0.1 Concepts of Solutions to 2-Player Matrix Games - Minimax Solutions

- The minimax strategy of a player is the strategy that maximises his/her expected payoff under the assumption that the other player aims to minimise this payoff.
- The minimax payoff of a player is the maximum expected payoff that he/she can guarantee him/hirself.
- In zero-sum games such a concept seems very reasonable, since by minimising the expected reward of the opponent, an individual maximises his/her expected reward.

0.1.1 Minimax Solutions

One way of finding a players minimax strategy is to use linear programming. Suppose players choose one of two actions. In this case you can think of the strategy of a player as the probability of playing the first action, i.e. choice from the interval [0, 1]. The simplest method of solution is to use calculus. Denote the reward of Player i when Player 1 uses strategy p and Player 2 uses strategy q by Ri(p, q).

0.1.2 Minimax Solutions - Example 3.4.1

Find the minimax strategies and payoffs in the following matrix game

		Player 2	
		Keep Quiet	Confess
Player 1	Keep Quiet	(5,3)	(2,6)
	Confess	(3,2)	(4,5)

0.1.3 Minimax Solutions - Example 3.4.1

- Suppose Player 1 takes action A with probability p and Player 2 takes action A with probability q. The expected reward of Player 1 is given by

$$R1(p, q) = 5pq + 2p(1q) + 3(1p)q + 4(1p)(1q) = 4pq2pq + 4.$$

- Player 1 assumes that Player 2 will minimise her payoff.
- In order to derive which strategy of Player 2 minimises Player 1s payoff, we calculate $\frac{\partial R1(p,q)}{\partial q}$ (i.e. the rate of change of Player 1s expected payoff with respect to the strategy of Player 2).

$$\frac{\partial R1(p, q)}{\partial q} = 4p1.$$

- In general, this derivative will be a linear function of p.

0.1.4 Minimax Solutions - Example 3.4.1

There are three possibilities

1. $\frac{\partial R1(p,q)}{\partial q}$ ¿ 0 for all $p \in [0,1]$. In this case to minimise Player 1s payoff, Player 2 should play q = 0 (i.e. always take his second action). The minimax strategy of Player 1 is to take the action that maximises her reward when Player 2 takes his second action.
2. $\frac{\partial R1(p,q)}{\partial q}$ ¡ 0 for all $p \in [0,1]$. In this case to minimise Player 1s payoff, Player 2 should play q = 1 (i.e. always take his first action). The minimax strategy of Player 1 is to take the action that maximises her reward when Player 2 takes his first action.
3. There exists some $p \in [0,1]$ such that $\frac{\partial R1(p,q)}{\partial q} = 0$. When Player 1 uses this strategy, her expected payoff does not depend on the strategy used by Player 2. This is her minimax strategy.

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$$\partial R1(p,q)\partial q = 0 \to p = 14.$$

We have

$$R1(14,q) = 42p = 3.5$$

. It follows that the minimax strategy of Player 1 is to choose action A with probability 0.25. In this way she guarantees herself a payoff of 3.5

- It should be noted that if Player 1 chose p ¿ 1 4 , then $\frac{\partial R1(p,q)}{\partial q}$ ¿ 0 and Player 2 minimises Player 1s reward by choosing q = 0, i.e. Playing B.
- In this case the payoff of Player 1 would be

$$R1(p,0) = 42p < 3.5.$$

- Similarly, if Player 1 chose p ¡ 1 4 , then $\frac{\partial R1(p,q)}{\partial q}$ ¡ 0 and Player 2 minimises Player 1s reward by choosing q = 1, i.e. Playing A.
- In this case the payoff of Player 1 would be R1(p, 1) = 2p + 3 ¡ 3.5.
- It is thus clear that 3.5 is the maximum expected payoff that Player 1 can guarantee herself.

We can calculate the minimax strategy of Player 2 in an analogous way, by deriving his expected payoff as a function of p and q and differentiating with respect to p, the strategy of Player 1. We have

$$R2(p,q) = 3pq + 6p(1q) + 2(1p)q + 5(1p)(1q) = p3q + 5.$$

Hence,

$$\partial R2(p,q)\partial p = 1 > 0.$$

It follows that Player 1 always minimises Player 2s expected reward by playing p = 0, i.e. always taking action B.

0.1.5 Minimax Solutions - Example 3.4.1

- If Player 1 takes action B, then Player 2 should take action B. This ensures him a payoff of 5.
- Note: It can be seen from the payoff matrix that if Player 1s aim is to minimise Player 2s payoff, then her action B dominates action A.
- This is due to the fact that Player 2s reward is always smaller when Player 1 plays B, whatever action Player 2 takes.

Suppose both players use their minimax strategies, i.e. Player 1 chooses action A with probability 0.25 and Player 2 always chooses action B. The payoff of Player 1 is

$$R1(0.25A + 0.75B, B) = 0.252 + 0.754 = 3.5.$$

The payoff of Player 2 is

$$R2(0.25A + 0.75B, B) = 0.256 + 0.755 = 5.25,$$

i.e. the reward obtained by Player 2 is greater than his minimax payoff. In general, if a players minimax strategy is a pure strategy, when both players play their minimax strategies, he/she may obtain a greater expected payoff than his/her minimax payoff.

0.1.6 Concepts of Solutions to 2-Player Matrix Games - Pure Nash equilibria

A pair of actions $(A^*, (B^*)$ is a pure Nash equilibrium if

$$R1(A^*, (B^*) \geq R1(A, B^*)$$

$$R2(A^*, (B^*) \geq R2(A^*, B)$$

for any action A available to Player 1 and any action B available to Player 2.

- That is to say that a pair of actions is a Nash equilibrium if neither player can gain by unilaterally changing their action (i.e. changing their action whilst the other player does not change their action).
- The value of the game corresponding to an equilibrium is the vector of expected payoffs obtained by the players.

A pair of actions (A , B) is a strong Nash equilibrium if R1(A , B) ¿ R1(A, B); R2(A , B) ¿ R2(A , B). for any action A 6= A available to Player 1 and any action B 6= B available to Player 2. i.e. a pair of actions is a strong Nash equilibrium if both players would lose by unilaterally changing their action. Any Nash equilibrium that is not strong is called weak.

0.1.7 Concepts of Solutions to 2-Player Matrix Games - Mixed Nash equilibria

A pair of mixed strategies, denoted (M1, M2) is a mixed Nash equilibrium if

$$R1(M1, M2) \geq R1(A, M2); R2(M1, M2) \geq R2(M1, B)$$

for any action (pure strategy) A available to Player 1 and any action B available to Player 2.

- It should be noted that the expected reward Player 1 obtains when he plays a mixed strategy M against M2 is a weighted average of the expected rewards of playing his pure actions against M2, where the weights correspond to the probability of playing each action.
- It follows that player 1 cannot do better against M2 than by using the best pure action against M2, i.e. if Player 1 cannot gain by switching to a pure strategy, then she cannot gain by switching to any mixed strategy.

0.1.8 The Bishop-Cannings Theorem

- The support of a mixed strategy M1 is the set of actions that are played with a positive probability under M1.
- Suppose (M1, M2) is a Nash equilibrium pair of mixed strategies and the support of M1 is S. We have R1(A, M2) = R1(M1, M2), $\forall A \in S$; R1(B, M2) ¡ R1(M1, M2), $\forall B \in S$.
- This is intuitively clear, since if a player uses actions A and B under a mixed strategy, then at equilibrium these actions must give the same expected reward, otherwise one action would be preferred over the other.
- It follows that at such an equilibrium all actions in the support of M1 must give the same expected reward, which thus has to be equal to the expected reward of using M1 (which is calculated as a weighted average). Thus all mixed Nash equilibria are weak.

0.1.9 Nash Equilibria - Results

- Every matrix game has at least one Nash equilibrium.
- If there is a unique pure Nash equilibrium of a 2 2 game (i.e. a game in which both players have just 2 possible actions), then that is the only Nash equilibrium.
- If there are no or two strong Nash equilibria in such a game, then there is always a mixed Nash equilibrium.
- Mixed Nash equilibria can be found using the Bishop-Cannings theorem.

0.1.10 Symmetric Games

A game is symmetric if 1. Players all choose from the same set of actions. 2. R1(A, B) = R2(B, A). Note that the symmetric Hawk-Dove game satisfies these conditions. If (A, B) is a pure Nash equilibrium in a symmetric game, then (B, A) is also a pure Nash equilibrium. At a mixed Nash equilibrium or minimax solution of a symmetric 2 2 game, the players use the same strategy as each other.

0.1.11 Nash Equilibria - Example 3.4.2

Derive all the Nash equilibrium of the following game H D H (-2,-3) (4,0) D (0,4) (3,1) The pure Nash equilibria can be found by checking every possible pair of actions.

- (H, H) is not a Nash equilibrium as either player would prefer to unilaterally change their action to D (and hence obtain 0 rather than -2 or -3).

- (D, D) is not a Nash equilibrium as either player would prefer to unilaterally change their action to H (and hence obtain 4).
- (H, D) is a Nash equilibrium, since if Player 1 switches to D, she obtains 3 not 4. If Player 2 switches to H, he obtains -3 not 0.
- The value corresponding to this equilibrium is (4, 0).
- Similarly, (D, H) is a Nash equilibrium. The value corresponding to this equilibrium is (0, 4).
- To find the mixed Nash equilibrium, we use Bishop-Cannings theorem. Suppose the strategy of Player 1 at equilibrium is pH + (1 p)D .
- Player 2 must be indifferent between his two actions. Hence,

  $$R2(pH + (1p)D, H) = R2(pH + (1p)D, D) = v2,$$

  where v2 is the value of the game to Player 2 .
- We thus have

  $$3p + 4(1p) = 0p + 1(1p) = v2.$$

  Solving these equations gives p = 0.5, v2 = 0.5.

Similarly, suppose the strategy of Player 2 at equilibrium is qH + (1 q)D. Player 1 must be indifferent between his two actions. Hence,

$$R1(H, qH + (1q)D) = R1(D, qH + (1q)D) = v1.$$

We thus obtain 2q + 4(1 q) = 0q + 3(1 q) = v1. Solving these equations gives q = 1 3 , v1 = 2. Hence, the mixed equilibrium is (0.5H + 0.5D, 1/3H + 2/3D). The corresponding value is (2, 0.5).

0.1.12 Advantages of the Concept of Minimax Strategies

1. A player only has to know his own payoffs in order to determine his/her minimax strategy and payoff.
2. Apart from degenerate cases (e.g. two actions always give the same payoff), there is a unique minimax strategy.
3. In the case of fixed-sum games, it seems eminently reasonable to follow a minimax strategy.

0.1.13 Disadvantages of the Concept of Minimax Strategies

1. Although the minimax value of a game is well defined, when both players play their minimax strategy their vector of expected payoffs is not necessarily the minimax value of the game (i.e. when both players play the strategy that maximises their guaranteed expected reward, one or more of the players may obtain more than this guaranteed minimum).
2. Many situations cannot be described in terms of pure competition (i.e. in terms of a fixed-sum game). In such cases, the assumption that the aim of an opponent is to minimise a players expected reward may well be unreasonable.

0.1.14 Advantages of the Concept of Nash Equilibria

1. In fixed sum games, the unique Nash Equilibrium pair of strategies is equal to the pair of minimax strategies.

2. In non-fixed sum games, the Nash equilibrium concept makes the more reasonable assumption that both players wish to maximise their own reward.

3. When using the concept of Nash equilibrium it is normally assumed that players know the payoff functions of their opponents. However, in order to find a pure Nash equilibrium, it is only necessary to be able to order the preferences of opponents, not their actual payoffs.

0.1.15 Disadvantages of the Concept of Nash Equilibria

1. There may be multiple equilibria of a game, so the concept of Nash equilibrium should be strengthened in order to make predictions in such situations.

2. Unlike in the derivation of minimax strategies, it is assumed that the payoff functions of opponents are known. This information is necessary to derive a mixed Nash equilibrium.

3. The concept of a Nash equilibrium requires that a player maximises his/her reward given the behaviour of opponents. However, this behaviour is not known a priori.

0.1.16 3.5 Actions Dominated by Pure Strategies

- Suppose that by taking action Ai Player 1 always gets at least the same reward as by playing Aj , regardless of the action taken by Player 2, and for at least one action of Player 2 he obtains a greater reward.

- Action Ai of Player 1 is said to dominate action Aj . i.e. action Ai of Player 1 dominates Aj if

$$R1(Ai, Bk) \geq R1(Aj, Bk), k = 1, 2, ..., n$$

and for some k0,

$$R1(Ai, Bk0) > R1(Aj, Bk0).$$

Actions Dominated by Pure Strategies

Similarly, action Bi of Player 2 dominates action Bj if

$$R2(Ak, Bi) \geq R2(Ak, Bj), k = 1, 2, ..., m$$

and for some k0 R2(Ak0 , Bi) ¿ R2(Ak0 , Bj).

0.1.17 Actions Dominated by Randomised Strategies

When removing dominated strategies, it is easiest to first remove those that are dominated by pure strategies. Once that is done we then remove those that are dominated by randomised strategies. The following statements are important in determining which strategies may be dominated by a randomised strategy. If Ai is Player 1s unique best response to Bj , then Ai cannot be dominated by any strategy, pure or randomised. Thus, if Bk such that R1(Ai , Bk) ¿ R1(Aj , Bk), j 6= i, then Ai cannot be dominated. Similarly, if Bi is Player 2s unique best response to Aj , then Bi cannot be dominated. Player 1s mixed strategy p1Ai1 + p2Ai2 + . . . + plAil dominates Aj , j 6= is for any s = 1, 2, ...l, if for all Bk

$$R1(p1Ai1 + p2Ai2 + ... + plAil, Bk) \geq R1(Aj, Bk)$$

and this inequality is strict for at least one value of k. Similarly, Player 2s mixed strategy q1Bi1 + q2Bi2 + . . . + qlBil dominates Bj , j 6= is for any s = 1, 2, ...l, if for all Ak

$$R2(Ak, q1Bi1 + q2Bi2 + ... + qlBil) \geq R2(Ak, Bj)$$

and this inequality is strict for at least one value of k.

0.1.18 Successive removal of dominated actions

- It is clear that an individual should not use a dominated action. Hence, we may remove such actions from the payoff matrix without changing either the minimax strategies or the Nash equilibria.
- It should be noted that an action that was not previously dominated may become dominated after the removal of dominated strategies.
- Hence, we continue removing dominated strategies until there are no dominated strategies left in the reduced game (see tutorial and following example).