Cournot Duopoly with Homogeneous items: Linear Demand and Linear Costs

Let x_1 and x_2 be the quantities of homogeneous items produced by two firms with associated costs $C_1(x_1) = c_1x_1$ and $C_2(x_2) = c_2x_2$ respectively.
Items sell at $P = a - b(x_1 + x_2)$ each and it is assumed that all items produced are sold.
The profits made by the firms are then

$$\pi_1 = Px_1 - c_1x_1 = (a - c_1 - b(x_1 + x_2))\, x_1$$

$$\pi_2 = Px_2 - c_2x_2 = (a - c_2 - b(x_1 + x_2))\, x_2$$

respectively.
Maximising π_1 with respect to x_1

$$\begin{aligned} \frac{\partial \pi_1}{\partial x_1} &= a - c_1 - b(x_1 + x_2) - bx_1 \\ &\overset{set}{=} 0 \\ \Rightarrow x_1 &= \frac{a - c_1}{2b} - \frac{1}{2}x_2 \end{aligned} \tag{1}$$

Similarly maximising π_2 with respect to x_2 yields

$$x_2 = \frac{a - c_2}{2b} - \frac{1}{2}x_1 \tag{2}$$

Equations 1 and 2 are referred to as *Reaction Functions* or Best Response Functions - provided their solutions are nonnegative, which I'll assume in the following.
Solving equations 1 and 2 simultaneously gives the *equilibrium* values

$$x_1^* = \frac{a - 2c_1 + c_2}{3b}, \qquad x_2^* = \frac{a - 2c_2 + c_1}{3b}$$

At these equilibrium values

$$P^* = \frac{a + c_1 + c_2}{3}$$

and

$$\pi_1^* = \frac{(a - 2c_1 + c_2)^2}{9b}, \qquad \pi_2^* = \frac{(a - 2c_2 + c_1)^2}{9b} \tag{3}$$

Cournot duopoly is an example of a 2-player matrix form game with an infinite number of strategies available to both players (firms), i.e. the choice of x_1 and x_2 respectively. $\langle x_1^*, x_2^* \rangle$ is then a *Nash* equilibrium with payoffs π_1^* and π_2^* respectively.

Stackelberg Duopoly

Stackelberg duopoly is an example of a 2-player extensive form game in which Firm 1 moves first (the "Leader") and Firm 2 responds (the "Follower"). Irrespective of what the leader does, the follower will use the reaction function (Eq. 2) as it is its best response.

Knowing this, the leader seeks to maximise

$$\Pi_1 = \left(a - c_1 - b\left(x_1 + \frac{a-c_2}{2b} - \frac{1}{2}x_1\right)\right)x_1 = \left(a - c_1 - b\left(\frac{x_1}{2} + \frac{a-c_2}{2b}\right)\right)x_1$$

as a function of x_1.

$$\begin{aligned}
\frac{d\Pi_1}{dx_1} &= a - c_1 - b\left(\frac{x_1}{2} + \frac{a-c_2}{2b}\right) - b\frac{x_1}{2} \\
&\overset{set}{=} 0 \\
\Rightarrow x_1 &= \frac{a - 2c_1 + c_2}{2b} \qquad (4)
\end{aligned}$$

Denoting this optimal value by X_1^* and the corresponding value of x_2 by X_2^* (substitute Eq. 4 into Eq. 2) gives

$$X_1^* = \frac{a - 2c_1 + c_2}{2b}, \qquad X_2^* = \frac{a + 2c_1 - 3c_2}{4b}$$

At these equilibrium values

$$P^* = \frac{a + 2c_1 + c_2}{4}$$

and

$$\Pi_1^* = \frac{(a - 2c_1 + c_2)^2}{8b}, \qquad \Pi_2^* = \frac{(a + 2c_1 - 3c_2)^2}{16b} \qquad (5)$$