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Game theory

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Undergraduate study in
Economics, Management,
Finance and the Social Sciences

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THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

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This is one of a series of subject guides published by the University. We regret that due to pressure of work the author is unable to enter into any correspondence relating to, or arising from, the guide. If you have any comments on this subject guide, favourable or unfavourable, please use the form at the back of this guide.

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Chapter 1
Introduction to the guide

40 Game theory is a 300 half-course offered on the Economics, Management, Finance and the Social Sciences (EMFSS) suite of programmes.

Game theory is the formal study of conflict and cooperation. It is concerned with situations where 'players' interact, so that it matters to each player what the other players do. Game theory provides mathematical tools to model, structure and analyse such interactive scenarios. The player's may be, for example, competing firms, political voters, mating animals, or buyers and sellers on the internet. The language and concepts of game theory are widely used in economics, political science, biology, and computer science, to name just a few disciplines.

Game theory helps to understand effects of interaction that seem puzzling at first. For example, the famous 'prisoners' dilemma' explains why fishers can exhaust their resources by over-fishing. They hurt themselves collectively, but each fisher on his own cannot really change this and still profit by fishing as much as possible. Other insights come from the way of looking at interactive situations. Game theory treats players equally and recommends to each player how to play well, given what the other players do. This mind-set is useful in strategic questions of management, because 'you put yourself in your opponent's shoes'.

1.1 Background to this guide

The content and style of this subject guide originate from a course on game theory that I have taught for many years at the London School of Economics. It is offered as a third-year half-course in mathematics, and is one of the largest third-year courses at the LSE, of ever-increasing popularity. Its students come from mathematics, economics, management, decision science, philosophy and other study disciplines.

Game theory is fascinating as a topic because of its diverse applications. The ideas of game theory started with mathematicians, most notably the outstanding mathematician John von Neumann (1903–57). In the 1950s, a group of young researchers in mathematics at Princeton developed game theory further, among them John Nash, Harold Kuhn and Lloyd Shapley, and these pioneers can still, over 50 years later, be met at conferences on game theory. Most research in game theory is now done by economists and other social scientists.

My own interests are in the mathematics of games, so I see my own research in the tradition of early game theory. My research specialty is the connection of game theory to computer science. In particular, I develop methods to find equilibria of games, which make use of insights from geometry. That interest is partly reflected in the choice of topics in this guide. This guide has also a strong emphas is on methods, so you will learn alot of 'tricks' that allow you to understand games quickly. With these methods at hand, you will be in a position to analyse games that you can create for applications to manage mentor economics.

I hope that you enjoy studying this course.

1.2 Aims and objectives

The half-course is designed to:

- familiarise you with formal methods for strategic analysis
- develop the mathematical theory of games as used in economics.

1.3 Learning outcomes

At the end of this half-course, and having completed the exercises and activities, you should have:

- knowledge of fundamental concepts of non-cooperative game theory
- the ability to apply solution concepts to examples of games, andto state and explain them precisely
- the ability to solve unseen games that are variants of known examples.

1.4 How to use this subject guide

The aim of this subject guide is to help you to interpret the syllabus. It providesa self-contained, comprehensive text for each area of the syllabus, and a set of detailed exercises together with their solutions. Occasionally, it also suggests relevant readings to help you to understand the material.

Unlike many of the courses available on the International Programmes, there is no set textbook which you must read for this course. All of the information you need to learn and understand is contained in examples and activities within the subject guide itself, and in the exercises, which you should attempt to solve **before** you look at the solutions.

This guide is carefully written to introduce and illustrate each concept with examples, many pictures, and a minimum of notation. Not every concept is defined with a formal definition (although many concepts are), but instead explained by means of an example. It should in each case be clear how the concept applies in similar settings, and in general. On the other hand, this requires some maturity of you, the reader, in dealing with mathematical concepts, and in being able to generalise from examples.

Based on my experience of teaching in the classroom, I have tried to be particularly careful whenever the material becomes difficult. At the same time, I have always tried to maintain the 'big picture', and to be as clear as possible.

I would recommend that you work through the guide in the order of the chapters. One possible exception is chapter 2 on combinatorial games (the first 'proper' chapter of this guide), which is independent of the other material. On the other hand, this topic is deliberately put at the beginning because it is more mathematical and formal than the other topics, so that it can be used to test whether you can cope with the abstract parts of game theory, and with the mathematical style of this subject.

Chapters 3, 4, and 5 successively build on each other. The final chapter 6 on bargaining is partly independent of chapters 4 and 5, so to a large extent it can be understood after chapter 3.

The five main chapters of this guide are long, and require concentrated study. Each chapter is subdivided in to many sections. It is useful to pause after each section and reflect on your understanding of the material. What is the point made in this section? (I have tried to be explicit and clear in this respect wherever possible.) What would be a suitable example? Examples are given throughout the text, and you may want to modify an existing example to test your understanding. In that way, you may even anticipate some issues that are taken up later in the text, and it is most gratifying to gain understanding in such an active way.

If you don't understand something even at a repeated attempt, it is often helpful to leave the problem aside, read on, and repeat the reading after having gained a coarse understanding. You, as reader, should gain an overview over the piece of text that you currently study (the current paragraph, section, or chapter), and be in control of what to read next. This is a general study and reading skill.

In the text, some topic sare explained in great detail. If some explanation is too slow for you, you can easily skip to the next paragraph. But beware: every paragraph makes a point, which you should not miss.

At the end of each chapter, we repeat the learning objectives given at the beginning of the chapter. Once you have covered the material in the chapter and the associated exercises, use this as a checklist of the main points that you should understand, and if necessary revise those parts that you are not sure about.

1.5 Activities and exercises

Throughout the text, activities are indicated within a box, like in this example:

> ⇒ Do exercise 6.4 on page 146.

Such a reference to an exercise is typically given at the first timein the text where you can do the exercise. You are, of course, free to decide when to do the exercise, but you should definitely do all exercises once youhave finished a chapter.

The exercises are grouped together at the end of each chapter. They are tried andtested, and form an essential part of your study in order to understand the material. Complete and detailed solutions are also given for each exercise. However, do not cheat yourself by looking at the solution when you get stuck. The solutions hould only be a final check that your own attempt is correct. Often, the given solution is more detailed than what would be expected from a student submitting this as homework, or as an exam answer. Such a detailed solution is meant to provide an additional explanation of the topic.

Except for exercises that take a long time to answer, most of these exercises may also occur in some variation as exam questions.

Another type of activity asks you to review immediately a concept, or prove some very simple property. An example is the following:

> ⇒ Make sure you clearly understand the distinction between perfect information and perfect recall. How are the respective concepts defined?

The purpose of such an activity is to make you pause and reflect because you will usually not be able follow the subsequent material without an understanding of the current issue. There are also some activities that ask you to elaborate on a mathematical reasoning, as in the following example (see page 134):

> ⇒ Show that concavity of u implies the inequality $u(1-\varepsilon/2) \geq 1-\varepsilon/2$ stated in the preceding paragraph.

These are activities for which **no solution** is provided, because their very purpose is that you perform this reasoning yourself. All questions of this kind are very easy to answer. If you do not succeed in answering them, you may find them easier at a later stage when re-reading the chapter.

1.6 Structure of the guide

Game theory is presented in this guide in five main chapters 2–6 (apart from this introductory chapter).These chapters, in particular chapters 3–5, cover the main concepts and methods of non-co-operative game theory. Chapter 6 on bargaining is a particularly interesting application of that theory.

Each chapter has a first 'guide' section with sub-sections that detail the aims of the chapter, learning objectives, essential and further reading, and further guidelines. The next section contains an **introduction** and outline of the results of the chapter. These introductory sections provide detailed chapter summaries. We give here a coarse outline of each chapter.

Chapter 2 on combinatorial games is actually on playing and winning games with perfect information defined by rules, in particular a simple game called 'nim', which has a central role in that theory. This chapter introduces abstract mathematics with a fun topic. You would probably not learn its content otherwise, because it does not have a typical application in economics. However, every game theorist should know the basics of combinatorial games. In fact,this can be seen as the motto for the contents of this course: **what every game theorist should know**.

With the exception of imperfect information, the fundamentals of non-cooperative game theory are laid out in chapter 3. This part of game theory provides ways to model in detail the agents in an interactive situation, their possible actions, and their incentives. The model is called a **game** and the agents are called **players**. There are two types of games, called **game trees** and games in **strategic form**. The game tree (also called the **extensive form** of a game) describes in depth the actions that are available to the players, how these evolve over time, and what the players know or do not know about the game. (Games with imperfect information are treated in chapter 5.) The players' incentives are modelled by payoffs that these players want to maximise, which is the sole guiding principle in non-cooperative game theory. In contrast, cooperative game theory studies, for example, how players should split their proceeds when they decide to cooperate, but leaves it open how they enforce an agreement. A simple example of this cooperative approach is explained in chapter 6 on bargaining. This 'bargaining solution' is then given an incentive-based justification with a more detailed non-cooperative model.

Chapter 4 shows that in certain games it may be useful to leave your actions uncertain. A nice example is the football penalty kick, which serves as our introduction to zero-sum games (see figure 4.11). The striker should not always kick into the same corner, nor should the goalkeeper always jump into the same corner, even if they are better at scoring or saving a penalty there. It is better to be unpredictable! Game theory tells the players how to choose optimal **probabilities** for each of their available strategies, which are then used to mix these strategies randomly. With the help of mixed strategies, every game has an **equilibrium**. This is the central result of John Nash, who discovered the equilibrium concept in 1950 for general games. For zero-sum games, this was already found earlier by John von Neumann. It is easier to prove for zero-sum games that they have an equilibrium than for general games. In a logical progression of topics, in particular when starting from win/lose games like nim, we could have treated zero-sum games before general games.However, we choose to treat zero-sum games later, as special cases of general games, because the latter are much more important in economics. In a course on game theory, one could omit zero-sum games and their special properties, which is why we treat them in the last section of chapter 4. However, one could not omit the concept of Nash equilibrium, which is therefore given prominence early on.

Chapter 5 explains how to model the **information** that players have in a game. This is done by means of so-called information sets in game trees, introduced in 1953 by Harold Kuhn. The central result of this chapter is called Kuhn's theorem. Essentially, this result states that players can choose a 'behaviour strategy', which is away of playing the game that is not too complicated, provided they do not forget what they knew and did earlier. This result is typically considered as technical, and given short shrift in many game theory texts. We go in to great detail in explaining this result. The first reason is that it is a beautiful result of discrete mathematics, because elementary concepts like the game tree and the information sets are combined naturally to give a new result. Secondly, the result is used in other, more elaborate 'dynamic' games that develop overtime. For more advanced studies of game theory, it is therefore useful to have a solid understanding of game trees with imperfect information.

We have already outlined the purpose of chapter 5, which treats bargaining situations. A first model provides conditions – called axioms – that an acceptable 'solution' to bargaining situations should fulfil and shows that these axioms lead to a unique solution of this kind. A second model of 'alternating offers' is more detailed and uses, in particular, the analysis of game trees with perfect information introduced in chapter 3.

We often use the symbol □ to denote the end of a proof, where we have finished explaining why a particular result is true. This is just to make it clear where the proof ends and the following text begins.

1.7 Essential reading

There is **no** required text for this course because the subject guide is self-contained. For supplementary reading, the two books by Mendelson and by Osborne and Rubinstein listed in section 1.8 together cover most of the material in this guide.

Game theory, and this guide, use only a few **prerequisites** from elementary linear algebra, probability theory, and some analysis. You should know that vectors and point have a **geometric interpretation** (for example, as either points or vectors in three-dimensional space if the dimension is three). It should be clear how to multiply matrices, and how to multiply a matrix with a vector. The required notions from probability theory are that of **expected value** of a function (that is, function values weighted with their probabilities), and that **independent** events have a probability that is the product of the probabilities of the individual events. The concepts from analysis are those of a **continuous** function, which, for example, assumes its maximum on a compact (closed and bounded) domain. None of these concepts is difficult, and you are reminded of their basic ideas in the text whenever they are needed.

However, it will quickly become apparent that this subject guide is a mathematical text that requires some maturity and experience with mathematical concepts. Most importantly, concepts are defined formally so that they have a precise meaning. Statements need carefully stated assumptions, and a proof of their claim. The mentioned 'mathematical maturity' of you, the reader, means that you recognise a proof even when it is only outlined in a paragraph in the main text, and that you can generalise from examples.

1.8 Further reading

Please note that you are then free to read around the subject area in any text, paper or online resource. You will need to support your learning by reading as widely as possible and by thinking about how these principles apply in the real world. To help you read extensively, you have free access to the virtual learning environment (VLE) and University of London Online Library (see below).

No matter how much the author of a text strives for clarity, students often like to see an alternative description. Most of the material in this guide is covered in at least one of the following two books.

Mendelson, Elliot *Introducing Game Theory and Its Applications*. (Chapman & Hall/CRC, 2004) [ISBN 1584883006].

Unlike most other textbooks, the book by Mendelson gives a detailed treatment of combinatorial games. It also treats zero-sum games in much more depth than the present subject guide. On the other hand, general games are given much less attention.

Osborne, M.J. and A. Rubinstein *A Course in Game Theory*. (MIT Press, 1994) [ISBN 0262650401].

Osborne and Rubinstein treat game theory as it is used in economics. Rubinstein is also a pioneer of bargaining theory. He invented the alternating-offers model treated in chapter 6 of this guide. On the other hand, the book uses some non-standard descriptions. For example, Osborne and Rubinstein define games of perfect information via 'histories' and not game trees; we prefer the latter because they are less abstract.

Other useful texts are the following.

Bewersdorff, J. *Logic, Luck and White Lies*.(A.K. Peters, 2005) [ISBN 1568812108].

This book is not a textbook, but a study of parlour games from the mathematical viewpoint. Its ho s how concepts of logic, probability theory, and game theory can be used to play games well. The text is entertaining reading for the mathematically inclined. It is mathematically sound and a rich source of historical references.

Another entertaining read goes in the opposite direction to the book by Bewersdorff:

Dixit, Avinash K.,and Barry J.Nalebuff *Thinking Strategically: The Competitive Edge in Business, Politics, and Everyday Life*. (W.W.Norton,1993) [ISBN 0393310353].

This book can be viewed as a game theory textbook in disguise,but without the mathematics. Remarkably, it has been a business best seller that one can buy in airport bookstalls. This maybe a good way to get started in game theory, so that you are motivated to learn its mathematics from the present course.

Gibbons, R. *A Primer in Game Theory* [in the United States sold under the title *Game Theory for Applied Economists*].(Prentice Hall/Harvester Wheatsheaf, 1992) [ISBN 0745011594].

This book describes the theory and applications of games for someone who is already familiar with economic models, for example the Cournotduopoly of competing firm who have to decide on quantities of production.

A useful resource for game-theoretic books and other texts is the website www.gametheory.net

1.9 Online study resources

In addition to the subject guide and the reading, it is crucial that you take advantage of the study resources that are available online for this course, including the VLE and the Online Library.

You can access the VLE, the Online Library and your University of London email account via the Student Portal at:
http://my.londoninternational.ac.uk

You should have received your login details for the Student Portal with your official offer, which was emailed to the address that you gave on your application form. You have probably already logged in to the Student Portal in order to register! As soon as you registered, you will automatically have been granted access to the VLE, Online Library and your fully functional University of London email account.

If you forget your login details at any point, please email uolia.support@london.ac.uk quoting your student number.

The VLE

The VLE, which complements this subject guide, has been designed to enhance your learning experience, providing additional support and a sense of community. It forms an important part of your study experience with the University of London and you should access it regularly.

The VLE provides a range of resources for EMFSS courses:

- Self-testing activities: Doing these allows you to test your own understanding of subject material.
- Electronic study materials: The printed materials that you receive from the University of London are available to download, including updated reading lists and references.
- Past examination papers and *Examiners' commentaries*: These provide advice on how each examination question might best be answered.
- A student discussion forum: This is an open space for you to discuss interests and experiences, seek support from your peers, work collaboratively to solve problems and discuss subject material.
- Videos: There are recorded academic introductions to the subject, interviews and debates and, for some courses, audio-visual tutorials and conclusions.
- Recorded lectures: For some courses, where appropriate, the sessions from previous years' Study Weekends have been recorded and made available.
- Study skills: Expert advice on preparing for examinations and developing your digital literacy skills.
- Feedback forms.

Some of these resources are available for certain courses only, but we are expanding our provision all the time and you should check the VLE regularly for updates.

Making use of the Online Library

The Online Library contains a huge array of journal articles and other resources to help you read widely and extensively.

To access the majority of resources via the Online Library you will either need to use your University of London Student Portal login details, or you will be required to register and use an Athens login: http://tinyurl.com/ollathens

The easiest way to locate relevant content and journal articles in the Online Library is to use the **Summon** search engine.

If you are having trouble finding an article listed in a reading list, try removing any punctuation from the title, such as single quotation marks, question marks and colons.

For further advice, please see the online help pages: www.external.shl.lon.ac.uk/summon/about.php

1.10 Examination structure

Important: Please note that subject guides may be used for several years. Because of this we strongly advise you to always check both the current *Regulations*, for relevant information about the examination, and the VLE where you should be advised of any forthcoming changes. You should also carefully check the rubric/instructions on the paper you actually sit and follow those instructions.

The examination for this half-course is two hours in duration and you are expected to answer **four** questions, from a choice of **five**. The Examiner attempts to ensure that all of the topics covered in the syllabus and subject guide are examined. Some questions could cover more than one topic from the syllabus because the different topics are not self-contained, and to ensure that students do not avoid entire chapters (hence, for example, questions on combinatorial games will appear as parts of more than one exam question).

Two sample examination papers appear as an appendix to this guide, along with *Examiners' commentaries*. The *Examiners' commentaries* contain valuable information about how to approach the examination and so you are strongly advised to read them carefully. Past examination papers and the associated reports are valuable resources when preparing for the examination.

You should ensure that all four questions are answered, allowing an approximately equal amount of time for each question, and attempting all parts or aspects of a question.

Remember, it is important to check the VLE for:

- up-to-date information on examination and assessment arrangements for this course
- where available, past examination papers and *Examiners' commentaries* for this course which give advice on how each question might best be answered.

1.11 Examination advice

This is a quantitative course, and it is therefore mandatory (and much easier) to **understand** rather than memorise a topic. Examination questions always contain parts that are 'bookwork' or routine work, but they often serve as warm-ups to familiarise the student with the question.

Here are some hints on **exam preparation**:

- Understand the easy cases first. Be active in understanding the basics, by solving exercises and creating your own examples. This will often lead you on to the more complicated parts, and will at any rate give you the basis for the more advanced topics.
- Don't panic. Postpone a topic that you don't understand to a later time. Better to understand one concept clearly, precisely and thoroughly, than all concepts superficially.
- Practise exercises and questions from previous exams.During the exam,read each question carefully, in particular when the question looks familiar, and answer it in full. In order to test your understanding, almost every exam question has some new twist that forces you to think rather than rattle off some straight forward method.

During the exam:

- Concentrate on reading the question first and answer all parts of the question (otherwise you may miss out on easy points).
- Every sentence in a question is there for a reason. Answer the parts in order, because earlier parts will often lead on to the later parts.
- While keeping the time in mind, it is advisable to focus, in order, on a single question (out of four) at a time and not flit back and forth between questions.
- Be precise and concise in your answers, and **use words** and not just formulas or equations, to explain what you do. Short sentences, and sometimes even key words, suffice, as long as you convey clearly what you mean.
- Don't waffle. You will never get credit for ambiguous answers that maybe right or wrong.

1.12 Syllabus

If taken as part of a BSc degree, you must have passed **05a Mathematics 1** and **05b Mathematics 2** or **174 Calculus** before this half-course may be attempted.

This half-course is an introduction to game theory. At the end of this half-course, you should be familiar with the main concepts of non-cooperative game theory, and know how they are used in modelling and analysing an interactive situation. The key concepts are:

- Players are assumed to act out of self-interest (hence the term 'non-cooperative' game theory).This is not identical to monetary interest, but can be anything subjectively desirable. Mathematically, this is modeled by a utility function.
- Players should act strategically. This means that playing well does not mean being smarter than the rest, but assuming that everybody else is also 'rational' (acting out of self-interest). The game theorist's recommendation how to play must therefore be such that everybody would follow it. This is captured by the central concept of Nash equilibrium.
- It can be useful to randomise. In antagonistic situations, a player may play best by rolling a die that decides what to do next. In poker, for example, it maybe useful to occasionally bet high even on a weak hand ('to bluff') so that your opponent will take the bet even if you have a strong hand.

Topics covered are:

- Combinatorial games and nim.
- Game trees with perfect information, backward induction.
- Extensive and strategic form of a game.
- Nash equilibrium.
- Commitment.
- Mixed strategies and Nash equilibria in mixed strategies.
- Finding mixed-strategy equilibria for two-person games.
- Zero-sum games, max-min strategies.
- Extensive games with information sets, behaviour strategies, perfect recall.
- The Nash bargaining solution.
- Multistage bargaining.

Chapter 2

Nim and combinatorial games

2.1 Aims of the chapter

This chapter

- introduces the basics of combinatorial games, and explains the central role of the game nim.

A detailed summary of the chapter is given in section 2.5.

Furthermore, this chapter

- demonstrates the use of abstract mathematics in game theory.

This chapter is written more formally than the other chapters, in parts in the traditional mathematical style of definitions, theorems and proofs. One reason for doing this, and why we start with combinatorial games, is that this topic and style serves as a warning shot to those who think that game theory, and this course in particular, is 'easy'. If we started with the well-known 'prisoner's dilemma' (which makes its due appearance in Chapter 3), the less formally inclined student might be lulled into a false sense of familiarity and 'understanding'. We therefore start deliberately with an unfamiliar topic.

This is a mathematics course, with great emphasis on rigour and clarity, and on using mathematical notions precisely. As mathematical prerequisites, game theory requires only the very basics of linear algebra, calculus and probability theory. However, game theory provides its own conceptual tools that are used to model and analyse interactive situations. This course emphasises the mathematical structure of these concepts, which belong to 'discrete mathematics'. Learning a number of new mathematical concepts is exemplified by combinatorial game theory, and it will continue in the study of classical game theory in the later chapters.

2.2 Learning outcomes

After studying this chapter, you should be able to:

- play nim optimally;
- explain the concepts of game-sums, equivalent games, nim values and the mex rule;
- apply these concepts to play other impartial games like those described in the exercises.

2.3 Essential reading

This chapter of the guide.

2.4 Further reading

Very few textbooks on game theory deal with combinatorial games. An exception is chapter 1 of the following book:

- Mendelson, Elliot *Introducing Game Theory and Its Applications.* (Chapman & Hall / CRC, 2004) [ISBN 1584883006].

The winning strategy for the game nim based on the binary system was first described in the following article, which is available electronically from the JSTOR archive:

- Bouton, Charles 'Nim, a game with a complete mathematical theory.' *The Annals of Mathematics*, 2nd Ser., Vol. 3, No. 1/4 (1902), pp. 35–39.

The definitive text on combinatorial game theory is the set of volumes 'Winning Ways' by Berlekamp, Conway and Guy. The material of this chapter appears in the first volume:

- Berlekamp, Elwyn R., John H. Conway and Richard K. Guy *Winning Ways for Your Mathematical Plays, Volume 1*, second edition. (A. K. Peters, 2001) [ISBN 1568811306].

Some small pieces of that text have been copied here nearly verbatim, for example in Sections 2.7, 2.9, and 2.12 below.

The four volumes of 'Winning Ways' are beautiful books. However, they are not suitable reading for a beginner, because the mathematics is hard, and the reader is confronted with a wealth of material. The introduction to combinatorial game theory given here represents a very small fraction of that body of work, but may invite you to study it further.

A very informative and entertaining mathematical tour of parlour games is

- Bewersdorff, Jörg *Logic, Luck and White Lies.* (A. K. Peters, 2005) [ISBN 1568812108].

Combinatorial games are treated in part II of that book.

2.5 What is combinatorial game theory?

This chapter is on the topic of **combinatorial games**. These are games with two players, perfect information, and no chance moves, specified by certain rules. Familiar games of this sort are chess, go, checkers, tic-tac-toe, dots-and-boxes, and nim. Such games can be played perfectly in the sense that either one player can force a win or both can force a draw. In reality, games like chess and go are too complex to find an optimal strategy, and they derive their attraction from the fact that (so far) it is not known how to play them perfectly. We will, however, learn how to play nim perfectly.

There is a 'classical' game theory with applications in economics which is very different from combinatorial game theory. The games in classical game theory are typically formal models of conflict **and** co-operation which cannot only be lost or won, and in which there is often no perfect information about past and future moves. To the economist, combinatorial games are not very interesting. Chapters 3–6 of the course are concerned with classical game theory.

Why, then, study combinatorial games at all in a course that is mostly about classical game theory, and which aims to provide an insight into the theory of games as used in economics? The reason is that combinatorial games have a rich and interesting mathematical theory. We will explain the basics of that theory, in particular the central role of the game nim for impartial games. It is non-trivial mathematics, it is fun, and you, the student, will have learned something that you would most likely not have learned otherwise.

The first 'trick' from combinatorial game theory is how to win in the game nim, using the binary system. Historically, that winning strategy was discovered first (published by Charles Bouton in 1902). Only later did the central importance of nim, in what is known as the Sprague–Grundy theory of impartial games, become apparent. It also revealed why the binary system is important (and not, say, the ternary system, where numbers are written in base three), and learning that is more satisfying than just learning how to use it.

In this chapter, we first define the game nim and more general classes of games with perfect information. These are games where every player knows exactly the state of the game. We then define and study the concepts listed in the learning outcomes above, which are the concepts of game-sums, equivalent games, nim values and the mex rule. It is best to learn these concepts by following the chapter in detail. We give a brief summary here, which will make more sense, and should be re-consulted, after a first study of the chapter (so do not despair if you do not understand this summary).

Mathematically, any game is defined by other 'games' that a player can reach in his first move. These games are called the **options** of the game. This seemingly circular definition of a 'game' is sound because the options are **simpler** games, which need fewer moves in total until they end. The definition is therefore not circular, but **recursive**, and the mathematical tool to argue about such games is that of mathematical **induction**, which will be used extensively (it will also recur in chapter 3 as 'backward induction' for game trees). Here, it is very helpful to be familiar with mathematical induction for proving statements about natural numbers.

We focus here on **impartial** games, where the available moves are the same no matter whether player I or player II is the player to make a move. Games are 'combined' by the simple rule that a player can make a move in exactly one of the games, which defines a **sum** of these games. In a 'losing game', the first player to move loses (assuming, as always, that both players play as well as they can). An impartial game added to itself is always losing, because any move can be copied in the other game, so that the second player always has a move left. This is known as the 'copycat' principle (lemma 2.6). An important observation is that a losing game can be 'added' (via the game-sum operation) to any game without changing the winning or losing properties of the original game.

In section 2.11, the central theorem 2.10 explains the winning strategy in nim. The importance of nim for impartial games is then developed in section 2.12 via the beautiful mex rule. After the comparatively hard work of the earlier sections, we almost instantly obtain that any impartial game is equivalent to a nim heap (corollary 2.13).

At the end of the chapter, the sizes of these equivalent nim heaps (called nim values) are computed for some examples of impartial games. Many other examples are studied in the exercises.

Our exposition is distinct from the classic text 'Winning Ways' in the following respects: First, we only consider impartial games, even though many aspects carry over to more general combinatorial games. Secondly, we use a precise definition of **equivalent** games (see section 2.10), because a game where you are bound to lose against a smart opponent is not the **same** as a game where you have already lost. Two such games are merely equivalent, and the notion of equivalent games is helpful in understanding the theory. So this text is much more restricted, but to some extent more precise than 'Winning Ways', which should help make this topic accessible and enjoyable.

2.6 Nim – rules

The game nim is played with heaps (or piles) of chips (or counters, beans, pebbles, matches). Players alternate in making a move, by removing some chips from one of the heaps (at least one chip, possibly the entire heap). The first player who cannot move any more loses the game.

The players will be called, rather unimaginatively, player I and player II, with player I to start the game.

For example, consider three heaps of size $1,1,2$. What is a good move? Removing one of the chips from the heap with two chips will create the position $1,1,1$, then player II must move to $1,1$, then player I to 1, and then player II takes the last chip and wins. So this is not a good opening move. The winning move is to remove all chips from the heap of size 2, to reach position $1,1$, and then player I will win. Hence we call $1,1,2$ a **winning position**, and $1,1$ a **losing position**.

When moving in a winning position, the player to move can win by playing well, by moving to a losing position of the other player. In a losing position, the player to move will lose no matter what move she chooses, if her opponent plays well. This means that **all** moves from a losing position lead to a winning position of the opponent. In contrast, one needs only **one** good move from a winning position that goes to a losing position of the next player.

Another winning position consists of three nim heaps of sizes $1,1,1$. Here all moves result in the same position and player I always wins. In general, a player in a winning position must play well by picking the right move. We assume that players play well, forcing a win if they can.

Suppose nim is played with only two heaps. If the two heaps have equal size, for example in position $4,4$, then the first player to move loses (so this is a losing position), because player II can always **copy** player I's move by equalising the two heaps. If the two heaps have different sizes, then player I can equalise them by removing an appropriate number of chips from the larger heap, putting player II in a losing position. The rule for 2-heap nim is therefore:

Lemma 2.1 *The nim position m,n is winning if and only if $m \neq n$, otherwise losing, for all $m,n \geq 0$.*

This lemma applies also when $m = 0$ or $n = 0$, and thus includes the cases that one or both heap sizes are zero (meaning only one heap or no heap at all).

With three or more heaps, nim becomes more difficult. For example, it is not immediately clear if, say, positions $1,4,5$ or $2,3,6$ are winning or losing positions.

⇒ At this point, you should try exercise 2.1(a) on page 28.

2.7 Combinatorial games, in particular impartial games

The games we study in this chapter have, like nim, the following properties:

1. There are just two players.
2. There are several, usually finitely many, **positions**, and sometimes a particular **starting position**.
3. There are clearly defined **rules** that specify the **moves** that either player can make from a given position to the possible new positions, which are called the **options** of that position.
4. The two players move alternately, in the game as a whole.
5. In the **normal play** convention a player unable to move loses.
6. The rules are such that play will always come to an end because some player will be unable to move. This is called the **ending condition**. So there can be no games which are drawn by repetition of moves.
7. Both players know what is going on, so there is **perfect information**.
8. There are no **chance moves** such as rolling dice or shuffling cards.
9. The game is **impartial**, that is, the possible moves of a player only depend on the position but not on the player.

As a negation of condition 5, there is also the **misère play** convention where a player unable to move **wins**. In the surrealist (and unsettling) movie 'Last year at Marienbad' by Alain Resnais from 1962, misère nim is played, several times, with rows of matches of sizes $1, 3, 5, 7$. If you have a chance, try to watch that movie and spot when the other player (not the guy who brought the matches) makes a mistake! Note that this is misère nim, not nim, but you will be able to find out how to play it once you know how to play nim. (For games other than nim, normal play and misère versions are typically not so similar.)

In contrast to condition 9, games where the available moves depend on the player (as in chess where one player can only move white pieces and the other only black pieces) are called **partisan** games. Much of combinatorial game theory is about partisan games, which we do not consider to keep matters simple.

Chess, and the somewhat simpler tic-tac-toe, also fail condition 6 because they may end in a tie or draw. The card game poker does not have perfect information (as required in 7) and would lose all its interest if it had. The analysis of poker, although it is also a win-or-lose game, leads to the 'classical' theory of zero-sum games (with imperfect information) that we will consider later. The board game backgammon is a game with perfect information but with chance moves (violating condition 8) because dice are rolled.

We will be relatively informal in style, but our notions are precise. In condition 3 above, for example, the term **option** refers to a position that is reachable in one move from the current position; do not use 'option' when you mean 'move'. Similarly, we will later use the term **strategy** to define a plan of moves, one for every position that can occur in the game. Do not use 'strategy' when you mean 'move'. However, we will take some liberty in identifying a game with its starting position when the rules of the game are clear.

⇒ Try now exercises 2.2 and 2.3 starting on page 28.

2.8 Simpler games and notation for nim heaps

A game, like nim, is defined by its rules, and a particular starting position. Let G be such a particular instance of nim, say with the starting position $1,1,2$. Knowing the rules, we can identify G with its starting position. Then the options of G are $1,2$, and $1,1,1$, and $1,1$. Here, position $1,2$ is obtained by removing either the first or the second heap with one chip only, which gives the same result. Positions $1,1,1$ and $1,1$ are obtained by making a move in the heap of size two. It is useful to list the options systematically, considering one heap to move in at a time, so as not to overlook any option.

Each of the options of G is the starting position of another instance of nim, defining one of the new games H, J, K, say. We can also say that G is defined by the moves to these games H, J, K, and we call these **games** also the **options** of G (by identifying them with their starting positions; recall that the term 'option' has been defined in point 3 of section 2.7).

That is, we can define a game as follows: Either the game has no move, and the player to move loses, or a game is given by one or several possible moves to new games, in which the other player makes the initial move. In our example, G is defined by the possible moves to H, J, or K. With this definition, the entire game is completely specified by listing the initial moves and what games they lead to, because all subsequent use of the rules is encoded in those games.

This is a **recursive** definition because a 'game' is defined in terms of 'game' itself. We have to add the **ending** condition that states that every sequence of moves in a game must eventually end, to make sure that a game cannot go on indefinitely.

This recursive condition is similar to defining the set of natural numbers as follows: (a) 0 is a natural number; (b) if n is a natural number, then so is $n+1$; and (c) all natural numbers are obtained in this way, starting from 0. Condition (c) can be formalised by the principle of induction that says: if a property $P(n)$ is true for $n=0$, and if the property $P(n)$ implies $P(n+1)$, then it is true for all natural numbers.

We use the following **notation for nim heaps**. If G is a single nim heap with n chips, $n \geq 0$, then we denote this game by $*n$. This game is completely specified by its options, and they are:

$$\text{options of } *n: \quad *0,\ *1,\ *2, \ldots,\ *(n-1). \tag{2.1}$$

Note that $*0$ is the empty heap with no chips, which allows no moves. It is invisible when playing nim, but it is useful to have a notation for it because it defines the most basic losing position. (In combinatorial game theory, the game with no moves, which is the empty nim heap $*0$, is often simply denoted as 0.)

We could use (2.1) as the definition of $*n$; for example, the game $*4$ is defined by its options $*0, *1, *2, *3$. It is very important to include $*0$ in that list of options, because it means that $*4$ has a winning move. Condition (2.1) is a recursive definition of the game $*n$, because its options are also defined by reference to such games $*k$, for numbers k smaller than n. This game fulfils the ending condition because the heap gets successively smaller in any sequence of moves.

If G is a game and H is a game reachable by one or more successive moves from the starting position of G, then the game H is called **simpler** than G. We will often prove a property of games inductively, using the assumption that the property applies to all simpler games. An example is the – already stated and rather obvious –

property that one of the two players can force a win. (Note that this applies to games where winning or losing are the only two outcomes for a player, as implied by the 'normal play' convention in 5 above.)

Lemma 2.2 *In any game G, either the starting player I can force a win, or player II can force a win.*

Proof. When the game has no moves, player I loses and player II wins. Now assume that G does have options, which are simpler games. By inductive assumption, in each of these games one of the two players can force a win. If, in all of them, the starting player (which is player II in G) can force a win, then she will win in G by playing accordingly. Otherwise, at least one of the starting moves in G leads to a game G' where the second-moving player in G' (which is player I in G) can force a win, and by making that move, player I will force a win in G. □

If in G, player I can force a win, its starting position is a winning position, and we call G a **winning game**. If player II can force a win, G starts with a losing position, and we call G a **losing game**.

2.9 Sums of games

We continue our discussion of nim. Suppose the starting position has heap sizes $1,5,5$. Then the obvious good move is to option $5,5$, which is losing.

What about nim with four heaps of sizes $2,2,6,6$? This is losing, because $2,2$ and $6,6$ independently are losing positions, and any move in a heap of size 2 can be copied in the other heap of size 2, and similarly for the heaps of size 6. There is a second way of looking at this example, where it is not just two losing games put together: consider the game with heap sizes $2,6$. This is a winning game. However, two such winning games, put together to give the game $2,6,2,6$, result in a losing game, because any move in one of the games $2,6$, for example to $2,4$, can be copied in the other game, also to $2,4$, giving the new position $2,4,2,4$. So the second player, who plays 'copycat', always has a move left (the copying move) and hence cannot lose.

Definition 2.3 The **sum** of two games G and H, written $G+H$, is defined as follows: The player may move in either G or H as allowed in that game, leaving the position in the other game unchanged.

Note that $G+H$ is a notation that applies here to **games** and not to numbers, even if the games are in some way defined using numbers (for example as nim heaps). The result is a new game.

More formally, assume that G and H are defined in terms of their options (via moves from the starting position) $G_1, G_2, \ldots, G_k$ and $H_1, H_2, \ldots, H_m$, respectively. Then the options of $G+H$ are given as

$$\text{options of } G+H: \quad G_1+H, \ldots, G_k+H, G+H_1, \ldots, G+H_m. \tag{2.2}$$

The first list of options G_1+H, $G_2+H, \ldots$, G_k+H in (2.2) simply means that the player makes his move in G, the second list $G+H_1$, $G+H_2, \ldots$, $G+H_m$ that he makes his move in H.

We can define the game nim as a sum of nim heaps, where any single nim heap is recursively defined in terms of its options by (2.1). So the game nim with heaps of size $1,4,6$ is written as $*1+*4+*6$.

The 'addition' of games with the abstract $+$ operation leads to an interesting connection of combinatorial games with abstract algebra. If you are somewhat familiar with the concept of an abstract **group**, you will enjoy this connection; if not, you do not need to worry, because this connection it is not essential for our development of the theory.

A group is a set with a binary operation $+$ that fulfils three properties:

1. The operation $+$ is **associative**, that is, $G+(J+K)=(G+J)+K$ holds for all G,J,K.

2. The operation $+$ has a **neutral element** 0, so that $G+0=G$ and $0+G=G$ for all G.

3. Every element G has an **inverse** $-G$ so that $G+(-G)=0$.

Furthermore,

4. The group is called **commutative** (or 'abelian') if $G+H=H+G$ holds for all G,H.

Familiar groups in mathematics are, for example, the set of integers with addition, or the set of positive real numbers with multiplication (where the multiplication operation is written as $\cdot$, the neutral element is 1, and the inverse of G is written as G^{-1}).

The games that we consider form a group as well. In the way the sum of two games G and H is defined, $G+H$ and $H+G$ define the same game, so $+$ is commutative. Moreover, when one of these games is itself a sum of games, for example $H=J+K$, then $G+H$ is $G+(J+K)$ which means the player can make a move in exactly one of the games G, J, or K. This means obviously the same as the sum of games $(G+J)+K$, that is, $+$ is associative. The sum $G+(J+K)$, which is the same as $(G+J)+K$, can therefore be written unambiguously as $G+J+K$.

An obvious neutral element is the empty nim heap $*0$, because it is 'invisible' (it allows no moves), and adding it to any game G does not change the game.

However, there is no direct way to get an inverse operation because for any game G which has some options, if one adds any other game H to it (the intention being that H is the inverse $-G$), then $G+H$ will have some options (namely at least the options of moving in G and leaving H unchanged), so that $G+H$ is not equal to the empty nim heap.

The way out of this is to identify games that are 'equivalent' in a certain sense. We will see shortly that if $G+H$ is a losing game (where the first player to move cannot force a win), then that losing game is 'equivalent' to $*0$, so that H fulfils the role of an inverse of G.

2.10 Equivalent games

There is a neutral element that can be added to any game G without changing it. By definition, because it allows no moves, it is the empty nim heap $*0$:

$$G+*0=G. \tag{2.3}$$

However, other games can also serve as neutral elements for the addition of games. We will see that any losing game can serve that purpose, provided we consider certain games as equivalent according to the following definition.

Definition 2.4 Two games G,H are called **equivalent**, written $G \equiv H$, if and only if for any other game J, the sum $G+J$ is losing if and only if $H+J$ is losing.

In definition 2.4, we can also say that $G \equiv H$ if for any other game J, the sum $G+J$ is winning if and only if $H+J$ is winning. In other words, G is equivalent to H if, whenever G appears in a sum $G+J$ of games, then G can be replaced by H without changing whether $G+J$ is winning or losing.

One can verify easily that $\equiv$ is indeed an equivalence relation, meaning it is reflexive ($G \equiv G$), symmetric ($G \equiv H$ implies $H \equiv G$), and transitive ($G \equiv H$ and $H \equiv K$ imply $G \equiv K$; all these conditions hold for all games G,H,K).

Using $J = *0$ in definition 2.4 and (2.3), $G \equiv H$ implies that G is losing if and only if H is losing. The converse is not quite true: just because two games are winning does not mean they are equivalent, as we will see shortly. However, any two **losing** games are equivalent, because they are all equivalent to $*0$:

Lemma 2.5 *If G is a losing game (the second player to move can force a win), then* $G \equiv *0$.

Proof. Let G be a losing game. We want to show $G \equiv *0$ By definition 2.4, this is true if and only if for any other game J, the game $G+J$ is losing if and only if $*0+J$ is losing. According to (2.3), this holds if and only if J is losing.

So let J be any other game; we want to show that $G+J$ is losing if and only if J is losing. Intuitively, adding the losing game G to J does not change which player in J can force a win, because any intermediate move in G by his opponent is simply countered by the winning player, until the moves in G are exhausted.

Formally, we first prove by induction the simpler claim that for all games J, if J is losing, then $G+J$ is losing. (So we first ignore the 'only if' part.) Our inductive assumptions for this simpler claim are: for all losing games G'' that are simpler than G, if J is losing, then $G''+J$ is losing; and for all games J'' that are simpler than J, if J'' is losing, then $G+J''$ is losing.

So suppose that J is losing. We want to show that $G+J$ is losing. Any initial move in J leads to an option J' which is winning, which means that there is a corresponding option J'' of J' (by player II's reply) where J'' is losing. Hence, when player I makes the corresponding initial move from $G+J$ to $G+J'$, player II can counter by moving to $G+J''$. By inductive assumption, this is losing because J'' is losing. Alternatively, player I may move from $G+J$ to $G'+J$. Because G is a losing game, there is a move by player II from G' to G'' where G'' is again a losing game, and hence $G''+J$ is also losing, by inductive assumption, because J is losing. This completes the induction and proves the claim.

What is missing is to show that if $G+J$ is losing, so is J. If J was winning, then there would be a winning move to some option J' of J where J' is losing, but then, by our claim (the 'if' part that we just proved), $G+J'$ is losing, which would be a winning option in $G+J$ for player I. But this is a contradiction. This completes the proof. □

The preceding lemma says that any losing game Z, say, can be added to a game G without changing whether G is winning or losing (in lemma 2.5, Z is called G). That is, extending (2.3),

$$Z \text{ losing} \implies G+Z \equiv G. \tag{2.4}$$

As an example, consider $Z = *1 + *2 + *3$, which is nim with three heaps of sizes $1,2,3$. To see that Z is losing, we examine the options of Z and show that all of them are winning games. Removing an entire heap leaves two unequal heaps, which is a winning position by lemma 2.1. Any other move produces three heaps, two of which have equal size. Because two equal heaps define a losing nim game Z, they can be ignored by (2.4), meaning that all these options are like single nim heaps and therefore winning positions, too.

So $Z = *1 + *2 + *3$ is losing. The game $G = *4 + *5$ is clearly winning. By (2.4), the game $G+Z$ is equivalent to G and is also winning. However, verifying directly that $*1 + *2 + *3 + *4 + *5$ is winning would not be easy to see without using (2.4).

It is an easy exercise to show that in sums of games, games can be replaced by equivalent games, resulting in an equivalent sum. That is, for all games G,H,J,

$$G \equiv H \quad \Longrightarrow \quad G+J \equiv H+J. \tag{2.5}$$

Note that (2.5) is not merely a re-statement of definition 2.4, because equivalence of the games $G+J$ and $H+J$ means more than just that the games are either both winning or both losing (see the comments before lemma 2.9 below).

Lemma 2.6 (The copycat principle) *$G+G \equiv *0$ for any impartial game G.*

Proof. Given G, we assume by induction that the claim holds for all simpler games G'. Any option of $G+G$ is of the form $G'+G$ for an option G' of G. This is winning by moving to the game $G'+G'$ which is losing, by inductive assumption. So $G+G$ is indeed a losing game, and therefore equivalent to $*0$ by lemma 2.5. □

We now come back to the issue of inverse elements in abstract groups, mentioned at the end of section 2.9. If we identify equivalent games, then the addition $+$ of games defines indeed a group operation. The neutral element is $*0$, or any equivalent game (that is, a losing game).

The inverse of a game G, written as the negative $-G$, fulfils

$$G+(-G) \equiv *0. \tag{2.6}$$

Lemma 2.6 shows that for an impartial game, $-G$ is simply G itself.

Side remark: For games that are not impartial, that is, partisan games, $-G$ exists also. It is G but with the roles of the two players exchanged, so that whatever move was available to player I is now available to player II and vice versa. As an example, consider the game checkers (with the rule that whoever can no longer make a move loses), and let G be a certain configuration of pieces on the checkerboard. Then $-G$ is the same configuration with the white and black pieces interchanged. Then in the game $G+(-G)$, player II (who can move the black pieces, say), can also play 'copycat'. Namely, if player I makes a move in either G or $-G$ with a white piece, then player II copies that move with a black piece on the other board ($-G$ or G, respectively). Consequently, player II always has a move available and will win the game, so that $G+(-G)$ is indeed a losing game for the starting player I, that is, $G+(-G) \equiv *0$. However, we only consider impartial games, where $-G = G$.

The following condition is very useful to prove that two games are equivalent.

Lemma 2.7 *Two impartial games G,H are equivalent if and only if $G+H \equiv *0$.*

Proof. If $G \equiv H$, then by (2.5) and lemma 2.6, $G+H \equiv H+H \equiv *0$. Conversely, $G+H \equiv *0$ implies $G \equiv G+H+H \equiv *0+H \equiv H$. □

Sometimes, we want to prove equivalence inductively, where the following observation is useful.

Lemma 2.8 *Two games G and H are equivalent if all their options are equivalent, that is, for every option of G there is an equivalent option of H and vice versa.*

Proof. Assume that for every option of G there is an equivalent option of H and vice versa. We want to show $G+H \equiv *0$. If player I moves from $G+H$ to $G'+H$ where G' is an option in G, then there is an equivalent option H' of H, that is, $G'+H' \equiv *0$ by lemma 2.7. Moving there defines a winning move in $G'+H$ for player II. Similarly, player II has a winning move if player I moves to $G+H'$ where H' is an option of H, namely to $G'+H'$ where G' is an option of G that is equivalent to H'. So $G+H$ is a losing game as claimed, and $G \equiv H$ by lemma 2.5 and lemma 2.7. □

Note that lemma 2.8 states only a sufficient condition for the equivalence of G and H. Games can be equivalent without that property. For example, $G+G \equiv *0$, but $*0$ has no options whereas $G+G$ has many.

We conclude this section with an important point. Equivalence of two games is a finer distinction than whether the games are both losing or both winning, because that property has to be preserved in sums of games as well. Unlike losing games, winning games are in general **not** equivalent.

Lemma 2.9 *Two nim heaps are equivalent only if they have equal size:* $*n \equiv *m \Longrightarrow n = m$.

Proof. By lemma 2.7, $*n \equiv *m$ if and only if $*n + *m$ is a losing position. By lemma 2.1, this implies $n = m$. □

That is, different nim heaps are not equivalent. In a sense, this is due to the different amount of 'freedom' in making a move, depending on the size of the heap. However, all the relevant freedom in making a move in an impartial game can be captured by a nim heap. We will later show that **any impartial game** is equivalent to **some** nim heap.

2.11 Sums of nim heaps

Before we show how impartial games can be represented as nim heaps, we consider the game of nim itself. We show in this section how any nim game, which is a sum of nim heaps, is equivalent to a single nim heap. As an example, we know that $*1 + *2 + *3 \equiv *0$, so by lemma 2.7, $*1 + *2$ is equivalent to $*3$. In general, however, the sizes of the nim heaps cannot simply be added to obtain the equivalent nim heap (which by lemma 2.9 has a unique size). For example, as shown after (2.4), $*1 + *2 + *3 \equiv *0$, that is, $*1 + *2 + *3$ is a losing game and not equivalent to the nim heap $*6$. Adding the game $*2$ to both sides of the equivalence $*1 + *2 + *3 \equiv *0$ gives $*1 + *3 \equiv *2$, and in a similar way $*2 + *3 \equiv *1$, so any two heaps from sizes $1,2,3$ has the third size as its equivalent single heap. This rule is very useful in simplifying nim positions with small heap sizes.

If $*k \equiv *n + *m$, we also call k the **nim sum** of n and m, written $k = n \oplus m$. The following theorem states that for distinct powers of two, their nim sum is the ordinary sum. For example, $1 = 2^0$ and $2 = 2^1$, so $1 \oplus 2 = 1 + 2 = 3$.

Theorem 2.10 *Let* $n \geq 1$, *and* $n = 2^a + 2^b + 2^c + \cdots$, *where* $a > b > c > \cdots \geq 0$. *Then*

$$*n \equiv *(2^a) + *(2^b) + *(2^c) + \cdots. \tag{2.7}$$

We first discuss the implications of this theorem, and then prove it. The right-hand side of (2.7) is a sum of games, whereas n itself is represented as a sum of powers of two. Any n is uniquely given as such a sum. This amounts to the binary representation of n, which, if $n < 2^{a+1}$, gives n as the sum of all powers of two

$2^a, 2^{a-1}, 2^{a-2}, \ldots, 2^0$, each power multiplied with one or zero. These ones and zeros are then the digits in the binary representation of n. For example,

$$13 = 8+4+1 = 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0,$$

so that 13 in decimal is written as 1101 in binary. Theorem 2.10 uses only the powers of two $2^a, 2^b, 2^c, \ldots$ that correspond to the digits 'one' in the binary representation of n.

Equation (2.7) shows that $*n$ is equivalent to the game sum of many nim heaps, all of which have a size that is a power of two. Any other nim heap $*m$ is also a sum of such games, so that $*n + *m$ is a game sum of several heaps, where equal heaps cancel out in pairs. The remaining heap sizes are all distinct powers of two, which can be added to give the size of the single nim heap $*k$ that is equivalent to $*n + *m$. As an example, let $n = 13 = 8+4+1$ and $m = 11 = 8+2+1$. Then $*n + *m \equiv *8 + *4 + *1 + *8 + *2 + *1 \equiv *4 + *2 \equiv *6$, which we can also write as $13 \oplus 11 = 6$. In particular, $*13 + *11 + *6$ is a losing game, which would be very laborious to show without the theorem.

One consequence of theorem 2.10 is that the nim sum of two numbers never exceeds their ordinary sum. Moreover, if both numbers are less than some power of two, then so is their nim sum.

Lemma 2.11 *Let* $0 \le p, q < 2^a$. *Then* $*p + *q \equiv *r$ *where* $0 \le r < 2^a$, *that is,* $r = p \oplus q < 2^a$.

Proof. Both p and q are sums of distinct powers of two, all smaller than 2^a. By theorem 2.10, r is also a sum of such powers of two, where those that appear in both p and q cancel out, so that $r < 2^a$. □

The following proof may be best understood by considering it along with an example, say $n = 7$.

Proof of theorem 2.10. We proceed by induction. Consider some n, and assume that the theorem holds for all smaller n. Let $n = 2^a + q$ where $q = 2^b + 2^c + \cdots$. If $q = 0$, the claim holds trivially (n is just a single power of two), so let $q > 0$. We have $q < 2^a$. By inductive assumption, $*q \equiv *(2^b) + *(2^c) + \cdots$, so all we have to prove is that $*n \equiv *(2^a) + *q$ in order to show (2.7). We show this using lemma 2.8, that is, by showing that the options of the games $*n$ and $*(2^a) + *q$ are all equivalent. The options of $*n$ are $*0, *1, *2, \ldots, *(n-1)$.

The options of $*(2^a) + *q$ are of two kinds, depending on whether the player moves in the nim heap $*(2^a)$ or $*q$. The first kind of options are given by

$$\begin{aligned} *0 + *q &\equiv *r_0 \\ *1 + *q &\equiv *r_1 \\ &\vdots \\ *(2^a - 1) + *q &\equiv *r_{2^a-1} \end{aligned} \tag{2.8}$$

where the equivalence of $*i + *q$ with some nim heap $*r_i$, for $0 \le i < 2^a$, holds by inductive assumption. Moreover, by lemma 2.11 (which is a consequence of theorem 2.10 which can be used by inductive assumption), both i and q are less than 2^a so that also $r_i < 2^a$. On the right-hand side in (2.8), there are 2^a many nim heaps $*r_i$ for $0 \le i < 2^a$. We claim they are all different, so that these options form exactly the set $\{*0, *1, *2, \ldots, *(2^a - 1)\}$. Namely, by adding the game $*q$ to the heap $*r_i$, (2.5) implies $*r_i + *q \equiv *i + *q + *q \equiv *i$, so that $*r_i \equiv *r_j$ implies $*r_i + *q \equiv *r_j + *q$, that is, $*i \equiv *j$ and hence $i = j$ by lemma 2.9, for $0 \le i, j < 2^a$.

The second kind of options of $*(2^a)+*q$ are of the form

$$
\begin{aligned}
*(2^a)+*0 &\equiv *(2^a+0)\\
*(2^a)+*1 &\equiv *(2^a+1)\\
&\vdots\\
(2^a)+(q-1) &\equiv *(2^a+q-1),
\end{aligned}
$$

where the heap sizes on the right-hand sides are given again by inductive assumption. These heaps form the set $\{*(2^a), *(2^a+1), \ldots, *(n-1)\}$. Together with the first kind of options, they are exactly the options of $*n$. This shows that the options of $*n$ and of the game sum $*(2^a)+*q$ are indeed equivalent, which completes the proof. □

The nim sum of any set of numbers can be obtained by writing each number as a sum of distinct powers of two and then cancelling repetitions in pairs. For example,

$6 \oplus 4 = (4+2) \oplus 4 = 2$, or $11 \oplus 16 \oplus 18 = (8+2+1) \oplus 16 \oplus (16+2) = 8+1 = 9.$

This is usually described as 'write the numbers in binary and add without carrying', which comes to the same thing. In the following tables, the top row shows the powers of two needed in the binary representation for the numbers beneath; the bottom row gives the resulting nim sum.

	4	2	1
6 =	1	1	0
4 =	1	0	0
2 =	0	1	0

	16	8	4	2	1
11 =		1	0	1	1
16 =	1	0	0	0	0
18 =	1	0	0	1	0
9 =	0	1	0	0	1

However, using only the powers of two that are used and cancelling repetitions is easier to do in your head, and is less prone to error.

How does theorem 2.10 translate into playing nim? When the nim sum of the heap sizes is zero, then the player is in a losing position. (Such nim positions are sometimes called **balanced** positions.) All moves will lead to a winning position, and in practice the best advice may only be not to move to a winning position that is too obvious, like one where two heap sizes are equal, in the hope that the opponent makes a mistake.

If the nim sum of the heap sizes is not zero, it is some sum of powers of two, say $s = 2^a + 2^b + 2^c + \cdots$, like for example $11 \oplus 16 \oplus 18 = 9 = 8+1$ above. The winning move is then obtained as follows:

1. Identify a heap of size n which uses 2^a, the largest power of 2 in the nim sum; at least one such heap must exist. In the example, that heap has size 11 (so it is not always the largest heap).

2. Compute $n \oplus s$. In that nim sum, the power 2^a appears in both n and s, and it cancels out, so the result is some number m that is smaller than n. In the example, $m = 11 \oplus 9 = (8+2+1) \oplus (8+1) = 2$.

3. Reduce the heap of size n to size m, in the example from size 11 to size 2. The resulting heap $*m$ is equivalent to $*n + *s$, so when it replaces $*n$ in the original sum, $*s$ is added and cancels with $*s$, and the result is equivalent to $*0$, a losing position.

On paper, the binary representation may be easier to use. In step 2 above, computing $n \oplus s$ amounts to 'flipping the bits' in the binary representation of the heap n whenever the corresponding bit in the binary representation of the sum s is one. In this way, a player in a winning position moves always from an 'unbalanced' position (with nonzero nim sum) to a balanced position (nim sum zero), which is losing because any move will create again an unbalanced position. This is the way nim is usually explained. The method was discovered by Bouton who published it in 1902.

⇒ You are now in a position to answer all of exercise 2.1 on page 28.

So far, it is not fully clear why powers of two appear in the computation of nim sums. One reason is provided by the proof of theorem 2.10. The options of moving from $*(2^a) + *q$ in (2.8) neatly produce exactly the numbers $0, 1, \ldots, 2^a - 1$, which would not work when replacing 2^a with something else.

As a second reason, the copycat principle $G + G \equiv *0$ shows that the impartial games form a group where every element is its own inverse. There is essentially only one mathematical structure that has these particular properties, namely the addition of binary vectors. In each component, such vectors are separately added modulo two, where $1 \oplus 1 = 0$. Here, the binary vectors translate into binary numbers for the sizes of the nim heaps. The 'addition without carry' of binary vectors defines exactly the winning strategy in nim, as stated in theorem 2.10. However, the proof of this theorem is stated directly and without any recourse to abstract algebra and groups.

A third reason uses the construction of equivalent nim heaps for any impartial game, in particular a sum of two nim heaps, which we explain next; see also figure 2.2 below.

2.12 Poker nim and the mex rule

Poker nim is played with heaps of poker chips. Just as in ordinary nim, either player may reduce the size of any heap by removing some of the chips. But alternatively, a player may also **increase** the size of some heap by adding to it some of the chips he acquired in earlier moves. These two kinds of moves are the only ones allowed.

Let's suppose that there are three heaps, of sizes $3, 4, 5$, and that the game has been going on for some time, so that both players have accumulated substantial reserves of chips. It's player I's turn, who moves to $1, 4, 5$ because that is a good move in ordinary nim. But now player II adds 50 chips to the heap of size 4, creating position $1, 54, 5$, which seems complicated.

What should player I do? After a moment's thought, he just removes the 50 chips player II has just added to the heap, reverting to the previous position. Player II may keep adding chips, but will eventually run out of them, no matter how many she acquires in between, and then player I can proceed as in ordinary nim.

So a player who can win a position in ordinary nim can still win in poker nim. He replies to the opponent's reducing moves just as he would in ordinary nim, and reverses the effect of any increasing move by using a reducing move to restore the heap to the same size again. Strictly speaking, the ending condition (see condition 6 in section 2.7) is violated in poker nim because in theory the game could go on forever. However, a player in a winning position wants to end the game with his victory, and never has to put back any chips; then the losing player will eventually run out of chips that she can add to a heap, so that the game terminates.

Consider now an impartial game where the options of player I are games that are equivalent to the nim heaps $*0$, $*1$, $*2$, $*5$, $*9$. This can be regarded as a rather peculiar nim heap of size 3 which can be reduced to any of the sizes $0, 1, 2$, but which can also be increased to size 5 or 9. The poker nim argument shows that this extra freedom is in fact of no use whatsoever.

The **mex rule** says that if the options of a game G are equivalent to nim heaps with sizes from a set S (like $S = \{0, 1, 2, 5, 9\}$ above), then G is equivalent to a nim heap of size m, where m is the **smallest non-negative integer not contained in** S. This number m is written $\text{mex}(S)$, where mex stands for 'minimum excludant'. That is,

$$m = \text{mex}(S) = \min\{k \geq 0 \mid k \notin S\}. \tag{2.9}$$

For example, $\text{mex}(\{0,1,2,3,5,6\}) = 4$, $\text{mex}(\{1,2,3,4,5\}) = 0$, and $\text{mex}(\emptyset) = 0$.

⇒ Which game has the empty set $\emptyset$ as its set of options?

Theorem 2.12 (The mex rule) *Let the impartial game G have the set of options that are equivalent to $\{*s \mid s \in S\}$ for some set S of non-negative integers (assuming S is not the set of all non-negative integers, for example if S is finite). Then $G \equiv *(\text{mex}(S))$.*

Proof. Let $m = \text{mex}(S)$. We show $G + *m \equiv *0$, which proves the theorem by lemma 2.7. If player I moves from $G + *m$ to $G + *k$ for some $k < m$, then $k \in S$ and there is an option K of G so that $K \equiv *k$ by assumption, so player II can counter by moving from $G + *k$ to the losing position $K + *k$. Otherwise, player I may move to $K + *m$, where K is some option K which is equivalent to $*k$ for some $k \in S$. If $k < m$, then player II counters by moving to $K + *k$. If $k > m$, then player II counters by moving to $M + *m$ where M is the option of K that is equivalent to $*m$ (because $K \equiv *k$). The case $k = m$ is excluded by the definition of $m = \text{mex}(S)$. This shows $G + *m$ is a losing game. □

A special case of the preceding theorem is that $\text{mex}(S) = 0$, which means that all options of G are equivalent to positive nim heaps, so they are all winning positions, or that G has no options at all. Then G is a losing game, and indeed $G \equiv *0$.

Corollary 2.13 *Any impartial game G is equivalent to some nim heap $*n$.*

Proof. We can assume by induction that this holds for all games that are simpler than G, in particular the options of G. They are equivalent to nim heaps whose sizes form the set S (which we assume is not the set of all non-negative integers). Theorem 2.12 then shows $G \equiv *m$ for $m = \text{mex}(S)$. □

⇒ Do exercise 2.7 on page 31, which provides an excellent way to understand the mex rule.

2.13 Finding nim values

By corollary 2.13, any impartial game can be played like nim, provided the equivalent nim heaps of the positions of the game are known. This forms the basis of the **Sprague–Grundy** theory of impartial games, named after the independent discoveries of this principle by R. P. Sprague in 1936 and P. M. Grundy in 1939. Any sum of such games is then evaluated by taking the nim sum of the sizes of the corresponding nim heaps.

The nim values of the positions can be evaluated by the mex rule (theorem 2.12). This is illustrated in the 'rook-move' game in figure 2.1. Place a rook on a chess board of given arbitrary size. In one move, the rook is moved either horizontally to the left or vertically upwards, for any number of squares (at least one) as long as it stays on the board. The first player who can no longer move loses, when the rook is on the top left square of the board. We number the rows and columns of the board by $0, 1, 2, \ldots$ starting from the top left.

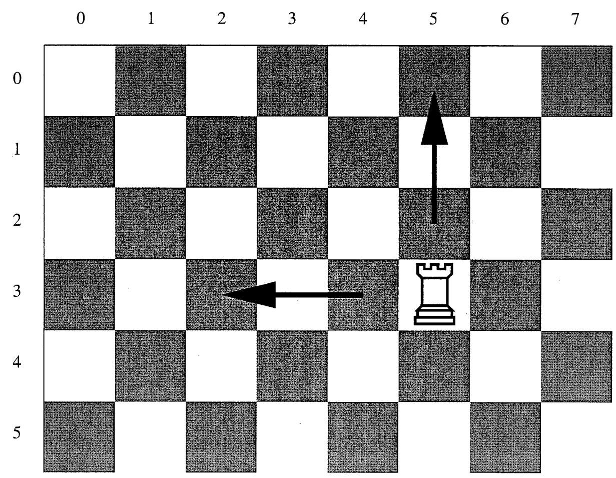


Figure 2.1 Rook move game, where the player may move the rook on the chess board in the direction of the arrows.

Figure 2.2 gives the nim values for the positions of the rook on the chess board. The top left square is equivalent to $*0$ because the rook can no longer move. The square below that allows only to reach the square with $*0$ on it, so it is equivalent to $*1$ because $\mathrm{mex}\{0\} = 1$. The square below gets $*2$ because its options are equivalent to $*0$ and $*1$. From any square in the leftmost column in figure 2.2, the rook can only move upwards, so any such square in row i corresponds obviously to a nim heap $*i$. Similarly, the topmost row has entry $*j$ in column j.

In general, a position on the board is evaluated knowing all nim values for the squares to the left and the top of it, which are the options of that position. As an example, consider the square in row 3 and column 2. To the left of that square, the entries $*3$ and $*2$ are found, and to the top the entries $*2, *3, *0$. So the square itself is equivalent to $*1$ because $\mathrm{mex}(\{0,2,3\}) = 1$. The square in row 3 and column 5, where the rook is placed in figure 2.1, gets entry $*6$.

The astute reader will have noticed that the rook move game is just nim with two heaps. A rook positioned in row i and column j can either diminish i by moving left, or j by moving up. So this position is the sum of nim heaps $*i + *j$. It is a losing position if and only if $i = j$, where the rook is on the diagonal leading to the top left square. Therefore, figure 2.2 represents the computation of nim heaps equivalent to $*i + *j$, or, by omitting the stars, the nim sums $i \oplus j$, for all $i, j \geq 0$.

The nim addition table figure 2.2 is computed by the mex rule, and does not require theorem 2.10. Given this nim addition table, one can conjecture (2.7). You may find it useful to go back to the proof of theorem 2.10 using figure 2.2 and check the options for a position of the form $*(2^a) + *q$ for $q < 2^a$, as a square in row 2^a and column q.

Another impartial game is shown in figure 2.3 where the rook is replaced by a queen, which may also move diagonally. The squares on the main diagonal are therefore no longer losing positions. This game can also be played with two heaps of chips where in one move, the player may either take chips from one heap as in nim, or reduce **both** heaps by the **same** number of chips (so this is no longer a sum of two games!). In order to illustrate that we are not just interested in the winning and losing squares, we add to this game a nim heap of size 4.

Figure 2.4 shows the equivalent nim heaps for the positions of the queen move game, determined by the mex rule. The square in row 3 and column 4 occupied by the

	0	1	2	3	4	5	6	7	8	9	10
0	*0	*1	*2	*3	*4	*5	*6	*7	*8	*9	*10
1	*1	*0	*3	*2	*5	*4	*7	*6	*9	*8	*11
2	*2	*3	*0	*1	*6	*7	*4	*5	*10	*11	*8
3	*3	*2	*1	*0	*7	*6	*5	*4	*11	*10	*9
4	*4	*5	*6	*7	*0	*1	*2	*3	*12	*13	*14
5	*5	*4	*7	*6	*1	*0	*3	*2	*13	*12	*15
6	*6	*7	*4	*5	*2	*3	*0	*1	*14	*15	*12
7	*7	*6	*5	*4	*3	*2	*1	*0	*15	*14	*13
8	*8	*9	*10	*11	*12	*13	*14	*15	*0	*1	*2
9	*9	*8	*11	*10	*13	*12	*15	*14	*1	*0	*3
10	*10	*11	*8	*9	*14	*15	*12	*13	*2	*3	*0

Figure 2.2 Equivalent nim heaps $*n$ for positions of the rook move game.

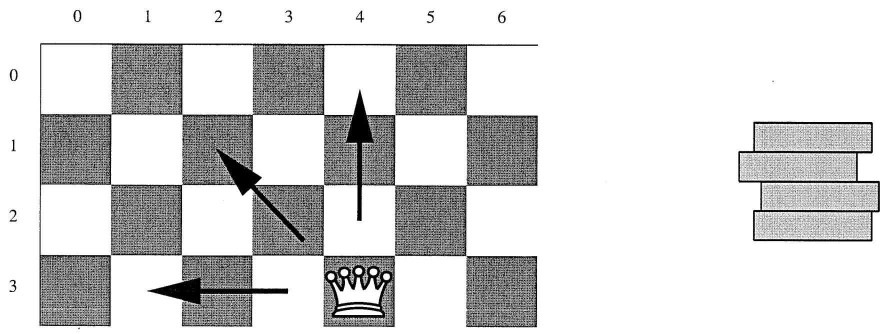


Figure 2.3 Sum of a queen move game and a nim heap. The player may **either** move the queen in the direction of the arrows **or** take some of the 4 chips from the heap.

queen in figure 2.3 has entry $*2$. So a winning move is to remove 2 chips from the nim heap to turn it into the heap $*2$, creating the losing position $*2 + *2$.

This concludes our introduction to combinatorial games. Further examples will be given in the exercises.

⇒ Do the remaining exercises 2.4–2.9, starting on page 30, which show how to use the mex rule and what you learned about nim and combinatorial games.

	0	1	2	3	4	5	6
0	$*0$	$*1$	$*2$	$*3$	$*4$		
1	$*1$	$*2$	$*0$	$*4$	$*5$		
2	$*2$	$*0$	$*1$	$*5$	$*3$		
3	$*3$	$*4$	$*5$	$*6$	$*2$		

Figure 2.4 Equivalent nim heaps for positions of the queen move game.

2.14 Reminder of learning outcomes

After studying this chapter, you should be able to:

- play nim optimally;
- explain the concepts of game-sums, equivalent games, nim values and the mex rule;
- apply these concepts to play other impartial games like those described in the exercises.

2.15 Exercises for chapter 2

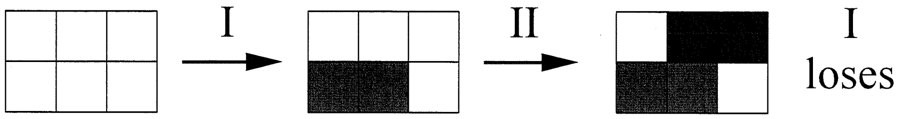
In this chapter, which is more abstract than the others, the exercises are particularly important. Exercise 2.1 is a standard question on nim, where part (a) can be answered even without the theory. Exercise 2.2 is an example of an impartial game, which can also be answered without much theory. Exercise 2.3 is difficult – beware not to rush into any quick and false application of nim values here; part (c) of this exercise is particularly challenging. Exercise 2.4 tests your understanding of the queen move game. For exercise 2.5, remember the concept of a sum of games, which applies here naturally. In exercise 2.6, try to see how nim heaps are hidden in the game. Exercise 2.7 is very instructive for understanding the mex rule. In exercise 2.8, it is essential that you understand nim values. It takes some work to investigate all the options in the game. Exercise 2.9 is an impartial game that is rather different from the previous games. In the challenging part (c) of that exercise, you should first formulate a conjecture and then prove it precisely.

Exercise 2.1 Consider the game nim with heaps of chips. The players alternately remove some chips from one of the heaps. The player to remove the last chip wins.

(a) For all positions with three heaps, where one of the heaps has only **one** chip, describe exactly the **losing** positions. Justify your answer, for example with a proof by induction, or by theorems on nim.
[*Hint: Start with the easy cases to find the pattern.*]

(b) Determine all initial winning moves for nim with three heaps of size 6, 10 and 15, using the theorem on nim where the heap sizes are represented as sums of powers of two.

Exercise 2.2 The game **dominos** is played on a board of $m \times n$ squares, where players alternately place a domino on the board which covers two adjacent squares that are free (not yet occupied by a domino), vertically or horizontally. The first player who cannot place a domino any more loses. Example play for a 2×3 board:



(a) Who will win in 3×3 dominos?
[*Hint: Use the symmetry of the game to investigate possible moves, and remember that it suffices to find one winning strategy.*]

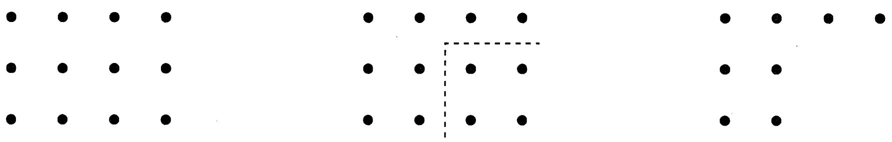
(b) Who will win in $m \times n$ dominos when both m and n are even?

(c) Who will win in $m \times n$ dominos when m is odd and n is even?

Justify your answers.

Note (not a question): Because of the known answers from (b) and (c), this game is more interesting for 'real play' on an $m \times n$ board where both m and n are odd. Play it with your friends on a 5×5 board, for example. The situation often decomposes into independent parts, like contiguous fields of 2, 3, 4, 5, 6 squares, that have a known winner, which may help you analyse the situation.

Exercise 2.3 Consider the following game chomp: A rectangular array of $m \times n$ dots is given, in m rows and n columns, like 3×4 in the next picture on the left. A dot in row i and column j is named (i, j). A move consists in picking a dot (i, j) and removing it and **all other dots to the right and below it**, which means removing all dots (i', j') with $i' \geq i$ and $j' \geq j$, as shown for $(i, j) = (2,3)$ in the middle picture, resulting in the picture on the right:



Player I is the first player to move, players alternate, and the last player who removes a dot **loses**.

An alternative way is to think of these dots as (real) cookies: a move is to eat a cookie and all those to the right and below it, but the top left cookie is poisoned. See also http://www.stolaf.edu/people/molnar/games/chomp/

(a) Assuming optimal play, determine the winning player and a winning move for chomp of size 2×2, size 2×3, size $2 \times n$, and size $m \times m$, where $m \geq 3$. Justify your answers.

(b) In the way described here, chomp is a misère game where the last player to make a move loses. Suppose we want to play the same game so that the normal play convention applies, where the last player to move wins. (This would be a boring game with the board as given, by simply taking the top left dot $(1,1)$.) Explain how this can be done by removing one dot from the initial array of dots.

(c) Show that when chomp is played for a game of any size $m \times n$, player I can always win.
[*Hint: You only have to show that a winning move* **exists**, *but you do not have to describe that winning move.*]

Exercise 2.4

(a) Complete the entries of equivalent nim heaps for the queen move game in columns 5 and 6, rows 0 to 3, in the table in figure 2.4.

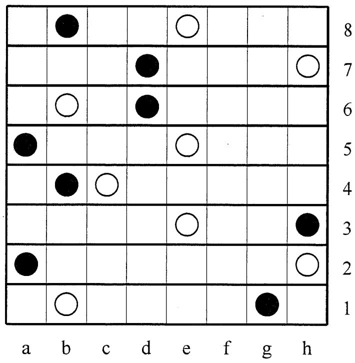
(b) Describe **all** winning moves in the game-sum of the queen move game and the nim heap in figure 2.3.

Exercise 2.5 Consider the game dominos from exercise 2.2, played on a $1 \times n$ board for $n \geq 2$. Let D_n be the nim value of that game, so that the starting position of the $1 \times n$ board is equivalent to a nim heap of size D_n. For example, $D_2 = 1$ because the 1×2 board is equivalent to $*1$.

(a) How is D_n computed from smaller values D_k for $k < n$?
[*Hint: Use sums of games and the mex rule. The notation for nim sums is* $\oplus$, *where* $a \oplus b = c$ *if and only if* $*a + *b \equiv *c$.]

(b) Give the values of D_n up to $n = 10$ (or more, if you are ambitious – higher values come at $n = 16$, and at some point they even repeat, but before you detect that you will probably have run out of patience). For which values of n, where $1 \leq n \leq 10$, is dominos on a $1 \times n$ board a losing game?

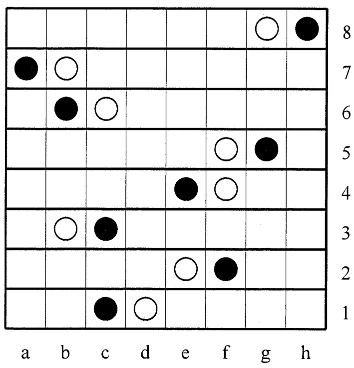
Exercise 2.6 Consider the following game on a rectangular board where a white and a black counter are placed in each row, like in this example:



(*)

Player I is white and starts, and player II is black. Players take turns. In a move, a player moves a counter of his colour to any other square within its row, but may not jump over the other counter. For example, in (*) above, in row 8 white may move from e8 to any of the squares c8, d8, f8, g8, or h8. The player who can no longer move loses.

(a) Who will win in the following position?

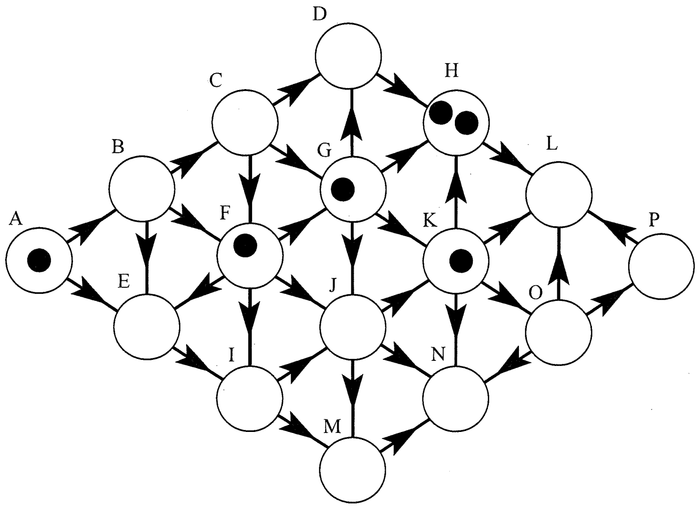


(b) Show that white can win in position (*) above. Give at least two winning moves from that position.

Justify your answers.

[*Hint: Compare this with another game that is not impartial and that also violates the ending condition, but that nevertheless is close to nim and ends in finite time when played well.*]

Exercise 2.7 Consider the following network (in technical terms, a directed graph or 'digraph'). Each circle, here marked with one of the letters A–P, represents a **node** of the network. Some of these nodes (here A, F, G, H, and K) have counters on them, which are allowed to share a node, like the two counters on H. In a move, one of the counters is moved to a neighbouring node in the direction of the arrow as indicated, for example from F to I (but not from F to C, nor directly from F to D, say). Players alternate, and the last player no longer able to move loses.

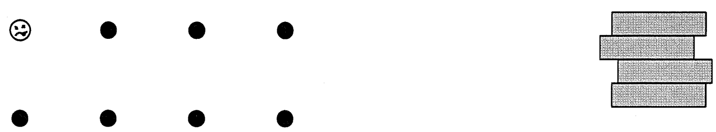


(a) Explain why this game fulfils the ending condition.

(b) Who is winning in the above position? If it is player I (the first player to move), describe **all** possible winning moves. Justify your answer.

(c) How does the answer to (b) change when the arrow from J to K is reversed so that it points from K to J instead?

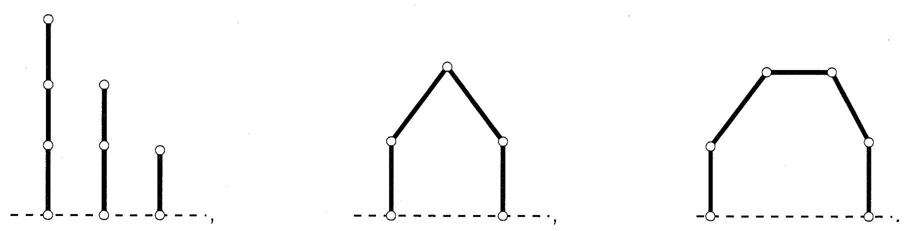
Exercise 2.8 Consider the game chomp from Exercise 2.3 of size 2×4, added to a nim heap of size 4.



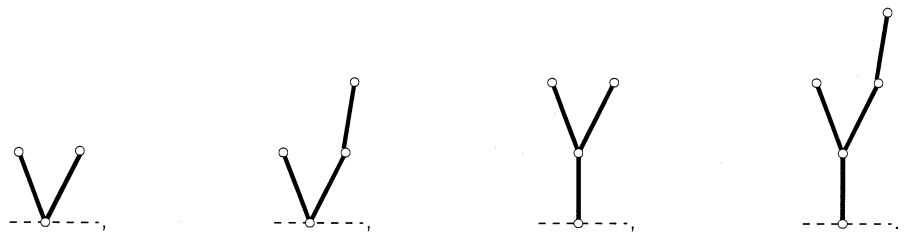
What are the winning moves of the starting player I, if any?
[*Hint: Represent chomp as a game in the normal play convention (see exercise 2.3(b), by changing the dot pattern), so that the losing player is not the player who takes the 'poisoned cookie', but the player who can no longer move. This will simplify finding the various nim values.*]

Exercise 2.9 The game **Hackenbush** is played on a figure consisting of dots connected with lines (called **edges**) that are connected to the ground (the dotted line in the pictures below). A move is to remove ('chop off') an edge, and with it all the edges that are then no longer connected to the ground. For example, in the leftmost figure below, one can remove any edge in one of the three stalks. Removing the second edge from the stalk consisting of three edges takes the topmost edge with it, leaving only a single edge. As usual, players alternate and the last player able to move wins.

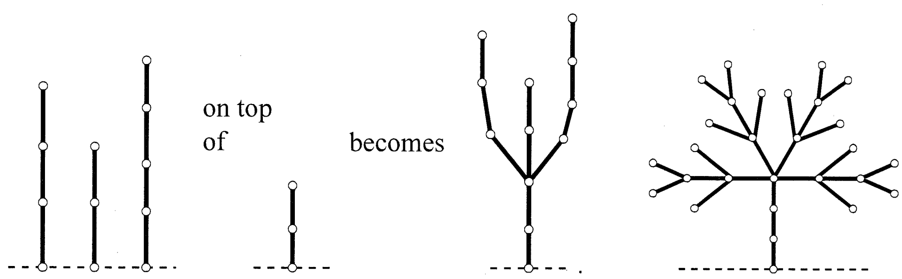
(a) Compute the nim values for the following three Hackenbush figures (using what you know about nim, and the mex rule):



(b) Compute the nim values for the following four Hackenbush figures:



(c) Based on the observation in (b), give a rule how to compute the nim value of a 'tree' which is constructed by putting several 'branches' with known nim values on top of a 'stem' of size n, for example $n = 2$ in the picture below, where three branches of height 3, 2, and 4 are put on top of the 'stem'. Prove that rule (you may find it advantageous to glue the branches together at the bottom, as in (b)). Use this to find the nim value of the rightmost picture.



Chapter 3

Games as trees and in strategic form

3.1 Aims of the chapter

This chapter

- introduces the main concepts of non-co-operative game theory: game trees and games in strategic form and the ways to analyse them.

3.2 Learning outcomes

After studying this chapter, you should be able to:

- interpret game trees and games in strategic form
- explain the concepts of a move in a game tree, strategy (and how it differs from a move), strategy profile, backward induction, symmetric games, dominance and weak dominance, dominance solvable, Nash equilibrium, reduced strategies and reduced strategic form, subgame perfect Nash equilibrium and commitment games
- apply these concepts to specific games.

3.3 Essential reading

This chapter of the guide. We occasionally refer to concepts discussed in chapter 2, such as 'game states', to clarify the connection, but chapter 2 is not a requirement to study this chapter.

3.4 Further reading

The presented concepts are standard in game theory. They can be found, for example, in the following book:

- Osborne, Martin J., and Ariel Rubinstein *A Course in Game Theory*. (MIT Press, 1994) [ISBN 0262650401].

However, as mentioned in section 1.8, the terminology used by Osborne and Rubinstein is not standard, so keep that in mind.

Another possible reference is:

- Gibbons, Robert *A Primer in Game Theory* [in the United States sold under the title *Game Theory for Applied Economists*]. (Prentice Hall / Harvester Wheatsheaf, 1992) [ISBN 0745011594].

In particular, this book looks at the commitment games of section 3.16 from the economic perspective of 'Stackelberg leadership'.

3.5 Introduction

In this chapter, we introduce several main concepts of non-co-operative game theory: **game trees** (with perfect information), which describe explicitly how a game evolves over time, and **strategies**, which describe a player's possible 'plans of action'. A game can be described in terms of strategies alone, which defines a game in **strategic form**.

Game trees can be solved by **backward induction**, where one finds optimal moves for each player, given that all future moves have already been determined. The central concept for games in strategic form is the **Nash equilibrium**, where each player chooses a strategy that is optimal given what the other players are doing. We will show that backward induction always produces a Nash equilibrium, also called 'subgame perfect Nash equilibrium' or SPNE.

The main difference between game trees and games in strategic form is that in a game tree, players act sequentially, being aware of the previous moves of the other players. In contrast, players in a game in strategic form move **simultaneously**. The difference between these two descriptions becomes striking when **changing** a two-player game in strategic form to a game with **commitment**, described by a game tree where one player moves first and the other second, but which otherwise has the same payoffs.

Every game tree can be converted to strategic form, which, however, is often much larger than the tree. A general game in strategic form can only be represented by a tree by modelling 'imperfect information' where a player is not aware of another player's action. Game trees with imperfect information are treated in chapter 5.

3.6 Definition of game trees

Figure 3.1 shows an example of a game tree. We always draw trees downwards, with the **root** at the top. (Conventions on drawing game trees vary. Sometimes trees are drawn from the bottom upwards, sometimes from left to right, and sometimes from the center with edges in any direction.)

The **nodes** of the tree denote game states. (In a combinatorial game, game states are called 'positions'.) Nodes are connected by lines, called **edges**. An edge from a node u to a successor node v (where v is drawn below u) indicates a possible **move** in the game. This may be a move of a 'personal' player, for example move X in figure 3.1 of player I. Then u is also called a **decision node**. Alternatively, u is a **chance node**. A chance node is drawn here as a **square**, like the node u that follows move b of player II in figure 3.1. The next node v is then determined by a random choice according to the probability associated with the edge leading from u to v. In figure 3.1, these probabilities are $\frac{1}{3}$ for the left move and $\frac{2}{3}$ for the right move.

Nodes without successors in the tree are called terminal nodes or **leaves**. At such a node, every player gets a **payoff**, which is a real number (in our examples often an integer). In figure 3.1, leaves are not explicitly drawn, but the payoffs given instead, with the top payoff to player I and the bottom payoff to player II.

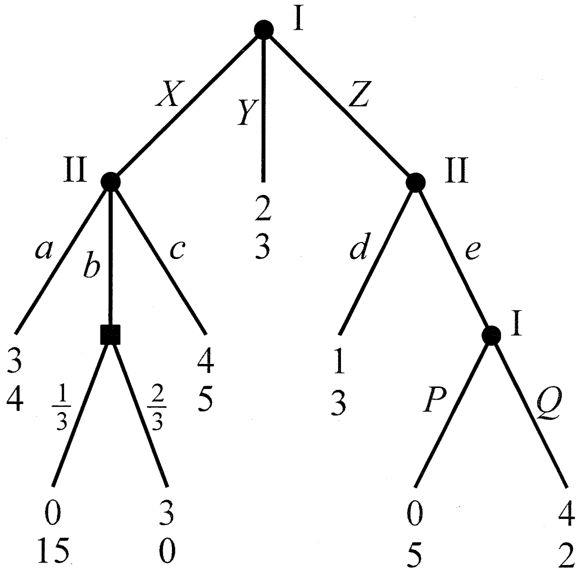


Figure 3.1 Example of a game tree. The square node indicates a chance move. At a leaf of the tree, the top payoff is to player I, the bottom payoff to player II.

The game tree, with its decision nodes, moves, chance probabilities and payoffs, is known to the players, and defines the game completely. The game is played by starting at the root. At a decision node, the respective player chooses a move which determines the next node. At a chance node, the move is made randomly according to the given probabilities. Play ends when a leaf is reached, where all players receive their payoffs.

Players are interested in maximising their own payoff. If the outcome of the game is random, then the players are assumed to be interested in maximising their **expected payoffs**. In figure 3.1, the expected payoff to player I after the chance move is $\frac{1}{3}\times 0+\frac{2}{3}\times 3=2$, and to player II it is $\frac{1}{3}\times 15+\frac{2}{3}\times 0=5$. In this game, the chance node could therefore be replaced by a leaf with payoff 2 for player I and payoff 5 for player II. These expected payoffs reflect the players' preferences for the random outcomes, including their attitude to the risk involved when facing such uncertain outcomes. This 'risk-neutrality' of payoffs does not necessarily hold for actual monetary payoffs. For example, if the payoffs to player II in figure 3.1 were millions of dollars, then in reality that player would probably prefer receiving 5 million dollars for certain to a lottery that gave her 15 million dollars with probability $1/3$ and otherwise nothing. However, payoffs can be adjusted to reflect the player's attitude to risk, as well as representing the 'utility' of an outcome like money so that one can take just the expectation. In the example, player II may consider a lottery that gives her 15 million dollars with probability $1/3$ and otherwise nothing only as a valuable as getting 2 million dollars for certain. In that case, '15 million dollars' may be represented by a payoff of 6 so that the said lottery has an expected payoff 2, assuming that '2 million dollars' are represented by a payoff of 2 and 'getting nothing' by a payoff of 0. This is discussed in greater detail in section 4.6.

For the game tree, it does not matter how it is drawn, but only its 'combinatorial' structure, that is, how its nodes are connected by edges. A tree can be considered as a special 'directed graph', given by a set of nodes, and a set of edges, which are pairs of nodes (u,v). A **path** in such a graph from node u to node v is a sequence of nodes $u_0u_1\ldots u_k$ so that $u_0=u$, $u_k=v$, and so that (u_i,u_{i+1}) is an edge for $0\le i<k$.

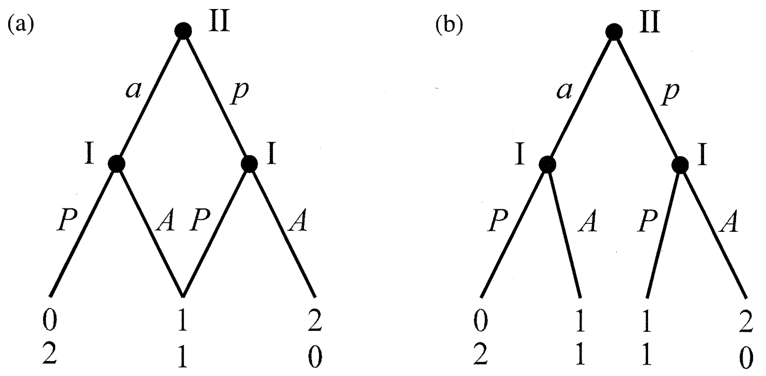


Figure 3.2 The left picture (a) is not a tree because the leaf where both players receive payoff 1 is reachable by two different paths. The correct tree representation is shown on the right in (b).

The graph is a tree if it has a distinguished node, the root, so that to each node of the graph there is a unique path from the root.

Figure 3.2 demonstrates the tree property that every node is reached by a unique path from the root. This fails to hold in the left picture (a) for the middle leaf. Such a structure may make sense in a game. For example, the figure could represent two people in a pub where first player II chooses to pay for the drinks (move p) or to accept (move a) that the first round is paid by player I. In the second round, player I may then decide to either accept not to pay (move A) or to pay (P). Then the players may only care how often, but not when, they have paid a round, with the middle payoff pair $(1,1)$ for two possible 'histories' of moves. The tree property prohibits game states with more than one history, because the history is represented by the unique path from the root. Even if certain differences in the game history do not matter, these game states are distinguished, mostly as a matter of mathematical convenience. The correct representation of the above game is shown in the right picture, figure 3.2(b).

Game trees are also called **extensive games** with **perfect information**. Perfect information means that a player always knows the game state and therefore the complete history of the game up to then. Game trees can be enhanced with an additional structure that represents 'imperfect information', which is the topic of chapter 5.

Note how game trees differ from the combinatorial games studied in chapter 2:

(a) A combinatorial game can be described very compactly, in particular when it is given as a sum of games. For general game trees, such sums are typically not considered.

(b) The 'rules' in a game tree are much more flexible: more than two players and chance moves are allowed, players do not have to alternate, and payoffs do not just stand for 'win' or 'lose'.

(c) The flexibility of game trees comes at a price, though: The game description is much longer than a combinatorial game. For example, a simple instance of nim may require a huge game tree. Regularities like the mex rule do not apply to general game trees.

3.7 Backward induction

Which moves should the players choose in a game tree? 'Optimal' play should maximise a player's payoff. This can be decided irrespective of other players' actions when the player is the **last** player to move. In figure 3.2(b), player I maximises his payoff by move A at both his decision nodes, because at the left node he receives 1 rather than 0 with that move, and at the right node payoff 2 rather than 1. Going backwards in time, player II has to make her move a or p at the root of the game tree, where she will receive either 1 or 0, assuming the described future behaviour of player I. Consequently, she will choose a.

This process is called **backward induction**: Starting with the decision nodes closest to the leaves, a player's move is chosen which maximises that player's payoff at the node. In general, a move is chosen in this way for each decision node provided all subsequent moves have already been decided. Eventually, this will determine a move for every decision node, and hence for the entire game. Backward induction is also known as **Zermelo's algorithm**. (This is attributed to an article by Zermelo (1913) on chess. Later, people decided that Zermelo proved something different, in fact a more complicated property, so that Zermelo's algorithm is sometimes called 'Kuhn's algorithm', according to a paper by Kuhn (1953), which we cite on page 94.)

The move selected by backward induction is not necessarily unique, if there is more than one move giving maximal payoff to the player. In the game in figure 3.1, backward induction chooses either move b or move c for player II, both of which give her payoff 5 (which is an expected payoff for move b) that exceeds her payoff 4 for move a. At the right-most node, player I chooses Q. This determines the preceding move d by player II which gives her the higher payoff 3 as opposed to 2 (via move Q). In turn, this means that player I, when choosing between X, Y, or Z at the root of the game tree, will get payoff 2 for Y and payoff 1 for Z; the payoff when he chooses X depends on the choice of player II: if that is b, then player I gets 2, and can choose either X or Y, both of which give him maximal payoff 2. If player II chooses c, however, then the payoff to player I is 4 when choosing X, so this is the unique optimal choice. To summarise, the possible combinations of moves that can arise in figure 3.1 by backward induction are, simply listing the moves for each player: (XQ, bd), (XQ, cd), and (YQ, bd). Note that Q and d are always chosen, but that Y can only be chosen in combination with the move b by player II.

The moves determined by backward induction are therefore, in general, not unique, and possibly interdependent.

Backward induction gives a **unique** recommendation to each player if there is always only one move that gives maximal payoff. This applies to **generic** games. A generic game is a game where the payoff parameters are real numbers that are in no special dependency of each other (like two payoffs being equal). In particular, it should be allowed to replace them with values nearby. For example, the payoffs may be given in some practical scenario which has some 'noise' that effects the precise value of each payoff. Then two such payoffs are equal with probability zero, and so this case can be disregarded. In generic games, the optimal move is always unique, so that backward induction gives a unique result.

3.8 Strategies and strategy profiles

Definition 3.1 In a game tree, a **strategy** of a player specifies a move for every decision node of that player. A **strategy profile** is a tuple of strategies, with one strategy for each player of the game.

If the game has only two players, a strategy profile is therefore a pair of strategies, with one strategy for player I and one strategy for player II.

In the game tree in figure 3.1, the possible strategies for player I are XP, XQ, YP, YQ, ZP, ZQ. The strategies for player II are ad, ae, bd, be, cd, ce. For simplicity of notation, we have thereby specified a strategy simply as a list of moves, one for each decision node of the player. When specifying a strategy in that way, this must be done with respect to a fixed order of the decision nodes in the tree, in order to identify each move uniquely. This matters when a move name appears more than once, for example in figure 3.2(b). In that tree, the strategies of player I are AA, AP, PA, PP, with the understanding that the first move refers to the left decision node and the second move to the right decision node of player I.

Backward induction defines a move for every decision node of the game tree, and therefore for every decision node of each player, which in turn gives a strategy for each player. The result of backward induction is therefore a strategy profile.

$\Rightarrow$ Exercise 3.1 on page 56 studies a game tree which is not unlike the game in figure 3.1. You can already answer part (a) of this exercise, and apply backward induction, which answers (e).

3.9 Games in strategic form

Assume a game tree is given. Consider a strategy profile, and assume that players move according to their strategies. If there are no chance moves, their play leads to a unique leaf. If there are chance moves, a strategy profile may lead to a probability distribution on the leaves of the game tree, with resulting **expected** payoffs. In general, any strategy profile defines an expected payoff to each player (which also applies to a payoff that is obtained deterministically, where the expectation is computed from a probability distribution that assigns probability one to a single leaf of the game tree).

Definition 3.2 The **strategic form** of a game is defined by specifying for each player the set of strategies, and the payoff to each player for each strategy profile.

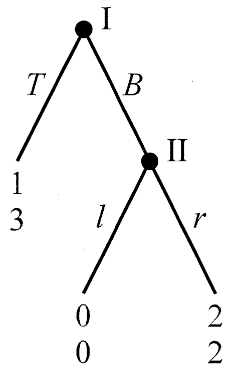
For two players, the strategic form is conveniently represented by a table. The rows of the table represent the strategies of player I, and the columns the strategies of player II. A strategy profile is a strategy pair, that is, a row and a column, with a corresponding cell of the table that contains two payoffs, one for player I and the other for player II.

If m and n are positive integers, then an $m \times n$ game is a two-player game in strategic form with m strategies for player I (the rows of the table) and n strategies for player II (the columns of the table).

Figure 3.3 shows an extensive game in (a), and its strategic form in (b). In the strategic form, T and B are the strategies of player I given by the top and bottom row of the table, and l and r are the strategies of player II, corresponding to the left and right column of the table. The strategic form for the game in figure 3.1 is shown in figure 3.4.

A game in strategic form can also be given directly according to definition 3.2, without any game tree that it is derived from. Given the strategic form, the game is played as follows: Each player chooses a strategy, independently from and simultaneously with the other players, and then the players receive their payoffs as given for the resulting strategy profile.

(a)



(b)

I \ II	l	r
T	1, 3	1, 3
B	0, 0	2, 2

Figure 3.3 Extensive game (a) and its strategic form (b). In a cell of the strategic form, player I receives the bottom left payoff, and player II the top right payoff.

I \ II	ad	ae	bd	be	cd	ce
XP	3, 4	3, 4	2, 5	2, 5	4, 5	4, 5
XQ	3, 4	3, 4	2, 5	2, 5	4, 5	4, 5
YP	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3
YQ	2, 3	2, 3	2, 3	2, 3	2, 3	2, 3
ZP	1, 3	0, 5	1, 3	0, 5	1, 3	0, 5
ZQ	1, 3	4, 2	1, 3	4, 2	1, 3	4, 2

Figure 3.4 Strategic form of the extensive game in figure 3.1. The less redundant reduced strategic form is shown in figure 3.13 below.

3.10 Symmetric games

Many game-theoretic concepts are based on the strategic form alone. In this section, we discuss the possible **symmetries** of such a game. We do this by presenting a number of standard games from the literature, mostly 2×2 games, which are the smallest games which are not just one-player decision problems.

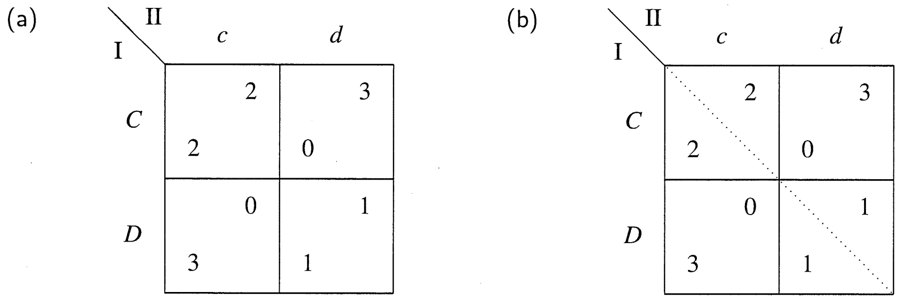


Figure 3.5 Prisoner's dilemma game (a), its symmetry shown by reflection along the dotted diagonal line in (b).

Figure 3.5(a) shows the well-known 'prisoner's dilemma' game. Each player has two strategies, called C and D for player I and c and d for player II. (We use upper case letters for player I and lower case letters for player II for easier identification of strategies and moves.) These letters stand for 'co-operate' and 'defect', respectively. The story behind the name 'prisoner's dilemma' is that of two prisoners held suspect of a serious crime. There is no judicial evidence for this crime except if one of the prisoners testifies against the other. If one of them testifies, he will be rewarded with immunity from prosecution (payoff 3), whereas the other will serve a long prison sentence (payoff 0). If both testify, their punishment will be less severe (payoff 1 for each). However, if they both 'co-operate' with each other by not testifying at all, they will only be imprisoned briefly for some minor charge that can be held against them (payoff 2 for each). The 'defection' from that mutually beneficial outcome is to testify, which gives a higher payoff no matter what the other prisoner does (which makes 'defect' a dominating strategy, as discussed in the next section). However, the resulting payoff is lower to both. This constitutes their 'dilemma'.

The prisoner's dilemma game is **symmetric** in the sense that if one exchanges player I with player II, and strategy C with c, and D with d, then the game is unchanged. This is demonstrated in figure 3.5(b) with the dotted line that connects the top left of the table with the bottom right: the payoffs remain the same when reflected along that dotted line. In order to illustrate the symmetry in this visual way, payoffs have to be written in diagonally opposite corners in a cell of the table.

This representation of payoffs in different corners of cells of the table is due to Thomas Schelling, which he used, for example, in his book *The Strategy of Conflict* (1961). Schelling modestly (and not quite seriously) calls the staggered payoff matrices his most important contribution to game theory, despite that fact that this book was most influential in applications of game theory to social science. Schelling received the 2005 Nobel prize in economics for his contribution to the understanding of conflict and co-operation using game theory.

The payoff to player I has to appear in the bottom left corner, the payoff to player II at the top right. These are also the natural positions for the payoffs that leave no ambiguity about which player each payoff belongs to, because the table is read from top to bottom and from left to right.

For this symmetry, the order of the strategies matters (which it does not for other aspects of the game), so that, when exchanging the two players, the first row is exchanged with the first column, the second row with the second column, and so on. A non-symmetric game can sometimes be made symmetric when re-ordering the strategies of one player, as illustrated in figure 3.7 below. Obviously, in a symmetric game, both players must have the same number of strategies.

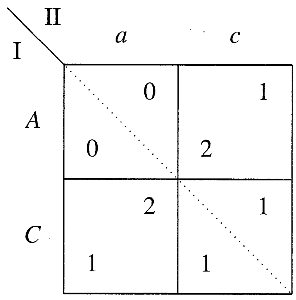


Figure 3.6 The game 'chicken', its symmetry indicated by the diagonal dotted line.

The game of 'chicken' is another symmetric game, shown in figure 3.6. The two strategies are A and C for player I and a and c for player II, which may be termed 'aggressive' and 'cautious' behaviour, respectively. The aggressive strategy is only advantageous (with payoff 2 rather than 0) if the other player is cautious, whereas a cautious strategy always gives payoff 1 to the player using it.

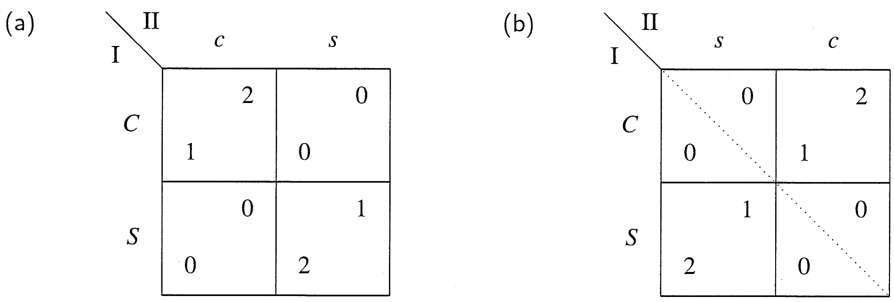


Figure 3.7 The 'battle of sexes' game (a), which is symmetric if the strategies of one player are exchanged, as shown in (b).

The game known as the 'battle of sexes' is shown in figure 3.7(a). In this scenario, player I and player II are a couple each deciding whether to go to a concert (strategies C and c) or to a sports event (strategies S and s). The players have different payoffs arising from which event they go to, but that payoff is zero if they have to attend the event alone.

This game is not symmetric when written as in figure 3.7(a), where strategy C of player I would be exchanged with strategy c of player II, and correspondingly S with s, because the payoffs for the strategy pairs (C,c) and (S,s) on the diagonal are not the same for both players, which is clearly necessary for symmetry. However, changing the order of the strategies of one player, for example of player II as shown in figure 3.7(b), makes this a symmetric game.

Figure 3.8 shows a 3×3 game known as 'rock-scissors-paper'. Both players choose simultaneously one of their three strategies, where rock beats scissors, scissors beat paper, and paper beats rock, and it is a draw otherwise. This is a **zero-sum** game because the payoffs in any cell of the table sum to zero. The game is symmetric, and because it is zero-sum, the pairs of strategies on the diagonal, here (R,r), (S,s), and (P,p), must give payoff zero to both players: otherwise the payoffs for these strategy pairs would not be the same for both players.

I \ II	r	s	p
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

Figure 3.8 The 'rock-scissors-paper' game.

3.11 Symmetries involving strategies*

In this section,[1] we give a general definition of symmetry that may apply to any game in strategic form, with possibly more than two players. This definition allows also for symmetries among strategies of a player, or across players. As an example, the rock-scissors-paper game also has a symmetry with respect to its three **strategies**: When cyclically replacing R by S, S by P, and P by R, and r by s, s by p, and p by r, which amounts to moving the first row and column in figure 3.8 into last place, respectively, then the game remains the same. (In addition, the players may also be exchanged as described.) The formal definition of a symmetry of the game is as follows.

[1] This is an additional section which can be omitted at first reading, and has non-examinable material, as indicated by the star sign * following the section heading.

Definition 3.3 Consider a game in strategic form with N as the finite set of players, and strategy set Σ_i for player $i \in N$, where $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$. The payoff to player i is given by $a_i((s_j)_{j\in N})$ for each strategy profile $(s_j)_{j\in N}$, where $s_j \in \Sigma_j$. Then a **symmetry** of the game is given by a bijection $\phi : N \to N$ and a bijection $\psi_i : \Sigma_i \to \Sigma_{\phi(i)}$ for each $i \in N$ so that

$$a_i((s_j)_{j\in N}) = a_{\phi(i)}((\psi_j(s_j))_{j\in N}) \tag{3.1}$$

for each player $i \in N$ and each strategy profile $(s_j)_{j\in N}$.

Essentially, a symmetry is a way of re-naming players and strategies that leaves the game unchanged. In definition 3.3, the strategy sets Σ_i of the players are assumed to be pairwise disjoint. This assumption removes any ambiguity when looking at a strategy profile $(s_j)_{j\in N}$, which defines one strategy s_j in Σ_j for each player j, because then the particular order of these strategies does not matter, because every strategy belongs to exactly one player. In our examples, strategy sets have been disjoint because we used upper-case letters for player I and lower-case letters for player II.

A symmetry as in definition 3.3 is, first, given by a re-naming of the players with the bijection ϕ, where player i is re-named as $\phi(i)$, which is either another player in the game or the same player. Secondly, a bijection ψ_i re-names the strategies s_i of player i as strategies $\psi_i(s_i)$ of player $\phi(i)$. Any strategy profile $(s_j)_{j\in N}$ thereby becomes a re-named strategy profile $(\psi_j(s_j))_{j\in N}$. With this re-naming, player $\phi(i)$ has taken the role of player i, so one should evaluate his payoff, normally given by the payoff function a_i, as $a_{\phi(i)}((\psi_j(s_j))_{j\in N})$ when applied to the re-named strategy profile. If this produces in all cases the same payoffs, as stated in (3.1), then the re-naming has produced the same game in strategic form, and is therefore called a symmetry of the game.

For two players, the reflection along the diagonal is obviously simpler to state and to observe. In this symmetry among the two players, with player set $N = \{\mathrm{I}, \mathrm{II}\}$ and with strategy sets Σ_{I} and Σ_{II}, we used $\phi(\mathrm{I}) = \mathrm{II}$ and thus $\phi(\mathrm{II}) = \mathrm{I}$, and a **single** bijection $\psi_{\mathrm{I}} : \Sigma_{\mathrm{I}} \to \Sigma_{\mathrm{II}}$ for re-naming the strategies, where ψ_{II} is the inverse of ψ_{I}. This last condition does not have to apply in definition 3.3, which makes this definition more general.

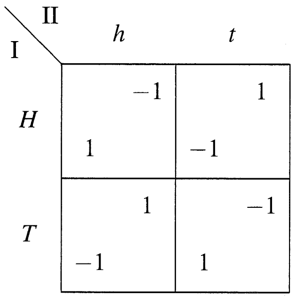


Figure 3.9 The 'matching pennies' game.

The game of 'matching pennies' in figure 3.9 illustrates the greater generality of definition 3.3. This zero-sum game is played with two players that each have a penny, which the player can choose to show as heads or tails, which is the strategy H or T for player I, and h or t for player II. If the pennies match, for the strategy pairs (H,h) and (T,t), then player II has to give her penny to player I; if they do not, for the strategy pairs (H,t) and (T,h), then player I loses his penny to player II. This game has an obvious symmetry in strategies where heads are exchanged with tails for both players, but where the players are not re-named, so that ϕ is the identity function on the player set. However, the game cannot be written so that it remains the same when reflected along the diagonal, because the diagonal payoffs would have to be zero, as in the rock-scissors-paper game, so it is not symmetric among players in the sense of any of the earlier examples. Definition 3.3 does capture the symmetry between the two players as follows (for example): ϕ exchanges I and II, and $\psi_{\mathrm{I}}(H) = h$, $\psi_{\mathrm{I}}(T) = t$, but $\psi_{\mathrm{II}}(h) = T$, $\psi_{\mathrm{II}}(t) = H$. That is, the sides of the penny keep their meaning for player I when he is re-named as player II, but heads and tails change their role for player II. In effect, this exchanges the players' preference for matching versus non-matching pennies, as required for the symmetry.

Please note: In the following discussion, when we call a game 'symmetric', we always mean the simpler symmetry of a two-player game described in the previous section 3.10 that can be seen by reflecting the game along the diagonal.

3.12 Dominance and elimination of dominated strategies

This section discusses the concepts of strict and weak dominance, which apply to strategies and therefore to games in strategic form.

Definition 3.4 Consider a game in strategic form, and let s and t be two strategies of some player P of that game. Then s **dominates** (or strictly dominates) t if for any fixed strategies of the other players, the payoff to player P when using s is higher than his payoff when using t. Strategy s **weakly dominates** t if for any fixed strategies of the other players, the payoff to player P when using s is at least as high as when using t, and in at least one case strictly higher.

In a two-player game, player P in definition 3.4 is either player I or player II. For player I, his strategy s dominates t if row s of the payoffs to player I is in each component larger than row t. If one denotes the payoff to player I for row i and column j by $a(i,j)$, then s dominates t if $a(s,j) > a(t,j)$ for each strategy j of player II. Similarly, if the payoff to player II is denoted by $b(i,j)$, and s and t are two strategies of player II, which are columns of the payoff table, then s dominates t if $b(i,s) > b(i,t)$ for each strategy i of player I.

In the prisoner's dilemma game in figure 3.5, strategy D of player I dominates strategy C, because, with the notation of the preceding paragraph, $a(D,c) = 3 > 2 = a(C,c)$ and $a(D,d) = 1 > 0 = a(C,d)$. Because the game is symmetric, strategy d of player II also dominates c.

It would be inaccurate to define dominance by saying that a strategy s dominates strategy t if s is 'always' better than t. This is only correct if 'always' means 'given the **same** strategies of the other players'. Even if s dominates t, it may happen that **some** payoff when playing s is worse than some other payoff when playing t. For example, $a(C,c) = 2 > 1 = a(D,d)$ in the prisoner's dilemma game, so this is a case where the dominating strategy D gives a lower payoff than C. However, the strategy used by the other player is necessarily different in that case. When s dominates t, then strategy s is better than t when considering any arbitrary (but same) fixed strategies of the other players.

Dominance is sometimes called strict dominance in order to distinguish it from weak dominance. Consider a two-player game with payoffs $a(i,j)$ to player I and $b(i,j)$ to player II when they play row i and column j. According to definition 3.4, strategy s of player I weakly dominates t if $a(s,j) \geq a(t,j)$ for all j and $a(s,j) > a(t,j)$ for at least one column j. The latter condition ensures that if the two rows s and t of payoffs to player I are equal, s is not said to weakly dominate t (because for the same reason, t could also be said to dominate s). Similarly, if s and t are strategies of player II, then s weakly dominates t if the column s of payoffs $b(i,s)$ to player II is in each component i at least as large as column t, and strictly larger, with $b(i,s) > b(i,t)$, in at least one row i. An example of such a strategy is l in figure 3.3(b), which is weakly dominated by r.

When a strategy s of player P dominates his strategy t, player P can always improve his payoff by playing s rather than t. This follows from the way a game in strategic form is played, where the players choose their strategies simultaneously, and the game is played only once.[2] Then player P may consider the strategies of the other players as fixed, and his own strategy choice cannot be observed and should not influence the choice of the others, so he is better off playing s rather than t.

[2] The game in strategic form is considered as a 'one-shot' game. Many studies concern the possible emergence of co-operation in the prisoner's dilemma when the game is repeated, which is a different context.

In consequence, one may study the game where all dominated strategies are **eliminated**. If some strategies are eliminated in this way, one then obtains a game that is simpler because some players have fewer strategies. In the prisoner's dilemma game, elimination of the dominated strategies C and c results in a game that has only one strategy per player, D for player I and d for player II. This strategy profile (D,d) may therefore be considered as a 'solution' of the game, a recommendation of a strategy for each player.

If one accepts that a player will never play a dominated strategy, one may eliminate it from the game and continue eliminating strategies that are dominated in the resulting game. This is called **iterated elimination of dominated strategies**. If this process ends in a unique strategy profile, the game is said to be **dominance solvable**, with the final strategy profile as its solution.

The 'quality game' in figure 3.10 is a game that is dominance solvable in this way. The game is nearly identical to the prisoner's dilemma game in figure 3.5, except for the payoff to player II for the top right cell of the table, which is changed from 3 to 1. The game may describe the situation of, say, player I as a restaurant owner,

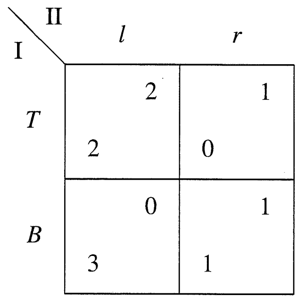


Figure 3.10 The 'quality game', with T and B as good or bad quality offered by player I, and l and r as buying or refusing to buy the product as strategies of player II.

who can provide food of good quality (strategy T) or bad quality (B), and a potential customer, player II, who may decide to eat there (l) or not (r). The customer prefers l to r only if the quality is good. However, whatever player II does, player I is better off by choosing B, which therefore dominates T. After eliminating T from the game, player II's two strategies l and r remain, but in this smaller game r dominates l, and l is therefore eliminated in a second iteration. The resulting strategy pair is (B, r).

When eliminating dominated strategies, the order of elimination does not matter, because if s dominates t, then s still dominates t in the game where some strategies (other than t) are eliminated. In contrast, when eliminating weakly dominated strategies, the order of elimination may matter. Moreover, a weakly dominated strategy, such as strategy l of player II in figure 3.3, may be just as good as the strategy that weakly dominates it, if the other player chooses some strategy, like T in figure 3.3, where the two strategies have equal payoff. Hence, there are no strong reasons for eliminating a weakly dominated strategy in the first place.

$\Rightarrow$ Exercise 3.5 on page 57 studies weakly dominated strategies, and what can happen when these are eliminated. You can answer this exercise except for (d) which you should do after having understood the concept of Nash equilibrium, treated in the next section.

3.13 Nash equilibrium

Not every game is dominance solvable, as the games of chicken and the battle of sexes demonstrate. The central concept of non-co-operative game theory is that of **equilibrium**, often called **Nash equilibrium** after John Nash, who introduced this concept in 1950 for general games in strategic form (the equivalent concept for zero-sum games was considered earlier). An equilibrium is a strategy profile where each player's strategy is a 'best response' to the remaining strategies.

Definition 3.5 Consider a game in strategic form and a strategy profile given by a strategy s_j for each player j. Then for player i, his strategy s_i is a **best response** to the strategies of the remaining players if no other strategy gives player i a higher payoff, when the strategies of the other players are unchanged. An **equilibrium** of the game, also called **Nash equilibrium**, is a strategy profile where the strategy of each player is a best response to the other strategies.

In other words, a Nash equilibrium is a strategy profile so that no player can gain by changing his strategy **unilaterally**, that is, with the remaining strategies kept fixed.

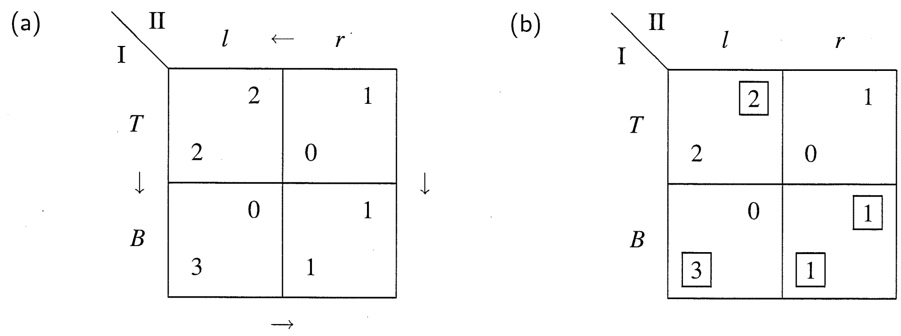


Figure 3.11 Indicating best responses in the quality game with arrows (a), or by putting best response payoffs in boxes (b).

Figure 3.11(a) shows the quality game of figure 3.10 where the best responses are indicated with arrows: The downward arrow on the left shows that if player II plays her left strategy l, then player I has best response B; the downward arrow on the right shows that the best response to r is also B; the leftward arrow at the top shows that player II's best response to T is l, and the rightward arrow at the bottom that the best response to B is r.

This works well for 2×2 games, and shows that a Nash equilibrium is where the arrows 'flow together'. For the quality game, player I's arrows both point downwards, which shows that B dominates T, and the bottom arrow points right, so that starting from any cell and following the arrows, one arrives at the equilibrium (B, r).

In figure 3.11(b), best responses are indicated by boxes drawn around the payoffs for each strategy that is a best response. For player I, a best response is considered against a strategy of player II. That strategy is a column of the game, so the best response payoff is the maximum of each column (which may occur more than once) of the payoffs of player I. Similarly, a best response payoff for player II is the maximum in each row of the payoffs to player II. Unlike arrows, putting best response payoffs into boxes can be done easily even when a player has more than two strategies. A Nash equilibrium is then given by a cell of the table where both payoffs are boxed.

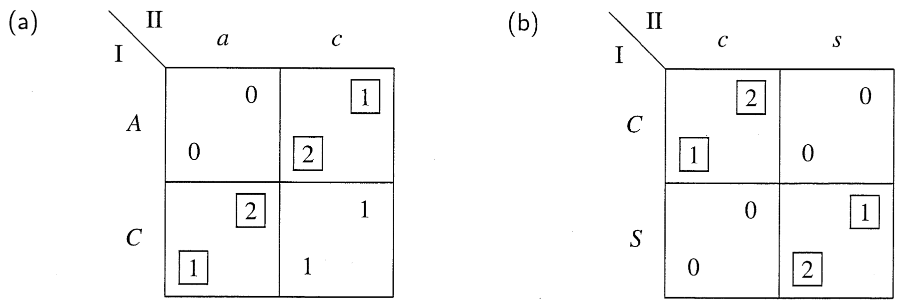


Figure 3.12 Best responses and Nash equilibria in the game of chicken (a), and in the battle of sexes game (b).

The game of chicken in figure 3.12(a) has two equilibria, (A, c) and (a, C). If a game is symmetric, like in this case, then an equilibrium is symmetric if it does not change under the symmetry (that is, when exchanging the two players). Neither of the two equilibria (A, c) and (a, C) is symmetric, but they map to each other under the symmetry (any non-symmetric equilibrium must have a symmetric counterpart; only symmetric equilibria map to themselves).

As figure 3.12(b) shows, the battle of sexes game has two Nash equilibria, (C,c) and (S,s). (When writing the battle of sexes game symmetrically as in figure 3.7(b), its equilibria are not symmetric either.) The prisoner's dilemma game has one equilibrium (D,d), which is symmetric.

It is clear that a dominated strategy is never a best response, and hence cannot be part of an equilibrium. Consequently, one can eliminate any dominated strategy from a game, and not lose any equilibrium. Moreover, this elimination cannot create additional equilibria because any best response in the absence of a dominated strategy remains a best response when adding the dominated strategy back to the game. By repeating this argument when considering the iterated elimination of dominated strategies, we obtain the following proposition.

Proposition 3.6 *If a game in strategic form is dominance solvable, its solution is the only Nash equilibrium of the game.*

$\Rightarrow$ In order to understand the concept of Nash equilibrium in games with more than two players, exercise 3.6 on page 58 is very instructive.

We have seen that a Nash equilibrium may not be unique. Another drawback is illustrated by the rock-scissors-paper game in figure 3.8, and the game of matching pennies in figure 3.9, namely that the game may have no equilibrium 'in pure strategies', that is, when the players may only use exactly one of their given strategies in a deterministic way. This drawback is overcome by allowing each player to use a 'mixed strategy', which means that the player chooses his strategy randomly according to a certain probability distribution. Mixed strategies are the topic of the next chapter.

In the following section, we return to game trees, which do have equilibria even when considering only non-randomised or 'pure' strategies.

3.14 Reduced strategies

The remainder of this chapter is concerned with the connection of game trees and their strategic form, and the Nash equilibria of the game.

We first consider a simplification of the strategic form. Recall figure 3.4, which shows the strategic form of the extensive game in figure 3.1. The two rows of the strategies XP and XQ in that table have identical payoffs for both players, as do the two rows YP and YQ. This is not surprising, because after move X or Y of player I at the root of the game tree, the decision node where player I can decide between P or Q cannot be reached, because that node is preceded by move Z, which is excluded by choosing X or Y. Because player I makes that move himself, it makes sense to replace both strategies XP and XQ by a less specific 'plan of action' $X*$ that precribes only move X. We **always** use the star '$*$' as a placeholder. It stands for an unspecified move at the respective unreachable decision node, to identify the node with its unspecified move in case the game has many decision nodes.

Leaving moves at unreachable nodes unspecified in this manner defines a **reduced strategy** according to the following definition. Because the resulting expected payoffs remain uniquely defined, tabulating these reduced strategies and the payoff for the resulting reduced strategy profiles gives the **reduced strategic form** of the game. The reduced strategic form of the game tree of figure 3.1 is shown in figure 3.13.

Definition 3.7 In a game tree, a **reduced strategy** of a player specifies a move for every decision node of that player, except for those decision nodes that are unreachable due to an earlier own move. A **reduced strategy profile** is a tuple of

I \ II	ad	ae	bd	be	cd	ce
$X*$	$\boxed{3}$, 4	3, 4	$\boxed{2}$, $\boxed{5}$	2, $\boxed{5}$	$\boxed{4}$, $\boxed{5}$	$\boxed{4}$, $\boxed{5}$
$Y*$	2, $\boxed{3}$	2, $\boxed{3}$	$\boxed{2}$, $\boxed{3}$	2, $\boxed{3}$	2, $\boxed{3}$	2, $\boxed{3}$
ZP	1, 3	0, $\boxed{5}$	1, 3	0, $\boxed{5}$	1, 3	0, $\boxed{5}$
ZQ	1, $\boxed{3}$	$\boxed{4}$, 2	1, $\boxed{3}$	$\boxed{4}$, 2	1, $\boxed{3}$	$\boxed{4}$, 2

Figure 3.13 Reduced strategic form of the extensive game in figure 3.1. The star $*$ stands for an arbitrary move at the second decision node of player I, which is not reachable after move X or Y.

reduced strategies, one for each player of the game. The **reduced strategic form** of a game tree lists all reduced strategies for each player, and tabulates the expected payoff to each player for each reduced strategy profile.

The preceding definition generalises definitions 3.1 and 3.2. It is important that the only moves that are left unspecified are at decision nodes which are unreachable due to an earlier **own** move of the player. A reduced strategy must not disregard a move because another player may not move there, because that possibility cannot be excluded by looking only at the player's own moves. In the game tree in figure 3.1, for example, no reduction is possible for the strategies of player II, because neither of her moves at one decision node precludes a move at her other decision node. Therefore, the reduced strategies of player II in that game are the same as her (unreduced, original) strategies.

In definition 3.7, reduced strategies are defined without any reference to payoffs, only by looking at the structure of the game tree. One could derive the reduced strategic form in figure 3.13 from the strategic form in figure 3.4 by identifying the strategies that have identical payoffs for both players and replacing them by a single reduced strategy. Some authors define the reduced strategic form in terms of this elimination of duplicate strategies in the strategic form. (Sometimes, dominated strategies are also eliminated when defining the reduced strategic form.) We define the reduced strategic form without any reference to payoffs because strategies may have identical payoffs by accident, due to special payoffs at the leaves of the game (this does not occur in generic games). Moreover, we prefer that reduced strategies refer to the structure of the game tree, not to some relationship of the payoffs, which is a different aspect of the game.

Strategies are combinations of moves, so for every additional decision node of a player, each move at that node can be combined with move combinations already considered. The number of move combinations therefore grows exponentially with the number of decision nodes of the player, because it is the product of the numbers of moves at each node. In the game in figure 3.14, for example, where m possible initial moves of player I are followed each time by two possible moves of player II, the number of strategies of player II is 2^m. Moreover, this is also the number of reduced strategies of player II because no reduction is possible. This shows that even the

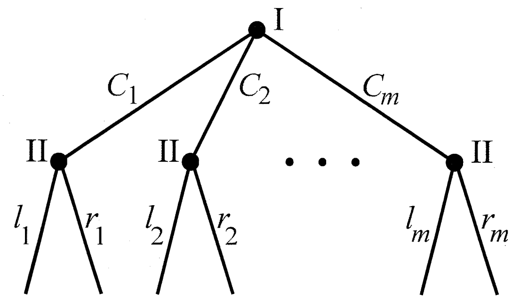


Figure 3.14 Extensive game with m moves $C_1, \ldots, C_m$ of player I at the root of the game tree. To each move C_i, player II may respond with two possible moves l_i or r_i. Player II has 2^m strategies.

reduced strategic form can be exponentially large in the size of the game tree. If a player's move is preceded by an own earlier move, the reduced strategic form is smaller because then that move can be left unspecified if the preceding own move is not made.

$\Rightarrow$ Exercise 3.4 on page 57 tests your understanding of reduced strategies.

In the reduced strategic form, a Nash equilibrium is defined, as in definition 3.5, as a profile of reduced strategies, each of which is a best response to the others.

In this context, we will for brevity sometimes refer to reduced strategies simply as 'strategies'. This is justified because, when looking at the reduced strategic form, the concepts of dominance and equilibrium can be applied directly to the reduced strategic form, for example to the table defining a two-player game. Then 'strategy' means simply a row or column of that table. The term 'reduced strategy' is only relevant when referring to the extensive form.

The Nash equilibria in figure 3.13 are identified as those pairs of (reduced) strategies that are best responses to each other, with both payoffs surrounded by a box in the respective cell of the table. These Nash equilibria in reduced strategies are $(X*, bd)$, $(X*, cd)$, $(X*, ce)$, and $(Y*, bd)$.

$\Rightarrow$ You are now in a position answer (b)–(e) of exercise 3.1 on page 56.

3.15 Subgame perfect Nash equilibrium (SPNE)

In this section, we consider the relationship between Nash equilibria of games in extensive form and backward induction.

The 2×2 game in figure 3.3(b) has two Nash equilibria, (T, l) and (B, r). The strategy pair (B, r) is also obtained, uniquely, by backward induction. We will prove shortly that backward induction always defines a Nash equilibrium.

The equilibrium (T, l) in figure 3.3(b) is not obtained by backward induction, because it prescribes the non-optimal move l for player II at her only decision node. Moreover, l is a weakly dominated strategy. Nevertheless, this is a Nash equilibrium because T is the best response to l, and, moreover, l is a best response to T because against T, the payoff to player II is no worse than when choosing r. This can also be seen when considering the game tree in figure 3.3(a): Move l is a best response to T because player II never has to make that move when player I chooses T, so player II's move does not affect her payoff, and she is indifferent between l and r. On the other hand, only when she makes move l is it optimal for player I to respond by choosing T, because against r player I would get more by choosing B.

The game in figure 3.3(a) is also called a 'threat game' because it has a Nash equilibrium (T,l) where player II 'threatens' to make the move l that is bad for both players, against which player I chooses T, which is then advantageous for player II compared to the backward induction outcome when the players choose (B,r). The threat works only because player II never has to execute it, given that player I acts rationally and chooses T.

The concept of Nash equilibrium is based on the strategic form because it applies to a strategy profile. When applied to a game tree, the strategies in that profile are assumed to be chosen by the players **before** the game starts, and the concept of best response applies to this given expectation of what the other players will do.

With the game tree as the given specification of the game, it is often desirable to keep its sequential interpretation. The strategies chosen by the players should therefore also express some 'sequential rationality' as expressed by backward induction. That is, the moves in a strategy profile should be optimal for any part of the game, including subtrees that cannot be reached due to earlier moves, possibly of other players, like the tree starting at the decision node of player II in the threat game in figure 3.3(a).

In a game tree, a **subgame** is any subtree of the game, given by a node of the tree as the root of the subtree and all its descendants. (Note: This definition of a subgame applies only to games with perfect information, which are the game trees considered so far. In extensive games with imperfect information, which we consider later, the subtree is a subgame only if all players **know** that they are in that subtree.) A strategy profile that defines a Nash equilibrium for every subgame is called a **subgame perfect Nash equilibrium** or SPNE.

⇒ You can now answer the final question (e) of exercise 3.1 on page 56, and exercise 3.2 on page 56.

Theorem 3.8 *Backward induction defines an SPNE.*

Proof. Recall the process of backward induction: Starting with the nodes closest to the leaves, consider a decision node u, say, with the assumption that all moves after u have already been selected. Among the moves at u, select a move that **maximises** the expected payoff to the player that moves at u. (Expected payoffs must be regarded if there are chance moves after u.) In that manner, a move is selected for every decision node, which determines an entire strategy profile.

We prove the theorem inductively: Consider a non-terminal node u of the game tree, which may be a decision node (as in the backward induction process), or a chance node. Suppose that the moves at u are $c_1, c_2, \ldots, c_k$, which lead to subtrees $T_1, T_2, \ldots, T_k$ of the game tree, and assume, as inductive hypothesis, that the moves selected so far define an SPNE in each of these trees. (As the 'base case' of the induction, this is certainly true if each of $T_1, T_2, \ldots, T_k$ is just a leaf of the tree, so that u is a 'last' decision node considered first in the backward induction process.) The induction step is completed if one shows that, by selecting the move at u, one obtains an SPNE for the subgame with root u.

First, suppose that u is a chance node, so that the next node is chosen according to the fixed probabilities specified in the game tree. Then backward induction does not select a move for u, and the inductive step holds trivially: For every player, the payoff in the subgame starting at u is the expectation of the payoffs for each subgame T_i (weighted with the probability for move c_i), and if a player could improve on that payoff, she would have to do so by changing her moves within at least one subtree T_i, which, by inductive hypothesis, she cannot.

Secondly, suppose that u is a decision node, and consider a player **other** than the player to move at u. Again, for that player, the moves in the subgame starting at u are completely specified, and, irrespective of what move is selected for u, she cannot improve her payoff because that would mean she could improve her payoff already in some subtree T_i.

Finally, consider the player to move at u. Backward induction selects a move for u that is best for that player, given the remaining moves. Suppose the player could improve his payoff by choosing some move c_i and additionally change his moves in the subtree T_i. But the resulting improved payoff would **only** be the improved payoff in T_i, that is, the player could already get a better payoff in T_i itself, contradicting the inductive assumption that the moves selected so far defined an SPNE for T_i. This completes the induction. □

This theorem has two important consequences. In backward induction, each move can be chosen deterministically, so that backward induction determines a profile of pure strategies. Theorem 3.8 therefore implies that game trees have Nash equilibria, and it is not necessary to consider randomised strategies. Secondly, subgame perfect Nash equilibria exist. For game trees, we can use 'SPNE' synonymously with 'strategy profile obtained by backward induction'.

Corollary 3.9 *Every game tree with perfect information has a pure-strategy Nash equilibrium.*

Corollary 3.10 *Every game tree has an SPNE.*

In order to describe an SPNE, it is important to consider **unreduced** (that is, fully specified) strategies. Recall that in the game in figure 3.1, the SPNE are (XQ, bd), (XQ, cd), and (YQ, bd). The reduced strategic form of the game in figure 3.13 shows the Nash equilibria $(X*, bd)$, $(X*, cd)$, and $(Y*, bd)$. However, we cannot call any of these an SPNE because they leave the second move of player I unspecified, as indicated by the $*$ symbol. In this case, replacing $*$ by Q results in all cases in an SPNE. The full strategies are necessary to determine if they define a Nash equilibrium in every subgame. As seen in figure 3.13, the game has, in addition, the Nash equilibrium $(X*, ce)$. This is not subgame perfect because it prescribes the move e, which is not part of a Nash equilibrium in the subgame starting with the decision node where player II chooses between d and e: When replacing $*$ by P, this would not be the best response by player I to e in that subgame, and when replacing $*$ by Q, player II's best response would not be e. This property can only be seen in the game tree, not even in the unreduced strategic form in figure 3.4, because it concerns an unreached part of the game tree due to the earlier move X of player I.

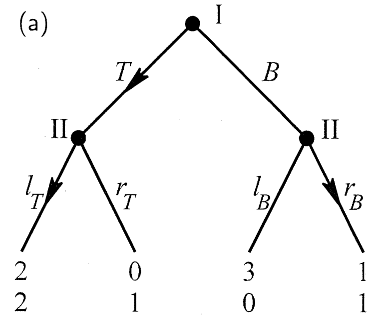
⇒ For understanding subgame perfection, exercise 3.3 on page 56 is very instructive. Exercise 3.7 on page 59 studies a three-player game.

3.16 Commitment games

The main difference between game trees and games in strategic form is that in a game tree, players act sequentially, being aware of the previous moves of the other players. In contrast, players in a game in strategic form move **simultaneously**. The difference between these two descriptions becomes striking when **changing** a two-player game in strategic form to a **commitment game**, as described in this section.

We start with a two-player game in strategic form, say an $m \times n$ game. The corresponding commitment game is an extensive game with perfect information. One of the players (we always choose player I) moves **first**, by choosing one of his m strategies. Then, unlike in the strategic form, player II is **informed** about the move of player I. For each move of player I, the possible moves of player II are the n strategies of the original strategic form game, with resulting payoffs to both players as specified in that game.

(a)



(b)

I \ II	$l_T l_B$	$l_T r_B$	$r_T l_B$	$r_T r_B$
T	2, $\boxed{2}$	$\boxed{2}$, $\boxed{2}$	0, 1	0, 1
B	$\boxed{3}$, 0	1, $\boxed{1}$	$\boxed{3}$, 0	$\boxed{1}$, $\boxed{1}$

Figure 3.15 The extensive game (a) is the **commitment** game of the quality game in figure 3.10. Player I moves first, and player II can **react** to that choice. The arrows show the backward induction outcome. Its strategic form with best-response payoffs is shown in (b).

An example is figure 3.15, which is the commitment game for the quality game in figure 3.10. The original strategies T and B of player I become his moves T and B. In general, if player I has m strategies in the given game, he also has m strategies in the commitment game. In the commitment game, each strategy l or r of player II becomes a move, which depends on the preceding move of player I, so we call them l_T and r_T when following move T, and l_B and r_B when following move B. In general, when player I has m moves in the commitment game and player II has n responses, then each combination of moves defines a pure strategy of player II, so she has n^m strategies in the commitment game. Figure 3.15(b) shows the resulting strategic form of the commitment game in the example. It shows that the given game and the commitment game are very different.

We then look for subgame perfect equilibria of the commitment game, that is, apply backward induction, which we can do because the game has perfect information. Figure 3.15(a) shows that the resulting unique SPNE is $(T, l_T r_B)$, which means that (going backwards) player II, when offered high quality T, chooses l_T, and r_B when offered low quality. In consequence, player I will then choose T because that gives him the higher payoff 2 rather than just 1 with B. Note that it is important that player I has the power to **commit** to T and cannot change later back to B after seeing that player II chose l; without that commitment power, the game would not be accurately reflected by the extensive game as described.

The commitment game in figure 3.15 also demonstrates the difference between an equilibrium, which is a strategy profile, and the **equilibrium path**, which is described by the moves that are actually played in the game when players use their equilibrium strategies. If the game has no chance moves, the equilibrium path is just a certain path in the game tree. In contrast, a strategy profile defines a move in every part of the game tree. Here, the SPNE $(T, l_T r_B)$ specifies moves l_T and r_B for both decision points of player II. Player I chooses T, and then only l_T is played. The equilibrium path is then given by move T followed by move l_T. However, it is **not sufficient** to simply call this equilibrium (T, l_T) or (T, l), because player II's move r_B must be specified to know that T is player I's best response.

We only consider SPNE when analysing commitment games. For example, the Nash equilibrium (B, r) of the original quality game in figure 3.10 can also be found in figure 3.15, where player II **always** chooses the move r corresponding to her equilibrium strategy in the original game, which is the strategy $r_T r_B$ in the commitment game. This defines the Nash equilibrium $(B, r_T r_B)$ in the commitment game, because B is a best response to $r_T r_B$, and $r_T r_B$ is a best response to B (which clearly holds as a general argument, starting with any Nash equilibrium of the original game). However, this Nash equilibrium of the commitment game is not subgame perfect, because it prescribes the suboptimal move r_T off the equilibrium path; this does not affect the Nash equilibrium property because T is not chosen by player I. In order to compare a strategic-form game with its commitment game in an interesting way, we consider only SPNE of the commitment game.

A practical application of a game-theoretic analysis may be to reveal the potential effects of changing the 'rules' of the game. This is illustrated by changing the quality game to its commitment version.

$\Rightarrow$ Now do exercise 3.8 on page 59.

Games in strategic form, when converted to a commitment game, typically have a **first-mover advantage**. A player in a game becomes a first mover or 'leader' when he can commit to a strategy as described, that is, choose a strategy irrevocably and inform the other players about it; this is a change of the 'rules of the game'. The first-mover advantage states that a player who can become a leader is not worse off than in the original game where the players act simultaneously. In other words, if one of the players has the power to commit, he or she should do so.

This statement must be interpreted carefully. For example, if more than one player has the power to commit, then it is not necessarily best to go first. For example, consider changing the game in figure 3.10 so that player II can commit to her strategy, and player I moves second. Then player I will always respond by choosing B because this is his dominant strategy in figure 3.10. Backward induction would then amount to player II choosing l, and player I choosing $B_l B_r$, with the low payoff 1 to both. Then player II is not worse off than in the simultaneous-choice game, as asserted by the first-mover advantage, but does not gain anything either. In contrast, making player I the first mover as in figure 3.15 is beneficial to both.

The first-mover advantage is also known as **Stackelberg leadership**, after the economist Heinrich von Stackelberg, who formulated this concept for the structure of markets in 1934. The classic application is to the duopoly model by Cournot, which dates back to 1838.

As an example, suppose that the market for a certain type of memory chip is dominated by two producers. The players can choose to produce a certain quantity of chips, say either high, medium, low, or none at all, denoted by H, M, L, N for player I and h, m, l, n for player II. The market price of the memory chips decreases with increasing total quantity produced by both companies. In particular, if both choose a high quantity of production, the price collapses so that profits drop to zero. The players know how increased production lowers the chip price and their profits. Figure 3.16 shows the game in strategic form, where both players choose their output level simultaneously. The symmetric payoffs are derived from Cournot's model, explained below.

This game is dominance solvable (see section 3.12 above). Clearly, no production is dominated by low or medium production, so that row N and column n in figure 3.16 can be eliminated. Then, high production is dominated by medium production, so that row H and column h can be omitted. At this point, only medium and low production remain. Then, regardless of whether the opponent produces medium or low, it is always better for each player to produce a medium quantity, eliminating L

I \ II	h	m	l	n
H	0, 0	12, 8	18, $\boxed{9}$	$\boxed{36}$, 0
M	8, 12	$\boxed{16}$, $\boxed{16}$	$\boxed{20}$, 15	32, 0
L	$\boxed{9}$, 18	15, $\boxed{20}$	18, 18	27, 0
N	0, $\boxed{36}$	0, 32	0, 27	0, 0

Figure 3.16 Duopoly game between two chip manufacturers who can decide between high, medium, low, or no production, denoted by H,M,L,N for player I and h,m,l,n for player II. Payoffs denote profits in millions of dollars.

and l. Only the strategy pair (M,m) remains. Therefore, the Nash equilibrium of the game is (M,m), where both players make a profit of \$16 million.

Consider now the commitment version of the game, with a game tree corresponding to figure 3.16 just as figure 3.15 is obtained from figure 3.10. We omit this tree to save space, but backward induction is easily done using the strategic form in figure 3.16: For each row H, M, L, or N, representing a possible commitment of player I, consider the best response of player II, as shown by the boxed payoff entry for player II **only** (the best responses of player I are irrelevant). The respective best responses are unique, defining the backward induction strategy $l_H m_M m_L h_N$ of player II, with corresponding payoffs 18, 16, 15, 0 to player I when choosing H, M, L, or N, respectively. Player I gets the maximum of these payoffs when he chooses H, to which player II will respond with l_H. That is, among the anticipated responses by player II, player I does best by announcing H, a high level of production. The backward induction outcome is thus that player I makes a profit \$18 million, as opposed to only \$16 million in the simultaneous-choice game. When player II must play the role of the follower, her profits fall from \$16 million to \$9 million.

The first-mover advantage comes from the ability of player I to credibly commit himself. After player I has chosen H, and player II replies with l, player I would like to be able switch to M, improving profits even further from \$18 million to \$20 million. However, once player I is producing M, player II would change to m. This logic demonstrates why, when the players choose their quantities simultaneously, the strategy combination (H,l) is not an equilibrium. The commitment power of player I, and player II's appreciation of this fact, is crucial.

The payoffs in figure 3.16 are derived from the following simple model due to Cournot. The high, medium, low, and zero production numbers are 6, 4, 3, and 0 million memory chips, respectively. The profit per chip is $12-Q$ dollars, where Q is the total quantity (in millions of chips) on the market. The entire production is sold. As an example, the strategy combination (H,l) yields $Q=6+3=9$, with a profit of \$3 per chip. This yields the payoffs of \$18 million and \$9 million for players I and II in the (H,l) cell in figure 3.16. Another example is player I acting as a monopolist (player II choosing n), with a high production level H of 6 million chips sold at a profit of \$6 each.

In this model, a monopolist would produce a quantity of 6 million even if numbers other than 6, 4, 3, or 0 were allowed, which gives the maximum profit of \$36 million. The two players could co-operate and split that amount by producing 3 million each, corresponding to the strategy combination (L,l) in figure 3.16. The equilibrium quantities, however, are 4 million for each player, where both players receive less. The central four cells in figure 3.16, with low and medium production in place of 'co-operate' and 'defect', have the structure of a prisoner's dilemma game (figure 3.5), which arises here in a natural economic context. The optimal commitment of a first mover is to produce a quantity of 6 million, with the follower choosing 3 million.

These numbers, and the equilibrium ('Cournot') quantity of 4 million, apply even when arbitrary quantities are allowed. That is, suppose x and y are the quantities of production for player I and player II, respectively. The payoffs $a(x,y)$ to player I and $b(x,y)$ to player II are defined as

$$\begin{aligned} a(x,y) &= x\cdot(12-x-y), \\ b(x,y) &= y\cdot(12-x-y). \end{aligned} \tag{3.2}$$

Then it is easy to see that player I's best response $x(y)$ to y is given by $6-y/2$, and player II's best response $y(x)$ to x is given by $6-x/2$. The pair of linear equations $x(y)=x$ and $y(x)=y$ has the solution $x=y=4$, which is the above Nash equilibrium (M,m) of figure 3.16. In the commitment game, player I maximises his payoff, assuming the unique best response $y(x)$ of player II in the SPNE, by maximising $a(x,y(x))$, which is $x\cdot(12-x-6+x/2)=x\cdot(12-x)/2$. That maximum is given for $x=6$, which happens to be the strategy H (high production) in figure 3.16. The best response of player II to that commitment is $6-6/2=3$, which we have named strategy l for low production.

⇒ Exercise 3.9 on page 60 studies a game with infinite strategy sets like (3.2), and makes interesting observations, in particular in (d), concerning the payoffs to the two players in the commitment game compared to the original simultaneous game.

3.17 Reminder of learning outcomes

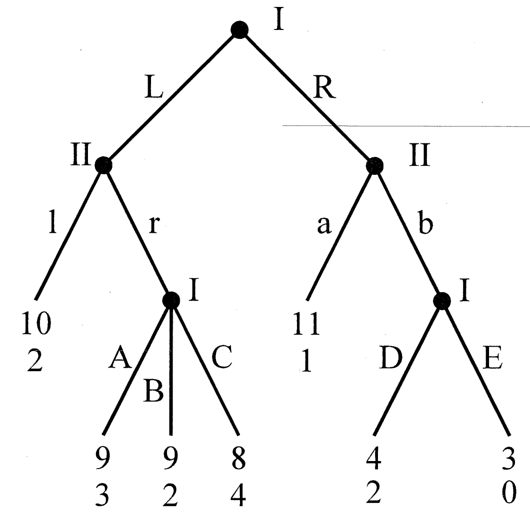
After studying this chapter, you should be able to:

- interpret game trees and games in strategic form;
- explain the concepts of move in a game tree, strategy (and how it differs from move), strategy profile, backward induction, symmetric games, dominance and weak dominance, dominance solvable, Nash equilibrium, reduced strategies and reduced strategic form, subgame perfect Nash equilibrium, and commitment games;
- apply these concepts to specific games.

3.18 Exercises for chapter 3

Exercises 3.1 and 3.2 ask you to apply the main concepts of this chapter to a simple game tree. A more detailed study of the concept of SPNE is exercise 3.3. Exercise 3.4 is on counting strategies and reduced strategies. Exercise 3.5 shows that iterated elimination of **weakly** dominated strategies is a problematic concept. Exercises 3.6 and 3.7 concern three-player games, which are very important to understand because the concepts of dominance and equilibrium require to keep the strategies of **all other** players fixed; this is not like just having another player in a two-player game. Exercise 3.8 is an instructive exercise on commitment games. Exercise 3.9 studies commitment games where in the original game both players have infinitely many strategies.

Exercise 3.1 Consider the following game tree. At a leaf, the top payoffs are for player I, the bottom payoffs are for player II.



(a) What is the number of strategies of player I and of player II?

(b) How many reduced strategies do they have?

(c) Give the reduced strategic form of the game.

(d) What are the Nash equilibria of the game in reduced strategies?

(e) What are the subgame perfect Nash equilibria of the game?

Exercise 3.2 Consider the game tree in figure 3.17. At a leaf, the top payoff is for player I, the bottom payoff for player II.

(a) What is the number of strategies of player I and of player II? How many reduced strategies does each of the players have?

(b) Give the reduced strategic form of the game.

(c) What are the Nash equilibria of the game in reduced strategies? What are the subgame perfect equilibria of the game?

(d) Identify every pair of reduced strategies where one strategy weakly or strictly dominates the other, and indicate if the dominance is weak or strict.

Exercise 3.3 Consider the game trees in figure 3.18.

(a) For the game tree in figure 3.18(a), find all Nash equilibria (in pure strategies). Which of these are subgame perfect?

(b) In the game tree in figure 3.18(b), the payoffs a, b, c, d are positive real numbers. For each of the following statements (i), (ii), (iii), decide if it is true or false,

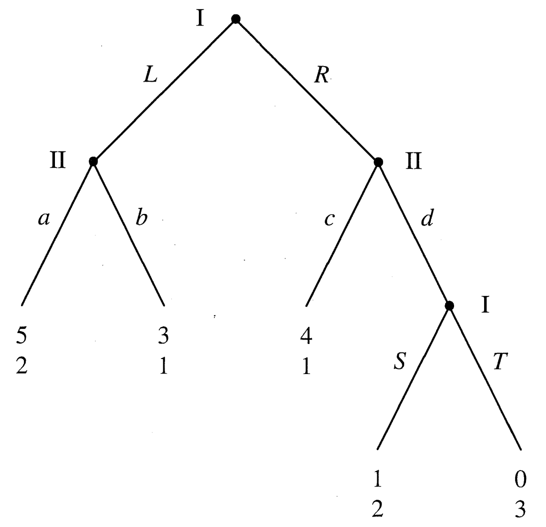


Figure 3.17 Game tree for exercise 3.2.

justifying your answer with an argument or counterexample; you may refer to any standard results. For any $a, b, c, d > 0$,

(i) the game always has a subgame perfect Nash equilibrium (SPNE);

(ii) the payoff to player II in any SPNE is always at least as high as her payoff in any Nash equilibrium;

(iii) the payoff to player I in any SPNE is always at least as high as his payoff in any Nash equilibrium.

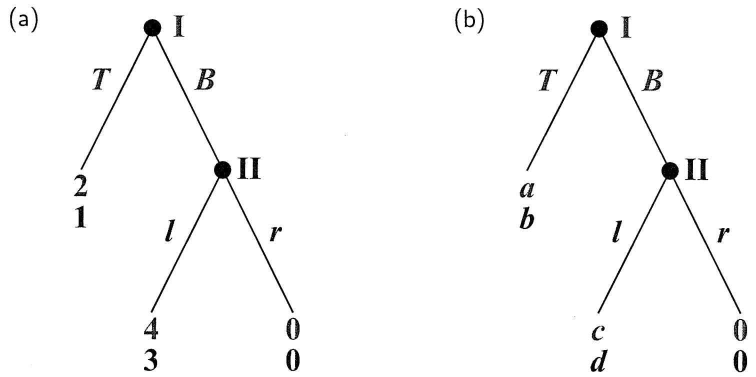


Figure 3.18 Game trees for exercise 3.3

Exercise 3.4 Consider the two game trees (a) and (b) in figure 3.19. In each case, how many strategies does each player have? How many reduced strategies?

Exercise 3.5 Consider the 3×3 game in figure 3.20.

(a) Identify all pairs of strategies where one strategy weakly dominates the other.

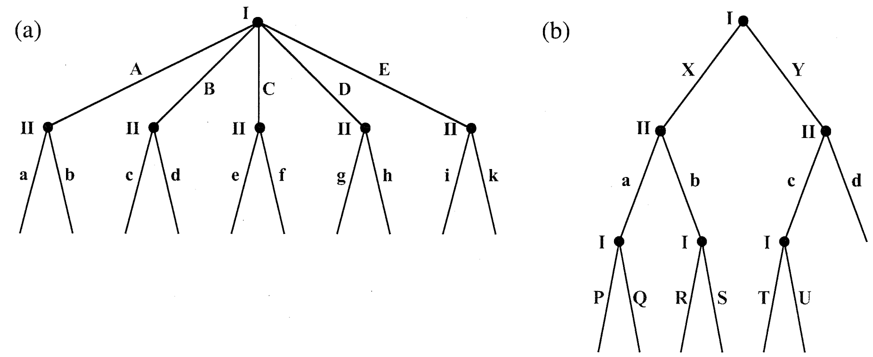


Figure 3.19 Game trees for exercise 3.4. Payoffs have been omitted because they are not relevant for the question.

I \ II	l	c	r
T	0 1	1 3	1 1
M	1 1	0 3	1 0
B	2 2	3 3	2 0

Figure 3.20 3×3 game for exercise 3.5.

(b) Assume you are allowed to remove a weakly dominated strategy of some player. Do so, and repeat this process (of iterated elimination of weakly dominated strategies) until you find a single strategy pair of the original game. (Remember that two strategies with identical payoffs do **not** weakly dominate each other!)

(c) Find such an iterated elimination of weakly dominated strategies that results in a strategy pair other than the one found in (b), where **both** strategies, and the payoffs to the players, are different.

(d) What are the Nash equilibria (in pure strategies) of the game?

Exercise 3.6 Consider the three-player game in strategic form in figure 3.21. Each player has two strategies: Player I has the strategies T and B (the top and bottom row below), player II has the strategies l and r (left or right column in each 2×2 panel), and player III has the strategies L and R (the right or left panel). The payoffs to the players in each cell are given as triples of numbers to players I, II, III.

(a) Identify all pairs of strategies where one strategy strictly, or weakly, dominates the other.
[*Hint: Make sure you understand what dominance means for more than two players. Be careful to consider the correct payoffs.*]

(b) Apply iterated elimination of strictly dominated strategies to this game. What are the Nash equilibria of the game?

I \ II	l	r
T	3,4,4	1,3,3
B	8,1,4	2,0,6

III: L

I \ II	l	r
T	4,0,5	0,1,6
B	5,1,3	1,2,5

III: R

Figure 3.21 Three-player game for exercise 3.6.

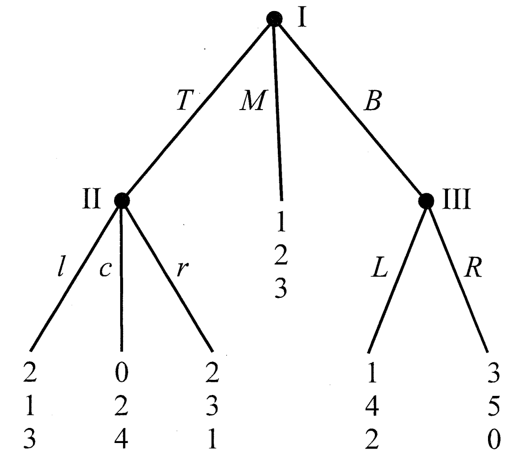


Figure 3.22 Game tree with three players for exercise 3.7. At a leaf, the topmost payoff is to player I, the middle payoff is to player II, and the bottom payoff is to player III.

Exercise 3.7 Consider the three-player game tree in figure 3.22.

(a) How many strategy **profiles** does this game have?

(b) Identify all pairs of strategies where one strategy strictly, or weakly, dominates the other.

(c) Find all Nash equilibria in pure strategies. Which of these are subgame perfect?

Exercise 3.8 Consider a game G in strategic form. Recall that the **commitment game** derived from G is defined by letting player I choose one of his pure strategies x which is then **announced** to player II, who can then in each case choose one of her strategies in G as a response to x. The resulting payoffs are as in the original game G.

(a) If G is an $m \times n$ game, how many strategies do player I and player II have, respectively, in the commitment game?

For each of the following statements, determine whether they are true or false; justify your answer by a short argument or counterexample.

(b) In an SPNE of the commitment game, player I never commits to a strategy that is strictly dominated in G.

(c) In an SPNE of the commitment game, player II never chooses a move that is a strictly dominated strategy in G.

Exercise 3.9 Let G be the following game: player I chooses a (not necessarily integer) non-negative number x, and player II in the same way a non-negative number y. The resulting (symmetric) payoffs are

$$\begin{array}{ll} x \cdot (4+y-x) & \text{for player I,} \\ y \cdot (4+x-y) & \text{for player II.} \end{array}$$

(a) Given x, determine player II's best response $y(x)$ (which is a function of x), and player I's best response $x(y)$ to y. Find a Nash equilibrium, and give the payoffs to the two players.

(b) Find an SPNE of the commitment game, and give the payoffs to the two players.

(c) Are the equilibria in (a) and (b) unique?

(d) Let G be a game where the best response $y(x)$ of player II to any strategy x of player I is always unique. Show that in any SPNE of the commitment game, the payoff to player I is at least as large as his payoff in any Nash equilibrium of the original game G.

Solutions to exercises

Solutions for chapter 2

Solution to exercise 2.1

(a) In analysing the games of three nim heaps where one heap has size one, we first look at some examples, and then use **mathematical induction** to prove what we conjecture to be the losing positions. A losing position is one where every move is to a winning position, because then the opponent will win. The point of this exercise is that you formulate precisely the statement to be proved.

Consider three heaps of sizes $1, m, n$, where $n \geq m \geq 1$. We observe the following: $1, 1, m$ is winning, by moving to $1, 1, 0$. Similarly, $1, m, m$ is winning, by moving to $0, m, m$. Next, $1, 2, 3$ is losing, and hence $1, 2, n$ for $n \geq 4$ is winning. $1, 3, n$ is winning for any $n \geq 3$ by moving to $1, 3, 2$. For $1, 4, 5$, reducing any heap produces a winning position, so this is losing.

The general pattern for the losing positions thus seems to be: $1, m, m+1$, for even numbers m. This includes also the case $m = 0$, which we can take as the base case for an induction. So we prove by induction that $1, m, m+1$, for even numbers m, are the losing positions: The move to $0, m, m+1$ produces a winning position. The move to $1, k, m+1$ for $k < m$ allows the following countermove to a losing position, by inductive hypothesis: If k is even, to $1, k, k+1$; if k is odd, to $1, k, k-1$. This covers all moves in the heap of size m. From the heap of size $m+1$, the move to $1, m, m$ is to a winning position, and to $1, m, k$ for $k < m$ as well, by inductive hypothesis: If k is even, then $k \leq m-2$ because m is even, so one can countermove to $1, k+1, k$, and if k is odd, to $1, k-1, k$. This completes the induction.

Alternatively, one can use the theorem on nim heap sizes represented as sums of powers of two: $*1 + *m + *n$ is losing if and only if, except for 2^0, the powers of two making up m and n come in pairs. So these must be the same powers of two, except for $1 = 2^0$, which occurs in only m or n, where we have assumed that n is the larger number: We have $m = 2^a + 2^b + 2^c + \cdots$ for $a > b > c > \cdots \geq 1$, so m is even, and, with the same $a, b, c, \ldots$, $n = 2^a + 2^b + 2^c + \cdots + 1 = m + 1$. Then $*1 + *m + *n \equiv *0$.

(b) We have $6 = 4 + 2$, $10 = 8 + 2$, and $15 = 8 + 4 + 2 + 1$. So $*6 + *10 + *15 = *(4+2) + *(8+2) + *(8+4+2+1) \equiv$ $*4 + *2 + *8 + *2 + *8 + *4 + *2 + *1 \equiv *2 + *1 \equiv *3$ by cancelling repetitions in pairs. So this is a winning position, which we have to change to a losing position by nim-adding $*2 + *1$ to a suitable heap. Here, all three heaps have the largest nim-heap $*2$ of the sum in their representation, so all three heaps can be suitably diminished to obtain the sum $*0$ for the new three heaps. Following steps 1–3 on page 23, we compute: $*6 + (*2 + *1) \equiv *4 + *1$, so the winning move is to reduce the heap of size 6 to size 5. By moving in the heap with 10 chips: $*10 + (*2 + *1) \equiv *8 + *1$, so another winning move is to reduce the heap of size 10 to size 9. Finally, by moving in the heap with 15 chips: $*15 + (*2 + *1) \equiv *8 + *4$, so the winning move is to reduce the heap of size 15 to size 12. (In particular, not all heaps are reduced by the same amount.)

In what amounts to the same computation, we get these moves by converting the heap sizes to binary:

$$\begin{array}{rcl} 6 & = & 0\,1\,1\,0 \\ 10 & = & 1\,0\,1\,0 \\ 15 & = & 1\,1\,1\,1 \\ \hline & & 0\,0\,1\,1 \end{array}$$

Because all three rows have a 1 in the leftmost odd 'sum' column, each row represents a heap where a winning move can be made. Hence, player I has three winning moves, namely: converting 0110 to a binary number where the binary digits in the last two columns (the 1's in 0011) are changed, giving the binary representation 0101 which is 5 in decimal. The second move is, similarly, obtained by converting 1010 to 1001, which means removing 1 chip from the heap of size 10 to give a heap of size 9. The third move is to convert 1111 to 1100, which means removing 3 chips from the heap of size 15, giving size 12.

Solution to exercise 2.2

(a) In 3×3 dominos, there are, up to symmetry, only two moves for player I, namely placing the domino such that it occupies a corner square or such that it occupies the centre square. In either case, player II can respond by placing her domino alongside the first domino, such that a 2×2 square in one corner is occupied, leaving the L-shaped remaining 5 squares. Then no matter what player I does, II will still be able to place the last domino, and player I loses. So 3×3 dominos is a win for II.

(b) When both m and n are even, player II will win in $m \times n$ dominos by playing 'copycat', using the **central symmetry** of the board. That is, whenever player I places his domino, player II responds by placing her domino on the square obtained by point-reflection on the centre of the board. Then the domino pattern of the board will have that central symmetry whenever player I makes his move, so that the response move of player II will always be possible. That is to say, the two adjacent squares in question are always empty because their symmetric counterparts have been empty when player I made his move.

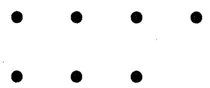
Be careful: Any 'copycat' strategy requires that the underlying symmetry of the situation is preserved. This fails when using the symmetry where the board is reflected along a line: here player I could place a domino on the line itself and II would not have a counter-move. So 'by copycat' is not a satisfactory answer.

(c) In $m \times n$ dominos when m is odd and n is even, player I will win by placing his domino on the centre pair of squares (in the middle row), and then playing copycat as described before for player II in (b).

Solution to exercise 2.3

(a) Player I always wins in these situations. We describe directly the winning move in the games with two rows of dots:

For $2 \times m$ chomp, the winning move is to remove the bottom right dot $(2, m)$, leaving a pattern like the following when $m = 4$:



Afterwards, player I can always re-create this pattern by removing the dot that is diagonally adjacent to the dot that player II removed. That is, any move of player II of the form $(1,i)$ for $i>1$ can be countered by $(2,i-1)$, and any move $(2,i)$ by $(1,i+1)$. So player I has always a move left and wins.

For square games of size $m\times m$, $m\geq 2$, the winning move is $(2,2)$. Then player I can respond to a move of type $(i,1)$ by removing the dot $(1,i)$ and vice versa, until player II is forced to take $(1,1)$ and loses.

(b) Remove the 'poisoned cookie', that is, the dot on $(1,1)$. Then the last player loses by not being able to move any more, exactly when before she would have had to take the poisoned cookie.

(c) [A beautiful argument; note that in the case $m=n$, (a) provides an explicit winning move.]

If removing the bottom right dot (m,n) is a winning move, then we are done. If not, there is a counter-move (i,j) by player II that would create a losing situation for player I. But then player I could make (i,j) to start with, creating the same losing situation for player II. So this game is a win for player I, even though we don't know the winning move.

Solution to exercise 2.4

(a) The completed table for columns 5 and 6 is

	0	1	2	3	4	5	6	
0	$*0$	$*1$	$*2$	$*3$	$*4$	$*5$	$*6$	
1	$*1$	$*2$	$*0$	$*4$	$*5$	$*3$	$*7$	
2	$*2$	$*0$	$*1$	$*5$	$*3$	$*4$	$*8$	
3	$*3$	$*4$	$*5$	$*6$	$*2$	$*0$	$*1$	

(b) The queen is on a square with nim value 2, so one winning move is to reduce the nim heap to size 2 to make this a losing position. However, there are also two squares with nim value 4 that the queen can reach, on row 3 column 1 and on row 0 column 4. These are the only such squares. Moving the queen to either of these also gives a losing position because it is equivalent to $*4+*4$. Any other move would produce a sum of two different nim heaps, which is not a losing position, so there are no other winning moves.

Solution to exercise 2.5

(a) Putting the domino anywhere on the board produces two independent boards of size $1\times k$ and $1\times(n-k-2)$. The two board lengths add up to $n-2$ because two squares are taken away by the domino. The resulting position is a sum of two games because the player can only move in one of them, which is equivalent to the sum of nim heaps $*D_k+*D_{n-k-2}$. Hence,

$$D_n=\operatorname{mex}(\{D_k\oplus D_{n-k-2}\mid 0\leq k\leq n-2\}).$$

(b) We use the result in (a). By symmetry, we only have to consider $k \le n/2 - 1$. With the simpler notation $\text{mex}(\ldots)$ instead of $\text{mex}(\{\ldots\})$,

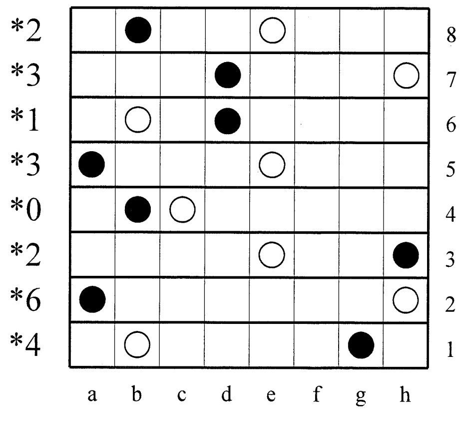
$$\begin{aligned}
D_0 &= D_1 = 0,\\
D_2 &= \text{mex}(D_0 \oplus D_0) = \text{mex}(0) = 1,\\
D_3 &= \text{mex}(D_0 \oplus D_1) = \text{mex}(0) = 1,\\
D_4 &= \text{mex}(D_0 \oplus D_2, D_1 \oplus D_1) = \text{mex}(1,0) = 2,\\
D_5 &= \text{mex}(D_0 \oplus D_3, D_1 \oplus D_2) = \text{mex}(1,1) = 0,\\
D_6 &= \text{mex}(D_0 \oplus D_4, D_1 \oplus D_3, D_2 \oplus D_2) = \text{mex}(2,1,0) = 3,\\
D_7 &= \text{mex}(D_0 \oplus D_5, D_1 \oplus D_4, D_2 \oplus D_3) = \text{mex}(0,2,0) = 1,\\
D_8 &= \text{mex}(D_0 \oplus D_6, D_1 \oplus D_5, D_2 \oplus D_4, D_3 \oplus D_3) = \text{mex}(3,0,3,0) = 1,\\
D_9 &= \text{mex}(D_0 \oplus D_7, D_1 \oplus D_6, D_2 \oplus D_5, D_3 \oplus D_4) = \text{mex}(1,3,1,3) = 0,\\
D_{10} &= \text{mex}(D_0 \oplus D_8, D_1 \oplus D_7, D_2 \oplus D_6, D_3 \oplus D_5, D_4 \oplus D_4) = \text{mex}(1,1,2,1,0) = 3.
\end{aligned}$$

For your possible interest, the sequence $D_0, D_1, \ldots$ is $0,0,1,1,2,0,3,1,1,0,3,3,2,2,4,0,\ 5,2,2,3,3,0,1,1,3,0,\ldots$ and eventually repeats with period 34, the highest occurring value being 9 (which requires that 8 occurs – explain! But no 6 occurs – how can that be?). Recognised from those n where $D_n = 0$, dominos on a $1 \times n$ board is a losing game for $n = 0,1,5,9,15$, and others.

Solution to exercise 2.6

(a) Here, black will win, by simply closing the gap to the white counter whenever white makes a move. So black always has a move left and wins.

(b) The close relationship, hopefully revealed by (a), is to poker nim, where a move that widens the gap between the two counters amounts to adding chips to the heap, whereas narrowing the gap is to reduce the heap size. That is, the gap between the two counters is the size of the nim heap corresponding to that row. The equivalent nim heaps are shown on the left:



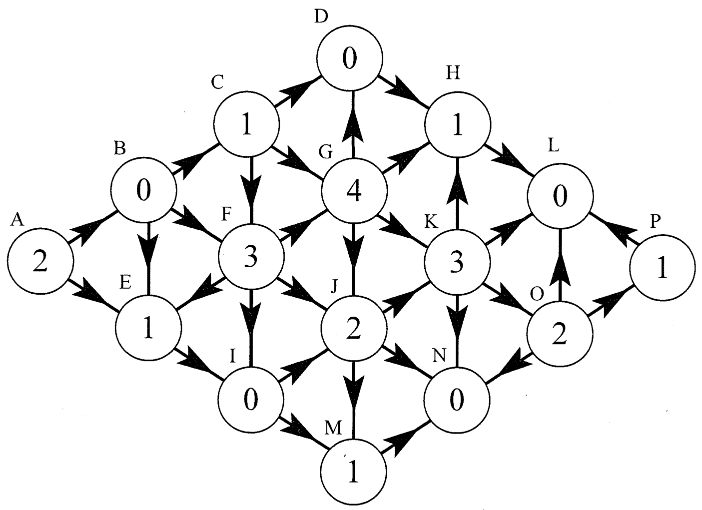
As an experienced nim player, you may spot that the top three rows $8,7,6$, equivalent to $*2$, $*3$, $*1$, sum to a losing position, and so do the bottom three rows $3,2,1$ that are equivalent to $*2$, $*6$, $*4$, and of course row 4 which is equivalent to $*0$. So one winning move is to turn any of the rows equivalent to $*3$ to $*0$, like by moving e5 to b5, or h7 to e7. Alternatively, you may spot that the equal-sized nim heaps in rows 8 and 3 (equivalent to $*2$) and in rows 7 and 5 (equivalent to $*3$) cancel out, leaving you with rows 6, 2, and 1 and equivalent nim heaps $*1$, $*6$, $*4$. Another winning move is then to reduce $*6$ to $*5$ because $*1 + *5 + *4$ is losing (see exercise 2.1(a)), with the counter moved from h2 to g2.

This game is known as **Northcott's game**. You can play this game interactively on the internet, see http://www.cut-the-knot.org/recurrence/Northcott.shtml , which also gives the explanation in terms of nim heaps. Now you should win whenever you can.

Solution to exercise 2.7

(a) This game fulfils the ending condition because no node can be visited again, because there are no 'cycles' in the digraph.

(b) Because the counters are moved independently, this is a sum of games, each corresponding to a counter. The nodes correspond to positions of a single game with one counter only. The options of a node are simply the nodes that the counter can move to. Every node is equivalent to a nim heap. Nodes L and N have no successor node, so these are nodes from which a counter cannot be moved any more. Consequently, they are equivalent to the empty nim heap $*0$. The nodes that have **only** L or N as successor are H, P, and M. They get nim value 1 as they are equivalent to $*1$. Next, those nodes get nim values that only have successors that already have nim values, computed as the mex of those values. For example, D has only H as successor, which has nim value 1, so D gets $\text{mex}(1) = 0$ as its nim value. Node O gets $\text{mex}(0,1) = 2$ as it nim value, then K gets 3 which is $\text{mex}(0,1,2)$. This continues in this fashion. The nim values of all nodes are given as follows:

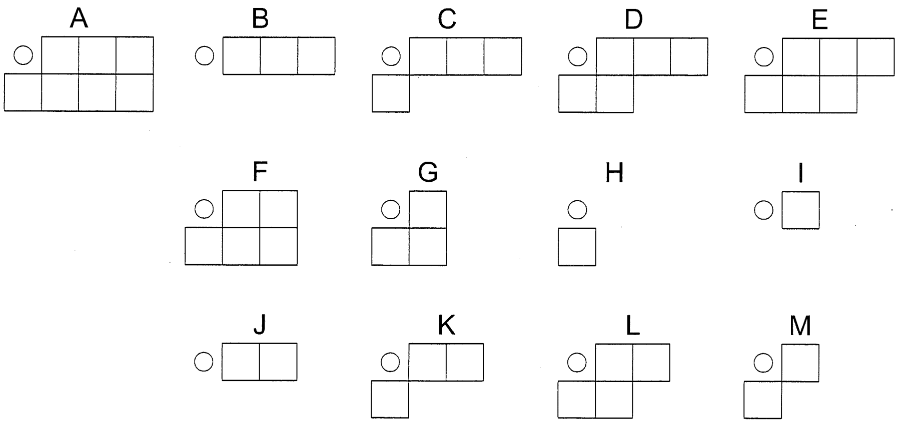


In order to determine if the sum of games determined by the counters is winning or losing, we have to take the nim sum of the respective nim values. We can ignore the two counters on H because they cancel each other. The resulting nim sum for the counters on A, F, G, K is $2 \oplus 3 \oplus 4 \oplus 3 = 2 \oplus 4 = 6$. This is nonzero, so this is a winning position for player I. There is one winning move, from G to J, because the counter on G has nim value 4, the highest power of two in the nim sum. The only other potential alternative would be to obtain a second 4 in the nim sum to cancel the existing 4 of the counter on G by moving from F to G, but this would not give a nim sum of 0, so moving from G to J is the **only** winning move.

(c) When the arrow from J to K is reversed so that it points from K to J instead, the nim value of J, which is 2, does not change, because 2 is still the mex of $\{0,1\}$, which is the same as the mex of the larger set $\{0,1,3\}$ when K is also a successor. Similarly, the nim value of K, which is $\text{mex}\{0,1,2\} = 3$, does not change, because the additional successor J has value 2, which is already in the set $\{0,1,2\}$. Because all nim values remain the same, the computation in (b) applies as well and the winning move is as before, so nothing changes.

Solution to exercise 2.8

In order to represent chomp in the normal play convention where the last player to move wins, we remove the top left dot, the 'poisoned cookie', indicated by a small circle in the picture below. Moreover, we replace the dots by squares. A move is, of course, to remove with any square all those to the right and below it. The following picture describes all possible positions of the game. The starting position is A.



Positions B–E correspond to the options of A by removing one of the squares in the bottom row, positions F, G, H to removing a square in the top row (the 'poisoned cookie' not being available as a removable square, of course). From F, options J, K, and L are reached by removing a bottom row square; the other options of F are G and H. From G, new reachable positions are H, I and M.

The simpler games have obvious nim values, partly because they represent a nim heap, like B which is $*3$, or because they are sums of nim heaps, like C which is $*1+*3$.

So the following nim values are computed directly: B $\equiv *3$, C $\equiv *1+*3 \equiv *2$, H $\equiv$ I $\equiv *1$, J $\equiv *2$, K $\equiv *1+*2 \equiv *3$, M $\equiv *1+*1 \equiv *0$.

Now to the more complicated pictures. The options of G are H, I, and M, and $\mathrm{mex}(1,1,0)=2$, so G $\equiv *2$. The options of L are G, H, J, K, with $\mathrm{mex}(2,1,2,3)=0$, so L $\equiv *0$, in agreement with exercise 2.3(a). The options of F are G, H, J, K, L with $\mathrm{mex}(2,1,2,3,0)=4$, so F $\equiv *4$. The options of D are B, C, G, H, L with $\mathrm{mex}(3,2,2,1,0)=4$, so D $\equiv *4$. Position E should be losing according to exercise 2.3(a), but let's compute its nim value. The options of E are B, C, D, F, G, H with $\mathrm{mex}(3,2,4,4,2,1)=0$, so indeed E $\equiv *0$. Finally, the options of A are B, C, D, E, F, G, H with $\mathrm{mex}(3,2,4,0,4,2,1)=5$, so A $\equiv *5$.

The nim heap of size four cannot be increased to a heap of size five. So the only possible winning moves in the game A $+*4$ are to move to an option of A that has nim value 4. The two options with that property are D and F.

Solution to exercise 2.9

(a) By the rules of Hackenbush, a single stalk of n edges is just a nim heap with n chips. So the leftmost picture is $*3+*2+*1$, which is equivalent to $*0$, a losing game. In the middle picture, a player can either remove one of the edges directly connected to the ground, leaving a nim heap $*3$, or one next to it, leaving

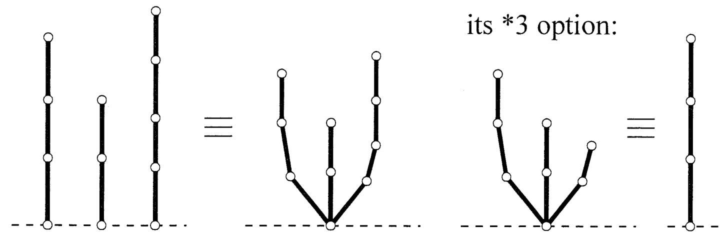
$*1+*2$, which is $*3$. Because $\text{mex}(3,3)=0$, this is a losing game with nim value 0. In the rightmost picture, the options are $*4$, $*1+*3 \equiv *2$, or $*2+*2 \equiv *0$. Because $\text{mex}(4,2,0)=1$, this is equivalent to $*1$.

(b) The first picture is just $*1+*1$, which is $*0$. Only edges are removed, not the dots connecting them, so it does not matter that the two edges are joined on the ground. Similarly, the second picture is $*1+*2$, which is $*3$. The options of the third picture are $*2$ by removing one of the top edges, or $*0$ by removing the bottom edge. Because $\text{mex}(2,0)=1$, that picture is equivalent to $*1$. In the last picture, the options are: $*3$ by removing the shorter branch, $*1$ (which is the third picture) by removing the end of the longer branch, or $*2$ by removing the entire long branch, and finally $*0$ by removing the bottom edge. Because $\text{mex}(3,1,2,0)=4$, that picture is equivalent to $*4$.

(c) In (b), two branches joined at the bottom with nim value 0 put on top of a stem of length 1 gives a tree with nim value 1. Two branches that together have nim value 3 put on top of a stem of length 1 gives a tree with nim value 4. Clearly, a single branch of length m put on top of a stem of length n produces a stalk of length $n+m$, which is a nim heap of size $n+m$. This gives rise to the following conjecture:

(*) **If the nim sum of the branches is m, and the stem has length n, then the tree obtained by putting the branches on top of the stem has nim value $n+m$.**

In the example shown in (c), the nim sum of the branches is $3\oplus2\oplus4=5$, and the stem has length 2, so the resulting tree should have nim value 7. This can be verified by examining the options of the tree, as was done in (b). We now give the general proof of (*), which proceeds exactly in that fashion. First, glue the branches together at the bottom, as in the following picture.



By assumption in (*), the nim sum of these branches is m, and hence all its options are equivalent to one of $*0, *1, \ldots, *(m-1)$ (as far as relevant for applying the mex rule, there may also be options with nim value k for $k>m$). The picture above shows an option with nim value $*3$. Each such option $*k$, for $0 \le k < m$, can be replaced by a single stalk of length k, as shown on the right above. When the branches are put on top of the stem (which has length n), these options, resulting from chopping off some edge in one of the branches, are equivalent to stalks of length $n+k$ for $0\le k<m$ because they are extended by the stem length. So the tree clearly has options equivalent to $*n, *(n+1), \ldots, *(n+m-1)$. It also does **not** have an option equivalent to $*(n+m)$ because that can only result in chopping off some edge in one of the branches, not in the stem. Removing any stem edge results in a stalk of any length $0,1,\ldots,n-1$. So the tree has all options of nim values $0,1,\ldots,n+m-1$, but not $n+m$, and hence is equivalent to a nim heap of size $n+m$, which proves (*).

The rightmost tree in (c) has four identical branches on top of a stem of length 3. The branches cancel out in pairs, so the nim value of the tree is just that of the stem, which is 3.

Solutions for chapter 3

Solution to exercise 3.1

(a) Recall that a strategy in a game tree of a player specifies a move for every decision node of that player. In a game, we assume a certain order of the decision nodes of the player, and list a move for each of them, to define a strategy. In the game tree given here, player I has three decision nodes. His strategies are therefore
$LAD, LAE, LBD, LBE, LCD, LCE, RAD, RAE, RBD, RBE, RCD, RCE$.
However, in order to count the number of strategies, it suffices to multiply the number of moves for all the decision nodes of the player. So number of strategies of player I is $2 \times 3 \times 2 = 12$, and of player II it is $2 \times 2 = 4$.

(b) The number of reduced strategies is found by listing them. Recall that a reduced strategy in a game tree of a player specifies a move for every decision node of that player, except for those moves that are unreachable due to an own earlier move. Given the order of the decision nodes, a reduced strategy is as before a list of moves, one for each decision node, except that any move unreachable due to an own earlier move, which is now unspecified, is replaced by a star $*$. In the game here, the third decision node of player I with moves D and E is unreachable when making move L, and the second decision node with moves A, B, C is unreachable after making move R. So the reduced strategies of player I are $LA*, LB*, LC*, R*D, R*E$.
So player I has 5 reduced strategies, and player II has 4, since for her no reduction is possible. (If the reduction leaves the strategies unchanged, strategies and reduced strategies are the same.)

(c) The reduced strategic form of the game is the following. Best response payoffs are boxed.

I \ II	la	lb	ra	rb
$LA*$	10, 2	$\boxed{10}$, 2	9, $\boxed{3}$	$\boxed{9}$, $\boxed{3}$
$LB*$	10, $\boxed{2}$	$\boxed{10}$, $\boxed{2}$	9, $\boxed{2}$	$\boxed{9}$, $\boxed{2}$
$LC*$	10, 2	$\boxed{10}$, 2	8, $\boxed{4}$	8, $\boxed{4}$
$R*D$	$\boxed{11}$, 1	4, $\boxed{2}$	$\boxed{11}$, 1	4, $\boxed{2}$
$R*E$	$\boxed{11}$, $\boxed{1}$	3, 0	$\boxed{11}$, $\boxed{1}$	3, 0

(d) The Nash equilibria of the game in reduced strategies are the pairs are $(LA*, rb)$, $(LB*, lb)$, $(LB*, rb)$, $(R*E, la)$, $(R*E, ra)$.

(e) Recall that subgame perfect equilibria are those obtained by backward induction, and have to be specified as complete (not reduced) strategy. Of the Nash equilibria in (a), the last two are clearly not subgame perfect, even when specifying the missing move $*$ as A, B, or C, since they prescribe the move E for player I which is not an optimal choice for player I when deciding between D and E.

The equilibrium $(LA*, rb)$ prescribes optimal moves for the players at their decision nodes after player I has chosen L. It looks as if we cannot decide whether move b for player II is optimal since we have not specified if player I plays D or E, since that is move is irrelevant after player I has chosen L. However, backward induction for player I chooses the move D and then move b of player II. So one subgame perfect equilibrium is (LAD, rb). By the same reasoning, (LBD, lb) and (LBD, rb) are the other two subgame perfect equilibria. Note that player II can choose to play l by backward induction if (and only if) player I chooses B.

Solution to exercise 3.2

(a) Player I has $2 \times 2 = 4$ pure strategies, and player II has $2 \times 2 = 4$ pure strategies. The number of reduced pure strategies for player I is 3, shown as rows in (b) below, and 4 (no reduction possible) for player II.

(b) Best responses are boxed:

I \ II	ac	ad	bc	bd
$L*$	$\boxed{5}$, $\boxed{2}$	$\boxed{5}$, $\boxed{2}$	3, 1	$\boxed{3}$, 1
RS	4, 1	1, $\boxed{2}$	$\boxed{4}$, 1	1, $\boxed{2}$
RT	4, 1	0, $\boxed{3}$	$\boxed{4}$, 1	0, $\boxed{3}$

(c) The Nash equilibria in reduced strategies are $(L*, ac)$, $(L*, ad)$. Using backward induction, the only subgame perfect equilibrium of the game is (LS, ad).

(d) For player I, RS weakly dominates RT.

For player II, ac weakly dominates bc, ad weakly dominates ac, ad strictly dominates bc, ad weakly dominates bd, bd weakly dominates bc.

Solution to exercise 3.3

(a) The strategic form looks as follows.

I \ II	l	r
T	2, $\boxed{1}$	$\boxed{2}$, $\boxed{1}$
B	$\boxed{4}$, $\boxed{3}$	0, 0

As shown by the boxed entries, the game has two pure strategy equilibria (T,r) and (B,l). The equilibrium (B,l) is obtained by backward induction and hence the unique subgame perfect equilibrium (SPNE).

(b) Of the stated conditions (i)–(iii),

(i) is true because every game tree has an SPNE obtained by backward induction.

(ii) is false, for example by changing the payoff 1 in (a) to 4, giving the Nash equilibrium (T,r) with higher payoff to player II.

(iii) is true: If $a > c$, then T is strictly dominating B, and T is always played by player I in equilibrium so that he gets payoff a, which is also the SPNE payoff when player II plays l (and player I plays T). If $a = c$, then player I also gets a in any equilibrium, and if $a < c$, then the unique SPNE is (B,l) with payoff c to player I, which is the highest payoff to player I in the game, and hence at least as high as any Nash equilibrium payoff.

Solution to exercise 3.4

(a) Player I has one decision node with five moves, and therefore five pure strategies. Player II has five decision nodes with two moves each, and therefore $2^5 = 32$ pure strategies. These are also the numbers of reduced pure strategies because no decision node of a player is preceded by an own earlier move.

(b) Here, player I has four decision nodes with two moves each, and therefore $2^4 = 16$ pure strategies. Player II has two decision nodes with two moves each, and therefore $2^2 = 4$ pure strategies. For player II, no reduction is possible, as no decision node is preceded by an own earlier move, so player II also has four reduced pure strategies. For player I, however, move X precedes the two decision nodes on the left with moves P, Q and R, S respectively, so the four combinations of these moves, preceded by X, form one set of reduced pure strategies (where no move on the right has to be specified). Similarly, the moves T and U are preceded by move Y, which in turn makes the move combinations with P or Q and R or S irrelevant. In total, these are six reduced pure strategies for player I. The list of these reduced strategies as described is $XPR*$, $XPS*$, $XQR*$, $XQS*$, $Y{*}{*}T$, $Y{*}{*}U$.

Solution to exercise 3.5

(a) Both T and B weakly dominate strategy M of player I. For player II, strategy r weakly dominates l.

(b) The diagram in figure 7.1 on page 159 gives an overview of all possible orders of eliminating weakly dominated strategies. One iterated elimination of weakly dominated strategies is to first eliminate M, then r, then T, then l, which

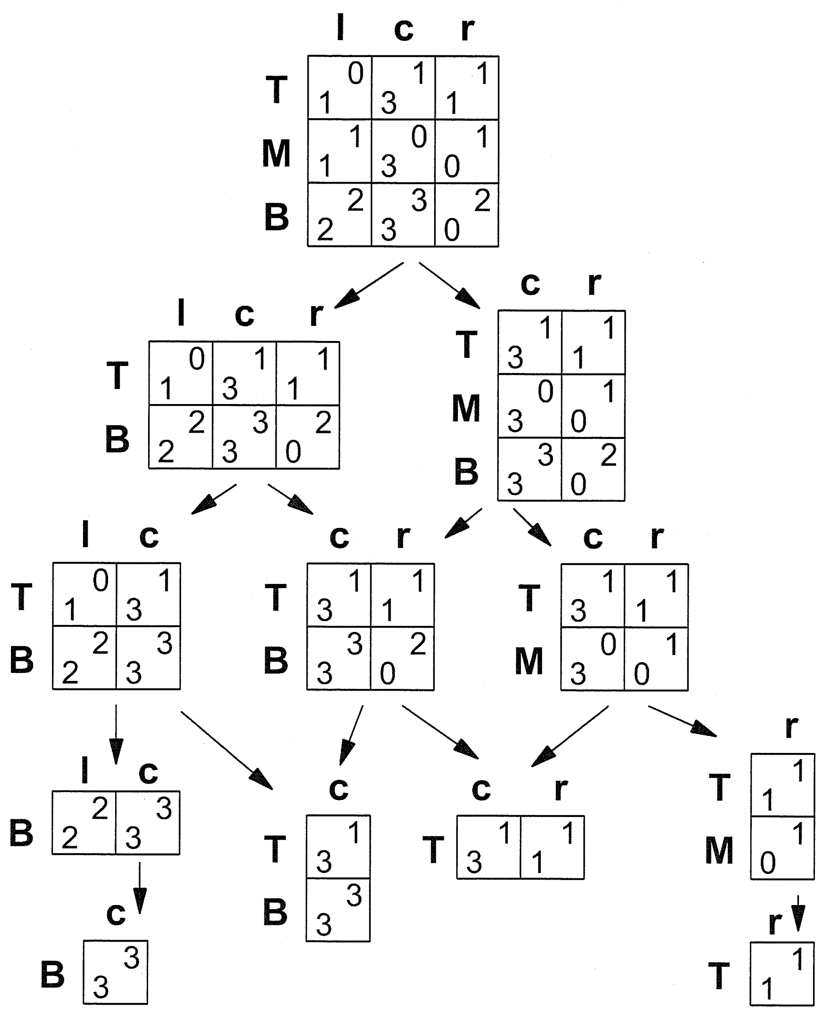


Figure 7.1 All possible sequences of eliminating weakly dominated strategies in the game in exercise 3.5.

terminates at the strategy pair (B,c). Note that eliminating M, r, l in that order leaves two strategies T and B with equal payoffs to player I, so neither of the two can be eliminated further.

(c) Another possible sequence of iterated elimination of weakly dominated strategies is to first eliminate l, then B, then c, then M, which terminates at the strategy pair (T,r).

(d) The Nash equilibria of the game are (T,c), (T,r), (B,c). Note that (T,c) is not among the strategy pairs found in (b) and (c) above.

Solution to exercise 3.6

(a) Only B strictly dominates T. Other dominances arise only after elimination of B as in (b), and do not count.

(b) After eliminating T, player III's L strictly dominates R. After eliminating R, player II's l strictly dominates r. The only remaining profile is (B,l,L), which is the sole Nash equilibrium of the game.

Solution to exercise 3.7

(a) Each player moves only once, so the number of a player's moves is the number of strategies. The number of strategy profiles is $3 \times 3 \times 2 = 18$.

(b) Figure 7.2 shows the strategic form, for which an example has been given earlier in exercise 3.6 on page 58. It is not necessary to give the strategic form, but it also simplifies the answer to (c). The boxes indicate best-response payoffs.

I \ II	l	c	r
T	$\boxed{2},1,\boxed{3}$	$0,2,\boxed{4}$	$\boxed{2},\boxed{3},\boxed{1}$
M	$1,\boxed{2},\boxed{3}$	$\boxed{1},\boxed{2},\boxed{3}$	$1,\boxed{2},\boxed{3}$
B	$1,\boxed{4},\boxed{2}$	$\boxed{1},\boxed{4},\boxed{2}$	$1,\boxed{4},\boxed{2}$

III: L

I \ II	l	c	r
T	$2,1,\boxed{3}$	$0,2,\boxed{4}$	$2,\boxed{3},\boxed{1}$
M	$1,\boxed{2},\boxed{3}$	$1,\boxed{2},\boxed{3}$	$1,\boxed{2},\boxed{3}$
B	$\boxed{3},\boxed{5},0$	$\boxed{3},\boxed{5},0$	$\boxed{3},\boxed{5},0$

III: R

Figure 7.2 Strategic form of the game in exercise 3.7(b).

No strategy strictly dominates another. For player I, B weakly dominates M. For player II, r weakly dominates l and c, and c weakly dominates l. For player III, L weakly dominates R.

(c) The Nash equilibria in pure strategies are (T,r,L), (M,c,L), (B,c,L). The only subgame perfect equilibrium is (T,r,L).

Solution to exercise 3.8

(a) The commitment game can be represented by a game tree where player I moves first, having m choices, and player II moves second, having n moves each. Thus, in the commitment game, player I has m and player II has n^m many strategies (n moves for each of the m commitments).

(b) This is false. An example is the quality game in figure 3.10, where in the commitment game player I commits to the strictly dominated strategy T.

(c) This is true. In an SPNE, player II's chosen move must always be a best response, which is never true for a strategy in G that is strictly dominated.

Solution to exercise 3.9

(a) The payoff to player II is quadratic in y and its derivative with respect to y is $4+x-2y$, which is zero if and only if $y = y(x) = 2+x/2$. This is a maximum because the second derivative -2 is negative. By symmetry, player I's best response is given by $x = x(y) = 2+y/2$. A Nash equilibrium is defined by $x = x(y)$ and $y = y(x)$, that is, $x = y = 4$, with payoff 16 to both players.

(b) In the commitment game, player I chooses x and player II a best response $y(x)$, so player I maximises the function $x \cdot (4 + y(x) - x) = x \cdot (6 - x/2)$. Its derivative with respect to x is $6 - x$, which is zero for $x = 6$, with best response $y(6) = 5$. The resulting payoffs to the two players are 18 for player I and 25 for player II.

(c) Yes, they are unique, as unique solutions to a linear equation for the Nash equilibrium and as a unique solution to a maximisation problem for the commitment game.

(d) In an SPNE, player I solves the maximisation problem of maximising his payoff, say $a(x,y)$, over x given that player II plays her best response $y = y(x)$. In any Nash equilibrium (x^*, y^*) of G, we have $y^* = y(x^*)$ because the best response is unique, so the corresponding payoff to player I certainly fulfils $a(x^*, y^*) = a(x^*, y(x^*)) \leq \max_x a(x, y(x))$, where the last term is the SPNE payoff of the commitment game.

In essence this says: Player I could always commit to his strategy in a Nash equilibrium and thereby get the payoff in that equilibrium, so the commitment payoff cannot be smaller.

Solutions for chapter 4

Solution to exercise 4.1

First, it is easy to see that the game has no Nash equilibrium in pure strategies. We find the mixed equilibria **without** using the upper-envelope method, to demonstrate the method described in section 4.13.

In a mixed strategy equilibrium, player II has to mix between l and r, each pure strategy getting positive probability. Let $(1-q, q)$ be the mixed strategy of player II.

Furthermore, let (a, b, c) be the mixed strategy of player I, where he plays T with probability a, M with probability b, B with probability c.

Suppose player I uses only T and M with positive probability, with the mixed strategy $(a, b, 0)$. Trying to make player II indifferent, this means

$$1 \cdot a + 0 \cdot b = 0 \cdot a + 2 \cdot b,$$

or $a = 2b = 2(1-a)$, meaning $a = 2/3$, $b = 1/3$. Then player II gets an expected payoff of $2/3$ for each of her pure strategies. In turn, she tries to play $(1-q, q)$ to make player I indifferent between T and M (because both have to be pure best responses to $(1-q, q)$ and hence must have equal payoff). This means $6q = 2(1-q) + 5q$ or $q = 2 - 2q$ or $3q = 2$, that is, the mixed strategy is $(1/3, 2/3)$. The expected payoff is 4 for both T and M, which is higher than the payoff 3 player I would get when playing B. So indeed, we have the equilibrium $((2/3, 1/3, 0), (1/3, 2/3))$.

Next, we try a mixed strategy $(a, 0, c)$ of player I that only uses T and B. Equating the expected payoffs to player II gives the equation $a + 3c = 4c$ or $a = c$, that is, the mixed strategy $(1/2, 0, 1/2)$ where player II gets expected payoff 2. Then player II has to play $(1-q, q)$ so that player I is indifferent between T and B, that is, $6q = 3$ or $q = 1/2$, with expected payoff 3 to player I. This looks like an equilibrium but in fact is **not**, because the expected payoff for M is 3.5, which is higher than 3. So T and B have equal payoff but are nevertheless not best responses to $(1/2, 1/2)$, which means we don't have an equilibrium.

Finally, consider the case that player I plays with a mixed strategy $(0, b, c)$ where only M and B have positive probability. Equal payoff to the columns of player II means