0.1 Poisson Formulae

A discrete random variable X is said to follow a Poisson distribution with parameter m, written $X \sim \text{Po}(m)$.

- Here the Poisson Mean is the expected number of occurence per unit period. It is usually denoted as either m or λ

Given the mean number of successes (m) that occur in a specified region, we can compute the Poisson probability based on the following formula.

Poisson Formulae

The probability that there will be k occurrences in a **unit time period** is denoted $P(X = k)$, and is computed as as below. Remark: This is known as the probability density function.

$$P(X = k) = \frac{m^k e^{-m}}{k!}$$

where

- λ is the expected value of the mean number of occurrences in any interval. (We often call this the Poisson mean)
- e=2.71828284
- $k = 0, 1, 2, \ldots$
- $m > 0$.

Worked Example 1

Given that there is on average 2 occurrences per hour, what is the probability of no occurences in the next hour?
i.e. Compute $P(X = 0)$ given that $m = 2$

$$P(X = 0) = \frac{2^0 e^{-2}}{0!}$$

- $2^0 = 1$
- $0! = 1$

The equation reduces to

$$P(X = 0) = e^{-2} = 0.1353$$

What is the probability of one occurrences in the next hour?
i.e. Compute $P(X = 1)$ given that $m = 2$

$$P(X = 1) = \frac{2^1 e^{-2}}{1!}$$

- $2^1 = 2$
- $1! = 1$

The equation reduces to

$$P(X = 1) = 2 \times e^{-2} = 0.2706$$

0.1.1 Changing the Unit Time.

- The number of arrivals, X, in an interval of length t has a Poisson distribution with parameter $\mu = mt$.
- m is the expected number of arrivals in a unit time period.
- μ is the expected number of arrivals in a time period t, that is different from the unit time period.
- Put simply : if we change the time period in question, we adjust the Poisson mean accordingly.
  * If we double the length of the time period, we double the value of the Poisson mean.
  * If we halve the length of the time period, we halve the value of the Poisson mean
- If 10 occurrences are expected in 1 hour, then 5 are expected in 30 minutes. Likewise, 20 occurrences are expected in 2 hours, and so on.
- (Remark : we will not use μ in this context anymore).

- If the Poisson mean $m = 8$ for a 15 minute period, what is the Poisson mean for a unit period of one hour?
- The answer is 32.

Worked Example 2

Given that there is on average 4 occurrences per day, what is the probability of one occurrences in a given day? i.e. Compute $P(X = 1)$ given that $m = 4$

$$P(X = 1) = \frac{4^1 e^{-4}}{1!}$$

The equation reduces to

$$P(X = 1) = 4 \times e^{-4} = 0.07326$$

What is the probability of one occurrences in a six hour period ?
i.e. Compute $P(X = 1)$ given that $m = 1$

$$P(X = 1) = \frac{1^1 e^{-1}}{1!}$$

0.1.2 Question on Poisson Distribution

Past experience shows that there, on average, are 2 traffic accidents on a particular stretch of road every week. What is the probability of:

- Four accidents during a randomly selected week?
- No accidents during a randomly selected week?
- The Poisson mean $\lambda = 2$ per week.
- (Unit period is 1 week for both questions)
- We use this following formula
$$P(X = k) = \frac{e^{-\lambda} \times \lambda^k}{k!}$$
- Using our value for the Poisson mean
$$P(X = k) = \frac{e^{-2} \times 2^k}{k!}.$$
- Probability of four accidents during a randomly selected week : $P(X = 4)$.
$$P(X = 4) = \frac{e^{-2} \times 2^4}{4!}$$
- Probability of no accidents during a randomly selected week : $P(X = 0)$.
$$P(X = 0) = \frac{e^{-2} \times 2^0}{0!}$$