DECISION TESTS

8.6. In a simple binary communication system, during every T seconds, one of two possible signals $s_0(t)$ and $s_1(t)$ is transmitted. Our two hypotheses are

$$H_0: \quad s_0(t) \text{ was transmitted.}$$
$$H_1: \quad s_1(t) \text{ was transmitted.}$$

We assume that

$$s_0(t) = 0 \qquad \text{and} \qquad s_1(t) = 1 \qquad 0 < t < T$$

The communication channel adds noise $n(t)$, which is a zero-mean normal random process with variance 1. Let $x(t)$ represent the received signal:

$$x(t) = s_i(t) + n(t) \qquad i = 0, 1$$

We observe the received signal $x(t)$ at some instant during each signaling interval. Suppose that we received an observation $x = 0.6$.

(a) Using the maximum likelihood test, determine which signal is transmitted.

(b) Find P_{I} and P_{II}.

(a) The received signal under each hypothesis can be written as

$$H_0: \quad x = n$$
$$H_1: \quad x = 1 + n$$

Then the pdf of x under each hypothesis is given by

$$f(x \mid H_0) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f(x \mid H_1) = \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}$$

The likelihood ratio is then given by

$$\Lambda(x) = \frac{f(x \mid H_1)}{f(x \mid H_0)} = e^{(x - 1/2)}$$

By Eq. (8.9), the maximum likelihood test is

$$e^{(x-1/2)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Taking the natural logarithm of the above expression, we get

$$x - \tfrac{1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} 0 \qquad \text{or} \qquad x \underset{H_0}{\overset{H_1}{\gtrless}} \tfrac{1}{2}$$

Since $x = 0.6 > \frac{1}{2}$, we determine that signal $s_1(t)$ was transmitted.

(b) The decision regions are given by

$$R_0 = \{x: x < \tfrac{1}{2}\} = (-\infty, \tfrac{1}{2}) \qquad R_1 = \{x: x > \tfrac{1}{2}\} = (\tfrac{1}{2}, \infty)$$

Then by Eqs. (*8.1*) and (*8.2*) and using Table A (Appendix A), we obtain

$$P_{\mathrm{I}} = P(D_1 \mid H_0) = \int_{R_1} f(x \mid H_0)\, dx = \frac{1}{\sqrt{2\pi}} \int_{1/2}^{\infty} e^{-x^2/2}\, dx = 1 - \Phi(\tfrac{1}{2}) = 0.3085$$

$$P_{\mathrm{II}} = P(D_0 \mid H_1) = \int_{R_0} f(x \mid H_1)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1/2} e^{-(x-1)^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1/2} e^{-y^2/2}\, dy = \Phi(-\tfrac{1}{2}) = 0.3085$$

8.7. In the binary communication system of Prob. 8.6, suppose that $P(H_0) = \frac{2}{3}$ and $P(H_1) = \frac{1}{3}$.

(*a*) Using the MAP test, determine which signal is transmitted when $x = 0.6$.

(*b*) Find P_{I} and P_{II}.

(*a*) Using the result of Prob. 8.6 and Eq. (*8.15*), the MAP test is given by

$$e^{(x-1/2)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)} = 2$$

Taking the natural logarithm of the above expression, we get

$$x - \tfrac{1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln 2 \qquad \text{or} \qquad x \underset{H_0}{\overset{H_1}{\gtrless}} \tfrac{1}{2} + \ln 2 = 1.193$$

Since $x = 0.6 < 1.193$, we determine that signal $s_0(t)$ was transmitted.

(*b*) The decision regions are given by

$$R_0 = \{x\colon x < 1.193\} = (-\infty, 1.193)$$
$$R_1 = \{x\colon x > 1.193\} = (1.193, \infty)$$

Thus, by Eqs. (*8.1*) and (*8.2*) and using Table A (Appendix A), we obtain

$$P_{\mathrm{I}} = P(D_1 \mid H_0) = \int_{R_1} f(x \mid H_0)\, dx = \frac{1}{\sqrt{2\pi}} \int_{1.193}^{\infty} e^{-x^2/2}\, dx = 1 - \Phi(1.193) = 0.1164$$

$$P_{\mathrm{II}} = P(D_0 \mid H_1) = \int_{R_0} f(x \mid H_1)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1.193} e^{-(x-1)^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.193} e^{-y^2/2}\, dy = \Phi(0.193) = 0.5765$$