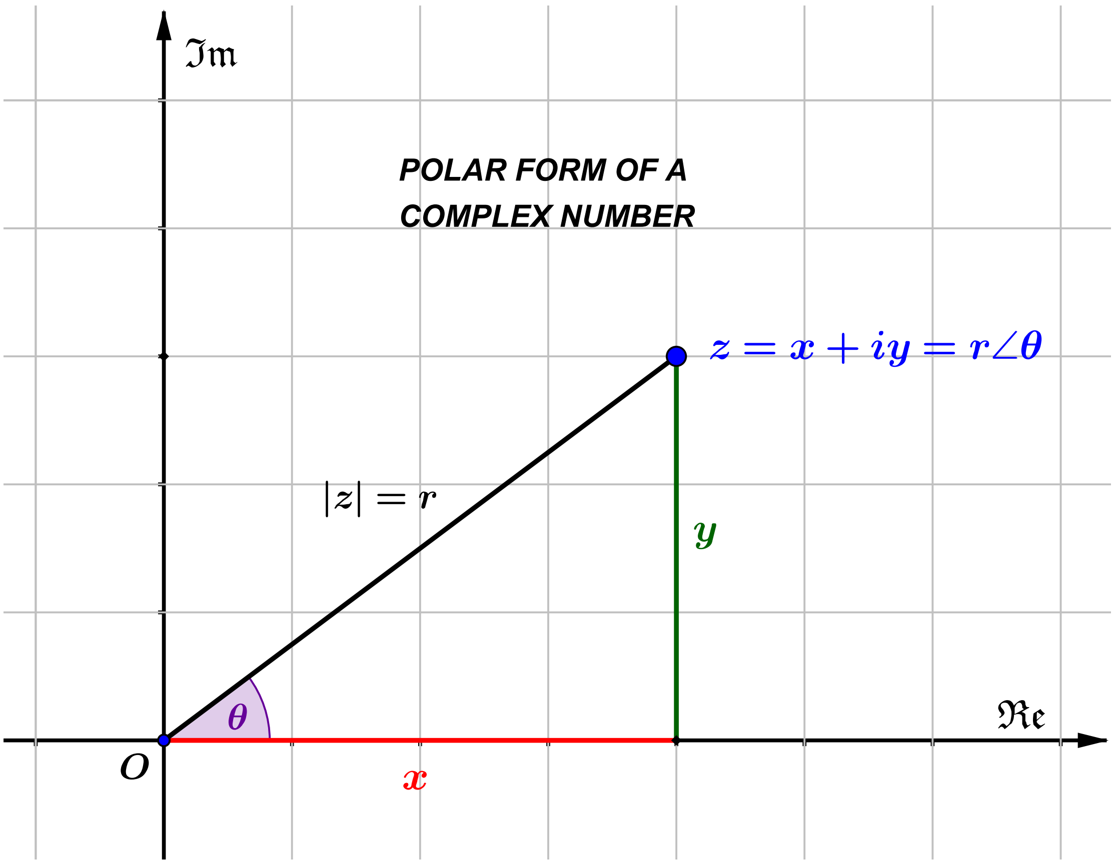
Polar Form of Complex numbers

A complex number z, when written in the form $z = x + iy$ references it to the coordinates (x, y) of the point in the Argand diagram (called the *Cartesian* or *rectangular* form of the number).

Alternatively we could specify the point in the Argand diagram by giving the distance r of z from the origin O, and the angle θ that Oz makes with the positive real-axis.

We write the complex number in polar form as $z = r\angle\theta$, read as 'r angle theta'.



Note that (from Pythagoras) $r = |z| = \sqrt{x^2 + y^2}$.

So $r \geqslant 0$.

Also $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$.

So $x = r\cos\theta$ and $y = r\sin\theta$.

Hence $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.

Hence $\angle\theta = \cos\theta + i\sin\theta$.

We can view this as the *definition* of $\angle\theta$.

See files *cartes1.ggb* and *polar2.ggb* to see how points in an Argand diagram "move around" differently using Cartesian and polar forms of the complex number.

$\angle\theta$ and the Unit Circle

First note that $\angle\theta$ has length 1, since

$$|\angle\theta|^2 = |\cos\theta + i\sin\theta|^2 = \cos^2\theta + \sin^2\theta = 1.$$

So $\angle\theta$ is the point on the unit circle (in an Argand diagram) corresponding to angle θ.

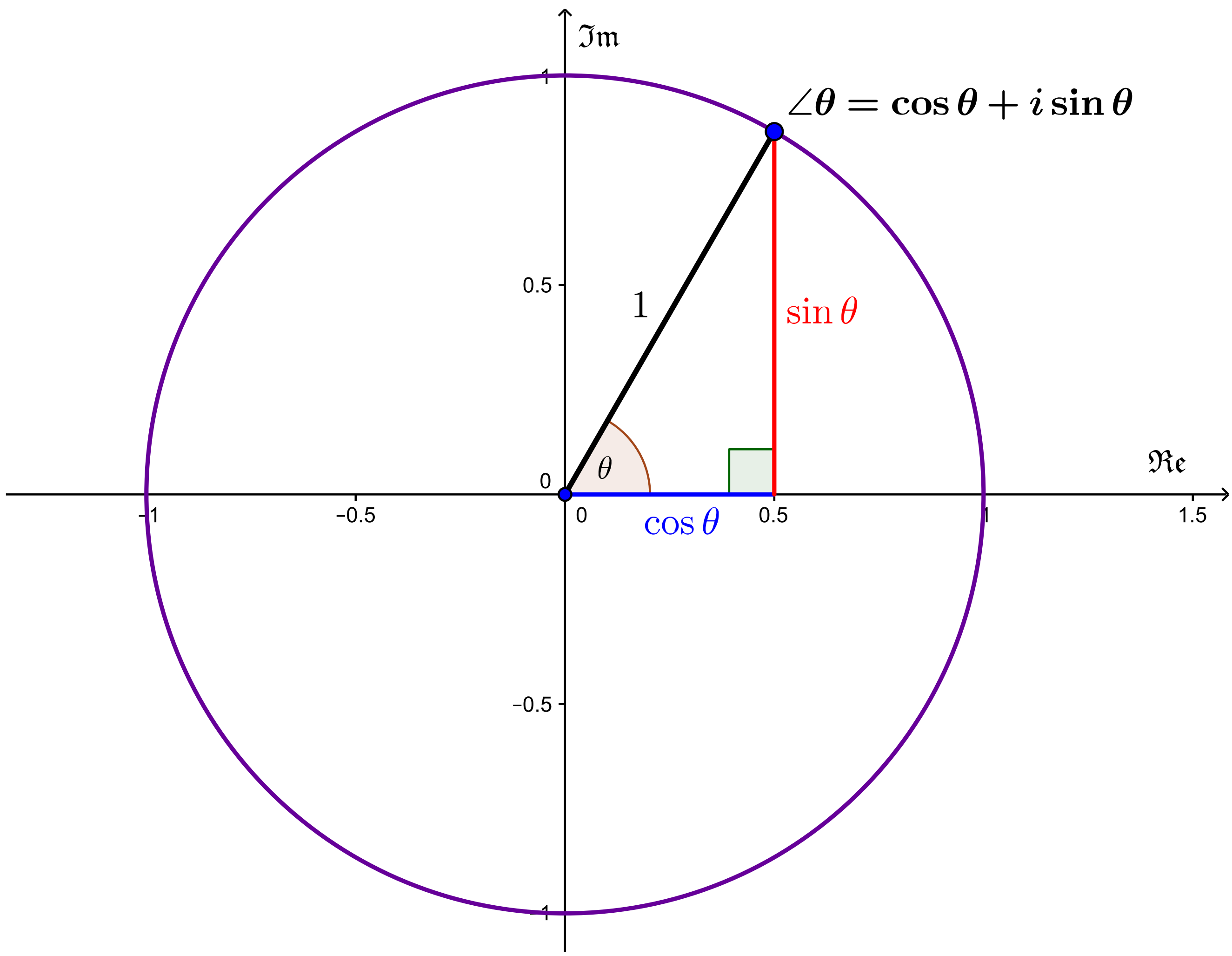
The number $\angle\theta$ shares many of the properties of cosine and sine (periodicity etc.). So we can import all our knowledge of the unit circle from trigonometry (MA4601, Week 1) and use it here.

Note that we will use *radian measure* for angles, as later when we encounter complex exponential it is essential.

Note that angles θ that differ by multiples of 2π are considered to be the same angle.

Note that θ is called the *polar angle* or sometimes the *phase* of the complex number z.

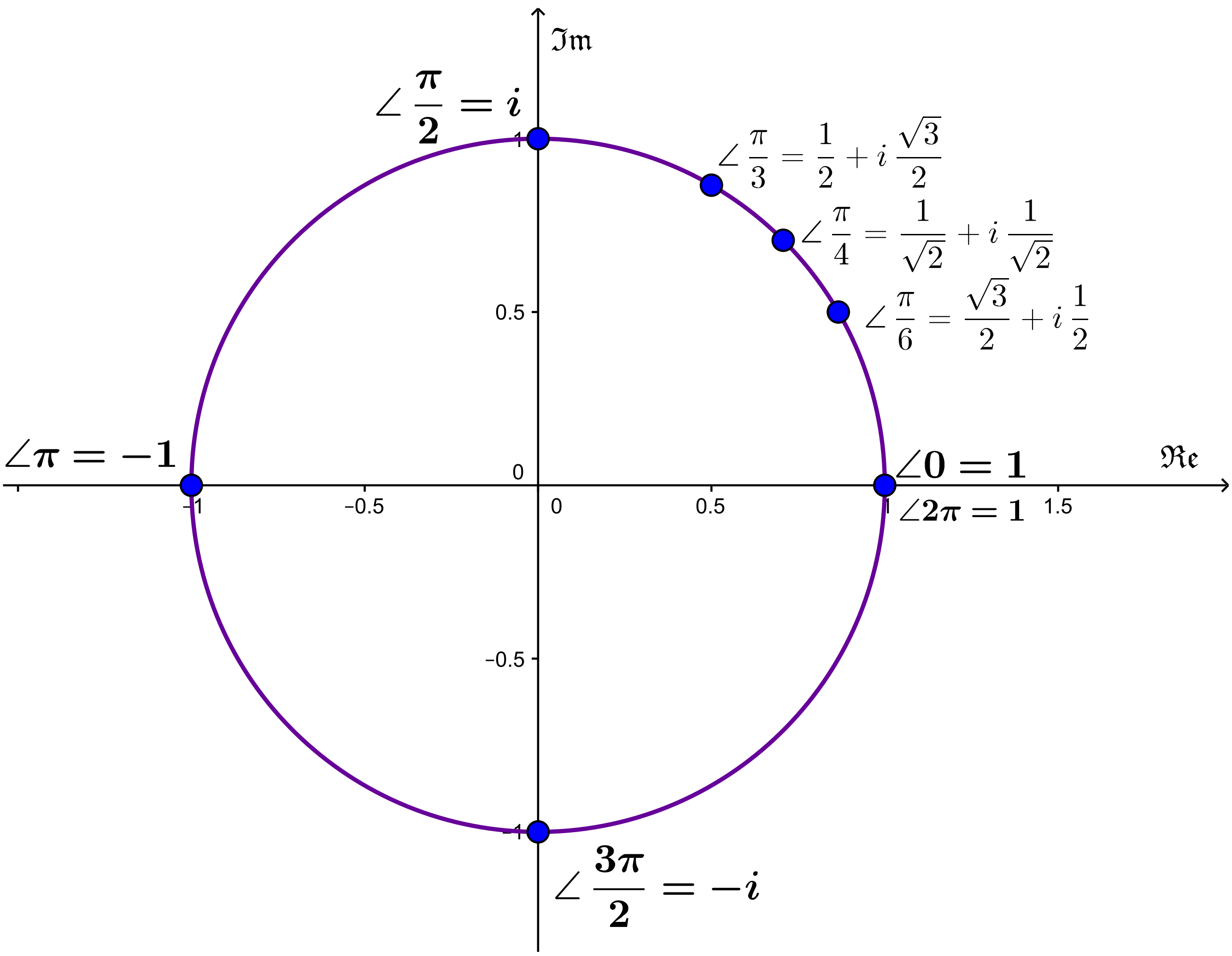
The number $\angle\theta$ in $z = r\angle\theta$ is sometimes called the *phase factor* of z.



Properties of $\angle\theta$

First some values:

- $\angle 0 \;=\; \cos 0 + i\sin 0 \;=\; 1;$
- $\angle \frac{\pi}{2} \;=\; \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \;=\; i;$
- $\angle \pi \;=\; \cos\pi + i\sin\pi \;=\; -1;$
- $\angle \frac{3\pi}{2} \;=\; \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} \;=\; -i;$
- $\angle 2\pi \;=\; \cos 2\pi + i\sin 2\pi \;=\; 1;$
- $\angle \frac{\pi}{4} \;=\; \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \;=\; \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}};$
- $\angle \frac{\pi}{6} \;=\; \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \;=\; \frac{\sqrt{3}}{2} + i\frac{1}{2};$
- $\angle \frac{\pi}{3} \;=\; \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \;=\; \frac{1}{2} + i\frac{\sqrt{3}}{2}.$



Important Properties:

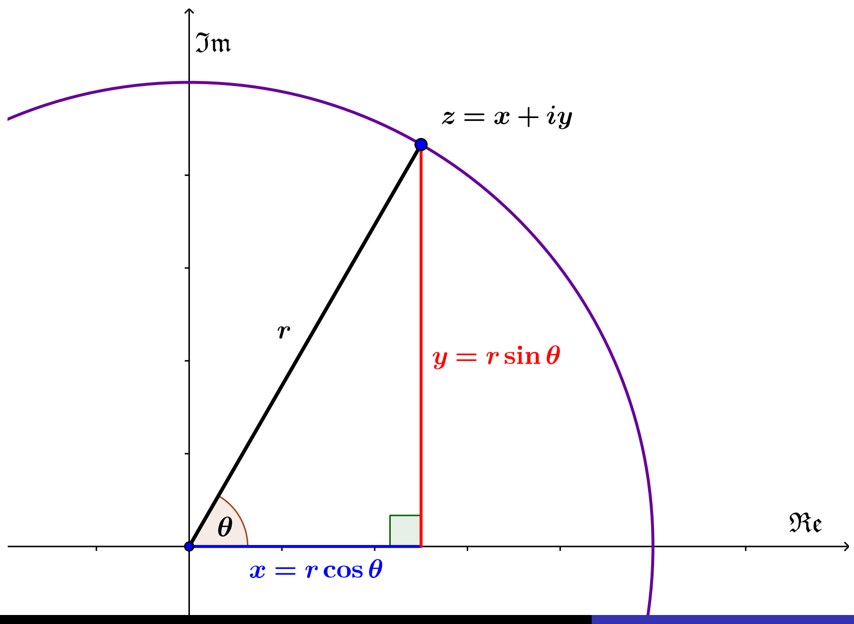
- $\angle(\theta + 2\pi) = \angle\theta$
  since $\cos(\theta + 2\pi) + i\sin(\theta + 2\pi) = \cos\theta + i\sin\theta$.
- $\angle(\theta + \pi) = -\angle\theta$.
  since $\cos(\theta + \pi) + i\sin(\theta + \pi) = -\cos\theta - i\sin\theta$.
- $\angle(-\theta) = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$
- It follows from this that $\angle(-\theta) = \overline{\angle\theta}$ (complex conjugate) and hence that $\angle\theta\angle(-\theta) = 1$.
  Hence $\angle(-\theta) = \dfrac{1}{\angle\theta}$. (We proved this before!)
- $|\angle\theta| = 1$. It is a point on the unit circle.

Converting from Cartesian to polar form

If $z = x + iy = r\angle\theta$

then, from triangle below, we get

$\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$.



So $x = r\cos\theta$ and $y = r\sin\theta$.

We can use these to write a complex number, expressed in polar form, in Cartesian form, that is, go from $(r, \theta) \longrightarrow (x, y)$.

But how do we go from Cartesian form to polar form?

That is, given (x, y), how do we get (r, θ)?

First, $r = |z| = \sqrt{x^2 + y^2}$.

To get θ, we proceed as follows:

(First assume $x \neq 0$. If $x = 0$, then z is pure imaginary, so $\theta = \pm\frac{\pi}{2}$.)

Get $\dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta$.

Then $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.

The problem with this is that your calculator only gives $\tan^{-1}$ in half the unit circle, namely $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (the first and fourth quadrants, that is $\mathfrak{Re}(z) > 0$).

If the complex number is in the second or third quadrants (that is $\mathfrak{Re}(z) < 0$), we have to add π radians.

Examples

Write $4\angle 5$ in Cartesian form.

Here $r = 4$ and $\theta = 5$ (radians).

So $x = 4\cos 5 = 1.135$ and $y = 4\sin 5 = -3.836$.

So $4\angle 5 = 1.135 - 3.836i$.

Write each of the following in polar form:

(a) $2 + 5i$ *(b)* $-3 + i$ *(c)* $6 - 3i$.

(a) $z = 2 + 5i$, so $r = \sqrt{2^2 + 5^2} = \sqrt{29} = 5.39$.

$\theta = \tan^{-1}\frac{5}{2} = \tan^{-1} 2.5 = 1.19$ (radians).

So $z = 5.39\angle 1.19$. (Correct as z is in the first quadrant.)

(b) $z = -3 + i$, so $r = \sqrt{(-3)^2 + 1^2} = \sqrt{10} = 3.16$.

$\theta = \tan^{-1}\frac{1}{-3} = \tan^{-1}(-0.3333) = -0.32$ (radians).

Since z is in the second quadrant, we must add $\pi = 3.14$ to the angle, that is $\theta = 3.14 - 0.32 = 2.82$.

So $z = 3.16\angle 2.82$.

(c) $z = 6 - 3i$, so $r = \sqrt{6^2 + (-3)^2} = \sqrt{45} = 6.71$.

Also $\theta = \tan^{-1}\frac{-3}{6} = \tan^{-1}(-0.5) = -0.46$ (radians).

Since z is in the fourth quadrant, this is correct.

So $z = 6.71\angle(-0.46)$.

We can also write $z = 6.71\angle 5.82$ (adding $2\pi = 6.28$).

Argument of a complex number $\arg(z)$

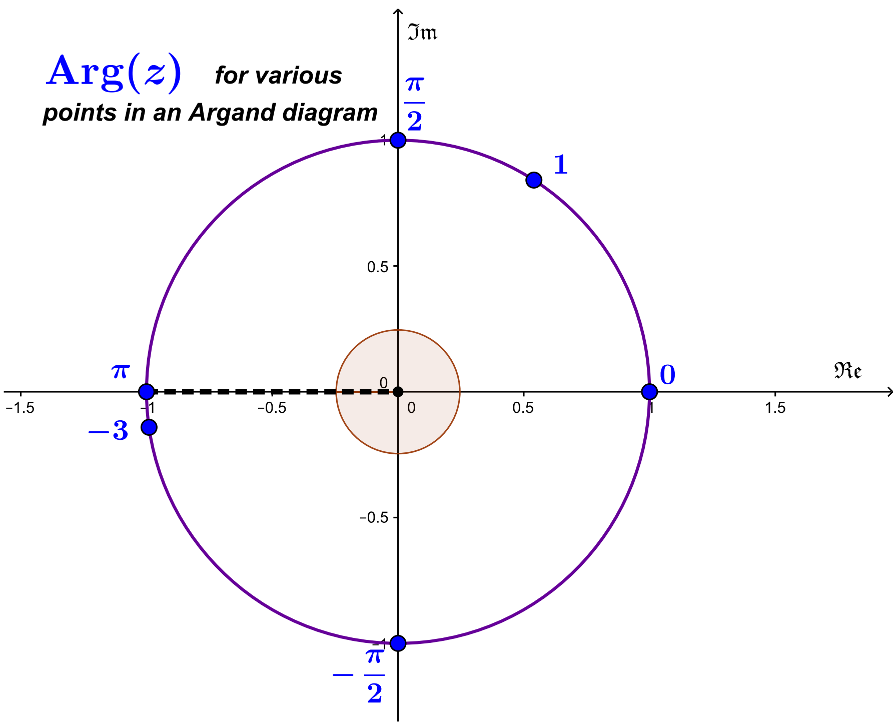
We have seen that we can write a complex number as $z = r\angle\theta = r(\cos\theta + i\sin\theta)$, where $\angle\theta$ is the point on the unit circle corresponding to angle θ.

We define $\arg(z) = \theta$.

The problem with this is that $\angle\theta$ and $\angle(\theta + 2n\pi)$ give the same complex number z, so $\arg(z)$ can have one of an infinite number of values, namely $\arg(z) = \theta + 2n\pi$, for any integer n.

For example, $\arg(3i) = \ldots -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$.

To get around this multi-valued answer, we fix a unique value by restricting $-\pi < \theta \leqslant \pi$. This gives what is called *the principal value of the argument* and is denoted by $\mathrm{Arg}(z)$. So $\mathrm{Arg}(3i) = \frac{\pi}{2}$.



So $-\pi < \mathrm{Arg}(z) \leqslant \pi$.

$\mathrm{Arg}(z)$ has a discontinuity as z crosses the negative real axis.

If $z = x + i0$ (real), then $\mathrm{Arg}(z) = 0$ if $x > 0$ and $\mathrm{Arg}(z) = \pi$ if $x < 0$.

If $z = 0 + iy$ (pure imaginary), then $\mathrm{Arg}(z) = \frac{\pi}{2}$ if $y > 0$ and $\mathrm{Arg}(z) = -\frac{\pi}{2}$ if $y < 0$.

Note that $z = r\angle\theta = |z|\angle \arg(z)$ are equivalent ways of writing z in polar form.

Exponential form of a complex number

A much more common notation for $\angle\theta$ is $e^{i\theta}$, that is

$$e^{i\theta} = \cos\theta + i\sin\theta = \angle\theta.$$

(This is called *Euler's formula*.)

This is the same exponential function as in exponential growth/decay, but with a complex number input. The connection is not obvious at this stage but as we derive its properties, we will see how it is compatible with e^x.

Writing $z = r\,e^{i\theta}$ gives the *exponential form* of z.

$e^{i\theta}$ is called the *phase factor* of the complex number z and θ is called the *phase*.

Since the exponential form is equivalent to $z = r\angle\theta$, all the properties of $\angle\theta$ listed previously apply to $e^{i\theta}$.

For example, some values:

$$e^{0i} = 1, \qquad e^{i\pi} = -1, \qquad e^{2\pi i} = 1, \qquad e^{2n\pi i} = 1,$$

$$e^{\frac{i\pi}{2}} = i, \qquad e^{\frac{3i\pi}{2}} = e^{-\frac{i\pi}{2}} = -i, \qquad e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

(The formula $e^{i\pi} + 1 = 0$ is called *Euler's identity* and is said to be the most beautiful formula in all mathematics).

Results from before are now written as: $\overline{e^{i\theta}} = e^{-i\theta} = \dfrac{1}{e^{i\theta}}$;

and $|e^{i\theta}| = 1$, that is $e^{i\theta}$ is a point on the unit circle.

A complex number can also be written in exponential form as $z = r\,e^{i\theta} = |z|e^{i\arg(z)}$.

Multiplying phase factors

Using trigonometric identities, one can show that, when we multiply two phase factors, we get $\angle\theta\angle\phi = \angle(\theta + \phi)$ that is the angles add!

(**Proof** $\angle\theta\angle\phi = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

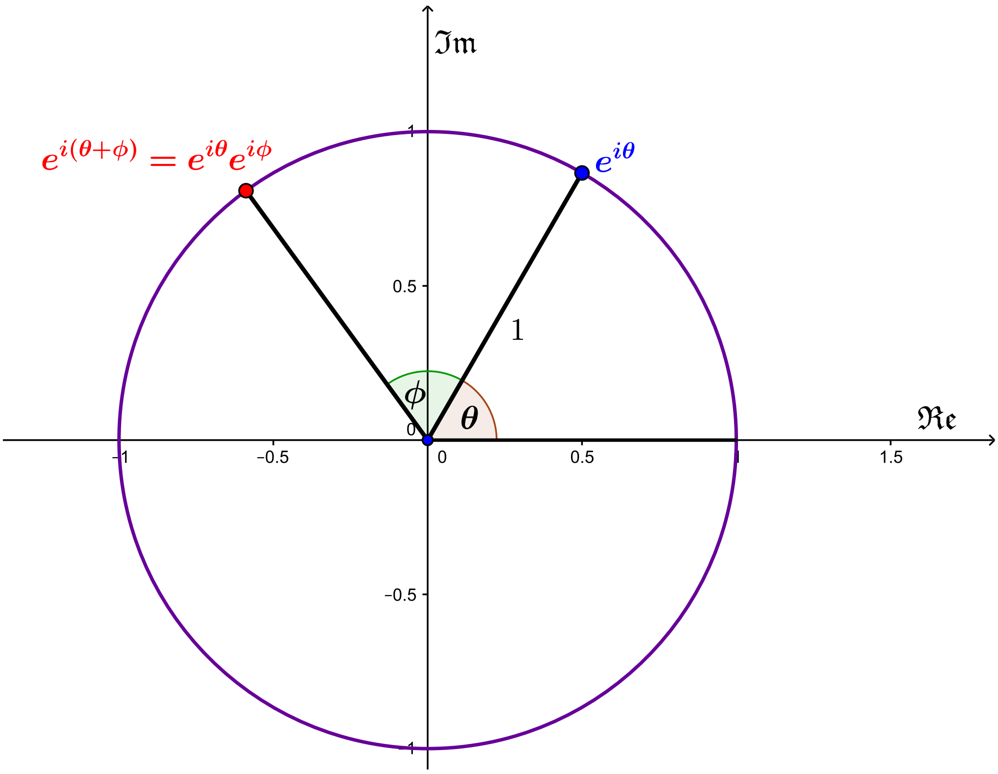
$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$= \cos(\theta + \phi) + i\sin(\theta + \phi) = \angle(\theta + \phi)$).

In exponential form this says $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$.

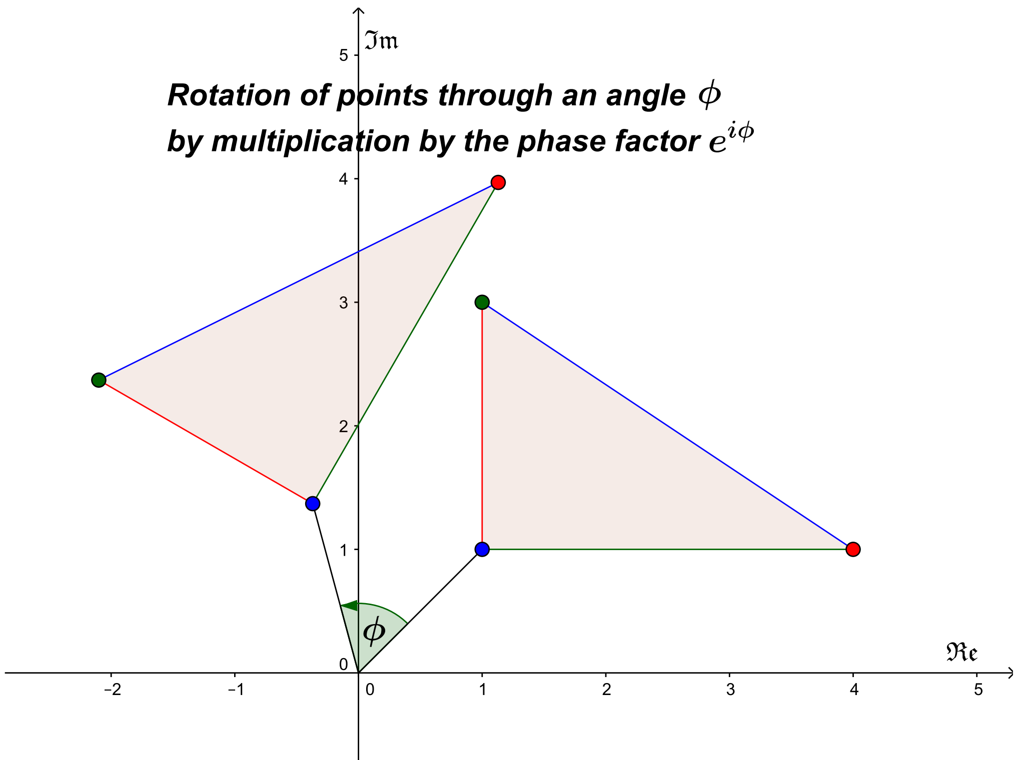
This is just the first law of exponents again!

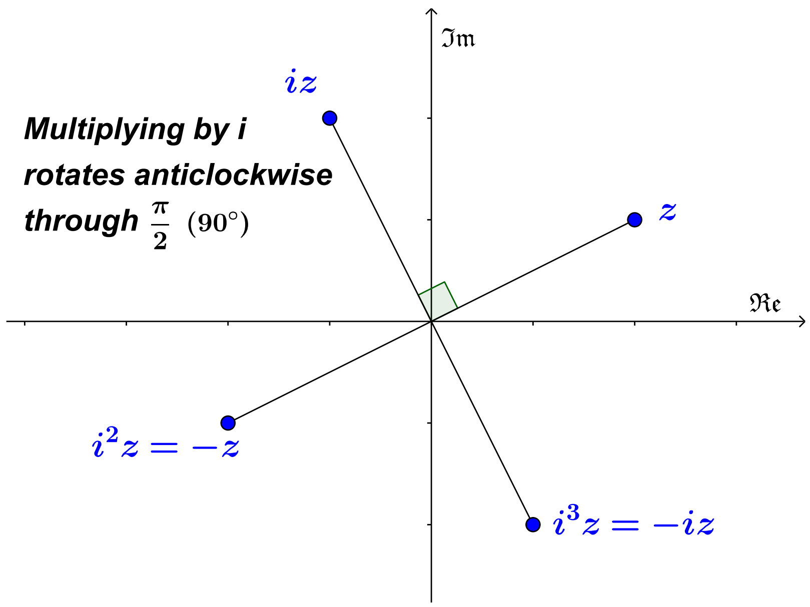
So to multiply two phase factors, we just add the angles!



Given any complex number $z = re^{i\theta}$, the complex number $e^{i\phi}z = e^{i\phi}re^{i\theta} = re^{i(\theta+\phi)}$. This is just z rotated anticlockwise about the origin through an angle ϕ!

That is, multiplying a set of complex numbers by the phase factor $e^{i\phi}$ just rotates the set through an angle ϕ.

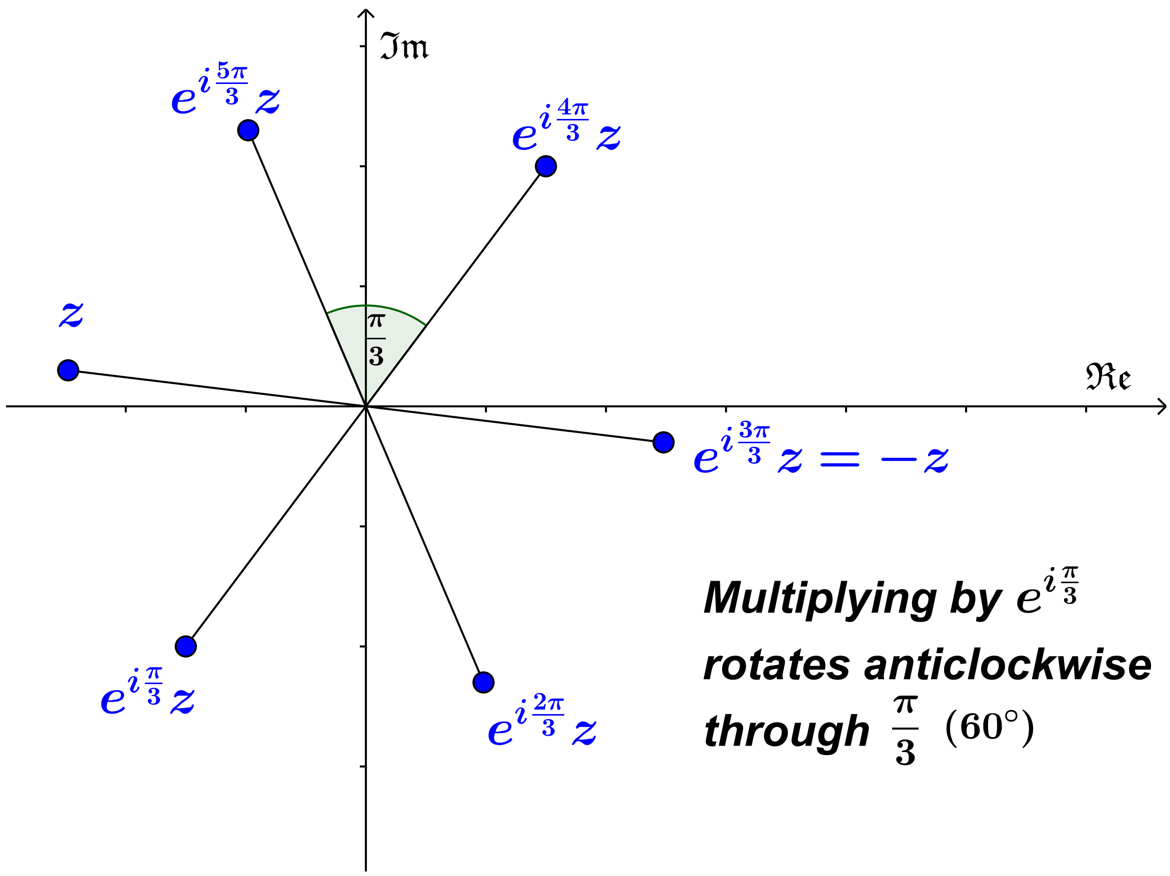




Note that

$i^3 = ii^2 = -i, \quad i^4 = \left(i^2\right)^2 = (-1)^2 = 1, \quad i^5 = i$, *etc.*

Also $i^{-1} = \dfrac{1}{i} = -i$.



These give the six “spokes” of a regular hexagon.

De Moivre's Theorem

It follows from $\angle\theta\angle\phi = \angle(\theta+\phi)$ that $\angle\theta\angle\theta = \angle(\theta+\theta) = \angle(2\theta)$.

(In exponential form we'd get $e^{i\theta}e^{i\theta} = e^{2i\theta}$.)

Consequently
$\angle\theta\angle\theta\angle\theta = \angle(2\theta)\angle\theta = \angle(2\theta+\theta) = \angle(3\theta)$.

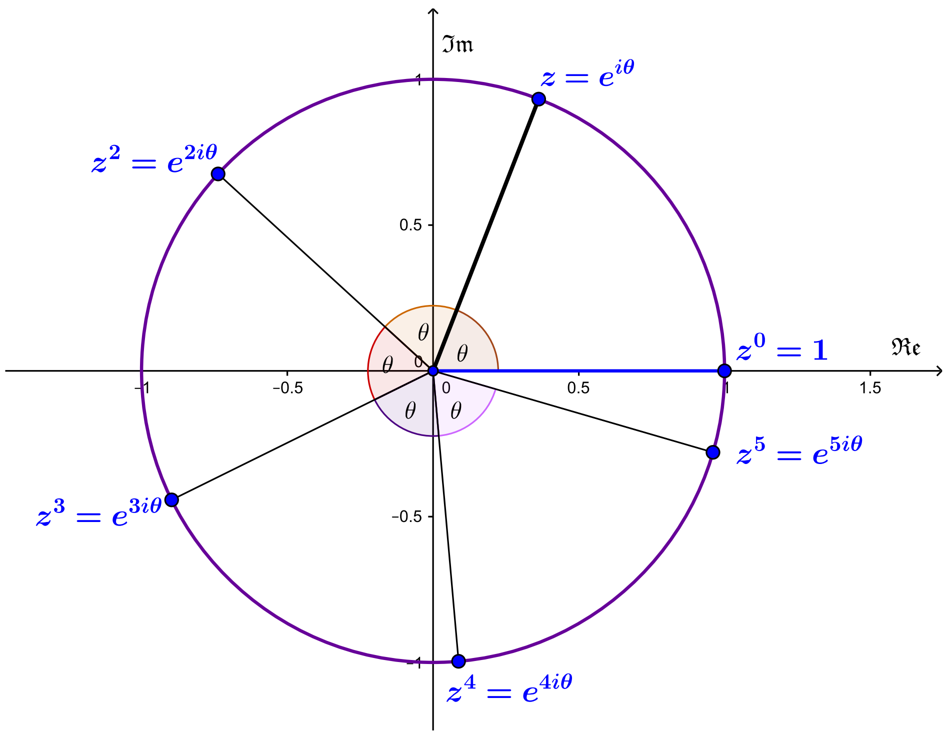
That is, $(\angle\theta)^3 = \angle(3\theta)$.

We can then prove by induction that $(\angle\theta)^n = \angle(n\theta)$, for any natural number n.

In exponential form, this gives $(e^{i\theta})^n = e^{in\theta}$, which is one of the laws of exponents again (power of power).

This is an important result, known as **de Moivre's Theorem** (DMT). Written out explicitly it states:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$



The most important applications of DMT allow us to take powers and roots of complex numbers, but they also allow us to derive various trigonometric identities involving multiple angles.

For example, $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$ gives $(\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta) = \cos(2\theta) + i\sin(2\theta)$.

Equating the real parts and imaginary parts of this equation gives the two identities:

$\cos(2\theta) = \cos^2\theta - \sin^2\theta,$ and $\sin(2\theta) = 2\cos\theta\sin\theta$.

These identities (called *double-angle formulas*) relate $\cos(2\theta)$ and $\sin(2\theta)$ to $\cos\theta$ and $\sin\theta$.

Negative powers

De Moivre's Theorem also works for negative powers since $(e^{i\theta})^{-1} = \dfrac{1}{e^{i\theta}} = e^{-i\theta}$.

So $(e^{i\theta})^{-n} = \dfrac{1}{(e^{i\theta})^n} = \dfrac{1}{e^{in\theta}} = e^{-in\theta},$ or equivalently

$(\angle\theta)^{-n} = \angle(-n\theta),$ or equivalently

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-n} &= \cos(-n\theta) + i\sin(-n\theta),\\ &= \cos(n\theta) - i\sin(n\theta)\end{aligned}$$

where n is a natural number.

Powers of a Complex Number

How do we calculate $(4+3i)^7$?

Multiplying out is painful.

However, if we convert the number into polar or exponential form first, we can use de Moivre's theorem!

If $z = r\angle\theta = re^{i\theta}$,

then $z^n = r^n(\angle\theta)^n = r^n(e^{i\theta})^n$

$= r^n\angle(n\theta) = r^n e^{in\theta}$ by DMT.

That is, $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

This formula is very useful in calculating powers.

In the example above, we find $4+3i = 5\angle 0.64350$ in polar form (exercise).

In exponential form, this becomes $4+3i = 5\,e^{0.64350i}$.

So $(4+3i)^7 = 5^7(e^{0.64350i})^7 = 5^7 e^{7\times 0.64350i} = 5^7 e^{4.50451i}$

$= 5^7(\cos(4.50451) + i\sin(4.50451))$

$= 78125(-0.20639 - 0.97847i) = -16124 - 76443i$.

More Examples

1. *Calculate* $(\sqrt{3}+i)^{15}$*.*

Put $z = \sqrt{3}+i$.

We want it in exponential form.

$r = |z| = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = 2.$

$\theta = \tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}.$

So $z = re^{i\theta} = 2e^{i\frac{\pi}{6}}$.

Then $z^{15} = 2^{15}(e^{i\frac{\pi}{6}})^{15} = 2^{15}e^{15i\frac{\pi}{6}}$ by DMT.

So $z = 2^{15}e^{i\frac{15\pi}{6}} = 2^{15}e^{i\frac{5\pi}{2}} = 2^{15}e^{i\frac{\pi}{2}} = 2^{15}i$.

2. *Calculate* $(-1+i)^{18}$*.*

Put $z = -1+i$.

We want it in exponential form.

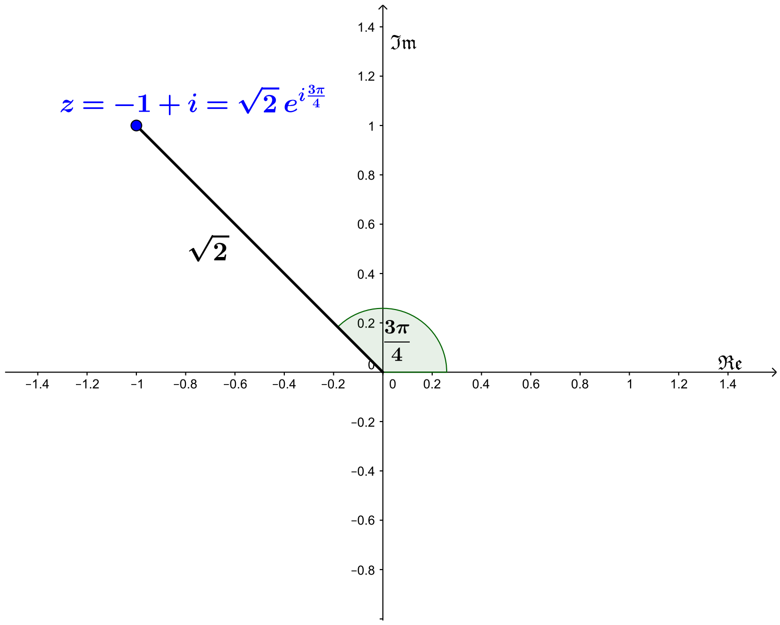
$r = |z| = \sqrt{(-1)^2+1^2} = \sqrt{1+1} = \sqrt{2}.$

$\theta = \tan^{-1}\dfrac{1}{-1} = \tan^{-1}-1 = -\dfrac{\pi}{4}.$

However since $-1+i$ is in the second quadrant, we need to correct this by adding π radians.

That is $\theta = -\dfrac{\pi}{4}+\pi = \dfrac{3\pi}{4}$.

So in exponential form $z = re^{i\theta} = \sqrt{2}\,e^{i\frac{3\pi}{4}}$.



So $(-1+i)^{18} = (\sqrt{2})^{18}(e^{i\frac{3\pi}{4}})^{18} = (2^{\frac{1}{2}})^{18}(e^{i\frac{3\pi}{4}})^{18}$

$$= 2^{18\times\frac{1}{2}}\ e^{18i\frac{3\pi}{4}} = 2^9 e^{i\frac{27\pi}{2}} = 2^9 e^{i(\frac{3\pi}{2}+12\pi)}$$

$$= 2^9 e^{\frac{3i\pi}{2}}\ e^{12\pi i} = 512\, e^{\frac{3i\pi}{2}}(1) = -512i.$$

Complex Roots of Unity: Cube roots

To find the nth roots of a complex number, we first address the problem of finding the nth roots of 1.

That is, we want solutions of $z^n = 1$.

The Fundamental Theorem of Algebra tells us that there are n of them. So what are they?

We know that $z^2 = 1$ has solutions $z = \pm 1$ $\checkmark$.

That is, $z = 1$ and $z = -1$ are the two square roots of unity (same as for the real numbers).

But what are the three solutions of $z^3 = 1$?

That is, what are the three complex cube roots of 1?

Certainly $z = 1$ is a root, but there two other complex roots. What are they?

The key idea is to 'slice up' the full circle with three equally-spaced 'spokes'. Since the full circle has 2π radians, each 'slice' will have $\frac{2\pi}{3}$ radians.

The phase factor $e^{i\frac{2\pi}{3}}$ rotates a point through $\frac{2\pi}{3}$ radians.

So three applications of $e^{i\frac{2\pi}{3}}$ takes us back to where we started since $e^{i\frac{2\pi}{3}} \times e^{i\frac{2\pi}{3}} \times e^{i\frac{2\pi}{3}} = e^{3i\frac{2\pi}{3}} = e^{2\pi i} = 1$.

This says that $\left(e^{i\frac{2\pi}{3}}\right)^3 = 1$.

So $\omega = e^{i\frac{2\pi}{3}}$ is a cube root of unity!

It satisfies $\omega^3 = 1$.

To get the third, we simply observe that $(\omega^2)^3 = (\omega^3)^2 = 1^2 = 1$.

So the third root is ω^2.

Indeed *any* integer power of ω is a cube root of unity.

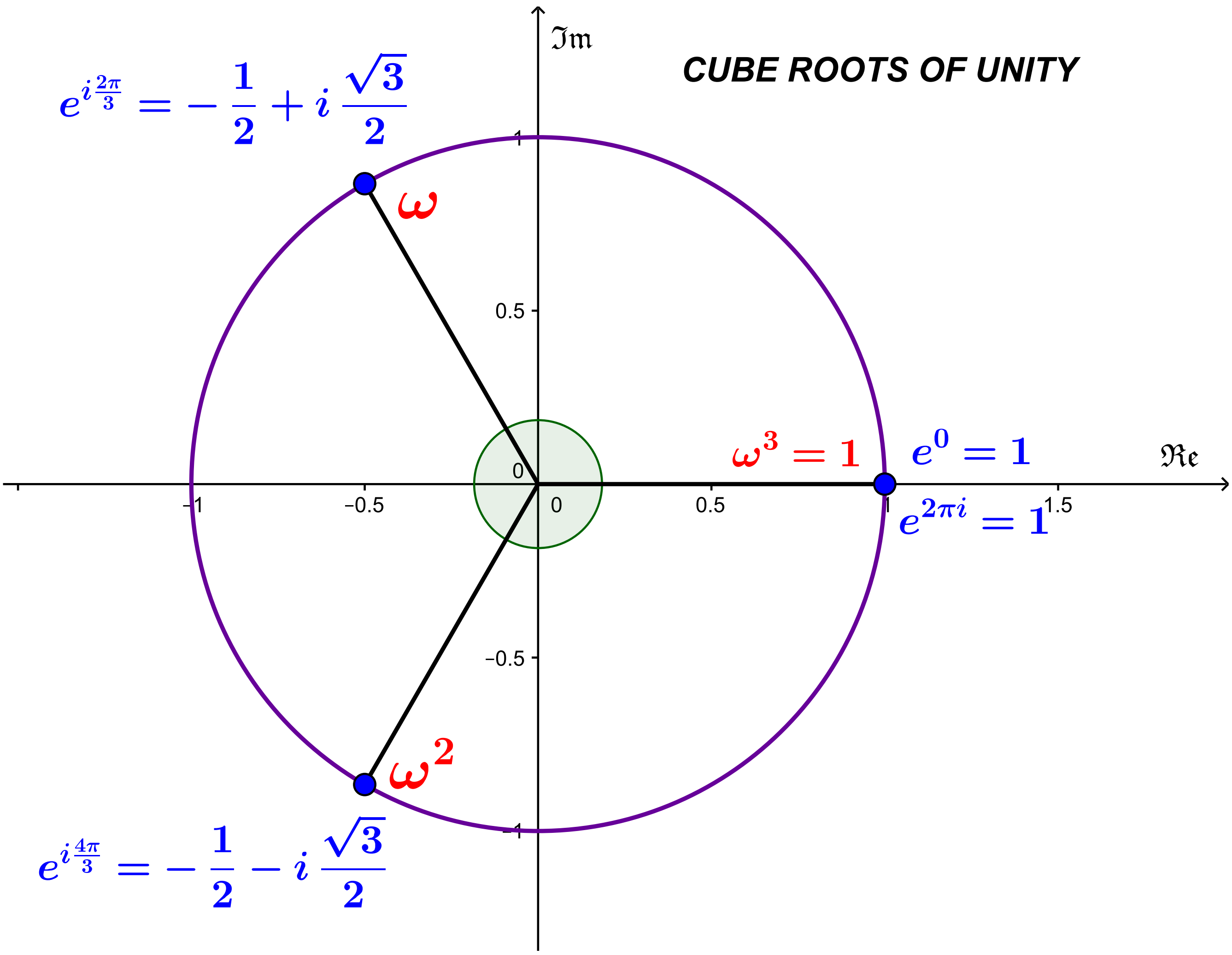
However, since $\omega^3 = 1$, the powers cycle with period three, that is we get

$$\omega^0 = 1, \quad \omega^1, \quad \omega^2, \quad \omega^3 = 1, \quad \omega^4 = \omega, \quad \omega^5 = \omega^2, \ldots.$$

So in summary, the three cube roots of unity are 1,

$\omega = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and
$\omega^2 = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

They give a 'three-spoked wheel'.



Complex Roots of Unity: general case

We call a complex number ω an nth root of unity

if it satisfies $z^n = 1$, that is $\omega^n = 1$.

Obviously $z = 1$ satisfies this, but how do we find another?

We use the same idea of dividing the 'wheel' with n equally-spaced spokes.

Since $\omega^n = 1,$ it follows that $|\omega|^n = 1,$ so $|\omega| = 1$ and hence $\omega = e^{i\theta}$ for some angle(s) θ.

Then $\omega^n = (e^{i\theta})^n = e^{in\theta} = 1$.

We can get a solution by putting $in\theta = 2\pi i,$ (since $e^{2\pi i} = 1$), that is $\theta = \frac{2\pi}{n}$.

So $\omega = e^{i\frac{2\pi}{n}}$ is a complex nth root of unity.

Having got one 'spoke' of the wheel, we can immediately get all the others, since if $\omega^n = 1,$ then
$(\omega^k)^n = (\omega^n)^k = 1^k = 1$.

So all powers of ω are also nth roots of unity.

So the n nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

When we get to $\omega^n,$ we're back to 1 again.

So the roots are $1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \ldots, e^{\frac{(2n-2)\pi i}{n}}$.

That is, the nth roots of unity are

$\omega^k = e^{\frac{2k\pi i}{n}} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},$ for $k = 0, 1, 2, \ldots, n-1$.

It helps to remember that $e^{2i\pi} = 1$.

So '$2i\pi$' is the full circle 'pizza'.

Think of 'two eye pie (the full pizza)'

The fourth roots of unity

We want the four complex numbers for which $z^4 = 1$.

Clearly $z = 1$ satisfies this.

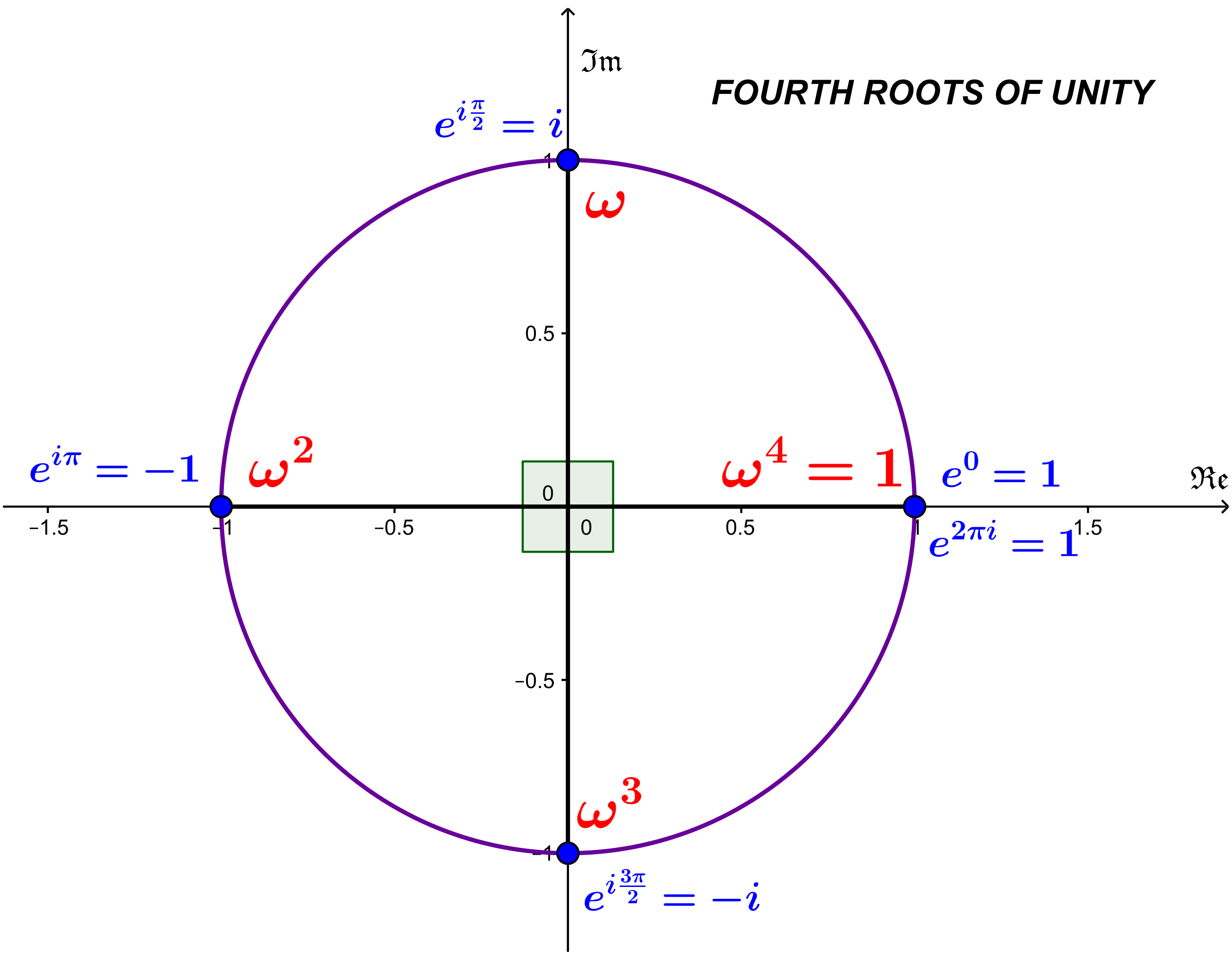
We get another root (the next 'spoke on the wheel') from $\omega = e^{\frac{2\pi i}{4}} = e^{\frac{i\pi}{2}} = i$.

This makes sense as $i^4 = 1$.

The four fourth roots of unity are then $1, \omega, \omega^2, \omega^3$.

That is $1, i, -1, -i$.

This makes sense as $0 = z^4 - 1 = (z^2 - 1)(z^2 + 1)$ gives $z^2 = 1$ or $z^2 = -1$, that is $z = \pm 1$ or $z = \pm i$.



The sixth roots of unity

We want the six complex numbers for which $z^6 = 1$.

Clearly $z = 1$ satisfies this.

We get another root (the next 'spoke on the wheel') from

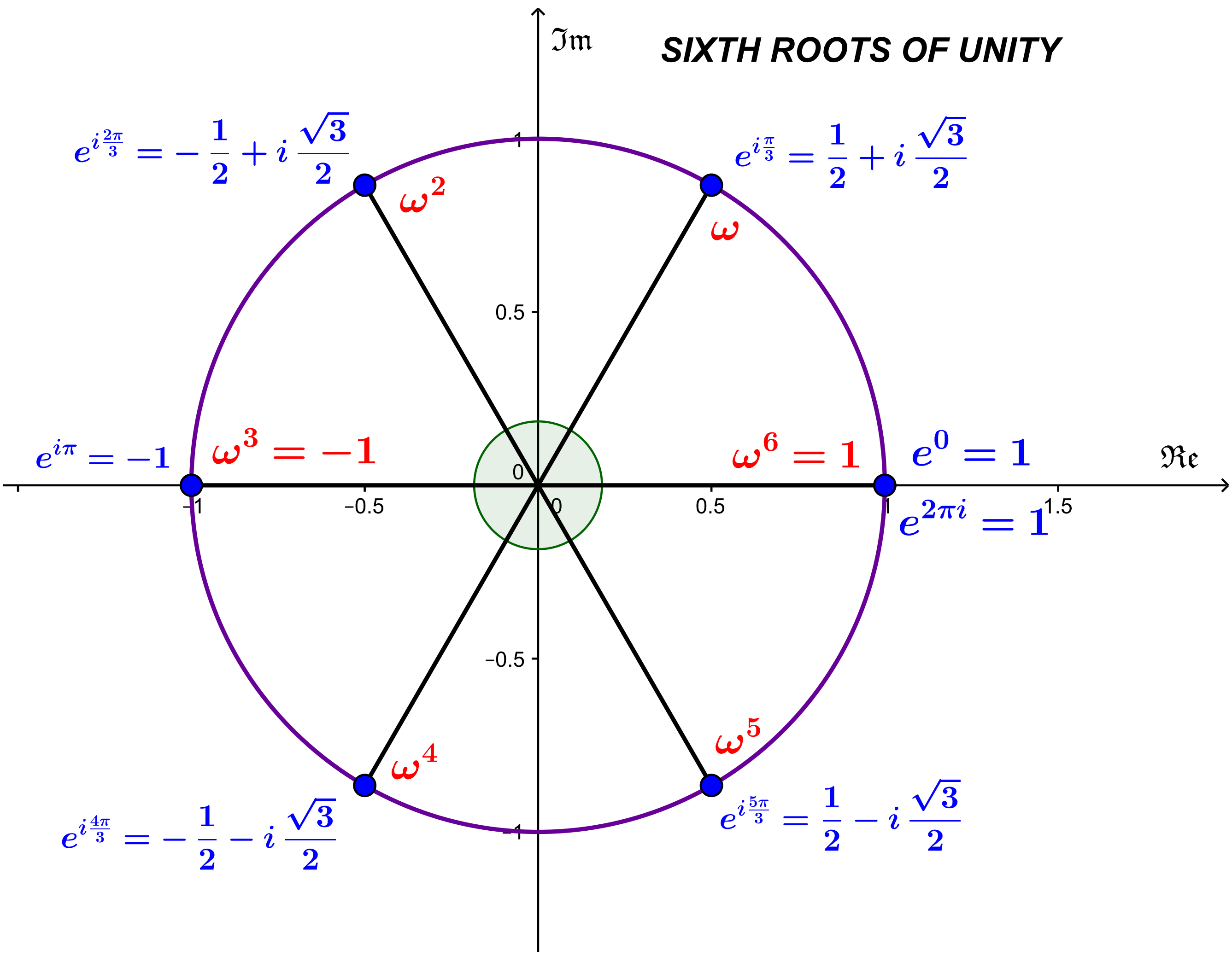
$\omega = e^{\frac{2\pi i}{6}} = e^{\frac{i\pi}{3}} = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Then $\omega^2 = (e^{\frac{i\pi}{3}})^2 = e^{\frac{2i\pi}{3}} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Next $\omega^3 = e^{\frac{3i\pi}{3}} = e^{i\pi} = -1$.

Finally $\omega^4 = (\omega^3)(\omega) = -\omega = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

and $\omega^5 = (\omega^3)(\omega^2) = -\omega^2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.



The eighth roots of unity

We want the eight complex numbers for which $z^8 = 1$.

Clearly $z = 1$ satisfies this.

We get another root (the next 'spoke on the wheel') from

$$\omega = e^{\frac{2\pi i}{8}} = e^{\frac{i\pi}{4}} = \cos(\tfrac{\pi}{4}) + i\sin(\tfrac{\pi}{4}) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}.$$

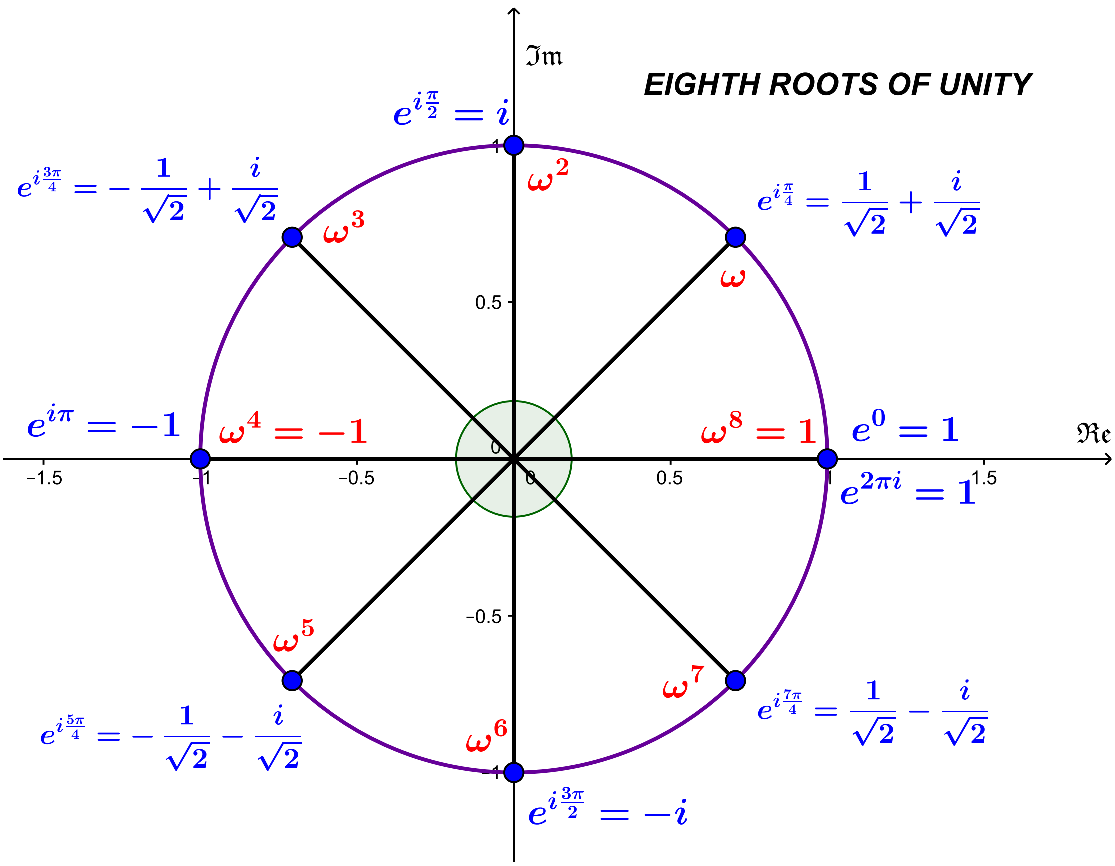
The next is $\omega^2 = (e^{\frac{i\pi}{4}})^2 = e^{\frac{2i\pi}{4}} = e^{\frac{i\pi}{2}} = i$.

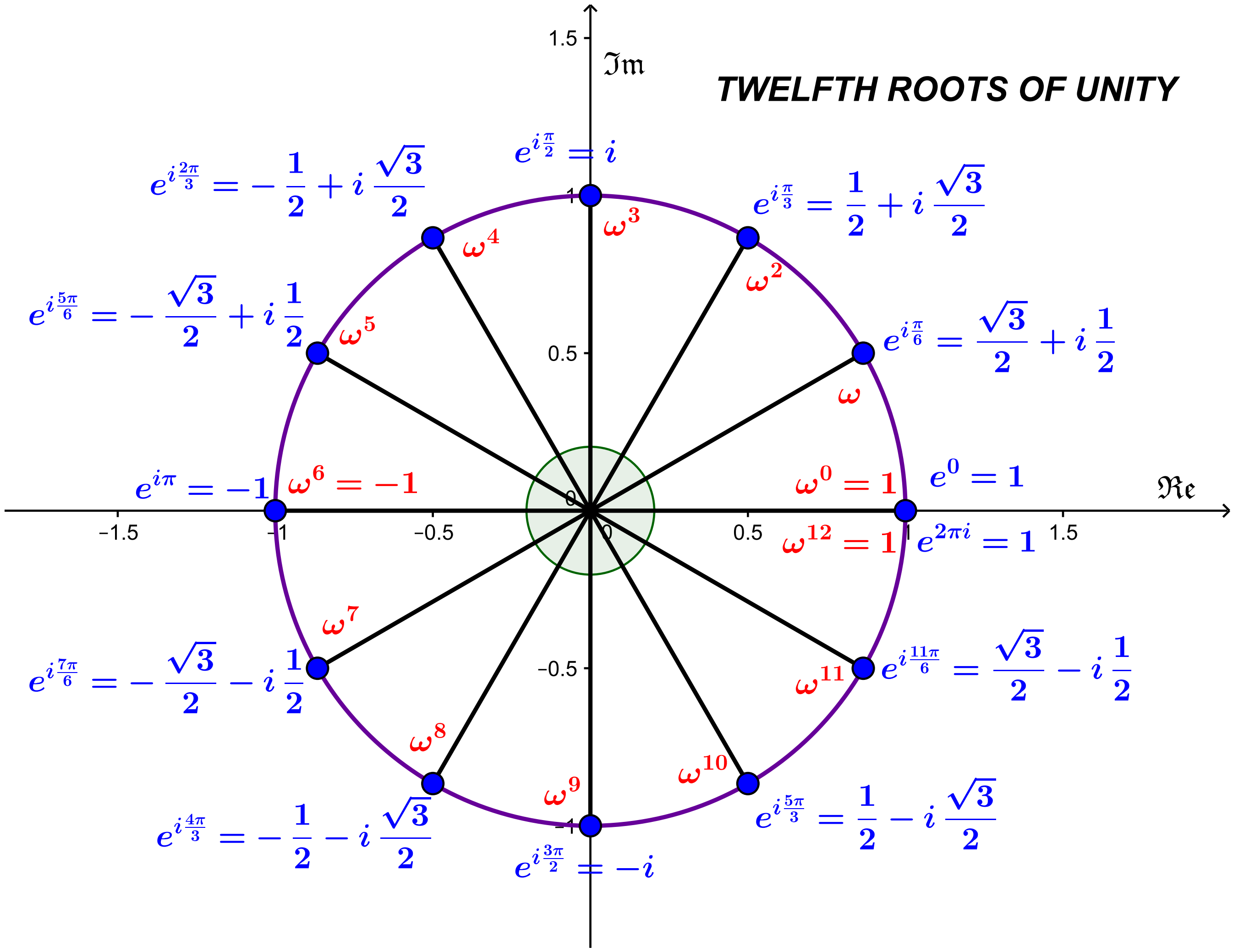
Then $\omega^3 = (\omega^2)(\omega) = i(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

Next $\omega^4 = (\omega^2)(\omega^2) = (i)(i) = -1$.

Since $\omega^4 = -1$, then $\omega^5 = -\omega$ and $\omega^6 = -\omega^2 = -i$ and $\omega^7 = -\omega^3$.

So the eight eighth roots are $z = \pm 1$ and $z = \pm i$ and $z = \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$.





Roots of any complex number

If we're given some non-zero complex number w, how do we find its n nth roots? We take a two-pronged approach:

- First find *any* nth root of w.
- Then multiply this root by the nth roots of unity to find the rest.

To see how this works, suppose z is an nth root of w, that is $z^n = w$.

Then if ω^k is an nth root of unity, then

$(z\,\omega^k)^n = z^n\,\omega^{kn} = (w)(\omega^n)^k = w(1^k) = w$.

That is, $z\,\omega^k = z\,e^{\frac{2ik\pi}{n}}$ is an nth root of w!

Example 1

Find the three cube roots of $-16+16i$ *in exponential form, that is the solutions of* $z^3 = -16+16i$.

We write $w = -16+16i$ in exponential form.

First, $|w| = r = \sqrt{16^2+16^2} = \sqrt{512} = \sqrt{2^9} = 2^{\frac{9}{2}}$.

Next, $\arg(w) = \theta = \tan^{-1}\dfrac{16}{-16} = \tan^{-1}(-1) = -\dfrac{\pi}{4}$.

Since w is in the second quadrant, we add π, so this becomes $\theta = \dfrac{3\pi}{4}$.

So, in exponential form, $w = 2^{\frac{9}{2}}\, e^{i\frac{3\pi}{4}}$.

Then a cube root is

$$z = w^{\frac{1}{3}} = \left(2^{\frac{9}{2}} e^{i\frac{3\pi}{4}}\right)^{\frac{1}{3}} = (2^{\frac{9}{2}})^{\frac{1}{3}}(e^{i\frac{3\pi}{4}})^{\frac{1}{3}} = 2^{\frac{9}{2}\times\frac{1}{3}}\, e^{i\frac{3\pi}{4}\times\frac{1}{3}}$$

$$= 2^{\frac{3}{2}}\, e^{i\frac{\pi}{4}} = 2\sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right) = 2(1+i) = 2+2i.$$

(Verify by multiplication that $(2+2i)^3 = -16+16i$.)

To get the other two, we use the cube roots of unity $1, \omega, \omega^2$ which are $1, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}$.

The cube roots of w are $z = 2^{\frac{3}{2}} e^{i\frac{\pi}{4}}$ times these;

that is $2^{\frac{3}{2}} e^{i\frac{\pi}{4}}, \quad 2^{\frac{3}{2}} e^{i\frac{\pi}{4}} e^{\frac{2i\pi}{3}}, \quad 2^{\frac{3}{2}} e^{i\frac{\pi}{4}} e^{\frac{4i\pi}{3}}$,

that is $2^{\frac{3}{2}} e^{i\frac{\pi}{4}}, \quad 2^{\frac{3}{2}} e^{\frac{11i\pi}{12}}, \quad 2^{\frac{3}{2}} e^{\frac{19i\pi}{12}}$.

Example 2

Find the four fourth roots of -81 *in Cartesian form, that is the solutions of* $z^4 = -81$.

We write $w = -81$ in exponential form.

First, $|w| = r = \sqrt{81^2 + 0^2} = 81$.

Next, $\arg(w) = \theta = \tan^{-1} \dfrac{0}{-81} = 0$.

Since w lies on the negative real axis, we must add π, so this becomes $\theta = \pi$.

So, in exponential form, $w = 81\, e^{i\pi}$.

Then a fourth root is

$z = w^{\frac{1}{4}} = (81e^{i\pi})^{\frac{1}{4}} = (81)^{\frac{1}{4}}(e^{i\pi})^{\frac{1}{4}} = 3\, e^{i\frac{\pi}{4}} = 3(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = \frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}$.

To get the other three, we use the fourth roots of unity $1,\ \omega,\ \omega^2,\ \omega^3$, which are $1,\ i,\ -1,\ -i$.

The fourth roots of w are $\frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}$ times these;

that is $\frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}},\ -\frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}},\ -\frac{3}{\sqrt{2}} - i\frac{3}{\sqrt{2}},\ \frac{3}{\sqrt{2}} - i\frac{3}{\sqrt{2}}$.

Example 3

Find the square roots of $2 - 2\sqrt{3}i$ in Cartesian form.

We write $w = 2 - 2\sqrt{3}i$ in exponential form.

First, $|w| = r = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4.$

Next, $\arg(w) = \theta = \tan^{-1}\frac{-2\sqrt{3}}{2} = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}.$

Since w lies in the fourth quadrant, this is correct.

So, in exponential form, $w = 4\,e^{-i\frac{\pi}{3}}$.

Then a square root is

$$z = w^{\frac{1}{2}} = \left(4e^{-i\frac{\pi}{3}}\right)^{\frac{1}{2}} = (4)^{\frac{1}{2}}(e^{-i\frac{\pi}{3}})^{\frac{1}{2}} = 2\,e^{-i\frac{\pi}{6}}$$

$$= 2(\cos\tfrac{\pi}{6} - i\sin\tfrac{\pi}{6}) = 2(\tfrac{\sqrt{3}}{2} - i\tfrac{1}{2}) = \sqrt{3} - i.$$

Since the square roots of 1 are 1 and -1, the other square root is $-z = -\sqrt{3} + i$.

So the square roots are $\sqrt{3} - i$ and $-\sqrt{3} + i$.

Comment: Note that since the square roots of 1 are 1 and $e^{i\pi} = -1$, then if z is the square root of a complex number, so is $-z$.

That is, if $z^2 = w$, then $(-z)^2 = w$ also.