0.1 What is Game Theory

Game theory is a branch of applied mathematics that uses models to study interactions with formalised incentive structures ("games").

Unlike decision theory, which also studies formalised incentive structures, game theory encompasses decisions that are made in an environment where various players interact strategically. In other words, game theory studies choice of optimal behavior when costs and benefits of each option are not fixed, but depend upon the choices of other individuals.

Game theory is the science of strategic reasoning, in such a way that it studies the behaviour of rational game players who are trying to maximise their utility, profits, gains, etc., in interaction with other players, and therefore in a context of strategic interdependence.

Game theory has applications in a variety of fields, including economics, international relations, evolutionary biology, political science, and military strategy. Game theorists study the predicted and actual behaviour of individuals in games, as well as optimal strategies. Seemingly different situations can have similar incentive structures, thus all exemplifying one particular game.

0.1.1 Types of Games

In game theory, the unscrupulous diner's dilemma (or just diner's dilemma) is an n-player prisoner's dilemma. The situation imagined is that several individuals go out to eat, and prior to ordering, they agree to split the check equally between all of them. Each individual must now choose whether to order the expensive or inexpensive dish. It is presupposed that the expensive dish is better than the cheaper, but not by enough to warrant paying the difference when eating alone. Each individual reasons that the expense s/he adds to their bill by ordering the more expensive item is very small, and thus the improved dining experience is worth the money. However, having all reasoned thus, they all end up paying for the cost of the more expensive meal, which by assumption, is worse for everyone than having ordered and paid for the cheaper meal.

0.2 Two-Person Zero-Sum Games: Basic Concepts

Game theory provides a mathematical framework for analyzing the decision-making processes and strategies of adversaries (or players) in different types of competitive situations. The simplest type of competitive situations are two-person, zero-sum games. These games involve only two players; they are called zero-sum games because one player wins whatever the other player loses.

A zero-sum game is a mathematical representation of a situation in which each participant's gain or loss of utility is exactly balanced by the losses or gains of the utility of the other participants.

0.3 Example: Odds and Evens

Consider the simple game called **odds and evens**. Suppose that player 1 takes evens and player 2 takes odds. Then, each player simultaneously shows either one finger or two fingers. If the number of fingers matches, then the result is even, and player 1 wins the bet ($2). If the number of fingers does not match, then the result is odd, and player 2 wins the bet ($2). Each player has two possible strategies: show one finger or show two fingers. The payoff matrix shown below represents the payoff to player 1.

Image

0.4 Basic Concepts of Two-Person Zero-Sum Games

This game of odds and evens illustrates important concepts of simple games.

A two-person game is characterized by the strategies of each player and the payoff matrix.

- The payoff matrix shows the gain (positive or negative) for player 1 that would result from each combination of strategies for the two players. Note that the matrix for player 2 is the negative of the matrix for player 1 in a zero-sum game.
- The entries in the payoff matrix can be in any units as long as they represent the utility (or value) to the player.
- There are two key assumptions about the behavior of the players. The first is that both players are rational. The second is that both players are greedy meaning that they choose their strategies in their own interest (to promote their own wealth).

0.5 Payoff Matrix

How to read a payoff matrix : Game Theory Eg Payoff matrix for a new technology game

IMAGE

0.5.1 Explanation

1. There are 2 firms A and B and they want to decide whether to Start a new campaign.
2. each firm will be affected by its competitors decision.
3. The above table shows the payoff to both firms. This table is called **payoff matrix**.
4. (a,b) -The first number in each cell is the payoff(profits) to A and second number in each cell is the payoff to B.
    - (10,5) shows the payoffs when both firms start a new campaign. Firm As profits are 10 and firm Bs are 5.
    - (15,0) shows the payoffs when firm A starts a new campaign abd Firm B does not. Firm As profits are 15 and firm Bs are 0.

0.5.2 Cooperative game

- A cooperative game is a game wherein two or more players do not compete, but rather strive toward a unique objective and therefore win or lose as a group.
- Cooperative games are rare, but still many exist. One example is "Stand Up", where a number of individuals sit down, link arms (all facing away from each other) and attempt to stand up. This objective becomes more difficult as the number of players increases.
- Another is the counting game, where the players, as a group, attempt to count to 20 with no two participants saying the same number twice. In a cooperative version of volleyball, the emphasis is on keeping the ball in the air for as long as possible.
- Cooperative games are rare in recreational gaming, where conflict between players is a powerful force. However, such scenarios can occur in real life (when the sense of the word "game" is extended beyond recreational games). For example, operation of a successful business is, at least in theory, a cooperative game, since all participants benefit if the business succeeds and suffer if it fails.
- Role-playing games are the most common form of cooperative game, though these games are not always purely cooperative. In such games, the players (who act through personae called "characters") usually strive toward intertwined and similar goals. However, each character has his or her own ambitions, and ultimately, individual goals. Hence conflict between characters often occurs in these games.

0.5.3 Winner's Curse

The winner's curse is a phenomenon that may occur in common value auctions with incomplete information. In short, the winner's curse says that in such an auction, the winner will tend to overpay. The winner may overpay or be "cursed" in one of two ways: 1) the winning bid exceeds the value of the auctioned asset such that the winner is worse off in absolute terms; or 2) the value of the asset is less than the bidder anticipated, so the bidder may still have a net gain but will be worse off than anticipated.[1] However, an actual overpayment will generally occur only if the winner fails to account for the winner's curse when bidding (an outcome that, according to the revenue equivalence theorem, need never occur).

0.6 Pure and Mixed Strategies

- A pure strategy determines all your moves during the game (and should therefore specify your moves for all possible other players' moves).
- A mixed strategy is a probability distribution over all possible pure strategies (some of which may get zero weight).

0.7 Pure and mixed strategies

A pure strategy provides a complete definition of how a player will play a game. In particular, it determines the move a player will make for any situation he or she could face. A player's strategy set is the set of pure strategies available to that player.

A mixed strategy is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy. Since probabilities are continuous, there are infinitely many mixed strategies available to a player.

Of course, one can regard a pure strategy as a degenerate case of a mixed strategy, in which that particular pure strategy is selected with probability 1 and every other strategy with probability 0.

A totally mixed strategy is a mixed strategy in which the player assigns a strictly positive probability to every pure strategy. (Totally mixed strategies are important for equilibrium refinement such as trembling hand perfect equilibrium.)

0.7.1 What is a pure strategy?

A pure strategy is an unconditional, defined choice that a person makes in a situation or game. For example, in the game of Rock-Paper-Scissors,if a player would choose to only play scissors for each and every independent trial, regardless of the other players strategy, choosing scissors would be the players pure strategy. The probability for choosing scissors equal to 1 and all other options (paper and rock) is chosen with the probability of 0. The set of all options (i.e. rock, paper, and scissors) available in this game is known as the strategy set.

0.7.2 What is a mixed strategy?

A mixed strategy is an assignment of probability to all choices in the strategy set. Using the example of Rock-Paper-Scissors, if a persons probability of employing each pure strategy is equal, then the probability distribution of the strategy set would be 1/3 for each option, or approximately 33

The definition of a mixed strategy does not rule out the possibility for an option(s)to never be chosen (eg. pscissors= 0.5, prock = 0.5, ppaper = 0). This means that in a way, a pure strategy can also be considered a mixed strategy at its extreme, with a binary probability assignment (setting one option to 1 and all others equal to 0). For this article, we shall say that pure strategies are not mixed strategies.

Dominance

In game theory, strategic dominance (commonly called simply dominance) occurs when one strategy is better than another strategy for one player, no matter how that player's opponents may play. Many simple games can be solved using dominance. The opposite, intransitivity, occurs in games where one strategy may be better or worse than another strategy for one player, depending on how the player's opponents may play.

0.7.3 Terminology

When a player tries to choose the "best" strategy among a multitude of options, that player may compare two strategies A and B to see which one is better. The result of the comparison is one of:

- B dominates A: choosing B always gives as good as or a better outcome than choosing A. There are 2 possibilities:
- B strictly dominates A: choosing B always gives a better outcome than choosing A, no matter what the other player(s) do.
- B weakly dominates A: There is at least one set of opponents' action for which B is superior, and all other sets of opponents' actions give B the same payoff as A.
- B and A are intransitive: B neither dominates, nor is dominated by, A. Choosing A is better in some cases, while choosing B is better in other cases, depending on exactly how the opponent chooses to play. For example, B is "throw rock" while A is "throw scissors" in Rock, Paper, Scissors.
- B is dominated by A: choosing B never gives a better outcome than choosing A, no matter what the other player(s) do. There are 2 possibilities:
- B is weakly dominated by A: There is at least one set of opponents' actions for which B gives a worse outcome than A, while all other sets of opponents' actions give A the same payoff as B. (Strategy A weakly dominates B).
- B is strictly dominated by A: choosing B always gives a worse outcome than choosing A, no matter what the other player(s) do. (Strategy A strictly dominates B).

This notion can be generalized beyond the comparison of two strategies.

Strategy B is strictly dominant if strategy B strictly dominates every other possible strategy. Strategy B is weakly dominant if strategy B dominates all other strategies, but some (or all) strategies are only weakly dominated by B. Strategy B is strictly dominated if some other strategy exists that strictly dominates B. Strategy B is weakly dominated if some other strategy exists that weakly dominates B.

0.7.4 A two person zero-sum game

The starting point in the mathematical theory of games is that the outcome of a game is determined by the strategies of the players. A two person zero-sum game is a game in which the winnings of one player equal the losses of the other for every combination of strategies. Taking winnings to be positive and losses to be negative gives a zero sum in each case. Viewing a game from one players point of view, we could represent the outcomes (called pay-offs) for each combination of strategies in a matrix. This is called the pay-off matrix for that player.

0.7.5 Example

A and B are two players in a zero-sum game. A uses one of two strategies, W or X, and B uses one of the strategies Y or Z. The table shows the pay-off matrix for A.

The pay-off matrix shows that if B adopts strategy Y then the pay-off for A will be 2 by using strategy W and 5 using strategy X.

On the other hand, if B adopts strategy Z then the pay-off for A will be 02 using strategy W and 04 using strategy X.

The idea is that neither player knows in advance which strategy the other will use.

The situation could equally be represented by the pay-off matrix for B. This would show corresponding values with opposite signs since this is a zero-sum game.

0.7.6 The play-safe strategy

The play-safe strategy for a player is the strategy for which the minimum pay-off is as high as possible. In the example above, the minimum pay-off for A using strategy W is 02, whereas the minimum pay-off using strategy X is 04. This means that strategy W is the play-safe strategy for player A.

Notice that finding the play-safe strategy for player A involves comparing the minimum values in the rows of the pay-off matrix for A.

Finding the play-safe strategy for B will involve comparing the values in the columns, remembering that Bs pay-offs are the negatives of the ones in the pay-off matrix for A.

The minimum value for B using strategy Y is 05 and the minimum value using strategy Z is 2. This means that the play-safe strategy for B is strategy Z.

The situation is shown in the pay-off matrices for A and B.

In this case, the maximum of the minimum pay-offs, for each player, is in the corresponding position in the two matrices. This represents the stable solution to the problem referred to as the saddle point (or minimax point).

The solution is stable in the sense that neither player can improve their pay-off by taking a different strategy, given that the other player doesnt change.

In other words, while B uses strategy Z, the best strategy for A is W and while A uses strategy W, the best strategy for B is Z.

KEY POINT - If the sum of the two values used to determine the play-safe strategies is not zero then the values cannot correspond to the same cell position in the play-off matrices. This means that there is no saddle point and the game has no stable solution.

Example The pay-off matrices for two players in a zero-sum game are given below. Show that there is no stable solution for the game.

Optimal strategies for games that are not stable The repeated use of the same strategy over a series of games is called a pure strategy. This provides the best results for both players in a game which has a stable solution. In the case where no stable solution exists, a mixed strategy is used in which each of the strategies is employed with a given probability to find the optimal solution.

Graphical representation Each graph corresponds to a strategy for Q (i.e. the opponent of P).

The diagram shows graphs of 9p 4 and 9p + 6 against values of p from 0 to 1. The point of intersection corresponds to the probability that gives the optimal mixed strategy for P in the last example.

KEY POINT - When Ps opponent has more strategies there will be more graphs with several points of intersection. You will need to identify the one that represents the optimal mixed strategy for P. This will be the highest point on or below each of the graphs.

This diagram represents a situation where Ps opponent has three strategies to choose from. The point representing the optimal mixed strategy for P is circled.

Notice how the problem of identifying the right vertex can be expressed as a linear programming problem in which the object is to maximise the expected gain for P subject to the constraints represented by the regions bounded by the straight line graphs.

This is, in fact, the approach used for higher dimensional problems. The conditions are formulated as a linear programming problem which may then be solved by the simplex algorithm.

KEY POINT - If the pay-offs for one strategy are always better than the corresponding pay-offs for some other strategy then the weaker one can be ignored when determining the probabilities for a mixed strategy. In this way, the pay-off matrix is reduced by what is called a dominance argument.