Chapter 8b:

Revision on Variables Separable,

First Order Linear,

and

Second Order Linear Constant Coefficient ODES

Variables Separable ODEs

Solve the first order variables separable ODE

$$\frac{dy}{dx} = \frac{x^2 + x}{y^2 + 1}.$$

To separate variables, we must bring all the y-stuff to the left along with the dy which is already there.

We must also multiply by dx so it is on the right with the dx-stuff.

$$(y^2 + 1)\, dy = (x^2 + x)\, dx.$$

Now insert integral signs:

$$\int (y^2 + 1)\, dy \;=\; \int (x^2 + x)\, dx.$$

Now do the integrals (each wrt its own variable):

$$\frac{y^3}{3} + y \;=\; \frac{x^3}{3} + \frac{x^2}{2} + C.$$

This is the general solution, as required.

The Logistic Equation

The logistic differential equation for population growth is

$$\frac{dN}{dt} \;=\; rN\left(1 - \frac{N}{K}\right).$$

The solution with initial condition $N(0) = N_0$ *is*

$$N(t) \;=\; \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}.$$

(a) For what populations N *is the population (i) increasing? (ii) decreasing?*

(b) Explain why $N = K$ *is a stable equilibrium and interpret the constant* K *in terms of the model.*

(c) By differentiating $\frac{dN}{dt}$ *and using the Chain Rule, show that*

$$\frac{d^2N}{dt^2} = r^2 N\left(1 - \frac{N}{K}\right)\left(1 - \frac{2N}{K}\right).$$

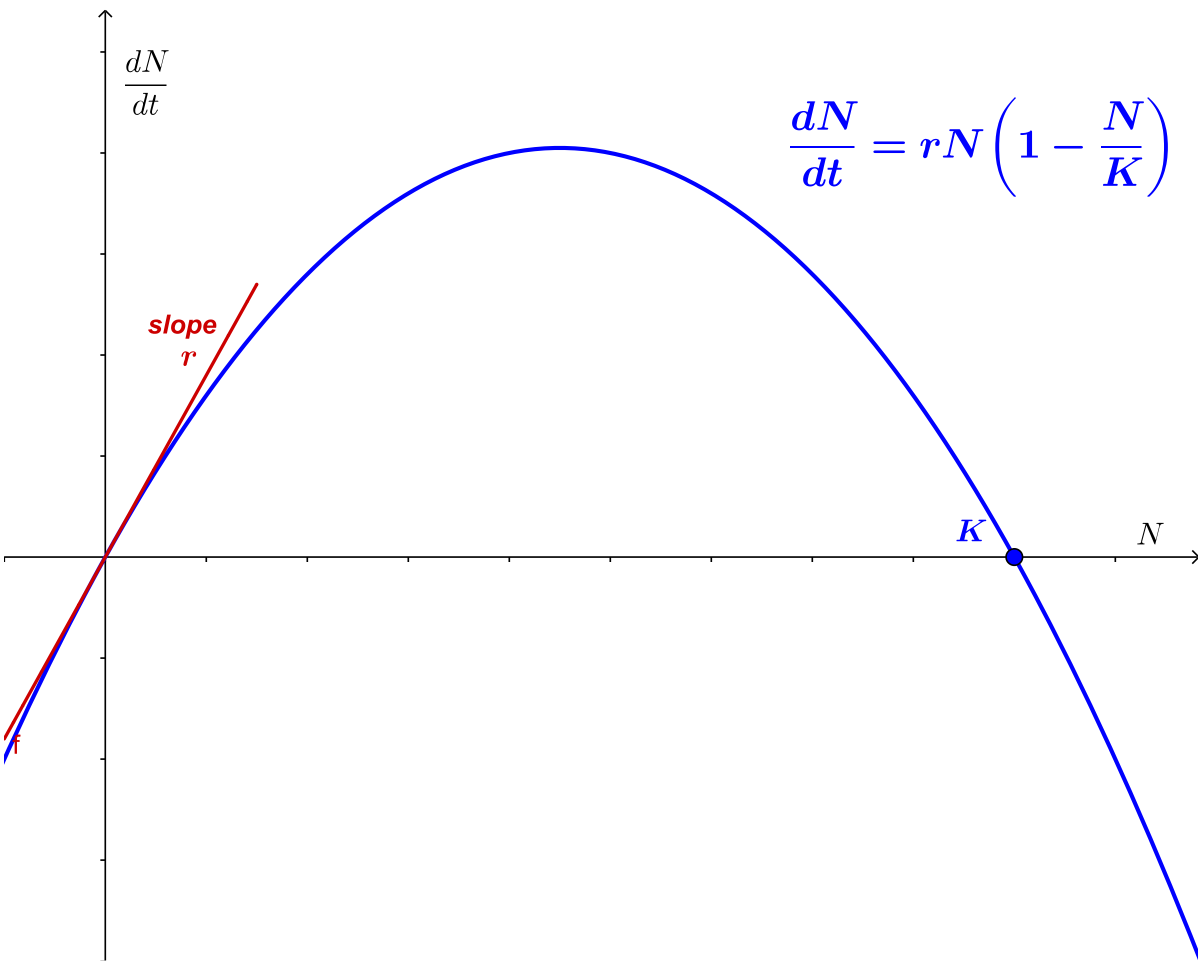
(d) Use this to show that there is a point of inflection when $N = \frac{K}{2}$ *and that this occurs at time* $\tau = \frac{1}{r}\ln\left(\frac{K - N_0}{N_0}\right)$.

(a) N is increasing when $\frac{dN}{dt} > 0$.

that is $N(K - N) > 0$, that is $0 < N < K$.

N is decreasing when $\frac{dN}{dt} < 0$,

that is $N(K - N) < 0$, that is $N > K$.



(b) At $N = K$, $\dfrac{dN}{dt} = 0$, so the population remains constant.

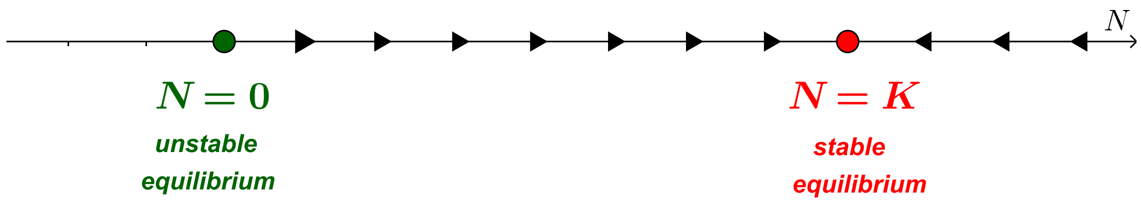
If N is below K, it is increasing, so moves towards K.

If N is above K, it is decreasing, so moves towards K.

So points near $N = K$ moves towards $N = K$, that is the equilibrium is stable.

$N = K$ is called the *carrying capacity* of the population.

It is also the long-term limit $\lim_{t\to\infty} N(t)$.



(c) $$\frac{d^2N}{dt^2} = \frac{d}{dt}\frac{dN}{dt} = \frac{d}{dt}\left(rN\left(1 - \frac{N}{K}\right)\right)$$

$$= r\frac{d}{dN}\left(N - \frac{N^2}{K}\right)\frac{dN}{dt} = r\left(1 - 2\frac{N}{K}\right)\left(rN\left(1 - \frac{N}{K}\right)\right)$$

$$= r^2N\left(1 - \frac{N}{K}\right)\left(1 - \frac{2N}{K}\right).$$

(d) When $N = \dfrac{K}{2}$, $\dfrac{d^2N}{dt^2} = 0$, and $\dfrac{dN}{dt} \neq 0$, so there is a point of inflection.

Suppose $N(\tau) = \dfrac{K}{2}$.

Then $\dfrac{K}{2} = \dfrac{KN_0}{N_0 + (K - N_0)e^{-r\tau}}$. We solve this for τ.

Divide both sides by K and multiply through by the denominator on the right to get:

$\frac{1}{2}(N_0 + (K - N_0)e^{-r\tau}) = N_0$.

Multiply by 2 and rearrange to get $(K - N_0)e^{-r\tau} = N_0$.

So $e^{-r\tau} = \dfrac{N_0}{K - N_0}$,

and hence (taking logs) $-r\tau = \ln\left(\dfrac{N_0}{K - N_0}\right)$.

So $\tau = \dfrac{1}{-r}\ln\left(\dfrac{N_0}{K - N_0}\right) = \dfrac{1}{r}\ln\left(\dfrac{K - N_0}{N_0}\right)$.

(Aside: in the last step we used $\ln\left(\dfrac{1}{x}\right) = -\ln x$.)

First Order Linear ODEs

A falling object of mass m has velocity v at time t. It satisfies the ODE $m\dfrac{dv}{dt} = mg - kv$, where g and k are constants.

(a) Show that the solution of this ODE with initial condition $v(0) = v_0$ is $v(t) = \dfrac{mg}{k} + \left(v_0 - \dfrac{mg}{k}\right)e^{-\frac{k}{m}t}$.

(b) Find the terminal velocity, *that is $\lim_{t\to\infty} v(t)$.*

(c) Take $m = 1$, $k = 0.1$ and $g = 10$. Write out an expression for $v(t)$ at all times t. Find the terminal velocity and find its speed after ten seconds.

(d) How long does it take for the speed to reach 90m/s?

(a) First we show it satisfies the differential equation:

$$\frac{dv}{dt} = 0 + \left(v_0 - \frac{mg}{k}\right)(-\frac{k}{m})e^{-\frac{k}{m}t} = (-\frac{k}{m})(v - \frac{mg}{k})$$

$$= -\frac{k}{m}v + g.$$

So $m\frac{dv}{dt} = -kv + mg. \quad \checkmark$

Next we check the initial condition:

$$v(0) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^0 = \frac{mg}{k} + v_0 - \frac{mg}{k} = v_0. \quad \checkmark$$

(b) $\lim_{t\to\infty} v(t) = \frac{mg}{k} + 0,$ since the exponential goes to zero.

(c) Putting in the numbers we get:

$$v(t) = \frac{10}{0.1} + \left(0 - \frac{10}{0.1}\right)e^{-\frac{0.1}{1}t} = 100(1 - e^{-0.1t}).$$

The terminal velocity is 100.

When $t = 10,$ we get $v(10) = 100(1 - e^{-1}) = 63.2$.

(d) When $v = 90,$ we get $90 = 100 - 100e^{-0.1t}$.

So $100e^{-0.1t} = 10,$ that is, $e^{-0.1t} = 0.1$.

Taking logs: $-0.1t = \ln 0.1 = -2.3$.

So $t = 23$ seconds.

Simple Harmonic Motion

Solve the simple harmonic oscillator *equation* $y'' + 25y = 0$, *with initial conditions* $y(0) = 3$ *and* $y'(0) = -5$.

Here $\omega^2 = 25$, so $\omega = 5$.

General solution: $y = C_1 \cos(5t) + C_2 \sin(5t)$.

Then $y' = -5C_1 \sin(5t) + 5C_2 \cos(5t)$.

Initial condition $y(0) = 3$ gives $3 = C_1$.

Initial condition $y'(0) = -5$ gives $-5 = 5C_2$, so $C_2 = -1$.

Solution: $y = 3\cos(5t) - \sin(5t)$.

Second-order Linear Homogeneous ODE

Solve the constant coefficient, second-order, linear, homogeneous ODE $y'' + 8y' + 15y = 0$, *subject to initial conditions* $y(0) = 4$ *and* $y'(0) = -8$.

Auxiliary equation is $m^2 + 8m + 15 = 0$.

This factors into $(m+3)(m+5) = 0$.

This has solutions $m = -3$ and $m = -5$.

We know each of these gives an "e^{mt}" solution.

So the general solution is $y = C_1 e^{-3t} + C_2 e^{-5t}$.

Differentiating this we get $y' = -3C_1 e^{-3t} - 5C_2 e^{-5t}$.

Applying initial condition $y(0) = 4$ gives $4 = C_1 + C_2$.

Applying initial condition $y'(0) = -8$ gives

$-8 = -3C_1 - 5C_2$.

Adding three times the first equation to the second, we get $12 - 8 = -2C_2$, so $C_2 = -2$.

The first equation then gives $C_1 = 6$.

So the solution is $y = 6e^{-3t} - 2e^{-5t}$.

Second-order Linear Homogeneous ODE

Solve the constant coefficient, second-order, linear, homogeneous ODE $y'' + 14y' + 49y = 0$, *subject to initial conditions* $y(0) = 1$ *and* $y'(0) = -5$.

Auxiliary equation is $m^2 + 14m + 49 = 0$.

This factors into $(m+7)(m+7) = 0$.

It has a double root $m = -7$.

So the general solution is $y = C_1 e^{-7t} + C_2\, t\, e^{-7t}$.

Differentiating (using the Product Rule) gives

$y' = -7C_1 e^{-7t} + C_2(e^{-7t} - 7t\, e^{-7t})$.

Applying initial condition $y(0) = 1$ gives $1 = C_1$.

Applying initial condition $y'(0) = -5$ gives
$-5 = -7C_1 + C_2 = -7 + C_2$.

So $C_2 = 2$.

So the solution is $y = e^{-7t} + 2t\,e^{-7t}$.

Second Order ODE with Forcing

The equation of motion of a damped mass spring system is $m\dfrac{d^2y}{dt^2} + \mu\dfrac{dy}{dt} + ky = f(t)$*.*
Consider a mass with $m = 1$ *and Hooke constant* $k = 10$ *and damping constant* $\mu = 2$*.*

The external force is $f(t) = 10t - 8$*.*

If the initial displacement is $y = 1$ *and the mass is released from rest at time* 0*, find the displacement at all times* $t \geqslant 0$*.*

Put in the given numbers to get the ODE

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 10t - 8,$$

with initial conditions $y(0) = 1$ and $y'(0) = 0$

(since the initial velocity is zero).

We write the full solution as $y = y_h + y_p$.

The homogeneous equation is $y_h'' + 2y_h' + 10y_h = 0$.

This has auxiliary equation $m^2 + 2m + 10 = 0$.

Solving this equation, we find it has complex roots

$m = -1 \pm 3i$.

The -1 gives e^{-t} and the $\pm 3i$ gives $e^{\pm 3it}$, which gives

$\cos(3t)$ and $\sin(3t)$.

So the homogeneous solution is

$y_h = C_1 e^{-t}\cos(3t) + C_2 e^{-t}\sin(3t)$.

We now look for a particular solution $y_p = \alpha t + \beta$.

Then $0 + 2\alpha + 10(\alpha t + \beta) = 10t - 8$.

That is $10\alpha t + 2\alpha + 10\beta = 10t - 8$.

So $10\alpha = 10$ and $2\alpha + 10\beta = -8$.

Solving these gives $\alpha = 1$ and $\beta = -1$.

So the particular solution is $y_p = t - 1$.

The general solution of the ODE is then

$y = C_1 e^{-t}\cos(3t) + C_2 e^{-t}\sin(3t) + t - 1$.

To apply the initial conditions, we also need the derivative.

Differentiating, using the Product Rule, gives

$$\begin{aligned} y' &= -C_1 e^{-t}\cos(3t) - 3C_1 e^{-t}\sin(3t) \\ &\qquad -C_2 e^{-t}\sin(3t) + 3C_2 e^{-t}\cos(3t) + 1. \end{aligned}$$

Applying the initial condition $y(0) = 1$ gives $1 = C_1 - 1$.

So $C_1 = 2$.

Applying the initial condition $y'(0) = 0$ gives

$0 = -C_1 + 3C_2 + 1$.

So $3C_2 = C_1 - 1 = 1$. That is $C_2 = \frac{1}{3}$.

So the solution is $y = 2e^{-t}\cos(3t) + \dfrac{1}{3}e^{-t}\sin(3t) + t - 1$.