8.11. Consider a binary decision problem with the following conditional pdf's:

$$f(x \mid H_0) = \tfrac{1}{2} e^{-|x|}$$
$$f(x \mid H_1) = e^{-2|x|}$$

The Bayes' costs are given by

$$C_{00} = C_{11} = 0 \qquad C_{01} = 2 \qquad C_{10} = 1$$

(*a*) Determine the Bayes' test if $P(H_0) = \frac{2}{3}$ and the associated Bayes' risk.

(*b*) Repeat (*a*) with $P(H_0) = \frac{1}{2}$.

(*a*) The likelihood ratio is

$$\Lambda(x) = \frac{f(x \mid H_1)}{f(x \mid H_0)} = \frac{e^{-2|x|}}{\frac{1}{2} e^{-|x|}} = 2e^{-|x|} \tag{8.33}$$

By Eq. (*8.21*), the Bayes' test is given by

$$2e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(1-0)\frac{2}{3}}{(2-0)\frac{1}{3}} = 1 \qquad \text{or} \qquad e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}$$

Thus, the decision regions are given by

$$R_0 = \{x: |x| > 0.693\} \qquad R_1 = \{x: |x| < 0.693\}$$

Then

$$P_1 = P(D_1 | H_0) = \int_{-0.693}^{0.693} \tfrac{1}{2} e^{-|x|}\, dx = 2\int_0^{0.693} \tfrac{1}{2} e^{-x}\, dx = 0.5$$

$$P_{II} = P(D_0 | H_1) = \int_{-\infty}^{-0.693} e^{2x}\, dx + \int_{0.693}^{\infty} e^{-2x}\, dx = 2\int_{0.693}^{\infty} e^{-2x}\, dx = 0.25$$

and by Eq. (*8.19*), the Bayes' risk is

$$\bar{C} = P(D_1 | H_0)P(H_0) + 2P(D_0 | H_1)P(H_1) = (0.5)(\tfrac{2}{3}) + 2(0.25)(\tfrac{1}{3}) = 0.5$$

(*b*) The Bayes' test is

$$2e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(1-0)\frac{1}{2}}{(2-0)\frac{1}{2}} = \frac{1}{2} \qquad \text{or} \qquad e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{4}$$

Again, taking the natural logarithm of both sides of the last expression yields

$$|x| \underset{H_0}{\overset{H_1}{\lessgtr}} -\ln(\tfrac{1}{4}) = 1.386$$

Thus, the decision regions are given by

$$R_0 = \{x: |x| > 1.386\} \qquad R_1 = \{x: |x| < 1.386\}$$

Then

$$P_1 = P(D_1 | H_0) = 2\int_0^{1.386} \tfrac{1}{2} e^{-x}\, dx = 0.75$$

$$P_{II} = P(D_0 | H_1) = 2\int_{1.386}^{\infty} e^{-2x}\, dx = 0.0625$$

and by Eq. (*8.19*), the Bayes' risk is

$$\bar{C} = (0.75)(\tfrac{1}{2}) + 2(0.0625)(\tfrac{1}{2}) = 0.4375$$