By Eq. (8.19), the Bayes' risk is

$$\bar{C} = C_{00} P(D_0 \mid H_0)P(H_0) + C_{10} P(D_1 \mid H_0)P(H_0) + C_{01}P(D_0 \mid H_1)P(H_1) + C_{11}P(D_1 \mid H_1)P(H_1)$$

Now we can express

$$P(D_i \mid H_j) = \int_{R_i} f(\mathbf{x} \mid H_j)\, d\mathbf{x} \qquad i = 0, 1; j = 0, 1 \tag{8.31}$$

Then $\bar{C}$ can be expressed as

$$\begin{aligned}\bar{C} = C_{00} P(H_0) \int_{R_0} f(\mathbf{x} \mid H_0)\, d\mathbf{x} + C_{10} P(H_0) \int_{R_1} f(\mathbf{x} \mid H_0)\, d\mathbf{x} \\ + C_{01}P(H_1) \int_{R_0} f(\mathbf{x} \mid H_1)\, d\mathbf{x} + C_{11}P(H_1) \int_{R_1} f(\mathbf{x} \mid H_1)\, d\mathbf{x}\end{aligned} \tag{8.32}$$

Since $R_0 \cup R_1 = S$ and $R_0 \cap R_1 = \phi$, we can write

$$\int_{R_0} f(\mathbf{x} \mid H_j)\, d\mathbf{x} = \int_{S} f(\mathbf{x} \mid H_j)\, d\mathbf{x} - \int_{R_1} f(\mathbf{x} \mid H_j)\, d\mathbf{x} = 1 - \int_{R_1} f(\mathbf{x} \mid H_j)\, d\mathbf{x}$$

Then Eq. (8.32) becomes

$$\bar{C} = C_{00} P(H_0) + C_{01}P(H_1) + \int_{R_1} \{[(C_{10} - C_{00})P(H_0)f(\mathbf{x} \mid H_0)] - [(C_{01} - C_{11})P(H_1)f(\mathbf{x} \mid H_1)]\}\, d\mathbf{x}$$

The only variable in the above expression is the critical region R_1. By the assumptions [Eq. (8.20)] $C_{10} > C_{00}$ and $C_{01} > C_{11}$, the two terms inside the brackets in the integral are both positive. Thus, $\bar{C}$ is minimized if R_1 is chosen such that

$$(C_{01} - C_{11})P(H_1)f(\mathbf{x} \mid H_1) > (C_{10} - C_{00})P(H_0)f(\mathbf{x} \mid H_0)$$

for all $\mathbf{x} \in R_1$. That is, we decide to accept H_1 if

$$(C_{01} - C_{11})P(H_1)f(\mathbf{x} \mid H_1) > (C_{10} - C_{00})P(H_0)f(\mathbf{x} \mid H_0)$$

In terms of the likelihood ratio, we obtain

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{10} - C_{00})P(H_0)}{(C_{01} - C_{11})P(H_1)}$$

which is Eq. (8.21).

8.11. Consider a binary decision problem with the following conditional pdf's:

$$f(x \mid H_0) = \tfrac{1}{2}e^{-|x|}$$
$$f(x \mid H_1) = e^{-2|x|}$$

The Bayes' costs are given by

$$C_{00} = C_{11} = 0 \qquad C_{01} = 2 \qquad C_{10} = 1$$

(a) Determine the Bayes' test if $P(H_0) = \frac{2}{3}$ and the associated Bayes' risk.

(b) Repeat (a) with $P(H_0) = \frac{1}{2}$.

(a) The likelihood ratio is

$$\Lambda(x) = \frac{f(x \mid H_1)}{f(x \mid H_0)} = \frac{e^{-2|x|}}{\frac{1}{2}e^{-|x|}} = 2e^{-|x|} \tag{8.33}$$

By Eq. (8.21), the Bayes' test is given by

$$2e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(1-0)\frac{2}{3}}{(2-0)\frac{1}{3}} = 1 \qquad \text{or} \qquad e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}$$