Chapter 2 (continued):

Optimisation of functions of two variables

Prerequisites: Optimisation from calculus of one variable. Solving linear equations. Factorising expressions.

Optimisation for functions of two variables

Review of functions of one variable

Optimisation means to maximise or minimise a function.

For a function of one variable $y = f(x)$, we saw that the maximum and minimum points occur where $f'(x) = \frac{dy}{dx} = 0$. The values of x that make the derivative zero are called *turning points* or alternatively are called *critical points*. We saw how the Second Derivative Test is used to classify critical points as maxima, minima or points of inflection. We now wish to extend these ideas to functions of two variables.

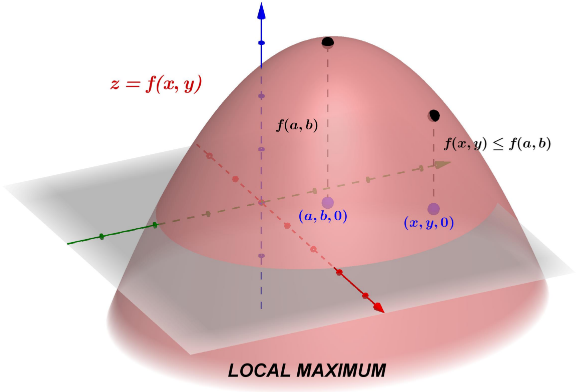
Functions of two variables

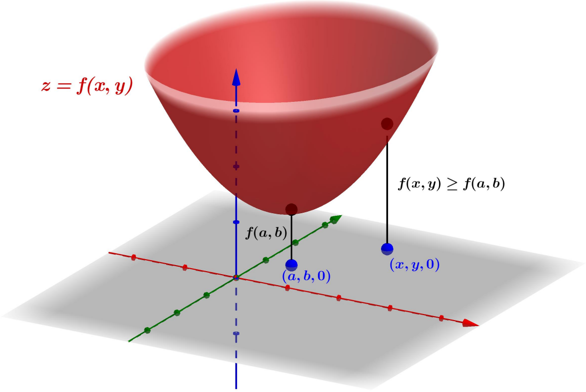
Consider a function of two variables: $z = f(x, y)$.

Thinking of $z = f(x, y)$ as a surface, it will have peaks and valleys, that is maxima and minima (of z).

The point (a, b) is called a **local maximum** of the function $z = f(x, y)$ if $f(x, y) \leqslant f(a, b)$ for all points (x, y) "near" (a, b).

The point (a, b) is called a **local minimum** of the function $z = f(x, y)$ if $f(x, y) \geqslant f(a, b)$ for all points (x, y) "near" (a, b).





LOCAL MINIMUM

We want to find numbers x and y that maximise (minimise) z.

The two definitions are not very practical so, as with a single variable function, we resort to calculus.

Note that there are two variables, each with its own partial derivative, so it is not surprising that the condition for a *critical point* of $z = f(x, y)$ is that

$$\frac{\partial z}{\partial x} = 0 \qquad \textbf{and} \qquad \frac{\partial z}{\partial y} = 0.$$

These are two **simultaneous equations** in x and y which must be solved to find (x, y).

We define a *critical point* of $z = f(x, y)$ to be a **pair** of numbers (x, y) that satisfies both these equations.

Local maxima and minima are examples of **critical points** of the function f.

There is a third type called a *saddle point*, which we will come back to.

At this stage, if we find a critical point, we cannot be sure what type it is (local maximum, local minimum or saddle point), though sometimes common sense will tell us.

Again, we will return to the issue of classifying critical points later, but for now let us do some examples of finding critical points.

Example 1

Find the critical points of $z = x^2 - 4xy + 3y^2 + 8y$.

First we need the partial derivatives of z:

$$\frac{\partial z}{\partial x} = 2x - 4y \qquad \frac{\partial z}{\partial y} = -4x + 6y + 8.$$

Next we set these equal to zero.

This gives the simultaneous equations:

$$\begin{cases} 2x - 4y &= 0 \qquad (\mathbf{A}) \\ -4x + 6y + 8 &= 0 \qquad (\mathbf{B}) \end{cases}$$

Then $2 \times (\mathbf{A}) + (\mathbf{B})$ gives $-2y + 8 = 0$, so $y = 4$.

Then equation $(\mathbf{A})$ gives $x = 8$.

So there is a unique critical point $(x, y) = (8, 4)$.

Example 2

Find the critical points of $z = x^2 - 2xy + 2x + 4y$.

We put the partial derivatives of z equal to zero:

$$\frac{\partial z}{\partial x} = 2x - 2y + 2 = 0 \qquad \frac{\partial z}{\partial y} = -2x + 4 = 0.$$

These give the simultaneous equations:

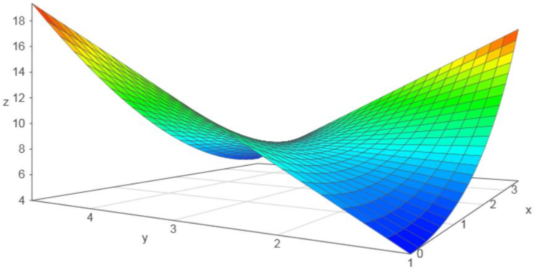
$$\begin{cases} 2x - 2y + 2 &= 0 \qquad (\mathbf{A}) \\ -2x + 4 &= 0 \qquad (\mathbf{B}) \end{cases}$$

$(\mathbf{B})$ gives $x = 2,$ and from $(\mathbf{A})$ we then get $y = 3$.

So there is a unique critical point $(x, y) = (2, 3)$.

If we want to plot this in three dimensions, we'll need $z = 2^2 - 2(2)(3) + 2(2) + 4(3) = 8,$ so the point is $(2, 3, 8)$.

This is a saddle point (-we'll meet these soon).



Example 3

Find the critical points of $z = x^3 + y^3 - 12x - 3y$.

We put the partial derivatives of z equal to zero:

$$\frac{\partial z}{\partial x} = 3x^2 - 12 = 0 \qquad \frac{\partial z}{\partial y} = 3y^2 - 3 = 0.$$

The first equation is $x^2 = 4$, with solutions $x = \pm 2$.

The second equation is $y^2 = 1$, with solutions $y = \pm 1$.

To find all solutions, we take each solution of the first equation with each solution of the second equation giving 2×2 solutions. So there are four critical points, namely $(2, 1)$, $(2, -1)$, $(-2, 1)$, $(-2, -1)$.

Example 4

Find the critical points of $z = x^4 - 2x^2y^2 + 8y^2$.

We put the partial derivatives of z equal to zero:

$$\frac{\partial z}{\partial x} = 4x^3 - 4xy^2 = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = -4x^2y + 16y = 0.$$

These are two simultaneous nonlinear equations in x and y.

Such equations can often be very difficult to solve.

A key to solving these is to factorise each equation (not losing any solutions) and combine the results!

A second key is that when the product of two or more factors gives zero, at least one of the factors must be zero.

The first equation factorises into $4x(x^2 - y^2) = 0$.

The second equation factorises into $4y(4 - x^2) = 0$.

From the second equation, we deduce that $y = 0$ or $x^2 = 4$.

That is, $y = 0$ or $x = 2$ or $x = -2$.

We treat these as three cases and use the first equation.

Case 1 $y = 0$.

The first equation gives $4x^3 = 0$, so $x = 0$.

This gives the critical point $(0, 0)$.

Case 2 $x = 2$.

The first equation gives $8(4 - y^2) = 0$, so $y = \pm 2$.

This gives two critical points $(2, 2)$ and $(2, -2)$.

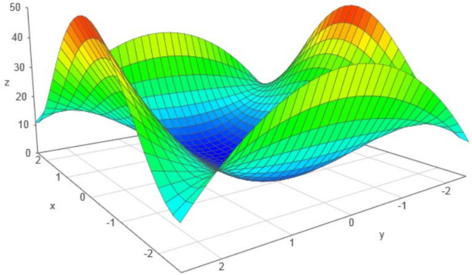
Case 3 $x = -2$.

The first equation gives $-8(4 - y^2) = 0$, so $y = \pm 2$.

This gives two critical points $(-2, 2)$ and $(-2, -2)$.

In conclusion, there are five critical points, namely $(0, 0)$, $(2, 2)$, $(2, -2)$, $(-2, 2)$, $(-2, -2)$.

It looks like $(0, 0)$ is a local minimum and the other four are saddle points (which we will meet soon).

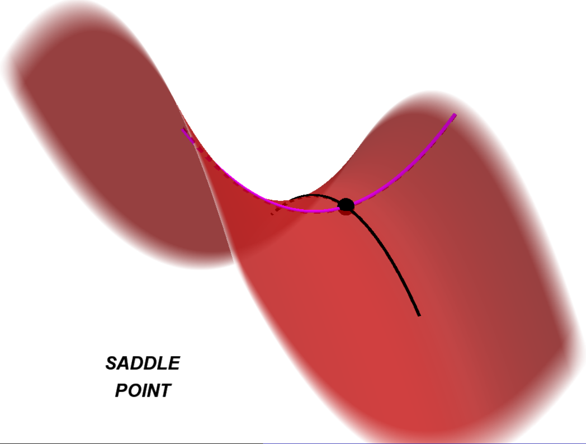


Classification of critical points

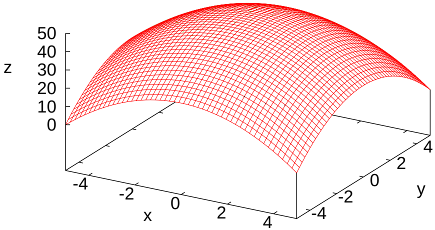
Just as for functions of a single variable, where the second derivative plays a crucial role in deciding whether a critical point is a local maximum a local minimum or a point of inflection, here too the second derivative plays a role but in a more complicated way.

The main difference is that for a function of two variables, we do not have points of inflection, but rather may have *saddle points*. A *saddle point* is a cross between and maximum and minimum.

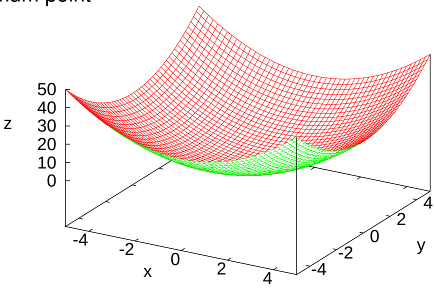
From one direction it looks like a maximum on a curve, from another it looks like a minimum.



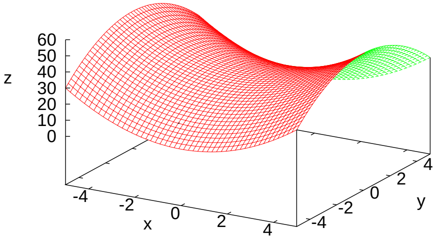
Maximum point



Minimum point



Saddle point



Conor Pass, Co. Kerry. Walking from hilltop to hilltop, the pass is a minimum. Crossing over the pass (by road) it is a maximum.

Ornak Pass, Tatras mountains, Poland.

The M-test (Second Derivative Test)

To tell if a critical point is a local maximum, a local minimum, a saddle point or something else, we use the M**-Test**.

The M-Test

Suppose (a, b) is a critical point of $f(x, y)$.

That is $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Define M to be the number got by evaluating $f_{xx}f_{yy} - f_{xy}^2$ at the point (a, b).

- If $M < 0$, then (a, b) is a saddle point.
- If $M > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local minimum.
- if $M > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local maximum.
- If $M = 0$, the test fails (does not apply).

Note: In the second and third cases, we can use $f_{yy}(a, b)$ in place of $f_{xx}(a, b)$ instead, if we wish.

Observe that the conditions for a local maximum or minimum resemble the Second Derivative Test for a function of one variable.

Examples

1. *Find and classify the critical points of* $f(x,y) = x^2 + xy + 4x + 2y$.

First we find the critical points by calculating the partial derivatives and then putting them equal to zero. That is we solve

$$\begin{aligned} f_x(x,y) &= 2x + y + 4 &= 0, \\ f_y(x,y) &= x + 2 &= 0. \end{aligned}$$

The second equation gives $x = -2$.

Substituting this into the first gives $y = 0$.

So there is a unique critical point, namely $(-2, 0)$.

To classify it, we need the second derivatives:

$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{xy} = 1.$

So $f_{xx}f_{yy} - f_{xy}^2 = (2)(0) - 1^2 = -1$.

This is constant, so evaluating it at $(-2, 0)$ still gives -1.

Therefore $M = -1 < 0$.

So by the M-test, $(-2, 0)$ is a saddle point.

2. *Find and classify the critical points of*

$$f(x, y) = 3x + 2y - 2x^2 - 5xy - 4y^2.$$

First we find the critical points by calculating the partial derivatives and then putting them equal to zero. That is we solve

$$\begin{aligned} f_x(x, y) &= \; 3 - 4x - 5y \; = 0, \qquad (\mathbf{A}) \\ f_y(x, y) &= \; 2 - 5x - 8y \; = 0. \qquad (\mathbf{B}) \end{aligned}$$

These are two simultaneous linear equations.

We eliminate x by taking $5 \times (\mathbf{A}) - 4 \times (\mathbf{A})$ to get $7 + 7y = 0$. So $y = -1$.

Substituting this into $(\mathbf{A})$ gives $8 - 4x = 0$, that is $x = 2$.

So there is just one critical point, namely $(2, -1)$.

To classify it, we need the second derivatives:

$f_{xx} = -4, \qquad f_{yy} = -8, \qquad f_{xy} = -5.$

So $f_{xx}f_{yy} - f_{xy}^2 = (-4)(-8) - (-5)^2 = 32 - 25 = 7.$

This is constant, so evaluating it at $(2, -1)$ still gives 7.

Therefore $M = 7 > 0$ and $f_{xx} = -4 < 0$.

So by the M-test, $(2, -1)$ is a local maximum..

3. *Find and classify the critical points of*

$$f(x,y) = 2x^4 - 8xy + y^2 - 3.$$

First we calculate the partial derivatives and put the answers equal to zero.

$$\begin{aligned} f_x(x,y) &= 8x^3 - 8y &= 0, \qquad (\mathbf{A}) \\ f_y(x,y) &= -8x + 2y &= 0. \qquad (\mathbf{B}) \end{aligned}$$

These are two simultaneous nonlinear equations.

Equation $(\mathbf{A})$ says $x^3 - y = 0$.

Equation $(\mathbf{B})$ is the simpler and gives $y = 4x$.

Now we can eliminate y by substituting this into $(\mathbf{A})$ to get $x^3 - 4x = 0$.

This is an equation in x only, which we now solve.

Factoring gives $x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2) = 0.$

This has three solutions $x = 0, \quad x = 2, \quad x = -2.$

Then equation $(\mathbf{B})$ gives the corresponding value of y.

When $x = 0,\ y = 0.$ When $x = 2,\ y = 8.$

When $x = -2,\ y = -8.$

So there are three critical points, namely $(0,0),\ (2,8),\ (-2,-8)$.

Comment: eliminating one variable is another key tool in solving systems of nonlinear equations.

Now we classify these points. First we need the second partial derivatives so we can apply the M-test.

$f_{xx} = 24x^2, \qquad f_{yy} = 2, \qquad f_{xy} = -8.$

So $f_{xx}f_{yy} - f_{xy}^2 = (24x^2)(2) - (-8)^2 = 48x^2 - 64.$

At $(0,0)$, we have $M = -64 < 0,$ so it is a saddle point.

At $(2,8)$, we have $M = (48)(4) - 64 > 0,$ so it is either a maximum or minimum.

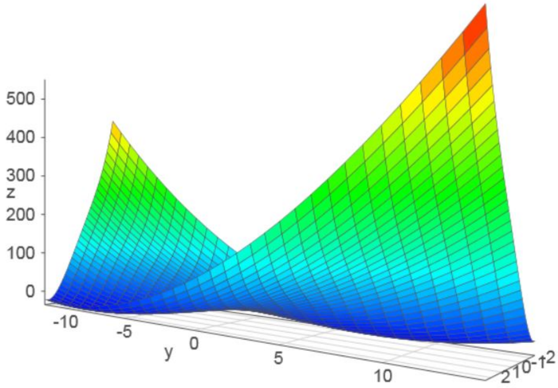
To decide, we look at $f_{xx}(2,8) = 24(2^2) > 0,$ so $(2,8)$ is a local minimum.

At $(-2,-8)$, we have $M = (48)(4) - 64 > 0.$

We also need $f_{xx}(-2,-8) = 24(-2)^2 > 0,$

so $(-2,-8)$ is also a local minimum.

In conclusion, $(0,0)$ is a saddle point, while $(2,8)$ and $(-2,-8)$ are local minima.



4. *Classify the five critical points found in the previous Example 4.*

The critical points we found were: $(0,0)$, $(2,2)$, $(2,-2)$, $(-2,2)$ and $(-2,-2)$.

These are the solutions of

$$\frac{\partial z}{\partial x} = 4x^3 - 4xy^2 = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = -4x^2y + 16y = 0.$$

To classify the points, we need the second derivatives:

$$z_{xx} = 12x^2 - 4y^2, \qquad z_{yy} = -4x^2 + 16, \qquad z_{xy} = -8xy.$$

So $z_{xx}\, z_{yy} - z_{xy}^2 = (12x^2 - 4y^2)(-4x^2 + 16) - (-8xy)^2$

$$= 16(3x^2 - y^2)(4 - x^2) - 64x^2y^2.$$

At $(0,0)$, $M = 0$, so the M-test fails!

It turns out that it is a local minimum.

At each of the other four points, $x^2 = 4$ and $y^2 = 4$, so $M = 16(12-4)(0) - 64(4)(4) < 0$, so they are all saddle points (as we predicted from the graph).

5. *Find and classify the critical points of* $f(x,y) = x^2y^2 - xy^2 - 4x^2 + 4x$.

To find the critical points, we put the two first partial derivatives equal to zero: That is we solve

$$f_x = 2xy^2 - y^2 - 8x + 4 = 0, \qquad (\mathbf{A})$$

$$f_y = 2x^2y - 2xy = 0. \qquad (\mathbf{B})$$

These are two simultaneous nonlinear equations.

Equation $(\mathbf{A})$ is messy, but equation $(\mathbf{B})$ factorises into $2(x)(y)(x-1) = 0$.

So $x = 0$ or $y = 0$ or $x = 1$.

This gives us three cases which we look at individually.

Case 1 $x = 0$.

Equation $(\mathbf{A})$ now gives $-y^2 + 4 = 0$, so $y = \pm 2$.

So we get two critical points $(0, 2)$ and $(0, -2)$.

Case 2 $y = 0$.

Equation $(\mathbf{A})$ now gives $-8x + 4 = 0$, so $x = \frac{1}{2}$.

So we get one critical point $(\frac{1}{2}, 0)$.

Case 3 $x = 1$.

Equation $(\mathbf{A})$ now gives $2y^2 - y^2 - 8 + 4 = 0$, that is $y^2 = 4$, so $y = \pm 2$.

So we get two critical points $(1, 2)$ and $(1, -2)$.

Next we need the second partial derivatives so we can classify them.

$$f_{xx} = 2y^2 - 8, \qquad f_{yy} = 2x^2 - 2x, \qquad f_{xy} = 4xy - 2y.$$

We make a table and evaluate each of these at the five points.

Then we calculate $M = f_{xx}f_{yy} - f_{xy}^2$ at each point and, if necessary, note the value of f_{xx}.

Crit.Pt.	f_{xx}	f_{yy}	f_{xy}	M	Classify
(1/2,0)	-8	-1/2	0	4	Local Max
(0,2)	0	0	-4	-16	Saddle
(0,-2)	0	0	4	-16	Saddle
(1,2)	0	0	4	-16	Saddle
(1,-2)	0	0	-4	-16	Saddle