1 Nash Equilibrium

Nash equilibrium is a solution concept of a non-cooperative game involving two or more players in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. If each player has chosen a strategy and no player can benefit by changing strategies while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitutes a Nash equilibrium.

- Nash Equilibrium recommends a strategy to each player that the player cannot improve upon unilaterally, as long as the other players follow the recommendation.
- Since the other olayers are assumed to be rational, it is reasonable to expect the opponents to follow the recommendation as well.

1.1 Informal definition

- Informally, a strategy profile is a Nash equilibrium if no player can do better by unilaterally changing their strategy. To see what this means, imagine that each player is told the strategies of the others. Suppose then that each player asks themselves: *"Knowing the strategies of the other players, and treating the strategies of the other players as set in stone, can I benefit by changing my strategy?"*
- If any player could answer "Yes", then that set of strategies is not a Nash equilibrium. But if every player prefers not to switch (or is indifferent between switching and not) then the strategy profile is a Nash equilibrium. Thus, each strategy in a Nash equilibrium is a best response to all other strategies in that equilibrium.
- The Nash equilibrium may sometimes appear non-rational in a third-person perspective. This is because it may happen that a Nash equilibrium is not Pareto optimal.
- The Nash equilibrium may also have non-rational consequences in sequential games because players may "threaten" each other with non-rational moves. For such games the subgame perfect Nash equilibrium may be more meaningful as a tool of analysis.

1.2 Finding Nash equilibria: games with a finite number of actions for each player

Consider, for example, the game

		Player 2	
		Left	Right
Player 1	Top	(2,2)	(0,3)
	Bottom	(3,0)	(1,1)

There are four action profiles $((T, L), (T, R), (B, L), and(B, R))$; we can examine each in turn to check whether it is a Nash equilibrium.

- **(T,L)** By choosing B rather than T, player 1 obtains a payoff of 3 rather than 2, given player 2's action. Player 2 also can increase their payoff (from 2 to 3) by choosing R rather than L. Thus (T,L) is not a Nash equilibrium.
- **(T,R)** By choosing B rather than T, player 1 obtains a payoff of 1 rather than 0, given player 2's action. Thus (T,R) is not a Nash equilibrium.
- **(B,L)** By choosing R rather than L, player 2 obtains a payoff of 1 rather than 0, given player 1's action. Thus (B,L) is not a Nash equilibrium.
- **(B,R)** Neither player can increase their payoff by choosing an action different from their current one. Thus this action profile is a Nash equilibrium.

We conclude that the game has a unique Nash equilibrium, (B,R). Notice that in this equilibrium both players are worse off than they are in the action profile (T,L). Thus they would like to achieve (T,L); but their individual incentives point them to (B,R).

2 The Nash Equilibrium

Nash Equilibrium is an outcome reached that, once achieved, means no player can increase payoff by changing decisions unilaterally. It can also be thought of as "no regrets," in the sense that once a decision is made, the player will have no regrets concerning decisions considering the consequences.

The Nash Equilibrium is reached over time, in most cases. However, once the Nash Equilibrium is reached, it will not be deviated from. After we learn how to find the Nash Equilibrium, take a look at how a unilateral move would affect the situation. Does it make any sense? It shouldn't, and that's why the Nash Equilibrium is described as "no regrets."

3 How to find a Nash Equilibrium in a 2X2 matrix

3.1 Dominant strategy method

Check each column for Row players highest payoff, this is their best choice given Column players choice. (if there are two high choices, then the result will be a mixed strategy outcome). Now check to see if Rows choice for 1) would also be their choice given any choice by Column player. If Row always sticks with their choice regardless of Columns choice, this is their dominant strategy. Repeat for Column player, and the Nash equilibrium is where the dominant strategies intersect.

3.2 rule of thumb method

Choose one opponents choice and see if the player has an incentive to change their choice. If no, circle that payoff, if yes; check another cell within the same choice by the opponent. Repeat for all choices for both players. The Nash equilibrium (could be more than 1) occur where both payoffs are circled.

3.3 example

Constructing the payoff matrix, rules: Consider a market dominated by two companies. The total market share is 10 Cost of advertising is 4 for high, 2 for low. If firms both choose the same advertising level they split the market, if one firm chooses high and the other low, than the firm that chose high advertising gets the entire market.

		Company 2	
		High	Low
Company 1	High	(1,1)	(6,-2)
	Low	(-2,6)	(3,3)

3.4 Example of finding Nash equilibrium using the dominant strategy method:

We can first look at Row players payoffs to see that if column chooses high, it is in rows best interest to choose high because 1¿-2, and if column choose low, row will also choose high because 6¿3. So choosing high is rows dominant strategy. We can do the same analysis for column player to get the same result. Since both players have a dominant strategy of choosing high, this will be a Nash equilibrium.

3.5 Example of finding Nash equilibrium using rule of thumb method:

Lets start with the first cell, and see if row player wants to switch choices. Since 1¿-2, row player doesnt want to switch, so we can circle that payoff . The same method for column player shows that they would not want to switch as well so we can circle their payoff . We can do the same analysis with each choice, to see where all of the circles should go. The cell with both payoffs circled is a Nash equilibrium. Remember that it is possible to have a payoff matrix with no Nash equilibrium.

- A Nash equilibrium of a strategic game is an action profile (list of actions, one for each player) with the property that no player can increase their payoff by choosing a different action, given the other players' actions.
- Note that nothing in the definition suggests that a strategic game necessarily has a Nash equilibrium, or that if it does, it has a single Nash equilibrium.
- A strategic game may have no Nash equilibrium, may have a single Nash equilibrium, or may have many Nash equilibria.

3.6 Finding a Nash Equilibria

		Player 2	
		Left	Right
Player 1	Up	(1,3)	(4,2)
	Down	(3,2)	(3,1)

Step One: Determine player one's best response to player two's actions. When examining the choices that may maximize a player's payout, we must look at how player one should respond to each of the options player two has. An easy way to do this visually is to cover up the choices of player two. Consider the matrix portrayed at the beginning of this article as we apply this method.

		Player 2	
		Left	Right
Player 1	Up	(1,...)	(4,...)
	Down	(3,...)	(3,...1)

- Player one has two possible choices to play: "up" or "down." Player two also has two choices to play: "left" or "right." In this step of determining Nash Equilibrium, we look at responses to player two's actions.
- If player two chooses to play "left," we can play "up" with the payoff of one, or play "down" with the payoff of three. Since three is greater than one, we will bold the 3 indicating the option to play "down" here.
- If player two chooses to play "right," we can either choose to play 'up' for a payoff of four or play "down" for a playoff of three. Since four is greater than three, we bold the four to indicate the option to play "up" here. The bold outcomes are shown below on the full matrix.

		Player 2	
		Left	Right
Player 1	Left	(1, 3)	(4, 2)
	Right	(3, 2)	(3, 1)

3.7 Step Two: Determine player two's best response to player one's actions.

As we did before with the player two payoffs for player one, we will hide the payoffs of player one when determining the best responses for player two.

		Player 2	
		Left	Right
Player 1	Left	(..., 3)	(..., 2)
	Right	(..., 2)	(..., 1)

Just as when looking at player one, each player has two choices to play. If player one chooses to play "up," we can play "left," with a payoff of three, or "right," with a payoff of two. Since three is greater than two, we bold the three to show the option to play "left" here. If player one chooses to play "down," we can play "left," for a payoff of two, or "right," for a payoff of one. Since two is greater than one, we bold the two indicating the option to play "left" here. The bold outcomes are shown below on the full matrix.

		Player 2	
		Left	Right
Player 1	Left	(1, 3)	(4, 2)
	Right	(4, 2)	(3, 1)

3.8 Step Three: Determine which outcomes have both payoffs bold.

That particular outcome is the Nash Equilibrium. Now, we combine the bold options for both players onto the full matrix.

		Player 2	
		Up	Down
Player 1	Up	(1,3)	(4,2)
	Down	(3,2)	(3,1)

Look for intersections where both payoffs are bold. In this case, we find the intersection of (Down , Left) with the payoff of (3, 2) fits our criteria. This indicates our Nash Equilibrium.

This method of finding Nash Equilibrium is well-suited to finding equilibria in games that are simultaneous since we are looking at how a player would respond independently of how the other acts. This scenario of a simultaneous game is often played out in businesses such as airlines. Below is an example, similar to the game above, of how airline pricing may play out. The payouts are in thousands of dollars. Remember, these are the payouts, not the prices. The method we applied previously is already applied to show where the Nash Equilibrium appears.

		Airline 2	
		Low Price	High Price
Player 1	Low Price	(3,000, 3,000)	(4,000, 2,000)
	High Price	(2,000, 4,000)	(3,500, 3,500)

- Looking at just A1's choices we can see that if A2 chooses to play low price, we choose between Low Price for 3,000 or high price for 2,000. We choose "low," since 3,000¿2,000.
- We do the same thing for A2 playing High Price and see that we play "low" because 4,000¿3,500. Conversely, looking just at A2's choices, we can see that if A1 chooses to play low price, we choose between "low price" for 3,000 and "high price" for 2,000.
- Since 3,000¿2,000, we choose the "low price" option here.
- If A1 plays high price, we can charge a low price for 4,000 or high price for 3,500. Since 4,000¿3,500, we choose to play *"low price"* here.
- The Nash Equilibrium is that both airlines will charge a low price (shown when choices for each party are highlighted). If both airlines charged a high price, they would each be better off than they are at the Nash Equilibrium.
- So why don't they agree to do this? First off, it's illegal to collude. Second, if this were to occur, a unilateral action on behalf of one airline to charge a low price would be beneficial, resulting in that airline making more money in turn. This logic also shows how the Nash Equilibrium is reached, and why it is not beneficial to deviate from it once it is reached.

3.9 Multiple Nash Equilibria & How The Nash Equilibrium Plays Out

- Generally, there can be more than one equilibrium in a game. However, this usually occurs in games with more complex elements than two choices by two players. In simultaneous games that are repeated over time, one of these multiple equilibria is reached after some trial and error.
- This scenario of differing choices over time before reaching equilibrium is the most often played out in the business world when two firms are determining prices for very interchangeable products, such as airfare or soda pop.

  With these advanced methods, more real-world situations can be modeled and solved.
- The different kinds of Nash Equilibrium we discussed are the most commonly found solutions to real-world modeled games. A working knowledge of Game Theory can help you form a strategy, whether playing a friend playing tic-tac-toe or vying for the largest profits.

3.10 Nash Equilibrium and Dominant Strategies

Nash Equilibrium is a term used in game theory to describe an equilibrium where each player's strategy is optimal given the strategies of all other players. A Nash Equilibrium exists when there is no unilateral profitable deviation from any of the players involved. In other words, no player in the game would take a different action as long as every other player remains the same. Nash Equilibria are self-enforcing; when players are at a Nash Equilibrium they have no desire to move because they will be worse off.

Necessary Conditions

The following game doesn't have payoffs defined:

L R T a,b c,d B e,f g,h In order for (T,L) to be an equilibrium in dominant strategies (which is also a Nash Equilibrium), the following must be true:

a ¿ e c ¿ g b ¿ d f ¿ h In order for (T,L) to be a Nash Equilibrium, only the following must be true: a ¿ or = e b ¿ or = d

3.11 Prisoners' Dilemma (Again)

If every player in a game plays his dominant pure strategy (assuming every player has a dominant pure strategy), then the outcome will be a Nash equilibrium. The Prisoners' Dilemma is an excellent example of this. It was reviewed in the introduction, but is worth reviewing again. Here's the game (remember that in the Prisoners' Dilemma, the numbers represent years in prison):

Jack C NC Tom C -10,-10 0,-20 NC -20,0 -5,-5

In this game, both players know that 10 years is better than 20 and 0 years is better than 5; therefore, C is their dominant strategy and they will both choose C (cheat). Since both players chose C, (10,10) is the outcome and also the Nash Equilibrium. To check whether this is a Nash Equilibrium, check whether either player would like to deviate from this position. Jack wouldn't want to deviate, because if he chose NC and Tom stayed at C, Jack would increase his prison time by 10 years.

3.12 Iterated Deletion of Dominated Strategies

Here's another game that doesn't have dominant pure strategies, but that we can solve by iterated deletion of dominated strategies. In other words, we can eliminate strategies that are dominated until we come to a conclusion:

2 Left Middle Right 1 Up 1,0 1,2 0,1 Down 0,3 0,1 2,0

Let's find the dominant strategies. The first strategy that is dominated, is Right. Player 2 will always be better off by playing Middle, so Right is dominated by Middle. At this point the column under Right can be eliminated since Right is no longer an option. This will be shown by crossing out the column:

		Player 2		
		Heads	Tails	Right
Player 1	Heads	1	-1	
	Tails	-1	1	

2 Left Middle Right 1 Up 1,0 1,2 0,1 Down 0,3 0,1 2,0 Remember that both players understand that player 2 has no reason to play Right–player 1 understands that player 2 is trying to find

an optimum, so he also no longer considers the payoffs in the Right column. With the Right column gone, Up now dominates Down for player 1. Whether player 2 plays Left or Middle, player 1 will get a payoff of 1 as long as he chooses Up. So now we no longer consider Down:

2 Left Middle Right

		Player 2		
		Heads	Tails	Right
Player 1	Heads	1	-1	
	Tails	-1	1	

1 Up 1,0 1,2 0,1 Down 0,3 0,1 2,0

Now we know that player 1 will choose Up, and player 2 will choose Left or Middle. Since Middle is better than Left (a payoff of 2 vs. 0), player 2 will choose Middle and we have solved the game for the Nash Equilibrium:

		Player 2		
		Heads	Tails	Right
Player 1	Heads	1	-1	
	Tails	-1	1	

2 Left Middle Right 1 Up 1,0 1,2 0,1 Down 0,3 0,1 2,0

To ensure that this answer (Up, Middle) is a Nash Equilibrium, check to see whether either player would like to deviate. As long as player 1 has chosen Up, player 2 will choose Middle. On the other hand, as long as player 2 has chosen Middle, player 1 will choose up.

3.13 Multiple Nash Equilibria

Here's a game that demonstrates multiple Nash Equilibria: Two drivers are traveling towards each other on a road. Should they drive on the left or the right side? They don't want to wreck...

Driver 2 Left Right Driver 1 Left 1,1 -1,-1 Right -1,-1 1,1 Both (Left,Left) and (Right,Right) are Nash Equilibria. As long as they're on opposite sides of the road, the drivers are happy and don't want to deviate. Games like this are often solved by social convention–beforehand all the players agree on a strategy so that everyone is better off. Of course, everyone knows that the right side is the best side to drive on, so the game should look more like this:

Driver 2 Left Right Driver 1 Left 1,1 -1,-1 Right -1,-1 2,2 In this case, the game itself gives the players a clue as to where the other player will be, even though there are two Nash Equilibria.

Here's a game with three Nash Equilibria and no dominated strategies:

2 a b c 1 A 1,1 2,0 3,0 B 0,2 3,3 0,0 C 0,3 0,0 10,10 The Nash Equilibria are (A,a), (B,b), and (C,c).

3.14 Example of an iterated deletion of dominated strategy equilibrium

- Consider the following game to better understand the concept of iterated elimination of strictly dominated strategies.

- Player 1 has two strategies and player 2 has three. S1=up,down and S2=left,middle,right. For player 1, neither up nor down is strictly dominated. Up is better than down if 2 plays left (since 1¿0), but down is better than up if 2 plays right (since 2¿0). For player 2, however, right is strictly dominated by middle (since 2¿1 and 1¿0), so player 2 being rational will not play right.

- Thus if player 1 knows that player 2 is rational then player 1 can eliminate right from player 2's strategy space. So, if player 1 knows that player 2 is rational then player 1 can play the game as if it was the game depicted below.

- In the figure above, down is strictly dominated by up for player 1 , and so if player 1 is rational (and player 1 knows that player 2 is rational, so that the second game applies) then player 1 will not play down. Consequently, if player 2 knows that player 1 is rational, and player 2 knows that player 1 knows that player 2 is rational (so that player 2 knows that the second game applies) then player 2 can eliminate down from player 1's strategy space, leaving the game looking like below.

- And now left is strictly dominated by middle for player 2 , leaving (up,middle) as the outcome of the game. This is process is called the iterated elimination of strictly dominated strategies.