Then, by setting $C_{00} = C_{11} = 0$, $C_{01} = 2$, and $C_{10} = 1$ in Eq. (*8.19*), the minimum Bayes' risk $\bar{C}^*$ can be expressed as a function of $P(H_0)$ as

$$\begin{aligned}\bar{C}^*[P(H_0)] &= P(H_0)\int_{-\delta}^{\delta} \tfrac{1}{2}e^{-|x|}\,dx + 2[1 - P(H_0)]\left[\int_{-\infty}^{-\delta} e^{2x}\,dx + \int_{\delta}^{\infty} e^{-2x}\,dx\right] \\ &= P(H_0)\int_0^{\delta} e^{-x}\,dx + 4[1 - P(H_0)]\int_{\delta}^{\infty} e^{-2x}\,dx \\ &= P(H_0)(1 - e^{-\delta}) + 2[1 - P(H_0)]e^{-2\delta} \end{aligned} \tag{8.35}$$

From the definition of δ [Eq. (*8.34*)], we have

$$e^{\delta} = \frac{4[1 - P(H_0)]}{P(H_0)}$$

Thus

$$e^{-\delta} = \frac{P(H_0)}{4[1 - P(H_0)]} \quad \text{and} \quad e^{-2\delta} = \frac{P^2(H_0)}{16[1 - P(H_0)]^2}$$

Substituting these values into Eq. (*8.35*), we obtain

$$\bar{C}^*[P(H_0)] = \frac{8P(H_0) - 9P^2(H_0)}{8[1 - P(H_0)]}$$

Now the value of $P(H_0)$ which maximizes $\bar{C}^*$ can be obtained by setting $d\bar{C}^*[P(H_0)]/dP(H_0)$ equal to zero and solving for $P(H_0)$. The result yields $P(H_0) = \frac{2}{3}$. Substituting this value into Eq. (*8.34*), we obtain the following minimax test:

$$|x| \underset{H_0}{\overset{H_1}{\lessgtr}} \ln \frac{4(1 - \frac{2}{3})}{\frac{2}{3}} = \ln 2 = 0.69$$

8.13. Suppose that we have n observations X_i, $i = 1, \ldots, n$, of radar signals, and X_i are normal iid r.v.'s under each hypothesis. Under H_0, X_i have mean μ_0 and variance σ^2, while under H_1, X_i have mean μ_1 and variance σ^2, and $\mu_1 > \mu_0$. Determine the maximum likelihood test.

By Eq. (*2.52*) for each X_i, we have

$$f(x_i \mid H_0) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}(x_i - \mu_0)^2\right]$$

$$f(x_i \mid H_1) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}(x_i - \mu_1)^2\right]$$

Since the X_i are independent, we have

$$f(\mathbf{x} \mid H_0) = \prod_{i=1}^{n} f(x_i \mid H_0) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2\right]$$

$$f(\mathbf{x} \mid H_1) = \prod_{i=1}^{n} f(x_i \mid H_1) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_1)^2\right]$$

With $\mu_1 - \mu_0 > 0$, the likelihood ratio is then given by

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} = \exp\left\{\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} 2(\mu_1 - \mu_0)x_i - n(\mu_1{}^2 - \mu_0{}^2)\right]\right\}$$

Hence, the maximum likelihood test is given by

$$\exp\left\{\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} 2(\mu_1 - \mu_0)x_i - n(\mu_1{}^2 - \mu_0{}^2)\right]\right\} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$