0.1 Summary of Normal Distribution

- The Standardisation Formula

$$P(X \leq Xo) = P(Z \leq Zo)$$

when

$$Zo = \frac{Xo - \mu}{\sigma}$$

- The Complement Rule

$$P(Z \leq z_o) = 1 - P(Z \geq z_o) \tag{1}$$

- The Symmetry Rule

$$P(Z \leq -z_o) = P(Z \geq z_o) \tag{2}$$

- The Interval Rule. Where L and U are the lower and upper bounds of an interval.

$$P(L \leq Z \leq U) = P(Z \geq L) - P(Z \geq U) \tag{3}$$

0.2 Normal Distribution : Arab Horses Worked Example

The mass of Arab horses is normally distributed with mean 900 lbs and standard deviation of 50lbs.

- Calculate the probability that an Arab horse weighs more than 940 lbs.
- Calculate the probability than an Arab horse weighs between 880 lbs and 960 lbs.

Solution

- Let X be mass of Arab horses.
- We have to find $P(X \leq 940)$. (Remark "equality component" is included as a formality, but it is not important)
- Find the Z value that corresponds to 940

$$Zo = \frac{Xo - \mu}{\sigma} = \frac{940 - 900}{50} = 0.8$$

$$P(X \leq 940) = P(Z \leq 0.8)$$

- From Murdoch Barnes tables 3, we find that $P(Z \leq 0.8) = 0.2119$

Solution

$$P(880 \leq X \leq 960).$$

What proportion of horses are between 880 lbs and 960 lbs?

- Find out the probability of the complement event.
- The complement event is the combination of being too high or too low for this interval.
- Inside interval $P(880 \leq X \leq 960)$.
- Outside interval $P(X \leq 880) + P(X \geq 960)$
- Complement Rule $P(880 \leq X \leq 960) = 1 - [P(X \leq 880) + P(X \geq 960)]$

Find the probability of being too high?

$$Zo = \frac{Xo - \mu}{\sigma} = \frac{960 - 900}{50} = 1.2$$

$$P(X \leq 960) = P(Z \leq 1.2) = 0.1151$$

Find the probability of being too low?

$$Zo = \frac{Xo - \mu}{\sigma} = \frac{880 - 900}{50} = -0.4$$

$$P(X \leq 880) = P(Z \leq -0.4)$$

How to compute $P(Z \leq -0.4)$
Symmetry: $P(Z \leq -0.4) = P(Z \geq 0.4) = 0.3446$

- Outside Interval = 0.4596 (0.3446 + 0.1151)
- Inside Interval = 0.5404

What weight is exceeded by 97.5% of Arab horses?
Find Xo such that P(XXo) = 0.975

- P(Z1.96) = 0.025 [From Tables]
- P(Z-1.96) = 0.025 [Symmetry]
- P(Z-1.96) = 0.975
- $$-1.96 = \frac{Xo - 900}{50}$$
- Xo= 802 lbs [Answer]

0.3 Exam Question:MA4413 Autumn 2008 paper

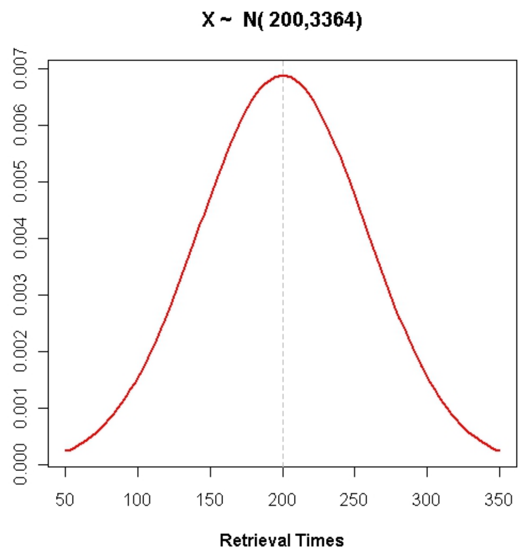
A model of an on-line computer system gives a mean times to retrieve a record from a direct access storage system device of 200 milliseconds, with a standard deviation of 58 milliseconds. If it can assumed that the retrieval times are normally distributed:

(i) What proportion of retrieval times will be greater than 75 milliseconds?

(ii) What proportion of retrieval times will be between 150 and 250 milliseconds?

(iii) What is the retrieval time below which 10% of retrieval times will be?

Normal Distribution



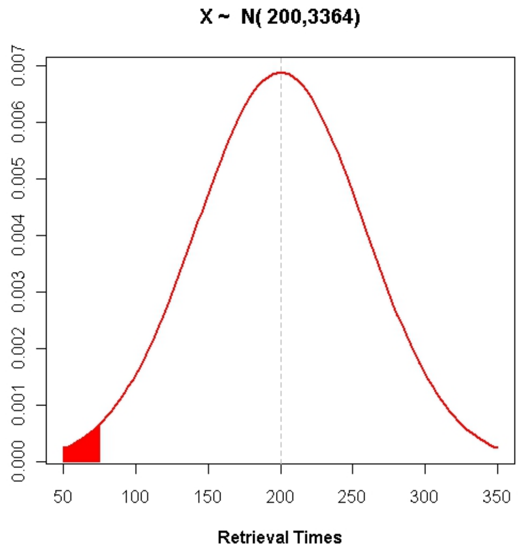
0.3.1 Solution to part 1)

What proportion of retrieval times will be greater than 75 milliseconds?

- Let X be the retrieval times, with $X \sim \mathrm{N}(200, 58^2)$.
- The first question asks us to find $P(X \geq 75)$.
- First compute the z score.

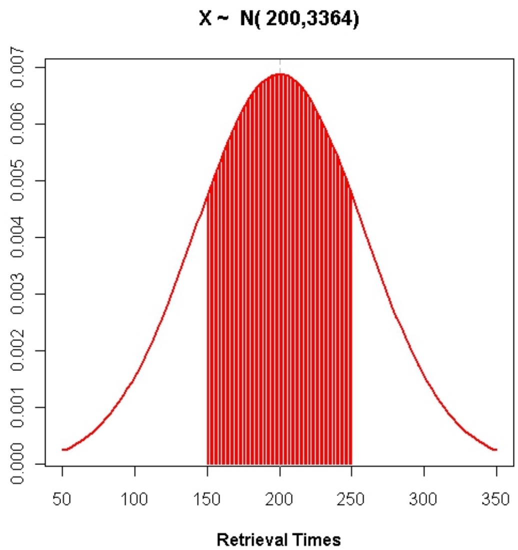
$$z_o = \frac{x_o - \mu}{\sigma} = \frac{75 - 200}{58} = -2.15$$

Normal Distribution



In this case, the probability of interest $P(X \geq 75)$, is represented by the white area under the curve.

- We can say
$$P(X \geq 75) = P(Z \geq -2.15)$$
- Using symmetry rule and complement rule
$$P(Z \geq -2.15) = P(Z \leq 2.15) = 1 - P(Z \geq 2.15)$$
- From tables $P(Z \geq 2.15) = 0.0158$
- Therefore $P(Z \leq 2.15) = 0.9842$
- Furthermore $P(X \geq 75) = \mathbf{0.9842}$ [Answer].



0.3.2 Solution to part 2)

- What proportion of retrieval times will be between 150 and 250 milliseconds?
- Find $P(150 \leq X \leq 250)$
- Use the 'Too Low / Too High ' approach.
- Too low $P(X \leq 150)$
- Too high $P(X \geq 250)$
- Find the z-scores for each.

$$z_{150} = \frac{150 - 200}{58} = -0.86$$
$$z_{250} = \frac{250 - 200}{58} = 0.86$$

- We can now say

$$1. P(X \leq 150) = P(Z \leq -0.86)$$
$$2. P(X \geq 250) = P(Z \geq 0.86)$$

- By symmetry rule, $P(Z \leq -0.86) = P(Z \geq 0.86)$

$$P(X \leq 150) = P(X \geq 250)$$

- Let's compute $P(X \geq 250)$. Using tables

$$P(X \geq 250) = P(Z \geq 0.86) = 0.1949$$

Using the Interval Rule

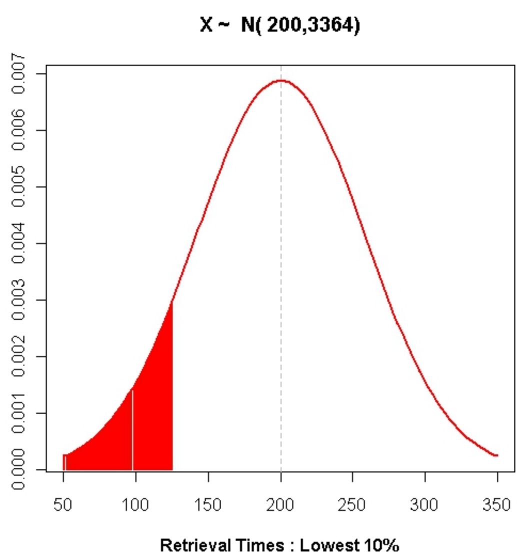
- Too high: $P(X \geq 250) = 0.1949$
- Too low: $P(X \leq 150) = 0.1949$
- Probability of being inside interval:

$$P(150 \leq X \leq 250) = 1 - [P(X \leq 150) + P(X \geq 250)]$$

- $P(150 \leq X \leq 250) = 1 - [0.1949 + 0.1949] = \mathbf{0.6102}$

0.3.3 Solution to part 3

- What is the retrieval time below which 10% of retrieval times will be?
- Find A such that $P(X \leq A) = 0.10$.
- What z-score would correspond to A? Lets call it z_A.
- $P(Z \leq z_A) = 0.10$
- Remark: z_A could be negative.
- Using symmetry $P(Z \geq -z_A) = 0.10$
- Remark: $-z_A$ could be positive.



- Use the Murdoch Barnes tables to get an approximate value for $-z_A$.

- The nearest value we can get is 1.28. ($P(Z \geq 1.28) = 0.1003$).
- If $-z_A = 1.28$, then $z_A = -1.28$
- We can now say
$$P(X \leq A) = P(Z \leq -1.28)$$
- Necessarily A and Z_A are related by the standardization formula
- Recall that $\mu = 200$ and $\sigma = 58$.
$$-1.28 = \frac{A - 200}{58}$$
- Re-arranging (multiply both sides by 58)
$$-74.24 = A - 200$$
- Re-arranging again (Add 200 to both sides)
$$125.76 = A$$
- Now we know the retrieval time below which 10% of retrieval times will be.
- $P(X \leq 125.76) = 0.10$ [Answer].

0.3.4 May 2012 Question 4 Normal Distribution / Theory

Parameter Values

Given the parameters of the normal distribution X in the question.

- Normal Mean $\mu = 73$ points
- Normal Standard Deviation $\sigma = 8$ points
- $P(X \leq 91)$
- $P(65 \leq X \leq 89)$

Find the Z score for X = 91.

$$Z = \frac{x - \mu}{\sigma} = \frac{91 - 73}{8} = \frac{18}{8} = 2.25$$

Therefore we can say :
$P(X \leq 91) = P(Z \leq 2.25)$

From the tables $P(Z \leq 2.25) = 0.9877$ Therefore the probability of getting a grade lower than 91 is 0.9877 (i.e 98.77%)

What is the probability of getting a score between 65 and 89. Writing this mathematically:

$$P(65 \leq X \leq 89)$$

- How many people get a score greater than 89? ($P(X \geq 89)$)
- How many people get a score less than 65? ($P(X \leq 65)$)

To compute $P(X \geq 89)$ first compute the Z-score.

$$Z = \frac{x-\mu}{\sigma} = \frac{89-73}{8} = \frac{16}{8} = 2$$

$P(X \geq 89) = P(Z \geq 2) = 0.0225$.
To compute $P(X \leq 65)$ first compute the Z-score.

$$Z = \frac{x-\mu}{\sigma} = \frac{65-73}{8} = \frac{-8}{8} = -1$$

$P(X \leq 65) = P(Z \leq -1)$

- We use the **symmetry rule**
$$P(Z \leq -1) = P(Z \geq +1)$$
- so we can say $P(X \leq 65) = P(Z \geq +1)$
- From the statistical tables $P(Z \geq +1) = 0.1583$.

0.3.5 May 2012 Question 5 Normal Distribution

Given

- X is the variable of interest.
- Normal Mean $\mu = 25.5$ mpg
- Normal Standard Deviation $\sigma = 4.5$ mpg
- Find x such that $P(X \geq x) = 0.30$

Solution
From the Standard Normal Tables, find the value of z that would give us

$$P(Z \geq z) = 0.30$$

Or if you are using the other type of tables

$$P(Z \leq z) = 0.70$$

0.3.6 May 2013 Question 3 Normal Distributions

Important Information from the Question

- Normal Mean $\mu = 1000$ units
- Normal Standard Deviation $\sigma = 200$ units

Objectives Compute the following :

- $P(X \geq 1400)$ More than 1400
- $P(X \leq 500)$ Less than 500

Part 1 - More than 1400

Firstly compute the z score for 1400.

$$Z_{1400} = \frac{X - \mu}{\sigma} = \frac{1400 - 1000}{200} = \frac{400}{200} = 2$$

So the **Z-score** in this case is 2.
This much we can say

$$P(X \geq 1400) = P(Z \geq 2)$$

$P(Z \geq 2)$ can be determined using statistical tables. Depending on which statistical tables you are using, you will get one of the following answers. (Note the second and third statements are examples of complementary probabilities.)

- $P(0 \leq Z \leq 2) = 0.4775$
- $P(Z \leq 2) = 0.9775$
- $P(Z \geq 2) = 0.0225$

The last expression is useful here. Recall that $P(X \geq 1400) = P(Z \geq 2)$. Therefore

$$P(X \geq 1400) = 0.0225$$

Objectives

Compute the following :

- $P(X \geq 1400)$ More than 1400
- $P(X \leq 500)$ Less than 500

Part 1 - More than 1400

Firstly compute the z score for 1400.

$$Z_{1400} = \frac{X - \mu}{\sigma} = \frac{1400 - 1000}{200} = \frac{400}{200} = 2$$

So the **Z-score** in this case is 2.

This much we can say

$$P(X \geq 1400) = P(Z \geq 2)$$

$P(Z \geq 2)$ can be determined using statistical tables. Depending on which statistical tables you are using, you will get one of the following answers. (Note the second and third statements are examples of complementary probabilities.)

- $P(0 \leq Z \leq 2) = 0.4775$
- $P(Z \leq 2) = 0.9775$
- $P(Z \geq 2) = 0.0225$

The last expression is useful here. Recall that $P(X \geq 1400) = P(Z \geq 2)$. Therefore

$$P(X \geq 1400) = 0.0225$$

0.4 Normal Distribution - Short Questions

1. 95% of students at school weigh between 62 kg and 90 kg. Assuming this data is normally distributed, what are the mean and standard deviation?
2. A machine produces electrical components. 99.7% of the components have lengths between 1.176 cm and 1.224 cm. Assuming this data is normally distributed, what are the mean and standard deviation?
3. 68% of the marks in a test are between 52 and 64 Assuming this data is normally distributed, what are the mean and standard deviation?

0.5 Question D10 - Exponential Distribution

Assume that the length of injected moulded plastic components are normally distributed with a mean of 10mm and a standard deviation of 2mm. Draw a rough sketch and then calculate corresponding probability for the following measurements occurring on an individual component:

(i) Between 10 and 12.4mms

(ii) Less than 9.7 mms

(iii) Between 9.8 and 10.1 mms

(iv) Less than 10.3 mms

1. The Fresha Tea Company pack tea in bags marked as 250 g A large number of packs of tea were weighed and the mean and standard deviation were calculated as 255 g and 2.5 g respectively. Assuming this data is normally distributed, what percentage of packs are underweight?
2. Students pass a test if they score 50% or more.

   The marks of a large number of students were sampled and the mean and standard deviation were calculated as 42% and 8% respectively.

   Assuming this data is normally distributed, what percentage of students pass the test?

0.6 Normal - example

In an examination the scores of students who attend schools of type A are normally distributed about a mean of 55 with a standard deviation of 6. The scores of students who attend type B schools are normally distributed about a mean of 60 with a standard deviation of 5.

Which type of school would have a higher proportion of students with marks above 70?

- $\mu_A = 55$
- $\sigma_A = 8$
- $\mu_B = 60$
- $\sigma_B = 5$

We have to fins $P(X_A \geq 70)$ and $P(X_B \geq 70)$.
using the standardisation formula $Z_A = \frac{70-55}{6} = \frac{15}{6} = 2.5$
$Z_B = \frac{70-60}{5} = \frac{10}{5} = 2$

0.7 Worked Example 3 - with Solutions

Assume that the number of weekly study hours for students at a certain university is approximately normally distributed with a mean of 22 and a standard deviation of 6.

1. Find the probability that a randomly chosen student studies less than 12 hours.
2. Estimate the percentage of students that study more than 37 hours.

solution $X \sim (22, 6^2)$ (in form $X \sim (\mu, \sigma^2)$

Part 1: $P(X \leq 12)$

$Z_1 = \frac{12-22}{6} = \frac{-10}{6} = -1.66$

Part 2: $P(X \geq 37)$

$Z_2 = \frac{37-22}{6} = \frac{15}{6} = 2.5$

0.8 Worked Example 4 - with Solutions

The mean is 550kg, with standard deviation 150kg, and we are interested in the area that is greater than 600kg.

$$Z = \frac{X - \mu}{\sigma} \tag{4}$$

Here X = 600kg, μ , the mean = 550kg σ, the standard deviation = 150kg

- $z = (600 - 550)/150$
- $z = 50/150$
- $z = 0.33$

Look in the table down the left hand column for z = 0.3, and across under 0.03. The number in the table is the tail area for z=0.33 which is 0.3707. This is the probability that the weight will exceed 600kg.

Question 12 Solution

Question 1(a)
Upper Limit ; $U = \mu + 3\sigma$ Lower Limit ; $L = \mu - 3\sigma$
Standardisation Apply the standardisation formula $Z = \frac{x-\mu}{\sigma}$ to both limits

$$Z_U = \frac{U - \mu}{\sigma} = \frac{(\mu + 3\sigma) - \mu}{\sigma} = 3$$

Similary
$Z_l = -3$
Probability of point being above Upper Limit

From Murdoch Barnes Tables (page 13) $P(Z \geq 3) = 0.00135$
Probability of point being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore
Conclusion: Probability of point being outside the 3 Sigma limits is
+ =0.00270 (i.e. 0.27
Question 1(b)
Upper Limit ; LowerLimit ;
Standardisation Apply the standardisation formula $Z = \frac{x-\mu}{\sigma}$ to both limits
Similary
Probability of point being above Upper Limit
From Murdoch Barnes Tables (page 13)
Probability of point being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore
Conclusion: Probability of point being outside the 3 Sigma limits is
+ =0.04550 (i.e. 4.55
Question 2C
Upper Limit ; 80.64 Mean Lower Limit ; 75.36 Standard Deviation
Standardisation Apply the standardisation formula to both limits
Similary
Probability of being above Upper Limit
From Murdoch Barnes Tables (page 13)
Probability of being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore
Conclusion: Probability of point being outside the specification limits
+ is equal to
+ =0.2584 (i.e. 26
Question 3A
Upper Limit ; 90 Mean Lower Limit ; 50 Standard Deviation
Standardisation Apply the standardisation formula $Z = \frac{x-\mu}{\sigma}$ to both limits
Similary
Probability of being above Upper Limit
From Murdoch Barnes Tables (page 13)
Probability of being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore
Conclusion: Probability of point being outside the specification limits is
+ is equal to
+ =0.01478 (i.e. 1.5
Question 3B

Upper Limit ; 90 Mean Lower Limit ; 50 Standard Deviation
Standardisation Apply the standardisation formula $Z = \frac{x-\mu}{\sigma}$ to both limits
Similary
Probability of being above Upper Limit
From Murdoch Barnes Tables (page 13)
Probability of being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore
Conclusion: Probability of point being outside the specification limits is
+ is equal to
+ =0.01099 (i.e. 1.1

0.8.1 Solutions 1

1. Assume that the number of weekly study hours for students at a certain university is approximately normally distributed with a mean of 22 and a standard deviation of 6.

   (a) Find the probability that a randomly chosen student studies less than 12 hours.

   (b) Estimate the percentage of students that study more than 37 hours.

   $X \sim (22, 6^2)$
   $P(X \leq 12)$
   $P(X \geq 37)$
   $Z_1 = \frac{12-22}{6} = \frac{-10}{6} = -1.66$
   $Z_2 = \frac{37-22}{6} = \frac{15}{6} = 2.5$

0.8.2 Worked Examples : Spring 2005

An important manufacturing process produces cylindrical component parts for the automotive industry. The diameter of these parts is normally distributed with a mean of 5 millimeters and a standard deviation of 0.1 millimeters.

(vi) What is the probability that a part will have a diameter greater than 5.24mm?

(vii) What is the probability a part will have diameter measuring between 4.78mm and 4.85mm?

(iii) The diameter of 99% of the parts is below what value?

Question 1(a)
Upper Limit ; $U = \mu + 3\sigma$ Lower Limit ; $L = \mu - 3\sigma$
Standardisation Apply the standardisation formula $Z = \frac{x-\mu}{\sigma}$ to both limits

$$Z_U = \frac{U-\mu}{\sigma} = \frac{(\mu+3\sigma)-\mu}{\sigma} = 3$$

Similary
$Z_l = -3$

Probability of point being above Upper Limit

From Murdoch Barnes Tables (page 13) $P(Z \geq 3) = 0.00135$
Probability of point being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore
Conclusion: Probability of point being outside the 3 Sigma limits is
+ =0.00270 (i.e. 0.27
Question 1(b)
Upper Limit ; LowerLimit ;
Standardisation Apply the standardisation formula to both limits
Similary

- Probability of point being above Upper Limit
- From Murdoch Barnes Tables (page 13)
- Probability of point being below Lower Limit
- To find we use the Property of Symmetry
- Property of Symmetry - for any value A
- Therefore

Conclusion: Probability of point being outside the 3 Sigma limits is
+ =0.04550 (i.e. 4.55%) Question 1(b)
Question 2C
Upper Limit ; 80.64 Mean Lower Limit ; 75.36 Standard Deviation
Standardisation Apply the standardisation formula to both limits
Similary
Probability of being above Upper Limit
From Murdoch Barnes Tables (page 13)
Probability of being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore

Conclusion

Probability of point being outside the specification limits
+ is equal to
+ =0.2584 (i.e. 26

Question 3A
Upper Limit ; 90 Mean Lower Limit ; 50 Standard Deviation
Standardisation Apply the standardisation formula to both limits
Similary
Probability of being above Upper Limit
From Murdoch Barnes Tables (page 13)

Probability of being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore

Conclusion

Probability of point being outside the specification limits is
+ is equal to
+ =0.01478 (i.e. 1.5
Question 3B
Upper Limit ; 90 Mean Lower Limit ; 50 Standard Deviation
Standardisation Apply the standardisation formula $Z = \frac{x-\mu}{\sigma}$ to both limits
Similary
Probability of being above Upper Limit
From Murdoch Barnes Tables (page 13)
Probability of being below Lower Limit
To find we use the Property of Symmetry
Property of Symmetry - for any value A
Therefore

Conclusion

Probability of point being outside the specification limits is
+ is equal to
+ =0.01099 (i.e. 1.1

Return on Investment Question

- The company needs to recover its investment in one year (i.e. make 50000).
- As each product sells for 2 dollars profit, the company needs to sell 25,000 units to recover its investment.
- we need to compute the probability of selling more than 25,000 units.

$$P(X \geq 25000)$$

- We are told the normal mean for demand $\mu = 20000$ and the normal standard deviation $\sigma = 4000$.
- The first step is to compute the ***z-score***

$$z = \frac{x-\mu}{\sigma} = \frac{25000-20000}{4000} = \frac{5000}{4000} = 1.25$$

0.9 Worked Example 8 - with Solutions (Tyres)

A tyre manufacturer claims that under normal driving conditions, the tread life of a certain tyre follows a normal distribution with mean 50,000 miles and standard deviation 5000 miles.

(i) If your tyres wear out at 45,000 miles, would you consider this unusual? Support your answer with an appropriate probability calculation using the normal curve. [10 marks]

(ii) If the manufacturer sells 100,000 of these tyres and warrants them to last at least 40,000 miles, about how many tyres will wear out before the warranty expires? [10 marks]

Part (i) Solution

- Test Value : 45,000km
- Mean 50,000 km
- Standard Deviation 5000 km

Find
Standardisation Apply the standardisation formula to test value
i.e. =
To find we use the Property of Symmetry
Property of Symmetry - for any value A
From Murdoch Barnes Tables (page 13)
Therefore = 0.1587
Complement Rule =1- for any given value A
=1- = 0.8413
Conclusion

- 15.87% of Tyres are expected to last less than 45,000km
- 84.13% of Tyres are expected to last longer than 45,000km

Part (i) Solution

- Lower Limit ; 40,000km Mean km
- Standard Deviation km

Find $P(X \leq 40000)$
Standardisation Apply the standardisation formula to limit
i.e. =
To find we use the Property of Symmetry
Property of Symmetry - for any value A
From Murdoch Barnes Tables (page 13)
Therefore $P(X \leq 40000) = 0.02275$
Conclusion 2.275% of Tyres are expected to last less than 40,000km
Of a Batch of 100,000 tyres, 2270 tyres will wear out before the warranty expires.