Solving ODEs using Laplace transforms

Solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 1 + t$$

given $y'(0) = 0$ and $y(0) = 0$

REMARK: THESE Boundary Conditions may be non-zero.

$$y''(t) + 2y'(t) + y = 1 + t.$$

LHS

- $L(y'') = s^2 Y(s) - s y'(0) - y(0)$
  $= s^2 Y(s)$
- $L(y') = s Y(s) - y(0)$
  $= s Y(s)$ $(\times 2)$
- $L(y) = Y(s)$

LHS :

$$(s^2+2s+1)\,Y(s).$$

RHS :

$$\mathcal{L}[1+t] = \frac{1}{s} + \frac{1}{s^2}.$$

$$= \frac{s+1}{s^2}.$$

$$Y(s)\,(s^2+2s+1) = \frac{s+1}{s^2}$$

$$Y(s) = \frac{(s+1)}{(s+1)^2(s^2)}$$

$$= \frac{1}{(s+1)(s^2)}$$

$$Y(s) = \frac{1}{(s+1)(s^2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2}$$

$$= \frac{As^2 + Bs^2 + Bs + Cs + C}{(s+1)(s^2)}$$

$$= \frac{(A+B)s^2 + (B+C)s + C}{(s+1)(s^2)}$$

$\therefore$ $C = 1$

$B + C = 0$ $\quad\therefore$ $B = -1$

$A + B = 0$ $\quad\therefore$ $A = 1$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}$$

$$y(t) = e^{-t} - 1 + t$$

$$= \underline{\underline{e^{-t} + t - 1}} \qquad t \geq 0.$$