Consider a binary decision problem with the following conditional p.d.fs.

- $f(x|H_0) = \frac{1}{2}e^{-|x|}$
- $f(x|H_1) = \frac{1}{2}e^{-2|x|}$

The Bayes' costs are given by

$C_{00} = 0$	$C_{01} = 2$
$C_{10} = 1$	$C_{11} = 0$

- Determine the Bayes test if $P(H_0) = 2/3$.
- Compute the associated Bayes' Risk.

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1) Compute the likelihood ratio.

$$\Lambda(x) = \frac{f(x|H_1)}{f(x|H_0)} = \frac{e^{-2|x|}}{\frac{1}{2}e^{-|x|}}$$

REMARK $e^{2K} = (e^{K})^2$.

$$\Lambda(x) = \frac{\left[e^{-|x|}\right]^2}{\frac{1}{2}\left[e^{-|x|}\right]} = 2e^{-|x|}$$

- $\Lambda(x) > \eta$: Decide H_1
- $\Lambda(x) < \eta$: Decide H_0

where

$$\eta = \frac{(C_{10} - C_{00})\, P(H_0)}{(C_{01} - C_{11})\, P(H_1)}$$

Therefore we need to compute η

$$\eta = \frac{(1-0)\,2/3}{(2-0)\,1/3} = \frac{2/3}{2/3} = 1.$$

$$\therefore\; 2e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \quad \text{or} \quad e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} 1/2.$$

Logarithm of both side then negate

$$\ln\left[e^{-|x|}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left[1/2\right]$$

$$|x| \underset{H_0}{\overset{H_1}{\lessgtr}} -\ln\left[1/2\right]$$

therefore

$$|x| \underset{H_0}{\overset{H_1}{\lessgtr}} 0.693$$

CAREFUL

operator switches Direction

$\therefore$ 0.693 is our threshold. The decision Regions are

- $R_0 : \{x : |x| > 0.693\}$
- $R_1 : \{x : |x| < 0.693\}$

$$P(D_1|H_0) = \int_{-0.693}^{0.693} \frac{1}{2} e^{-|x|} dx$$

$$= 2\int_0^{0.693} \frac{1}{2} e^{-x} dx = \int_0^{0.693} e^{-x} dx.$$

$$= 0.5$$

see note about integration of absolute value functions.

$$P(D_0|H_1) = \int_{-\infty}^{-0.693} e^{2x}\,dx + \int_{0.693}^{\infty} e^{-2x}\,dx$$

$$= 2\int_{0.693}^{\infty} e^{-2x}\,dx = 0.25.$$

Bayes Risk:

$$\bar{C} = P(D_1|H_0)P(H_0) \leftarrow \text{cost} = 1$$

$$+\, 2\,P(D_0|H_1)P(H_1) \leftarrow \text{cost} = 2$$

(other costs are zero)

$$= (0.5)\tfrac{2}{3} + 2(0.25)\tfrac{1}{3}$$

$$= \underline{0.5}$$

(see workings page 7)

modification of previous question

Consider a binary decision problem with the following conditional pdfs.

- $f(x \mid H_0) = \frac{1}{2} e^{-|x|}$
- $f(x \mid H_1) = e^{-2|x|}$

$C_{00} = C_{11} = 0$ $\quad C_{01} = 2 \quad C_{10} = 1.$

- Determine the Bayes test if $P(H_0) = \frac{1}{2}$
- Compute the associated Bayes risk.

$$\Lambda(x) = \frac{f(x|H_1)}{f(x|H_0)} = 2e^{-|x|}$$

$$\eta = \frac{(1-0)^{1/2}}{(2-0)^{1/2}} = \frac{1}{2}.$$

$$\therefore e^{-|x|} \underset{H_0}{\gtrless} \frac{1}{4}.$$

$$\boxed{|x| \underset{H_0}{\overset{H_1}{\lessgtr}} 1.386}$$

decision operator switches direction.

The Decision Regions Are therefore

$$R_0 : \{ x : |x| > 1.386 \}$$

$$R_1 : \{ x : |x| < 1.386 \}$$

$$P(D_1 | H_0) = 2 \int_0^{1.386} \tfrac{1}{2} e^{-x} \, dx = \underline{0.75}$$

see note about integration of functions with absolute values.

$$P(H_0 | D_1) = 2 \int_{1.386}^{\infty} e^{-2x} \, dx = 0.0625$$

Bayes Risk

$$\bar{C} = (0.75 \times \tfrac{1}{2}) + 2(0.0625)(1/2) = 0.4375.$$

$$\boxed{\text{REMARK. } P(D_0, H_1) = P(D_0 | H_1) \times P(H_1) \quad \text{i.e conditional prob. formula}}$$

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* WORKINGS from Page 3

$$\int e^{-x}\,dx = -1 \times e^{-x}$$

$$2\int_0^{0.693} \frac{e^{-x}}{2}\,dx = \frac{2}{2}\left[-e^{-0.693} - \left[-e^{0}\right]\right]$$

double negative

$$= \frac{2}{2}\left[-\frac{1}{2} + 1\right].$$

$$\approx \underline{\underline{0.5}}$$

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* WORKINGS from Page 4

$$\int e^{-2x}\,dx = \frac{e^{-2x}}{-2}.$$

$$2\int_{0.693}^{\infty} e^{-2x}\,dx = 2\left[e^{-\infty} - \left(-\frac{e^{-2(0.693)}}{2}\right)\right]$$

$$= 2\left[0 + \frac{0.25}{2}\right]$$

$$= \underline{\underline{0.25}}$$