Contents

1 LME Introduction

Henderson [1984a] shows that the solution to the mixed model equations

$$\begin{pmatrix} X'R^{-1}X & X'R^{-1}Z \\ Z'R^{-1}X & Z'R^{-1}Z + G^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} X'R^{-1}y \\ Z'R^{-1}y \end{pmatrix} \tag{1}$$

gives a solution for which the estimate $\boldsymbol{b}$ of the fixed effects is equal to that obtained by generalized least squares.

$$\hat{\beta} = (\boldsymbol{X'V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X'V}^{-1}\boldsymbol{y} \tag{2}$$

? showed that the BLUP of $\boldsymbol{b}$ is

$$\hat{\boldsymbol{b}} = (\boldsymbol{DZ'V})(\boldsymbol{y} - \boldsymbol{X}\hat{\beta}) \tag{3}$$

1.0.1 Henderson's equations

Because of the dimensionality of V (i.e. $n \times n$) computing the inverse of V can be difficult. As a way around the this problem **?????** offered a more simpler approach of jointly estimating $\hat{\beta}$ and $\hat{b}$. **?** made the (ad-hoc) distributional assumptions $y|b \sim \mathrm{N}(X\beta + Zb, \Sigma)$ and $b \sim \mathrm{N}(0, D)$, and proceeded to maximize the joint density of y and b

$$\left|\begin{pmatrix} D & 0 \\ 0 & \Sigma \end{pmatrix}\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\begin{pmatrix} b \\ y - X\beta - Zb \end{pmatrix}'\begin{pmatrix} D & 0 \\ 0 & \Sigma \end{pmatrix}^{-1}\begin{pmatrix} b \\ y - X\beta - Zb \end{pmatrix}\right\}, \tag{4}$$

with respect to β and b, which ultimately requires minimizing the criterion

$$(y - X\beta - Zb)'\Sigma^{-1}(y - X\beta - Zb) + b'D^{-1}b. \tag{5}$$

This leads to the mixed model equations

$$\begin{pmatrix} X'\Sigma^{-1}X & X'\Sigma^{-1}Z \\ Z'\Sigma^{-1}X & X'\Sigma^{-1}X + D^{-1} \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} = \begin{pmatrix} X'\Sigma^{-1}y \\ Z'\Sigma^{-1}y \end{pmatrix}. \tag{6}$$

Using these equations, obtaining the estimates requires the inversion of a matrix of dimension $p + q \times p + q$, considerably smaller in size than V. **?** shows that these mixed model equations do not depend on normality and that $\hat{\beta}$ and $\hat{b}$ are the BLUE and BLUP under general conditions, provided D and Σ are known.

? points out that although **?** initially referred to the estimates $\hat{\beta}$ and $\hat{b}$ from (12) as "joint maximum likelihood estimates", **?** later advised that these estimates should not be referred to as "maximum likelihood" as the function being maximized in (11) is a joint density rather than a likelihood function. **?** remarks that it is clear that Henderson used joint estimation for computational purposes, without recognizing the theoretical implications.

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$$(y - X\beta - Zb)'\Sigma^{-1}(y - X\beta - Zb) + b'D^{-1}b. \tag{8}$$

This leads to the solutions

$$\begin{pmatrix} X'\Sigma^{-1}X & X'\Sigma^{-1}Z \\ Z'\Sigma^{-1}X & X'\Sigma^{-1}X + D^{-1} \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} = \begin{pmatrix} X'\Sigma^{-1}y \\ Z'\Sigma^{-1}y \end{pmatrix}. \tag{9}$$

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