0.1   5. The Extensive Form of a Game

The strategic form of a game is a compact way of describing the mathematical aspects of a game. In addition, it allows a straightforward method of analysis, at least in principle. However, the flavor of many games is lost in such a simple model. Another mathematical model of a game, called the extensive form, is built on the basic notions of position and move, concepts not apparent in the strategic form of a game. In the extensive form, we may speak of other characteristic notions of games such as bluffing, signaling, sandbagging, and so on. Three new concepts make their appearance in the extensive form of a game: the game tree, chance moves, and information sets.

0.2   5.1 The Game Tree.

The extensive form of a game is modelled using a directed graph. A directed graph is a pair (T,F) where T is a nonempty set of vertices and F is a function that gives for each $x \in T$ a subset, F(x) of T called the followers of x. When a directed graph is used to represent a game, the vertices represent positions of the game. The followers, F(x), of a position, x, are those positions that can be reached from x in one move. A path from a vertex t0 to a vertex t1 is a sequence, $x_0, x_1, ..., x_n$, of vertices such that x0 = t0, xn = t1 and xi is a follower of xi1 for $i = 1, ..., n$. For the extensive form of a game, we deal with a particular type of directed graph called a tree.

   Definition. A tree is a directed graph, (T,F) in which there is a special vertex, t0, called the root or the initial vertex, such that for every other vertex $t \in T$, there is a unique path beginning at t0 and ending at t.

   The existence and uniqueness of the path implies that a tree is connected, has a unique initial vertex, and has no circuits or loops. In the extensive form of a game, play starts at the initial vertex and continues along one of the paths eventually ending in one of the terminal vertices. At terminal vertices, the rules of the game specify the payoff. For n-person games, this would be an n-tuple of payoffs. Since we are dealing with two-person zero-sum games, we may take this payoff to be the amount Player I wins from Player II. For the nonterminal vertices there are three possibilities. Some nonterminal vertices are assigned to Player I who is to choose the move at that position. Others are assigned to Player II. However, some vertices may be singled out as positions from which a chance move is made.

0.2.1   Chance Moves.

Many games involve chance moves. Examples include the rolling of dice in board games like monopoly or backgammon or gambling games such as craps, the dealing of cards as in bridge or poker, the spinning of the wheel of fortune, or the drawing of balls out of a cage in lotto. In these games, chance moves play an important role. Even in chess, there is generally a chance move to determine which player gets the white pieces (and therefore the first move which is presumed to be an advantage). It is assumed that the players are aware of the probabilities of the various outcomes resulting from a chance move. II 49 Information. Another important aspect we must consider in studying the extensive form of games is the amount of information available to the players about past moves of the game. In poker for example, the first move is the chance move of shuffling and dealing the cards, each player is aware of certain aspects of the outcome of this move (the cards he received)

but they is not informed of the complete outcome (the cards received by the other players). This leads to the possibility of bluffing.

0.3   5.2 Basic Endgame in Poker.

- One of the simplest and most useful mathematical models of a situation that occurs in poker is called the classical betting situation by Friedman (1971) and basic endgame by Cutler (1976). These papers provide explicit situations in the game of stud poker and of lowball stud for which the model gives a very accurate description.

- This model is also found in the exercises of the book of Ferguson (1967).

- Since this is a model of a situation that occasionally arises in the last round of betting when there are two players left, we adopt the terminology of Cutler and call it Basic Endgame in poker. This will also emphasize what we feel is an important feature of the game of poker, that like chess, go, backgammon and other games, there is a distinctive phase of the game that occurs at the close, where special strategies and tactics that are analytically tractable become important.

Basic Endgame is played as follows. Both players put 1 dollar, called the ante, in the center of the table. The money in the center of the table, so far two dollars, is called the pot. Then Player I is dealt a card from a deck. It is a winning card with probability 1/4 and a losing card with probability 3/4. Player I sees this card but keeps it hidden from Player II. (Player II does not get a card.) Player I then checks or bets. If they checks, then his card is inspected; if they has a winning card they wins the pot and hence wins the 1 dollar ante from II, and otherwise they loses the 1 dollar ante to II. If I bets, they puts 2 dollars more into the pot. Then Player II not knowing what card Player I has must fold or call. If they folds, they loses the 1 dollar ante to I no matter what card I has. If II calls, they adds 2 dollars to the pot. Then Player Is card is exposed and I wins 3 dollars (the ante plus the bet) from II if they has a winning card, and loses 3 dollars to II otherwise. Let us draw the tree for this game. There are at most three moves in this game: (1) the chance move that chooses a card for I, (2) Is move in which they checks or bets, and (3) IIs move in which they folds or calls. To each vertex of the game tree, we attach a label indicating which player is to move from that position. Chance moves we generally refer to as moves by nature and use the label N. See Figure 1.

Each edge is labelled to identify the move. (The arrows are omitted for the sake of clarity. Moves are assumed to proceed down the page.) Also, the moves leading from a vertex at which nature moves are labelled with the probabilities with which they occur. At each terminal vertex, we write the numerical value of Is winnings (II s losses). There is only one feature lacking from the above figure. From the tree we should be able to reconstruct all the essential rules of the game. That is not the case with the tree of Figure 1 since we have not indicated that at the time II makes their decision they does not know which card I has received. That is, when it is IIs turn to move, they does not know at which of their two possible positions they is. We indicate this on the diagram by encircling the II 50 3 1 1 3 1 1 N I II II I winning losing 1/4 3/4 bet check bet check call fold call fold Figure 1. two positions in a closed curve, and we say that these two vertices constitute an information set. The two vertices at which I is to move constitute two separate information sets since he is told the outcome of the chance move. To be complete, this must also be indicated on the diagram by drawing small circles about these vertices. We may delete one of the labels indicating IIs vertices since they belong to the same information set. It is really the information set that must be labeled. The completed game tree becomes 3 1 1 3 1 1 N I I II winning losing 1/4 3/4 bet check bet check call fold call fold Figure 2. The diagram now contains all the essential rules of the game.

0.4   5.3 The Kuhn Tree.

The game tree with all the payoffs, information sets, and labels for the edges and vertices included is known as the Kuhn Tree. We now give the formal definition of a Kuhn tree. Not every set of vertices can form an information set. In order for a player not to be aware of which vertex of a given information set the game has come to, each vertex in that information set must have the same number of edges leaving it. Furthermore, it is important that the edges from each vertex of an information set have the same set of labels. The player moving from such an information set really chooses a label. It is presumed that a player makes just one choice from each information set.

II 51

Definition. A finite two-person zero-sum game in extensive form is given by

1. a finite tree with vertices T,

2. a payoff function that assigns a real number to each terminal vertex,

3. a set T0 of non-terminal vertices (representing positions at which chance moves occur) and for each $t \in T0$, a probability distribution on the edges leading from t,

4. a partition of the rest of the vertices (not terminal and not in T0) into two groups of information sets T11, T12,...,T1k1 (for Player I) and T21, T22,...,T2k2 (for Player II), and

5. for each information set Tjk for player j, a set of labels Ljk, and for each $t \in Tjk$, a one-to-one mapping of Ljk onto the set of edges leading from t.

The information structure in a game in extensive form can be quite complex. It may involve lack of knowledge of the other players moves or of some of the chance moves. It may indicate a lack of knowledge of how many moves have already been made in the game (as is the case With Player II in Figure 3). 0 1 1 1 1 0 2 2 1 1 2 I II II I A B a b c d e DE de DE a b c D E Figure 3. It may describe situations in which one player has forgotten a move they has made earlier (as is the case With Player I in Figures 3 or 4). In fact, one way to try to model the game of bridge as a two-person zero-sum game involves the use of this idea. In bridge, there are four individuals forming two teams or partnerships of two players each. The interests of the members of a partnership are identical, so it makes sense to describe this as a two-person game. But the members of one partnership make bids alternately based on cards that one member knows and the other does not. This may be described as a single player who alternately remembers and forgets the outcomes of some of the previous random moves. Games in which players remember all past information they once knew and all past moves they made are called games of perfect recall. A kind of degenerate situation exists when an information set contains two vertices which are joined by a path, as is the case with Is information set in Figure 5. II 52 1 1 0 0 1 2 I II I f g ab ab cd cd Figure 4. We take it as a convention that a player makes one choice from each information set during a game. That choice is used no matter how many times the information set is reached. In Figure 5, if I chooses option a there is no problem. If I chooses option b, then in the lower of Is two vertices the a is superfluous, and the tree is really equivalent to Figure 6. Instead of using the above convention, we may if we like assume in the definition of a game in extensive form that no information set contains two vertices joined by a path. 2 0 0 1 2 I II a b cd cd a b 2 0 1 2 I I II a b cd cd b Figure 5. Figure 6. Games in which both players know the rules of the game, that is, in which both players know the Kuhn tree, are called games of

complete information. Games in which one or both of the players do not know some of the payoffs, or some of the probabilities of chance moves, or some of the information sets, or even whole branches of the tree, are called games with incomplete information, or pseudogames. We assume in the following that we are dealing with games of complete information.

0.4.1   5.4 The Representation of a Strategic Form Game in Extensive Form.

- The notion of a game in strategic form is quite simple. It is described by a triplet (X, Y, A) as in Section 1. The extensive form of a game on the other hand is quite complex. It is described by the game tree with each non-terminal vertex labeled as a chance move or as a move of one of the players, with all information sets specified, with probability distributions given for all chance moves, and with a payoff attached to each terminal vertex.

- It would seem that the theory of games in extensive is much more comprehensive than the theory of games in strategic form.

- However, by taking a game in extensive form and considering only the strategies and average payoffs, we may reduce the game to strategic form. First, let us check that a game in strategic form can be put into extensive form. In the strategic form of a game, the players are considered to make their choices simultaneously, while in the extensive form of a game simultaneous moves are not allowed.

- However, simultaneous moves may be made sequentially as follows. We let one of the players, say Player I, move first, and then let player II move without knowing the outcome of Is move.

- This lack of knowledge may be described by the use of an appropriate information set. The example below illustrates this.

Matrix Form Equivalent Extensive Form Player I has 2 pure strategies and Player II has 3. We pretend that Player I moves first by choosing row 1 or row 2. Then Player II moves, not knowing the choice of Player I. This is indicated by the information set for Player II. Then Player II moves by choosing column 1, 2 or 3, and the appropriate payoff is made.

0.4.2   5.5 Reduction of a Game in Extensive Form to Strategic Form.

- To go in the reverse direction, from the extensive form of a game to the strategic form, requires the consideration of pure strategies and the usual convention regarding random payoffs. Pure strategies.

- Given a game in extensive form, we first find X and Y , the sets of pure strategies of the players to be used in the strategic form. A pure strategy for Player I is a rule that tells him exactly what move to make in each of their information sets. Let T11,...,T1k1 be the information sets for Player I and let L11,...,L1k1 be the corresponding sets of labels. A pure strategy for I is a k1-tuple x = (xl, ..., xk1) where for each i, xi is one of the elements of L1i. If there are mi elements in L1i, the number of such kl-tuples and hence the number of I s pure strategies is the product m1m2 mk.

- The set of all such strategies is X.

- Similarly, if T21,...,T2k2 represent IIs information sets and L21,...,L2k2 the corresponding sets of labels, a pure strategy for II is a k2-tuple, y = (y1,...,yk2) where $yj \in L2j$ for each j. Player II has n_1, n_2nk2 pure strategies if there are nj elements in L2j . Y denotes the set of these strategies. Random payoffs. A

referee, given $x \in X$ and $y \in Y$, could play the game, playing the appropriate move from x whenever the game enters one of Is information sets, playing the II 54 appropriate move from y whenever the game enters one of IIs information sets, and playing the moves at random with the indicated probabilities at each chance move.

- The actual outcome of the game for given $x \in X$ and $y \in Y$ depends on the chance moves selected, and is therefore a random quantity.

- Strictly speaking, random payoffs were not provided for in our definition of games in normal form.

- However, we are quite used to replacing random payoffs by their average values (expected values) when the randomness is due to the use of mixed strategies by the players.

- We adopt the same convention in dealing with random payoffs when the randomness is due to the chance moves. The justification of this comes from utility theory.

Convention. If for fixed pure strategies of the players, $x \in X$ and $y \in Y$, the payoff is a random quantity, we replace the payoff by the average value, and denote this average value by $A(x, y)$. For example, if for given strategies $x \in X$ and $y \in Y$, Player I wins 3 with probability 1/4, wins 1 with probability 1/4, and loses 1 with probability 1/2, then their average payoff is 1 4 (3) + 1 4 (1) + 1 2 (1) = 1/2 so we let $A(x,y)$=1/2. Therefore, given a game in extensive form, we say (X, Y, A) is the equivalent strategic form of the game if X and Y are the pure strategy spaces of players I and II respectively, and if $A(x,y)$ is the average payoff for $x \in X$ and $y \in Y$. 5.6 Example. Let us find the equivalent strategic form to Basic Endgame in Poker described in the Section 5.2, whose tree is given in Figure 2. Player I has two information sets. In each set they must make a choice from among two options. they therefore has 22=4 pure strategies. We may denote them by

- (b, b): bet with a winning card or a losing card.

- (b, c): bet with a winning card, check with a losing card.

- (c, b): check with a winning card, bet with a losing card.

- (c, c): check with a winning card or a losing card.

Therefore, X = (b, b),(b, c),(c, b),(c, c). We include in X all pure strategies whether good or bad (in particular, (c, b) seems a rather perverse sort of strategy.) Player II has only one information set. Therefore, Y = c, f, where c: if I bets, call. f: if I bets, fold. Now we find the payoff matrix. Suppose I uses (b, b) and II uses c. Then if I gets a winning card (which happens with probability 1/4), they bets, II calls, and I wins 3 dollars. But if I gets a losing card (which happens with probability 3/4), they bets, II calls, and I loses 3 dollars. Is average or expected winnings is A((b, b), c) = 1 4 (3) + 3 4 (3) = 3 2 . II 55 This gives the upper left entry in the following matrix. The other entries may be computed similarly and are left as exercises.   c f (b, b) 3/2 1 (b, c) 0 1/2 (c, b) 2 1 (c, c) 1/2 1/2   Let us solve this 4 by 2 game. The third row is dominated by the first row, and the fourth row is dominated by the second row. In terms of the original form of the game, this says something you may already have suspected: that if I gets a winning card, it cannot be good for him to check. By betting they will win at least as much, and maybe more. With the bottom two rows eliminated the matrix becomes 0 1/2

, whose solution is easily found. The value is V = 1/4. Is optimal strategy is to mix (b, b) and (b, c) with probabilities 1/6 and 5/6 respectively, while IIs optimal strategy is to mix c and f with equal probabilities 1/2 each. The strategy (b, b) is Player Is bluffing strategy. Its use entails betting with a losing hand. The strategy

5

(b, c) is Player Is honest strategy, bet with a winning hand and check with a losing hand. Is optimal strategy requires some bluffing and some honesty. In Exercise 4, there are six information sets for I each with two choices. The number of Is pure strategies is therefore $26 = 64$. II has 2 information sets each with two choices. Therefore, II has $22 = 4$ pure strategies. The game matrix for the equivalent strategic form has dimension 64 by 4. Dominance can help reduce the dimension to a 2 by 3 game! (See Exercise 10(d).)

0.5   5.7 Games of Perfect Information.

- Now that a game in extensive form has been defined, we may make precise the notion of a game of perfect information. Definition. A game of perfect information is a game in extensive form in which each information set of every player contains a single vertex.

- In a game of perfect information, each player when called upon to make a move knows the exact position in the tree. In particular, each player knows all the past moves of the game including the chance ones.

- Examples include tic-tac-toe, chess, backgammon, craps, etc. Games of perfect information have a particularly simple mathematical structure. The main result is that every game of perfect information when reduced to strategic form has a saddle point; both players have optimal pure strategies.

- Moreover, the saddle point can be found by removing dominated rows and columns. This has an interesting implication for the game of chess for example. Since there are no chance moves, every entry of the game matrix for chess must be either $+1$ (a win for Player I), or 1 (a win for Player II), or 0 (a draw).

- A saddle point must be one of these numbers. Thus, either Player I can guarantee himself a win, or Player II can guarantee himself a win, or both players can assure themselves at least a draw. From the game-theoretic viewpoint, chess is a very simple game. One needs only to write down the matrix of the game. If there is a row of all $+1$s, Player I can win.

- If there is a column of all 1s, then Player II can win. Otherwise, there is a row with all $+1$s and 0s and a column with all 1s and 0s, and so the game is drawn with best play. Of course, the real game of chess is so complicated, there is virtually no hope of ever finding an optimal strategy. In fact, it is not yet understood how humans can play the game so well.