Question 6

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix}$$

- Last Class - We found the Eigenvalues of A
- Characteristic Equation $(\lambda - 1)(\lambda + 1)(\lambda + 2) = 0$
- Eigenvalues $\lambda = \{-1, -2, 1\}$
- To find Eigenspaces
- (Tutorial 6 Question 5 is a good example for this question)

$$A = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 2/3 & 1 \\ 0 & -2/3 & -3 \end{pmatrix}$$

- Find the Eigenspaces : Solve $(\lambda I - A)e = 0$
- $(\lambda I - A)$ is computed below. First $\lambda = -2$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

- The Eigenspace weightings are the solutions to the following. (you can let $e_1 = 1$, then re-weight if necessary)

$$\begin{pmatrix} -3 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- The solution is $e_1 = 1$, $e_2 = -3$ $e_3 = 6$
- For $\lambda = -1$ and $\lambda = 1$. The matrices are

$$\begin{pmatrix} -3 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- The P matrix is therefore

$$\begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & 0 \\ 6 & -2 & 0 \end{pmatrix}$$

- Normalising the columns

$$\begin{pmatrix} 1/\sqrt{46} & -1/\sqrt{9} & 1/\sqrt{1} \\ -3/\sqrt{46} & 2/\sqrt{9} & 0/\sqrt{1} \\ 6/\sqrt{46} & -2/\sqrt{9} & 0/\sqrt{1} \end{pmatrix} = \begin{pmatrix} 0.147 & -0.333 & 1 \\ -0.442 & 0.666 & 0 \\ 0.885 & -0.666 & 0 \end{pmatrix}$$

- Normalization : divide each element by magnitude of column vector ($\sqrt{1^2 + (-3)^2 + 6^2} = \sqrt{46}$)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Boundary conditions

$$X(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Linear Transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \left(\begin{pmatrix} 1/\sqrt{46} & -1/3 & 1 \\ -3/\sqrt{46} & 2/3 & 0 \\ 6/\sqrt{46} & -2/3 & 0 \end{pmatrix} \right) \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix}$$

- Let $\|A\|$ represent the norm of A.
- The condition number is defined as $\|A\| \times \|A^{-1}\|$
- Condition number is the measure of how well conditioned a matrix is. The smaller the value of $\kappa(A)$, the more accurate the solution of Ax=b
- Inverse of A found yesterday, using Elementary Row Operations.
- Can quickly compute the inverse of L and U by the same method.
- Recall $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -5 & 4 \\ 2 & -2 & 11 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} -47 & 31 & -7 \\ -14 & 9 & -2 \\ 6 & -4 & 1 \end{pmatrix}$$

- $\|A\| = \text{Max}\{(1 + |-3| + 1, 2 + |-5| + 4, 2 + |-2| + 11\} = \max\{5, 11, 15\} = 15$
- $\|A^-1\| = \text{Max}\{(|-47| + 31 + |-7|, |-14| + 9 + |-2|, 6 + |-4| + 1\} = \max\{85, 25, 11\}$
- $\kappa(A) = 15 \times 85 = 1275$

Question 4

$$a_o + a_1x + a_2x^2 = \alpha_1(1+2x) + \alpha_2(2x+x^2) + \alpha_3(4+2x-3X^2)$$

$$= (\alpha_1 + 4\alpha_3) + (2\alpha_1 + 2\alpha_2 + 2\alpha_3)x + (\alpha_2 - 3\alpha_3)x^2$$

- $a_o = \alpha_1 + 4\alpha_3$
- $a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3$
- $a_2 = \alpha_2 - 3\alpha_3$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 2 & 2 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 & 4 \\ 2 & 2 & 2 \\ 0 & 1 & -3 \end{vmatrix} = 0?\text{Yes}$$