0.1 Important rules for Normal distribution

0.2 Complement Rule

For some value A, and for any continuous distribution X (including any normal distribution and the Z distribution) we can say.

$$P(X \leq x) = 1 - P(X \geq x)$$

This rule is very simple intuitive. If 70% of the population is below a certain height, that means that 30% of the population is above that height. Having said that, it is very powerful in solving normal probability problems.

Complement rule Common to all continuous random variables

$$P(Z \geq k) = 1 - P(Z \leq k)$$

Similarly

$$P(X \leq k) = 1 - P(X \geq k)$$

$$P(Z \leq 1.28) = 1 - P(Z \geq 1.28) = 1 - 0.1003 = 0.8997$$

Any normal distribution problem can be solved with some combination of the following rules.

This rule applies to all continuous probability distributions, i.e. the normal distribution, the exponential distribution and the uniform distribution. (Common to all continuous random variables)

Example

$$P(Z \leq 1.96) = 1 - P(Z \geq 1.96) = 1 - 0.0250 = 0.9750$$

- $P(Z \leq 1.27) = 1 - P(Z \geq 1.27)$
- $P(Z \geq 1.27) = 1 - P(Z \leq 1.27)$
- The Complement rule (Common to all continuous random variables)

  $$P(Z \geq k) = 1 - P(Z \leq k)$$

  Similarly

  $$P(X \geq k) = 1 - P(X \leq k)$$

0.3 Symmetry Rule

For the standard normal (Z) distribution only, where

$$Z \sim \mathcal{N}(0, 1^2)$$

and z and $-z$ are both values drawn from that distribution, we can say

$$P(Z \leq -z) = P(Z \geq z)$$

or conversely

$$P(Z \geq -z) = P(Z \leq z)$$

Important - the symmetry rule can be applied to z-values.

- This rule is based on the property of symmetry mentioned previously.
- Only the probabilities corresponding to values between 0 and 4 are tabulated in Murdoch Barnes.
- If we have a negative value of k, we can use the symmetry rule.

$$P(Z \leq -k) = P(Z \geq k)$$

by extension, we can say

$$P(Z \geq -k) = P(Z \leq k)$$

- **Complement rule** is common to all continuous random variables

$$P(X \geq k) = 1 - P(X \leq k)$$

0.4 Symmetry Rules

- This rule is based on the property of symmetry mentioned previously.
- Only the probabilities corresponding to values between 0 and 4 are tabulated in Murdoch Barnes.
- If we have a negative value of k, we can use the symmetry rule.

$$P(Z \leq -k) = P(Z \geq k)$$

by extension, we can say

$$P(Z \geq -k) = P(Z \leq k)$$

- $P(Z \leq -1.27) = P(Z \geq 1.27)$
- $P(Z \geq -1.27) = P(Z \leq 1.27)$

Example
Find $P(Z \leq -1.8)$
Solution:
$P(Z \leq -1.80) = P(Z \geq 1.80) = 0.0359$

0.5 Complement and Symmetry Rules

Any normal distribution problem can be solved (partially at least) with some combination of the following rules.

0.5.1 Z Scores: Example 1

Find $P(Z \geq -1.28)$
Solution

- Remark
$$P(Z \leq 1.28) = 1 - P(Z \geq 1.28) = 1 - 0.1003 = 0.8997$$
- Using the symmetry rule
$$P(Z \geq -1.28) = P(Z \leq 1.28)$$
- Using the complement rule
$$P(Z \geq -1.28) = 1 - P(Z \geq 1.28)$$
$$P(Z \geq -1.28) = 1 - 0.1003 = 0.8997$$

0.6 Interval Rules

- We are often interested in the probability of being inside an interval, with lower bound L and upper bound U.
- It is often easier to compute the probability of the complement event, being outside the interval.
$$P(\text{Inside}) = 1 - P(\text{Outside})$$
- The probability of being inside this interval is the **complement** of being outside the interval.
- The event of being outside the interval is the union of two disjoint events.
  - The probability of X being too low for the interval (i.e. less than the interval minimum L)
  $$P(X \leq L)$$
  - The probability of X being too high for the interval (i.e. less than the interval maximum U)
  $$P(X \geq U)$$
- Being outside the interval is the conjunction of being too low and too high.
$$P(\text{Outside}) = P(\text{Too Low}) + P(\text{Too High})$$
- Therefore we can say
$$P(\text{Inside}) = 1 - [P(\text{Too Low}) + P(\text{Too High})]$$
- $P(\text{Too Low}) = P(X \leq L)$
- $P(\text{Too High}) = P(X \geq U)$

Suppose we have an interval for the random variable X defined by the

- the lower bound L
- the upper bound U

$$L \leq X \leq U$$

$$P(U \leq X \leq L) = 1 - (P(X \leq L) + P(X \geq U))$$

The Interval rule is defined.

$$P(L \leq Z \leq U) = P(Z \geq L) - P(Z \geq U) \tag{1.5c}$$

where L and U are the lower and upper bounds of the interval

0.6.1 Example

Find the probability of a "z" random variable being between -1.8 and 1.96? i.e. Compute $P(-1.8 \leq Z \leq 1.96)$
Solution

- Consider the complement event of being in this interval: a combination of being too low or too high.
- The probability of being too low for this interval is $P(Z \leq -1.80) = 0.0359$ (check)
- The probability of being too high for this interval is $P(Z \geq 1.96) = 0.0250$ (check)
- Therefore the probability of being **outside** the interval is 0.0359 + 0.0250 = 0.0609.
- Therefore the probability of being **inside** the interval is 1- 0.0609 = 0.9391 $P(-1.8 \leq Z \leq 1.96) = 0.9391$
- **Intervals**:
  $$P(L \leq Z \leq U) = 1 - [P(Z \leq L) + P(Z \geq U)]$$
  where L and U are the lower and upper bounds of an interval.
- Probability of having a value too low for the interval : $P(Z \leq L)$
- Probability of having a value too high for the interval : $P(Z \geq U)$

0.7 Example

Given that the mean $\mu = 100$ and that the standard deviation $\sigma = 2.5$, what is the "z-value" for normal random variable x = 106?

$$z = \frac{x - \mu}{\sigma} = \frac{106 - 100}{2.5} = \frac{6}{2.5} = 2.40$$

Relationship between "x value" and "z value"
[VERY IMPORTANT]

$$If z = \frac{x - \mu}{\sigma}$$

(z and x are some values)
then we can say

$$P(X \geq x) = \Pr(Z \geq z)$$

or equivalently

$$P(X \leq x) = \Pr(Z \leq z)$$

From previous example

$$P(X \geq 106) = \Pr(Z \geq 2.40)$$

From Murdoch Barnes Table 3,

$$\P(Z \geq 2.40) = 0.00820$$

Therefore

$$P(X \geq 106) = 0.00820$$

Find the probability of a "z" random variable being between -1.8 and 1.96? i.e. Compute $P(-1.8 \leq Z \leq 1.96)$ Solution

- Consider the complement event of being in this interval: a combination of being too low or two high.
- The probability of being too low for this interval is $P(Z \leq -1.80) = 0.0359$ (from before)
- The probability of being too high for this interval is $P(Z \geq 1.96) = 0.0250$ (from before)
- Therefore the probability of being **outside** the interval is 0.0359 + 0.0250 = 0.0609.
- Therefore the probability of being **inside** the interval is 1- 0.0609 = 0.9391 $P(-1.8 \leq Z \leq 1.96) = 0.9391$

Example Find the probability of a "z" random variable greater than (or equal to) -1.8? Find $P(Z \geq -1.8)$ Solution
(From a previous question, $P(Z \leq -1.8) = 0.0359$)

$$P(Z \geq -1.8) = 1 - P(Z \leq -1.8)$$

$$P(Z \geq -1.8) = 1 - 0.0359 = 0.9641$$

0.8 Complement and Symmetry Rules

For a normally distributed random variable with mean $\mu = 1000$ and standard deviation $\sigma = 100$, compute $P(X \geq 873)$.

- First, find the Z-value using the standardization formula.

$$z_{873} = \frac{x_o - \mu}{\sigma} = \frac{873 - 1000}{100} = \frac{-127}{100} = -1.27$$

- We can say $P(X \geq 873) = P(Z \geq -1.27)$.
- Use complement rule and symmetry rule to evaluate $P(Z \geq -1.27)$.
- $P(Z \geq -1.27) = P(Z \leq 1.27) = 1 - P(Z \geq 1.27) =$ 1 - 0.1020 = 0.8980.

0.9 Summary

- **Complement Rule**
  For some value A, and for any continuous distribution X (including any normal distribution and the Z distribution) we can say.
  $$P(X \leq a) = 1 - P(X \geq A)$$
- **Symmetry Rule**
  For the standard normal (Z) distribution only, we can say
  $$P(Z \leq -A) = P(Z \geq A)$$
  or conversely
  $$P(Z \geq -A) = P(Z \leq A)$$

0.10 Symmetric intervals

$$P(-A \leq Z \leq A) = 1 - 2 \times P(Z \geq A) \tag{1}$$

$P(-1.96 \leq Z \leq 1.96) = 1 - 2 \times P(Z \geq 1.96) = 1 - (2 \times 0.025) = 0.95$

0.11 Normal Distributions

$$Z_o = \frac{X_o - \mu}{\sigma} \tag{2}$$

$$P(Z \geq Z_o) = P(X \geq X_o) \tag{3}$$

0.11.1 Using Murdoch Barnes tables 3

- $P(Z \geq 1.64) = 0.505$
- $P(Z \geq 1.65) = 0.495$

- $P(Z \geq 1.645)$ is approximately the average value of $P(Z \geq 1.64)$ and $P(Z \geq 1.65)$.
- $P(Z \geq 1.645) = (0.0495 + 0.0505)/2 = 0.0500$. (i.e. 5%)

0.12 Symmetric Intervals

$$P(-A \leq Z \leq A) = 1 - 2 \times P(Z \geq A) \tag{4}$$

$P(-1.96 \leq Z \leq 1.96) = 1 - 2 \times P(Z \geq 1.96) = 1 - (2 \times 0.025) = 0.95$

$$P(-Z_o \leq Z \leq Z_o) = 1 - 2P(Z \geq Z_o) \tag{1.6}$$

0.12.1 Solving using the Z distribution

When we have a normal distribution with any mean μ and any standard deviation σ , we answer probability questions about the distribution by first converting all values to corresponding values of the standard normal ("z") distribution. The formula used to convert any random variable "X" (with mean μ and standard deviation σ specified) to the standard normal ("z") distribution is given as follows.

$$Z_o = \frac{X_o - \mu}{\sigma}$$

Z is the standard normal random variable with a mean of zero and a standard deviation of 1. It can be thought of as a measure of how many standard deviations that a value "x" is from mean μ .
Remarks

- A value of x equal to mean μ results in a z -value of 0

$$z = \frac{\mu - \mu}{\sigma} = \frac{0}{\sigma} = 0$$

- Thus we can see that a value of "x" corresponding to its mean μ corresponds to a z-value at its mean , which is 0.
- A value of "x" that is one standard deviation above its mean (i.e. $x = \mu + \sigma$), we see that the corresponding z value is 1.
- Thus a value of x that is one standard deviation away from it's mean yields a z-value of 1.

Solution
Example Find the probability of a "z" random variable greater than (or equal to) -1.8? Find
Solution

0.13 Normal Distribution : Solving problems

Recap:

- We must know the normal mean μ and the normal standard deviation σ.
- The normal random variable is $X \sim \mathrm{N}(\mu, \sigma^2)$.
- (If we don't, we usually have to determine them, given the information in the question.)
- The standard normal random variable is $Z \sim \mathrm{N}(0, 1^2)$.
- The standard normal distribution is well described in Murdoch Barnes Table 3, which tabulates $P(Z \geq z_o)$ for a range of Z values.
- For the given value x_o from the variable X, we compute the corresponding z-score z_o.

$$z_o = \frac{x_o - \mu}{\sigma}$$

- When z_o corresponds to x_o, the following identity applies:

$$P(X \geq x_o) = P(Z \geq z_o)$$

- Alternatively $P(X \leq x_o) = P(Z \leq z_o)$