Chapter 5:

Second Order Ordinary Differential Equations

Prerequisites: Properties of derivatives and integrals. Derivatives and integrals of standard functions. Properties of trigonometric functions and exponential functions.

Second Order Linear Differential Equations

A *second order ODE* is one that has $\dfrac{d^2y}{dt^2}$ in it

(as well as maybe $\dfrac{dy}{dt},\ y,\ t$ too).

Second order ODEs generally require doing two integrals which bring in two constants of integration. So the general solution to a second order ODE typically involves two unknown constants $C_1,\ C_2$.

A second order ODE is called *linear* if it can be written as

$$\frac{d^2y}{dt^2} + a(t)\frac{dy}{dt} + b(t)y = f(t)$$

for some functions $a(t),\ b(t)$ and $f(t)$.

If a and b are constants, it is called a *second order linear constant coefficient ODE* and is of the form

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = f(t) \qquad (\star)$$

where a and b are known constants (typically fixed parameters that describe the "physics" of some system, such as mass, spring stiffness etc.)

The function $f(t)$ is some known function that typically represents some external force applied to a system.

There are lots of examples of this. Many come from Newton's second law of motion (NewtonII):

$F = ma$ (force equals mass times acceleration),

since by definition the acceleration $a = \dfrac{d^2y}{dt^2}$.

So NewtonII becomes $\dfrac{d^2y}{dt^2} = \dfrac{F}{m}$.

When the force F is some constant linear combination of $\dfrac{dy}{dt}$, y, $f(t)$, we get an equation like $(\star)$.

The Homogeneous Case

If the 'external force' $f(t) = 0$ in $(\star)$, then the equation is said to be **homogeneous** and $(\star)$ becomes

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0.$$

We will solve this in full but for the moment, for simplicity, we will assume $a = 0$ and consider the resulting special case $\frac{d^2y}{dt^2} + by = 0 \qquad (\star\star)$.

We now solve this in the three only possible cases (i) $b = 0$, (ii) $b < 0$, and (iii) $b > 0$.

Homogeneous Second Order ODEs: Case 1: $b = 0$.

Here equation $(\star)$ becomes $\frac{d^2y}{dt^2} = 0$.

This means 'no force' (internal or external).

We write this ODE as $\frac{d}{dt}\left(\frac{dy}{dt}\right) = 0$.

Integrating this we get $\frac{dy}{dt} = C_1$,

where C_1 is a constant of integration.

Integrating this again, we get $y(t) = C_1 t + C_2$,

where C_2 is another constant of integration.

In the Newton's Law context, the solution is $y = v_0 t + y_0$ where y_0 is thought of as the initial position of a particle moving with constant speed v_0. That is, we put $C_1 = v_0, C_2 = y_0$.

This equation represents a particle moving in a straight line with constant velocity v_0 (resembling something like constant linear motion in deep outer space, where there are no forces.)

Homogeneous Second Order ODEs: Case 2: $b < 0$.

Observation: every negative number $b < 0$ can be written as $b = -\omega^2$, where $\omega > 0$.

We put this into equation $(\star\star)$, so in case (ii), we are dealing with the equation $\dfrac{d^2y}{dt^2} - \omega^2 y = 0$.

We write this as $\dfrac{d^2y}{dt^2} = \omega^2 y$.

We solve this by "guessing" two independent functions that satisfy it!

Recall the function $y = e^{\omega t}$ has the following properties:

$$\frac{d}{dt}(e^{\omega t}) = \omega e^{\omega t} \quad \text{and}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(\omega e^{\omega t}) = \omega(\omega\, e^{\omega t}) = \omega^2 e^{\omega t} = \omega^2 y.$$

So $y = e^{\omega t}$ satisfies the ODE.

Similarly $C_1 e^{\omega t}$ is also a solution for any constant C_1.

In the same way, you can check that $y = e^{-\omega t}$ also satisfies the ODE and hence so too does $C_2 e^{-\omega t}$ for any constant C_2 (since $(-\omega)^2 = \omega^2$).

You can also check in the same way that

$$y(t) = C_1 e^{\omega t} + C_2 e^{-\omega t}$$

is a solution for arbitrary constants C_1, C_2.

This is the general solution of the ODE.

It is a combination of exponential growth and decay.

Example 1

Find the general solution of the ODE $\dfrac{d^2y}{dt^2} - 4y = 0.$

This gives $\dfrac{d^2y}{dt^2} = 4y.$

We identify $\omega^2 = 4,$ and hence $\omega = 2.$

So the general solution is $y(t) = C_1 e^{2t} + C_2 e^{-2t}.$

Before we do the next example, we take a short diversion.

Notation

Sometimes $\frac{dy}{dt}$ is written as y' or as $\dot{y}$.

Likewise, $\frac{d^2y}{dt^2}$ is written as y'' or as $\ddot{y}$.

So, for example, equation ($\star$) could be written as

$$y'' + ay' + by = f(t) \qquad (\star)$$

So, for example, when we write $y'(0) = 3,$ we mean $y' = 3$ (that is, $\frac{dy}{dt} = 3$) when $t = 0$.

Example 2

Find the solution of the ODE $y'' = 9y,$ with initial conditions $y(0) = 5$ and $y'(0) = 3$.

First we identify $\omega^2 = 9,$ so $\omega = 3$.

So the general solution is $y(t) = C_1e^{3t} + C_2e^{-3t}$.

We now use the initial conditions to find the constants C_1, C_2.

Since $y(0) = 5,$ we get $5 = C_1 + C_2$ (using $e^0 = 1$).

To use the second initial condition, we must first differentiate $y(t)$.

Differentiating $y(t)$ gives $y'(t) = 3C_1e^{3t} - 3C_2e^{-3t}$.

So the condition $y'(0) = 3$ means $3 = 3C_1 - 3C_2$.

Hence $C_1 - C_2 = 1$.

The initial conditions therefore give two simultaneous equations:

$C_1 + C_2 = 5$

$C_1 - C_2 = 1$

Adding these gives $2C_1 = 6$, so $C_1 = 3$.

Then the first equation gives $C_2 = 2$.

Therefore the solution is $y(t) = 3e^{3t} + 2e^{-3t}$.

You can check that this satisfies both the differential equation and the initial conditions.

Homogeneous Second Order ODEs: Case 3: $b > 0$. Simple Harmonic Motion

Observation: every positive number $b > 0$ can be written as $b = \omega^2$, where $\omega > 0$.

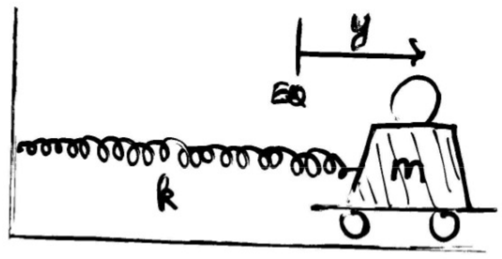
We put this into equation $(\star\star)$, so in case (ii), we are dealing with the equation $\frac{d^2y}{dt^2} + \omega^2 y = 0$.

We can also write this as $\frac{d^2y}{dt^2} = -\omega^2 y$.

This type of equation arises in modelling an oscillating mass attached to a spring (simple harmonic motion).

See *Animation Mass-Spring*.

Simple Harmonic Motion in an undamped Mass-Spring system



A mass m is attached to a spring, with spring constant k.

The mass is displaced from equilibrium a distance $y(t)$ at time t.

NewtonII says $F = m\frac{d^2y}{dt^2}$

and Hooke's Law (for a spring) says: $F = -ky$.

These combine to give the ODE $m\frac{d^2y}{dt^2} = -ky$.

That is, $\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$.

That is, $\frac{d^2y}{dt^2} + \omega^2 y = 0$, where $\omega = \sqrt{\frac{k}{m}}$.

This is known as the *Simple Harmonic Oscillator (SHO) equation*.

It very important for many applications outside of the mass-spring setting.

To solve it, we want to find functions for which
$\frac{d^2(?)}{dt^2} = -\omega^2(?)$.

Again we 'guess' two independent functions that satisfy it!

First note $\frac{d}{dt}(\cos \omega t) = -\omega \sin \omega t$.

Consequently, $\frac{d^2}{dt^2}(\cos \omega t) = \frac{d}{dt}(-\omega \sin \omega t) = -\omega^2 \cos \omega t$.

So $y = \cos \omega t$ solves the equation, and hence so does $y = C_1 \cos \omega t$, for any constant C_1.

Similarly $y = \sin \omega t$ also solves the equation, as does $y = C_2 \sin \omega t$.

So the general solution of SHO equation is

$$y(t) = C_1 \cos \omega t + C_2 \sin \omega t.$$

Notice that $y(t)$ is periodic, with period $T = \frac{2\pi}{\omega}$, that is, $y(t+T) = y(t)$.

Here ω is called the *angular frequency* of the oscillation.

Some comments on the general SHO solution

$y(t) = C_1 \cos \omega t + C_2 \sin \omega t$.

It is possible to rearrange this so it is in the form $y = A \sin(\omega t + \phi)$, called the *phase-amplitude form*.

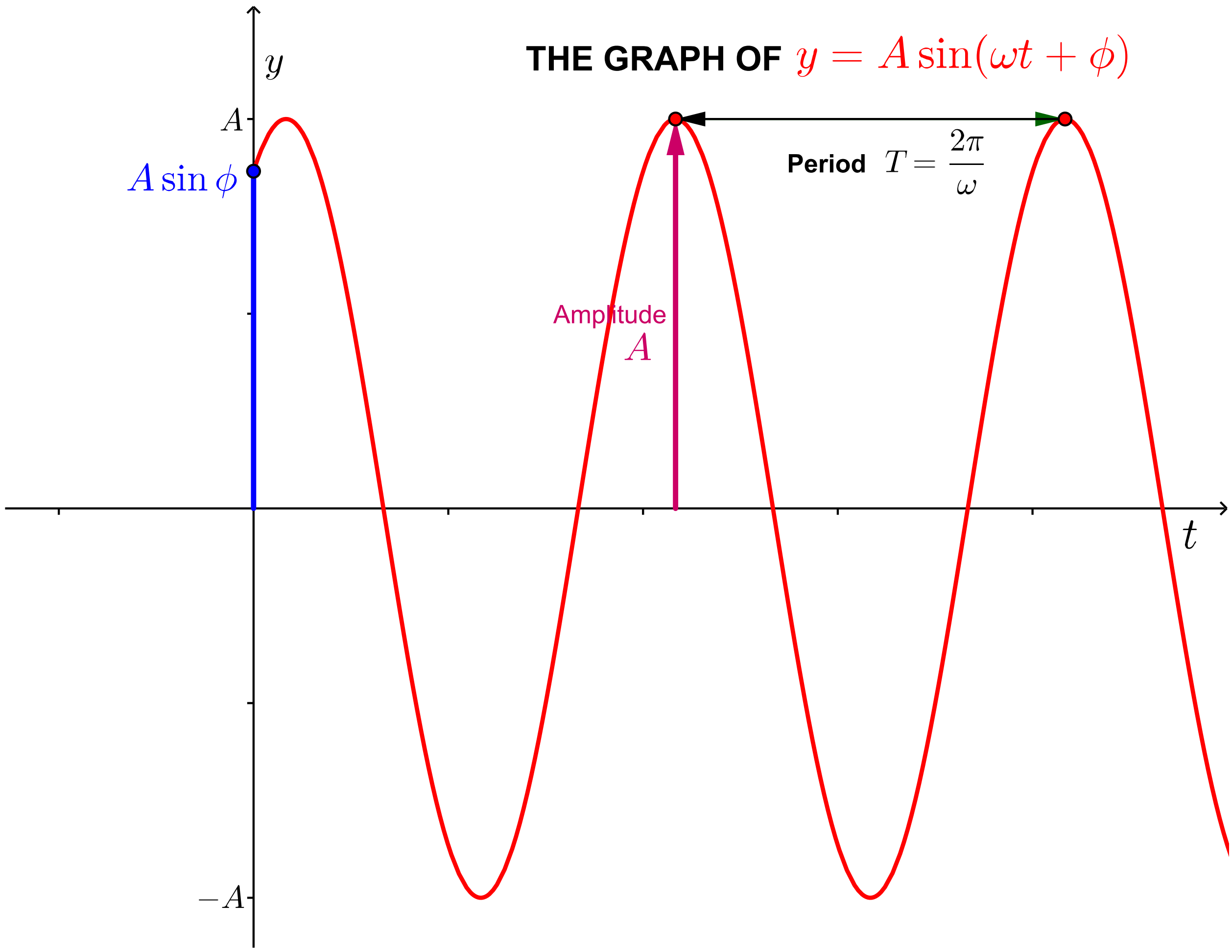
It is just a sine wave (sinusoid).

Here A is called the *amplitude*. It is related to the previous form by $A = \sqrt{C_1^2 + C_2^2}$.

The number of oscillations (cycles) per second is called the *frequency* f and it is defined by $f = \frac{1}{T} = \frac{\omega}{2\pi}$.

It is measured is cycles/second or *Hertz*.

Here ϕ is called the *phase*. It determines where the oscillation starts.



Examples

Example 1 *Find the general solution of the SHO equation* $y'' + 16y = 0$.

We identify $\omega^2 = 16$ and hence $\omega = \sqrt{16} = 4$.

So the general solution is $y(t) = C_1 \cos(4t) + C_2 \sin(4t)$.

Example 2 *Solve* $y'' + 9y = 0$, *subject to the initial conditions* $y(0) = 3$, $y'(0) = 12$.

To solve this, we first identify $\omega = \sqrt{9} = 3$ and write down the general solution $y(t) = C_1 \cos(3t) + C_2 \sin(3t)$.

Now apply the initial conditions: $y(0) = 3$ means $y = 3$ when $t = 0$, so $3 = y(0) = C_1 + 0$, so $C_1 = 3$.

Before we apply the second initial condition, we must get the derivative of y.

That is $y'(t) = -3C_1 \sin(3t) + 3C_2 \cos(3t)$.

So $12 = y'(0) = 0 + 3C_2$, which tells us that $C_2 = \frac{12}{3} = 4$.

Having found C_1, C_2, we can write down the solution to this problem: $y(t) = 3\cos(3t) + 4\sin(3t)$.

Aside: The period is $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{3}$
and the amplitude is $A = \sqrt{C_1^2 + C_2^2} = \sqrt{3^2 + 4^2} = 5$.

Example 3 *Suppose a mass of* 2*kg is attached to a spring with spring constant* $k = 8$*. The mass starts from rest (at time* 0*) and is initially extended by* 5*m from its rest/equilibrium position. Determine the subsequent (simple harmonic) motion when it is released, that is the displacement* $y(t)$ *at all times* t*.*

The equation of motion of an undamped mass-spring system is $\dfrac{d^2y}{dt^2} + \omega^2 y = 0$, where $\omega = \sqrt{\dfrac{k}{m}}$.

Here $m = 2$ and $k = 8$, so $\omega = \sqrt{\dfrac{8}{2}} = 2$.

So the general solution is $y(t) = C_1 \cos(2t) + C_2 \sin(2t)$.

One initial condition gives $y(0) = 5$, so $C_1 + 0 = 5$.

The other initial condition gives $y'(0) = 0$.

First we need $y'(t) = -2C_1 \sin(2t)C_1 + 2C_2 \cos(2t)$.

So $0 = y'(0) = 0 + 2C_2$, and hence $C_2 = 0$.

Therefore, $C_1 = 5$ and $C_2 = 0$, so the subsequent motion is described by $y(t) = 5\cos(2t)$.

Application: Vibrations of a Diatomic Molecule

In a classical (non-quantum mechanical) model of a diatomic molecule, such as CO, it will vibrate (oscillate) according to the ODE $m_r x'' = -kx$, where m_r is the 'reduced mass' of the pair of atoms (in kg) and k is the bond constant of the atoms in N/m.

(See animation *diatomic.gif*)

This gives $x'' + \frac{k}{m_r}x = 0,$ that is $x'' + \omega^2 x = 0,$

where $\omega = \sqrt{\frac{k}{m_r}}$ is the angular frequency.

The solution is $x(t) = C_1 \cos \omega t + C_2 \sin \omega t$.

The period of the motion is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_r}{k}}$.

The frequency of the vibration is $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m_r}}$.

For $CO,\ m_r = 1.139 \times 10^{-26}\,kg$ and $k = 1860\,N/m$.

So

$$f = \frac{1}{2\pi}\sqrt{\frac{1860}{1.139 \times 10^{-26}}} = 6.43 \times 10^{13} Hz = 64,300\,GHz.$$

(Note: GHz (GigaHertz) is a billion cycles per second.)