- Continuous Random Variables
- The Normal Distribution
- Characteristics of the Normal Distribution
- The Standard Normal (Z) Distribution
- Using Murdoch Barnes Table 3
- Standardization Formula
- Important Formulae

Normal Distribution

- Normal distributions are a family of distributions that have the same general shape.
- They are symmetric with scores more concentrated in the middle than in the tails. Normal distributions are sometimes described as bell shaped.
- Examples of normal distributions are shown below. Notice that they differ in how spread out they are. The area under each curve is the same.
- The height of a normal distribution can be specified mathematically in terms of two parameters: the mean (μ) and the standard deviation (σ).

Normal Distribution

The standard normal distribution is a normal distribution with a mean of 0 and a standard deviation of 1. Normal distributions can be transformed to standard normal distributions by the formula:

$$Z = \frac{X - \mu}{\sigma}$$

where X is a score from the original normal distribution, μ is the mean of the original normal distribution, and σ is the standard deviation of original normal distribution. The standard normal distribution is sometimes called the Z distribution. A z score always reflects the number of standard deviations above or below the mean a particular score is. For instance, if a person scored a 68 on a test with a mean of 50 and a standard deviation of 9, then they scored 2 standard deviations above the mean. Converting the test scores to z scores, an X of 70 would be:

$$Z = \frac{68 - 50}{9}$$

So, a Z score of 2 means the original score was 2 standard deviations above the mean. Note that the z distribution will only be a normal distribution if the

The Normal Distribution

The probability density function of the normal distribution is given as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We will not use this formula.

Characteristics of the Normal probability distribution

1. The highest point on the normal curve is at the mean, which is also the median and mode of the distribution.
2. [VERY IMPORTANT] The normal probability curve is bell-shaped and symmetric, with the shape of the curve to the left of the mean a mirror image of the shape of the curve to the right of the mean.
3. The standard deviation determines the width of the curve. Larger values of the the standard deviation result in wider flatter curves, showing more dispersion in data.
4. The total area under the curve for the normal probability distribution is 1.

Characteristics of the Normal probability distribution

- The interval defined by **the mean** $\pm 1\times$ standard deviation includes 68% of the observations ,leaving 16% (approx) in each tail.
- The interval defined by **the mean** $\pm 1.96\times$ standard deviation includes 95% of the observations ,leaving 2.5% (approx) in each tail.
- The interval defined by **the mean** $\pm 2.58\times$ standard deviation includes 99% of the observations ,leaving 0.5% (approx) in each tail.

The Standardized Value

- Suppose that mean $\mu = 80$ and that standard deviation $\sigma = 8$.
- What is the Z value for $X = 100$?

$$Z_{100} = \frac{X_0 - \mu}{\sigma} = \frac{100 - 80}{8} = \frac{20}{8} = 2.5$$

- Therefore $Z_{100} = 2.5$

The Standardization Formula

$$Z_o = \frac{X_o - \mu}{\sigma}$$

All normally distributed random variables have correspondinng Z values

- We can find a probability associated with a value, that is from a normally distribution, by computing the Z value.

$$z_0 = \frac{x_0 - \mu}{\sigma}$$

  - X_o - Some random value from the population of X values.
  - μ - The mean of the population of X values.
  - σ - The variance of the population of X values.
  - Z_o - The Z value that corresponds to X_o

The Standard Normal Distribution

- The standard normal distribution (commonly called the Z distribution) is a special case of the ***normal distribution***.
- It is characterized by the following
  - The mean μ is always equal to 0.
  - The standard deviation σ is always equal to 1.
  - The variance σ^2 is therefore equal to 1 also .

The Standard Normal (Z) Distribution

- A random variable that has a normal distribution with a mean of zero and a standard deviation of one is said to have a standard normal probability distribution. It is often nick-named the ”z” distribution.
- Importantly, probabilities relating to the z distribution are comprehensively tabulated in Murdoch Barnes table 3.
- Given a value of k (with k usually between 0 and 4), the probability of a standard normal ”z” random variable being greater than (or equal to) k is given in Murdoch Barnes table 3 (page 71).

Complement and Symmetry Rules

Any normal distribution problem can be solved with some combination of the following rules.

- The Complement rule (Common to all continuous random variables)

$$P(Z \geq k) = 1 - P(Z \leq k)$$

Similarly

$$P(X \geq k) = 1 - P(X \leq k)$$

Complement and Symmetry Rules

- This rule is based on the property of symmetry mentioned previously.
- Only the probabilities corresponding to values between 0 and 4 are tabulated in Murdoch Barnes.
- If we have a negative value of k, we can use the symmetry rule.

$$P(Z \leq -k) = P(Z \geq k)$$

by extension, we can say

$$P(Z \geq -k) = P(Z \leq k)$$

Example

Find $P(Z \geq -1.28)$ **Solution**

- Using the symmetry rule

$$P(Z \geq -1.28) = P(Z \leq 1.28)$$

- Using the complement rule

$$P(Z \geq -1.28) = 1 - P(Z \geq 1.28)$$

$$P(Z \geq -1.28) = 1 - 0.1003 = 0.8997$$

Find the probability of a “z” random variable being between -1.8 and 1.96? i.e. Compute $P(-1.8 \leq Z \leq 1.96)$ Solution

- Consider the complement event of being in this interval: a combination of being too low or two high.
- The probability of being too low for this interval is $P(Z \leq -1.80) = 0.0359$ (from before)
- The probability of being too high for this interval is $P(Z \geq 1.96) = 0.0250$ (from before)
- Therefore the probability of being **outside** the interval is 0.0359 + 0.0250 = 0.0609.
- Therefore the probability of being **inside** the interval is 1- 0.0609 = 0.9391 $P(-1.8 \leq Z \leq 1.96) = 0.9391$

Solving using the Z distribution

When we have a normal distribution with any mean μ and any standard deviation σ , we answer probability questions about the distribution by first converting all values to corresponding values of the standard normal (”z”) distribution. The formula used to convert any random variable ”X” (with mean μ and standard deviation σ specified) to the standard normal (”z”) distribution is given as follows.

$$Z_o = \frac{X_o - \mu}{\sigma}$$

Z is the standard normal random variable with a mean of zero and a standard deviation of 1. It can be thought of as a measure of how many standard deviations that a value ”x” is from mean μ .

The Standard Normal Distribution

- Special case of the normal distributions
- The distribution is well described in statistical tables
- rahter than computing probabilities from first principles, X values

The Standardized Value

- The first step in solving the problem is to compute the standardized value, also known as the ‘Z’ value.
- We must know the value of the mean μ and the standard deviation σ.
- To find the ‘Z’ value Z_0 for a particular quantity X_0.

$$Z_0 = \frac{X_0 - \mu}{\sigma}$$

Find $P(Z \geq 0.60)$

	0.00	0.01	0.02	0.03	...	...
...	...	...	...	...	...	...
0.4	0.3446	0.3409	0.3372	0.3336	...	...
0.5	0.3085	0.3050	0.3015	0.2981	...	...
0.6	0.2743	0.2709	0.2676	0.2643	...	...
0.7	0.2420	0.2389	0.2358	0.2327	...	...
...	...	...	...	...	...	...

Find $P(Z \geq 1.28)$

	...	...	0.006	0.07	0.08	0.09
...	...	...	...	...	...	...
1.0	...	...	0.1446	0.1423	0.1401	0.1379
1.1	...	...	0.1230	0.1210	0.1190	0.1170
1.2	...	...	0.1038	0.1020	0.1003	0.0985
1.3	...	...	0.0869	0.0853	0.0838	0.0823
...	...	...	...	...	...	...

Find $P(Z \geq 1.65)$ and $P(Z \geq 1.65)$

	...	0.04	0.05	0.06	0.07	...
...	...	...	...	...	...	...
1.5	...	0.0630	0.0618	0.0606	0.0594	...
1.6	...	0.0516	0.0505	0.0495	0.0485	...
1.7	...	0.0418	0.0409	0.0401	0.0392	...
...	...	...	...	...	...	...

Estimate $P(Z \geq 1.645)$

	$\ldots$	0.04	0.05	0.06	0.07	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
1.5	$\ldots$	0.0630	0.0618	0.0606	0.0594	$\ldots$
1.6	$\ldots$	0.0516	0.0505	0.0495	0.0485	$\ldots$
1.7	$\ldots$	0.0418	0.0409	0.0401	0.0392	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$