0.1 Introduction to Matrix Games

Every matrix A defines a game as follows

(1) There are two players, one called R (or player 1), and the other called C (or player 2).

(2) A play of the game consists of R choosing a row of matrix A and, simultaneously, C's choosing of a column of matrix A.

(3) After each play of the game, R receives from C an amount equal to the entry in the chosen cell. A negative entry denotes a payment from R to C.

0.1.1 Example 1.1

Consider the following matrix games G_1 and G_2:

G_1 :

3	−1	4
−1	1	−2

G_2;

0	−5
1	2

- In the first game, i.e. G_1, if R consistently plays the first row, hoping to win an amount 3 or 4, then C can play the second column and so win an amount 1. However if C consistently plays the second column then R can play the second row and so win an amount 1.
- The we see that if either player should consistently play a particular row or column, the other player can take advantage of that fact.
- In the second game,i.e. G_2 R can guarantee winning 1 or more by consistently playing row 2. Player C can minimize their loss by playing column 1. Thus in this case it is "best" if R plays a given rown and C a given column.

On the other hand, suppose a competitive situation is as follows:

(i) there are two people (or organizations, or competitors etc)

(ii) each competitor has a finite number of courses of action.

(iii) both competitors choose a course of action simultaneously.

(iv) for each pair i, j of choices, there is a payment b_{ij} from one given player to the other.

The the above competitive situation is equivalent to the matrix game determined by the matrix $B = b_i$

0.1.2 Example 1.2 : Odds and Evens

Each of the two competitors R and C simultaneously show 1 or 2 fingers. If the sum of the fingers is even, then R wins the sum from C, of the sum is odd, the R loses the sum to C. The matrix of this game is as follows:

$A =$

3	0	−4	−3
2	3	**1**	2
−4	2	−1	3

We shall show later that this game is favourable to C.

We remark that matrix games and the above competition games which can be represented by matrix games are termed *two-person zero sum games.* The zero sum corresponds to the fact that the sum of the winnings and losses of all of the players is zero after each play. There also exists a theory for n-person games and non-zero sum games but that lies beyond the scope of this course.

0.1.3 Strategies

- By a *strategy* for R in a matrix game, we mean a decision by R to play the various rows with a given probability distribution, for example to play row 1 with a probability of p_1, to play row 2 with a probability of p_2 and so on.
- The strategy for R is formally denoted by the probability vector $\boldsymbol{p} = \{p_1, p_2, \ldots, p_n\}$.
- For example, if the matrix game has two rows and r tosses a coin to decide which row to play, then their strategy is the probability vector $p = (1/2, 1/2)$.
- Similarly By a strategy for C in a matrix game, we mean a decision by C to play the various columns with a given probability distribution, for example to play columns 1 with a probability of q_1, to play column 2 with a probability of q_2 and so on.
- The strategy for C is formally denoted by the probability vector $\boldsymbol{q} = \{q_{1,2}, \ldots, q_n\}$.

A strategy which contains a 1 as a component and 0 everywhere else, i.e. where R decides to play consistently a given row or C decides to play consistently a given column, is called a ***pure strategy***, otherwise it is called a ***mixed strategy***.

0.2 Optimum Strategies and the Value of the game

For simplicity, we shall first consider a 2×2 matrix game, say

$$A = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

Suppose player R adopts the strategy $p = (p_1, p_2)$ and player C adopts a strategy $q = (q_1, q_2)$. Then R play row 1 with probability p_1 and C plays column 1 with probability q_1, and so the entry a will occur with probability $p_1 \times q_1$, entry b will occur with probability $p_1 \times q_2$ and so on.

The expected winnings for player R

$$E(p, q) = a(p_1 \times q_1) + b(p_1 \times q_2) + c(p_2 \times q_1) + d(p_2 \times q_2)$$

$$E(p, q) = (p_1, p_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = (pAq^t)$$

You can show that E(p,q) can be expressed as qA^tp^t. Similarly if A is any $k \times m$ matrix and player R adopts the strategy $p = ()$ and C adopts the strategy $q = ()$ then the expected winding of R is given b

$$E(p, q) = pAq^t = qA^tp^t$$

Now suppose that v_1 is the maximum numer for which there is a strategy p^0 for R such that expected winnings of R is at least v_1 for any strategy adopted by C:

$$E \text{ for every strategy q of C}$$

Since v_1 is the maximum of such numbers of such numbers, the strategy p^0 is in a certain sense a best strategy that R can adopt; we called p^0 an optimum strategy for R. We show in problem 25.9 page 233 that (1) is equivalent to the condition

$$A \geq \{v_1, v1, \ldots, v_1\}$$

On the other hand, suppose v_2 is the minimum number for which there is a strategy q^0 for player C such that the expected loss of C is not greater than v_2 for any strategy adopted by$R$ Then the strategy q^0 is in a certain sense a best strategy that C can adopt, we call q^0 an optimum strategy or C. The fundamental theorem on game theory by Von Neumann states that v_1 and v_2 exist for any matrix game and are equal to each other : $v = v_1 = v_2$. This number v is called the value of the game, and the game is said to be fair if $v = 0$.ann's th We observe that (1) and (2) imply that $v_1 leq E(p^0, q^0) \leq v_2$, hence by von Neumann's Theorem we have the equality relation

$$E(p^0, q^0) = p^0 A(q^0)^t = q^0 A^t (p^0)^t = v$$

WE summarize and formally define the terms introduced above.

0.2.1 Definition

In a matrix game A, a strategy P^0 for player R is an optimum strategy, a strategy q^0 for player C is an optimum strategy, and a number v is the value ofthe game if p^0,q^0 and v are related as follows.

- $p^0 A \geq (v, v, \ldots v)$
- $q^0 A^t \leq (v, v, \ldots v)$

if $v = 0$ the game is said to be fair.

We point out that the solution of a matrix means find optimum strategies for the players and the value of the game.

We remark that the solution of a matrix game means finding optimum strategies for the players and the value of the game. Before we give the solution to an arbitrary matrix game by means of the simplex method We shall discuss two special cases which occur frequently and which offer simple solutions: striclty determined games and 2×2 games.

0.3 Strictly Determined Games

A matrix game is called strictly determined if the matrix has an entry which is a minimum in its row and a maximum in its column. Such an entry is called a "saddle point".

Theorem Let v be a saddle point of a strictly determined game. Then an optimum strategy for R is always to play the row containing v, an optimum strategy for C is always to play the column containing v, and v is the value of the game.

Thus in a strictly determined game a pure strategy for each player is an optimum strategy.

$A =$

3	0	−4	−3
2	3	1	2
−4	2	−1	3

If we choose the minimum entry in each row,

$A =$

3	0	**−4**	−3
2	3	**1**	2
−4	2	−1	3

and the maximum entry in each column,

$A =$

3	0	−4	−3
2	**3**	**1**	2
−4	2	−1	**3**

we obtain the following.

$A =$

3	0	−4	−3
2	3	**1**	2
−4	2	−1	3

We now show that p^0, q^0 and v do satisfy the required properties to be optimum strategies and the value of the game:

1.

$A =$

3	0	−4	−3
2	3	**1**	2
−4	2	−1	3

2.

$A =$

3	0	−3
2	3	2
−4	−1	3
−4	−1	3

We remark that the proof of Theorem 25.1 is similar to the proof above the special case.

0.3.1 2×2 Matrix Games

If A is strictly determined, then the solution is indicated in the preceding section. Thus we need only considered the case in which A is non-strictly determined. We first state a useful criterion to determine whether or not a 2×2 matrix game is strictly determined.

Lemma The following theorem applies

Lemma

Suppose that the above matrix A is strictly determined. Then $P^0 = (x_1, x_2)$ is an optimum strategy for player R, $Q^0 = (y_1, y_2)$ is an optimum strategy for player C, and y is the value of the game.

$$x_1 = \frac{d-c}{a+d-b-c}, x_2 = \frac{a-b}{a+d-b-c}$$

$$y_1 = \frac{d-b}{a+d-b-c}, y_2 = \frac{a-c}{a+d-b-c}$$

$$v = \frac{ad-bc}{a+d-b-c}$$

Note that the five quotients above have the same denominator, and that the numerator of the formula for v is the determinant of the matrix A.

Example 3.1

Consider the following matrix game (see Example 1.2).

2	−3
−3	4

Use the above formula, we can obtain the value of the game.

$$v = \frac{(2 \times 4) - (-3 \times -3)}{2+4+3+3} = \frac{-1}{12}$$

Thus the game is not fair and is in favour of the column player C. Optimum strategies p^0 for R and q^0 for C are as follows: $p^0 = (7/12, 5/12)$ and $q^0 = (7/12, 5/12)$.

0.3.2 Recessive Rows and Columns

- Let A be a matrix game. Suppose A contains a row r_i such that $r_i \leq rj$ for some other row r_j.
- Recall that $r_i \leq r_j$ means that every entry of r_i is less than or equal to the corresponding entry of r_j.
- The r_i is called a ***recessive row***, and r_j is said to dominate it.
- (Important) Cleary player R would always play row j than row i since they are guaranteed to win the same or a greater amount in each possible play of the game. Accordingly a recessive row can be omitted from the game.
- On the other hand, suppose A contains a column c_i such that $c \geq c_i$ for some other column c_i. Then c_i is called a recessive column and c_j is said to dominate it.
- For analogous reasons player C would always rather play column c_j than c_i. Hence the recessive column c_i can be omitted from the game.
- We emphasize the a recessive row contains numbers smaller than those of another row, whereas a recessive column contains numbers than those of another column.

0.3.3 Example 4.1

Consider the game A.

2	-1	2
-2	3	4
-2	3	4

.

Note the $(-5,-3,1) \leq (2,-1,2)$, i.e. every entry is the first row is $\leq$ the corresponding entry in the second row. Thus the first row is recessive and can be omitted from the game may be reduced to

2	-1	2
-2	3	4

.

Now observe that the third row is recessive since each entry is greater than corresponding entry in the second column. Thus the game can be reduced to a 2×2 game.

2	-1
-2	3

.

The solution to A^* can be found by using the Theorem and is

$$v = \frac{4}{8} = 12; p_0^* = (5/8, 3/8); q_0^* = (1/2, 1/2)$$

Then the solution to the original game A is

$$v = \frac{1}{2}; p_0^* = (0, 5/8, 3/8); q_0^* = (1/2, 1/2, 0)$$