(*b*) Let

$$Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then by Eqs. (*4.108*) and (*4.112*), and the result of Prob. 5.60, we see that Y is a normal r.v. with zero mean and variance $1/n$ under H_0, and is a normal r.v. with mean 1 and variance $1/n$ under H_1. Thus

$$P_I = P(D_1 \mid H_0) = \int_{R_1} f_Y(y \mid H_0)\,dy = \frac{\sqrt{n}}{\sqrt{2\pi}}\int_{1/2}^{\infty} e^{-(n/2)y^2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{n}/2}^{\infty} e^{-z^2/2}\,dz = 1 - \Phi(\sqrt{n}/2)$$

$$P_{II} = P(D_0 \mid H_1) = \int_{R_0} f_Y(y \mid H_1)\,dy = \frac{\sqrt{n}}{\sqrt{2\pi}}\int_{-\infty}^{1/2} e^{-(n/2)(y-1)^2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{n}/2} e^{-z^2/2}\,dz = \Phi(-\sqrt{n}/2) = 1 - \Phi(\sqrt{n}/2)$$

Note that $P_I = P_{II}$. Using Table A (Appendix A), we have

$$P_I = P_{II} = 1 - \Phi(1.118) = 0.1318 \qquad \text{for } n = 5$$
$$P_I = P_{II} = 1 - \Phi(1.581) = 0.057 \qquad \text{for } n = 10$$

8.16. In the binary communication system of Prob. 8.6, suppose that $s_0(t)$ and $s_1(t)$ are arbitrary signals and that n observations of the received signal $x(t)$ are made. Let n samples of $s_0(t)$ and $s_1(t)$ be represented, respectively, by

$$\mathbf{s}_0 = [s_{01}, s_{02}, \ldots, s_{0n}]^T \qquad \text{and} \qquad \mathbf{s}_1 = [s_{11}, s_{12}, \ldots, s_{1n}]^T$$

where T denotes "transpose of." Determine the MAP test.

For each X_i, we can write

$$f(x_i \mid H_0) = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}(x_i - s_{0i})^2\right]$$

$$f(x_i \mid H_1) = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}(x_i - s_{1i})^2\right]$$

Since the noise components are independent, we have

$$f(\mathbf{x} \mid H_j) = \prod_{i=1}^{n} f(x_i \mid H_j) \qquad j = 0, 1$$

and the likelihood ratio is given by

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} = \frac{\prod_{i=1}^{n}\exp\left[-\tfrac{1}{2}(x_i - s_{1i})^2\right]}{\prod_{i=1}^{n}\exp\left[-\tfrac{1}{2}(x_i - s_{0i})^2\right]}$$

$$= \exp\left[\sum_{i=1}^{n}(s_{1i} - s_{0i})x_i - \tfrac{1}{2}(s_{1i}^2 - s_{0i}^2)\right]$$

Thus, by Eq. (*8.15*), the MAP test is given by

$$\exp\left[\sum_{i=1}^{n}(s_{1i} - s_{0i})x_i - \tfrac{1}{2}(s_{1i}^2 - s_{0i}^2)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{P(H_0)}{P(H_1)}$$

Taking the natural logarithm of both sides of the above expression yields

$$\sum_{i=1}^{n}(s_{1i} - s_{0i})x_i \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left[\frac{P(H_0)}{P(H_1)}\right] + \tfrac{1}{2}(s_{1i}^2 - s_{0i}^2) \tag{8.38}$$