Chapter 8

Decision Theory

8.1 INTRODUCTION

There are many situations in which we have to make decisions based on observations or data that are random variables. The theory behind the solutions for these situations is known as *decision theory* or *hypothesis testing*. In communication or radar technology, decision theory or hypothesis testing is known as (signal) detection theory. In this chapter we present a brief review of the binary decision theory and various decision tests.

8.2 HYPOTHESIS TESTING

A. Definitions:

A *statistical hypothesis* is an assumption about the probability law of r.v.'s. Suppose we observe a random sample $(X_1, \ldots, X_n)$ of a r.v. X whose pdf $f(\mathbf{x}; \theta) = f(x_1, \ldots, x_n; \theta)$ depends on a parameter θ. We wish to test the assumption $\theta = \theta_0$ against the assumption $\theta = \theta_1$. The assumption $\theta = \theta_0$ is denoted by H_0 and is called the *null hypothesis*. The assumption $\theta = \theta_1$ is denoted by H_1 and is called the *alternative hypothesis*.

$$
\begin{aligned}
H_0&: \quad \theta = \theta_0 \quad \text{(Null hypothesis)} \\
H_1&: \quad \theta = \theta_1 \quad \text{(Alternative hypothesis)}
\end{aligned}
$$

A hypothesis is called *simple* if all parameters are specified exactly. Otherwise it is called *composite*. Thus, suppose $H_0: \theta = \theta_0$ and $H_1: \theta \neq \theta_0$; then H_0 is simple and H_1 is composite.

B. Hypothesis Testing and Types of Errors:

Hypothesis testing is a decision process establishing the validity of a hypothesis. We can think of the decision process as dividing the observation space R^n (*Euclidean* n-space) into two regions R_0 and R_1. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the observed vector. Then if $\mathbf{x} \in R_0$, we will decide on H_0; if $\mathbf{x} \in R_1$, we decide on H_1. The region R_0 is known as the *acceptance region* and the region R_1 as the *rejection* (or *critical*) *region* (since the null hypothesis is rejected). Thus, with the observation vector (or data), one of the following four actions can happen:

1. H_0 true; accept H_0
2. H_0 true; reject H_0 (or accept H_1)
3. H_1 true; accept H_1
4. H_1 true; reject H_1 (or accept H_0)

The first and third actions correspond to correct decisions, and the second and fourth actions correspond to errors. The errors are classified as

1. Type I error: Reject H_0 (or accept H_1) when H_0 is true.
2. Type II error: Reject H_1 (or accept H_0) when H_1 is true.

Let P_{I} and P_{II} denote, respectively, the probabilities of Type I and Type II errors:

$$P_{\text{I}} = P(D_1 \mid H_0) = P(\mathbf{x} \in R_1; H_0) \tag{8.1}$$

$$P_{\text{II}} = P(D_0 \mid H_1) = P(\mathbf{x} \in R_0; H_1) \tag{8.2}$$