Chapter 1

CHAPTER - GAME THEORY

1.1 The Concept of a Game

Game Theory (GT) is the study of strategic interdependence.

A game is defined by

(a) The set of players (at least two "***rational***" players).

(b) The set of actions available to each player (each has at least two possible actions).

(c) A payoff function, which gives the (expected) payoff of each player given the actions used by the players. The (expected) payoff of a player does depend on the actions used by the other players.

The typical "game" consists of players, actions, strategies and payoffs. The standard modes of analysis are once-played games in either

(i) matrix or strategic form where players' choice of actions are made simultaneously,

or

(ii) extensive form where choice of actions are made sequentially.

- According to such a definition, roulette is not a mathematical game, since the expected winnings of a player only depend on how they play.
- The lottery is a mathematical game. Although the probability of winning the jackpot is independent of the choices of other players, a player can maximise his/her expected reward by choosing combinations of numbers that other individuals do not choose (such an individual is not more likely to win the jackpot, but when they win, they win more on average).

1.2 3.2 The Matrix Form of a 2-Player Game

- Assume that each player has a finite set of actions to choose from.
- In the matrix form of a 2-player game, each row corresponds to an action of Player 1 and each column corresponds to an action of Player 2.
- Each cell of the payoff matrix is associated with a payoff vector. The i-th component of this vector gives the payoff to Player i.

For example, the following describes a so called ***Hawk-Dove*** game

		Player 2	
		Hawk	Dove
Player 1	Hawk	(-2,-2)	(4,0)
	Dove	(0,4)	(2,2)

For example, when Player 1 plays H and Player 2 plays D, Player 1 obtains a payoff of 4 and Player 2 obtains a payoff of 0.

In general, a 2-player matrix game can be described by

1. The set of actions available to Player 1,
$$A = \{a_1, a_2, \ldots, a_m\}$$
.
2. The set of actions available to Player 2,
$$B = \{b_1, b_2, \ldots, b_n\}$$
.
3. The m n matrix of 2-dimensional vectors giving the payoffs of both players for each of the m n possible combinations of actions.
4. The payoff of Player k when Player 1 plays ai and Player 2 plays bj will be denoted Rk (ai , bj).

- One advantage of the matrix form of a game is its simplicity.
- One disadvantage is that it is assumed that players choose their actions simultaneously.
- That is to say, the actions may be interpreted as strategies which are chosen at the beginning of a game.
- As we continue through this chapter, we will differentiate between the concept of action and the concept of strategy.

1.3 Zero-sum and Constant-sum Games

- A game is said to be a constant sum game, if the sum of the payoffs obtained by the players is fixed, regardless of the combination of strategies used.
- A game is said to be a zero-sum game, if the sum of payoffs obtained by the players is always 0, regardless of the combination of strategies used.
- A game of constant sum k is essentially the same as a zero-sum game. A referee could pay both of the players k/2 and require that they play the game in which each payoff is reduced by k/2 (this would be a zero-sum game).
- 2-player constant-sum games are games of pure conflict. Whatever one player gains, the other will lose.

1.4 The Extensive Form of a Game

- In many games, e.g. chess, each player makes a sequence of moves. Such a game can be represented by a tree.
- Each node represents a position (state) in which a player must make a move.
- Each edge coming out of a node represents a move that the player can make in that particular state.
- Each terminal point of the tree is associated with a payoff vector.

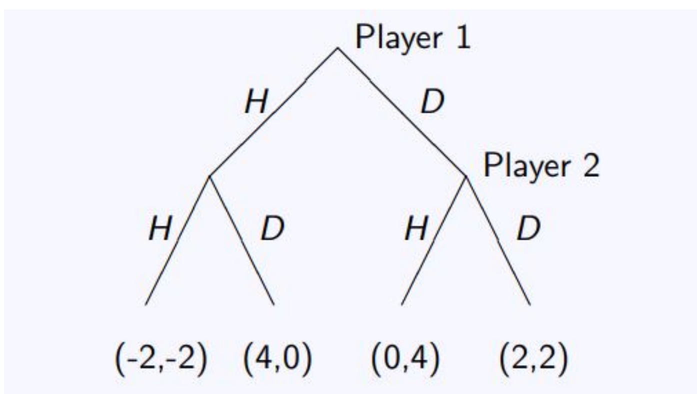

Figure 1.1:

- In such a game, Player 1 first decides whether to play H and D.
- After observing the decision of Player 1, Player 2 then decides whether to play H and D.
- For example, if Player 1 chooses H and Player 2 chooses D, Player 1 obtains a payoff of 4 and Player 2 obtains a payoff of 0.

1.4.1 The Asymmetric Hawk-Dove Game

- It might appear that this game is equivalent to the matrix form of the Hawk-Dove given above.
- In order to see the difference between these games, we should differentiate between strategies and actions.
- In a game represented in extensive form, a strategy is a rule that defines which action a player should play at all the nodes where he/she makes a move.
- An action is the observed behaviour resulting from such a strategy.
- Here Player 1 first chooses between H and D. This is her only choice.
- Thus Player 1s strategies correspond exactly to her possible actions (H and D). However, since Player 2 observes the move made by Player 1, he can condition his move on the move made by Player 1.
- For each of the moves made by Player 1, Player 2 has 2 possible moves. Hence, he has $2 \times 2 = 4$ possible strategies. These are listed on the following slide.

In the extended game described above, Player 2s strategies are

1. $H|H$, $H|D$, i.e. play H when Player 1 plays H and play H when Player 1 plays D, in other words always play H.
2. $H|H$, $D|D$, i.e. take the same action as Player 1.
3. D—H, $D|D$, i.e. always play D.
4. D—H, $H|D$, i.e. play D when Player 1 plays H and play H when Player 1 plays D.

In the matrix game presented earlier, Player 2 cannot condition his action on the action made by Player 1. Hence, he only has 2 strategies corresponding to the two actions H and D.

1.4.2 Simultaneous Moves in Extensive Form Games

- Although the extensive form of the game is designed to describe games in which a sequence of moves are made, they can be adapted to describe games in which moves are made simultaneously.
- This is done using information sets. Suppose two nodes represent states in which one player moves and that player cannot differentiate between the states.
- These two nodes are said to be in the same information set. An information set is represented by a box.
- A player may only condition an action on the information set, he/she is presently in.

In this case both players have 2 possible strategies: H and D.

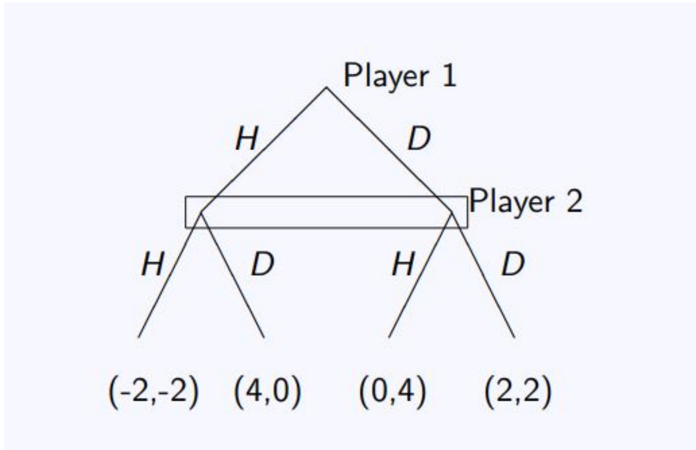

Figure 1.2:

1.4.3 Translating from Extensive Form into Matrix Form

In order to translate the description of a game from extensive form to matrix form, we

1. List the possible strategies of both players.
2. In the matrix describing the game, the rows correspond to the strategies of Player 1 and the columns correspond to the strategies of Player 2.
3. We then derive the payoff vector corresponding to each possible strategy pair.

- It should be noted that each game in extensive form has a unique definition as a matrix game (apart from possible differences in the labelling of the strategies).
- However, there may be various games in extensive form corresponding to a game given in matrix form.
- Thus it is generally not possible to translate a game in matrix form into a game in extensive form.
- This is unsurprising, since the extensive form of a game gives more detailed description of how the game is actually played.
- Since Player 1 has 2 strategies and Player 2 has 4 strategies, the payoff matrix will be of dimension 2×4.
- Player 1 can play H or D.
- Player 2 can play
$$(H|H, H|D), (H|H, D|D), (D|H, D|D) or (D|H, H|D).$$

We now consider the payoff vectors associated with each strategy pair.

Suppose Player 1 plays H.

(a) If Player 2 plays $(H|H, H|D)$ or $(H|H, D|D)$, then both players take the action H and the resulting payoff vector is (2, 2).

(b) If Player 2 plays (D—H, D—D) or $(D|H, H|D)$, then Player 1 takes the action H and Player 2 takes the action D and the resulting payoff vector is (4, 0).

Now suppose Player 1 plays D.

(a) If Player 2 plays $(D|H, D|D)$ or $(H|H, D|D)$, then both players take the actions D and the resulting payoff vector is (2, 2).

(b) If Player 2 plays $(H|H, H|D)$ or $(D|H, H|D)$, then Player 1 takes the action D and Player 2 takes the action H and the resulting payoff vector is (0, 4).

It follows that the matrix form of the asymmetric game is given by

$$(H|H, H|D)(H|H, D|D)(D|H, D|D)(D|H, H|D)$$

H (-2,-2) (-2,-2) (4,0) (4,0) D (0,4) (2,2) (2,2) (0,4)

1.4.4 Moves by Nature

- Extensive form can also be used to describe moves by nature, i.e. random events, the roll of a die, dealing cards.
- Whenever nature is called to make a move at a given node, the edges from this node correspond to the possible results. The probability of each result should also be given.
- When such a game is written in matrix form, we only consider the strategies that the players can use.
- In order to define the vector of expected payoffs given the combination of strategies used, we take expectations with respect to the moves of nature (i.e. nature is assumed to be **random** or **non-rationa**l).

1.4.5 Example

- Suppose that player 1 can first choose either A or B. Player 2 does not know this choice.
- Afterwards Player 2 observes the result of a coin toss. Regardless of the result of the coin toss, Player 2 can play A or B.
- If the coin toss results in heads, when both players choose the same action Player 1 obtains \$1 from Player 2, but gives Player 2 \$1 if each chooses different actions.
- If the coin toss results in tails, when each player chooses different actions Player 1 obtains \$2 from Player 2, otherwise he gives Player 2 \$2 (note this is a zero-sum game).

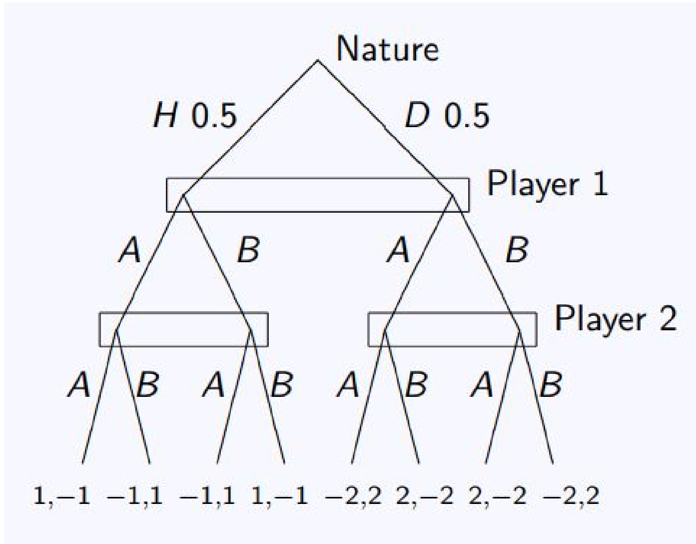

Figure 1.3:

- When writing the game in extensive form, we can often shift a move to an different position in the tree than its natural position, in order to make a more legible tree.
- The important thing here is to correctly describe what information each player has when making a move.
- In this example, Player 1 does not know the result of the coin toss (or Player 2s move).
- Player 2 does not know Player 1s move, but knows the result of the coin toss.
- Hence, Player 2s move must come lower down the tree than the coin toss. We can order the moves as follows: nature, Player 1, Player 2.

- In order to describe the game in matrix form, note that Player 1 has no information regarding the coin toss or Player 2s move. It follows that she has 2 strategies: A and B.
- Player 2 has no information regarding Player 1s move, but knows the result of the coin toss.
- Hence, his action can be made conditional on the result of the coin toss. He has 4 strategies:

$$(A|H, A|T), (A|H, B|T), (B|H, B|T), (B|H, A|T).$$

- Suppose Player 1 plays A.
- When Player 2 plays (A—H, A—T), i.e. always A, Player 1 wins 1 if the result of the coin toss was H and loses 2 if the result of the coin toss was tails. Player 1s expected reward is $0.510.52 = 0.5$.
- Since the game is zero-sum, Player 2s expected payoff is 0.5.

- When Player 2 plays (B—H, B—T), i.e. always B, Player 1 loses 1 if the result of the coin toss was H and wins 2 if the result of the coin toss was tails. Player 1s expected reward is 0.520.51 = 0.5.
- The expected rewards under all the other possible pairs of strategies can be calculated in a similar way.

1.4.6 Example

The matrix form of this game is given by (A—H, A—T) (A—H, B—T) (B—H, B—T) (B—H, A—T) A (-0.5,0.5) (1.5,-1.5) (0.5,-0.5) (-1.5,1.5) B (0.5,-0.5) (-1.5,1.5) (-0.5,0.5) (1.5,-1.5)

1.4.7 Extensive Forms of Games with a Continuum of Strategies

Consider the following game:

- Player 1 chooses a number x between 0 and 1. Having observed the choice of Player 1, Player 2 chooses a number y between 0 and 1.
- The payoff of Player 1 is given by $R1(x, y) = 4xyx$.
- The payoff of Player 2 is given by $R2(x, y) = 4xyy$.
- In order to depict choice from a continuum, instead of using branches we can use a triangle with a horizontal base. The strategy set is described alongside the corresponding triangle.
- If a move is unobserved by the next player to move, the base of the triangle should be enclosed in a box denoting an information set.
- Suppose the second player can observe whether the move of Player 1 belongs to certain intervals. Each of these intervals corresponds to an information set.
- The possible moves of Player 2 corresponding to each information set should be depicted by triangles extending out of these sets.

1.4.8 Example 3.3.1

The extensive form of the game described above is
(4xy x, 4xy y)

1.4.9 The Concepts of Complete Information and Perfect Information

- A game is a game with complete information if both players know both the actions available to the other players and the payoffs obtained by other players under all the possible combinations of strategies.
- In addition, if each player always knows which node of the extensive form of the game he/she is at when he/she has to make a move, then such a game is a game with perfect information.
- For example, chess is a game of perfect information (given a payoff structure of 1 point for a win, 0.5 for a draw, 0 for a loss), since the state of the game is always known, each player knows which moves are available to the opponent.

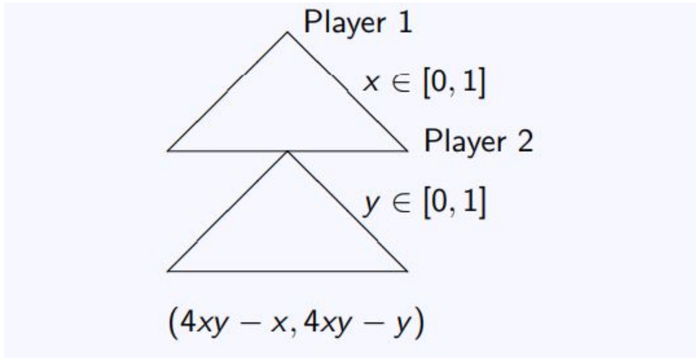

Figure 1.4:

- Bridge is definitely not a game of perfect information, as e.g. the bidder does not know how the cards are split between his opponents.
- However, it may be argued that it is a game of complete information, since the scoring system is well defined and players know what others are allowed to play given their hand and the bidding (*a full description of a strategy would however be very complex*).

1.4.10 Solution of Games with Perfect Information

- In games of perfect information, moves are made in succession and each of the previous moves are known to each player.
- Such games can be solved by recursion based on the extensive form of the game.
- Consider the asymmetric hawk-dove game given above. The final move is made by Player 2. Given the move made by Player 1 (H or D), Player 2 simply maximises his expected reward.
- This defines Player 2s optimal action after H and his optimal action after D.
- Working backwards, Player 1 assumes that Player 2 will use his best response to her action.
- This defines the action that Player 1 should take. It follows that there always exists a solution to a game with perfect information.
- Unless at any node a player is indifferent between the actions he/she can take, this solution will be unique.
- Any vector of payoffs corresponding to the solution of such a game is called a value of the game.

1.4.11 The Asymmetric Hawk-Dove Game

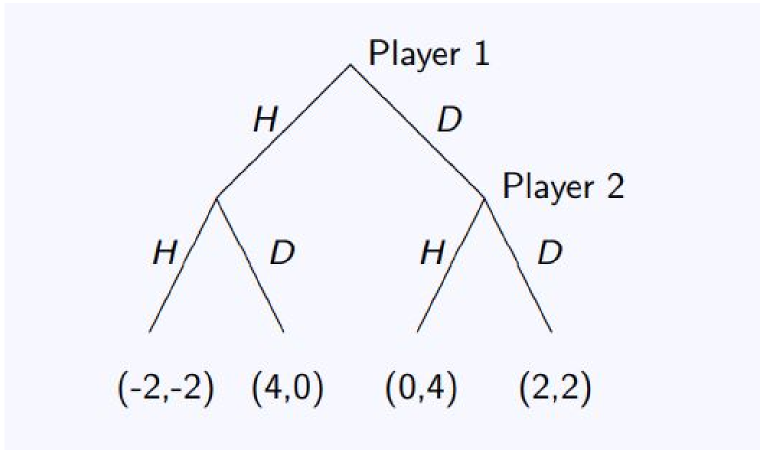

Figure 1.5:

Player 2 (-2,-2) (4,0) (0,4) (2,2)

1.4.12 Solution of Games with Perfect Information

- Suppose Player 1 has played H. Player 2 obtains -2 by playing H and 0 by playing D. Hence, his best response to H is to play D. Suppose Player 1 has played D.
- Player 2 obtains 4 by playing H and 2 by playing D. Hence, his best response to D is H. Now we consider the action of Player 1.
- If she plays H, then Player 2 will respond by playing D. Player 1 obtains a reward of 4.
- If she plays D, then Player 2 will respond by playing H. Player 1 obtains a reward of 0.
- It follows that Player 1 should play H. Player 2 follows by playing D.
- The value of the game is (4, 0).

1.4.13 Equilibrium Path and Subgame Perfect Equilibria

- The set of actions observed at a solution of an extensive form game with perfect information is called an equilibrium path.
- In the Hawk-Dove game considered above, the equilibrium path is (H, D).
- However, such a path does not describe how players should react to mistakes, i.e. how should individuals act off the equilibrium path.

- An equilibrium is said to be subgame perfect, if starting from any node on the game tree, players play an equilibrium pair of strategies, i.e. an equilibrium is played in any subgame of the game in question.
- In this case, the subgame perfect equilibrium is defined by giving the optimal response of Player 2 to each action of Player 1 and the optimal action of Player 1.
- These were derived during the recursive solution of the game. Player 2 should respond to H by playing D and respond to D by playing H. Hence, Player 2s subgame perfect equilibrium strategy is (D—H, H—D).
- The subgame perfect equilibrium strategy of Player 2 is H. If nature has moves in a game of perfect information, then each player is assumed to maximise his/her expected reward at each stage of the recursion procedure.

1.5 Concepts of Pure and Mixed Strategies

In games of perfect information, it is clear that unless a player is indifferent between two actions, then he/she should never randomise. However, in games of imperfect information (e.g. when moves are made simultaneously like rock-scissors-paper) it is clear that one player may not want the other to guess which action he/she is going to take. In such cases individuals choose the action they take at random, i.e. they use a mixed strategy.

The matrix form of the rock-scissors-paper game is:

$$R_1(\pi_1^*, \pi_2^*) > R_1(\pi_1, \pi_2^*), \quad \forall \pi_1 \neq \pi_1^*$$
$$R_2(\pi_1^*, \pi_2^*) > R_2(\pi_1^*, \pi_2) \quad \forall \pi_2 \neq \pi_2^*$$

Figure 1.6:

Intuitively, at equilibrium both players choose each action with a probability of 1/3 (see tutorial sheet for details).

1.5.1 Concepts of Pure and Mixed Strategies

- If a player always chooses the same action in a matrix game, then he/she is using a so called pure strategy.
- It is normally assumed that players choose their actions independently of each other (i.e. there is no communication).
- Later we will consider games in which players can communicate, i.e. the actions they take may be correlated.
- Suppose that in the rock-scissors-paper game, Player 1 plays rock, scissors and paper with probability pR, pS and pP, respectively.
- Player 2 plays rock, scissors and paper with probability qR, qS and qP. The probability distribution over the set of strategy pairs is

When players use mixed strategies, the expected rewards of the players can be calculated by taking expectations with respect to the probability distribution over the set of strategy pairs. Hence,

$R1(M1, M2) = pRqRR1(R, R) + pRqSR1(R, S) + pRqPR1(R, P) + + pSqRR1(S, R) + pSqSR1(S, S) + pSqPR1(S, P) + + pPqRR$

	R	S	P
R	$p_R q_R$	$p_R q_S$	$p_R q_P$
S	$p_S q_R$	$p_S q_S$	$p_S q_P$
P	$p_P q_R$	$p_P q_S$	$p_P q_P$

Figure 1.7:

Chapter 2

Glossary

2.1 Glossary

- **Normal form game** A game in normal form is a function:

  $\pi\ :\prod_{i\in\mathrm{N}}\Sigma^{\ i}\to\mathbb{R}^{\mathrm{N}}\pi\ :\prod_{i\in\mathrm{N}}\Sigma^{\ i}\to\mathbb{R}^{\mathrm{N}}$ Given the tuple of strategies chosen by the players, one is given an allocation of payments (given as real numbers).

  A further generalization can be achieved by splitting the game into a composition of two functions:

  $\pi\ :\prod_{i\in\mathrm{N}}\Sigma^{\ i}\to\Gamma\ \pi\ :\prod_{i\in\mathrm{N}}\Sigma^{\ i}\to\Gamma$ the outcome function of the game (some authors call this function "the game form"), and:

  $\nu\ :\Gamma\ \to\mathbb{R}^{\mathrm{N}}\nu\ :\Gamma\ \to\mathbb{R}^{\mathrm{N}}$ the allocation of payoffs (or preferences) to players, for each outcome of the game.

- **Extensive form game** This is given by a tree, where at each vertex of the tree a different player has the choice of choosing an edge. The outcome set of an extensive form game is usually the set of tree leaves.

- **Cooperative game** A game in which players are allowed to form coalitions (and to enforce coalitionary discipline). A cooperative game is given by stating a value for every coalition:

  $\nu\ :2^{\mathbb{P}(N)}\to\mathbb{R}\nu\ :2^{\mathbb{P}(N)}\to\mathbb{R}$ It is always assumed that the empty coalition gains nil. Solution concepts for cooperative games usually assume that the players are forming the grand coalition NN, whose value $\nu(N)\nu(N)$ is then divided among the players to give an allocation.

- **Simple game** A Simple game is a simplified form of a cooperative game, where the possible gain is assumed to be either '0' or '1'. A simple game is couple (N, W), where W is the list of "winning" coalitions, capable of gaining the loot ('1'), and N is the set of players.

- Finite game is a game with finitely many players, each of which has a finite set of strategies. Grand coalition refers to the coalition containing all players. In cooperative games it is often assumed that the grand coalition forms and the purpose of the game is to find stable imputations. Mixed strategy for player i is a probability distribution P on $\Sigma^{\ i}\Sigma^{\ i}$. It is understood that player i chooses a strategy randomly according to P. Mixed Nash Equilibrium Same as Pure Nash Equilibrium, defined on the space of mixed strategies. Every finite game has Mixed Nash Equilibria.

- Pareto efficiency An outcome a of game form is (strongly) pareto efficient if it is undominated under all preference profiles.

- Preference profile is a function $\nu : \Gamma \to \mathbb{R}^{\mathrm{N}} \nu : \Gamma \to \mathbb{R}^{\mathrm{N}}$. This is the ordinal approach at describing the outcome of the game. The preference describes how 'pleased' the players are with the possible outcomes of the game. See allocation of goods.

- Pure Nash Equilibrium An element $\sigma = (\sigma_i)_{i\in \mathrm{N}} \sigma = (\sigma_i)_{i\in \mathrm{N}}$ of the strategy space of a game is a pure nash equilibrium point if no player i can benefit by deviating from his strategy $\sigma_i \sigma_i$, given that the other players are playing in $\sigma\ \sigma\$.Formally : \forall i \in \mathrm{N} \quad \forall \tau_i \in \Sigma^i \quad \pi(\tau, \sigma_{-i}) \leq \pi(\sigma) \forall i \in \mathrm{N} \quad \forall \tau_i \in \Sigma^i \quad \pi(\tau, \sigma_{-i}) \leq \pi(\sigma)$. No equilibrium point is dominated. Say

- Value A value of a game is a rationally expected outcome. There are more than a few definitions of value, describing different methods of obtaining a solution to the game.

- Veto A veto denotes the ability (or right) of some player to prevent a specific alternative from being the outcome of the game. A player who has that ability is called a veto player. Antonym: Dummy.

- Weakly acceptable game is a game that has pure nash equilibria some of which are pareto efficient.

- Zero sum game is a game in which the allocation is constant over different outcomes. Formally: $\forall \gamma \in \Gamma \displaystyle\sum_{i\in \mathrm{N}} \nu_i(\gamma) = con$ $\Gamma \sum_{i\in \mathrm{N}} \nu_i(\gamma) = const$. w.l.g. we can assume that constant to be zero. In a zero sum game, one player's gain is another player's loss. Most classical board games (e.g. chess, checkers) are zero sum.

Chapter 3

Mark Burke

Game Theory

3.1 Analysis of (finite) Matrix Games

One of the best known games is

Prisoner's Dilemma (PD)

Player 2

Player 1		Keep Quiet	Confess
	Keep Quiet	(-1,-1)	(-12,0)
	Confess	(0,-12)	(-8,-8)

The solution is $\langle$ confess, confess $\rangle$. (See IESDS).
Why don't the players coordinate to get $\langle$ Keep Quiet, Keep Quiet $\rangle$?

The exact payoffs are irrelevant, the game can also be represented by the order of players' preferences - most preferred (p1) to least preferred (p4) :

Player 2

Player 1		Keep Quiet	Confess
	Keep Quiet	(p2,p2)	(p4,p1)
	Confess	(p1,p4)	(p3,p3)

Other examples of PD-like games are

- War with strategies Defend, Attack respectively.
- Arms Race with strategies Pass, Build respectively.
- Free Trade/ Protection with strategies No Tax, Tax respectively.
- Advertising with strategies No Ads, Ads respectively.

Deadlock is another game (success is to fail!):

		Player 2	
		Try	Fail
Player 1	Try	(0,0)	(-1,1)
	Fail	(1,-1)	(0,0)

Using preferences, we can consider the more general version of deadlock

		Player 2	
		Left	Right
Player 1	Up	(p2,p2)	(p1,p4)
	Down	(p4,p1)	(p3,p3)

The solution is $\langle$ Up, Left $\rangle$. (See IESDS). Neither player gets hir first choice unless the other makes a mistake.

3.1.1 Strict Dominance

Strategy X strictly dominates strategy Y for a player if X gives a bigger (more preferred) payoff than Y no matter what the other players do. Players never rationally choose strictly dominated strategies.

Reduce the matrix by **Iterated Elimination of Strictly Dominated Strategies** (IESDS) - see PD and Deadlock above. The order of elimination is irrelevant. Another example is the Dance Club game:

		Boon docks	
		Salsa	Hip Hop
Downtown	Salsa	(80,0)	(60,40)
	Hip Hop	(40,60)	(40,0)

The solution is ⟨ Salsa, Hip Hop ⟩. Salsa dominates Hip Hop for Club Downtown, then Boonies choses Hip Hop.

Some more examples

		Player 2		
		Left	Centre	Right
Player 1	Up	(13,3)	(1,4)	(7, 3)
	Middle	(4,1)	(3,3)	(6,2)
	Down	(-1,9)	(2,8)	(8,-1)

Denoting strict dominance by $>$, then in the order given C $>$ R, M $>$ D, C $>$ L, M $>$ U. Thus the solution is ⟨ Middle, Centre ⟩.

Cournot Duopoly game. Firm 1 can produce i units at a cost of €1 each. Similarly firm 2 can produce j units at a cost of €1 each. The units sell on the market at a price of €$[8-2(i+j)]^+$ each, where $[.]^+$ is the positive part of [.] The payoff to firm 1 is the profit gained which is €$[8-2(i+j)]^+i-i$. Similarly the payoff to firm 2 is €$[8-2(i+j)]^+j-j$. The game matrix is

		Firm 2			
		$j=0$	$j=1$	$j=2$	$j=3$
Firm 1	$i=0$	(0,0)	(0,5)	(0, 6)	(0,1)
	$i=1$	(5,0)	(3,3)	(1,2)	(-1,-3)
	$i=2$	(6,0)	(2,1)	(-2,-2)	(-2,-3)
	$i=3$	(1,0)	(-3,-1)	(-3,-2)	(-3,-3)

The solution is $\langle i=1, j=1\rangle$. What is the sequence of eliminations that leads to this?

All routes lead to the same result (proof?):

		Player 2	
		Left	Right
Player 1	Up	(1,-1)	(4,2)
	Middle	(0,2)	(3,3)
	Down	(-2,-2)	(2,-1)

The solution is ⟨ Up, Right ⟩.

3.1.2 Weak Dominance

Strategy X weakly dominates strategy Y for a player if X gives at least as big a payoff as Y no matter what the other players do and there is one at least one X payoff than is strictly greater than the corresponding Y payoff.

Iterated Elimination of Weakly Dominated Strategies (IEWDS) is in general not a valid technique. Consider the game:

		Player 2	
		Left	Right
	Up	(0,1)	(-4,2)
Player 1	Middle	(0,3)	(3,3)
	Down	(-2,2)	(3,-1)

If we proceed as before and denoting weak dominance by $\geq$, we might argue that M $\geq$ U, L $\geq$ R. The solution is then $\langle$ Middle, Left $\rangle$. Alternatively we might argue that M $\geq$ D, R $\geq$ L which leads to the solution $\langle$ Middle, Right $\rangle$.

3.2 Best Response & Nash Equilibrium

Stag Hunt(SH) - it requires cooperation to catch a stag!

Player 2

Player 1		Stag	Hare
	Stag	$(3^*, 3^*)$	(0,2)
	Hare	(2,0)	$(1^*, 1^*)$

For this game there is no SDS nor WDS.

A **Nash equilibrium** (NE) is a set of strategies, one for each player, from which there is no incentive for any one player to deviate if all the other players play these strategies, i.e. no player can gain by changing, also called a "No regrets" choice. The *Best Response* of a player to another player's choice of strategy is the strategy that gives the largest or best payoff. We'll denote this by placing an * beside the payoff, e.g. in the stag game above , "stag" with associated payoff 3 is the best response of player 1 to player 2 playing "stag". Hence if both parts of a payoff pair have asterisks beside them, this must be a pure strategy *Nash* equilibrium (PSNE) - a pair of strategies where both players are playing deterministic strategies as opposed to a mixed strategy *Nash* equilibrium (MSNE), where players are randomly mixing between the strategies available to them.

Thus in the stag game, the PSNE solutions are $\langle$ stag, stag $\rangle$ and $\langle$ hare, hare $\rangle$. Notice that without efficient coordination, either solution is possible.

Consider the "Good buddies prisoner's dilemma" game:

Player 2

Player 1		Keep Quiet	Confess
	Keep Quiet	(p1,p1)	(p4,p2)
	Confess	(p2,p4)	(p3,p3)

This is identical to the stag hunt game.

An alternative definition of a NE is "mutual best response". Some further examples:

Traffic Lights(TL)

Car 2

Car 1		Go	Stop
	Go	(-5,-5)	(1,0)
	Stop	(0,1)	(-1,-1)

The PSNE solutions are $\langle$ go, stop $\rangle$ and $\langle$ stop, go $\rangle$.

Generals, Armies & Battles. Each general commands 3 armies. No battle occurs if either general puts 0 armies in the field. Otherwise the general with more armies wins the day. With i and$j$ standing for the number of armies of General 1 and General 2 respectively in the battlefield, one possible game matrix is

		General 2			
		$j=0$	$j=1$	$j=2$	$j=3$
General 1	$i=0$	(0,0)	(0,0)	(0, 0)	(0,0)
	$i=1$	(0,0)	(0,0)	(-1,1)	(-1,1)
	$i=2$	(0,0)	(1,-1)	(0,0)	(-1,1)
	$i=3$	(0,0)	(1,-1)	(1,-1)	(0,0)

What are the PSNE(s) ?

3.2.1 Dominance & NE

If IESDS results in a unique solution then it is a (unique) NE. [proof by appeal to "no regrets"]. IESDS does not remove any NE.

IEWDS may lose NE. It is necessary to check using e.g. Best Responses. If you have a choice eliminate SDS before WDS. Examples:

- The example of Section 3.1.2 has two NE, each obtained by a different sequence of IEWDS.
- Consider the game

		Player 2	
		Left	Right
Player 1	Up	(2,3)	(4,3)
	Down	(3,3)	(1,1)

  Using IEWDS, L $\geq$ R, then D $>$ U. Hence the solution is $\langle$ Down, Left $\rangle$. But using Best Responses, another NE is $\langle$ Up, Right $\rangle$.

- The game

		Player 2		
		Left	Centre	Right
Player 1	Up	(2,2)	(4,2)	(4, 3)
	Middle	(2,4)	(5,5)	(7,3)
	Down	(3,4)	(3,7)	(6,6)

  Using IEWDS C $\geq$ L, then M $>$ U, M $>$ D and C$>$ R. Hence the solution is $\langle$ Middle, Centre $\rangle$. This is the only NE.

3.3 MB2-MSNE

Matching Pennies(MP) is an example of a game with no PSNE.

		Player 2	
		Heads	Tails
Player 1	Heads	(1,-1)	(-1,1)
	Tails	(-1,1)	(1,-1)

Other names for this game are

- Goalkeeeper v. Penalty Taker
- Offense v. Defense (American Football)
- Fastball v. Curveball (Baseball)
- Attack A or B v. Defend A or B

Nash proved that every finite [1] game has a NE.
This game has the MSNE $\langle$ 1/2 Heads + 1/2 Tails, 1/2 Heads + 1/2 Tails $\rangle$.

More generally, MSNEs can be found using the *Bishop-Cannings* theorem. Consider the following game:

		Player 2	
		Left	Right
Player 1	Up	(3,-3)	(-2,2)
	Down	(-1,1)	(0,0)

Again note that there is no PSNE. We'll use the notation $R_1\langle s_1, s_2\rangle$ and $R_2\langle s_1, s_2\rangle$ to stand for the payoffs to Player 1 and 2 respectively when Player 1 plays strategy s_1 and Player 2 strategy s_2.
In the mixed strategy game let p be the probability that Player 1 plays Up, and similarly q the probability that Player 2 plays Left. Then the payoff to player 2 by playing Left against Player 1's mixed strategy is

$$R_2\langle p\text{Up} + (1-p)\text{Down}, \text{Left}\rangle = -3p + 1(1-p)$$

Again the payoff to player 2 by playing Right against Player 1's mixed strategy is

$$R_2\langle p\text{Up} + (1-p)\text{Down}, \text{Right}\rangle = 2p + 0(1-p)$$

If the two payoff values are different then Player 2 has a definite preference between the two and so would not want to randomise. If for instance Player 2 were to choose Left, then Player 1 would abandon his mixed strategy and choose Up instead. Similarly if Player 2 were to choose Right instead, Player 1 would change from randomising to playing Down. In either case, the mixed strategies go out the window and no NE exists. To get a MSNE requires that Player 2 be indifferent between the two payoffs i.e. requires that the two payoffs be

[1] finite number of players, finite number of pure strategies

the same and since the the payoff at any mixed strategy is a linear combination of the pure strategy payoffs, this must also have the same value. Equating the two payoffs gives

$$\begin{aligned} -3p + 1(1-p) &= 2p + 0(1-p) \\ \Rightarrow p &= 1/6 \end{aligned}$$

and the value of the payoff is

$$v_2 = R_2 \left\langle \frac{1}{6}\text{Up} + \frac{5}{6}\text{Down}, \text{Left or Right} \right\rangle = \frac{1}{3}$$

Similarly the payoffs to Player 1 by playing Up (respectively Down) against Player 2's mixed strategy are

$$R_1\langle \text{Up}, q\text{Left} + (1-q)\text{Right}\rangle = 3q + (-2)(1-q)$$

$$R_1\langle \text{Down}, q\text{Left} + (1-q)\text{Right}\rangle = (-1)q + 0(1-q)$$

respectively. To be indifferent between the two requires

$$\begin{aligned} 3q + (-2)(1-q) &= (-1)q + 0(1-q) \\ \Rightarrow q &= 1/3 \end{aligned}$$

and the value of the payoff is

$$v_1 = R_1 \left\langle \text{Up or Down}, \frac{1}{3}\text{Left} + \frac{2}{3}\text{Right} \right\rangle = -\frac{1}{3}$$

Exercise: Show that the stag hunt game has a MSNE at ⟨ 1/2 stag + 1/2 hare, 1/2 stag + 1/2 hare ⟩.

We can have partial MSNE where (at least) one player has a pure strategy and (at least) one player has a mixed strategy.

Examples of games with MSNE:

- Chicken (aka Snowdrift) :

		Player 2	
		Continue / Stay in car	Swerve / Shovel
Player 1	Continue / Stay in car	(-10,-10)	(2,-2)
	Swerve / Shovel	(-2,2)	(0,0)

  PSNEs occur at ⟨ Continue, Swerve ⟩ and at ⟨ Swerve, Continue ⟩. There is also a MSNE at ⟨ 1/5 continue + 4/5 swerve, 1/5 continue + 4/5 swerve ⟩. Show that $v_1 = v_2 = -2/5$.

- Battle of the Sexes:

		Her	
		Ballet	Fight
Him	Ballet	(1,2)	(-1,1)
	Fight	(-1,1)	(2,1)

PSNEs occur at $\langle$ Ballet, Ballet $\rangle$ and at $\langle$ Fight, Fight $\rangle$. Show that there is a MSNE at $\langle$ 1/3 ballet + 2/3 fight, 2/3 ballet + 1/3 fight $\rangle$ with $v_1 = 2/3 = v_2$. Compare and contast this with the payoffs of the PSNEs.

3.4 MB1-MSNE

Matching Pennies(MP) is an example of a game with no PSNE.

		Player 2	
		Heads	Tails
Player 1	Heads	(1,-1)	(-1,1)
	Tails	(-1,1)	(1,-1)

Other names for this game are

- Goalkeeeper v. Penalty Taker
- Offense v. Defense (American Football)
- Fastball v. Curveball (Baseball)
- Attack A or B v. Defend A or B

Nash proved that every finite [2] game has a NE.
This game has the MSNE ⟨ 1/2 Heads + 1/2 Tails, 1/2 Heads + 1/2 Tails ⟩.

More generally, MSNEs can be found using the *Bishop-Cannings* theorem. Consider the following game:

		Player 2	
		Left	Right
Player 1	Up	(3,-3)	(-2,2)
	Down	(-1,1)	(0,0)

Again note that there is no PSNE. We'll use the notation $R_1\langle s_1, s_2\rangle$ and $R_2\langle s_1, s_2\rangle$ to stand for the payoffs to Player 1 and 2 respectively when Player 1 plays strategy s_1 and Player 2 strategy s_2.
In the mixed strategy game let p be the probability that Player 1 plays Up, and similarly q the probability that Player 2 plays Left. Then the payoff to player 2 by playing Left against Player 1's mixed strategy is

$$R_2\langle p\text{Up} + (1-p)\text{Down}, \text{Left}\rangle = -3p + 1(1-p)$$

Again the payoff to player 2 by playing Right against Player 1's mixed strategy is

$$R_2\langle p\text{Up} + (1-p)\text{Down}, \text{Right}\rangle = 2p + 0(1-p)$$

If the two payoff values are different then Player 2 has a definite preference between the two and so would not want to randomise. If for instance Player 2 were to choose Left, then Player 1 would abandon his mixed strategy and choose Up instead. Similarly if Player 2 were to choose Right instead, Player 1 would change from randomising to playing Down. In either case, the mixed strategies go out the window and no NE exists. To get a MSNE requires that Player 2 be indifferent between the two payoffs i.e. requires that the two payoffs be

[2] finite number of players, finite number of pure strategies

the same and since the the payoff at any mixed strategy is a linear combination of the pure strategy payoffs, this must also have the same value. Equating the two payoffs gives

$$\begin{aligned} -3p + 1(1-p) &= 2p + 0(1-p) \\ \Rightarrow p &= 1/6 \end{aligned}$$

and the value of the payoff is

$$v_2 = R_2 \left\langle \frac{1}{6}\text{Up} + \frac{5}{6}\text{Down}, \text{Left or Right} \right\rangle = \frac{1}{3}$$

Similarly the payoffs to Player 1 by playing Up (respectively Down) against Player 2's mixed strategy are

$$R_1\langle \text{Up}, q\text{Left} + (1-q)\text{Right}\rangle = 3q + (-2)(1-q)$$

$$R_1\langle \text{Down}, q\text{Left} + (1-q)\text{Right}\rangle = (-1)q + 0(1-q)$$

respectively. To be indifferent between the two requires

$$\begin{aligned} 3q + (-2)(1-q) &= (-1)q + 0(1-q) \\ \Rightarrow q &= 1/3 \end{aligned}$$

and the value of the payoff is

$$v_1 = R_1 \left\langle \text{Up or Down}, \frac{1}{3}\text{Left} + \frac{2}{3}\text{Right} \right\rangle = -\frac{1}{3}$$

Exercise: Show that the stag hunt game has a MSNE at ⟨ 1/2 stag + 1/2 hare, 1/2 stag + 1/2 hare ⟩.

We can have partial MSNE where (at least) one player has a pure strategy and (at least) one player has a mixed strategy.

Examples of games with MSNE:

- Chicken (aka Snowdrift) :

		Player 2	
		Continue / Stay in car	Swerve / Shovel
Player 1	Continue / Stay in car	(-10,-10)	(2,-2)
	Swerve / Shovel	(-2,2)	(0,0)

  PSNEs occur at ⟨ Continue, Swerve ⟩ and at ⟨ Swerve, Continue ⟩. There is also a MSNE at ⟨ 1/5 continue + 4/5 swerve, 1/5 continue + 4/5 swerve ⟩. Show that $v_1 = v_2 = -2/5$.

- Battle of the Sexes:

		Her	
		Ballet	Fight
Him	Ballet	(1,2)	(-1,1)
	Fight	(-1,1)	(2,1)

PSNEs occur at $\langle$ Ballet, Ballet $\rangle$ and at $\langle$ Fight, Fight $\rangle$. Show that there is a MSNE at $\langle$ 1/3 ballet + 2/3 fight, 2/3 ballet + 1/3 fight $\rangle$ with $v_1 = 2/3 = v_2$. Compare and contast this with the payoffs of the PSNEs.

3.4.1 MSNE and Dominance

A SDS cannot be played with positive probability in a MSNE (otherwise a higher payoff can be obtained by not playing the SDS when the strategy says to play it).

3.4.2 Strict Dominance in Mixed Strategies

Consider the game:

		Player 2	
		Left	Right
Player 1	Up	(3,-1)	(-1,1)
	Middle	(0,0)	(0,0)
	Down	(-1,2)	(2,-1)

This game has no SDS dominated by a pure strategy nor any PSNE. If a mixture of two pure strategies dominates another, then that strategy is a SDS.
In the above game 1/2 Up + 1/2 Down > Middle. Remove Middle to get

		Player 2	
		Left	Right
Player 1	Up	(3,-1)	(-1,1)
	Down	(-1,2)	(2,-1)

Show that a MSNE exists at ⟨ (3/5)Up + (2/5)Down, (3/7)Left + (4/7)Right ⟩.

Another example:

		Player 2		
		Left	Centre	Right
Player 1	Up	(-3,6)	(9,1)	(0, 2)
	Middle	(3,-4)	(2,4)	(4,1)
	Down	(4,7)	(3,2)	(-3,2)

Using IESDS, we get (1/4)L + (3/4)C > R, D > M, L > C, D > U. Thus the solution is ⟨ Down, Left ⟩.

3.4.3 Atypical Matrix Games

Almost all matrix games have an odd number of solutions (Wilson 1971). Examples of non generic games follow. Weak dominance is usually the culprit.

- Player 2

Player 1		Left	Right
	Up	(1,1)	(0,0)
	Down	(0,0)	(0,0)

  There are two PSNEs.

- Player 2

Player 1		Left	Right
	Up	(2,2)	(9,0)
	Down	(2,3)	(5,-1)

  Using IESDS, L > R. Player 1 can choose Up or Down as pure strategies or any mixture of the two. All strategies yield a payoff of 2: an infinite number of strategies.
  If tempted to use IEWDS, U $\geq$ D, leading to the solution $\langle$ Up, Right $\rangle$. Caveat emptor!

- Example 2 of Section 3.2.1 had two PSNEs. It also has an infinite number of MSNEs at $\langle$ Up, qLeft + $(1-q)$Right $\rangle$ for $q \leq 3/4$.

- *Selten*'s Horse:

  Player 2

Player 1		Left	Right
	Up	(3,1)	(0,0)
	Down	(2,2)	(2,2)

  There are two PSNEs at $\langle$ Up,Left $\rangle$ and $\langle$ Down,Right $\rangle$ and an infinite number of MSNEs at $\langle$ Down, qLeft + $(1-q)$Right $\rangle$ for $q \leq 2/3$.

- Take or Share - TV game show:

  Player 2

Player 1		Share	Take
	Share	(4,4)	(0,8)
	Take	(8,0)	(0,0)

  There are three PSNEs at all but $\langle$ Share, Share $\rangle$. If one player chooses Take, then the other is indifferent between Share and Take which leads to an infinity of MSNEs.

3.4.4 Atypical Matrix Games

Almost all matrix games have an odd number of solutions (Wilson 1971). Examples of non generic games follow. Weak dominance is usually the culprit.

- Player 2 (columns), Player 1 (rows):

	Left	Right
Up	(1,1)	(0,0)
Down	(0,0)	(0,0)

  There are two PSNEs.

- Player 2 (columns), Player 1 (rows):

	Left	Right
Up	(2,2)	(9,0)
Down	(2,3)	(5,-1)

  Using IESDS, L $>$ R. Player 1 can choose Up or Down as pure strategies or any mixture of the two. All strategies yield a payoff of 2: an infinite number of strategies.
  If tempted to use IEWDS, U $\geq$ D, leading to the solution $\langle$ Up, Right $\rangle$. Caveat emptor!

- Example 2 of Section 3.2.1 had two PSNEs. It also has an infinite number of MSNEs at $\langle$ Up, qLeft $+ (1-q)$Right $\rangle$ for $q \leq 3/4$.

- *Selten*'s Horse:

  Player 2 (columns), Player 1 (rows):

	Left	Right
Up	(3,1)	(0,0)
Down	(2,2)	(2,2)

  There are two PSNEs at $\langle$ Up,Left $\rangle$ and $\langle$ Down,Right $\rangle$ and an infinite number of MSNEs at $\langle$ Down, qLeft $+ (1-q)$Right $\rangle$ for $q \leq 2/3$.

- Take or Share - TV game show:

  Player 2 (columns), Player 1 (rows):

	Share	Take
Share	(4,4)	(0,8)
Take	(8,0)	(0,0)

  There are three PSNEs at all but $\langle$ Share, Share $\rangle$. If one player chooses Take, then the other is indifferent between Share and Take which leads to an infinity of MSNEs.

3.5 Minimax Game Solution

An alternative to “solving ” matrix games using the concept of *Nash* Equilibrium is the Minimax approach. It was originally devised for 2-player “Zero Sum” or “Constant Sum” games whereby what one player gains the other player loses. Each player attempts to maximise hir payoff assuming that hir opponent is attempting to minimise it.

Consider the constant sum game