0.0.1 3.6 Refinements of the concept of Nash equilibrium

We consider 3 refinements of the concept of Nash equilibrium, which are useful when there are multiple equilibria.

1. Subgame perfect Nash equilibria.
2. Payoff dominant Nash equilibria.
3. Risk domninant Nash equilibria.

0.0.2 Subgame Perfect Equilibria in the Context of Matrix Games

- We have considered the concept of subgame perfection in extended games.
- In the asymmetric Hawk-Dove game, Player 1 will take action H and Player 2 will take action D under the following strategy pairs (H, [D—H, H—D]) and (H, [D—H, D—D]).
- However, when Player 2 plays [D—H, D—D] he does not use his best response when Player 1 makes a mistake and plays D. Hence, the second pair of actions does not define a subgame perfect Nash equilibrium.
- We will now consider the matrix form of this game (this was derived earlier).

Subgame Perfect Equilibria in the Context of Matrix Games

	$(H\|H, H\|D)$	$(H\|H, D\|D)$	$(D\|H, D\|D)$	$(D\|H, H\|D)$
H	(-2,-2)	(-2,-2)	(4,0)	(4,0)
D	(0,4)	(2,2)	(2,2)	(0,4)

Figure 1:

It can be seen that (H, [D—H, D—D]) is a (weak) Nash equilibrium in this game, since neither player can increase their payoff by unilaterally switching strategy.

- However, the strategy [D—H, H—D] dominates the action [D—H, D—D], since Player 2 always does as well by playing [D—H, H—D] rather than [D—H, D—D] and does better when Player 1 plays D.
- It can be shown that (H, [D—H, H—D]) is the only Nash equilibrium left after the removal of dominated strategies.

0.0.3 Payoff Dominant Nash Equilibria

- A payoff vector (v1, v2) is said to Pareto dominate payoff vector (x1, x2) if v1 x1, v2 x2 and inequality is strict in at least one of the cases.
- That is to say, Nash Equilibrium 1 of a game Pareto dominates Nash Equilibrium 2 if no player prefers Equilibrium 1 to Equilibrium 2 and at least one player prefers Equilibrium 1.
- A Nash equilibrium is payoff dominant if the value of the game corresponding to this equilibrium Pareto dominates all the values of the game corresponding to other equilibria.

0.0.4 Example

Consider the following game A B A (4,4) (0,0) B (0,0) (2,2) Such a game is called a coordination game, as both players would like to take the same action.

0.0.5 Example

There are 3 Nash equilibria 1. (A, A) - Value (4,4). 2. (B, B) - Value (2,2). 3. (1/3A + 2/3B, 1/3A + 2/3B) - Value (4 3 , 4 3). The first equilibrium Pareto dominates the other two equilibria, whilst the second equilibrium Pareto dominates the third. (A, A) is the payoff dominant equilibrium.

0.0.6 Risk Dominance

- Suppose there are pure Nash equilibria (A, C) and (B, D).
- The risk factor associated with strategy A, FA, is the probability with which Player 2 should play C (which is used at the Nash equilibrium where Player 1 plays A), in order to make Player 1 indifferent between playing A or B.
- A high risk factor indicates that Player 1 must be relatively sure that Player 2 will play C for Player 1 to prefer A to B.
- Similarly, the risk factor associated with strategy B is the probability with which Player 2 should play D, in order to make Player 1 indifferent between playing A or B.
- The risk factors associated with C and D can be calculated in a similar way.
- The Nash equilibrium (A, C) risk dominates (B, D) if FA FB , FC FD and there is strictly inequality in at least one case.

0.0.7 Example

Consider the following symmetric game A B A (4,4) (-1000,0) B (0,-1000) (2,2) It is clear that the equilibrium (A, A) payoff dominates the equilibrium (B, B). However, there is a large risk associated with playing action A (the possibility of obtaining a payoff of -1000). From the symmetry of the game, the risk factors of the two strategies are the same for both players. The risk factor associated with A is given by the solution of 4p 1000(1 p) = 0p + 2(1 p) p = 1002 1006 . The risk factor associated with B is given by the solution to 2p + 0(1 p) = 1000p + 4(1 p) p = 4 1006 . It follows that B risk dominates A.

0.0.8 Conclusion

- If an equilibrium both payoff and risk dominates another, it seems clear that this should be the one chosen.
- In other cases, it is not clear what equilibrium should be played.
- The concept of risk domination is important in evolutionary game theory (see later).

0.0.9 3.7 2-Player Games with a Continuum of Strategies and Simultaneous Moves

- Assume that Player i chooses an action from a finite interval Si .
- The payoff to Player i when Player 1 takes action x1 and Player 2 takes action x2 is given by Ri(x1, x2).
- It is assumed that the payoff functions are differentiable.

0.0.10 The Symmetric Cournot Game

- Assume that two firms produce an identical good. Firm i produces xi units per time interval.
- The price of the good is determined by total supply and all production is sold at this clearing price. It is assumed that p = A B[x1 + x2], (A, B ¿ 0).
- The costs of producing x units of the good are assumed to be C + Dx for both firms.
- The payoff of a firm is taken to be the profit obtained (revenue minus costs). Revenue is simply production times price.

0.0.11 The Symmetric Cournot Game

The payoff obtained by Firm 1 is given by

$$R1(x1, x2) = px1CDx1 = (AD)x1Bx21Bx1x2C.$$

By symmetry, the payoff obtained by Player 2 is

$$R2(x1, x2) = px2CDx2 = (AD)x2Bx22Bx1x2C.$$

It should be noted that it clearly does not pay firms to produce more than the amount xmax that guarantees that the price is equal to the unit (marginal) cost of production. Since xmax is finite, we may assume that firms choose their strategy (production level) from a finite interval.

0.0.12 Best Response Functions

Given the output of the opponent, we can calculate the optimal response of a player using calculus. Let B1(x2) denote the best response of Player 1 to x2. Let B2(x1) denote the best response of Player 2 to x1. 47 / 61 Nash Equilibria in Games with a Continuum of Strategies and Simultaneous Actions At a Nash equilibrium (x 1 , x 2), we have x 1 = B1(x 2); x 2 = B2(x 1). Thus, at a Nash equilibrium Player 1 plays her best response to Player 2s strategy and vice versa.

0.0.13 The Cournot Game

Suppose A = 3, B = 1 1000 , C = 100 and D = 1. We have R1(x1, x2) = 2x1 x 2 1 1000 x1x2 1000 100. In order order to derive the best response of Player 1 to Player 2s action, we differentiate Player 1s payoff function with respect to x1, his action. We have

$$R1(x1, x2)x1 = 22x1 + x21000.$$

It should be noted that this derivative is decreasing in x1, hence any stationary point must be a maximum. The optimal response is given by 2 2x1 + x2 1000 = 0 x1 = 1000 x2 2 . Thus B1(x2) = 1000 x2 2 .

0.0.14 The Cournot Game

It should be noted that this solution is valid as long as x2 2000 (production cannot be negative). If x2 ¿ 2000, then R1(x1,x2) x1 is negative for all non-negative values of x1. In this case the best response is not to produce anything. By symmetry the best response of Player 2 is given by B2(x1) = max0, 1000 x1 2 . We look for an equilibrium at which both firms are producing. In this case x 1 = 1000 x 2 2 ; x 2 = 1000 x 1 2 . It follows that at Nash equilibrium both firms must produce 2000 3 units. Intuitively, from the form of the game both firms should produce the same amount. The value of the game can be found by substituting these values into the payoff functions. R1(x 1 , x 2) = R2(x 1 , x 2) = 2 2000 3 2 1000 9 100 344. The equilibrium price is 3 2 2 3 = 5 3 .

0.0.15 The Cournot Game

- There cannot be an equilibrium at which one of the firms does not produce. The argument is as follows.
- If one firm does not produce, then the optimal response to this is to produce 1000 units.
- (0,1000) cannot be a Nash equilibrium, since the best response of Firm 1 to x2 = 1000 is to choose x1 = 500.

0.1 The Stackelberg Model

- This is identical to the Cournot model, except that it is assumed that one of the firms is a market leader and chooses its production level before the second firm chooses.
- The second firm observes the production level of the first.

0.1.1 Games with a Continuum of Strategies and Sequential Moves

- Suppose Player 2 moves after Player 1 and observes the action taken by Player 1. The equilibrium is derived by recursion.
- Player 2 should choose the optimum action given the action of Player 1.
- Hence, we first need to solve R2(x1, x2) x2 = 0.
- This gives the optimal response of Player 2 as a function of the strategy of Player 1, x2 = B2(x1).

0.1.2 Games with a Continuum of Strategies and Sequential Moves

- We now calculate the optimal strategy of the first player to move.
- If Player 1 plays x1, Player 1 responds by playing B2(x1). Hence, we can express the payoff of Player 1 as a function simply of x1, i.e. R1(x1, B2(x1)).
- In order to find the optimal action of Player 1, we differentiate this function with respect to x1.
- Having calculated the optimal value of x1, we can derive x2.

0.1.3 Example

Derive the equilibrium of the Stackelberg version of the previous example. We have R2(x1, x2)=2x2 x 2 2 1000 x1x2 1000 100. R2(x1, x2) x2 =2 x2 500 x1 1000 .

0.1.4 Example

It follows that the best response of Player 2 is given by 2 x2 500 x1 1000 = 0 x2 = 1000 x1 2 . Hence, R1(x1, B2(x1))=R1(x1, 1000 x1 2) =2x1 x 2 1 1000 x1(1000 x1/2) 1000 100 =x1 x 2 1 2000 100.

$$
\begin{aligned}
R_1(x_1, B_2(x_1)) = & R_1\left(x_1, 1000 - \frac{x_1}{2}\right) \\
& = 2x_1 - \frac{x_1^2}{1000} - \frac{x_1(1000 - x_1/2)}{1000} - 100 \\
& = x_1 - \frac{x_1^2}{2000} - 100.
\end{aligned}
$$

Figure 2:

0.1.5 Example

Differentiating

$$R1(x1, B2(x1))x1 = 1x11000.$$

It follows that Firm 1 maximises its profit by producing 1000 units. The best response of Firm 2 is B2(x1) = 1000 x1 2 = 500 units.

- The Stackelberg equilibrium is (1000, 500).
- Hence, the leader produces more than at the Cournot equilibrium and the follower produces less.
- Total production is greater than at the Cournot equilibrium, i.e. the equilibrium price is lower.

0.1.6 Example

The profit of Firm 1 at this equilibrium is

$$R1(1000, 500) = 210001000500100 = 400.$$

The profit of Firm 2 at this equilibrium

$$R2(1000, 500) = 2500250500100 = 150.$$

It is clear that Firm 1 gains by being the leader. Firm 2 loses. The sum of profits is lower than at the Cournot equilibrium. This seems somewhat counter-intuitive, as the market would seem to be more competitive under the Cournot model.