Now both hypotheses are simple. We make a Type II error if $X > 5$ when in fact $p = 0.02$. Hence, by Eq. (2.37),

$$P_{II} = P(D_0 | H_1) = P(X > 5; p = 0.02)$$
$$= \sum_{k=6}^{\infty} \binom{200}{k} (0.02)^k (0.98)^{200-k}$$

Again using the Poisson approximation with $\lambda = np = 200(0.02) = 4$, we obtain

$$P_{II} \approx 1 - \sum_{k=0}^{5} e^{-4} \frac{4^k}{k!} = 0.215$$

8.3. Let $(X_1, \ldots, X_n)$ be a random sample of a normal r.v. X with mean μ and variance 100. Let

$$H_0: \quad \mu = 50$$
$$H_1: \quad \mu = \mu_1 \ (>50)$$

and sample size $n = 25$. As a decision procedure, we use the rule to reject H_0 if $\bar{x} \geq 52$, where $\bar{x}$ is the value of the sample mean $\bar{X}$ defined by Eq. (7.27).

(a) Find the probability of rejecting H_0: $\mu = 50$ as a function of μ (> 50).

(b) Find the probability of a Type I error α.

(c) Find the probability of a Type II error β (i) when $\mu_1 = 53$ and (ii) when $\mu_1 = 55$.

(a) Since the test calls for the rejection of H_0: $\mu = 50$ when $\bar{x} \geq 52$, the probability of rejecting H_0 is given by

$$g(\mu) = P(\bar{X} \geq 52; \mu) \tag{8.27}$$

Now, by Eqs. (4.112) and (7.27), we have

$$\mathrm{Var}(\bar{X}) = \sigma_{\bar{X}}^2 = \frac{1}{n}\sigma^2 = \frac{100}{25} = 4$$

Thus, $\bar{X}$ is $N(\mu; 4)$, and using Eq. (2.55), we obtain

$$g(\mu) = P\left(\frac{\bar{X} - \mu}{2} \geq \frac{52 - \mu}{2}; \mu\right) = 1 - \Phi\left(\frac{52 - \mu}{2}\right) \qquad \mu \geq 50 \tag{8.28}$$

The function $g(\mu)$ is known as the *power function of the test*, and the value of $g(\mu)$ at $\mu = \mu_1$, $g(\mu_1)$, is called the *power* at μ_1.

(b) Note that the power at $\mu = 50$, $g(50)$, is the probability of rejecting H_0: $\mu = 50$ when H_0 is true—that is, a Type I error. Thus, using Table A (Appendix A), we obtain

$$\alpha = P_I = g(50) = 1 - \Phi\left(\frac{52 - 50}{2}\right) = 1 - \Phi(1) = 0.1587$$

(c) Note that the power at $\mu = \mu_1$, $g(\mu_1)$, is the probability of rejecting H_0: $\mu = 50$ when $\mu = \mu_1$. Thus, $1 - g(\mu_1)$ is the probability of accepting H_0 when $\mu = \mu_1$—that is, the probability of a Type II error β.

(i) Setting $\mu = \mu_1 = 53$ in Eq. (8.28) and using Table A (Appendix A), we obtain

$$\beta = P_{II} = 1 - g(53) = \Phi\left(\frac{52 - 53}{2}\right) = \Phi(-\tfrac{1}{2}) = 1 - \Phi(\tfrac{1}{2}) = 0.3085$$

(ii) Similarly, for $\mu = \mu_1 = 55$ we obtain

$$\beta = P_{II} = 1 - g(55) = \Phi\left(\frac{52 - 55}{2}\right) = \Phi(-\tfrac{3}{2}) = 1 - \Phi(\tfrac{3}{2}) = 0.0668$$

Notice that clearly, the probability of a Type II error depends on the value of μ_1.