1 Outcomes

- An outcome is a situation which results from a combination of player's strategies. Every combination of strategies (one for each player) is an outcome of the game. A primary purpose of game theory is to determine which outcomes are stable according to a solution concept (e.g. Nash equilibria).
- In a game where chance or a random event is involved, the outcome is not known from only the set of strategies, but is only realized when the random event(s) are realized.
- A set of payoffs can be considered a set of N-tuples, where N is the number of players in the game, and the cardinality of the set is equal to the total number of possible outcomes when the strategies of the players are varied. The payoff set can thus be partially ordered, where the partial ordering comes from the value of each entry in the tuple.

2 Strictly Determined Games

A strictly determined game is a two-player zero-sum game that has at least one **Nash equilibrium** with both players using pure strategies. The value of a strictly determined game is equal to the value of the equilibrium outcome.

If there is an entry in the payoff matrix that is simultaneously the smallest entry in its row and the largest entry in its column, we call that point a saddle point. If a payoff matrix has a saddle point, we say that the game is ***strictly determined***. The saddle point is also called the value of the game.

- If the value of the game is positive, then the game favors the row player.
- If the value of the game is negative, then the game favors the column player.
- If the value of the game is zero, then the game is fair.

3 Dominance

In game theory, dominance occurs when one strategy is better than another strategy for one player, no matter how that player's opponents may play. Many simple games can be solved using dominance. The opposite, ***intransitivity***, occurs in games where one strategy may be better or worse than another strategy for one player, depending on how the player's opponents may play.

3.1 Terminology

When a player tries to choose the "best" strategy among a multitude of options, that player may compare two strategies A and B to see which one is better. The result of the comparison is one of:

- B dominates A: choosing B always gives as good as or a better outcome than choosing A. There are 2 possibilities:
  1. B strictly dominates A: choosing B always gives a better outcome than choosing A, no matter what the other player(s) do.
  2. B weakly dominates A: There is at least one set of opponents' action for which B is superior, and all other sets of opponents' actions give B the same payoff as A.
- B and A are intransitive: B neither dominates, nor is dominated by, A. Choosing A is better in some cases, while choosing B is better in other cases, depending on exactly how the opponent chooses to play.
  For example, B is "throw rock" while A is "throw scissors" in Rock, Paper, Scissors.
- B is dominated by A: choosing B never gives a better outcome than choosing A, no matter what the other player(s) do. There are 2 possibilities:
  1. B is weakly dominated by A: There is at least one set of opponents' actions for which B gives a worse outcome than A, while all other sets of opponents' actions give A the same payoff as B. (Strategy A weakly dominates B).
  2. B is strictly dominated by A: choosing B always gives a worse outcome than choosing A, no matter what the other player(s) do. (Strategy A strictly dominates B).

This notion can be generalized beyond the comparison of two strategies.

- Strategy B is strictly dominant if strategy B strictly dominates every other possible strategy.
- Strategy B is weakly dominant if strategy B dominates all other strategies, but some (or all) strategies are only weakly dominated by B.
- Strategy B is strictly dominated if some other strategy exists that strictly dominates B.
- Strategy B is weakly dominated if some other strategy exists that weakly dominates B.

4 Nash Equilibrium

Nash equilibrium is a solution concept of a non-cooperative game involving two or more players in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. If each player has chosen a strategy and no player can benefit by changing strategies while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitutes a Nash equilibrium.

- Nash Equilibrium recommends a strategy to each player that the player cannot improve upon unilaterally, as long as the other players follow the recommendation.
- Since the other players are assumed to be rational, it is reasonable to expect the opponents to follow the recommendation as well.

4.1 Informal definition

- Informally, a strategy profile is a Nash equilibrium if no player can do better by unilaterally changing their strategy. To see what this means, imagine that each player is told the strategies of the others. Suppose then that each player asks themselves: *"Knowing the strategies of the other players, and treating the strategies of the other players as set in stone, can I benefit by changing my strategy?"*
- If any player could answer "Yes", then that set of strategies is not a Nash equilibrium. But if every player prefers not to switch (or is indifferent between switching and not) then the strategy profile is a Nash equilibrium. Thus, each strategy in a Nash equilibrium is a best response to all other strategies in that equilibrium.
- Nash Equilibrium can also be thought of as "no regrets," in the sense that once a decision is made, the player will have no regrets concerning decisions considering the consequences.
- The Nash Equilibrium is reached over time, in most cases. However, once the Nash Equilibrium is reached, it will becoming self-enforcing.

5 Finding a Nash Equilibria

Consider the following matrix game.

		Player 2	
		Left	Right
Player 1	Up	(1,3)	(4,2)
	Down	(3,2)	(3,1)

5.1 Step One: Determine player one's best response to player two's actions.

When examining the choices that may maximize a player's payout, we must look at how player one (i.e. you) should respond to each of the options player two has. An easy way to do this visually is to cover up the choices of player two. Consider the matrix portrayed above as we apply this method.

		Player 2	
		Left	Right
Player 1	Up	(1,...)	(4,...)
	Down	(3,...)	(3,...)

- Player one has two possible choices to play: "up" or "down." Player two also has two choices to play: "left" or "right." In this step of determining Nash Equilibrium, we look at responses to player two's actions.
- If player two chooses to play "left," we can play "up" with the payoff of one, or play "down" with the payoff of three. Since three is greater than one, we will bold the 3 indicating the option to play "down" here.
- If player two chooses to play "right," we can either choose to play 'up' for a payoff of four or play "down" for a playoff of three. Since four is greater than three, we bold the four to indicate the option to play "up" here. The bold outcomes are shown below on the full matrix.

		Player 2	
		Left	Right
Player 1	Up	(1, 3)	(**4**, 2)
	Down	(**3**, 2)	(3, 1)

5.2 Step Two: Determine Player Two's best response to Player one's actions.

As we did before with the player two payoffs for player one, we will hide the payoffs of player one when determining the best responses for player two.

		Player 2	
		Left	Right
Player 1	Up	(..., 3)	(..., 2)
	Down	(..., 2)	(..., 1)

- Just as when looking at player one, each player has two choices to play. If player one chooses to play "up," Player 2 can play "left," with a payoff of three, or "right," with a payoff of two.
- Since three is greater than two, we bold the three to show the option to play "left" here.
- If player one chooses to play "down," player 2 can play "left," for a payoff of two, or "right," for a payoff of one.
- Since two is greater than one, we bold the two indicating the option to play "left" here.
- The bold outcomes are shown below on the full matrix.

		Player 2	
		Left	Right
Player 1	Up	(1, **3**)	(4, 2)
	Down	(3, **2**)	(3, 1)

5.3 Step Three: Determine which outcomes have both payoffs bold.

That particular outcome is the Nash Equilibrium.Now, we combine the bold options for both players onto the full matrix.

		Player 2	
		Up	Down
Player 1	Up	(1,3)	(4,2)
	Down	(**3,2**)	(3,1)

Look for intersections where both payoffs are bold. In this case, we find the intersection of (Down , Left) with the payoff of (3, 2) fits our criteria. This indicates our Nash Equilibrium.

5.4 Airline Example

Below is an example, similar to the game above, of how airline pricing may play out. The payouts are in thousands of dollars. Remember, these are the payouts, not the prices. The method we applied previously is already applied to show where the Nash Equilibrium appears.

		Airline 2	
		Low Price	High Price
Player 1	Low Price	(3,000, 3,000)	(4,000, 2,000)
	High Price	(2,000, 4,000)	(3,500, 3,500)

- Looking at just A1's choices we can see that if A2 chooses to play low price, we choose between Low Price for 3,000 or high price for 2,000. We choose "low," since $3,000 > 2,000$.
- We do the same thing for A2 playing High Price and see that we play "low" because $4,000 > 3,500$.
- Conversely, looking just at A2's choices, we can see that if A1 chooses to play low price, we choose between "low price" for 3,000 and "high price" for 2,000.
- Since $3,000 > 2,000$, we choose the "low price" option here.
- If A1 plays high price, we can charge a low price for 4,000 or high price for 3,500. Since $4,000 > 3,500$, we choose to play *"low price"* here.

		Airline 2	
		Low Price	High Price
Player 1	Low Price	**(3,000, 3,000)**	(4,000, 2,000)
	High Price	(2,000, 4,000)	(3,500, 3,500)

- The Nash Equilibrium is that both airlines will charge a low price (shown when choices for each party are highlighted). If both airlines charged a high price, they would each be better off than they are at the Nash Equilibrium.
- So why don't they agree to do this? *(In reality, it's illegal to collude.)* If this were to occur, a unilateral action on behalf of one airline to charge a low price would be beneficial, resulting in that airline making more money in turn.
- This logic also shows how the Nash Equilibrium is reached, and why it is not beneficial to deviate from it once it is reached.

6 Finding Nash Equilibria - Another Case

Consider, for example, the game

		Player 2	
		Left	Right
Player 1	Top	(2,2)	(0,3)
	Bottom	(3,0)	(1,1)

There are four action profiles $((T,L),(T,R),(B,L),and(B,R))$; we can examine each in turn to check whether it is a Nash equilibrium.

- **(T,L)** By choosing B rather than T, player 1 obtains a payoff of 3 rather than 2, given player 2's action. Player 2 also can increase their payoff (from 2 to 3) by choosing R rather than L. Thus (T,L) is not a Nash equilibrium.
- **(T,R)** By choosing B rather than T, player 1 obtains a payoff of 1 rather than 0, given player 2's action. Thus (T,R) is not a Nash equilibrium.
- **(B,L)** By choosing R rather than L, player 2 obtains a payoff of 1 rather than 0, given player 1's action. Thus (B,L) is not a Nash equilibrium.
- **(B,R)** Neither player can increase their payoff by choosing an action different from their current one. Thus this action profile is a Nash equilibrium.

We conclude that the game has a unique Nash equilibrium, (B,R). Notice that in this equilibrium both players are worse off than they are in the action profile (T,L). Thus they would like to achieve (T,L); but their individual incentives point them to (B,R).

- A Nash equilibrium of a strategic game is an action profile (list of actions, one for each player) with the property that no player can increase their payoff by choosing a different action, given the other players' actions.
- Note that nothing in the definition suggests that a strategic game necessarily has a Nash equilibrium, or that if it does, it has a single Nash equilibrium.
- A strategic game may have no Nash equilibrium, may have a single Nash equilibrium, or may have many Nash equilibria.