GAME THEORY 1. The Strategic Form of a Game. The individual most closely associated with the creation of the theory of games is John von Neumann, one of the greatest mathematicians of the 20th century. Although others preceded him in formulating a theory of games - notably Emile Borel - it was von Neumann who published in 1928 the paper that laid the foundation for the theory of two-person zero-sum games. Von Neumanns work culminated in a fundamental book on game theory written in collaboration with Oskar Morgenstern entitled Theory of Games and Economic Behavior, 1944. Other discussions of the theory of games relevant for our present purposes may be found in the text book, Game Theory by Guillermo Owen, 2nd edition, Academic Press, 1982, and the expository book, Game Theory and Strategy by Philip D. Straffin, published by the Mathematical Association of America, 1993. The theory of von Neumann and Morgenstern is most complete for the class of games called two-person zero-sum games, i.e. games with only two players in which one player wins what the other player loses. In Part Player 2, we restrict attention to such games. We will refer to the players as Player 1 and Player Player 2.

0.0.1 1.1 Strategic Form.

The simplest mathematical description of a game is the strategic form, mentioned in the introduction. For a two-person zero-sum game, the payoff function of Player Player 2 is the negative of the payoff of Player 1, so we may restrict attention to the single payoff function of Player 1, which we call here A.

Definition 1. The strategic form, or normal form, of a two-person zero-sum game is given by a triplet (X, Y, A), where

1. X is a nonempty set, the set of strategies of Player 1
2. Y is a nonempty set, the set of strategies of Player Player 2
3. A is a real-valued function defined on $X \times Y$. (Thus, $A(x, y)$ is a real number for every $x \in X$ and every $y \in Y$.)

The interpretation is as follows. Simultaneously, Player 1 chooses $x \in X$ and Player Player 2 chooses $y \in Y$, each unaware of the choice of the other. Then their choices are made known and I wins the amount $A(x, y)$ from Player 2. Depending on the monetary unit involved, $A(x, y)$ will be cents, dollars, pesos, beads, etc. If A is negative, I pays the absolute value of this amount to Player 2. Thus, $A(x, y)$ represents the winnings of I and the losses of Player 2. This is a very simple definition of a game; yet it is broad enough to encompass the finite combinatorial games and games such as tic-tac-toe and chess. This is done by being sufficiently broadminded about the definition of a strategy. A strategy for a game of chess, Player 2 4 for example, is a complete description of how to play the game, of what move to make in every possible situation that could occur. It is rather time-consuming to write down even one strategy, good or bad, for the game of chess. However, several different programs for instructing a machine to play chess well have been written. Each program constitutes one strategy. The program Deep Blue, that beat then world chess champion Gary Kasparov in a match in 1997, represents one strategy.

The set of all such strategies for Player 1 is denoted by X. Naturally, in the game of chess it is physically impossible to describe all possible strategies since there are too many; in fact, there are more strategies than there are atoms in the known universe. On the other hand, the number of games of tic-tac-toe is rather small, so that it is possible to study all strategies and find an optimal strategy for each player. Later, when we study the extensive form of a game, we will see that many other types of games may be modeled and described in strategic form. To illustrate the notions involved in games, let us consider the simplest non-trivial case when both X and Y consist of two elements. As an example, take the game called Odd-or-Even.

0.0.2 1.2 Example: Odd or Even.

Players I and Player 2 simultaneously call out one of the numbers one or two. Player 1s name is Odd; they wins if the sum of the numbers is odd. Player Player 2s name is Even; they wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in dollars. To put this game in strategic form we must specify X, Y and A. Here we may choose X = 1, 2, $Y = \{1, 2\}$, and A as given in the following table. Player 2 (even) y

$A(x, y)$ = Is winnings = Player 2s losses. It turns out that one of the players has a distinct advantage in this game. Can you tell which one it is? Let us analyze this game from Player 1s point of view. Suppose they calls one 3/5ths of the time and two 2/5ths of the time at random. In this case, 1. If Player 2 calls one, I loses 2 dollars 3/5ths of the time and wins 3 dollars 2/5ths of the time; on the average, they wins 2(3/5) + 3(2/5) = 0 (he breaks even in the long run). 2. If Player 2 call two, I wins 3 dollars 3/5ths of the time and loses 4 dollars 2/5ths of the time; on the average they wins 3(3/5) 4(2/5) = 1/5. That is, if I mixes their choices in the given way, the game is even every time Player 2 calls one, but I wins 20/c on the average every time Player 2 calls two. By employing this simple strategy, I is assured of at least breaking even on the average no matter what Player 2 does. Can Player 1 fix it so that they wins a positive amount no matter what Player 2 calls? Player 2 5 Let p denote the proportion of times that Player 1 calls one. Let us try to choose p so that Player 1 wins the same amount on the average whether Player 2 calls one or two. Then since Is average winnings when Player 2 calls one is 2p + 3(1 p), and their average winnings when Player 2 calls two is 3p 4(1 p) Player 1 should choose p so that

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\begin{aligned}
2p + 3(1p) &= 3p4(1p) && (1)\\
35p &= 7p4 && (2)\\
12p &= 7 && (3)\\
p &= 7/12. && (4)\\
& && (5)
\end{aligned}
$$

- Hence, Player 1 should call one with probability 7/12, and two with probability 5/12. On the average, Player 1 wins 2(7/12) + 3(5/12) = 1/12, or 8 1 3 cents every time they plays the game, no matter what Player 2 does.
- Such a strategy that produces the same average winnings no matter what the opponent does is called an equalizing strategy. Therefore, the game is clearly in Player 1s favor. Can they do better than 8 1 3 cents per game on the average? The answer is: Not if Player 2 plays properly. Player 1n fact, Player 2 could use the same procedure: call one with probability 7/12 call two with probability 5/12.
- If Player 1 calls one, Player 2s average loss is 2(7/12) + 3(5/12) = 1/12.
- If Player 1 calls two, Player 2s average loss is 3(7/12) 4(5/12) = 1/12.
- Hence, Player 1 has a procedure that guarantees him at least 1/12 on the average, and Player 2 has a procedure that keeps her average loss to at most 1/12. 1/12 is called the value of the game, and the procedure each uses to insure this return is called an optimal strategy or a minimax strategy.

- If instead of playing the game, the players agree to call in an arbitrator to settle this conflict, it seems reasonable that the arbitrator should require Player 2 to pay 8 1 3 cents to Player 1. For Player 1 could argue that they should receive at least 8 1 3 cents since their optimal strategy guarantees him that much on the average no matter what Player 2 does.
- On the other hand Player 2 could argue that they should not have to pay more than 8 1 3 cents since they has a strategy that keeps her average loss to at most that amount no matter what Player 1 does.

0.0.3 1.3 Pure Strategies and Mixed Strategies.

- It is useful to make a distinction between a pure strategy and a mixed strategy. We refer to elements of X or Y as pure strategies. The more complex entity that chooses among the pure strategies at random in various proportions is called a mixed strategy. Thus, Is optimal strategy in the game of Odd-or-Even is a mixed strategy; it mixes the pure strategies one and two with probabilities 7/12 and 5/12 respectively.
- Of course every pure strategy, $x \in X$, can be considered as the mixed strategy that chooses the pure strategy x with probability 1. In our analysis, we made a rather subtle assumption. We assumed that when a player uses a mixed strategy, they are only interested in their average return. He does not care about his Player 2 6 maximum possible winnings or losses only the average.
- This is actually a rather drastic assumption. We are evidently assuming that a Player 1s indifferent between receiving 5 million dollars outright, and receiving 10 million dollars with probability 1/2 and nothing with probability 1/2. I think nearly everyone would prefer the $5,000,000 outright. This is because the utility of having 10 megabucks is not twice the utility of having 5 megabucks.

Utility Theory

The main justification for this assumption comes from utility theory and is treated in Appendix 1. The basic premise of utility theory is that one should evaluate a payoff by its utility to the player rather than on its numerical monetary value. Generally a players utility of money will not be linear in the amount. The main theorem of utility theory states that under certain reasonable assumptions, a players preferences among outcomes are consistent with the existence of a utility function and the player judges an outcome only on the basis of the average utility of the outcome. However, utilizing utility theory to justify the above assumption raises a new difficulty. Namely, the two players may have different utility functions. The same outcome may be perceived in quite different ways. This means that the game is no longer zero-sum. We need an assumption that says the utility functions of two players are the same (up to change of location and scale). This is a rather strong assumption, but for moderate to small monetary amounts, we believe it is a reasonable one. A mixed strategy may be implemented with the aid of a suitable outside random mechanism, such as tossing a coin, rolling dice, drawing a number out of a hat and so on. The seconds indicator of a watch provides a simple personal method of randomization provided it is not used too frequently. For example, Player 1 of Odd-or-Even wants an outside random event with probability 7/12 to implement their optimal strategy. Since $7/12 = 35/60$, they could take a quick glance at their watch; if the seconds indicator showed a number between 0 and 35, they would call one, while if it were between 35 and 60, he would call two.

0.1 1.4 The Minimax Theorem.

A two-person zero-sum game (X, Y, A) is said to be a finite game if both strategy sets X and Y are finite sets. The fundamental theorem of game theory due to von Neumann states that the situation encountered in the game of Odd-or-Even holds for all finite two-person zero-sum games.

The Minimax Theorem. For every finite two-person zero-sum game, (1) there is a number V , called the value of the game, (2) there is a mixed strategy for Player 1 such that Is average gain is at least V no matter what Player 2 does, and (3) there is a mixed strategy for Player Player 2 such that Player 2s average loss is at most V no matter what I does.

If V is zero we say the game is fair. If V is positive, we say the game favors Player 1, while if V is negative, we say the game favors Player Player 2. Player 2 7

0.1.1 1.5 Exercises.

1. Consider the game of Odd-or-Even with the sole change that the loser pays the winner the product, rather than the sum, of the numbers chosen (who wins still depends on the sum). Find the table for the payoff function A, and analyze the game to find the value and optimal strategies of the players. Is the game fair?

2. Player 1 holds a black Ace and a red 8. Player Player 2 holds a red 2 and a black 7. The players simultaneously choose a card to play. If the chosen cards are of the same color, Player 1 wins. Player Player 2 wins if the cards are of different colors. The amount won is a number of dollars equal to the number on the winners card (Ace counts as 1.) Set up the payoff function, find the value of the game and the optimal mixed strategies of the players.

3. Sherlock Holmes boards the train from London to Dover in an effort to reach the continent and so escape from Professor Moriarty. Moriarty can take an express train and catch Holmes at Dover. However, there is an intermediate station at Canterbury at which Holmes may detrain to avoid such a disaster. But of course, Moriarty is aware of this too and may himself stop instead at Canterbury. Von Neumann and Morgenstern (loc. cit.) estimate the value to Moriarty of these four possibilities to be given in the following matrix (in some unspecified units). What are the optimal strategies for Holmes and Moriarty, and what is the value? (Historically, as related by Dr. Watson in The Final Problem in Arthur Conan Doyles The Memoires of Sherlock Holmes, Holmes detrained at Canterbury and Moriarty went on to Dover.)

4. The entertaining book The Compleat Strategyst by John Williams contains many simple examples and informative discussion of strategic form games. Here is one of his problems.

   > I know a good game, says Alex. We point fingers at each other; either one finger or two fingers. If we match with one finger, you buy me one Daiquiri, If we match with two fingers, you buy me two Daiquiris. If we dont match I let you off with a payment of a dime. Itll help pass the time. Olaf appears quite unmoved. That sounds like a very dull game at least in its early stages. His eyes glaze on the ceiling for a moment and their lips flutter briefly; they returns to the conversation with: Now if youd care to pay me 42 cents before each game, as a partial

> compensation for all those 55-cent drinks Ill have to buy you, then Id be happy to pass the time with you.

Olaf could see that the game was inherently unfair to him so they insisted on a side payment as compensation. Does this side payment make the game fair? What are the optimal strategies and the value of the game? Player 2 8