0.1 Other useful Continuous Distributions

Pareto distribution for a single such quantity whose log is exponentially distributed; the prototypical power law distribution

Log-normal distribution for a single such quantity whose log is normally distributed

Weibull distribution

Continuous Distributions

(a) The continuous uniform distribution

(b) The exponential distribution

(c) The Weibull distribution

(d) The Pareto distribution

0.2 Gamma Distribution

The Gamma distribution is very important for technical reasons, since it is the parent of the exponential distribution and can explain many other distributions.

The probability distribution function is:

$$f_x(x) = \begin{cases} \frac{1}{a^p\Gamma(p)} x^{p-1}e^{-x/a}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad a, p > 0$$

Where $\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}\,dt$ is the Gamma function. The cumulative distribution function cannot be found unless p=1, in which case the Gamma distribution becomes the exponential distribution. The Gamma distribution of the stochastic variable X is denoted as $X \in \Gamma(p, a)$.

Alternatively, the gamma distribution can be parameterized in terms of a shape parameter $\alpha = k$ and an inverse scale parameter $\beta = 1/\theta$, called a rate parameter:

$$g(x; \alpha, \beta) = Kx^{\alpha-1}e^{-\beta\, x} \text{ for } x > 0.$$

where the K constant can be calculated setting the integral of the density function as 1:

$$\int_{-\infty}^{+\infty} g(x; \alpha, \beta)\mathrm{d}t = \int_0^{+\infty} Kx^{\alpha-1}e^{-\beta\, x}\mathrm{d}x = 1$$

following:

$$K\int_0^{+\infty} x^{\alpha-1}e^{-\beta\, x}\mathrm{d}x = 1$$

$$K = \frac{1}{\int_0^{+\infty} x^{\alpha-1}e^{-\beta\, x}\mathrm{d}x}$$

and, with change of variable $y = \beta x$:

$$
\begin{aligned}
K &= \frac{1}{\int_0^{+\infty} \frac{y^{\alpha-1}}{\beta^{\alpha-1}} e^{-y} \frac{\mathrm{d}y}{\beta}} && (1)\\
&= \frac{1}{\frac{1}{\beta^{\alpha}} \int_0^{+\infty} y^{\alpha-1} e^{-y} \mathrm{d}y} && (2)\\
&= \frac{\beta^{\alpha}}{\int_0^{+\infty} y^{\alpha-1} e^{-y} \mathrm{d}y} && (3)\\
&= \frac{\beta^{\alpha}}{\Gamma(\alpha)} && (4)
\end{aligned}
$$

following:

$$g(x;\alpha,\beta) = x^{\alpha-1} \frac{\beta^{\alpha}\, e^{-\beta\, x}}{\Gamma(\alpha)} \text{ for } x > 0.$$

0.2.1 Probability Density Function

We first check that the total integral of the probability density function is 1.

$$\int_{-\infty}^{\infty} \frac{1}{a^p \Gamma(p)} x^{p-1} e^{-x/a} dx$$

Now we let y=x/a which means that dy=dx/a

$$\frac{1}{\Gamma(p)} \int_0^{\infty} y^{p-1} e^{-y} dy$$

$$\frac{1}{\Gamma(p)} \Gamma(p) = 1$$

Mean

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{a^p \Gamma(p)} x^{p-1} e^{-x/a} dx$$

Now we let y=x/a which means that dy=dx/a.

$$\mathrm{E}[X] = \int_0^{\infty} ay \cdot \frac{1}{\Gamma(p)} y^{p-1} e^{-y} dy$$

$$\mathrm{E}[X] = \frac{a}{\Gamma(p)} \int_0^{\infty} y^p e^{-y} dy$$

$$\mathrm{E}[X] = \frac{a}{\Gamma(p)} \Gamma(p+1)$$

We now use the fact that $\Gamma(z+1) = z\Gamma(z)$

$$\mathrm{E}[X] = \frac{a}{\Gamma(p)} p\Gamma(p) = ap$$

Variance[edit] We first calculate $E[X^2]$

$$\mathrm{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{a^p\Gamma(p)} x^{p-1}e^{-x/a}dx$$

Now we let y=x/a which means that dy=dx/a.

$$\mathrm{E}[X^2] = \int_{0}^{\infty} a^2y^2 \cdot \frac{1}{a\Gamma(p)} y^{p-1}e^{-y}ady$$

$$\mathrm{E}[X^2] = \frac{a^2}{\Gamma(p)} \int_{0}^{\infty} y^{p+1}e^{-y}dy$$

$$\mathrm{E}[X^2] = \frac{a^2}{\Gamma(p)}\Gamma(p+2) = pa^2(p+1)$$

Now we use calculate the variance

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

$$\mathrm{Var}(X) = pa^2(p+1) - (ap)^2 = pa^2$$

Gamma Distribution[edit] Gamma Probability density function Probability density plots of gamma distributions Cumulative distribution function Cumulative distribution plots of gamma distributions Parameters

```
\scriptstyle k \;>\; 0 shape
\scriptstyle \theta \;>\; 0\, scale
Support\scriptstyle x \;\in\; (0,\, \infty)\!
PDF\scriptstyle \frac{1}{\Gamma(k) \theta^k} x^{k \,-\, 1} e^{-\frac{x}{\theta}}\,\!
CDF\scriptstyle \frac{1}{\Gamma(k)} \gamma\left(k,\, \frac{x}{\theta}\right)\!
Mean\scriptstyle \operatorname{E}[ X] = k \theta \!
\scriptstyle \operatorname{E}[\ln X] = \psi(k) +\ln(\theta)\!
(see digamma function)
MedianNo simple closed form
Mode\scriptstyle (k \,-\, 1)\theta \text{ for } k \;>\; 1\,\!
Variance\scriptstyle\operatorname{Var}[ X] = k \theta^2\,\!
\scriptstyle\operatorname{Var}[\ln X] = \psi_1(k)\!
(see trigamma function )
Skewness\scriptstyle \frac{2}{\sqrt{k}}\,\!
Ex. kurtosis\scriptstyle \frac{6}{k}\,\!
Entropy\scriptstyle \begin{align}
\scriptstyle k &\scriptstyle \,+\, \ln\theta \,+\, \ln[\Gamma(k)]\\
\scriptstyle   &\scriptstyle \,+\, (1 \,-\, k)\psi(k)
\end{align}
```

0.3 Gamma Distribution

Applications
The gamma distribution can be used a range of disciplines including queuing models, climatology, and financial services.

- The amount of rainfall accumulated in a reservoir
- The size of loan defaults or aggregate insurance claims
- The flow of items through manufacturing and distribution processes
- The load on web servers

0.4 Other useful Continuous Distributions

0.4.1 Weibull Distribution

The Weibull distribution is used

- in survival analysis,
- in reliability engineering and failure analysis,
- in industrial engineering to represent manufacturing and delivery times,
- in extreme value theory,
- in weather forecasting.

The probability density function of a Weibull random variable x is:

$$f(x; \lambda, k) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k} & x \geq 0, \\ 0 & x < 0, \end{cases}$$

where $k > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter of the distribution.

0.4.2 Gamma Distribution

There are three different parametrizations in common use:

- With a shape parameter k and a scale parameter θ.
- With a shape parameter $\alpha = k$ and an inverse scale parameter $\beta = 1/\theta$, called a rate parameter.
- With a shape parameter k and a mean parameter $\mu = k/\beta$.

Gamma Distribution: Probability density function (pdf)

$$\frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$$

$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

0.5 The General Pareto Distribution

- As with many other distributions, the Pareto distribution is often generalized by adding a scale parameter. Thus, suppose that Z has the basic Pareto distribution with shape parameter a¿0.
- If b¿0, the random variable X=bZ has the Pareto distribution with shape parameter a and scale parameter b. Note that X takes values in the interval $[b, \infty)$.
- Analogies of the results given above follow easily from basic properties of the scale transformation.
- The probability density function is

$$f(x) = abaxa + 1, b \leq x < \infty$$

- The distribution function is

$$F(x) = 1 - (bx)a, b \leq x < \infty$$

- The quantile function is

$$F - 1(p) = b(1 - p)1/a, 0 \leq p < 1$$

0.5.1 Cumulative distribution function

From the definition, the cumulative distribution function of a Pareto random variable with parameters α and xm is

$$F_X(x) = \begin{cases} 1 - \left(\frac{x_\mathrm{m}}{x}\right)^\alpha & \text{for } x \geq x_\mathrm{m}, \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

When plotted on linear axes, the distribution assumes the familiar J-shaped curve which approaches each of the orthogonal axes asymptotically. All segments of the curve are self-similar (subject to appropriate scaling factors). When plotted in a log-log plot, the distribution is represented by a straight line.

0.5.2 Probability density function

It follows (by differentiation) that the probability density function is

$$f_X(x) = \begin{cases} \alpha \, \dfrac{x_\mathrm{m}^\alpha}{x^{\alpha+1}} & \text{for } x \geq x_\mathrm{m}, \\ \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

The expected value of a random variable following a Pareto distribution is

$$E(X) = \begin{cases} \infty & \text{if } \alpha \leq 1, \\ \frac{\alpha x_\mathrm{m}}{\alpha - 1} & \text{if } \alpha > 1. \end{cases}$$

The variance of a random variable following a Pareto distribution is

$$\mathrm{Var}(X) = \begin{cases} \infty & \text{if } \alpha \in (1, 2], \\ \left(\frac{x_\mathrm{m}}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2} & \text{if } \alpha > 2. \end{cases} (If \alpha \leq 1, the variance does not exist.)$$

The Pareto distribution is a continuous distribution with the probability density function (pdf):

$$f(x; \alpha, \beta) = \alpha\beta\alpha/x\alpha + 1$$

For shape parameter $\alpha > 0$, and location parameter $\beta > 0$, and $\alpha > 0$.

0.5.3 Cumulative distribution function

From the definition, the cumulative distribution function of a Pareto random variable with parameters α and x_m is

$$F_X(x) = \begin{cases} 1 - \left(\frac{x_\mathrm{m}}{x}\right)^\alpha & \text{for } x \ge x_\mathrm{m}, \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

- It follows (by differentiation) that the probability density function is

$$f_X(x) = \begin{cases} \alpha \, \dfrac{x_\mathrm{m}^\alpha}{x^{\alpha+1}} & \text{for } x \ge x_\mathrm{m}, \\ \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

- The expected value of a random variable following a Pareto distribution is

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- The Pareto distribution is a continuous distribution with the probability density function (pdf):

$$f(x; \alpha, \beta) = \alpha\beta\alpha/x\alpha + 1$$

- For shape parameter $\alpha > 0$, and location parameter $\beta > 0$, and $\alpha > 0$.
- The Pareto distribution often describes the larger compared to the smaller.
- A classic example is that 80% of the wealth is owned by 20% of the population.

0.6 The General Pareto Distribution

- As with many other distributions, the Pareto distribution is often generalized by adding a scale parameter. Thus, suppose that Z has the basic Pareto distribution with shape parameter a¿0.
- If b¿0, the random variable X=bZ has the Pareto distribution with shape parameter a and scale parameter b. Note that X takes values in the interval $[b, \infty)$.
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- The probability density function is

$$f(x) = abaxa + 1, b \leq x < \infty$$

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- The quantile function is

$$F - 1(p) = b(1 - p)1/a, 0 \leq p < 1$$

Cumulative distribution function From the definition, the cumulative distribution function of a Pareto random variable with parameters α and xm is

$$F_X(x) = \begin{cases} 1 - \left(\frac{x_\mathrm{m}}{x}\right)^\alpha & \text{for } x \geq x_\mathrm{m}, \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

When plotted on linear axes, the distribution assumes the familiar J-shaped curve which approaches each of the orthogonal axes asymptotically. All segments of the curve are self-similar (subject to appropriate scaling factors). When plotted in a log-log plot, the distribution is represented by a straight line.

0.6.1 Probability density function

It follows (by differentiation) that the probability density function is

$$f_X(x) = \begin{cases} \alpha\,\dfrac{x_\mathrm{m}^\alpha}{x^{\alpha+1}} & \text{for } x \geq x_\mathrm{m}, \\ \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

The expected value of a random variable following a Pareto distribution is

$$E(X) = \begin{cases} \infty & \text{if } \alpha \leq 1, \\ \frac{\alpha x_\mathrm{m}}{\alpha - 1} & \text{if } \alpha > 1. \end{cases}$$

The variance of a random variable following a Pareto distribution is

$$\mathrm{Var}(X) = \begin{cases} \infty & \text{if } \alpha \in (1, 2], \\ \left(\frac{x_\mathrm{m}}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2} & \text{if } \alpha > 2. \end{cases} (If \alpha \leq 1, the variance does not exist.)$$

0.6.2 Cumulative distribution function

From the definition, the cumulative distribution function of a Pareto random variable with parameters α and x_m is

$$F_X(x) = \begin{cases} 1 - \left(\frac{x_\mathrm{m}}{x}\right)^\alpha & \text{for } x \ge x_\mathrm{m}, \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

- It follows (by differentiation) that the probability density function is

$$f_X(x) = \begin{cases} \alpha \dfrac{x_\mathrm{m}^\alpha}{x^{\alpha+1}} & \text{for } x \ge x_\mathrm{m}, \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

- The expected value of a random variable following a Pareto distribution is

$$E(X) = \begin{cases} \infty & \text{if } \alpha \le 1, \\ \frac{\alpha x_\mathrm{m}}{\alpha-1} & \text{if } \alpha > 1. \end{cases}$$

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$$\mathrm{Var}(X) = \begin{cases} \infty & \text{if } \alpha \in (1,2], \\ \left(\frac{x_\mathrm{m}}{\alpha-1}\right)^2 \frac{\alpha}{\alpha-2} & \text{if } \alpha > 2. \end{cases} (If \alpha \le 1, the variance does not exist.)$$

- The Pareto distribution is a continuous distribution with the probability density function (pdf):

$$f(x; \alpha, \beta) = \alpha\beta\alpha/x\alpha + 1$$

- For shape parameter $\alpha > 0$, and location parameter $\beta > 0$, and $\alpha > 0$.

0.7 The Pareto Distribution Worked Example

Suppose the distribution of monthly salaries of full-time workers in the UK has a Pareto distribution with minimum monthly salary $x_m = 1000$ and concentration factor $\alpha = 3$.
The Pareto Distribution

1. Calculate the mean monthly salary of UK full-time workers.
2. Calculate the probability that a UK full-time worker earns more than 2000 per month.
3. Calculate the median monthly salary of UK full-time workers.

The expected value of a random variable following a Pareto distribution is

$$E(X) = \begin{cases} \infty & \text{if } \alpha \le 1, \\ \frac{\alpha x_\mathrm{m}}{\alpha-1} & \text{if } \alpha > 1. \end{cases}$$

Because $\alpha = 3$, we will use this

$$E(X) = \frac{\alpha x_\mathrm{m}}{\alpha - 1}$$

Recall that $X_m = 1000$.

The cumulative distribution function of a Pareto random variable with parameters α and x_m is

$$F_X(x) = \begin{cases} 1 - \left(\frac{x_\mathrm{m}}{x}\right)^\alpha & \text{for } x \geq x_\mathrm{m}, \\ 0 & \text{for } x < x_\mathrm{m}. \end{cases}$$

Using values for this example:

$$F_X(x) = \begin{cases} 1 - \left(\frac{1000}{x}\right)^3 & \text{for } x \geq 1000, \\ 0 & \text{for } x < 1000. \end{cases}$$

Calculate the probability that a UK full-time worker earns ***more than*** 2000 per month.

$$F_X(x) = \begin{cases} 1 - \left(\frac{1000}{x}\right)^3 & \text{for } x \geq 1000, \\ 0 & \text{for } x < 1000. \end{cases}$$

The Pareto Distribution

Calculate the median monthly salary of UK full-time workers.

$$\text{Median} : F_X(x) = 0.50$$

$$F_X(x) = \begin{cases} 1 - \left(\frac{1000}{x}\right)^3 & \text{for } x \geq 1000, \\ 0 & \text{for } x < 1000. \end{cases}$$

$$F_X(x) = 0.5 \qquad \rightarrow \qquad 1 - \left(\frac{1000}{x}\right)^3 = 0.50$$

$\sqrt[3]{0.5} = 0.7937$

$$\frac{1000}{0.7937} = 1259.92$$

0.7.1 The Weibull Distribution

- The two-parameter Weibull distribution is the most widely used distribution for life data analysis. Apart from the 2-parameter Weibull distribution, the 3-parameter and the 1-parameter Weibull distribution are often used for detailed analysis.
- The 2-parameter Weibull cumulative distribution function (CDF), has the explicit equation: