0.1 3.7 Evolutionary Game Theory

The Hawk-Dove game is used in biology to explain why aggression is present within populations, but is not always observed. Consider the symmetric Hawk-Dove game. The intuition is that two indistinguishable individuals must decide whether to share a resource or demand the resource for themselves. If only one individual demands the resource, then he/she obtains that resource (of value v). If both demand the resource, they fight. Each wins with probability 0.5 (thus obtaining the resource). The loser pays a cost of c (due to injuries incurred). It is assumed that c ¿ v. 1 / 46

General Form of the Symmetric Hawk-Dove Game The general form of the symmetric Hawk-Dove game is H D H (0.5[v c], 0.5[v c]) (v, 0) D (0, v) 0.5(v, v) 2 / 46 Evolutionarily Stable Strategies in Symmetric 2-Player Matrix Games (ESSs) In a symmetric 2-player game, we may define the payoff to a player using strategy 1 against a player using strategy 2 to be R(1, 2). It is assumed that each player plays a sequence of games, with each opponent being chosen at random from the population as a whole. Each individual reproduces at a rate proportional to the average reward gained in these games (or proportional to some increasing function of this average reward). It is assumed that the population size is very large (essentially infinite). 3 / 46

Evolutionarily Stable Strategies in Symmetric 2-Player Matrix Games (ESSs) An ESS, of a symmetric game is a strategy that will be selected for when adopted by virtually the whole population. In mathematical terms, is an ESS if Either 1) R(,) ¿ R(,), 6= or 2) R(,) R(,), and if R(,) = R(,) for some 6= , then R(,) ¡ R(,). 4 / 46

0.1.1 Weak and Strong ESSs

- Any ESS which satisfies Condition 1 is called a strong ESS.
- A strong, symmetric Nash equilibrium of a symmetric game Strong ESS of symmetric game.
- Any ESS which does not satisfy Condition 1, but satisfies Condition 2 is called a weak ESS.

0.1.2 Evolutionarily Stable Strategies in Symmetric 2-Player Matrix Games (ESSs)

The first condition states that if a player knows that an opponent will play , then he/she does strictly better by playing than by playing any other strategy. i.e. Any individual playing a different strategy from would be selected against. The second condition states that if there exists an individual mutant (individual not using) who is not selected against, then when the frequency of such mutants is close to zero (but positive), then such mutants are selected against.

0.1.3 Evolutionarily Stable Strategies in Symmetric 2-Player Matrix Games (ESSs)

To see this is the case, suppose that Condition 2 is satisfied and a proportion 1ϵ play and a proportion ϵ play , where ϵ is small. Since each opponent is chosen from the population as a whole, the average reward obtained by players is

$$R(,(1\epsilon)+\epsilon)=(1\epsilon)R(,)+\epsilon R(,).$$

The average reward obtained by players is

$$R(\pi,(1\epsilon)pi+\epsilon\pi)=(1\epsilon)R(,)+\epsilon R(,)<R(,(1\epsilon)\pi+\epsilon\pi).$$

If Condition 1 is satisfied, then there is a value $\epsilon_0>0$ such that this inequality is satisfied for all $\epsilon<\epsilon_0$.

0.1.4 Evolutionarily Stable Strategies in Symmetric 2-Player Matrix Games (ESSs)

- Due to the linearity of the payoff function when a player faces an opponent using a mixed strategy, when checking the stability conditions we may assume that any mutant uses a pure strategy.
- It should be noted that at an ESS of a symmetric game, each individual should use the same strategy. i.e. an ESS of a symmetric game is given by a single strategy, while a Nash equilibrium is given by a pair of strategies.
- Hence, the non-symmetric Nash equilibria [(H, D) and (D, H)] of the Hawk-Dove game do not correspond to an evolutionarily stable strategy of the Hawk-Dove game.

0.1.5 Relation Between ESSs and Nash Equilibria in Symmetric Games

An ESS of a symmetric game must correspond to a symmetric Nash equilibrium of that game (an ESS of a symmetric game is a Nash equilibrium in which both players play the same strategy). Hence, the concept of an ESS is a refinement of the concept of a Nash equilibrium. Any non-trivial 22 symmetric matrix game has an ESS. Non-trivial - the payoff is essentially dependent on the pair of actions taken.

0.1.6 ESSs of Symmetric Matrix Games

To derive the ESSs of a symmetric matrix game, we 1. Find all the symmetric pure Nash equilibria (i.e. Nash equilibria where both players always use the same action). These are always ESSs (called pure ESSs). 2. Find the mixed equilibria and check whether they satisfy the equilibrium condition (such ESSs are called mixed ESSs).

0.1.7 ESSs of the Hawk-Dove Game

First we look for a pure ESS of the game. (H, H) is not a pure ESS, since both players obtain a negative expected payoff and by changing unilaterally to D, either one can increase their payoff to 0. (D, D) is not a pure ESS, since both players obtain v/2. By changing unilaterally to H, a player can gain an expected reward of v. There is no pure ESS. We thus look for a mixed ESS. 11 / 46

0.1.8 ESSs of the Hawk-Dove Game

- We first look for a mixed Nash equilibrium using the Bishop-Cannings theorem.
- Note any mixed equilibrium of a symmetric 22 game is symmetric.
- Suppose the mixed Nash equilibrium is [pH + (1 p)D, pH + (1 p)D]. When an opponent plays this strategy a player is indifferent between H and D.

Hence, R(H, pH + (1 p)D)=R(D, pH + (1 p)D) 0.5p(v c) + (1 p)v=0.5(1 p)v p = v c 12 / 46

- Hence, the probability of playing H (level of aggression) is increasing in the value of the resource and decreasing in the costs incurred in losing a fight.
- Note that any mixed ESS is by definition a weak ESS.

- This is due to the fact that when all the population use a strategy corresponding to a mixed Nash equilibrium, an individual using an action in the support of this equilibrium (or mixture thereof) will obtain the same expected payoff as a member of the general population.
- We now check that Condition 2 for an ESS is satisfied.
- This condition states that the ESS strategy must do better against any pure strategy than the pure strategy does against itself.

Firstly, we check whether an individual using the ESS does better against a hawk than a hawk does against another hawk, i.e. R(pH + (1 p)D, H) ¿ R(H, H). We have

$$R(pH + (1p)D, H) = 0.5p(vc) > R(H, H) = 0.5(vc),$$

since p ¡ 1 and v c ¡ 0.

0.1.9 ESSs of the Hawk-Dove Game

Secondly, we check whether an individual using the ESS does better against a dove than a dove does against another dove, i.e. R(pH + (1 p)D, D) ¿ R(D, D). We have

$$R(pH + (1p)D, D) = pv + (1p)v/2 > R(D, D) = v/2.$$

Hence, pH + (1 p)D is an ESS. 15 / 46

0.1.10 Interpretation of Mixed ESSs

- One may ask the following question: does the existence of a mixed ESS in which all individuals play H with probability p and play D with probability 1 p mean that there is an equilibrium in which a proportion p of individuals always play H and a proportion 1 p always play D?
- Such an equilibrium is called a stable polymorphism (different individuals use different strategies).
- In the types of game we consider, the answer to the above question is Yes. Hence, we can think of a population following a mixed strategy as being equivalent to a population in which the proportion of individuals using each pure action is equal to the probability of using that action under the mixed strategy.

0.1.11 Coordination and Anti-Coordination Games

The Hawk-Dove game is an example of an anti-coordination game. An anti-coordination game is a symmetric 22 game in which there are 2 pure equilibria where the players take differing actions. In such games the mixed equilibrium is the unique ESS. In co-ordination games there are 2 pure, symmetric Nash equilibria (i.e. players take the same action at a Nash equilibrium).

0.1.12 ESSs in a Co-ordination Game

Suppose two individuals play the following game.

It is simple to check that (A, A) and (B, B) are strong Nash equilibria. Hence, both A and B are ESSs. There is a mixed Nash equilibrium [pA + (1 p)B, pA + (1 p)B], where R(A, pA + (1 p)B)=R(B, pA + (1 p)B) 5p = (1 p) p = 1 6 .

	A	B
A	(5, 5)	(0, 0)
B	(0, 0)	(1, 1)

Figure 1:

0.1.13 ESSs in a Co-ordination Game

We now check whether 1 6 A + 5 6 B satisfies Condition 2 for an ESS. We first check whether this mixed strategy does better against A than A does against itself. We have R(1 6 A + (1 p)B, A) = 5 6 ¡ R(A, A) = 5. Hence, 1 6 A + 5 6 B is not an ESS. A group of mutants playing A would obtain a higher expected payoff (i.e. invade such a population).

0.1.14 Replicator Dynamics

When there are multiple ESSs, it is natural to ask which one would be favoured by natural selection. It is clear that this depends on the initial population. If in the co-ordination game most of the population are playing A, then the population would evolve so that eventually all the population plays A. Similarly, if nearly all of the population are playing B, then the population would evolve so that eventually all the population plays B. Replicator dynamics are used to simulate the evolution of such a population.

0.1.15 3.8 Replicator Dynamics in Symmetric Games

Suppose players choose one of m actions. The payoff of a player when he/she plays Ai and the opponent plays Aj is R(Ai , Aj), where R(Ai , Aj) 0. It is assumed that individuals use pure strategies, i.e. they always use the same action. Let pi,n be the proportion of individuals using action i in generation n.

0.1.16 Replicator Dynamics in Symmetric Games

Denote the average reward of an individual using action i in generation n by Ri,n. We have

$$Ri, n = R(Ai, p1, nA1 + p2, nA2 + ... + pm, nAm).$$

Denote the average reward in the population as a whole in generation n as Rn. We have Rn = Xm i=1 pi,nRi,n.

0.1.17 Replicator Dynamics in Symmetric Games

Individuals using action i are assumed to reproduce in proportion to the average reward of these individuals. It follows that the proportion of individuals using action i in generation i + 1 is given by pi,n+1 = pi,nRi,n Rn Varying i from 1 to m 1, we obtain the replicator dynamic equations. Note that pm,n = 1 Pm1 i=1 pi,n. 24 / 46

0.1.18 Fixed Points of the Replicator Dynamic Equations

A fixed point (p1, p2, . . . , pm) of the replicator dynamic equations satisfies pi = piRi R , i = 1, 2, . . . , m. where Ri and R denote the average reward of individuals taking action i and of the population, respectively. 25 / 46

0.1.19 Fixed Points of the Replicator Dynamic Equations

- It should be noted that all actions that occur with positive probability have to obtain the same reward at a fixed point. pi = 1, for some i such that 1 i m and pj = 0, j 6= i always defines a fixed point of the replicator dynamic equations.
- A fixed point of the replicator dynamics corresponds to an ESS if the fixed point is an attractor.
- A fixed point of a dynamic process is an attractor if in some neighbourhood of the fixed point, evolution of the system leads to the process getting ever closer to that fixed point.