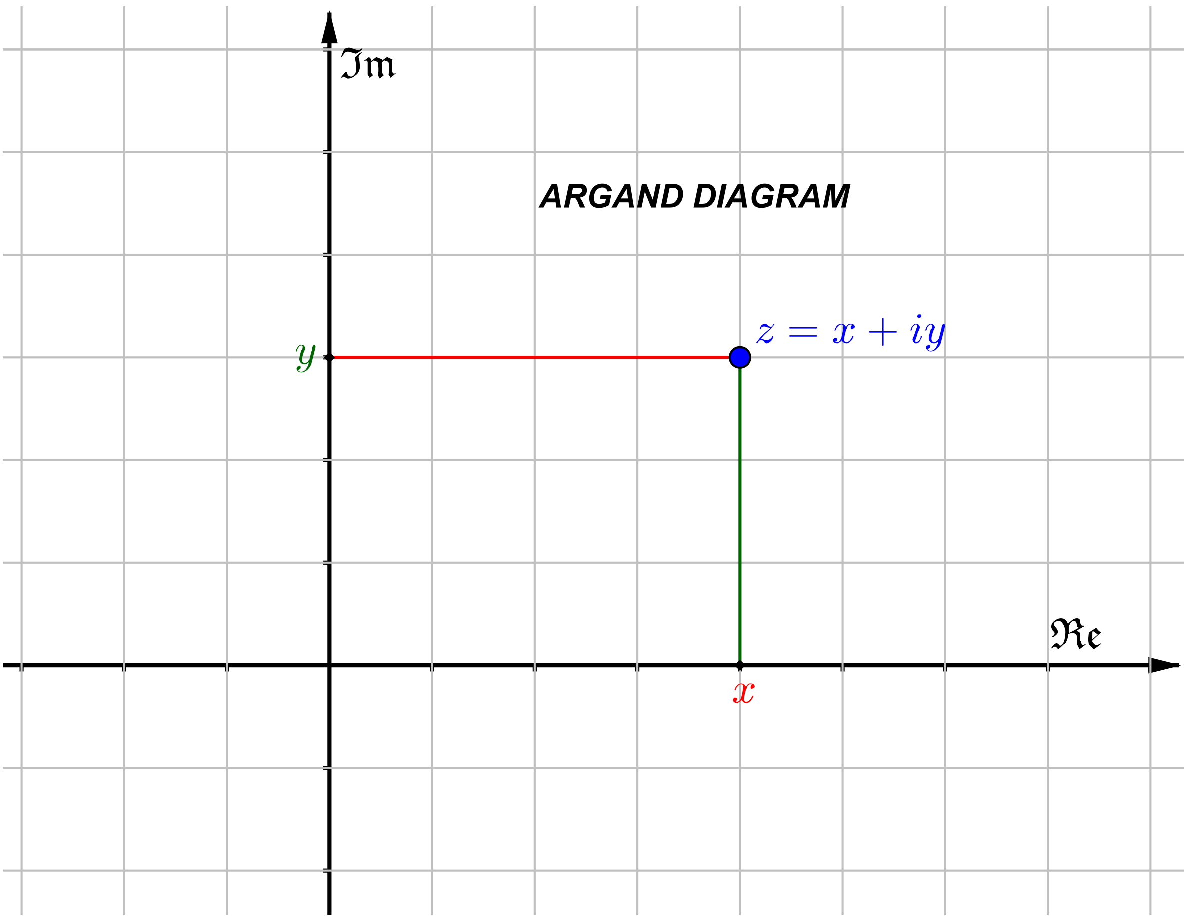
Division, however, is not so simple, as we will soon see.

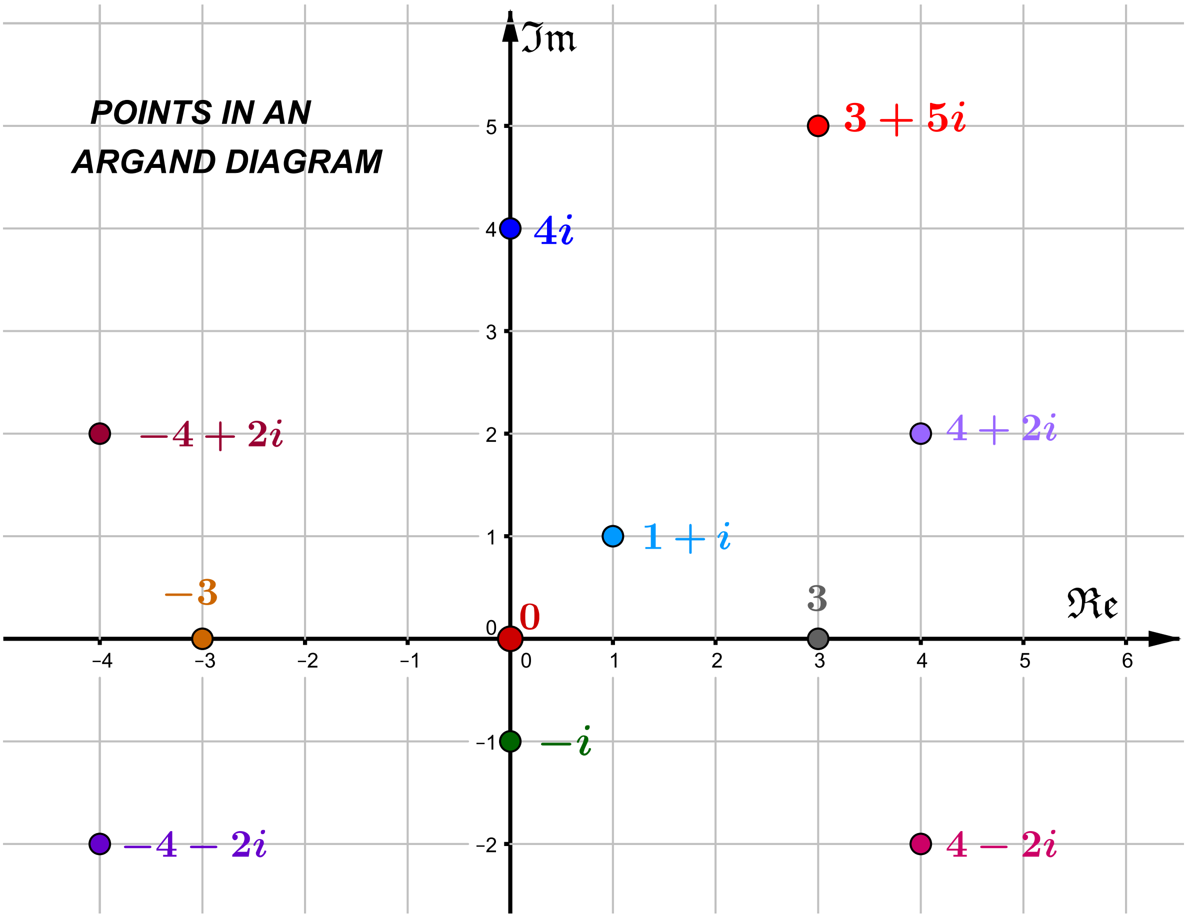
Argand Diagram

By making the association $z = x + iy \longleftrightarrow (x, y)$, we can represent the complex number z as the point in the plane with coordinates (x, y).

This gives an **Argand Diagram**.

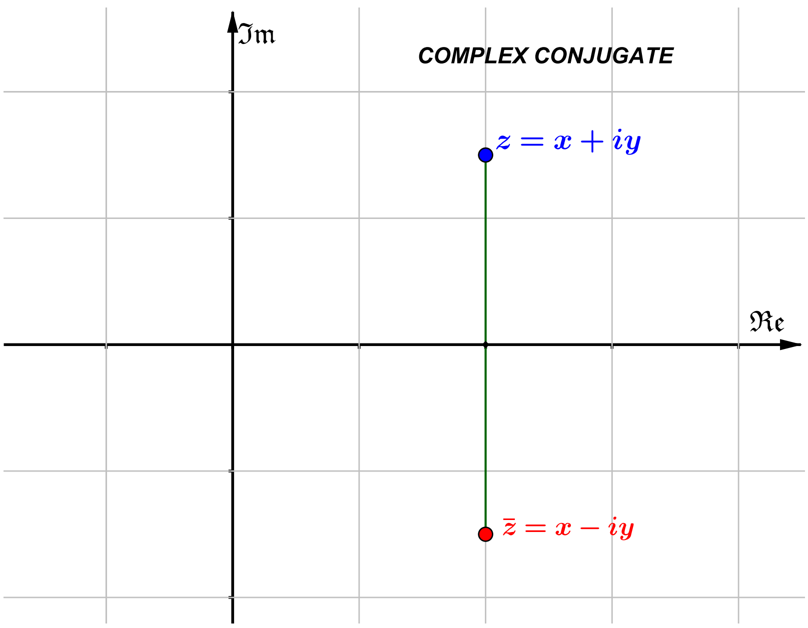
The x-axis is called the *real axis* and the y-axis is called the *imaginary axis.*





Complex Conjugate

The *complex conjugate* of the complex number $z = x + iy$ is defined to be $\bar{z} = x - iy$. That is, $\bar{z}$ is the mirror image of z under reflection in the real axis.



Properties of Complex Conjugate

- $\overline{\overline{z}} = z$
- $\overline{z} = z$ if z is real.
- $\overline{z} = -z$ if z is pure imaginary. (To see this, put $z = iy$ for some real y. Then $\overline{z} = -iy = -z$.)
- $(\overline{z+w}) = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- A very useful fact is that

  $\overline{z}z = (x - iy)(x + iy) = x^2 + ixy - ixy - i^2y^2 = x^2 + y^2$.

  So $\overline{z}z = z\overline{z}$ is real, and moreover $z\overline{z} \geqslant 0$.

Example Verify that $\overline{zw} = \overline{z}\,\overline{w}$ for the complex numbers $z = 1 + 2i$ and $w = 4 - 3i$.

First

$zw = (1+2i)(4-3i) = 4+8i-3i-6i^2 = 4+5i+6 = 10+5i$.

So its complex conjugate is $\overline{zw} = 10 - 5i$.

Next $\overline{z} = 1 - 2i$ and $\overline{w} = 4 + 3i$.

So $\overline{z}\,\overline{w} = (1 - 2i)(4 + 3i) = 4 - 8i + 3i - 6i^2 = 10 - 5i \quad \checkmark$.

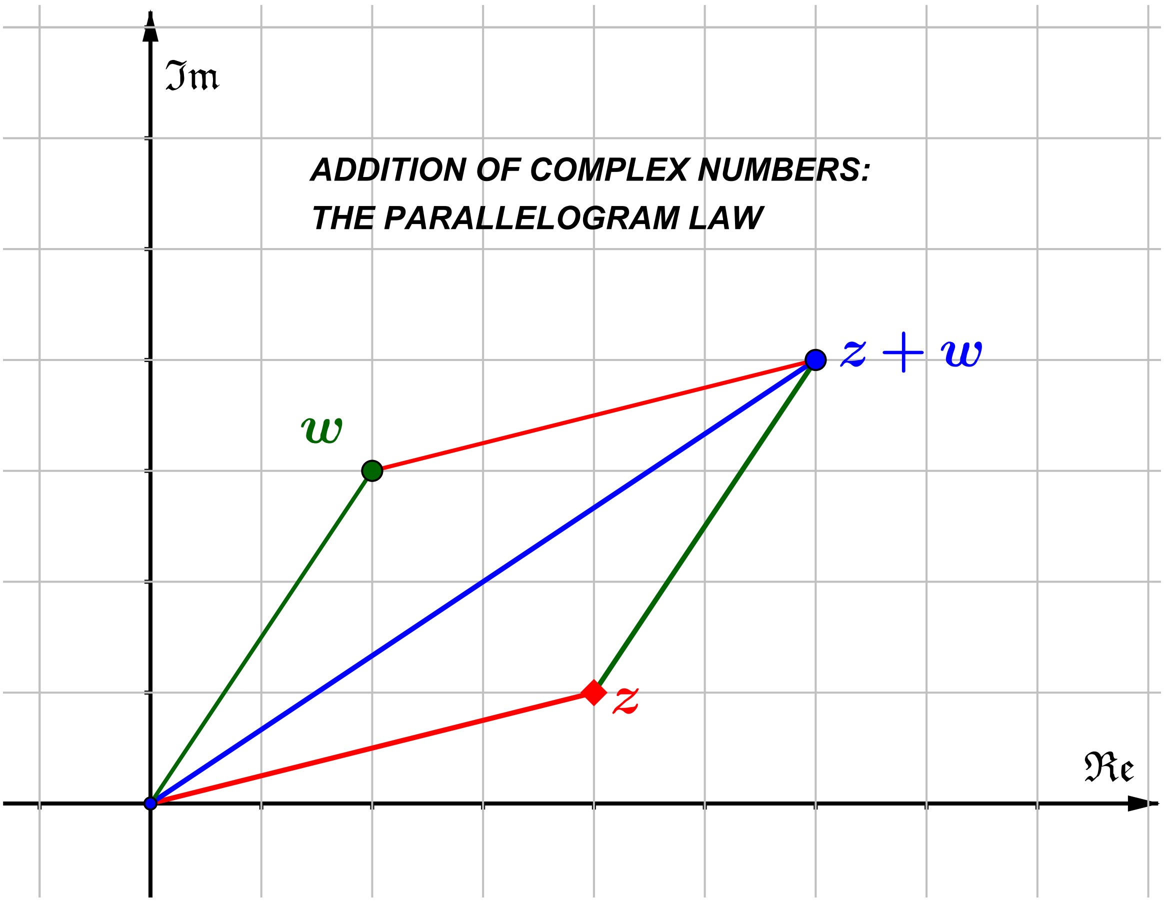
Addition of Complex Numbers (Argand Diagram)

The complex number $z = a + ib$ in an Argand diagram can be thought of as associated with the **vector** $\mathbf{v} = \langle a, b \rangle$ in $\mathbb{R}^2$.

Addition of complex numbers is then analogous to vector addition.

To add the complex numbers z and w, place them "head-to-tail", thereby forming a parallelogram. The sum $z + w$ is then the diagonal of the parallelogram.

This is called the **Parallelogram Law**.



Absolute Value of z

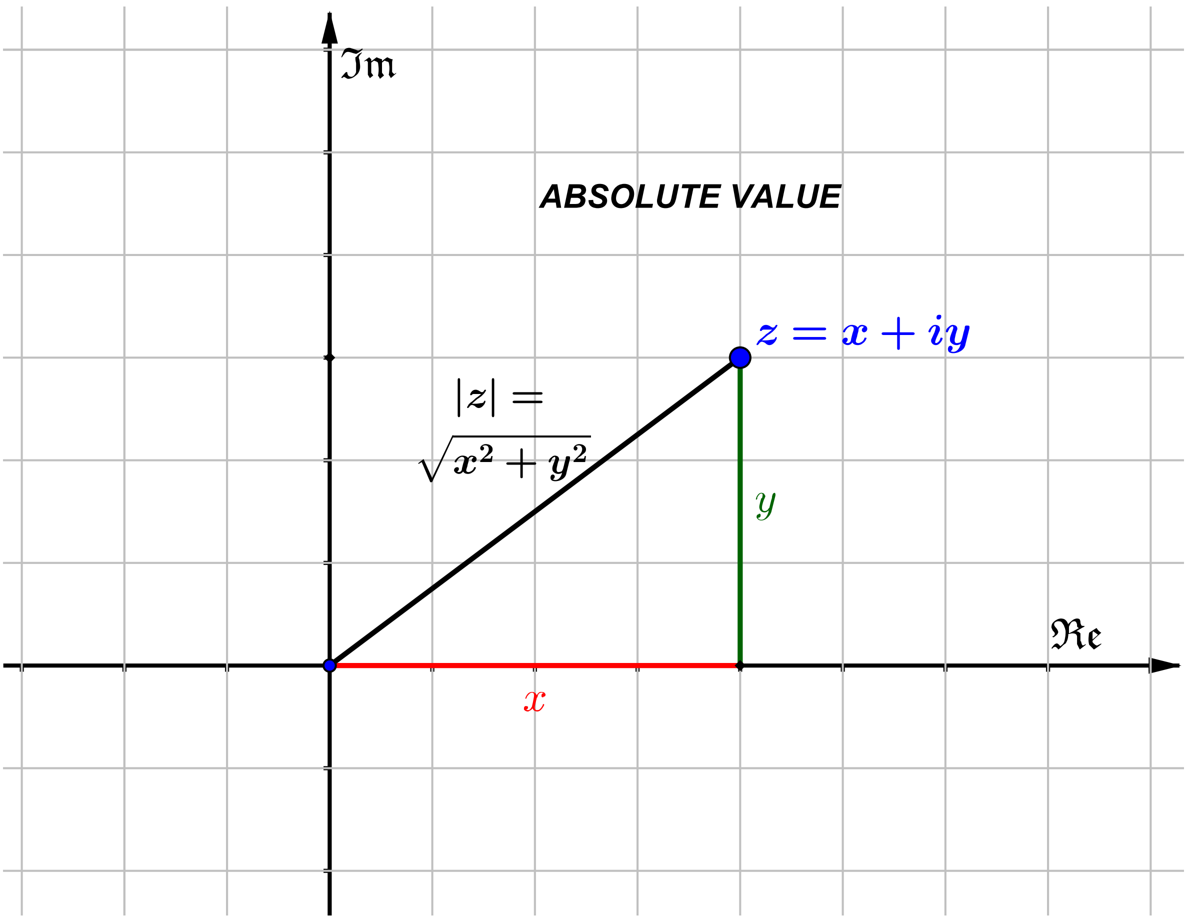
If $z = x + iy$, we define its absolute value to be $|z| = \sqrt{x^2 + y^2}$.

Notice that i does not appear in this definition, that is the absolute value is a *real number*. (The absolute value is sometimes called the *modulus* of z.)

For example, $|4 + 3i| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$.

The absolute value represents the *distance* of the complex number z from the origin in an Argand diagram.

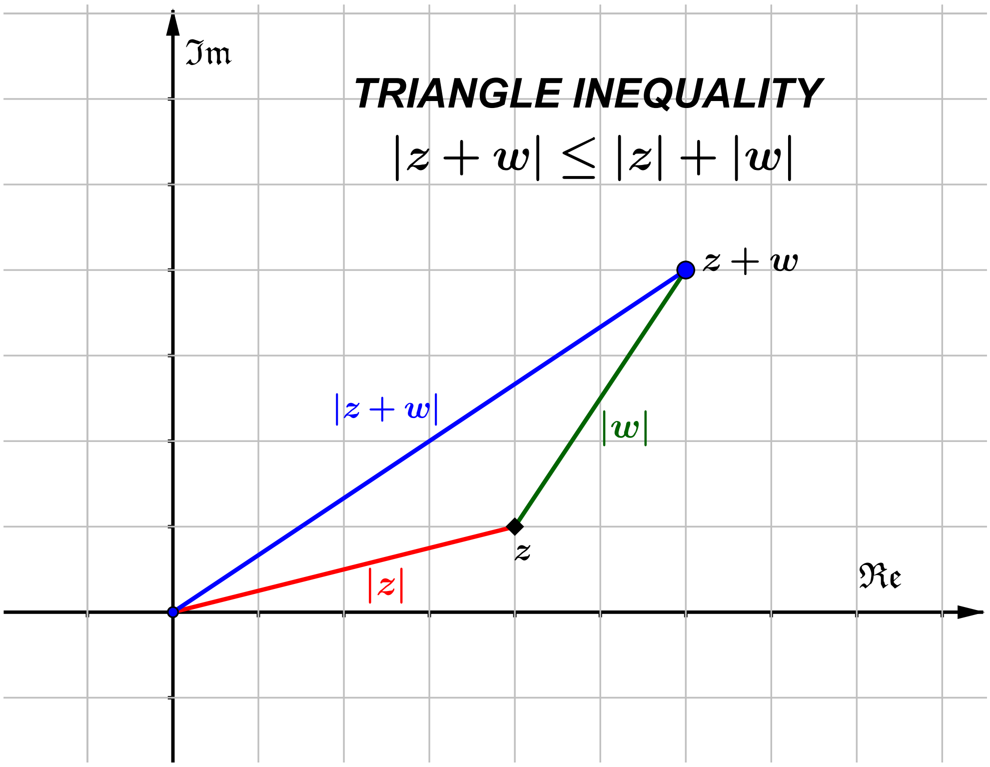
By Pythagoras's theorem, we see $x^2 + y^2 = |z|^2$.



Properties of Absolute Value

1. $|z| \geqslant 0$. Indeed if $z \neq 0$, then $|z| > 0$.
2. The "Triangle Inequality" which is $|z + w| \leqslant |z| + |w|$. (See next page for picture which justifies this.)
3. $|zw| = |z||w|$.
4. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ if $w \neq 0$.
5. $z\bar{z} = |z|^2$.

   This follows since we showed that $z\bar{z} = x^2 + y^2$, which by definition equals $|z|^2$.



The length of the blue side of the triangle is less than or equal to the sum of the lengths of the red and green sides.

Division of Complex Numbers

What does it mean to *divide* by a complex number?

For example, $(1+3i)(3+4i) = -9+13i$,

so it makes sense to say $\dfrac{-9+13i}{3+4i} = 1+3i$.

But how do we do this in general?

The key comes from the formula we had above: $z\bar{z} = |z|^2$.

We can use this to *define* $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$, when $z \neq 0$.

(Note the division here is by the *real number* $|z|^2$.)

For example, if $z = 3+4i$, then $\bar{z} = 3-4i$ and

$|z|^2 = 3^2 + 4^2 = 25$, so $\dfrac{1}{3+4i} = \dfrac{3-4i}{25} = \dfrac{3}{25} - \dfrac{4}{25}i$.

To do complex division, we use this as follows:

we write $\dfrac{w}{z} = w\left(\dfrac{1}{z}\right) = w\left(\dfrac{\bar{z}}{|z|^2}\right) = \dfrac{w\bar{z}}{|z|^2}$.

Here the denominator is real so division is done in the usual way for real numbers.

Procedure for Division of Complex Numbers

To evaluate the quotient $\dfrac{w}{z}$, we just

multiply the numerator and the denominator by $\bar{z}$,

that is by the complex conjugate of the denominator!

That is, $\dfrac{w}{z} = \dfrac{w}{z} \times \dfrac{\bar{z}}{\bar{z}} = \dfrac{w\bar{z}}{z\bar{z}} = \dfrac{w\bar{z}}{|z|^2}$.

Examples (Note that the denominators will all involve $z\bar{z}$. If $z = x + iy$, we saw that $z\bar{z} = x^2 + y^2$.)

Write each of the following in the form $a + ib$:

(In each case, we'll multiply above and below by the complex conjugate of the denominator.)

1. $$\frac{5+12i}{2-3i} = \frac{(5+12i)(2+3i)}{(2-3i)(2+3i)} = \frac{(5+12i)(2+3i)}{2^2+(-3)^2}$$

$$= \frac{10+24i+15i+36i^2}{4+9} = \frac{10+39i-36}{13} = \frac{-26+39i}{13}$$

$$= -\frac{26}{13} + \frac{39i}{13} = -2+3i.$$

2. $$\frac{-9+13i}{3+4i} = \frac{(-9+13i)(3-4i)}{(3+4i)(3-4i)}$$

$$= \frac{-27+36i+39i-52i^2}{9-16i^2} = \frac{-27+75i+52}{9+16}$$

$$= \frac{25+75i}{25} = 1+3i.$$

3. *Prove that* $(\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta)$.

Using odd/even properties of cos/sin, this is the same as asking to prove that $\dfrac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$.

$$LHS = \frac{(1)(\cos\theta - i\sin\theta)}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{(\cos\theta)^2 + (\sin\theta)^2} = \frac{\cos\theta - i\sin\theta}{1} \quad \checkmark$$

Solving equations with complex numbers

Sometimes we can solve equations with complex variables in much the same way we solve equations with real variables.

For example, solve $2z - 2 - 5i = iz + 4 + 3i$.

Rearrange to get $2z - iz = 2 + 5i + 4 + 3i = 6 + 8i$.

So $(2 - i)z = 6 + 8i$ and hence

$$z = \frac{6 + 8i}{2 - i} = \frac{(6 + 8i)(2 + i)}{(2 - i)(2 + i)} = \frac{4 + 22i}{5} = \frac{4}{5} + \frac{22}{5}i.$$

Often a complex equation can be treated as two real equations, namely the real and imaginary parts.

Example 1 *Solve* $z + 2i\bar{z} = 5 + 7i$.

Write $z = x + iy$.

Then $(x + iy) + 2i(x - iy) = 5 + 7i$.

That is, $x + iy + 2ix + 2y = 5 + 7i$.

That is, $x + 2y + i(y + 2x) = 5 + 7i$.

Equating real parts and imaginary parts, we get two equations: $x + 2y = 5$ and $y + 2x = 7$.

Solving these two simultaneous linear equations gives $x = 3$ and $y = 1$. So the solution is $z = 3 + i$.

Example 2 *Find all real numbers t such that $i(t-3i)^2$ is real.*

Expand to get $i(t-3i)^2 = i(t^2-6it-9) = it^2-6i^2t-9i$ $= it^2+6t-9i = 6t+i(t^2-9)$.

For this number to be real, its imaginary part must be zero.

That is, $t^2-9=0$.

That is, $t^2=9$. So $t=\pm 3$.

Solving Quadratic Equations

Complex numbers allow us to solve *all* quadratic equations.

For example, $x^2+4x+13=0$ gives

$$x = \frac{-4\pm\sqrt{4^2-4(1)(13)}}{2(1)} = \frac{-4\pm\sqrt{16-52}}{2}$$

$$= \frac{-4\pm\sqrt{-36}}{2} = \frac{-4\pm 6i}{2} = -2\pm 3i.$$

Notice the roots are complex conjugates of each other.

We can also solve quadratics with complex coefficients.

For example, $z^2 + (6+i)z + 11 + 3i = 0$ gives

$$z = \frac{-(6+i) \pm \sqrt{(6+i)^2 - 4(1)(11+3i)}}{2(1)}$$

$$= \frac{-6 - i \pm \sqrt{36 + 12i + i^2 - 44 - 12i}}{2} =$$

$$\frac{-6 - i \pm \sqrt{36 - 1 - 44}}{2} = \frac{-6 - i \pm \sqrt{-9}}{2} = \frac{-6 - i \pm 3i}{2}.$$

So the roots are $z = \dfrac{-6 - i + 3i}{2} = -3 + i$ and $z = \dfrac{-6 - i - 3i}{2} = -3 - 2i$. Notice the roots are **not** complex conjugates. Also solving this kind of equation usually requires taking the square root of a complex number, which we can't do (yet).

The Fundamental Theorem of Algebra

Consider now some general nth degree polynomial with complex coefficients:

$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0,$ where $a_k \in \mathbb{C}$.

(For example a quadratic or cubic or quartic *etc.*)

The **roots** are the numbers α for which $P(\alpha) = 0$.

The **Fundamental Theorem of Algebra** (FTA) states that $P(z) = 0$ has n solutions in the complex numbers, that is n roots, counting multiplicities.

So for example, **all** quadratics have two roots in the complex numbers.

The FTA allows us to write $P(z)$ as a product of factors:
$P(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n),$ where
$\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ are the roots.

More often we encounter **real** polynomials with real coefficients, that is
$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$ where $a_k \in \mathbb{R}$.

The Fundamental Theorem still applies, but now we can say more: namely if α is a complex root, then so is its complex conjugate $\bar{\alpha}$. That is, (non-real) complex roots occur in complex conjugate pairs!

For example, the real quartic $x^4 + 4x^3 + x^2 - 16x - 20$ has roots $\pm 2,$ and $-2 \pm i$.

We can prove the result using the fact that $\overline{(\alpha^k)} = (\bar{\alpha})^k$.

If $P(\alpha) = 0,$ then $\overline{P(\alpha)} = 0$, so
$\overline{(\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0)} = 0.$

So $\bar{\alpha}^n + a_{n-1}\bar{\alpha}^{n-1} + \cdots + a_1\bar{\alpha} + a_0 = 0$
(since the a_ks are all real).

So $P(\bar{\alpha}) = 0,$ that is $\bar{\alpha}$ is a root.

Example 1 Given that $2+3i$ is a root of
$P(x) = x^3 - 3x + 52,$ find the other two roots.

Since the coefficients are real and $2+3i$ is a root, then $2-3i$ is also a root.

So $(x-2-3i)(x-2+3i)$ is a factor, which after multiplying out says that $x^2 - 4x + 13$ is a factor.

So $x^3 - 3x + 52 = (x^2 - 4x + 13)(x - \alpha)$,

where α is the third root.

Comparing the constant terms, we get $52 = -13\alpha$, and hence $\alpha = -4$.

You can check that $x^3 - 3x + 52 = (x^2 - 4x + 13)(x + 4)$ is correct.

So the other roots are $2 - 3i$ and -4.

Example 2 Find the roots of $x^4 - 4$ and write it as a product of linear factors.

We factorise the difference of squares:

So $x^4 - 4 = (x^2 - 2)(x^2 + 2) = 0$,

giving $x^2 - 2 = 0$ or $x^2 + 2 = 0$.

That is $x^2 = 2$ or $x^2 = -2$.

So $x = \pm\sqrt{2}$ or $x = \pm\sqrt{2}\,i$ are the roots.

The factorisation is

$x^4 - 4 = (x - \sqrt{2})(x + \sqrt{2})(x - i\sqrt{2})(x + i\sqrt{2})$.

Example 3 Verify that $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$ and hence write it as a product of linear factors.

First $(x^2 + 2x + 2)(x^2 - 2x + 2) \;=\; (x^2 + 2 + 2x)(x^2 + 2 - 2x)$

$= \;(x^2 + 2)^2 - (2x)^2 \;=\; x^4 + 4x^2 + 4 - 4x^2 \;=\; x^4 + 4 \quad \checkmark.$

Now $x^2 + 2x + 2 = 0$ has solutions $x = -1 \pm i$.

So $x^2 + 2x + 2 \;=\; (x + 1 + i)(x + 1 - i)$.

Similarly $x^2 - 2x + 2 \;=\; (x - 1 + i)(x - 1 - i)$.

So the full factorisation is

$x^4 + 4 \;=\; (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.