Taking the natural logarithm of both sides of the above expression yields

$$\frac{\mu_1 - \mu_0}{\sigma^2} \sum_{i=1}^{n} x_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{n(\mu_1^2 - \mu_0^2)}{2\sigma^2}$$

or

$$\frac{1}{n} \sum_{i=1}^{n} x_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\mu_1 + \mu_0}{2} \tag{8.36}$$

Equation (8.36) indicates that the statistic

$$s(X, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

provides enough information about the observations to enable us to make a decision. Thus, it is called the *sufficient statistic* for the maximum likelihood test.

8.14. Consider the same observations X_i, $i = 1, \ldots, n$, of radar signals as in Prob. 8.13, but now, under H_0, X_i have zero mean and variance σ_0^2, while under H_1, X_i have zero mean and variance σ_1^2, and $\sigma_1^2 > \sigma_0^2$. Determine the maximum likelihood test.

In a similar manner as in Prob. 8.13, we obtain

$$f(\mathbf{x} \mid H_0) = \frac{1}{(2\pi\sigma_0^2)^{n/2}} \exp\left(-\frac{1}{2\sigma_0^2} \sum_{i=1}^{n} x_i^2\right)$$

$$f(\mathbf{x} \mid H_1) = \frac{1}{(2\pi\sigma_1^2)^{n/2}} \exp\left(-\frac{1}{2\sigma_1^2} \sum_{i=1}^{n} x_i^2\right)$$

With $\sigma_1^2 - \sigma_0^2 > 0$, the likelihood ratio is

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left[\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2}\right) \sum_{i=1}^{n} x_i^2\right]$$

and the maximum likelihood test is

$$\left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left[\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2}\right) \sum_{i=1}^{n} x_i^2\right] \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Taking the natural logarithm of both sides of the above expression yields

$$\sum_{i=1}^{n} x_i^2 \underset{H_0}{\overset{H_1}{\gtrless}} n\left[\ln\left(\frac{\sigma_1}{\sigma_0}\right)\right]\left(\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2}\right) \tag{8.37}$$

Note that in this case,

$$s(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^2$$

is the sufficient statistic for the maximum likelihood test.

8.15. In the binary communication system of Prob. 8.6, suppose that we have n independent observations $X_i = X(t_i)$, $i = 1, \ldots, n$, where $0 < t_1 < \cdots < t_n \le T$.

(a) Determine the maximum likelihood test.

(b) Find P_I and P_{II} for $n = 5$ and $n = 10$.

(a) Setting $\mu_0 = 0$ and $\mu_1 = 1$ in Eq. (8.36), the maximum likelihood test is

$$\frac{1}{n} \sum_{i=1}^{n} x_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}$$