8.4. Consider the binary decision problem of Prob. 8.3. We modify the decision rule such that we reject H_0 if $x \geq c$.

(a) Find the value of c such that the probability of a Type I error $\alpha = 0.05$.

(b) Find the probability of a Type II error β when $\mu_1 = 55$ with the modified decision rule.

(a) Using the result of part (b) in Prob. 8.3, c is selected such that [see Eq. (8.27)]

$$\alpha = g(50) = P(\bar{X} \geq c;\ \mu = 50) = 0.05$$

However, when $\mu = 50$, $\bar{X} = N(50;\ 4)$, and [see Eq. (8.28)]

$$g(50) = P\left(\frac{\bar{X} - 50}{2} \geq \frac{c - 50}{2};\ \mu = 50\right) = 1 - \Phi\left(\frac{c - 50}{2}\right) = 0.05$$

From Table A (Appendix A), we have $\Phi(1.645) = 0.95$. Thus

$$\frac{c - 50}{2} = 1.645 \qquad \text{and} \qquad c = 50 + 2(1.645) = 53.29$$

(b) The power function $g(\mu)$ with the modified decision rule is

$$g(\mu) = P(\bar{X} \geq 53.29;\ \mu) = P\left(\frac{\bar{X} - \mu}{2} \geq \frac{53.29 - \mu}{2};\ \mu\right) = 1 - \Phi\left(\frac{53.29 - \mu}{2}\right)$$

Setting $\mu = \mu_1 = 55$ and using Table A (Appendix A), we obtain

$$\beta = P_{\text{II}} = 1 - g(55) = \Phi\left(\frac{53.29 - 55}{2}\right) = \Phi(-0.855)$$
$$= 1 - \Phi(0.855) = 0.1963$$

Comparing with the results of Prob. 8.3, we notice that with the change of the decision rule, α is reduced from 0.1587 to 0.05, but β is increased from 0.0668 to 0.1963.

8.5. Redo Prob. 8.4 for the case where the sample size $n = 100$.

(a) With $n = 100$, we have

$$\text{Var}(\bar{X}) = \sigma_{\bar{X}}^2 = \frac{1}{n}\sigma^2 = \frac{100}{100} = 1$$

As in part (a) of Prob. 8.4, c is selected so that

$$\alpha = g(50) = P(\bar{X} \geq c;\ \mu = 50) = 0.05$$

Since $\bar{X} = N(50;\ 1)$, we have

$$g(50) = P\left(\frac{\bar{X} - 50}{1} \geq \frac{c - 50}{1};\ \mu = 50\right) = 1 - \Phi(c - 50) = 0.05$$

Thus
$$c - 50 = 1.645 \qquad \text{and} \qquad c = 51.645$$

(b) The power function is

$$g(\mu) = P(\bar{X} \geq 51.645;\ \mu)$$
$$= P\left(\frac{\bar{X} - \mu}{1} \geq \frac{51.645 - \mu}{1};\ \mu\right) = 1 - \Phi(51.645 - \mu)$$

Setting $\mu = \mu_1 = 55$ and using Table A (Appendix A), we obtain

$$\beta = P_{\text{II}} = 1 - g(55) = \Phi(51.645 - 55) = \Phi(-3.355) \approx 0.0004$$