General Non-homogeneous Cases

We now return to dealing with the most general second order linear constant coefficient ODE which is now non-homogeneous:

$$y'' + ay' + by = f(t) \qquad (\star)$$

The function $f(t)$ on the right hand side is called the **driving force** for the ODE. In a mass-spring system, we can view it as an external force acting on the mass.

To solve ($\star$), we break the solution into two pieces:

$$y = y_h + y_p.$$

Here y_h is the *homogeneous solution*, that is the general solution of the homogeneous equation $y_h'' + ay_h' + by_h = 0$. This is the unforced problem (no $f(t)$), which we have studied solving in great detail.

The solution involves two constants (of integration) C_1, C_2, which we will find later when we impose initial conditions.

The second part y_p is any solution to the ODE ($\star$), which we usually get by 'intelligent guesswork'. We will return to this later.

We see that $y = y_h + y_p$ solves ($\star$) since

$$(y_h+y_p)''+a(y_h+y_p)'+b(y_h+y_p) \;=\; y_h''+y_p''+ay_h'+ay_p'+by_h+by_p$$

$$=\;(y_h''+ay_h'+by_h)+(y_p''+ay_p'+by_p)\;=\;0+f(t). \quad \checkmark$$

So how do we find a particular solution y_p?

The answer is that it should 'look like' $f(t)$. That is, whatever function(s) are in $f(t)$ should appear in y_p.

This will usually involve introducing some extra parameters to get the right "combination" that satisfies ($\star$).

Though there are a variety of functional forms for the driving force $f(t)$ that can be readily solved, we restrict our attention to just three:

$f(t)$	$y_p(t)$
k	α
kt	$\alpha t+\beta$
ke^{at}	αe^{at} (same a)

So for example, if $4t$ appears on the right side of ($\star$), we look for a particular solution of the form $y_p = \alpha t+\beta$.

That is, we want to find numbers α, β so that when we substitute this y_p into the left side of ($\star$), we reproduce $4t$ on the right.

Example 1

Find the general solution of each ODE below:

(a) $y'' + 3y' + 2y = 8.$

(b) $y'' + 3y' + 2y = 4t.$

(c) $y'' + 3y' + 2y = 6e^{-3t}.$

In each case, the homogeneous equation is

$y'' + 3y' + 2y = 0.$

The associated auxiliary equation is

$m^2 + 3m + 2 = (m + 2)(m + 1) = 0.$

So $m_1 = -1$ and $m_2 = -2$ (Case (I)).

This gives homogeneous solution $y_h(t) = C_1e^{-t} + C_2e^{-2t}$.

Example 1(a)

Here $f(t) = 8$, (a constant), so we try for a particular solution of the form $y_p = \alpha$.

Substituting this trial solution into (1(a)) (noting that $y_p'' = y_p' = 0,$) we get $0 + 3(0) + 2\alpha = 8$, so $\alpha = 4$.

So $y_p = 4$ is a particular solution.

Since the general solution is $y = y_h + y_p$, we find that the general solution to 1(a) is $y(t) = C_1e^{-t} + C_2e^{-2t} + 4$.

To find the constants C_1, C_2 we would need to specify initial conditions.

Example 1(b)

Here $f(t) = 4t$ (a constant), so we try for a particular solution of the form $y_p = \alpha t + \beta$.

Note that we must include a constant term here too, as when we differentiate $\alpha t,$ we generate constant terms, that must be cancelled, so we end up with $4t$.

To substitute $y_p = \alpha t + \beta$ into 1(b), we need its derivatives:

$y_p' = \alpha$ and $y_p'' = 0$.

Then 1(b) gives $0 + 3(\alpha) + 2(\alpha t + \beta) = 4t$.

That is, $2\alpha t + (3\alpha + 2\beta) = 4t$.

We require this equation to be *true for all* t, so the t-terms must match and the constant terms must match.

So we get two equations: $2\alpha = 4$ and $3\alpha + 2\beta = 0,$ since there is no constant term on the right.

So $\alpha = 2$ and $\beta = -3,$ and hence $y_p = 2t - 3$.

So the full general solution is $y(t) = y_h + y_p$

$$= C_1 e^{-t} + C_2 e^{-2t} + 2t - 3.$$

Example 1(c)

Here $f(t) = 6e^{-3t},$ so we try for a particular solution of the same type: $y_p = \alpha e^{-3t}$.

(Notice it is important we don't change the -3.)

We need its derivatives: $y_p' = -3\alpha e^{-3t}$ and $y_p'' = 9\alpha e^{-3t}$.

Then 1(c) gives $9\alpha e^{-3t} + 3(-3\alpha e^{-3t}) + 2\alpha e^{-3t} = 6e^{-3t}$.

So $2\alpha e^{-3t} = 6e^{-3t}$, and hence $2\alpha = 6$, so $\alpha = 3$.

The particular solution is then $y_p = 3e^{-3t}$.

The general solution of 1(c) is $y(t) = y_h + y_p$

$$= C_1 e^{-t} + C_2 e^{-2t} + 3e^{-3t}.$$

Example 2

Solve $y'' + 5y' + 6y = 6e^{-t}$,

subject to initial conditions $y(0) = 5, \quad y'(0) = 0$.

We first get the general solution in the form $y = y_h + y_p$.

The homogeneous equation is $y_h'' + 5y_h' + 6y_h = 0$.

It has auxiliary equation

$m^2 + 5m + 6 = (m+2)(m+3) = 0$.

The roots are $m_1 = -2$ and $m_2 = -3$ (Case (I)).

So the homogeneous solution is $y_h = C_1 e^{-2t} + C_2 e^{-3t}$.

We try as particular solution $y_p = \alpha e^{-t}$.

Its derivatives are $y_p' = -\alpha e^{-t}$ and $y_p'' = +\alpha e^{-t}$.

Putting these into the ODE gives

$\alpha e^{-t} + 5(-\alpha e^{-t}) + 6(\alpha e^{-t}) = 6e^{-t}$.

So $(1 - 5 + 6)\alpha e^{-t} = 6e^{-t}$.

So $2\alpha = 6$ and hence $\alpha = 3$.

So $y_p = 3e^{-t}$ is a particular solution.

The general solution is then

$y = y_h + y_p = C_1 e^{-2t} + C_2 e^{-3t} + 3e^{-t}$.

To find the constants C_1, C_2 we use the initial conditions.

First $y(0) = C_1 + C_2 + 3 = 5$, so $C_1 + C_2 = 2$.

Now we must calculate the derivative:

$y'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} - 3e^{-t}$.

So $y'(0) = -2C_1 - 3C_2 - 3 = 0$.

So $2C_1 + 3C_2 = -3$.

Solving the pair of equations $C_1 + C_2 = 2$ and $2C_1 + 3C_2 = -3$, we get $C_2 = -7$ and $C_1 = 9$.

So the solution is $y(t) = 9e^{-2t} - 7e^{-3t} + 3e^{-t}$.

Example 3

Solve $4y'' + 4y' + y = t + 6,$

subject to initial conditions $y(0) = 0, \quad y'(0) = 3.$

We first get the general solution in the form $y = y_h + y_p$.

The homogeneous equation is $4y_h'' + 4y_h' + y_h = 0$.

It has auxiliary equation

$4m^2 + 4m + 1 = (2m+1)(2m+1) = 0$.

There is a double root $m_1 = -\frac{1}{2}$. (Case (II)).

So the homogeneous solution is $y_h = C_1 e^{-\frac{1}{2}t} + C_2\, t\, e^{-\frac{1}{2}t}$.

We try as particular solution $y_p = \alpha t + \beta$.

Its derivatives are $y_p' = \alpha$ and $y_p'' = 0$.

Putting these into the ODE gives

$4(0) + 4(\alpha) + (\alpha t + \beta) = t + 6$.

So $\alpha t + (4\alpha + \beta) = t + 6$.

Equating coefficients of t and equating the constant terms, we get two equations: $\alpha = 1$ and $4\alpha + \beta = 6$.

Solving these, we get $\alpha = 1$ and $\beta = 2$.

So $y_p = t + 2$ is a particular solution.

The general solution is then

$y = y_h + y_p = C_1 e^{-\frac{t}{2}} + C_2 t\, e^{-\frac{t}{2}} + t + 2.$

To find the constants C_1, C_2 we use the initial conditions.

First $y(0) = C_1 + 2 = 0$, so $C_1 = -2$.

Now we must calculate the derivative:

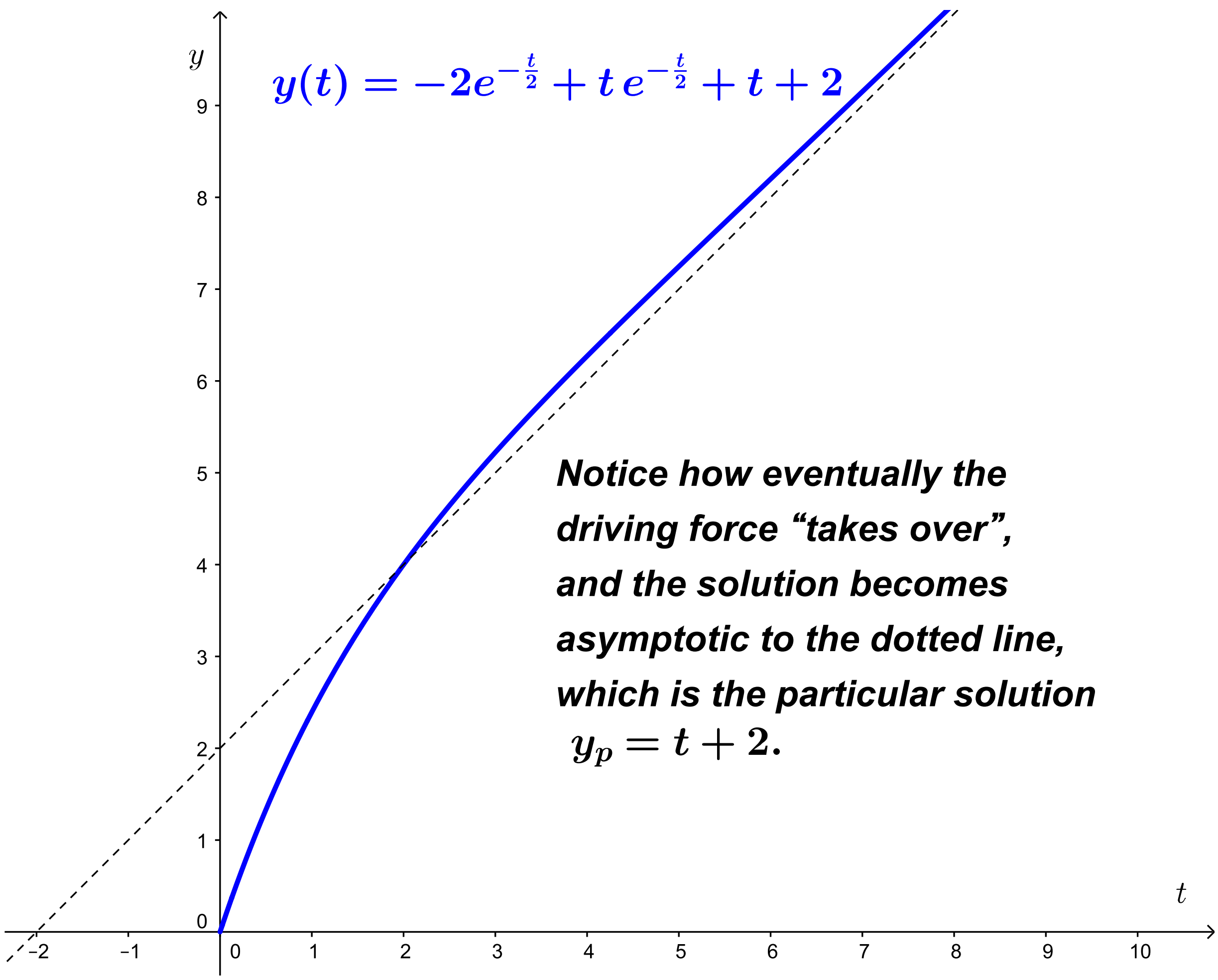
$$y'(t) = -\frac{1}{2}C_1 e^{-\frac{t}{2}} + C_2(e^{-\frac{t}{2}} - \frac{t}{2}e^{-\frac{t}{2}}) + 1.$$

So $y'(0) = -\frac{1}{2}C_1 + C_2 + 1 = 3$.

So $-C_1 + 2C_2 = 4$.

But $C_1 = -2$, so $C_2 = 1$.

So the solution is $y(t) = -2e^{-\frac{t}{2}} + t\,e^{-\frac{t}{2}} + t + 2$.



Example 4

Solve $y'' + 2y' + 5y = 25t$,

subject to initial conditions $y(0) = 3, \quad y'(0) = 4$.

We first get the general solution in the form $y = y_h + y_p$.

The homogeneous equation is $y_h'' + 2y_h' + 5y_h = 0$.

It has auxiliary equation $m^2 + 2m + 5 = 0$.

Solving this, we get

$$m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

The roots are $m_1 = -1 + 2i, \quad m_2 = -1 - 2i \quad$ (Case (III)).

So the homogeneous solution is

$y_h = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$.

We try as particular solution $y_p = \alpha t + \beta$.

Its derivatives are $y_p' = \alpha$ and $y_p'' = 0$.

Putting these into the ODE gives

$0 + 2(\alpha) + 5(\alpha t + \beta) = 25t$.

So $5\alpha t + (2\alpha + 5\beta) = 25t$.

Equating coefficients of t and equating the constant terms, we get two equations: $5\alpha = 25$ and $2\alpha + 5\beta = 0$.

Solving these, we get $\alpha = 5$ and $\beta = -2$.

So $y_p = 5t - 2$ is a particular solution.

The general solution is then

$$y = y_h + y_p = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t) + 5t - 2.$$

To find the constants C_1, C_2 we use the initial conditions.

Applying the first initial condition gives

$3 = y(0) = C_1 - 2,$ so $C_1 = 5$.

For the second initial condition we need the derivative.

We calculated the homogeneous part of this in a previous Example 3 and got $y'(t) =$

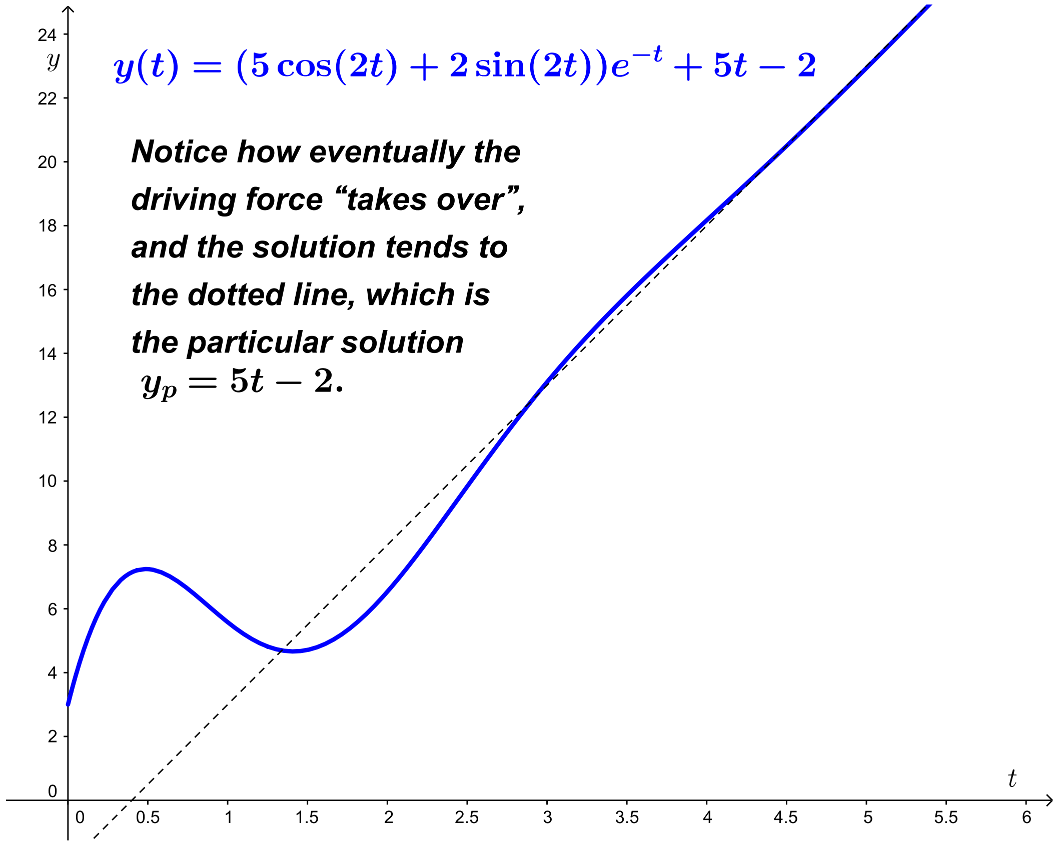
$$-C_1 e^{-t}(2\sin(2t) + \cos(2t)) + C_2 e^{-t}(2\cos(2t) - \sin(2t)) + 5.$$

So $4 = y'(0) = -C_1 + 2C_2 + 5 = -5 + 2C_2 + 5 = 2C_2$.

So $C_2 = 2$.

The solution to the original problem is then

$$y = 5e^{-t}\cos(2t) + 2e^{-t}\sin(2t) + 5t - 2.$$



Resonance effects in periodically-driven forced harmonic oscillators

Let us consider a simple harmonic oscillator, say the solution of $y'' + 9y = 0$,

with initial conditions $y(0) = 0, \quad y'(0) = 3$.

Here $\omega = 3$, and it's easy show the solution is the sinusoid $y = \sin(3t)$.

Now let us force this oscillator with a periodic driving force, with a different frequency, say $f(t) = \sin(2t)$.

That is we want to solve the problem:

Solve $y'' + 9y = \sin(2t)$,

subject to initial conditions $y(0) = 0, \quad y'(0) = 3$.

As usual, we write $y = y_h + y_p$.

The homogeneous equation is $y_h'' + 9y_h = 0$.

It has auxiliary equation $m^2 + 9 = 0$.

So $m^2 = -9$.

Solving this, we get the roots $m = \pm 3i$.

So $y_h = C_1 \cos(3t) + C_2 \sin(3t)$.

Now we look for a particular solution of the form $y_p = \alpha \sin(2t)$.

Its derivatives are $y_p' = 2\alpha \cos(2t)$ and $y_p'' = -4\alpha \sin(2t)$.

Now putting these into the differential equation, we get

$-4\alpha\sin(2t) + 9(\alpha\sin(2t)) = \sin(2t)$.

So $5\alpha = 1$, and hence $\alpha = 0.2$.

So the general solution is

$y(t) = C_1\cos(3t) + C_2\sin(3t) + 0.2\sin(2t)$.

Applying the first initial condition gives $0 = y(0) = C_1$, so $C_1 = 0$.

For the second initial condition we need the derivative:

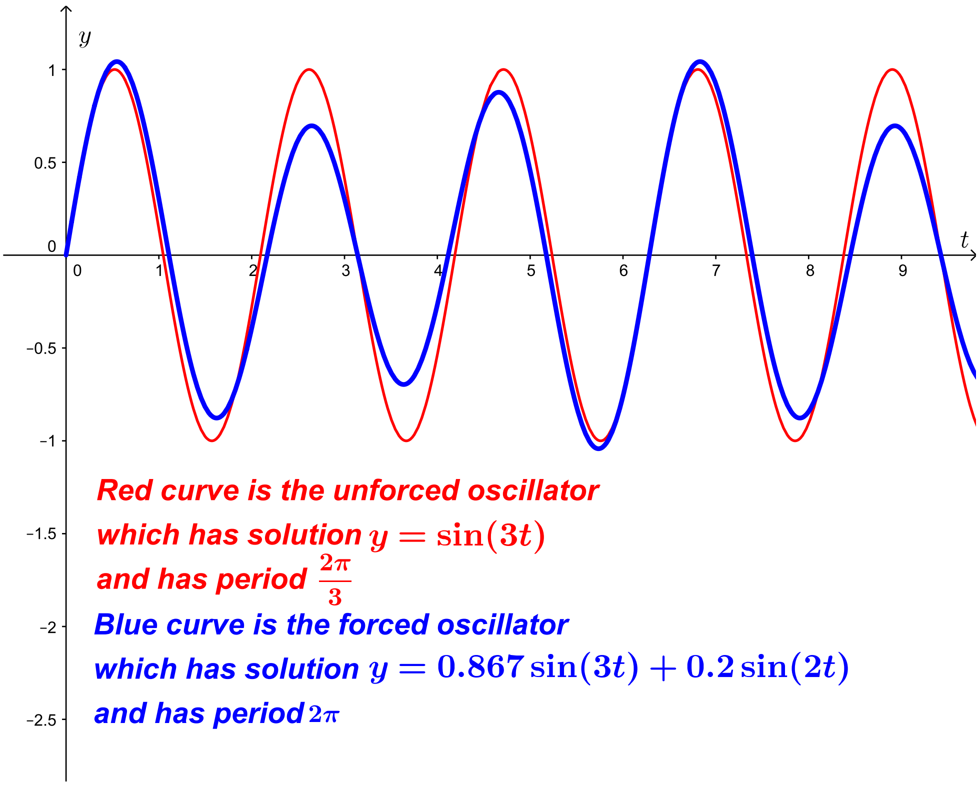
$y'(t) = 3C_2\cos(3t) + 0.2(2)\cos(2t)$.

So the second initial condition gives

$3 = y'(0) = 3C_2 + 0.4$. So $C_2 = \frac{2.6}{3} \approx 0.867$.

So the solution to the forced problem is

$y(t) = 0.867\sin(3t) + 0.2\sin(2t)$.



We now generalise the problem and solve

Solve $y'' + 9y = \sin(\rho t)$,

subject to initial conditions $y(0) = 0, \quad y'(0) = 3.$

We have introduced a parameter ρ that allows us to adjust the period (frequency) of the driving force. We leave everything else the same. (The case we solved above was $\rho = 2$.)

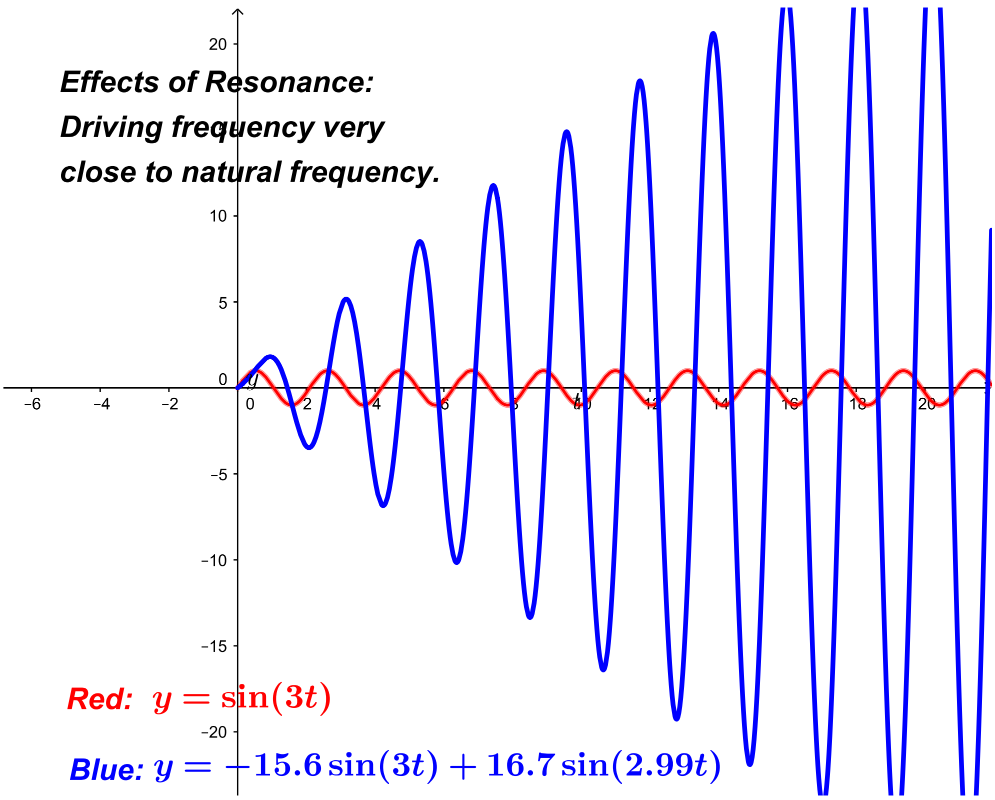
One can show that the solution of this problem is

$$y(t) = \left(1 - \frac{\rho}{3(9-\rho^2)}\right)\sin(3t) + \frac{1}{9-\rho^2}\sin(\rho t).$$

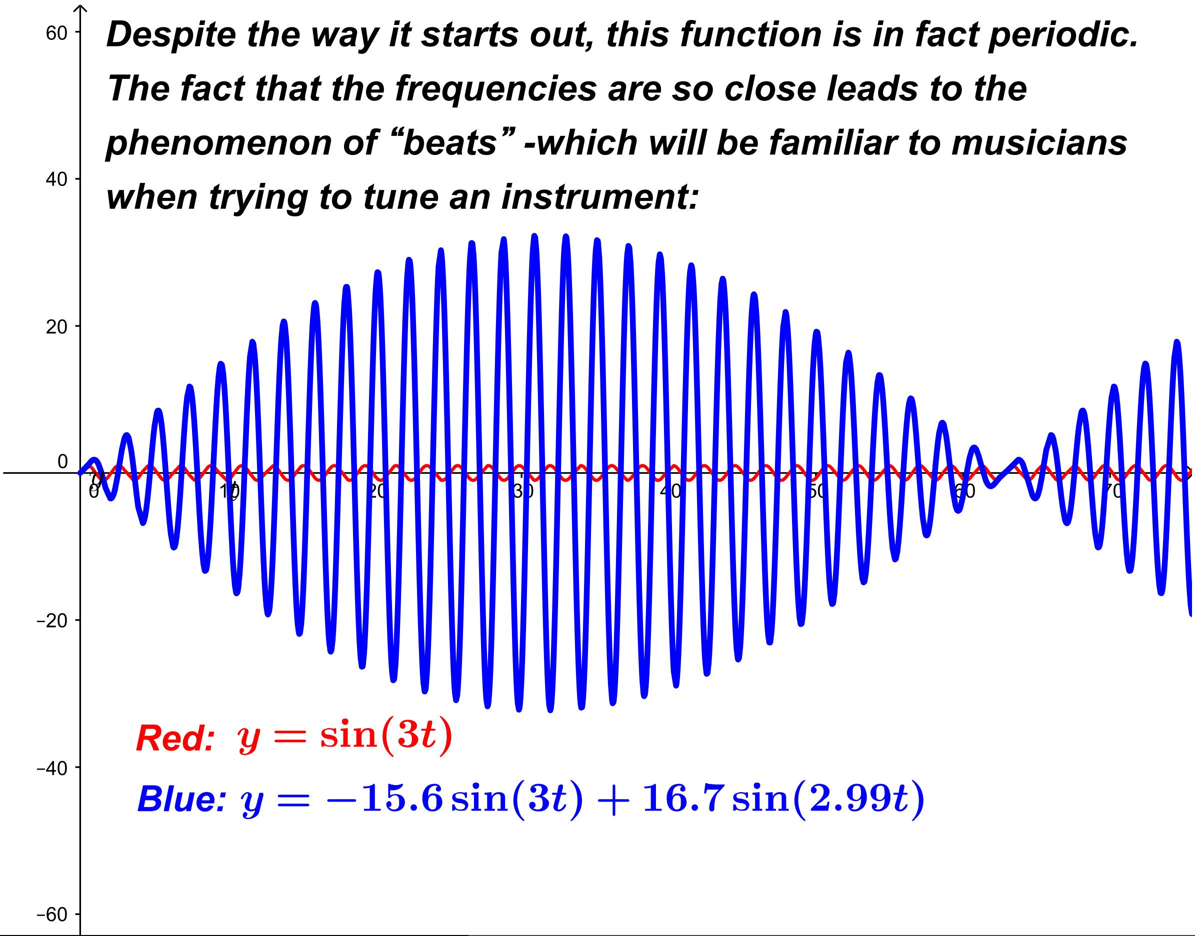
Let us investigate what happens when the period (frequency) of the driving force gets close to the natural period (frequency) of the spring, that is as $\rho \to 3$.

Let us take $\rho = 2.99$.

Then the solution is $y(t) = -15.6\sin(3t) + 16.7\sin(2.99t)$.



Despite the way it starts out, this function is in fact periodic. The fact that the frequencies are so close leads to the phenomenon of "beats" -which will be familiar to musicians when trying to tune an instrument:



When the period (frequency) of the driving force is very close to the natural period (frequency) of the spring, we see initially the amplitude of the oscillations grows linearly with time.

Resonance occurs when the period (frequency) of the driving force matches (equals) the natural period (frequency) of the spring.

Resonance can lead to instability, sometimes with catastrophic effects (collapse of Tacoma Narrows Bridge) or be used constructively, such as in Nuclear Magnetic Resonance (NMR) Spectroscopy.

So how do we solve the problem when $\rho = 3$? :

Solve $y'' + 9y = \sin(3t)$,

subject to initial conditions $y(0) = 0, \quad y'(0) = 3$.

The difficulty is that a particular solution of the type $y_p = \alpha \sin(3t)$ cannot reproduce the right side, since when we plug it in on the left, we get zero!

We need to introduce terms like $t \sin(3t)$. These terms are oscillatory but the amplitude grows linearly with time.

The full solution of the problem above is
$y(t) = \frac{19}{18} \sin(3t) - \frac{1}{6} t \cos(3t)$.

It is not periodic.