0.1 Stern Strategies in Prisoners Dilemma Type Games

The stern strategy S cooperates until the other player defects and then always defects. After defecting against a stern player, it is clear that the defector should carry on defecting (since the stern player will always defect from then onwards and defection is the best response to this). In order to check whether S is a Nash equilibrium, we only have to check that R1(S, S) R1(D, S). 24 / 50 Stern Strategies in Prisoners Dilemma Type Games It follows that if tit-for-tat is a Nash equilibrium, then stern is also a Nash equilibrium. Note that in a population comprised purely of stern and tit-for-tat players, everyone will always cooperate. One advantage of tit-for-tat occurs when with a small probability players make a mistake when choosing an action or perceiving the action taken by the other player. If players use stern, then after a defection cooperation breaks down. Using tit-for-tat (or a similar reactive but forgiving strategy), cooperation may well be re-established. 25 / 50

0.1.1 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

The repeated version of the Cournot game takes into account that firms will face a sequence of production decisions and take the previous behaviour of competitors into account. What strategies in the repeated symmetric Cournot game would correspond to C, D and S in the repeated prisoners dilemma game? The corresponding strategy to C would be the strategy which when followed by both firms would maximise the sum of the profits (these would be split evenly between the firms). The corresponding strategy to D would be the strategy both firms follow at the unique Nash equilibrium of the Cournot game. 26 / 50

0.1.2 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

The corresponding strategy to S would be the strategy: follow the profit maximisation strategy C as long as the other firm does the same, always play the strategy corresponding to the standard Cournot game after the other firm has deviated from the profit maximisation strategy. For simplicity, we do not consider the analogous strategy to tit-for-tat. Such defection only pays if the short-term reward from the initial defection outweighs the later loss from the lack of collusion. The only deviation from tit-for-tat we need to consider is the best response to the collusive strategy followed by repeating the Cournot equilibrium strategy. 27 / 50

0.1.3 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

We assume that the payoff obtained in round i is discounted by a factor of i1 compared to the payoff obtained in round 1 (as in the second interpretation of the iterated prisoners dilemma game). The method of finding the values of for which collusion is a Nash equilibrium is similar to the method used for the prisoners dilemma. 1. We derive the symmetric action pair (i.e. both firms choose the same production level) that maximises the sum of profits. The corresponding action pair is (C, C). 2. We derive the Cournot equilibrium of the single-shot game and the associated profits. The corresponding action pair is (D, D),

0.1.4 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

3. We derive the best response to C and the payoffs obtained in this case. The corresponding action pair is (C, D). 4. We compare the discounted reward of a defector playing against a firm using S with the discounted reward of a firm playing S against another playing S. 5. Without loss of generality, we may assume that defection occurs in round 1.

0.1.5 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

Consider the repeated Cournot game where the production of firm i in a period is xi . The price in a period is given by p = 3 x1+x2 1000 . The costs incurred by firm i in a period are given by ci(x) = 100 + xi . 30 / 50

0.1.6 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

Note that this is the problem considered in Section 3.6. Player 1s best response to x2 is 1000 x2 2 , when x2 2000. The Cournot equilibrium for this game is for both firm to produce 2000 3 units per period. The profits obtained by both players at the Cournot equilibrium are approximately 344. 31 / 50

0.1.7 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

We need to derive the total production xtot = x1 + x2 which maximises the total reward of the two firms. The total costs incurred by the firms are c1(x1) + c2(x2) = 200 + x1 + x2 = 200 + xtot The total revenue is pxtot = (3 xtot 1000)xtot = 3xtot x 2 tot 1000 . It follows that the total profit is Rtot = pxtot c1(x1) c2(x2) = 2xtot x 2 tot 1000 200. 32 / 50

0.1.8 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

Differentiating with respect to xtot, we obtain Rtot xtot = 2 xtot 500 . It follows that the optimal joint production is 1000, i.e. each firm produces 500 units per period. The total profit is 800, i.e. each firm makes a profit of 400 units at this collusive solution. 33 / 50 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game We now derive the optimal response to this collusive level of production. The analysis leading to the Cournot equilibrium shows that the optimal response of a firm when the other produces 500 units is to produce 1000 500 2 = 750 units. We now calculate the profit obtained by the defector in this case. The price when the strategy pair is (500,750) is given by p = 3 1250 1000 = 1.75. 34 / 50 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game The defector obtains R2(500, 750) = 750 1.75 100 750 = 462.5. We can now find the values of for which collusion is a Nash equilibrium in the repeated game. 35 / 50 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game The stern strategy, S, is to produce 500 units in a period as long as the other firm produces 500 units in a period and then always produce 2000 3 units in a period. A defector, denoted D, produces 750 units in the first period and then produces 2000 3 units in a period. Two firms playing S obtain 400 units per period. Their discounted reward is R1(S, S) = 400 + 400 + 400 2 + . . . = 400 1   36 / 50

0.1.9 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

The payoff of a defector against a stern player is 462.5 in round 1 and 344 in subsequent rounds. Hence, R1(D, S) = 462.5 + 344 + 344 2 + . . . = 462.5 + 344 1 . Tit-for-tat is a Nash equilibrium in the repeated game if 400 1   462.5 + 344 1   0.5274. 37 / 50

0.1.10 Implicit Collusion in the Repeated Version of the Symmetric Cournot Game

The concept of collusion can be applied to asymmetric games. However, it is not so clear how to split the gains from collusion. One possibility is that both firms split the gains from collusion equally between each other. Another possibility is that the ratio between the profits of the firms at a collusive solution is equal to the ratio between the profits of the firms at the equilibrium of the corresponding one-shot Cournot (or Stackelberg) game. These two possibilities are equivalent in the case of symmetric games. 38 / 50

0.1.11 Equilibria in infinitely Repeated Games Without Discounting

In the Hawk-Dove game playing the pair of actions (D, H) and (H, D) in alternate rounds is a Nash equilibrium in any n-repeated Hawk-Dove game, i.e. each pure Nash equilibrium pair is played half the time. Compare this solution to the egalitarian correlated equilibrium toss a coin if the result is heads play (H, D) and (D, H). If the game is repeated infinitely often, any solution where (H, D) is played a proportion p of the time and (D, H) is played a proportion 1 p of the time is a Nash equilibrium. Hence, any expected payoff vector at a correlated equilibrium which is a randomisation over the set of Nash equilibria can be achieved at a Nash equilibrium in an infinitely repeated game (in the sense of mean payoff per round). 39 / 50

0.1.12 Set of Possible Payoff Rates in Infinitely Repeated Games

Over an infinite horizon the mean (undiscounted) reward of the players per period is given by a weighted mean of all the payoff vectors over all the possible action pairs. The weights are the frequencies with which each action pair is played. Hence, the set of attainable mean rewards is the same as the set of attainable expected rewards using a correlated equilibrium. 40 / 50

0.1.13 The Folk Theorem

The Folk Theorem: In an infinitely repeated game, for any mean payoff vector (per round) in which both players obtain at least their corresponding minimax reward there is a Nash equilibrium. There is no equilibrium at which at least one player obtains less than their minimax reward. Such a Nash equilibrium is defined by a pair of strategies that give the required mean payoff vector plus the threat that if the other player defects from this strategy pair, then a player will play so as to minimise the payoff of the defector. 41 / 50

0.1.14 The Folk Theorem

It follows that in such an infinitely repeated game a Nash equilibrium which satisfies any sensible optimality condition must be Pareto optimal. This follows since if a solution is not Pareto optimal, we can always find a Nash equilibrium at which one player obtains a greater mean reward without reducing the mean reward of the other player. 42 / 50 Example Consider the infinitely repeated version of the chicken game A B A (6,6) (2,8) B (8,2) (0,0) 43 / 50 Example First we derive the minimax payoffs in the one-shot game (note that the game is symmetric). Suppose Player 1 takes action A with probability p and Player 2 takes action A with probability q. R1(M1, M2) = 6pq + 2p(1 q) + 8(1 p)q = 2p + 8q 4pq. 44 / 50

0.1.15 Example

We wish to find the strategy of Player 2 which minimises Player 1s expected reward. Hence, we calculate the derivative of Player 1s reward with respect to q (the decision of Player 2). R1(M1, M2) q = 8 4p. This is positive for all p. Hence, to minimise Player 1s payoff, Player 2 should choose q to be as small as possible. It follows that Player 2 minimises Player 1s reward by always taking action B. 45 / 50 Example If Player 2 plays B, then Player 1 should play A. It follows that Player 1s minimax reward is 2. By symmetry, this is the minimax payoff of Player 2. Hence, at any Nash equilibrium of the infinitely repeated game, both players must obtain a payoff of at least 2. 46 / 50 The Set of Attainable Mean Payoff Vectors R2 R1 0 (8,2) (2,8) (6,6) S 47 / 50 Example Suppose we wish to find the Nash equilibrium which a) maximises the mean payoff of Player 1, b) maximises the sum of the mean payoffs. a) The mean payoff vector at such a solution is the Pareto optimal mean payoff vector which gives Player 2 at least 2 and maximises the mean payoff of Player 1. This is the mean payoff vector (8, 2). The players have to play the action pair (B, A) in each round at such an equilibrium. At such an equilibrium the threat, that a player will play B if the other deviates from this action pair, is essentially meaningless. Player 1 cannot gain by deviating and Player 2 deviating would simply lead to Player 1 playing B and so Player 1 should then play A (i.e. they return to the same action pair). 48 / 50 Example b) The mean payoff vector at such a solution is the Pareto optimal mean payoff vector which gives the greatest sum of mean payoffs while ensuring both of the players at least 2. Since the set of Pareto optimal solutions is piecewise linear (it is made up of the lines between (2,8) and (6,6) and between (6,6) and (8,2), the maximum sum must come at one of the endpoints of one of these line segments. The appropriate mean payoff vector is thus (6,6). At such an equilibrium, the players should always play A. If either deviates from this action, then the other should always play B. 49 / 50 Example It should be noted that this solution cannot be attained using a correlated equilibrium. It follows that the ability to react to the strategy of other players in a repeated game is a stronger force than communication without the power of making a contract (i.e. the conditions required for attaining a correlated equilibrium), in the sense that the set of mean payoffs possible at an equilibrium of a repeated game contains the set of expected payoffs using a correlated equilibrium in a one-shot game. 50 / 50