Continuous Uniform Distribution

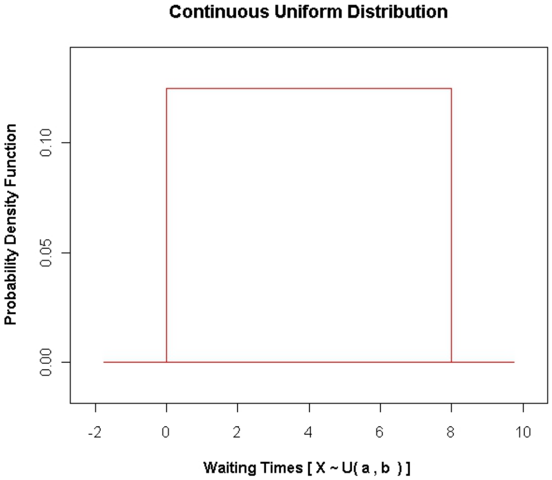
A random variable X is called a continuous uniform random variable over the interval (a,b) if it's probability density function is given by

$$f_X(x) = \frac{1}{b-a} \qquad \text{when } a \leq x \leq b \text{ (otherwise } f_X(x) = 0)$$

The corresponding cumulative density function is

$$F_x(x) = \frac{x-a}{b-a} \qquad \text{when } a \leq x \leq b$$

The Continuous Uniform Distribution



Continuous Uniform Distribution

- The continuous distribution is very simple to understand and implement, and is commonly used in computer applications (e.g. computer simulation).
- It is also known as the ‘Rectangle Distribution’ for obvious reasons.
- We specify the word “continuous” so as to distinguish it from it’s discrete equivalent: the discrete uniform distribution.
- Remark; the dice distribution is a discrete distribution with lower and upper limits 1 and 6 respectively.

Uniform Distribution Parameters

The continuous uniform distribution is characterized by the following parameters

- The lower limit a
- The upper limit b
- We denote a uniform random variable X as $X \sim U(a,b)$

It is not possible to have an outcome that is lower than a or larger than b.

$$P(X < a) = P(X > b) = 0$$

Interval Probability

- We wish to compute the probability of an outcome being within a range of values.
- We shall call this lower bound of this range L and the upper bound U.
- Necessarily L and U must be possible outcomes.
- The probability of X being between L and U is denoted $P(L \leq X \leq U)$.

$$P(L \leq X \leq U) = \frac{U - L}{b - a}$$

- (This equation is based on a definite integral).

Uniform Distribution: Cumulative Distribution

- For any value "c" between the minimum value a and the maximum value b, we can say
- $P(X \geq c)$
$$P(X \geq c) = \frac{b-c}{b-a}$$
here b is the upper bound while c is the lower bound
- $P(X \leq c)$
$$P(X \leq c) = \frac{c-a}{b-a}$$
here c is the upper bound while a is the lower bound.

Uniform Distribution: Mean and Variance

- The mean of the continuous uniform distribution, with parameters a and b is

$$E(X) = \frac{a+b}{2}$$

- The variance is computed as

$$V(X) = \frac{(b-a)^2}{12}$$

Uniform Distribution: Example

- Suppose there is a platform in a subway station in a large large city.
- Subway trains arrive **every three minutes** at this platform.
- What is the shortest possible time a passenger would have to wait for a train?
- What is the longest possible time a passenger will have to wait?

Uniform Distribution: Example

- What is the shortest possible time a passenger would have to wait for a train?
- If the passenger arrives just before the doors close, then the waiting time is zero.

$$a = 0 \text{ minutes}$$

Uniform Distribution: Example

- What is the longest possible time a passenger will have to wait?
- If the passenger arrives just after the doors close, and missing the train, then he or she will have to wait the full three minutes for the next one.

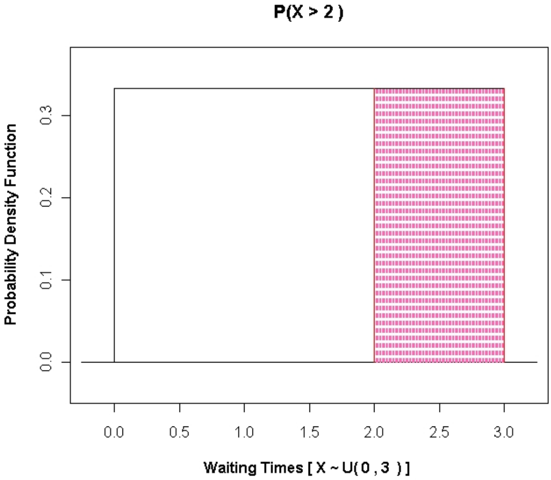
$$b = 3 \text{ minutes } \; = 180 \text{ seconds}$$

Uniform Distribution: Example

- What is the longest probability that he will have to wait longer than 2 minutes?

$$P(X \geq 2) = \frac{3-2}{3-0} = 1/3 = 0.33333$$

The Continuous Uniform Distribution

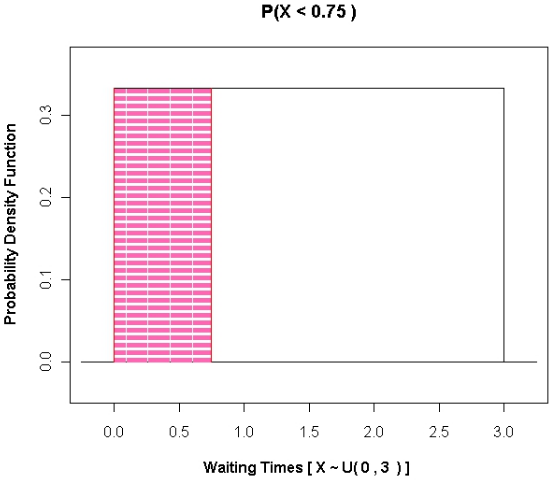


Uniform Distribution: Example

- What is the longest probability that he will have to wait less than than 45 seconds (i.e. 0.75 minutes)?

$$P(X \leq 0.75) = \frac{0.75 - 0}{3 - 0} = 0.75/3 = 0.250$$

The Continuous Uniform Distribution



Uniform Distribution: Expected Value

We are told that, for waiting times, the lower limit a is 0, and the upper limit b is 3 minutes.

The expected waiting time $\mathrm{E}[X]$ is computed as follows

$$\mathrm{E}[X] = \frac{b+a}{2} = \frac{3+0}{2} = 1.5 \text{ minutes}$$

Uniform Distribution: Variance

The variance of the continuous uniform distribution, denoted $\mathrm{V}[X]$, is computed using the following formula

$$\mathrm{V}[X] = \frac{(b-a)^2}{12}$$

For our previous example this is

$$\mathrm{V}[X] = \frac{(3-0)^2}{12} = \frac{3^2}{12} = \frac{9}{12} = 0.75$$