Contents

Chapter 1

Method Comparison Studies

1.1 What is a method comparison study?

The problem of assessing the agreement between two or more methods of measurement is ubiquitous in scientific research, and is commonly referred to as a 'method comparison study'. **?** states that the purpose of comparing two measurements "of a continuous biological variable" is to uncover systematic differences, not to point to similarities". The need to compare the results of two different measurement techniques is common in medical statistics. Published examples of method comparison studies can be found in disciplines as diverse as pharmacology (**?**), anaesthesia (**?**), and cardiac imaging methods (**?**).

To illustrate the characteristics of a typical method comparison study consider the data in Table I (**?**). In each of twelve experimental trials, a single round of ammunition was fired from a 155mm gun and its velocity was measured simultaneously (and independently) by three chronographs devices, identified here by the labels 'Fotobalk', 'Counter' and 'Terma'.

An important aspect of the these data is that all three methods of measurement are assumed to have an attended measurement error, and the velocities reported in Table 1.1 can not be assumed to be 'true values' in any absolute sense.

Round	Fotobalk [F]	Counter [C]	Terma [T]
1	793.8	794.6	793.2
2	793.1	793.9	793.3
3	792.4	793.2	792.6
4	794.0	794.0	793.8
5	791.4	792.2	791.6
6	792.4	793.1	791.6
7	791.7	792.4	791.6
8	792.3	792.8	792.4
9	789.6	790.2	788.5
10	794.4	795.0	794.7
11	790.9	791.6	791.3
12	793.5	793.8	793.5

Table 1.1.1: Velocity measurement from the three chronographs (Grubbs 1973).

A method of measurement should ideally be both accurate and precise. **?** describes agreement as being a broader term that contains both of those qualities. An accurate measurement method will give results close to the unknown 'true value'. The precision of a method is indicated by how tightly measurements obtained under identical conditions are distributed around their mean measurement value. A precise and accurate method will yield results consistently close to the true value. Of course a method may be accurate, but not precise, if the average of its measurements is close to the true value, but those measurements are highly dispersed. Conversely a method that is not accurate may be quite precise, as it consistently indicates the same level of inaccuracy. The tendency of a method of measurement to consistently give results above or below the true value is a source of systematic bias. The smaller the systematic bias, the greater the accuracy of the method.

In the context of the agreement of two methods, there is also a tendency of one

measurement method to consistently give results above or below the other method. Lack of agreement is a consequence of the existence of 'inter-method bias'. For two methods to be considered in good agreement, the inter-method bias should be in the region of zero. A simple estimation of the inter-method bias can be calculated using the differences of the paired measurements. The data in Table 1.2 are a good example of possible inter-method bias; the 'Fotobalk' consistently recording smaller velocities than the 'Counter' method. Consequently one would conclude that there is lack of agreement between the two methods.

The absence of inter-method bias by itself is not sufficient to establish whether two measurement methods agree. The two methods must also have equivalent levels of precision. Should one method yield results considerably more variable than those of the other, they can not be considered to be in agreement. With this in mind a methodology is required that allows an analyst to estimate the inter-method bias, and to compare the precision of both methods of measurement.

Round	Fotobalk (F)	Counter (C)	F-C
1	793.8	794.6	-0.8
2	793.1	793.9	-0.8
3	792.4	793.2	-0.8
4	794.0	794.0	0.0
5	791.4	792.2	-0.8
6	792.4	793.1	-0.7
7	791.7	792.4	-0.7
8	792.3	792.8	-0.5
9	789.6	790.2	-0.6
10	794.4	795.0	-0.6
11	790.9	791.6	-0.7
12	793.5	793.8	-0.3

Table 1.1.2: Difference between Fotobalk and Counter measurements.

An important aspect of the these data is that all three methods of measurement are assumed to have an attended measurement error, and the velocities reported in Table 1.1 can not be assumed to be 'true values' in any absolute sense.

A method of measurement should ideally be both accurate and precise. **?** describes agreement as being a broader term that contains both of those qualities. An accurate measurement method will give results close to the unknown 'true value'. The precision of a method is indicated by how tightly measurements obtained under identical conditions are distributed around their mean measurement value. A precise and accurate method will yield results consistently close to the true value. Of course a method may be accurate, but not precise, if the average of its measurements is close to the true value, but those measurements are highly dispersed. Conversely a method that is not accurate may be quite precise, as it consistently indicates the same level of inaccuracy. The tendency of a method of measurement to consistently give results above or below the true value is a source of systematic bias. The smaller the systematic bias, the greater the accuracy of the method.

In the context of the agreement of two methods, there is also a tendency of one measurement method to consistently give results above or below the other method. Lack of agreement is a consequence of the existence of 'inter-method bias'. For two methods to be considered in good agreement, the inter-method bias should be in the region of zero. A simple estimation of the inter-method bias can be calculated using the differences of the paired measurements. The data in Table 1.2 are a good example of possible inter-method bias; the 'Fotobalk' consistently recording smaller velocities than the 'Counter' method. Consequently one would conclude that there is lack of agreement between the two methods.

1.2 Agreement

? defined perfect agreement as the case where all of the pairs of rater data lie along the line of equality, where the line of equality is defined as the 45 degree line passing through the origin , or or X=Y on a XY plane.

To carry their idea a step further, we define a specific numerical measure of agreement as twice the expected squared perpendicular distance of the pair of random variables (X_1, X_2) to the line of equality or agreement in the (X_1, X_2)-plane, that is, $E(X_1 - X_2)2$, where X1 and X2 denote the continuous measurements of rater 1 and rater 2, respectively.

Obviously, other L_p norms may be considered for the purpose of numerically measuring agreement and warrant future consideration. Note that we will use the term rater and measuring device interchangeably throughout this article.

Agreement is the extent to which the measure of the variable of interest, under a constant set of experimental conditions, yields the same result on repeated trials (Sanchez et al). The more consistent the results, the more reliable the measuring procedure.

In particular, in medicine, new methods or devices that are cheaper, easier to use, or less invasive, are routinely developed. Agreement between a new method and either a traditional reference or gold standard must be evaluated before the new one is put into practice. Various methodologies have been proposed for this purpose in recent years.

1.3 Equivalence and Interchangeability

Limits of agreement are intended to analyse equivalence. How this is assessed is the considered judgement of the practitioner. In **?** an example of good agreement is cited. For two methods of measuring 'oxygen saturation', the limits of agreement

are calculated as (-2.0,2.8).A practitioner would ostensibly find this to be sufficiently narrow.

If the limits of agreement are not clinically important, which is to say that the differences tend not to be substantial, the two methods may be used interchangeably. **?** takes issue with the notion of 'equivalence', remarking that while agreement indicated equivalence, equivalence does not reflect agreement.

1.4   Purpose of Method Comparison Studies

? provides a review of many descriptions of the purpose of Method Comparison studies, several of which are reproduced here.

> "The question being answered is not always clear, but is usually epxressed as an attempt to quantify the agreement between two methods" (**?**).

> "Some lack of agreement between different methods of measurement is inevitable. What matters is the amount by which they disagree. We want to know by how much the new method is likely to differ from the old, so that it is not enough to cause problems in the mathematical interpretation we can preplace the old method by the new, or even use the two interchangeably" (**?**).

> "It often happens that the same physical and chemical property can be measured in different ways. For example, one can determine For example, one can determine sodium in serum by flame atomic emission spectroscopy or by isotope dilution mass spectroscopy. The question arises as to which method is better" (Mandel, 1991).

> "In areas of inter-laboratory quality control, method comparisons, assay validations and individual bio-equivalence, etc, the agreement between observations and target (reference) values is of interest" (**?**).

"The purpose of comparing two methods of measurement of a continuous biological variable is to uncover systematic differences, not to point to similarities'" (**?**).

"In the pharmaceutical industry, measurement methods that measure the quantity of prdocuts are regulated. The FDA (U.S. Food and Drug Administration) requires that the manufacturer show equivalency prior to approving the new or alternatice method in quality control" (Tan & Inglewicz, 1999).

While several major commonalities are present in each definitions, there is a different emphasis for each, which will inevitably give rise to confusion. **?** seems to endorse a simple phrasing of the research question that is proposed by **?**, i.e. *"do the two methods of measurement agree sufficiently closely?"* with **?** expressing the view that other considerations (for example, the "equivalence" of two methods) to be treated as separate research questions. As such, we will revert to other research questions, such as "equivalence of methods" later, focussing on agreement and repeatability of methods.

1.5   Discussion on Method Comparison Studies

The need to compare the results of two different measurement techniques is common in medical statistics.

In particular, in medicine, new methods or devices that are cheaper, easier to use, or less invasive, are routinely developed. Agreement between a new method and a traditional reference or gold standard must be evaluated before the new one is put into practice. Various methodologies have been proposed for this purpose in recent years.

1.6 Detection of Outliers

Indications on how to deal with outliers in Bland Altman plots

We wish to determine how outliers should be treated in a Bland Altman Plot

In their 1983 paper they merely state that the plot can be used to 'spot outliers'.

In their 1986 paper, Bland and Altman give an example of an outlier. They state that it could be omitted in practice, but make no further comments on the matter.

In Bland and Altmans 1999 paper, we get the clearest indication of what Bland and Altman suggest on how to react to the presence of outliers. Their recommendation is to recalculate the limits without them, in order to test the difference with the calculation where outliers are retained.

BAExample - The span has reduced from 77 to 59 mmHg, a noticeable but not particularly large reduction.

However, they do not recommend removing outliers. Furthermore, they say:

We usually find that this method of analysis is not too sensitive to one or two large outlying differences.

We ask if this would be so in all cases. Given that the limits of agreement may or may not be disregarded, depending on their perceived suitability, we examine whether it would possible that the deletion of an outlier may lead to a calculation of limits of agreement that are usable in all cases?

Should an Outlying Observation be omitted from a data set? In general, this is not considered prudent.

Also, it may be required that the outliers are worthy of particular attention themselves.

Classifying outliers and recalculating We opted to examine this matter in more detail. The following points have to be considered

how to suitably identify an outlier (in a generalized sense)

Would a recalculation of the limits of agreement generally results in a compacted range between the upper and lower limits of agreement?

1.7 Normality of Case-wise Differences

The difference are assumed to be normally distributed, although the measurements themselves are not assumed to follow any distribution.Therefore the authors argue that the 95% of differences are expected to lie within these limits. This assumption is justified because variation between subjects has been removed, leaving only measurement error (**?**). There are formal methodologies to test whether this assumption holds.

The problem of assessing the agreement between two or more methods of measurement is ubiquitous in scientific research, and is commonly referred to as a 'method comparison study'. Published examples of method comparison studies can be found in disciplines as diverse as Pharmacology (**?**), Anaesthesia (**?**), and cardiac imaging methods (**?**).

To illustrate the characteristics of a typical method comparison study consider the data in Table I, taken from **?**. In each of twelve experimental trials a single round of ammunition was fired from a 155mm gun, and its velocity was measured simultaneously (and independently) by three chronographs devices, referred to here as 'Fotobalk', 'Counter' and 'Terma'.

1.8 Inferences on Bland-Altman estimates

?advises on how to calculate confidence intervals for the inter-method bias and limits of agreement. For the inter-method bias, the confidence interval is a simply that of a mean: $\bar{d} \pm t_{(0.5\alpha, n-1)} S_d / \sqrt{n}$. The confidence intervals and standard error for the limits of agreement follow from the variance of the limits of agreement, which is shown to be

$$\text{Var}(LoA) = (\frac{1}{n} + \frac{1.96^2}{2(n-1)})s_d^2.$$

If n is sufficiently large this can be following approximation can be used

$$\text{Var}(LoA) \approx 1.71^2 \frac{s_d^2}{n}.$$

Consequently the standard errors of both limits can be approximated as 1.71 times the standard error of the differences.

A 95% confidence interval can be determined, by means of the t distribution with $n-1$ degrees of freedom. However **?** comment that such calculations may be 'somewhat optimistic' on account of the associated assumptions not being realized.

1.9   Lack Of Agreement

Constant Bias

This is a form of systematic deviations estimated as the average difference between the test and the reference method

Proportional Bias

Two methods may agree on average, but they may exhibit differences over a range of measurements.

1.10   Statement of a Model

? presents a useful formulation for comparing two methods X and Y, in their measurement of item i, where the unknown 'true value' is τ_i. Other authors, such as **?**, present similar formulations of the same model, as well as modified models to account for multiple measurements by each methods on each item, known as replicate measurements.

$$X_i = \tau_i + \delta_i, \qquad \delta_i \sim \mathcal{N}(0, \sigma_\delta^2) \tag{1.1}$$

$$Y_i = \alpha + \beta\tau_i + \epsilon_i, \qquad \epsilon_i \sim \mathcal{N}(0, \sigma_\epsilon^2) \tag{1.2}$$

In some types of analysis, such as the conversion problems described by **?**, an estimate for the scaling factor β may also be sought. For the time being, we will restrict ourselves to problems where β is assumed to be 1.

$$X_i = \tau_i + \delta_i, \qquad \delta_i \sim \mathcal{N}(0, \sigma_\delta^2) \tag{1.3}$$

$$Y_i = \alpha + \beta\tau_i + \epsilon_i, \qquad \epsilon_i \sim \mathcal{N}(0, \sigma_\epsilon^2) \tag{1.4}$$

In this formulation, α represents the inter-method bias, and can be estimated as $E(X - Y)$. That is to say, a simple estimate of the inter-method bias is given by the differences between pairs of measurements. Table **??** is a good example of possible inter-method bias; the 'Fotobalk' consistently recording smaller velocities than the 'Counter' method. A cursory inspection of the table will indicate a systematic tendency for the Counter method to result in higher measurements than the Fotobalk method.

Chapter 2

Review of Current Methodologies:Bland-Altman Methodology

2.1 Bland-Altman methodology

The issue of whether two measurement methods comparable to the extent that they can be used interchangeably with sufficient accuracy is encountered frequently in scientific research. Historically comparison of two methods of measurement was carried out by use of paired sample $t-$test, correlation coefficients or simple linear regression. Simple linear regression is unsuitable for method comparison studies because of the required assumption that one variable is measured without error. In comparing two methods, both methods are assume to have attendant random error.

Statisticians Martin Bland and Douglas Altman recognized the inadequacies of these analyzes and articulated quite thoroughly the basis on which of which they are unsuitable for comparing two methods of measurement (**?**). Furthermore they proposed their simple methodology specifically constructed for method comparison studies. They acknowledge the opportunity to apply other valid, but complex, methodologies, but ar-

gue that a simple approach is preferable, especially when the results must be 'explained to non-statisticians'.

Notwithstanding previous remarks about linear regression, the first step recommended, which the authors argue should be mandatory, is construction of a simple scatter plot of the data. The line of equality should also be shown, as it is necessary to give the correct interpretation of how both methods compare. In the case of good agreement, the observations would be distributed closely along the line of equality. A scatter plot of the Grubbs data is shown in Figure 1.1. Visual inspection confirms the previous conclusion that there is an inter-method bias present, i.e. Fotobalk device has a tendency to record a lower velocity.

? notes that scatter plots were very seldom presented in the Annals of Clinical Biochemistry. This apparently results from the fact that the 'Instructions for Authors' dissuade the use of regression analysis, which conventionally is accompanied by a scatter plot.

2.2   Bland-Altman Approach

The issue of whether two measurement methods comparable to the extent that they can be used interchangeably with sufficient accuracy is encountered frequently in scientific research. Historically, comparison of two methods of measurement was carried out by use of paired sample $t-$test, correlation coefficients or simple linear regression. However, simple linear regression is unsuitable for method comparison studies due to the assumption that one variable is measured without error. In comparing two methods, both methods are assume to have attendant random error.

? highlighted the inadequacies of these approaches for comparing two methods of measurement, and proposed methodologies with this specific application in mind. Although the authors also acknowledge the opportunity to apply other, more complex, approaches, but argue that simpler approaches is preferable, especially when the results must be 'explained to non-statisticians'.

Notwithstanding previous remarks about linear regression, the first step recommended, which the authors argue should be mandatory, is the construction of a scatter plot of the data. Scatterplots can facilitate an initial judgement and helping to identify potential outliers, with the addition of the line of equality. In the case of good agreement, the observations would be distributed closely along this line. However, they are not useful for a thorough examination of the data. ? notes that data points will tend to cluster around the line of equality, obscuring interpretation.

A scatter plot of the Grubbs data is shown in Figure 1.1. Visual inspection confirms the previous conclusion that inter-method bias is present, i.e. the Fotobalk device has a tendency to record a lower velocity.

2.3   Bland-Altman plots for the Grubbs data

In the case of the Grubbs data the inter-method bias is -0.61 metres per second, and is indicated by the dashed line on Figure 1.2. By inspection of the plot, it is also possible
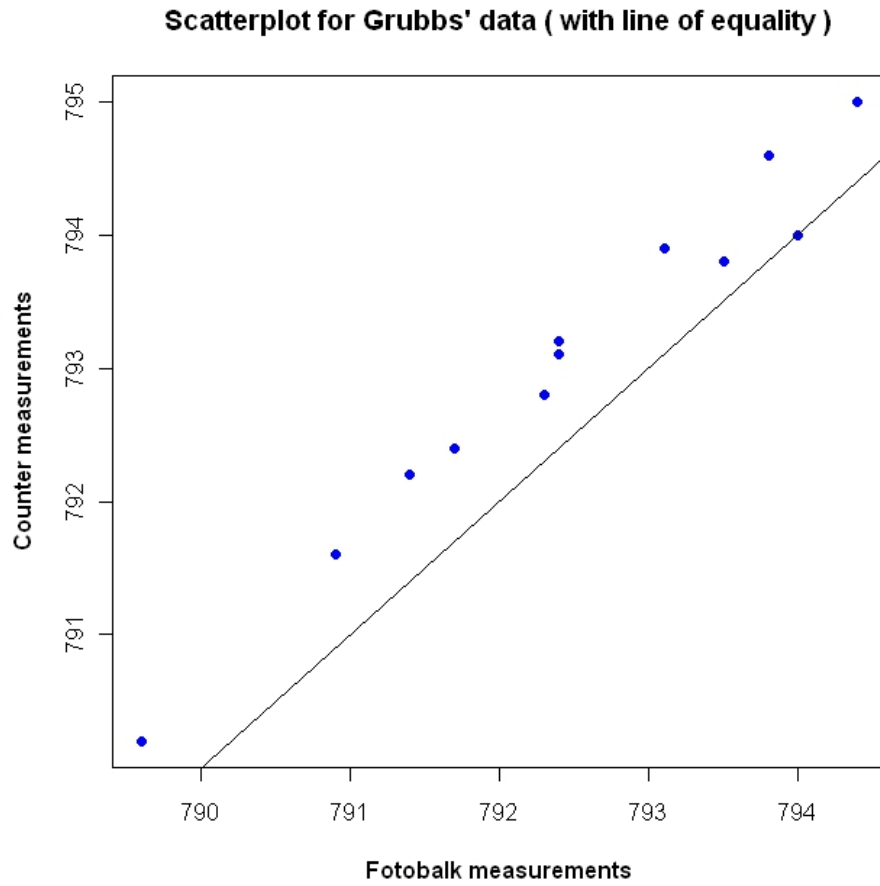
Figure 2.2.1: Scatter plot for Fotobalk and Counter methods.

to compare the precision of each method. Noticeably the differences tend to increase as the averages increase.

The Bland-Altman plot for comparing the 'Fotobalk' and 'Counter' methods, which shall henceforth be referred to as the 'F vs C' comparison, is depicted in Figure 1.2, using data from Table 1.3. The presence and magnitude of the inter-method bias is indicated by the dashed line.

Round	Fotobalk [F]	Counter [C]	Differences [F-C]	Averages [(F+C)/2]
1	793.8	794.6	-0.8	794.2
2	793.1	793.9	-0.8	793.5
3	792.4	793.2	-0.8	792.8
4	794.0	794.0	0.0	794.0
5	791.4	792.2	-0.8	791.8
6	792.4	793.1	-0.7	792.8
7	791.7	792.4	-0.7	792.0
8	792.3	792.8	-0.5	792.5
9	789.6	790.2	-0.6	789.9
10	794.4	795.0	-0.6	794.7
11	790.9	791.6	-0.7	791.2
12	793.5	793.8	-0.3	793.6

Table 2.3.1: Fotobalk and Counter methods: differences and averages.

Round	Fotobalk [F]	Terma [T]	Differences [F-T]	Averages [(F+T)/2]
1	793.8	793.2	0.6	793.5
2	793.1	793.3	-0.2	793.2
3	792.4	792.6	-0.2	792.5
4	794.0	793.8	0.2	793.9
5	791.4	791.6	-0.2	791.5
6	792.4	791.6	0.8	792.0
7	791.7	791.6	0.1	791.6
8	792.3	792.4	-0.1	792.3
9	789.6	788.5	1.1	789.0
10	794.4	794.7	-0.3	794.5
11	790.9	791.3	-0.4	791.1
12	793.5	793.5	0.0	793.5

Table 2.3.2: Fotobalk and Terma methods: differences and averages.

In Figure 1.3 Bland-Altman plots for the 'F vs C' and 'F vs T' comparisons are shown, where 'F vs T' refers to the comparison of the 'Fotobalk' and 'Terma' methods. Usage of the Bland-Altman plot can be demonstrate in the contrast between these comparisons. By inspection, there exists a larger inter-method bias in the 'F vs C'

Figure 2.3.2: Bland-Altman plot For Fotobalk and Counter methods.

comparison than in the 'F vs T' comparison. Conversely there appears to be less precision in 'F vs T' comparison, as indicated by the greater dispersion of covariates.
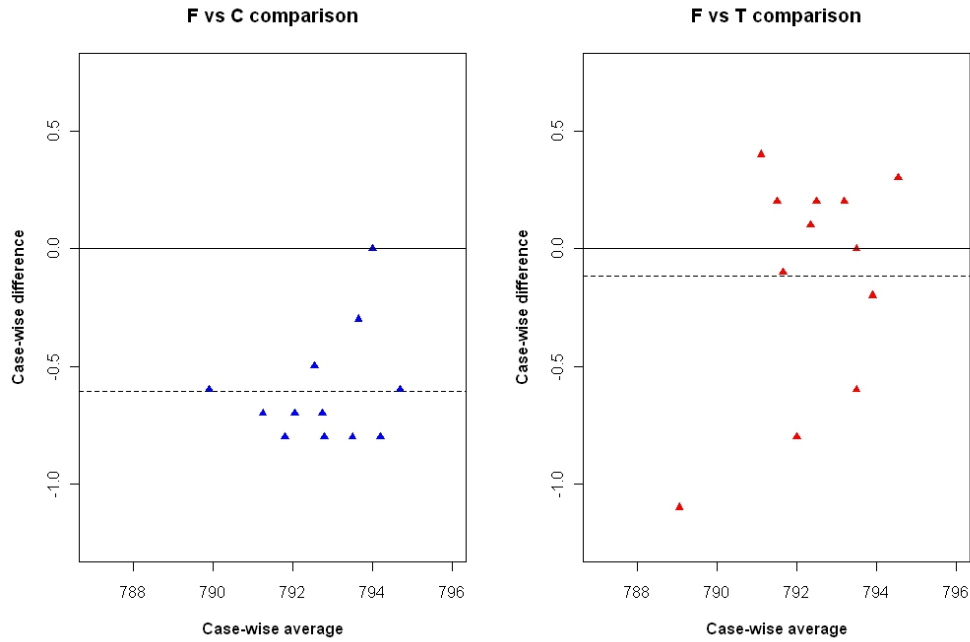
Figure 2.3.3: Bland-Altman plots for Grubbs' F vs C and F vs T comparisons.

2.4   Bland-Altman plots for the Grubbs data

In the case of the Grubbs data the inter-method bias is -0.61 metres per second, and is indicated by the dashed line on Figure 1.2. By inspection of the plot, it is also possible to compare the precision of each method. Noticeably the differences tend to increase as the averages increase.

The Bland-Altman plot for comparing the 'Fotobalk' and 'Counter' methods, which shall henceforth be referred to as the 'F vs C' comparison, is depicted in Figure 1.2, using data from Table 1.3. The presence and magnitude of the inter-method bias is indicated by the dashed line.

Round	Fotobalk [F]	Counter [C]	Differences [F-C]	Averages [(F+C)/2]
1	793.8	794.6	-0.8	794.2
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8	792.3	792.8	-0.5	792.5
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10	794.4	795.0	-0.6	794.7
11	790.9	791.6	-0.7	791.2
12	793.5	793.8	-0.3	793.6

Table 2.4.3: Fotobalk and Counter methods: differences and averages.

Round	Fotobalk [F]	Terma [T]	Differences [F-T]	Averages [(F+T)/2]
1	793.8	793.2	0.6	793.5
2	793.1	793.3	-0.2	793.2
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Table 2.4.4: Fotobalk and Terma methods: differences and averages.

In Figure 1.3 Bland-Altman plots for the 'F vs C' and 'F vs T' comparisons are shown, where 'F vs T' refers to the comparison of the 'Fotobalk' and 'Terma' methods. Usage of the Bland-Altman plot can be demonstrate in the contrast between these comparisons. By inspection, there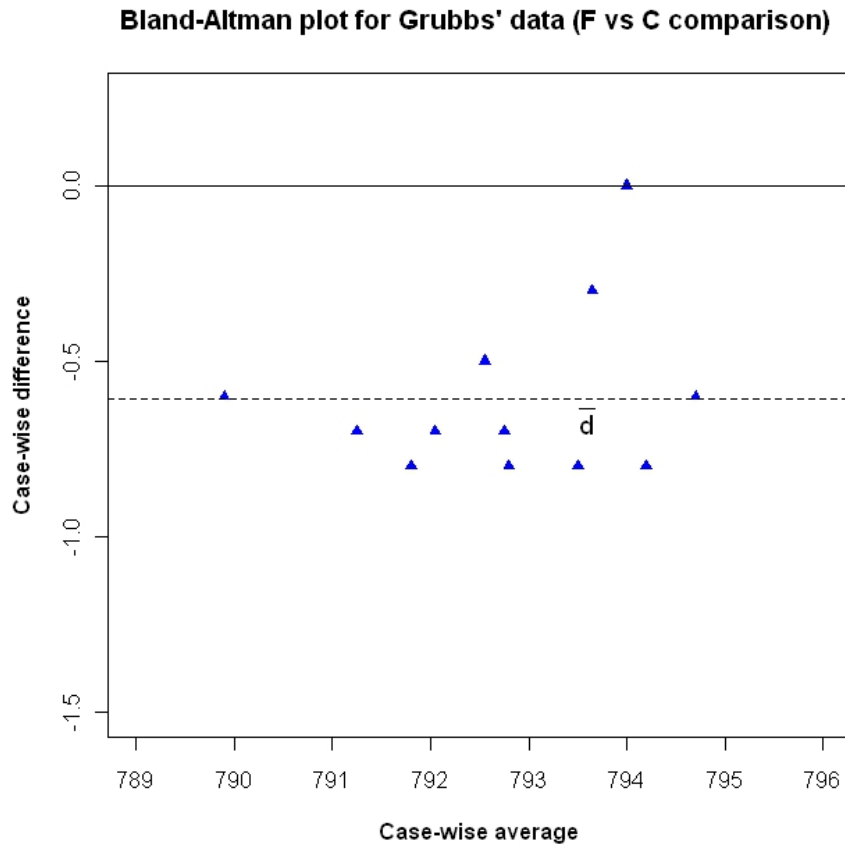 exists a larger inter-method bias in the 'F vs C' comparison than in the 'F vs T' comparison. Conversely there appears to be less precision in 'F vs T' comparison, as indicated by the greater dispersion of covariates.

2.5   Inspecting the Data

Bland-Altman plots are a powerful graphical methodology for making a visual assessment of the data. **?** express the motivation for this plot thusly:

> "From this type of plot it is much easier to assess the magnitude of disagreement (both error and bias), spot outliers, and see whether there is any trend, for example an increase in (difference) for high values. This way of plotting the data is a very powerful way of displaying the results of a method comparison study."

Figures 1.3 1.4 and 1.5 are three Bland-Altman plots derived from simulated data, each for the purpose of demonstrating how the plot would inform an analyst of trends that would adversely affect use of the recommended methodology. Figure 1.3 demonstrates how the Bland Altman plot would indicate increasing variance of differences over the measurement range. Figure 1.4 is an example of cases where the inter-method bias changes over the measurement range. This is known as proportional bias (**?**).
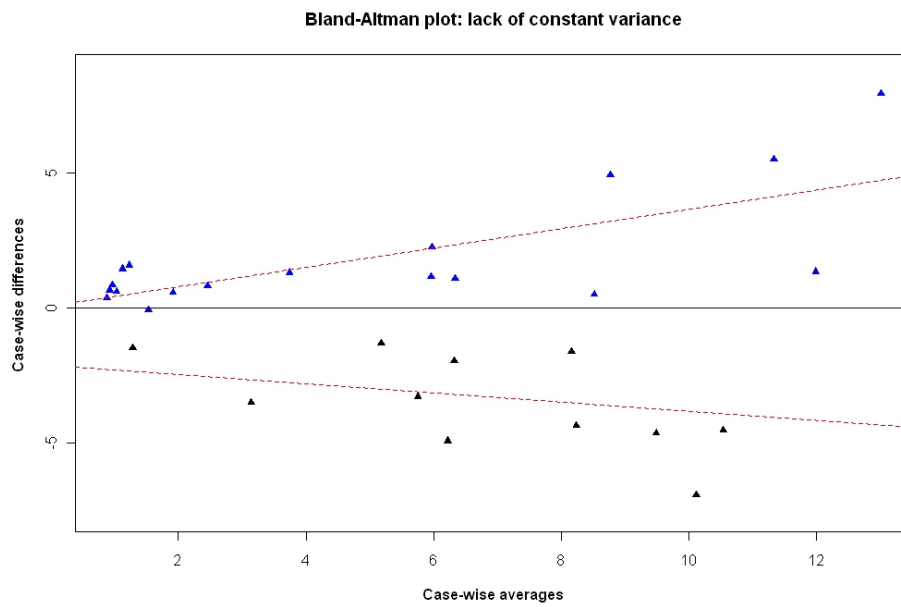
Figure 2.5.4: Bland-Altman Plot demonstrating the increase of variance over the range
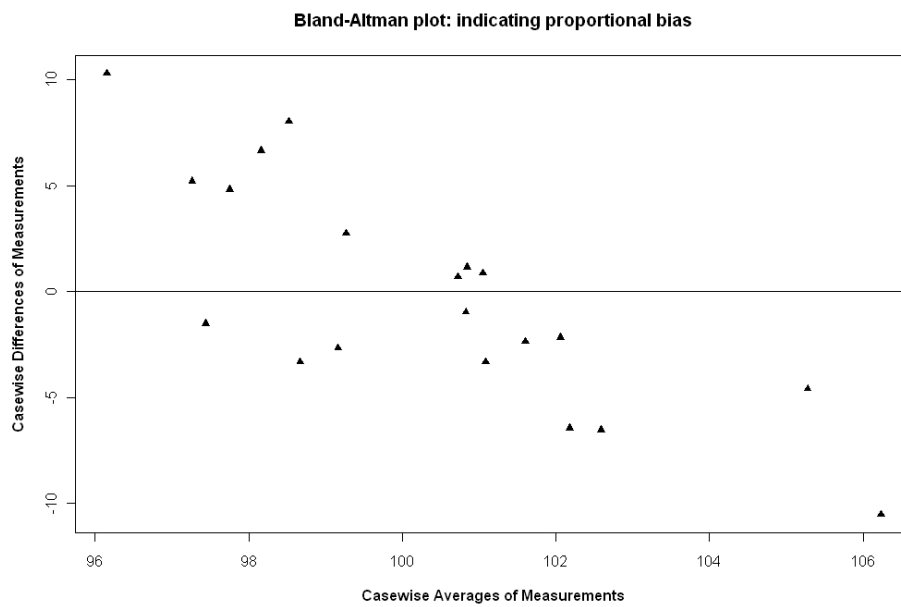
Figure 2.5.5: Bland-Altman Plot indicating the presence of proportional bias

29

Figure 1.4 is an example of cases where the inter-method bias changes over the measurement range. This is known as proportional bias (Ludbrook, 1997). Both of these cases violate the assumptions necessary for further analysis using limits of agreement ,which shall be discussed later. The plot also can be used to identify outliers. An outlier is an observation that is numerically distant from the rest of the data. Classification thereof is a subjective decision in any analysis, but must be informed by the logic of the formulation. Figure 1.5 is a Bland Altman plot with two conspicuous observations, at the extreme left and right of the plot respectively.
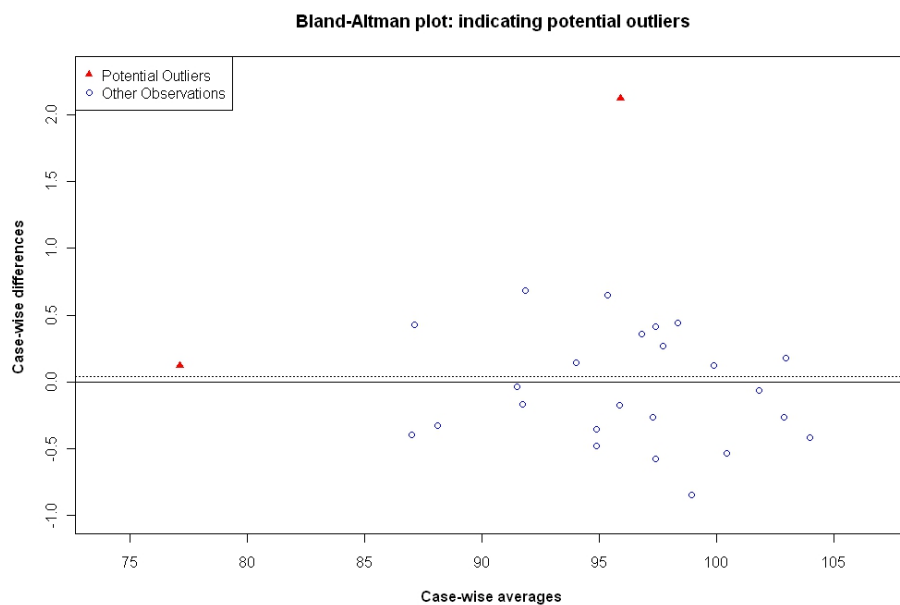
Figure 2.5.6: Bland-Altman Plot indicating the presence of Outliers

In the Bland-Altman plot, the horizontal displacement of any observation is supported by two independent measurements. Hence any observation , such as the one on the extreme right of figure 1.5, should not be considered an outlier on the basis of a noticeable horizontal displacement from the main cluster. The one on the extreme left should be considered an outlier, as it has a noticeable vertical displacement from the rest of the observations.

? do not recommend excluding outliers from analyses. However recalculation of the inter-method bias estimate , and further calculations based upon that estimate, are useful for assessing the influence of outliers.(**?**) states that *"We usually find that this method of analysis is not too sensitive to one or two large outlying differences."*

2.6   The Bland Altman Plot

In 1986 Bland and Altman published a paper in the Lancet proposing the difference plot for use for method comparison purposes. It has proved highly popular ever since. This is a simple, and widely used , plot of the differences of each data pair, and the corresponding average value. An important requirement is that the two measurement methods use the same scale of measurement.

2.7   Bland Altman Plots

The issue of whether two measurement methods comparable to the extent that they can be used interchangeably with sufficient accuracy is encountered frequently in scientific research. Historically comparison of two methods of measurement was carried out by use of correlation coefficients or simple linear regression. Bland and Altman recognized the inadequacies of these analyses and articulated quite thoroughly the basis on which of which they are unsuitable for comparing two methods of measurement (**?**).

Furthermore they proposed their simple methodology specifically constructed for method comparison studies. They acknowledge that there are other valid, but complex, methodologies, and argue that a simple approach is preferable to this complex approaches, *especially when the results must be explained to non-statisticians* (**?**).

Notwithstanding previous remarks about regression, the first step recommended ,which the authors argue should be mandatory,is construction of a simple scatter plot of the data. The line of equality $(X = Y)$ should also be shown, as it is necessary to give the correct interpretation of how both methods compare. A scatter plot of the Grubbs

data is shown in figure 2.1. A visual inspection thereof confirms the previous conclusion that there is an inter method bias present, i.e. Fotobalk device has a tendency to record a lower velocity.
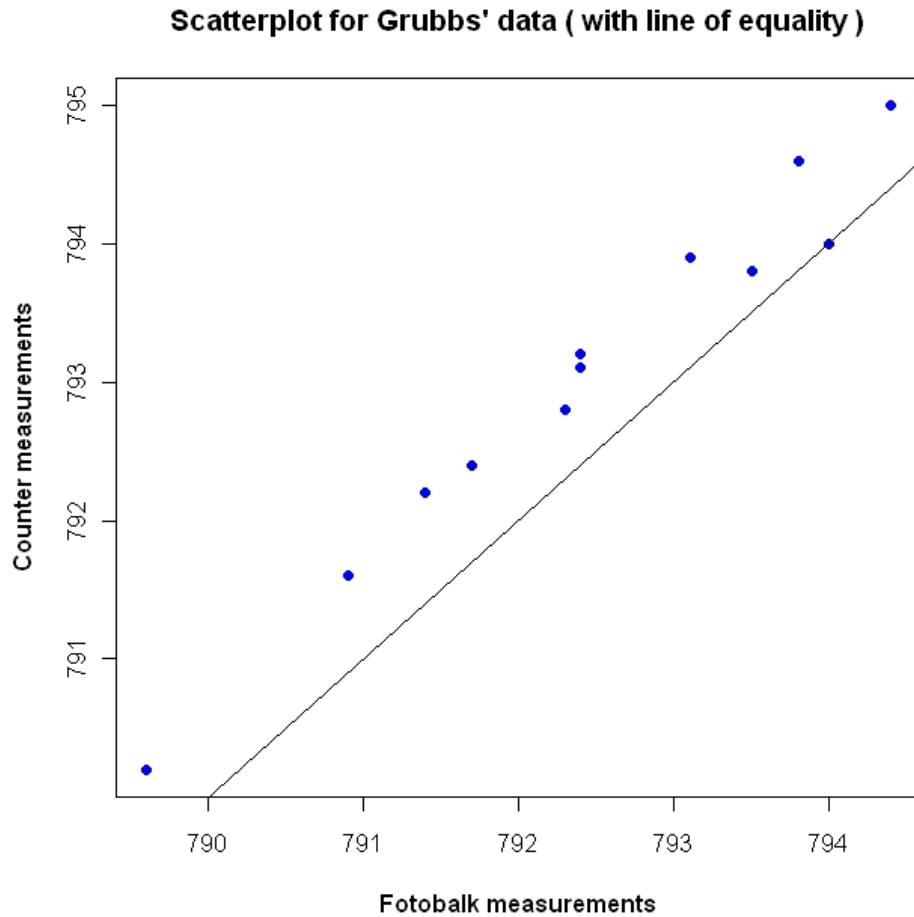
Figure 2.7.7: Scatter plot For Fotobalk and Counter Methods

In light of shortcomings associated with scatterplots, **?** recommend a further analysis of the data. Firstly differences of measurements of two methods on the same subject should be calculated, and then the average of those measurements (Table 2.1). These differences and averages are then plotted (Figure 2.2).

The dashed line in Figure 2.2 alludes to the inter method bias between the two methods, as mentioned previously. Bland and Altman recommend the estimation of

inter method bias by calculating the average of the differences. In the case of Grubbs data the inter method bias is -0.6083 metres per second.

Round	Fotobalk [F]	Counter [C]	Differences [F-C]	Averages [(F+C)/2]
1	793.80	794.60	-0.80	794.20
2	793.10	793.90	-0.80	793.50
3	792.40	793.20	-0.80	792.80
4	794.00	794.00	0.00	794.00
5	791.40	792.20	-0.80	791.80
6	792.40	793.10	-0.70	792.80
7	791.70	792.40	-0.70	792.00
8	792.30	792.80	-0.50	792.50
9	789.60	790.20	-0.60	789.90
10	794.40	795.00	-0.60	794.70
11	790.90	791.60	-0.70	791.20
12	793.50	793.80	-0.30	793.60

Table 2.7.5: Fotobalk and Counter Methods: Differences and Averages

By inspection of the plot, it is also possible to compare the precision of each method. Noticeably the differences tend to increase as the averages increase.

2.8   Bland Altman Plot

Bland Altman have recommended the use of graphical techniques to assess agreement. Principally their method is calculating , for each pair of corresponding two methods of measurement of some underlying quantity, with no replicate measurements, the difference and mean. Differences are then plotted against the mean.

Hopkins argued that the bias in a subsequent Bland-Altman plot was due, in part, to using least-squares regression at the calibration phase.

measurements

2.9   Bland Altman Plot

Bland Altman have recommended the use of graphical techniques to assess agreement. Principally their method is calculating , for each pair of corresponding two methods of measurement of some underlying quantity, with no replicate measurements, the difference and mean. Differences are then plotted against the mean.

Hopkins argued that the bias in a subsequent Bland-Altman plot was due, in part, to using least-squares regression at the calibration phase.

2.10   Bland-Altman Plots

In light of shortcomings associated with scatterplots, **?** recommend a further analysis of the data. Firstly case-wise differences of measurements of two methods $d_i = y_{1i} - y_{2i}$, for $i = 1, 2, \ldots, n$, on the same subject should be calculated, and then the average of those measurements, $(a_i = (y_{1i} + y_{2i})/2$ for $i = 1, 2, \ldots, n$.

Following a technique known as the Tukey mean-difference plot, as noted by **?** **?** proposed that a_i should be plotted against d_i, a plot now widely known as the Bland-Altman plot, and motivated this plot as follows:

> "From this type of plot it is much easier to assess the magnitude of disagreement (both error and bias), spot outliers, and see whether there is any trend, for example an increase in (difference) for high values. This way of plotting the data is a very powerful way of displaying the results of a method comparison study."

The case wise-averages capture several aspects of the data, such as expressing the range over which the values were taken, and assessing whether the assumptions of constant variance holds. Case-wise averages also allow the case-wise differences to be presented on a two-dimensional plot, with better data visualization qualities than a one dimensional plot. **?** cautions that it would be the difference against either measurement value instead of their average, as the difference relates to both value. This approach

has proved very popular, and the Bland-Altman plots is widely regarded as powerful graphical tool for making a visual assessment of the data.

The magnitude of the inter-method bias between the two methods is simply the average of the differences $\bar{d}$. This inter-method bias is represented with a line on the Bland-Altman plot. As the objective of the Bland-Altman plot is to advise on the agreement of two methods, the individual case-wise differences are also particularly relevant. The variances around this bias is estimated by the standard deviation of these differences S_d.

2.11 Limits of Agreement

A third element of the Bland-Altman approach, an interval known as 'limits of agreement' is introduced in **?** (sometimes referred to in literature as 95% limits of agreement). Limits of agreement are used to assess whether the two methods of measurement can be used interchangeably. **?** refer to this as the 'equivalence' of two measurement methods. The specific question to which limits of agreement are intended as the answer to must be established clearly. **?** comment that the limits of agreement show 'how far apart measurements by the two methods were likely to be for most individuals', a definition echoed in their 1999 paper:

> "We can then say that nearly all pairs of measurements by the two methods will be closer together than these extreme values, which we call 95% limits of agreement. These values define the range within which most differences between measurements by the two methods will lie."

The limits of agreement (LoA) are computed by the following formula:

$$LoA = \bar{d} \pm 1.96s_d$$

with $\bar{d}$ as the estimate of the inter method bias, s_d as the standard deviation of the differences and 1.96 (sometimes rounded to 2) is the 95% quantile for the standard

normal distribution. The limits of agreement methodology assumes a constant level of bias throughout the range of measurements. Importantly the authors recommend prior determination of what would constitute acceptable agreement, and that sample sizes should be predetermined to give an accurate conclusion. However **?** highlight inadequacies in the correct application of limits of agreement, resulting in contradictory estimates of limits of agreement in various papers.

For the Grubbs 'F vs C' comparison, these limits of agreement are calculated as -0.132 for the upper bound, and -1.08 for the lower bound. Figure 1.9 shows the resultant Bland-Altman plot, with the limits of agreement shown in dashed lines.
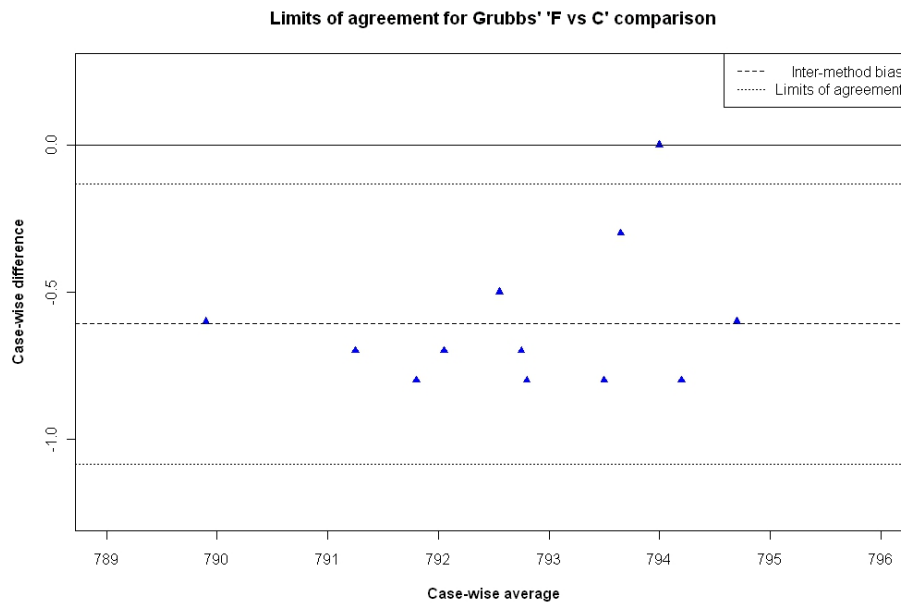
Figure 2.11.8: Bland-Altman plot with limits of agreement

2.12   Limits Of Agreement

Bland and Altman proposed a pair of Limits of agreement. These limits are intended to demonstrate the range in which 95% of the sample data should lie. The Limits

of agreement centre on the average difference line and are 1.96 times the standard deviation above and below the average difference line.

How this relates the overall population is unclear. It seems that it depends on an expert to decide whether or not the range of differences is acceptable. In a study A Bland-Altman plots compare two assay methods. It plots the difference between the two measurements on the Y axis, and the average of the two measurements on the X axis.

The bias is computed as the average of the difference of paired assays.

If one method is sometimes higher, and sometimes the other method is higher, the average of the differences will be close to zero. If it is not close to zero, this indicates that the two assay methods are producing different results systematically.

2.13   Appropriate Use of Limits of Agreement

Importantly **?** makes the following point:

> These estimates are meaningful only if we can assume bias and variability
> are uniform throughout the range of measurement, assumptions which can
> be checked graphically.

The import of this statement is that , should the Bland Altman plot indicate that these assumptions are not met, then their entire methodology, as posited thus far, is inappropriate for use in a method comparison study. Again, in the context of potential outlier in the Grubbs data (figure 1.2), this raises the question on how to correctly continue.

Carstensen attends to the issue of repeated data, using the expression replicate to express a repeated measurement on a subject by the same methods. Carstensen formulates the data as follows Repeated measurement - Arrangement of data into groups, based on the series of results of each subject.

2.14 Formal definition of limits of agreement

? note the similarity of limits of agreement to confidence intervals, but are clear that they are not the same thing. Interestingly, they describe the limits as 'being like a reference interval'.

Limits of agreement have very similar construction to Shewhart control limits. The Shewhart chart is a well known graphical methodology used in statistical process control. Consequently there is potential for misinterpreting the limits of agreement as they were Shewhart control limits.

? regards the limits of agreement as a prediction interval for the difference between future measurements with the two methods on a new individual, but states that it does not fit the formal definition of a prediction interval, since the definition does not consider the errors in estimation of the parameters. Prediction intervals, which are often used in regression analysis, are estimates of an interval in which future observations will fall, with a certain probability, given what has already been observed. **?** offers an alternative formulation, a 95% prediction interval for the difference

$$\bar{d} \pm t_{(0.025, n-1)} s_d \sqrt{1 + \frac{1}{n}}$$

where n is the number of subjects. Carstensen is careful to consider the effect of the sample size on the interval width, adding that only for 61 or more subjects is the quantile less than 2.

? offers an alternative description of limits of agreement, this time as tolerance limits. A tolerance interval for a measured quantity is the interval in which a specified fraction of the population's values lie, with a specified level of confidence. **?** describes them as a probability interval, and offers a clear description of how they should be used; 'if the absolute limit is less than an acceptable difference d_0, then the agreement between the two methods is deemed satisfactory'.

The prevalence of contradictory definitions of what limits of agreement strictly are will inevitably attenuate the poor standard of reporting using limits of agreement, as mentioned by **?**.

2.15 Inferences on Bland-Altman estimates

? advises on how to calculate confidence intervals for the inter-method bias and limits of agreement. For the inter-method bias, the confidence interval is a simply that of a mean: $\bar{d} \pm t_{(\alpha/2, n-1)} S_d / \sqrt{n}$. The confidence intervals and standard error for the limits of agreement follow from the variance of the limits of agreement, which is shown to be

$$\text{Var}(LoA) = (\frac{1}{n} + \frac{1.96^2}{2(n-1)}) s_d^2.$$

If n is sufficiently large this can be following approximation can be used

$$\text{Var}(LoA) \approx 1.71^2 \frac{s_d^2}{n}.$$

Consequently the standard errors of both limits can be approximated as 1.71 times the standard error of the differences.

A 95% confidence interval can be determined, by means of the t distribution with $n-1$ degrees of freedom. However, **?** comment that such calculations may be 'somewhat optimistic' on account of the associated assumptions not being realized.

2.16 Adverse features

Estimates for inter-method bias and variance of differences are only meaningful if there is uniform inter-bias and variability throughout the range of measurements. Fulfilment of these assumptions can be checked by visual inspection of the plot.The prototype Bland-Altman plots depicted in Figures 1.4, 1.5 and 1.6 are derived from simulated data, for the purpose of demonstrating how the plot would inform an analyst of features that would adversely affect use of the recommended methodology.

Figure 1.4 demonstrates how the Bland-Altman plot would indicate increasing variance of differences over the measurement range. Fitted regression lines, for both the upper and lower half of the plot, has been added to indicate the trend. Figure 1.5 is an example of cases where the inter-method bias changes over the measurement range.

This is known as proportional bias, and is defined by **?** as meaning that 'one method gives values that are higher (or lower) than those from the other by an amount that is proportional to the level of the measured variable'. In both Figures 1.4 and 1.5, the assumptions necessary for further analysis using the limits of agreement are violated.

Application of regression techniques to the Bland-Altman plot, and subsequent formal testing for the constant variability of differences is informative. The data set may be divided into two subsets, containing the observations wherein the difference values are less than and greater than the inter-method bias respectively. For both of these fits, hypothesis tests for the respective slopes can be performed. While both tests can be considered separately, multiple comparison procedures, such as the Benjamini-Hochberg (**?**) test, should be also be used.

2.17 Outliers

The Bland-Altman plot also can be used to identify outliers. An outlier is an observation that is conspicuously different from the rest of the data that it arouses suspicion that it occurs due to a mechanism, or conditions, different to that of the rest of the observations. **?** do not recommend excluding outliers from analyzes, but remark that recalculation of the inter-method bias estimate, and further calculations based upon that estimate, are useful for assessing the influence of outliers. The authors remark that 'we usually find that this method of analysis is not too sensitive to one or two large outlying differences'. Figure 1.6 demonstrates how the Bland-Altman plot can be used to visually inspect the presence of potential outliers.

As a complement to the Bland-Altman plot, **?** proposes the use of a bivariate confidence ellipse, constructed for a predetermined level. **?** provides the relevant calculations for the ellipse. This ellipse is intended as a visual guidelines for the scatter plot, for detecting outliers and to assess the within- and between-subject variances.

The minor axis relates to the between subject variability, whereas the major axis relates to the error mean square, with the ellipse depicting the size of both relative to each other. Consequently Bartko's ellipse provides a visual aid to determining the relationship between variances. If $var(a)$ is greater than $var(d)$, the orientation of the ellipse is horizontal. Conversely if $var(a)$ is less than $var(d)$, the orientation of the ellipse is vertical.

The Bland-Altman plot for the Grubbs data, complemented by Bartko's ellipse, is depicted in Figure 1.7. The fourth observation is shown to be outside the bounds of the ellipse, indicating that it is a potential outlier.

The limitations of using bivariate approaches to outlier detection in the Bland-Altman plot can demonstrated using Bartko's ellipse. A covariate is added to the 'F vs C' comparison that has a difference value equal to the inter-method bias, and an average value that markedly deviates from the rest of the average values in the comparison, i.e. 786. Table 1.8 depicts a 95% confidence ellipse for this manipulated

data set. By inspection of the confidence interval, a conclusion would be reached that this extra covariate is an outlier, in spite of the fact that this observation is wholly consistent with the conclusion of the Bland-Altman plot.

Importantly, outlier classification must be informed by the logic of the data's formulation. In the Bland-Altman plot, the horizontal displacement of any observation is supported by two independent measurements. Any observation should not be considered an outlier on the basis of a noticeable horizontal displacement from the main cluster, as in the case with the extra covariate. Conversely, the fourth observation, from the original data set, should be considered an outlier, as it has a noticeable vertical displacement from the rest of the observations.

In classifying whether a observation from a univariate data set is an outlier, many formal tests are available, such as the Grubbs test for outliers. In assessing whether a covariate in a Bland-Altman plot is an outlier, this test is useful when applied to the case-wise difference values treated as a univariate data set. The null hypothesis of the Grubbs test procedure is the absence of any outliers in the data set. Conversely, the alternative hypotheses is that there is at least one outlier present.

The test statistic for the Grubbs test (G) is the largest absolute deviation from the sample mean divided by the standard deviation of the differences,

$$G = \max_{i=1,\ldots,n} \frac{\left|d_i - \bar{d}\right|}{S_d}.$$

For the 'F vs C' comparison it is the fourth observation gives rise to the test statistic, $G = 3.64$. The critical value is calculated using Student's t distribution and the sample size,

$$U = \frac{n-1}{\sqrt{n}} \sqrt{\frac{t^2_{\alpha/(2n),n-2}}{n - 2 + t^2_{\alpha/(2n),n-2}}}.$$

For this test $U = 0.75$. The conclusion of this test is that the fourth observation in the 'F vs C' comparison is an outlier, with $p-$value $= 0.003$, according with the previous result using Bartko's ellipse.

2.18 Adverse features

Estimates for inter-method bias and variance of differences are only meaningful if there is uniform inter-bias and variability throughout the range of measurements. Fulfilment of these assumptions can be checked by visual inspection of the plot.The prototype Bland-Altman plots depicted in Figures 1.4, 1.5 and 1.6 are derived from simulated data, for the purpose of demonstrating how the plot would inform an analyst of features that would adversely affect use of the recommended approach.

Figure 1.4 demonstrates how the Bland-Altman plot would indicate increasing variance of differences over the measurement range. Fitted regression lines, for both the upper and lower half of the plot, has been added to indicate the trend. Figure 1.5 is an example of cases where the inter-method bias changes over the measurement range. This is known as proportional bias, and is defined by **?** as meaning that 'one method gives values that are higher (or lower) than those from the other by an amount that is proportional to the level of the measured variable'. In both Figures 1.4 and 1.5, the assumptions necessary for further analysis using the limits of agreement are violated.

Application of regression techniques to the Bland-Altman plot, and subsequent formal testing for the constant variability of differences is informative. The data set may be divided into two subsets, containing the observations wherein the difference values are less than and greater than the inter-method bias respectively. For both of these fits, hypothesis tests for the respective slopes can be performed. While both tests could be considered separately, multiple comparison procedures, such as the Benjamini-Hochberg (**?**) test, are advisable.
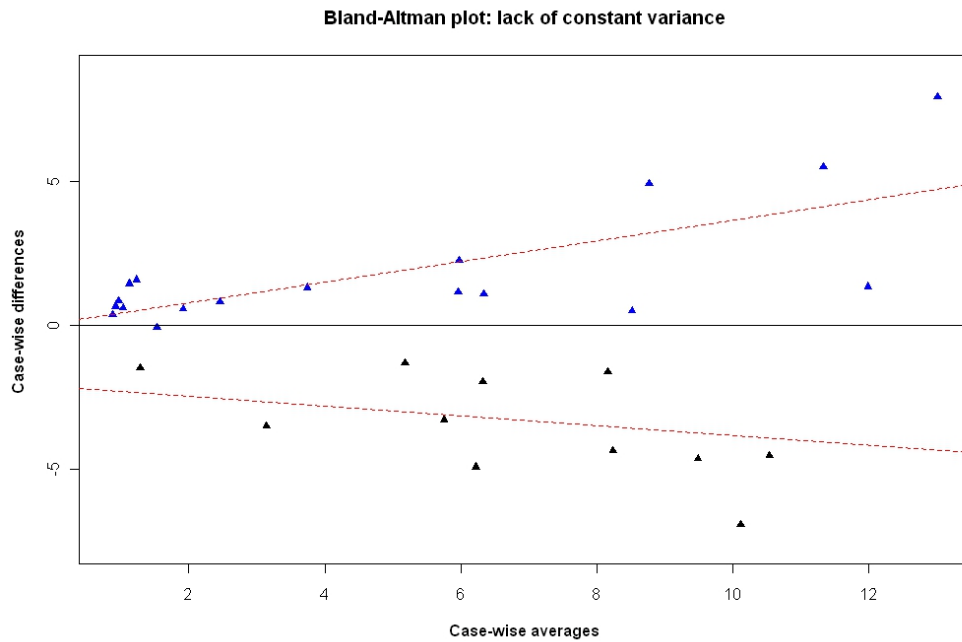
Figure 2.18.9: Bland-Altman plot demonstrating the increase of variance over the range.

The Bland-Altman plot also can be used to identify outliers. An outlier is an observation that is conspicuously different from the rest of the data that it arouses suspicion that it occurs due to a mechanism, or conditions, different to that of the rest of the observations. ? do not recommend excluding outliers from analyses, but remark that recalculation of the inter-method bias estimate, and further calculations based upon that estimate, are useful for assessing the influence of outliers. The authors remark that 'we usually find that this method of analysis is not too sensitive to one or two large outlying differences'. Figure 1.6 demonstrates how the Bland-Altman plot can be used to visually inspect the presence of potential outliers.

As a complement to the Bland-Altman plot, ? proposes the use of a bivariate confidence ellipse, constructed for a predetermined level. ? provides the relevant calculations for the ellipse. This ellipse is intended as a visual guidelines for the scatter plot, for detecting outliers and to assess the within- and between-subject variances.

Bland-Altman plot: indicating proportional bias

Figure 2.18.10: Bland-Altman plot indicating the presence of proportional bias.

Bland-Altman plot: indicating potential outliers
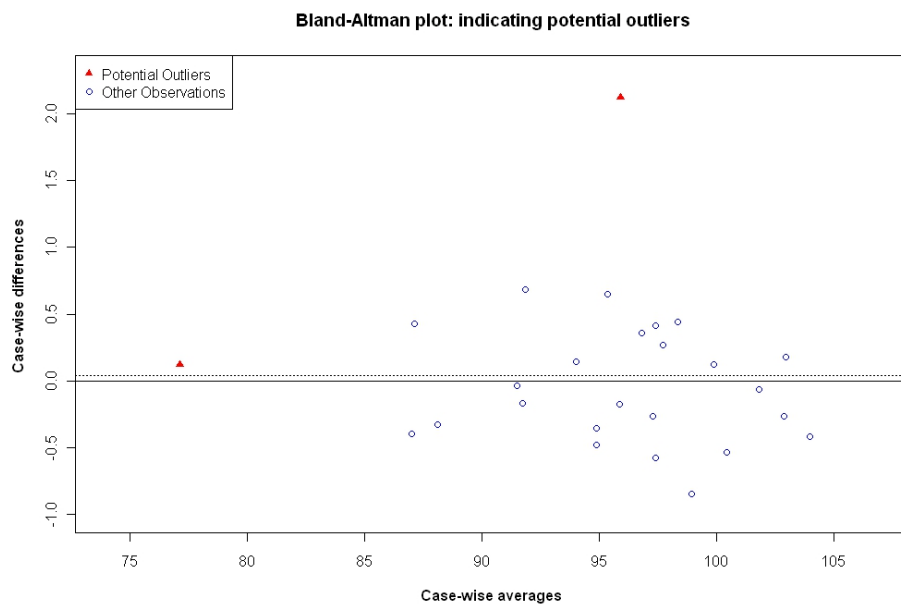
Figure 2.18.11: Bland-Altman plot indicating the presence of potential outliers.

The minor axis relates to the between subject variability, whereas the major axis relates to the error mean square, with the ellipse depicting the size of both relative to each other. Consequently Bartko's ellipse provides a visual aid to determining the relationship between variances. If var(a) is greater than var(d), the orientation of the ellipse is horizontal. Conversely if var(a) is less than var(d), the orientation of the ellipse is vertical.

The Bland-Altman plot for the Grubbs data, complemented by Bartko's ellipse, is depicted in Figure 1.7. The fourth observation is shown to be outside the bounds of the ellipse, indicating that it is a potential outlier.

Figure 2.18.12: Bartko's Ellipse for Grubbs' data.

The limitations of using bivariate approaches to outlier detection in the Bland-Altman plot can demonstrated using Bartko's ellipse. A covariate is added to the 'F vs C' comparison that has a difference value equal to the inter-method bias, and an average value that markedly deviates from the rest of the average values in the comparison, i.e. 786. Table 1.8 depicts a 95% confidence ellipse for this manipulated data set. By inspection of the confidence interval, we would conclude that this extra

covariate is an outlier, in spite of the fact that this observation is very close to the inter-method bias as determined by this approach.

Bartko's ellipse for Grubbs' F vs C comparison - Extra covariate
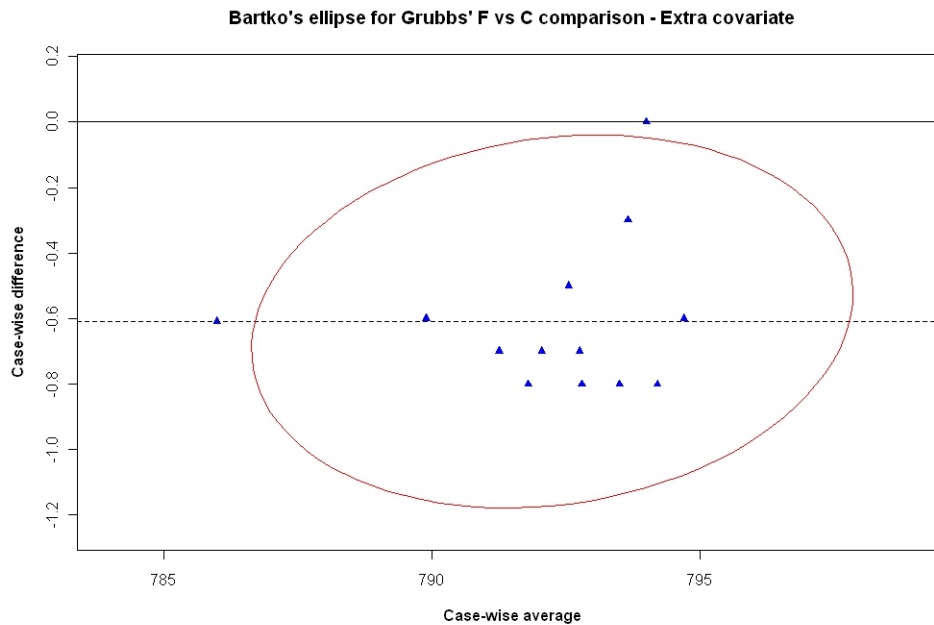
Figure 2.18.13: Bartko's Ellipse for Grubbs' data, with an extra covariate.

Importantly, outlier classification must be informed by the logic of the mechanism that produces the data. In the Bland-Altman plot, the horizontal displacement (i.e. the average) of any observation is supported by two separate measurements. Any observation should not be considered an outlier on the basis of a noticeable horizontal displacement from the main cluster, as in the case with the extra covariate. Conversely, the fourth observation, from the original data set, should be considered an outlier, as it has a noticeable vertical displacement from the rest of the observations.

In classifying whether a observation from a univariate data set is an outlier, many formal tests are available, such as the Grubbs test for outliers. In assessing whether a covariate in a Bland-Altman plot is an outlier, this test is useful when applied to the case-wise difference values treated as a univariate data set. The null hypothesis of the Grubbs test procedure is the absence of any outliers in the data set. Conversely, the alternative hypotheses is that there is at least one outlier present.

48

The test statistic for the Grubbs test (G) is the largest absolute deviation from the sample mean divided by the standard deviation of the differences,

$$G = \max_{i=1,\ldots,n} \frac{\left|d_i - \bar{d}\right|}{S_d}. \tag{2.1}$$

For the 'F vs C' comparison it is the fourth observation gives rise to the test statistic, $G = 3.64$. The critical value is calculated using Student's t distribution and the sample size,

$$U = \frac{n-1}{\sqrt{n}} \sqrt{\frac{t^2_{\alpha/(2n),n-2}}{n-2+t^2_{\alpha/(2n),n-2}}}.$$

For this test $U = 0.75$. The conclusion of this test is that the fourth observation in the 'F vs C' comparison is an outlier, with $p-$value $= 0.003$, in accordance with the previous result of Bartko's ellipse.

2.19 Agreement

Bland and Altman (1986) defined perfect agreement as the case where all of the pairs of rater data lie along the line of equality, where the line of equality is defined as the 45 degree line passing through the origin(i.e. the $X = Y$ line).

Bland and Altman (1986)expressed this in the terms *we want to know by how much the new method is likely to differ from the old; if this is not enough to cause problems in clinical interpretation we can replace the old method by the new or use the two interchangeably. How far apart measurements can be without causing difficulties will be a question of judgment. Ideally, it should be defined in advance to help in the interpretation of the method comparisonand to choose the sample size .*

2.20 Bias

Bland and Altman define bias a *a consistent tendency for one method to exceed the other* and propose estimating its value by determining the mean of the differences. The

variation about this mean shall be estimated by the standard deviation of the differences. Bland and Altman remark that these estimates are based on the assumption that bias and variability are constant throughout the range of measures.

2.21   scatter plots

The authors advise the use of scatter plots to identify outliers, and to determine if there is curvilinearity present. In the region of linearity ,simple linear regression may yield results of interest.

2.22   Effect of Outliers

Another argument against the use of model I regression is based on outliers. Outliers can adversely influence the fitting of a regression model. Cornbleet and Cochrane compare a regression model influenced by an outlier with a model for the same data set, with the outlier excluded from the data set. A demonstration of the effect of outliers was made in Bland Altman's 1986 paper. However they discourage the exclusion of outliers.

2.23   Precision of Limits of Agreement

The limits of agreement are estimates derived from the sample studied, and will differ from values relevant to the whole population. A different sample would give different limits of agreement. **?** advance a formulation for confidence intervals of the inter-method bias and the limits of agreement. These calculations employ quantiles of the 't' distribution with $n-1$ degrees of freedom.

2.24   Variations of the Bland-Altman Plot

Referring to the assumption that bias and variability are constant across the range of measurements, **?** address the case where there is an increase in variability as the magnitude increases. They remark that it is possible to ignore the issue altogether, but the limits of agreement would wider apart than necessary when just lower magnitude measurements are considered. Conversely the limits would be too narrow should only higher magnitude measurements be used. To address the issue, they propose the logarithmic transformation of the data. The plot is then formulated as the difference of paired log values against their mean. Bland and Altman acknowledge that this is not easy to interpret, and may not be suitable in all cases.

? offers two variations of the Bland-Altman plot that are intended to overcome potential problems that the conventional plot would inappropriate for. The first variation is a plot of case-wise differences as percentage of averages, and is appropriate when there is an increase in variability of the differences as the magnitude increases. The second variation is a plot of case-wise ratios as percentage of averages. This will remove the need for *log* transformation. This approach is useful when there is an increase in variability of the differences as the magnitude of the measurement increases. **?** proposed such a ratio plot, independently of Bland and Altman. **?** commented on the reception of this article by saying 'Strange to say,this report has been overlooked'.

2.25   The Bland Altman Plot - Variations

Variations of the Bland Altman plot is the use of ratios, in the place of differences.

$$D_i = X_i - Y_i \tag{2.2}$$

Altman and Bland suggest plotting the within subject differences $D = X_1 - X_2$ on the ordinate versus the average of x_1 and x_2 on the abscissa.

2.26   Variants of the Bland-Altman Plot

In light of some potential pitfalls associated with the conventional difference plot, a series of alternative formulations for the Bland-Altman approach have been proposed.

Referring to the assumption that bias and variability are constant across the range of measurements, **?** address the case where there is an increase in variability as the magnitude increases. They remark that it is possible to ignore the issue altogether, but the limits of agreement would be wider apart than necessary when just lower magnitude measurements are considered. Conversely the limits would be too narrow should only higher magnitude measurements be used. To address the issue, they propose the logarithmic transformation of the data. The plot is then formulated as the difference of paired log values against their mean. Bland and Altman acknowledge that this is not easy to interpret, and may not be suitable in all cases.

2.27   Prevalence of the Bland-Altman plot

?, which further develops the Bland-Altman methodology, was found to be the sixth most cited paper of all time by the **?**. **?** describes the rate at which prevalence of the Bland-Altman plot has developed in scientific literature. **?** reviewed the use of Bland-Altman plots by examining all articles in the journal 'Clinical Chemistry' between 1995 and 2001. This study concluded that use of the Bland-Altman plot increased over the years, from 8% in 1995 to 14% in 1996, and 31-36% in 2002.

The Bland-Altman Plot has since become expected, and often obligatory, approach for presenting method comparison studies in many scientific journals (**?**). Furthermore **?** recommend its use in papers pertaining to method comparison studies for the journal of the British Hypertension Society.

2.28 Bland Altman Plots In Literature

? contains a study the use of Bland Altman plots of 44 articles in several named journals over a two year period. 42 articles used Bland Altman's limits of agreement, wit the other two used correlation and regression analyses. **?** remarks that 3 papers, from 42 mention predefined maximum width for limits of agreement which would not impair medical care.

The conclusion of **?** is that there are several inadequacies and inconsistencies in the reporting of results ,and that more standardization in the use of Bland Altman plots is required. The authors recommend the prior determination of limits of agreement before the study is carried out. This contention is endorsed by **?**, which makes a similar recommendation for the sample size, noting that *sample sizes required either was not mentioned or no rationale for its choice was given.*

> In order to avoid the appearance of "data dredging", both the sample size and the (limits of agreement) should be specified and justified before the actual conduct of the trial. (**?**)

? remarks that the limits of agreement should be compared to a clinically acceptable difference in measurements.

2.29 Alternative Agreement Indices

As an alternative to limits of agreement, **?** proposes the use of the mean square deviation in assessing agreement. The mean square deviation is defined as the expectation of the squared differences of two readings. The MSD is usually used for the case of two measurement methods X and Y, each making one measurement for the same subject, and is given by

$$MSDxy = E[(x-y)^2] = (\mu_x - \mu_y)^2 + (\sigma_x - \sigma_y)^2 + 2\sigma_x\sigma_y(1 - \rho_{xy}).$$

? advises the use of a predetermined upper limit for the MSD value, MSD_{ul}, to define satisfactory agreement. However, a satisfactory upper limit may not be easily determinable, thus creating a drawback to this methodology.

Alternative indices, proposed by **?**, are the square root of the MSD and the expected absolute difference (EAD).

$$EAD = E(|x - y|) = \frac{\sum |x_i - y_i|}{n}$$

Both of these indices can be interpreted intuitively, since their units are the same as that of the original measurements. Also they can be compared to the maximum acceptable absolute difference between two methods of measurement d_0. For the sake of brevity, the EAD will be considered solely.

The EAD can be used to supplement the inter-method bias in an initial comparison study, as the EAD is informative as a measure of dispersion, is easy to calculate and requires no distributional assumptions. A consequence of using absolute differences is that high variances would result in a higher EAD value.

	U	V	$U - V$	$|U - V|$
1	98.05	99.53	-1.49	1.49
2	99.17	96.53	2.64	2.64
3	100.31	97.55	2.75	2.75
4	100.35	96.03	4.32	4.32
5	99.51	99.00	0.51	0.51
6	98.50	100.76	-2.26	2.26
7	100.66	99.37	1.29	1.29
8	99.66	108.87	-9.21	9.21
9	99.70	105.16	-5.45	5.45
10	101.55	94.31	7.24	7.24

Table 2.29.6: Example data set

To illustrate the use of EAD, consider table 2.29.6. The inter-method bias is 0.03,
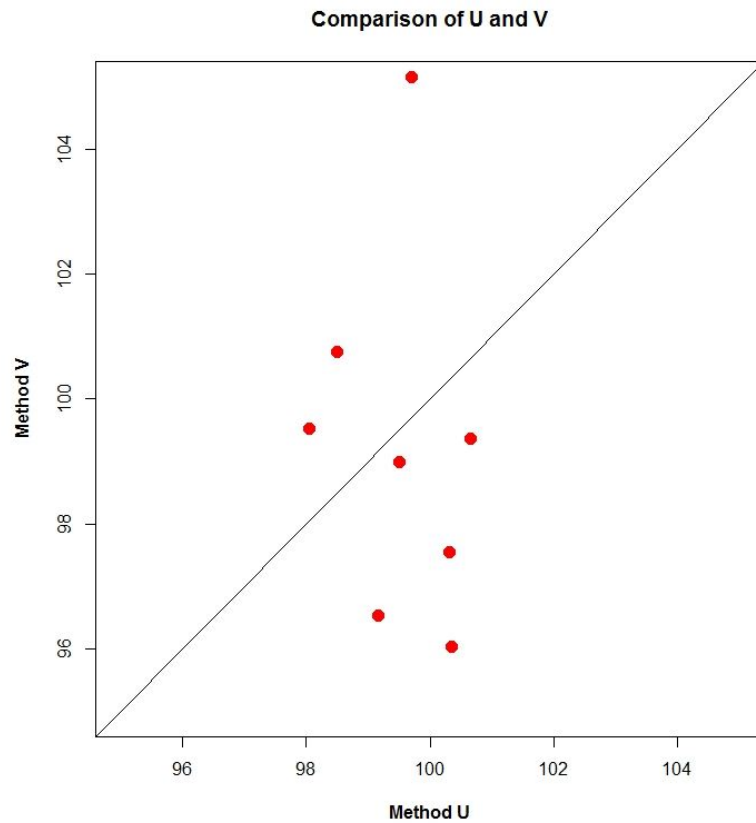
Figure 2.29.14: Identity Plot for example data

which is quite close to zero, which is desirable in the context of agreement. However, an identity plot would indicate very poor agreement, as the points are noticeably distant from the line of equality.

The limits of agreement are $[-9.61, 9.68]$, a wide interval for this data. As with the identity plot, this would indicate lack of agreement. As with inter-method bias, an EAD value close to zero is desirable. However, from table 2.29.6, the EAD can be computed as 3.71. The Bland-Altman plot remains a useful part of the analysis. In 2.29.15, it is clear there is a systematic decrease in differences across the range of measurements.

? remarks that a comparison of EAD and MSD , using simulation studies, would be interesting, while further adding that '*It will be of interest to investigate the benefits*
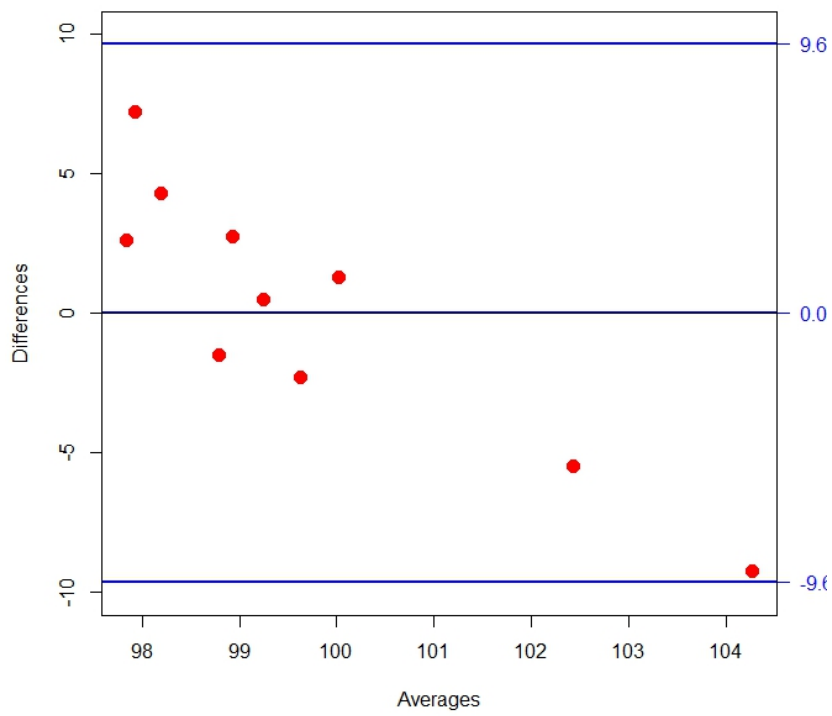
Figure 2.29.15: Bland-Altman Plot for UV comparison

of these possible new unscaled agreement indices'. For the Grubbs' 'F vs C' and 'F vs T' comparisons, the inter-method bias, difference variances, limits of agreement and EADs are shown in Table 1.5. The corresponding Bland-Altman plots for 'F vs C' and 'F vs T' comparisons were depicted previously on Figure 1.3. While the inter-method bias for the 'F vs T' comparison is smaller, the EAD penalizes the comparison for having a greater variance of differences. Hence the EAD values for both comparisons are much closer.

	F vs C	F vs T
Inter-method bias	-0.61	0.12
Difference variance	0.06	0.22
Limits of agreement	(-1.08, -0.13)	(-0.81,1.04)
EAD	0.61	0.35

Table 2.29.7: Agreement indices for Grubbs' data comparisons.

Further to **?** and **?**, individual agreement between two measurement methods may be assessed using the the coverage probability (CP) criteria or the total deviation index (TDI). If d_0 is predetermined as the maximum acceptable absolute difference between two methods of measurement, the probability that the absolute difference of two measures being less than d_0 can be computed. This is known as the coverage probability (CP).

$$CP = P(|x_i - y_i| \leq d_0) \tag{2.3}$$

If π_0 is set as the predetermined coverage probability, the boundary under which the proportion of absolute differences is π_0 may be determined. This boundary is known as the 'total deviation index' (TDI). Hence the TDI is the $100\pi_0$ percentile of the absolute difference of paired observations.

2.30   Alternative agreement indices

As an alternative to limits of agreement, **?** proposes the use of the mean square deviation is assessing agreement. The mean square deviation is defined as the expectation of the squared differences of two readings . The MSD is usually used for the case of two measurement methods X and Y , each making one measurement for the same subject, and is given by

$$MSDxy = E[(x - y)^2] = (\mu_x - \mu_y)^2 + (\sigma_x - \sigma_y)^2 + 2\sigma_x\sigma_y(1 - \rho_{xy}).$$

? advises the use of a predetermined upper limit for the MSD value, MSD_{ul}, to define satisfactory agreement. However, a satisfactory upper limit may not be properly determinable, thus creating a drawback to this methodology.

? proposes both the use of the square root of the MSD or the expected absolute difference (EAD) as an alternative agreement indices. Both of these indices can be interpreted intuitively, being denominated in the same units of measurements as the original measurements. Also they can be compare to the maximum acceptable absolute difference between two methods of measurement d_0.

$$EAD = E(|x - y|) = \frac{\sum |x_i - y_i|}{n}$$

The EAD can be used to supplement the inter-method bias in an initial comparison study, as the EAD is informative as a measure of dispersion, is easy to calculate and requires no distributional assumptions.

? remarks that a comparison of EAD and MSD , using simulation studies, would be interesting, while further adding that 'It will be of interest to investigate the benefits of these possible new unscaled agreement indices'. For the Grubbs' 'F vs C' and 'F vs T' comparisons, the inter-method bias, difference variances, limits of agreement and EADs are shown in Table 1.5. The corresponding Bland-Altman plots for 'F vs C' and 'F vs T' comparisons were depicted previously on Figure 1.3. While the inter-method bias for the 'F vs T' comparison is smaller, the EAD penalizes the comparison for

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	F vs C	F vs T
Inter-method bias	-0.61	0.12 3
Difference variances	0.06	0.22
Limits of agreement	(-1.08, -0.13)	(-0.81,1.04)
EAD	0.61	0.35

Table 2.30.8: Agreement indices for Grubbs' data comparisons.

Further to **?** and **?**, individual agreement between two measurement methods may
be assessed using the the coverage probability (CP) criteria or the total deviation
index (TDI). If d_0 is predetermined as the maximum acceptable absolute difference
between two methods of measurement, the probability that the absolute difference of
two measures being less than d_0 can be computed. This is known as the coverage
probability (CP).

$$CP = P(|x_i - y_i| \leq d_0) \tag{2.4}$$

If π_0 is set as the predetermined coverage probability, the boundary under which the
proportion of absolute differences is π_0 may be determined. This boundary is known as
the 'total deviation index' (TDI). Hence the TDI is the $100\pi_0$ percentile of the absolute
difference of paired observations.

2.31   Lin's Reproducibility Index

Lin proposes the use of a reproducibility index, called the Concordance Correlation
Coefficent (CCC).While it is not strictly a measure of agreement as such, it can form
part of an overall method comparision methodology.

Chapter 3

Improper MCS Techniques

3.1 Inappropriate Methodologies

Use of the Pearson Correlation Coefficient , although seemingly intuitive, is not appropriate approach to assessing agreement of two methods. Arguments against its usage have been made repeatedly in the relevant literature. It is possible for two analytical methods to be highly correlated, yet have a poor level of agreement.

3.2 Inappropriate assessment of Agreement

3.3 Paired T tests

This method can be applied to test for statistically significant deviations in bias. This method can be potentially misused for method comparison studies.

It is a poor measure of agreement when the rater's measurements are perpendicular to the line of equality[Hutson et al]. In this context, an average difference of zero between the two raters, yet the scatter plot displays strong negative correlation.

3.4 Pearson's Correlation Coefficient

It is well known that Pearson's correlation coefficient is a measure of the linear association between two variables, not the agreement between two variables (e.g., see Bland and Altman 1986)..This is a well known as a measure of linear association between two variables.Nonetheless this is not necessarily the same as Agreement. This method is considered wholly inadequate to assess agreement because it only evaluates only the association of two sets of observations.

3.5   Inappropriate use of the Correlation Coefficient

It is intuitive when dealing with two sets of related data, i.e the results of the two raters, to calculate the correlation coefficient (r). Bland and Altman attend to this in their 1999 paper.

They present a data set from two sets of meters, and an accompanying scatterplot. An hypothesis test on the data set leads us to conclude that there is a relationship between both sets of meter measurements. The correlation coeffiecient is determined to be r =0.94.However, this high correlation does not mean that the two methods agree. It is possible to determine from the scatterplot that the intercept is not zero, a requirement for stating both methods have high agreement. Essentially, should two methods have highly correlated results, it does not follow that they have high agreement.

Intra-class correlation coefficient

- The ICC, which takes on values between 0 and 1, is based on analysis of variance techniques. It is close to 1 when the differences between paired measurements is very small compared to the differences between subjects. Of these three procedures–t test, correlation coefficient, intra-class correlation coefficient–the ICC is best because it can be large only if there is no bias and the paired measurements are in good agreement, but it suffers from the same faults ii and iii as ordinary correlation coefficients. The magnitude of the ICC can be manipulated by the choice of samples to split and says nothing about the magnitude of the paired differences.

Regression Methods

- Regression analysis is typically misused by regressing one measurement on the other and declare them equivalent if and only if the confidence interval for the

regression coefficient includes 1. Some simple mathematics shows that if the measurements are comparable, the population value of the regression coefficient will be equal to the correlation coefficient between the two methods. The population correlation coefficient may be close to 1, but is never 1 in practice. Thus, the only things that can be indicated by the presence of 1 in the confidence interval for the regression coefficient is (1) that the measurements are comparable but there weren't enough observations to distinguish between 1 and the population regression coefficient, or (2) the population regression coefficient is 1 and therefore, the measurements aren't comparable.

- There is a line whose slope will be 1 if the measurements are comparable. It is known as a structural equation and is the method advanced by Kelly (1985). Altman and Bland (1987) criticize it for a reason that should come as no surprise: Knowing the data are consistent with a structural equation with a slope of 1 says something about the absence of bias but *nothing* about the variability about $Y = X$ (the difference between the measurements), which, as has already been stated, is all that really matters.

3.6 Variations and Alternative Graphical Methods

In this section, we will look at some variations and enhancements of the Bland-Altman plot, as well as some alternative graphcial techniques. Strictly speaking, the Identity Plot is advised by Bland and Altman as a prior analysis to the Bland-Alman plot, and therefore is neither a variant nor an alternative approach. However it is worth mentioning, as it is a simple, powerful and elegant technique that is often overlooked in method comparison studies. The identity plot is a simple scatter-plot approach of measurements for both methods on either axis, with the line of equality (the $X = Y$ line, i.e. the 45 degree line through the origin). This plot can gives the analyst a cursory examination of how well the measurement methods agree. In the case of good agreement, the covariates of the plot accord closely with the line of equality.

Bland and Altman's Percentage and Ratio Plots

? offer two variations of the Bland-Altman plot intended to overcome situations where the conventional plot is inappropriate. The first variation is a plot of casewise differences as percentage of averages, and is appropriate when the variability of the differences increase as the magnitude increases.

The second variation is a plot of casewise ratios as percentage of averages. This will remove the need for logarithmic transformation. This approach is useful when there is an increase in variability of the differences as the magnitude of the measurement increases. **?** proposed such a ratio plot, independently of Bland and Altman. **?** commented on the reception of this article by saying '*Strange to say, this report has been overlooked*'.

3.7   Survival-Agreement Plot

A graphical technique for method comparison studies, that is entirely different to the Bland-Altman plot, was proposed by **?**. This approach, known as the survival-agreement plot, is used to determine the degree of agreement using the Kaplan-Meier method, a well known graphical technique in the area of Survival Analysis. Furthermore **?** propose that commonly used survival analysis techniques should complement this method, *providing a new analytical insight for agreement.* Two survival?agreement plots are used to detect the bias between to measurements of the same variable. The presence of inter-method bias is tested with the log-rank test, and its magnitude with Cox regression.

The degree of agreement (or disagreement) of a measure is expressed as a function of several limits of tolerance, using the Kaplan-Meier method, where the failures occur exactly at absolute values of the differences between the two methods of measurement.

According to Luiz et al, the survival-agreement plot is a step function of a typical survival analysis without censored data, where the Y axis represents the proportion of

discordant cases. This is equivalent to a step function where the X axis represents the absolute observed differences and the Y axis is the proportion of the cases with at least the observed difference (x_i).

3.8   Mountain Plot

Krouwer and Monti have proposed a folded empirical cumulative distribution plot, otherwise known as a Mountain plot.

They argue that it is suitable for detecting large, infrequent errors. This is a non-parametric method that can be used as a complement with the Bland Altman plot. Mountain plots are created by computing a percentile for each ranked difference between a new method and a reference method. (Folded plots are so called because of the following transformation is performed for all percentiles above 50: percentile = 100 - percentile.) These percentiles are then plotted against the differences between the two methods.

Krouwer and Monti argue that the mountain plot offers some following advantages. It is easier to find the central 95% of the data, even when the data are not normally distributed. Also, comparison on different distributions can be performed with ease.

3.9   Formal Models and Tests

While the Bland-Altman plot is useful for inspection of data, **?** notes the lack of formal testing offered by this methodology. Furthermore, **?** formulates a model for single measurement observations as a linear mixed effects model, i.e. a model that additively combines fixed effects and random effects:

$$Y_{ij} = \quad \mu + \beta_j + u_i + \epsilon_{ij} \qquad i = 1, \ldots, n \qquad j = 1, 2$$

The true value of the measurement is represented by μ while the fixed effect due to method j is β_j. For simplicity these terms can be combined into single terms;

$\mu_1 = \mu + \beta_1$ and $\mu_2 = \mu + \beta_2$. The inter-method bias is the difference of the two fixed effect terms, $\beta_1 - \beta_2$. Each individual is assumed to give rise to a random error, represented by u_i. This random effects term is assumed to have mean zero and be normally distributed with variance σ^2. There is assumed to be an attendant error for each measurement on each individual, denoted ϵ_{ij}. This is also assumed to have mean zero. The variance of measurement error for both methods are not assumed to be identical for both methods variance, hence it is denoted σ_j^2. The set of observations (x_i, y_i) by methods X and Y are assumed to follow a bivariate normal distribution with expected values $E(x_i) = \mu_i$ and $E(y_i) = \tau_i$ respectively. The variance covariance of the observations $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 + \sigma_1^2 & \sigma^2 \\ \sigma^2 & \sigma^2 + \sigma_2^2 \end{bmatrix}$$

? demonstrates the estimation of the variance terms and relative precisions relevant to a method comparison study, with attendant confidence intervals for both. The measurement model introduced by **??** provides a formal procedure for estimating the variances σ^2, σ_1^2 and σ_2^2. **?** offers estimates, commonly known as Grubbs estimators, for the various variance components. These estimates are maximum likelihood estimates, which shall be revisited in due course.

$$\hat{\sigma^2} = \sum \frac{(x_i - \bar{x})(y_i - \bar{y})}{n-1} = Sxy$$
$$\hat{\sigma_1^2} = \sum \frac{(x_i - \bar{x})^2}{n-1} = S^2x - Sxy$$
$$\hat{\sigma_2^2} = \sum \frac{(y_i - \bar{y})^2}{n-1} = S^2y - Sxy$$

? defines $\Delta_j = \sigma^2/\sigma_j^2, j = 1, 2$, to be a measure of the relative precision of the measurement methods, and demonstrates how to make statistical inferences about Δ_j.

Based on the following identities,

$$C_x = (n-1)S_x^2,$$

$$C_{xy} = (n-1)S_{xy},$$

$$C_y = (n-1)S_y^2,$$

$$|A| = C_x \times C_y - (C_{xy})^2,$$

the confidence interval limits of Δ_1 are

$$\frac{C_{xy} - t(\frac{|A|}{n-2}))^{\frac{1}{2}}}{C_x - C_{xy} + t(\frac{|A|}{n-2}))^{\frac{1}{2}}} < \Delta_1 < \frac{C_{xy} + t(\frac{|A|}{n-2}))^{\frac{1}{2}}}{C_x - C_{xy} - t(\frac{|A|}{n-1}))^{\frac{1}{2}}}$$

The value t is the $100(1 - \alpha/2)\%$ upper quantile of Student's t distribution with $n-2$ degrees of freedom (**?**). The confidence limits for Δ_2 are found by substituting C_y for C_x in (1.2). Negative lower limits are replaced by the value 0.

The case-wise differences and means are calculated as $d_i = x_i - y_i$ and $a_i = (x_i + y_i)/2$ respectively. Both d_i and a_i are assumed to follow a bivariate normal distribution with $E(d_i) = \mu_d = \mu_1 - \mu_2$ and $E(a_i) = \mu_a = (\mu_1 + \mu_2)/2$, and the variance matrix $\Sigma_{(a,d)}$ is

$$\Sigma_{(a,d)} = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & \frac{1}{2}(\sigma_1^2 - \sigma_2^2) \\ \frac{1}{2}(\sigma_1^2 - \sigma_2^2) & \sigma^2 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2) \end{bmatrix}. \tag{3.1}$$

3.10 Measurement Error Models

DunnSEME proposes a measurement error model for use in method comparison studies. Consider n pairs of measurements X_i and Y_i for $i = 1, 2, ...n$.

$$X_i = \tau_i + \delta_i \tag{3.2}$$

$$Y_i = \alpha + \beta\tau_i + \epsilon_i$$

In the above formulation is in the form of a linear structural relationship, with τ_i and $\beta\tau_i$ as the true values, and δ_i and ϵ_i as the corresponding measurement errors. In the case where the units of measurement are the same, then $\beta = 1$.

$$E(X_i) = \tau_i \tag{3.3}$$

$$E(Y_i) = \alpha + \beta\tau_i$$

$$E(\delta_i) = E(\epsilon_i) = 0$$

The value α is the inter-method bias between the two methods.

$$z_0 = d = 0 \tag{3.4}$$

$$z_{n+1} = z_n^2 + c \tag{3.5}$$

3.11 Inappropriate assessment of Agreement

Chapter 4

Formal Testing Procedures

4.1 Formal Models and Tests

The Bland-Altman plot is a simple tool for inspection of data, and **?** comments on the lack of formal testing offered by that methodology. **?** formulates a model for single measurement observations for a method comparison study as a linear mixed effects model, i.e. model that additively combine fixed effects and random effects.

$$Y_{ij} = \quad \mu + \beta_j + u_i + \epsilon_{ij} \qquad i = 1, \dots, n \qquad j = 1, 2$$

The true value of the measurement is represented by μ while the fixed effect due to method j is β_j. For simplicity these terms can be combined into single terms; $\mu_1 = \mu + \beta_1$ and $\mu_2 = \mu + \beta_2$. The inter-method bias is the difference of the two fixed effect terms, $\beta_1 - \beta_2$. Each of the i individuals are assumed to give rise to random error, represented by u_i. This random effects terms is assumed to have mean zero and be normally distributed with variance σ^2. There is assumed to be an attendant error for each measurement on each individual, denoted ϵ_{ij}. This is also assumed to have mean zero. The variance of measurement error for both methods are not assumed to be identical for both methods variance, hence it is denoted σ_j^2. The set of observations (x_i, y_i) by methods X and Y are assumed to follow the bivariate normal distribution with expected values $E(x_i) = \mu_i$ and $E(x_i) = \mu_i$ respectively. The variance covariance

69

of the observations Σ is given by

$$\Sigma = \begin{bmatrix} \sigma^2 + \sigma_1^2 & \sigma^2 \\ \sigma^2 & \sigma^2 + \sigma_2^2 \end{bmatrix}$$

The inter-method bias is the difference of the two fixed effect terms, $\beta_1 - \beta_2$.

? demonstrates the estimation of the variance terms and relative precisions relevant to a method comparison study, with attendant confidence intervals for both. The measurement model introduced by **??** provides a formal procedure for estimate the variances σ^2, σ_1^2 and σ_2^2 devices. **?** offers estimates, commonly known as Grubbs estimators, for the various variance components. These estimates are maximum likelihood estimates, a statistical concept that shall be revisited in due course.

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? defines Δ_j to be a measure of the relative precision of the measurement methods, with $\Delta_j = \sigma^2/\sigma_j^2$. Thompson also demonstrates how to make statistical inferences about Δ_j. Based on the following identities,

$$
\begin{aligned}
C_x &= (n-1)S_x^2, \\
C_{xy} &= (n-1)S_{xy}, \\
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|A| &= C_x \times C_y - (C_{xy})^2,
\end{aligned}
$$

the confidence interval limits of Δ_1 are

$$\Delta_1 > \frac{C_{xy} - t(\frac{|A|}{n-2}))^{\frac{1}{2}}}{C_x - C_{xy} + t(\frac{|A|}{n-2}))^{\frac{1}{2}}} \tag{4.1}$$

$$\Delta_1 > \frac{C_{xy} + t(\frac{|A|}{n-2}))^{\frac{1}{2}}}{C_x - C_{xy} - t(\frac{|A|}{n-1}))^{\frac{1}{2}}}$$

The value t is the $100(1 - \alpha/2)\%$ upper quantile of Student's t distribution with $n - 2$ degrees of freedom (?). The confidence limits for Δ_2 are found by substituting C_y for C_x in (1.3). Negative lower limits are replaced by the value 0.

The case-wise differences and means are calculated as $d_i = x_i - y_i$ and $a_i = (x_i + y_i)/2$ respectively. Both d_i and a_i are assumed to follow a bivariate normal distribution with $E(d_i) = \mu_d = \mu_1 - \mu_2$ and $E(a_i) = \mu_a = (\mu_1 + \mu_2)/2$. The variance matrix $\Sigma_{(a,d)}$ is

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$$\Delta_1 > \frac{C_{xy} + t(\frac{|A|}{n-2}))^{\frac{1}{2}}}{C_x - C_{xy} - t(\frac{|A|}{n-1}))^{\frac{1}{2}}}$$

The value t is the $100(1-\alpha/2)\%$ upper quantile of Student's t distribution with $n-2$ degrees of freedom (?). The confidence limits for Δ_2 are found by substituting C_y for C_x in (1.3). Negative lower limits are replaced by the value 0.

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4.3 Formal Testing

The Bland Altman plot is a simple tool for inspection of the data, but in itself it offers no formal testing procedure in this regard. To this end, the approach proposed by **?** is a formal test on the Pearson correlation coefficient of casewise differences and means (ρ_{AD}). According to the authors, this test is equivalent to a well established tests for equality of variances, known as the 'Pitman Morgan Test' (**??**).

For the Grubbs data, the correlation coefficient estimate (r_{AD}) is 0.2625, with a 95% confidence interval of (-0.366, 0.726) estimated by Fishers 'r to z' transformation (**?**). The null hypothesis ($\rho_{AD} = 0$) would fail to be rejected. Consequently the null hypothesis of equal variances of each method would also fail to be rejected.

There has no been no further mention of this particular test in the subsequent article published by Bland and Altman, although **?** refers to Spearmans' rank correlation coefficient.

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There has no been no further mention of this particular test in the subsequent article published by Bland and Altman, although **?** refers to Spearmans' rank correlation coefficient.

4.5 Thompson

? presents three relations that hold simultaneously with probability $1 - 2\alpha$ where $2\alpha = 0.01$ or 0.05.

$$|\sigma^2 - C_{xy}K| \leqslant M(C_x C_y)^{\frac{1}{2}} \tag{4.5}$$

$$|\sigma_1^2 - (C_x - C_{xy})K| \leqslant M(C_x(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}$$

$$|\sigma_2^2 - (C_y - C_{xy})K| \leqslant M(C_y(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}$$

The case-wise differences and means are $D_i = Y_{i1} - Y_{i2}$ and $A_i = (Y_{i1} + Y_{i2})/2$ respectively. Both D_i and A_i follow a bivariate normal distribution with $E(D_i) = \mu_D = \mu_1 - \mu_2$ and $E(A_i) = \mu_A = (\mu_1 + \mu_2)/2$. The variance matrix Σ is

$$\Sigma = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & \frac{1}{2}(\sigma_1^2 - \sigma_2^2) \\ \frac{1}{2}(\sigma_1^2 - \sigma_2^2) & \sigma_S^2 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2) \end{bmatrix} \tag{4.6}$$

? demonstrates how the Grubbs estimators for the error variances can be calculated using the difference values, providing a worked example on a data set.

$$\hat{\sigma_1^2} = \sum (y_{i1} - \bar{y}1)(D_i - \bar{D}) \tag{4.7}$$
$$\hat{\sigma_2^2} = \sum (y_{i2} - \bar{y}_2)(D_i - \bar{D})$$

4.6 Thompson 1963

? defines Δ_j to be a measure of the relative precision of the measurement methods, with $\Delta_j = \sigma_S^2/\sigma_j^2$ (where $j = 1, 2$). Confidence intervals for Δ_j are also presented.

$$\Delta_1 > \frac{C_{xy} - t(\frac{|A|}{n-1}))^{\frac{1}{2}}}{C_x - C_{xy} + t(\frac{|A|}{n-1}))^{\frac{1}{2}}}, \tag{4.8}$$

where

$$C_x = (n-1)S_x^2,$$
$$C_{xy} = (n-1)S_{xy},$$
$$C_y = (n-1)S_y^2,$$
$$A = C_x \times C_y - (C_{xy})^2.$$

The value t is the $100(1 - \alpha/2)\%$ quantile of Student's t distribution with $n - 2$ degrees of freedom. The ratio Δ_2 can be found by interchanging C_y and C_x. A lower confidence limit can be found by calculating the square root. The inequality in equation 1.10 may also be used for hypothesis testing.

For the interval estimates for the variance components, **?** presents three relations that hold simultaneously with probability $1 - 2\alpha$ where $2\alpha = 0.01$ or 0.05.

$$|\sigma^2 - C_{xy}K| \leqslant M(C_xC_y)^{\frac{1}{2}}$$
$$|\sigma_1^2 - (C_x - C_{xy})K| \leqslant M(C_x(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}$$
$$|\sigma_2^2 - (C_y - C_{xy})K| \leqslant M(C_y(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}$$

? contains tables for K and M.

? offers a formal simultaneous hypothesis test for the mean and variance of two paired data sets. Using simple linear regression of the differences of each pair against the sums, a line is fitted to the model, with estimates for intercept and slope ($\hat{\beta}_0$ and $\hat{\beta}_1$). The null hypothesis of this test is that the mean (μ) and variance (σ^2) of both data sets are equal if the slope and intercept estimates are equal to zero(i.e $\sigma_1^2 = \sigma_2^2$ and $\mu_1 = \mu_2$ if and only if $\beta_0 = \beta_1 = 0$)

A test statistic is then calculated from the regression analysis of variance values (**?**) and is distributed as 'F' random variable. The degrees of freedom thereof are $\nu_1 = 2$ and $\nu_1 = n - 2$ (where n is the number of pairs). The critical value is chosen for $\alpha\%$ significance with those same degrees of freedom. **?** amends this methodology for use in method comparison studies, using the averages of the pairs, as opposed to the sums, and their differences. This approach can facilitate simultaneous usage of test with the Bland-Altman methodology. Bartko's test statistic take the form:

$$F.test = \frac{(\Sigma d^2) - SSReg}{2MSReg} \qquad (4.9)$$

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Averages	1	0.04	0.04	0.74	0.4097
Residuals	10	0.60	0.06		

Table 4.6.1: Regression ANOVA of case-wise differences and averages for Grubbs Data

For the Grubbs data, $\Sigma d^2 = 5.09$, $SSReg = 0.60$ and $MSreg = 0.06$ Therefore the test statistic is 37.42, with a critical value of 4.10. Hence the means and variance of the Fotobalk and Counter chronometers are assumed to be simultaneously equal.

Importantly, this methodology determines whether there is both inter-method bias and precision present, or alternatively if there is neither present. It has previously been demonstrated that there is a inter-method bias present, but as this procedure does not

allow for separate testing, no conclusion can be drawn on the comparative precision of both methods.

4.7 Thompson 1963

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$$\Delta_1 > \frac{C_{xy} - t(\frac{|A|}{n-1}))^{\frac{1}{2}}}{C_x - C_{xy} + t(\frac{|A|}{n-1}))^{\frac{1}{2}}}, \tag{4.10}$$

where

$$
\begin{aligned}
C_x &= (n-1)S_x^2, \\
C_{xy} &= (n-1)S_{xy}, \\
C_y &= (n-1)S_y^2, \\
A &= C_x \times C_y - (C_{xy})^2.
\end{aligned}
$$

The value t is the $100(1 - \alpha/2)\%$ quantile of Student's t distribution with $n - 2$ degrees of freedom. The ratio Δ_2 can be found by interchanging C_y and C_x. A lower confidence limit can be found by calculating the square root. The inequality in equation 1.10 may also be used for hypothesis testing.

For the interval estimates for the variance components, ? presents three relations that hold simultaneously with probability $1 - 2\alpha$ where $2\alpha = 0.01$ or 0.05.

$$
\begin{aligned}
|\sigma^2 - C_{xy}K| &\leqslant M(C_x C_y)^{\frac{1}{2}} \\
|\sigma_1^2 - (C_x - C_{xy})K| &\leqslant M(C_x(C_x + C_y - 2C_{xy}))^{\frac{1}{2}} \\
|\sigma_2^2 - (C_y - C_{xy})K| &\leqslant M(C_y(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}
\end{aligned}
$$

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A test statistic is then calculated from the regression analysis of variance values (**?**) and is distributed as 'F' random variable. The degrees of freedom thereof are $\nu_1 = 2$ and $\nu_1 = n - 2$ (where n is the number of pairs). The critical value is chosen for $\alpha\%$ significance with those same degrees of freedom. **?** amends this methodology for use in method comparison studies, using the averages of the pairs, as opposed to the sums, and their differences. This approach can facilitate simultaneous usage of test with the Bland-Altman methodology. Bartko's test statistic take the form:

$$F.test = \frac{(\Sigma d^2) - SSReg}{2MSReg} \tag{4.11}$$

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4.8   Model Formulation and Formal Testing

? formulates a model for un-replicated observations for a method comparison study as a mixed model.

$$Y_{ij} = \mu_j + S_i + \epsilon_{ij} \quad i = 1, 2...n \quad j = 1, 2 \tag{4.12}$$

$$S \sim N(0, \sigma_s^2) \qquad \epsilon_{ij} \sim N(0, \sigma_j^2)$$

As with all mixed models, the variance of each observation is the sum of all the associated variance components.

$$var(Y_{ij}) = \sigma_s^2 + \sigma_j^2 \tag{4.13}$$

$$cov(Y_{i1}, Y_{i2}) = \sigma_s^2$$

? offers maximum likelihood estimators, commonly known as Grubbs estimators, for the various variance components:

$$\hat{\sigma_s^2} = \sum \frac{(x_i - \bar{x})(y_i - \bar{y})}{n-1} = Sxy \tag{4.14}$$

$$\hat{\sigma_1^2} = \sum \frac{(x_i - \bar{x})^2}{n-1} = S^2x - Sxy$$

$$\hat{\sigma_2^2} = \sum \frac{(y_i - \bar{y})^2}{n-1} = S^2y - Sxy$$

The standard error of these variance estimates are:

$$var(\sigma_1^2) = \frac{2\sigma_1^4}{n-1} + \frac{\sigma_S^2\sigma_1^2 + \sigma_S^2\sigma_2^2 + \sigma_1^2\sigma_2^2}{n-1} \tag{4.15}$$

$$var(\sigma_2^2) = \frac{2\sigma_2^4}{n-1} + \frac{\sigma_S^2\sigma_1^2 + \sigma_S^2\sigma_2^2 + \sigma_1^2\sigma_2^2}{n-1}$$

?presents confidence intervals for the relative precisions of the measurement methods, $\Delta_j = \sigma_S^2/\sigma_j^2$ (where $j = 1, 2$), as well as the variances σ_S^2, σ_1^2 and σ_2^2.

$$\Delta_1 > \frac{C_{xy} - t(|A|/n - 2))^{\frac{1}{2}}}{C_x - C_{xy} + t(|A|/n - 2))^{\frac{1}{2}}} \tag{4.16}$$

where

$$C_x = (n-1)S_x^2$$

$$C_{xy} = (n-1)S_{xy}$$

$$C_y = (n-1)S_y^2$$

$$A = C_x \times C_y - (C_{xy})^2$$

t is the $100(1 - \alpha/2)\%$ quantile of Student's t distribution with $n - 2$ degrees of freedom. Δ_2 can be found by changing C_y for C_x. A lower confidence limit can be found by calculating the square root. This inequality may also be used for hypothesis testing.

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$$S \sim N(0, \sigma_s^2) \qquad \epsilon_{ij} \sim N(0, \sigma_j^2)$$

As with all mixed models, the variance of each observation is the sum of all the associated variance components.

$$var(Y_{ij}) = \sigma_s^2 + \sigma_j^2 \tag{4.18}$$

$$cov(Y_{i1}, Y_{i2}) = \sigma_s^2$$

? offers maximum likelihood estimators, commonly known as Grubbs estimators, for the various variance components:

$$\hat{\sigma_s^2} = \sum \frac{(x_i - \bar{x})(y_i - \bar{y})}{n-1} = Sxy \tag{4.19}$$

$$\hat{\sigma_1^2} = \sum \frac{(x_i - \bar{x})^2}{n-1} = S^2x - Sxy$$

$$\hat{\sigma_2^2} = \sum \frac{(y_i - \bar{y})^2}{n-1} = S^2y - Sxy$$

The standard error of these variance estimates are:

$$var(\sigma_1^2) = \frac{2\sigma_1^4}{n-1} + \frac{\sigma_S^2\sigma_1^2 + \sigma_S^2\sigma_2^2 + \sigma_1^2\sigma_2^2}{n-1} \tag{4.20}$$

$$var(\sigma_2^2) = \frac{2\sigma_2^4}{n-1} + \frac{\sigma_S^2\sigma_1^2 + \sigma_S^2\sigma_2^2 + \sigma_1^2\sigma_2^2}{n-1}$$

?presents confidence intervals for the relative precisions of the measurement methods, $\Delta_j = \sigma_S^2/\sigma_j^2$ (where $j = 1, 2$), as well as the variances σ_S^2, σ_1^2 and σ_2^2.

$$\Delta_1 > \frac{C_{xy} - t(|A|/n-2))^{\frac{1}{2}}}{C_x - C_{xy} + t(|A|/n-2))^{\frac{1}{2}}} \tag{4.21}$$

where

$$C_x = (n-1)S_x^2$$

$$C_{xy} = (n-1)S_{xy}$$

$$C_y = (n-1)S_y^2$$

$$A = C_x \times C_y - (C_{xy})^2$$

t is the $100(1 - \alpha/2)\%$ quantile of Student's t distribution with $n - 2$ degrees of freedom. Δ_2 can be found by changing C_y for C_x. A lower confidence limit can be found by calculating the square root. This inequality may also be used for hypothesis testing.

For the interval estimates for the variance components, **?** presents three relations that hold simultaneously with probability $1 - 2\alpha$ where $2\alpha = 0.01$ or 0.05.

$$|\sigma^2 - C_{xy}K| \leqslant M(C_x C_y)^{\frac{1}{2}} \tag{4.22}$$

$$|\sigma_1^2 - (C_x - C_{xy})K| \leqslant M(C_x(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}$$

$$|\sigma_2^2 - (C_y - C_{xy})K| \leqslant M(C_y(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}$$

The case-wise differences and means are $D_i = Y_{i1} - Y_{i2}$ and $A_i = (Y_{i1} + Y_{i2})/2$ respectively. Both D_i and A_i follow a bivariate normal distribution with $E(D_i) = \mu_D = \mu_1 - \mu_2$ and $E(A_i) = \mu_A = (\mu_1 + \mu_2)/2$. The variance matrix Σ is

$$\Sigma = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & \frac{1}{2}(\sigma_1^2 - \sigma_2^2) \\ \frac{1}{2}(\sigma_1^2 - \sigma_2^2) & \sigma_S^2 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2) \end{bmatrix} \tag{4.23}$$

? demonstrates how the Grubbs estimators for the error variances can be calculated using the difference values, providing a worked example on a data set.

$$\begin{aligned} \hat{\sigma_1^2} &= \sum (y_{i1} - \bar{y}1)(D_i - \bar{D}) \\ \hat{\sigma_2^2} &= \sum (y_{i2} - \bar{y}_2)(D_i - \bar{D}) \end{aligned} \tag{4.24}$$

4.10   Model Formulation and Formal Testing

? formulates a model for un-replicated observations for a method comparison study as a mixed model.

$$\begin{aligned} Y_{ij} &= \mu_j + S_i + \epsilon_{ij} \quad i = 1, 2...n \quad j = 1, 2 \\ S &\sim N(0, \sigma_s^2) \qquad \epsilon_{ij} \sim N(0, \sigma_j^2) \end{aligned} \tag{4.25}$$

As with all mixed models, the variance of each observation is the sum of all the associated variance components.

$$\begin{aligned} var(Y_{ij}) &= \sigma_s^2 + \sigma_j^2 \\ cov(Y_{i1}, Y_{i2}) &= \sigma_s^2 \end{aligned} \tag{4.26}$$

? offers maximum likelihood estimators, commonly known as Grubbs estimators, for the various variance components:

$$\begin{aligned} \hat{\sigma_s^2} &= \sum \frac{(x_i - \bar{x})(y_i - \bar{y})}{n - 1} = Sxy \\ \hat{\sigma_1^2} &= \sum \frac{(x_i - \bar{x})^2}{n - 1} = S^2x - Sxy \\ \hat{\sigma_2^2} &= \sum \frac{(y_i - \bar{y})^2}{n - 1} = S^2y - Sxy \end{aligned} \tag{4.27}$$

The standard error of these variance estimates are:

$$\begin{aligned} var(\sigma_1^2) &= \frac{2\sigma_1^4}{n - 1} + \frac{\sigma_S^2\sigma_1^2 + \sigma_S^2\sigma_2^2 + \sigma_1^2\sigma_2^2}{n - 1} \\ var(\sigma_2^2) &= \frac{2\sigma_2^4}{n - 1} + \frac{\sigma_S^2\sigma_1^2 + \sigma_S^2\sigma_2^2 + \sigma_1^2\sigma_2^2}{n - 1} \end{aligned} \tag{4.28}$$

?presents confidence intervals for the relative precisions of the measurement methods, $\Delta_j = \sigma_S^2/\sigma_j^2$ (where $j = 1, 2$), as well as the variances σ_S^2, σ_1^2 and σ_2^2.

$$\Delta_1 > \frac{C_{xy} - t(|A|/n - 2))^{\frac{1}{2}}}{C_x - C_{xy} + t(|A|/n - 2))^{\frac{1}{2}}} \tag{4.29}$$

where

$$
\begin{aligned}
C_x &= (n-1)S_x^2 \\
C_{xy} &= (n-1)S_{xy} \\
C_y &= (n-1)S_y^2 \\
A &= C_x \times C_y - (C_{xy})^2
\end{aligned}
$$

t is the $100(1 - \alpha/2)\%$ quantile of Student's t distribution with $n - 2$ degrees of freedom. Δ_2 can be found by changing C_y for C_x. A lower confidence limit can be found by calculating the square root. This inequality may also be used for hypothesis testing.

For the interval estimates for the variance components, ? presents three relations that hold simultaneously with probability $1 - 2\alpha$ where $2\alpha = 0.01$ or 0.05.

$$|\sigma^2 - C_{xy}K| \leqslant M(C_x C_y)^{\frac{1}{2}} \tag{4.30}$$

$$|\sigma_1^2 - (C_x - C_{xy})K| \leqslant M(C_x(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}$$

$$|\sigma_2^2 - (C_y - C_{xy})K| \leqslant M(C_y(C_x + C_y - 2C_{xy}))^{\frac{1}{2}}$$

The case-wise differences and means are $D_i = Y_{i1} - Y_{i2}$ and $A_i = (Y_{i1} + Y_{i2})/2$ respectively. Both D_i and A_i follow a bivariate normal distribution with $E(D_i) = \mu_D = \mu_1 - \mu_2$ and $E(A_i) = \mu_A = (\mu_1 + \mu_2)/2$. The variance matrix Σ is

$$\Sigma = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & \frac{1}{2}(\sigma_1^2 - \sigma_2^2) \\ \frac{1}{2}(\sigma_1^2 - \sigma_2^2) & \sigma_S^2 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2) \end{bmatrix} \tag{4.31}$$

? demonstrates how the Grubbs estimators for the error variances can be calculated using the difference values, providing a worked example on a data set.

$$\hat{\sigma_1^2} = \sum (y_{i1} - \bar{y1})(D_i - \bar{D}) \tag{4.32}$$
$$\hat{\sigma_2^2} = \sum (y_{i2} - \bar{y_2})(D_i - \bar{D})$$

4.11   Identifiability

? highlights an important issue regarding using models such as these, the identifiability problem. This comes as a result of there being too many parameters to be estimated. Therefore assumptions about some parameters, or estimators used, must be made so that others can be estimated. For example in literature the variance ratio $\lambda = \frac{\sigma_1^2}{\sigma_2^2}$ must often be assumed to be equal to 1 (**?**).**?** considers methodologies based on two methods with single measurements on each subject as inadequate for a serious study on the measurement characteristics of the methods. This is because there would not be enough data to allow for a meaningful analysis. There is, however, a contrary argument that in many practical settings it is very difficult to get replicate observations when the measurement method requires invasive medical procedure.

? offers a formal simultaneous hypothesis test for the mean and variance of two paired data sets. Using simple linear regression of the differences of each pair against the sums, a line is fitted to the model, with estimates for intercept and slope ($\hat{\beta}_0$ and $\hat{\beta}_1$). The null hypothesis of this test is that the mean (μ) and variance (σ^2) of both data sets are equal if the slope and intercept estimates are equal to zero(i.e $\sigma_1^2 = \sigma_2^2$ and $\mu_1 = \mu_2$ if and only if $\beta_0 = \beta_1 = 0$)

A test statistic is then calculated from the regression analysis of variance values (**?**) and is distributed as 'F' random variable. The degrees of freedom are $\nu_1 = 2$ and $\nu_1 = n - 2$ (where n is the number of pairs). The critical value is chosen for $\alpha\%$ significance with those same degrees of freedom. **?** amends this methodology for use in method comparison studies, using the averages of the pairs, as opposed to the sums,

and their differences. This approach can facilitate simultaneous usage of test with the Bland-Altman methodology. Bartko's test statistic take the form:

$$F.test = \frac{(\Sigma d^2) - SSReg}{2MSReg}$$

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Averages	1	0.04	0.04	0.74	0.4097
Residuals	10	0.60	0.06		

Table 4.11.3: Regression ANOVA of case-wise differences and averages for Grubbs Data

For the Grubbs data, $\Sigma d^2 = 5.09$, $SSReg = 0.60$ and $MSreg = 0.06$ Therefore the test statistic is 37.42, with a critical value of 4.10. Hence the means and variance of the Fotobalk and Counter chronometers are assumed to be simultaneously equal.

Importantly, this methodology determines whether there is both inter-method bias and precision present, or alternatively if there is neither present. It has previously been demonstrated that there is a inter-method bias present, but as this procedure does not allow for separate testing, no conclusion can be drawn on the comparative precision of both methods.

4.12 Morgan Pitman

The test of the hypothesis that the variance of both methods are equal is based on the correlation value $\rho_{D,A}$ which is evaluated as follows;

$$\rho(D, A) = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{(\sigma_1^2 + \sigma_2^2)(4\sigma_S^2 + \sigma_1^2 + \sigma_2^2)}} \tag{4.33}$$

The correlation constant takes the value zero if, and only if, the two variances are equal. Therefore a test of the hypothesis $H : \sigma_1^2 = \sigma_2^2$ is equivalent to a test of the

hypothesis $H : \rho(D, A) = 0$. The corresponds to the well-known t test for a correlation coefficient with $n - 2$ degrees of freedom.

? describes the Morgan-Pitman test as identical to the test of the slope equal to zero in the regression of Y_{i1} on Y_{12}, adding that this result can be shown using straightforward algebra.

4.13 Bland-Altman correlation test

The approach proposed by **?** is a formal test on the Pearson correlation coefficient of case-wise differences and means (ρ_{AD}). According to the authors, this test is equivalent to the 'Pitman Morgan Test'. For the Grubbs data, the correlation coefficient estimate (r_{AD}) is 0.2625, with a 95% confidence interval of (-0.366, 0.726) estimated by Fishers 'r to z' transformation (**?**). The null hypothesis ($\rho_{AD} = 0$) fail to be rejected. Consequently the null hypothesis of equal variances of each method would also fail to be rejected. There has no been no further mention of this particular test in **?**, although **?** refers to Spearman's rank correlation coefficient. **?** comments 'we do not see a place for methods of analysis based on hypothesis testing'. **?** also states that consider structural equation models to be inappropriate.

4.14 Pitman & Morgan Test

This test assess the equality of population variances. Pitman's test tests for zero correlation between the sums and products.

Correlation between differences and means is a test statistics for the null hypothesis of equal variances given bivariate normality.

4.15 Paired T tests

This method can be applied to test for statistically significant deviations in bias. This method can be potentially misused for method comparison studies.

It is a poor measure of agreement when the rater's measurements are perpendicular to the line of equality[Hutson et al]. In this context, an average difference of zero between the two raters, yet the scatter plot displays strong negative correlation.

4.16 Morgan Pitman Testing

An early contribution to formal testing in method comparison was made by both **?** and **?**, in separate contributions. The basis of this approach is that the distribution of the original measurements is bivariate normal. Morgan and Pitman noted that the correlation coefficient depends upon the difference $\sigma_1^2 - \sigma_2^2$, being zero if and only if $\sigma_1^2 = \sigma_2^2$.

The classical Pitman-Morgan test is a hypothesis test for equality of the variance of two data sets; $\sigma_1^2 = \sigma_2^2$, based on the correlation value $\rho_{a,d}$,and is evaluated as follows;

$$\rho(a, d) = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{(\sigma_1^2 + \sigma_2^2)(4\sigma_S^2 + \sigma_1^2 + \sigma_2^2)}} \tag{4.34}$$

The correlation constant takes the value zero if, and only if, the two variances are equal. Therefore a test of the hypothesis $H : \sigma_1^2 = \sigma_2^2$ is equivalent to a test of the hypothesis $H : \rho(D, A) = 0$. This corresponds to the well-known t test for a correlation coefficient with $n - 2$ degrees of freedom. **?** describes the Morgan-Pitman test as identical to the test of the slope equal to zero in the regression of Y_{i1} on Y_{12}, a result that can be derived using straightforward algebra.

4.17 Paired sample T-test

? discusses the use of the well known paired sample t test to test for inter-method bias;

$H : \mu_D = 0$. The test statistic is distributed a t random variable with $n - 1$ degrees of freedom and is calculated as follows;

$$t^* = \bar{D}/\frac{S_D}{\sqrt{n}} \qquad (4.35)$$

where $\bar{D}$ and S_D is the average of the differences of the n observations.

- Paired t tests test only whether the mean responses are the same. Certainly, we want the means to be the same, but this is only a small part of the story. The means can be equal while the (random) differences between measurements can be huge.

- The correlation coefficient measures linear agreement–whether the measurements go up-and-down together. Certainly, we want the measures to go up-and-down together, but the correlation coefficient itself is deficient in at least three ways as a measure of agreement. The correlation coefficient can be close to 1 (or equal to 1!) even when there is considerable bias between the two methods. For example, if one method gives measurements that are always 10 units higher than the other method, the correlation will be 1 exactly, but the measurements will always be 10 units apart.

- The magnitude of the correlation coefficient is affected by the range of subjects/units studied.

- The correlation coefficient can be made smaller by measuring samples that are similar to each other and larger by measuring samples that are very different from each other. The magnitude of the correlation says nothing about the magnitude of the differences between the paired measurements which, when you get right down to it, is all that really matters.

- The usual significance test involving a correlation coefficient– whether the population value is 0–is irrelevant to the comparability problem. What is important is not merely that the correlation coefficient be different from 0. Rather, it should be close to (ideally, equal to) 1!

4.18   Paired sample t test

? discusses the use of the well known paired sample t test to test for inter-method bias;

$H : \mu_d = 0$. The test statistic is distributed a t random variable with $n - 1$ degrees of freedom and is calculated as follows,

$$t^* = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}} \tag{4.36}$$

where $\bar{d}$ and s_d is the average of the differences of the n observations. Only if the two methods show comparable precision then the paired sample student t-test is appropriate for assessing the magnitude of the bias.

Structural Equation Modelling

Authors, such as a **?**, **?** and **?**, strongly advocate the use of *Structural Equation Models* for the purposes of method comparison. Conversely **?** also states that consider structural equation models to be inappropriate.

Chapter 5

Regression Procedures

5.1 Constant and Proportional Bias

Constant Bias This is a form of systematic deviations estimated as the average difference between the test and the reference method.

Proportional Bias Two methods may agree on average, but they may exhibit differences over a range of measurements.

Proportional Bias is a difference in the two measures which is proportional to the scale of the measurement.

Using a naive estimation of bias, such as the mean of differences, it may incorrectly indicate absence of bias, by yielding a mean difference close to zero. This would be caused by positive differences in the measurements at one end of the range of measurements being canceled out by negative differences at the other end of the scale.

5.2 Regression Methods

Conventional regression models are estimated using the ordinary least squares (OLS) technique, and are referred to as 'Model I regression' (??). A key feature of Model I models is that the independent variable is assumed to be measured without error.

As often pointed out in several papers (**??**), this assumption invalidates simple linear regression for use in method comparison studies, as both methods must be assumed to be measured with error.

The use of regression models that assumes the presence of error in both variables X and Y have been proposed for use instead (**??**). These methodologies are collectively known as 'Model II regression'. They differ in the method used to estimate the parameters of the regression.

Regression estimates depend on formulation of the model. A formulation with one method considered as the X variable will yield different estimates for a formulation where it is the Y variable. With Model I regression, the models fitted in both cases will entirely different and inconsistent. However with Model II regression, they will be consistent and complementary.

Regression approaches are useful for a making a detailed examination of the biases across the range of measurements, allowing bias to be decomposed into fixed bias and proportional bias. Fixed bias describes the case where one method gives values that are consistently different to the other across the whole range. Proportional bias describes the difference in measurements getting progressively greater, or smaller, across the range of measurements. A measurement method may have either an attendant fixed bias or proportional bias, or both. (**?**). Determination of these biases shall be discussed in due course.

5.3   Other Types of Studies

? categorize method comparison studies into three different types. The key difference between the first two is whether or not a 'gold standard' method is used. In situations where one instrument or method is known to be 'accurate and precise', it is considered as the'gold standard' (**?**). A method that is not considered to be a gold standard is referred to as an 'approximate method'. In calibration studies they are referred to a criterion methods and test methods respectively.

1. Calibration problems. The purpose is to establish a relationship between methods, one of which is an approximate method, the other a gold standard. The results of the approximate method can be mapped to a known probability distribution of the results of the gold standard (**?**). (In such studies, the gold standard method and corresponding approximate method are generally referred to a criterion method and test method respectively.) **?** make clear that their methodology is not intended for calibration problems.

2. Comparison problems. When two approximate methods, that use the same units of measurement, are to be compared. This is the case which the Bland-Altman methodology is specfically intended for, and therefore it is the most relevant of the three.

3. Conversion problems. When two approximate methods, that use different units of measurement, are to be compared. This situation would arise when the measurement methods use 'different proxies', i.e different mechanisms of measurement. **?** deals specifically with this issue. In the context of this study, it is the least relevant of the three.

?, p.47 cautions that'gold standards' should not be assumed to be error free. 'It is of necessity a subjective decision when we come to decide that a particular method or instrument can be treated as if it was a gold standard'. The clinician gold standard , the sphygmomanometer, is used as an example thereof. The sphygmomanometer 'leaves considerable room for improvement' (**?**). **?** similarly addresses the issue of glod standards, 'well-established gold standard may itself be imprecise or even unreliable'.

The NIST F1 Caesium fountain atomic clock is considered to be the gold standard when measuring time, and is the primary time and frequency standard for the United States. The NIST F1 is accurate to within one second per 60 million years (**?**).

Measurements of the interior of the human body are, by definition, invasive medical procedures. The design of method must balance the need for accuracy of measurement with the well-being of the patient. This will inevitably lead to the measurement error as

described by **?**. The magnetic resonance angiogram, used to measure internal anatomy, is considered to the gold standard for measuring aortic dissection. Medical test based upon the angiogram is reported to have a false positive reporting rate of 5% and a false negative reporting rate of 8%. This is reported as sensitivity of 95% and a specificity of 92% (**?**).

In literature they are, perhaps more accurately, referred to as 'fuzzy gold standards' (**?**). Consequently when one of the methods is essentially a fuzzy gold standard, as opposed to a 'true' gold standard, the comparison of the criterion and test methods should be consider in the context of a comparison study, as well as of a calibration study.

5.4   Outline of Thesis

Thus the study of method comparison is introduced. The intention of this thesis is to progress the study of method comparison studies, using a statistical method known as Linear mixed effects models. Chapter two shall describe linear mixed effects models, and how the use of the linear mixed effects models have so far extended to method comparison studies. Implementations of important existing work shall be presented, using the `R` programming language.

Model diagnostics are an integral component of a complete statistical analysis. In chapter three model diagnostics shall be described in depth, with particular emphasis on linear mixed effects models, further to chapter two.

For the fourth chapter, important linear mixed effects model diagnostic methods shall be extended to method comparison studies, and proposed methods shall be demonstrated on data sets that have become well known in literature on method comparison. The purpose is to both calibrate these methods and to demonstrate applications for them. The last chapter shall focus on robust measures of important parameters such as agreement.

5.5   Bartko's Regression and Ellipse

? offers a formal simultaneous hypothesis test for the mean and variance of two paired data sets. Using simple linear regression of the differences of each pair against the sums, a line is fitted to the model, with estimates for intercept and slope (β_0 and β_1). The null hypothesis of this test is that the mean (μ) and variance (σ^2) of both data sets are equal if the slope and intercept estimates are equal to zero(i.e $\sigma_1^2 = \sigma_2^2$ and $\mu_1 = \mu_2$ if and only if $\beta_0 = \beta_1 = 0$)

A test statistic is then calculated from the regression analysis of variance values (**?**) and is distributed as 'F' random variable. The degrees of freedom thereof are $\nu_1 = 2$ and $\nu_1 = n - 2$ (where n is the number of pairs). The critical value is chosen for

$\alpha\%$ significance with those same degrees of freedom. **?** amends this metholodogy for calculation using the from the averages of the pairs, as opposed to the sums, and their differences. This would facilitate simultaneous usage of test with the Bland Altman methodology. Bartko's test statistic take the form:

$$F.test = \frac{(\Sigma D^2) - SSReg}{2MSReg} \tag{5.1}$$

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Averages	1	0.04	0.04	0.74	0.4097
Residuals	10	0.60	0.06		

Table 5.5.1: Regression ANOVA of case-wise differences and averages for Grubbs Data

For the Grubbs data, $\Sigma D^2 = 5.09$, $SSReg = 0.60$ and $MSreg = 0.06$ Therefore the test statistic is 37.42, with a critical value of 4.102821 (calculate using r code $qf(0.95, 2, 10)$). Hence the means and variance of the Fotobalk and Counter chronometers are assumed to be simultaneously equal.

Importantly, this methodology determines whether there is both inter-method bias and precision present, or alternatively if there is neither present. It has previously been demonstrated that there is a inter-method bias present, but as this procedure does not allow for seperate testing, no conclusion can be drawn on the comparative precision of both methods.

5.6 Bartko's Regression and Ellipse

? offers a formal simultaneous hypothesis test for the mean and variance of two paired data sets. Using simple linear regression of the differences of each pair against the sums, a line is fitted to the model, with estimates for intercept and slope (β_0 and β_1). The null hypothesis of this test is that the mean (μ) and variance (σ^2) of both data sets are equal if the slope and intercept estimates are equal to zero(i.e $\sigma_1^2 = \sigma_2^2$ and $\mu_1 = \mu_2$ if and only if $\beta_0 = \beta_1 = 0$)

A test statistic is then calculated from the regression analysis of variance values (**?**) and is distributed as 'F' random variable. The degrees of freedom thereof are $\nu_1 = 2$ and $\nu_1 = n - 2$ (where n is the number of pairs). The critical value is chosen for $\alpha\%$ significance with those same degrees of freedom. **?** amends this metholodgy for calculation using the from the averages of the pairs, as opposed to the sums, and their differences. This would facilitate simultaneous usage of test with the Bland Altman methodology. Bartko's test statistic take the form:

$$F.test = \frac{(\Sigma D^2) - SSReg}{2MSReg} \tag{5.2}$$

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Averages	1	0.04	0.04	0.74	0.4097
Residuals	10	0.60	0.06		

Table 5.6.2: Regression ANOVA of case-wise differences and averages for Grubbs Data

For the Grubbs data, $\Sigma D^2 = 5.09$, $SSReg = 0.60$ and $MSreg = 0.06$ Therefore the test statistic is 37.42, with a critical value of 4.102821 (calculate using r code $qf(0.95, 2, 10)$). Hence the means and variance of the Fotobalk and Counter chronometers are assumed to be simultaneously equal.

Importantly, this methodology determines whether there is both inter-method bias and precision present, or alternatively if there is neither present. It has previously been demonstrated that there is a inter-method bias present, but as this procedure does not allow for seperate testing, no conclusion can be drawn on the comparative precision of both methods.

5.7   Bartko's Ellipse

? offers a graphical complement to the Bland-Altman plot, in the form of a bivariate confidence ellipse. ? provides the relevant calculations.

5.8   Bartko's Ellipse

As an enhancement on the Bland Altman Plot, ? has expounded a confidence ellipse for the covariates. ? proposes a bivariate confidence ellipse as a boundary for dispersion. The stated purpose is to 'amplify dispersion', which presumably is for the purposes of outlier detection. The orientation of the the ellipse is key to interpreting the results. The minor axis is related to the between-item variability whereas the major axis is related to the mean squared error (referred to here as Error Mean Square).The ellipse illustrates the size of both relative to each other.
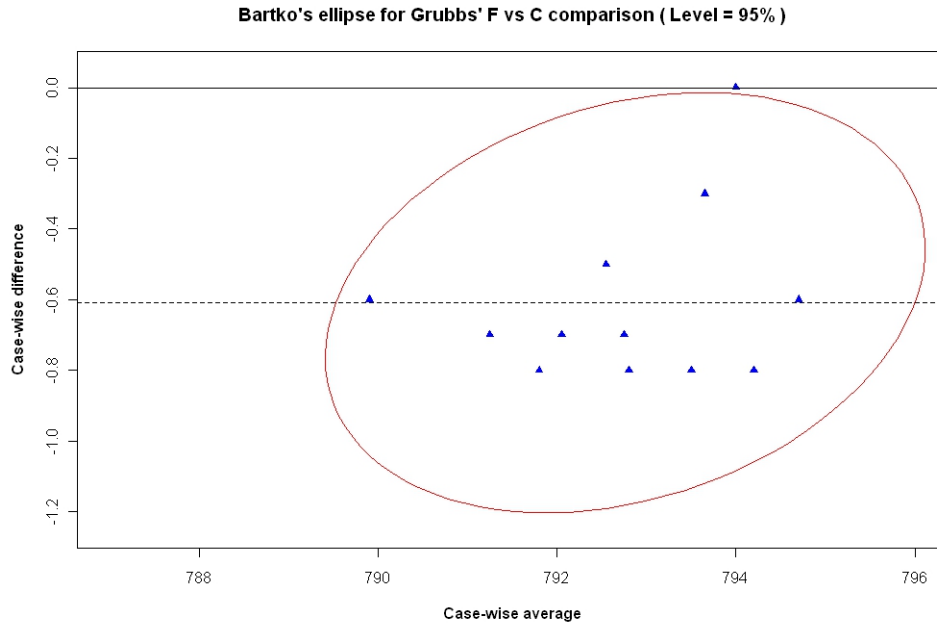
Figure 5.7.1: Bartko's Ellipse For Grubbs Data

Consequently Bartko's ellipse provides a visual aid to determining the relationship between variances. Furthermore, the ellipse provides a visual aid to determining the relationship between the variance of the means $Var(a_i)$ and the variance of the differences $Var(d_i)$. If var(a) is greater than var(d), the orientation of the ellipse is horizontal. Conversely if var(a) is less than var(d), the orientation of the ellipse is vertical. The more horizontal the ellipse, the greater the degree of agreement between the two methods being tested.

Bartko states that the ellipse can, inter alia, be used to detect the presence of outliers (furthermore **?** proposes formal testing procedures, that shall be discussed in due course). The Bland-Altman plot for the Grubbs data, complemented by Bartko's ellipse, is depicted in Figure 5.8.2. The fourth observation is shown to be outside the bounds of the ellipse, indicating that it is a potential outlier.

The limitations of using bivariate approaches to outlier detection in the Bland-Altman plot can demonstrated using Bartko's ellipse. A covariate is added to the 'F vs C' comparison that has a difference value equal to the inter-method bias, and
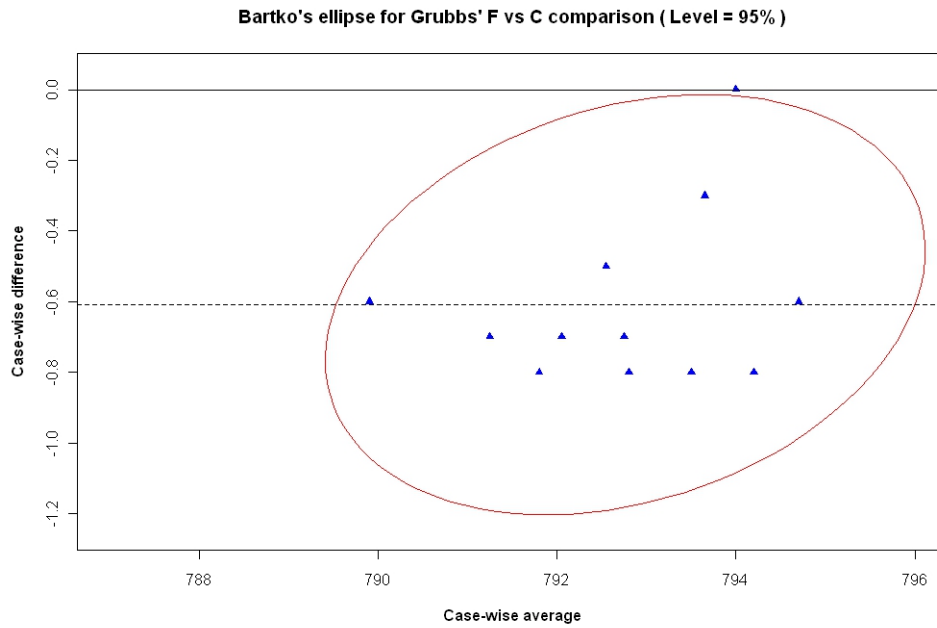
Figure 5.8.2: Bartko's Ellipse For Grubbs' Data.

an average value that markedly deviates from the rest of the average values in the comparison, i.e. 786. Table 1.8 depicts a 95% confidence ellipse for this manipulated data set. By inspection of the confidence interval, a conclusion would be reached that this extra covariate is an outlier, in spite of the fact that this observation is wholly consistent with the conclusion of the Bland-Altman plot.

Importantly, outlier classification must be informed by the logic of the data's formulation. In the Bland-Altman plot, the horizontal displacement of any observation is supported by two independent measurements. Any observation should not be considered an outlier on the basis of a noticeable horizontal displacement from the main cluster, as in the case with the extra covariate. Conversely, the fourth observation, from the original data set, should be considered an outlier, as it has a noticeable vertical displacement from the rest of the observations.

Figure 5.8.3: Bartko's Ellipse For Grubbs' Data, with an extra covariate.

In the Bland-Altman plot, the horizontal displacement of any point on the plot is supported by two independent measurements. Any point should not be considered an outlier on the basis of a noticeable horizontal displacement from the main cluster, as in the case with the extra co-variate. Conversely, the fourth point, from the original data set, should be considered an outlier, as it has a noticeable vertical displacement from the rest of the observations.

5.9 Blackwood Bradley Model

Bradley and Blackwood have developed a regression based approach assessing the agreement.

5.10 Blackwood Bradley Model

This is a regression based approach that performs a simultaneous test for the equivalence of means and variances of the respective methods.

We have identified this approach to be examined to see if it can be used as a foundation for a test perform a test on means and variances individually.

$$D = (X_1 - X_2) \tag{5.3}$$

$$M = (X_1 + X_2)/2 \tag{5.4}$$

The Bradley Blackwood Procedure fits D on M as follows:

$$D = \beta_0 + \beta_1 M \tag{5.5}$$

Both beta values, the intercept and slope, are derived from the respective means and standard deviations of their respective data sets.

We determine if the respective means and variances are equal if both beta values are simultaneously equal to zero. The Test is conducted using an F test, calculated from the results of a regression of D on M.

Russell et al have suggested this method be used in conjunction with a paired t-test , with estimates of slope and intercept. Bradley and Blackwood have developed a regression based approach assessing the agreement.

The Bradley Blackwood test is a simultaneous test for bias and precision. They propose a regression approach which fits D on M, where D is the difference and average of a pair of results.

5.11 Blackwood Bradley Model

This is a regression based approach that performs a simultaneous test for the equivalence of means and variances of the respective methods.

We have identified this approach to be examined to see if it can be used as a foundation for a test perform a test on means and variances individually.

$$D = (X_1 - X_2) \tag{5.6}$$

$$M = (X_1 + X_2)/2 \tag{5.7}$$

The Bradley Blackwood Procedure fits D on M as follows:

$$D = \beta_0 + \beta_1 M \tag{5.8}$$

Both beta values, the intercept and slope, are derived from the respective means and standard deviations of their respective data sets.

We determine if the respective means and variances are equal if both beta values are simultaneously equal to zero. The Test is conducted using an F test, calculated from the results of a regression of D on M.

Russell et al have suggested this method be used in conjunction with a paired t-test , with estimates of slope and intercept. Bradley and Blackwood have developed a regression based approach assessing the agreement.

The Bradley Blackwood test is a simultaneous test for bias and precision. They propose a regression approach which fits D on M, where D is the difference and average of a pair of results.

5.12 Blackwood -Bradley Model

? have developed a regression based procedure for assessing the agreement. This approach performs a simultaneous test for the equivalence of means and variances of the respective methods. Using simple linear regression of the differences of each pair against the sums, a line is fitted to the model, with estimates for intercept and slope

$(\hat{\beta}_0$ and $\hat{\beta}_1)$.

$$D = (X_1 - X_2) \tag{5.9}$$

$$M = (X_1 + X_2)/2 \tag{5.10}$$

The Bradley Blackwood Procedure fits D on M as follows:

$$D = \beta_0 + \beta_1 M \tag{5.11}$$

This technique offers a formal simultaneous hypothesis test for the mean and variance of two paired data sets. The null hypothesis of this test is that the mean (μ) and variance (σ^2) of both data sets are equal if the slope and intercept estimates are equal to zero(i.e $\sigma_1^2 = \sigma_2^2$ and $\mu_1 = \mu_2$ if and only if $\beta_0 = \beta_1 = 0$)

A test statistic is then calculated from the regression analysis of variance values (?) and is distributed as 'F' random variable. The degrees of freedom are $\nu_1 = 2$ and $\nu_1 = n - 2$ (where n is the number of pairs). The critical value is chosen for $\alpha\%$ significance with those same degrees of freedom. **?** amends this approach for use in method comparison studies, using the averages of the pairs, as opposed to the sums, and their differences. This approach can facilitate simultaneous usage of test with the Bland-Altman approach. Bartko's test statistic take the form:

$$F.test = \frac{(\Sigma d^2) - SSReg}{2MSReg}$$

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Averages	1	0.04	0.04	0.74	0.4097
Residuals	10	0.60	0.06		

Table 5.12.3: Regression ANOVA of case-wise differences and averages for Grubbs Data

For the Grubbs data, $\Sigma d^2 = 5.09$, $SSReg = 0.60$ and $MSreg = 0.06$ Therefore the test statistic is 37.42, with a critical value of 4.10. Hence the means and variance of the Fotobalk and Counter chronometers are assumed to be simultaneously equal.

Importantly, this approach determines whether there is both inter-method bias and precision present, or alternatively if there is neither present. It has previously been demonstrated that there is a inter-method bias present, but as this procedure does not allow for separate testing, no conclusion can be drawn on the comparative precision of both methods.

5.13 Bartko's Bradley-Blackwood Test

This is a regression based approach that performs a simulataneous test for the equivalence of means and variances of the respective methods.

$$D = (X_1 - X_2) \tag{5.12}$$

$$M = (X_1 + X_2)/2 \tag{5.13}$$

The Bradley Blackwood Procedure fits D on M as follows:

$$D = \beta_0 + \beta_1 M \tag{5.14}$$

Both beta values, the intercept and slope, are derived from the respective means and standard deviations of their respective data sets.

We determine if the respective means and variances are equal if both beta values are simultaneously equal to zero. The Test is conducted using an F test, calculated from the results of a regression of D on M.

We have identified this approach to be examined to see if it can be used as a foundation for a test perform a test on means and variances individually.

Russell et al have suggested this method be used in conjunction with a paired t-test , with estimates of slope and intercept.

subsectiont-test

5.14 Bartko's Bradley-Blackwood Test

This is a regression based approach that performs a simulataneous test for the equivalence of means and variances of the respective methods.

$$D = (X_1 - X_2) \tag{5.15}$$

$$M = (X_1 + X_2)/2 \tag{5.16}$$

The Bradley Blackwood Procedure fits D on M as follows:

$$D = \beta_0 + \beta_1 M \tag{5.17}$$

Both beta values, the intercept and slope, are derived from the respective means and standard deviations of their respective data sets.

We determine if the respective means and variances are equal if both beta values are simultaneously equal to zero. The Test is conducted using an F test, calculated from the results of a regression of D on M.

We have identified this approach to be examined to see if it can be used as a foundation for a test perform a test on means and variances individually.

Russell et al have suggested this method be used in conjunction with a paired t-test , with estimates of slope and intercept.

    subsectiont-test

5.15 Bland-Altman correlation test

The approach proposed by **?** is a formal test on the Pearson correlation coefficient of case-wise differences and means (ρ_{AD}). According to the authors, this test is equivalent to the 'Pitman Morgan Test'. For the Grubbs data, the correlation coefficient estimate (r_{AD}) is 0.2625, with a 95% confidence interval of (-0.366, 0.726) estimated by Fishers 'r to z' transformation (**?**). The null hypothesis $(\rho_{AD} = 0)$ fail to be rejected.

Consequently the null hypothesis of equal variances of each method would also fail to be rejected. There has no been no further mention of this particular test in **?**, although **?** refers to Spearman's rank correlation coefficient. **?** state that they ' do not see a place for methods of analysis based on hypothesis testing'.

5.16   Identifiability

? highlights an important issue regarding using models such as structural equation modelling, which is the identifiability problem. This comes as a result of there being too many parameters to be estimated. Therefore assumptions about some parameters, or estimators used, must be made so that others can be estimated. For example, in the literature, the variance ratio $\lambda = \frac{\sigma_1^2}{\sigma_2^2}$ must often be assumed to be equal to 1 (**?**). **?** considers approaches based on two methods with single measurements on each subject as inadequate for a serious study on the measurement characteristics of the methods. This is because there would not be enough data to allow for a meaningful analysis. There is, however, a counter-argument that in many practical settings it is very difficult to get replicate observations when, for example, the measurement method requires invasive medical procedure.

   ? offer a formal simultaneous hypothesis test for the mean and variance of paired data sets. This approach is based upon regressing the differences of each pair on the sum of each pair, a line is fitted to the model, with estimates for intercept and slope ($\hat{\beta}_0$ and $\hat{\beta}_1$). The null hypothesis of this test is that the mean (μ) and variance (σ^2) of both data sets are equal if the slope and intercept estimates are equal to zero (i.e $\sigma_1^2 = \sigma_2^2$ and $\mu_1 = \mu_2$ if and only if $\beta_0 = \beta_1 = 0$)

   A test statistic is then calculated from the regression analysis of variance values (**?**) and is distributed as 'F' random variable. The degrees of freedom are $\nu_1 = 2$ and $\nu_2 = n - 2$ (where n is the number of pairs). **?** amends this approach for use in method comparison studies, using the averages of the pairs, as opposed to the sums, and their differences. This approach can facilitate simultaneous usage of test with the

Bland-Altman approach. Bartko's test statistic take the form:

$$F.test = \frac{(\Sigma d^2) - SSReg}{2MSReg}$$

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Averages	1	0.04	0.04	0.74	0.4097
Residuals	10	0.60	0.06		

Table 5.16.4: Regression ANOVA of case-wise differences and averages for Grubbs Data

For the Grubbs data, $\Sigma d^2 = 5.09$, $SSReg = 0.60$ and $MSreg = 0.06$. Therefore the test statistic is 3.742, with a critical value of 4.10. Hence the means and variance of the Fotobalk and Counter chronometers are assumed to be simultaneously equal.

Importantly, this methodology determines whether there is both inter-method bias and precision present, or alternatively if there is neither present. It has previously been demonstrated that there is a inter-method bias present, but as this procedure does not allow for separate testing, no conclusion can be drawn on the comparative precision of both methods.

sectionBartko's Bradley-Blackwood Test This is a regression based approach that performs a simultaneous test for the equivalence of means and variances of the respective methods.We have identified this approach to be examined to see if it can be used as a foundation for a test perform a test on means and variances individually.

$$D = (X_1 - X_2) \tag{5.18}$$

$$M = (X_1 + X_2)/2 \tag{5.19}$$

The Bradley Blackwood Procedure fits D on M as follows:

$$D = \beta_0 + \beta_1 M \tag{5.20}$$

- The Bradley Blackwood test is a simultaneous test for bias and precision. They propose a regression approach which fits D on M, where D is the difference and average of a pair of results.

- Both beta values, the intercept and slope, are derived from the respective means and standard deviations of their respective data sets.

- We determine if the respective means and variances are equal if both beta values are simultaneously equal to zero. The Test is conducted using an F test, calculated from the results of a regression of D on M.

- We have identified this approach to be examined to see if it can be used as a foundation for a test perform a test on means and variances individually.

- Russell et al have suggested this method be used in conjunction with a paired t-test , with estimates of slope and intercept.

5.17 Bradley-Blackwood Test (Kevin Hayes Talk)

This work considers the problem of testing $\mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$ using a random sample from a bivariate normal distribution with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

The new contribution is a decomposition of the Bradley-Blackwood test statistic (*Bradley and Blackwood, 1989*)for the simultaneous test of $\mu_1 = \mu_2$; $\sigma_1^2 = \sigma_2^2$ as a sum of two statistics.

One is equivalent to the Pitman-Morgan (*Pitman, 1939; Morgan, 1939*) test statistic for $\sigma_1^2 = \sigma_2^2$ and the other one is a new alternative to the standard paired-t test of $\mu_D = \mu_1 = \mu_2 = 0$.

Surprisingly, the classic Student paired-t test makes no assumptions about the equality (or otherwise) of the variance parameters.

The power functions for these tests are quite easy to derive, and show that when $\sigma_1^2 = \sigma_2^2$, the paired t-test has a slight advantage over the new alternative in terms

of power, but when $\sigma_1^2 \neq \sigma_2^2$, the new test has substantially higher power than the paired-t test.

While Bradley and Blackwood provide a test on the joint hypothesis of equal means and equal variances their regression based approach does not separate these two issues.

The rejection of the joint hypothesis may be due to two groups with unequal means and unequal variances; unequal means and equal variances, or equal means and unequal variances. We propose an approach for resolving this (model selection) problem in a manner controlling the magnitudes of the relevant type I error probabilities.

5.18   Regression Methods for Method Comparison

Conventional regression models are estimated using the ordinary least squares (OLS) technique, and are referred to as 'Model I regression' (**??**). A key feature of Model I models is that the independent variable is assumed to be measured without error. However this assumption invalidates simple linear regression for use in method comparison studies, as both methods must be assumed to be measured with error (**??**).

The use of regression models that assumes the presence of error in both variables X and Y have been proposed for use instead (**??**). These methodologies are collectively known as 'Model II regression'. They differ in the method used to estimate the parameters of the regression.

Regression estimates depend on formulation of the model. A formulation with one method considered as the X variable will yield different estimates for a formulation where it is the Y variable. With Model I regression, the models fitted in both cases will entirely different and inconsistent. However with Model II regression, they will be consistent and complementary.

Regression approaches are useful for a making a detailed examination of the biases across the range of measurements, allowing bias to be decomposed into fixed bias and proportional bias. Fixed bias describes the case where one method gives values that are consistently different to the other across the whole range. Proportional bias describes

the difference in measurements getting progressively greater, or smaller, across the range of measurements. A measurement method may have either an attendant fixed bias or proportional bias, or both. (**?**). Determination of these biases shall be discussed in due course.

5.19   Deming Regression

As stated previously, the fundamental flaw of simple linear regression is that it allows for measurement error in one variable only. This causes a downward biased slope estimate.

Deming regression is a regression fitting approach that assumes error in both variables. Deming regression is recommended by **?** as the preferred Model II regression for use in method comparison studies. The sum of squared distances from measured sets of values to the regression line is minimized at an angles specified by the ratio λ of the residual variance of both variables. I When λ is one, the angle is 45 degrees. In ordinary linear regression, the distances are minimized in the vertical directions (**?**). In cases involving only single measurements by each method, λ may be unknown and is therefore assumes a value of one. While this will produce biased estimates, they are less biased than ordinary linear regression.

The Bland Altman Plot is uninformative about the comparative influence of proportional bias and fixed bias. Model II approaches, such as Deming regression, can provide independent tests for both types of bias.

For a given λ, **?** derived the following estimate that would later be used for the Deming regression slope parameter. The intercept estimate α is simply estimated in the same way as in conventional linear regression, by using the identity $\bar{Y} - \hat{\beta}\bar{X}$;

$$\hat{\beta} = \frac{S_{yy} - \lambda S_{xx} + [(S_{yy} - \lambda S_{xx})^2 + 4\lambda S_{xy}^2]^{1/2}}{2S_{xy}} \tag{5.21}$$

, with λ as the variance ratio. As stated previously λ is often unknown, and therefore must be assumed to equal one. **?** states that Deming regression is acceptable only

when the precision ratio (λ,in their paper as η) is correctly specified, but in practice this is often not the case, with the λ being underestimated. Several candidate models, with varying variance ratios may be fitted, and estimates of the slope and intercept are produced. However no model selection information is available to determine the best fitting model.

As with conventional regression methodologies, Deming regression calculates an estimate for both the slope and intercept for the fitted line, and standard errors thereof. Therefore there is sufficient information to carry out hypothesis tests on both estimates, that are informative about presence of fixed and proportional bias.

A 95% confidence interval for the intercept estimate can be used to test the intercept, and hence fixed bias, is equal to zero. This hypothesis is accepted if the confidence interval for the estimate contains the value 0 in its range. Should this be, it can be concluded that fixed bias is not present. Conversely, if the hypothesis is rejected, then it is concluded that the intercept is non zero, and that fixed bias is present.

Testing for proportional bias is a very similar procedure. The 95% confidence interval for the slope estimate can be used to test the hypothesis that the slope is equal to 1. This hypothesis is accepted if the confidence interval for the estimate contains the value 1 in its range. If the hypothesis is rejected, then it is concluded that the slope is significant different from 1 and that a proportional bias exists.

For convenience, a new data set shall be introduced to demonstrate Deming regression. Measurements of transmitral volumetric flow (MF) by doppler echocardiography, and left ventricular stroke volume (SV) by cross sectional echocardiography in 21 patients with aortic valve disease are tabulated in **?**. This data set features in the discussion of method comparison studies in **?**, p.398 .

? states that Deming's regression is acceptable only when the precision ratio (λ, in their paper as η) is correctly specified, but in practice this is often not the case, with the λ being underestimated.

Patient	MF (cm^3)	SV (cm^3)	Patient	MF (cm^3)	SV (cm^3)	Patient	MF (cm^3)	SV (cm^3)
1	47	43	8	75	72	15	90	82
2	66	70	9	79	92	16	100	100
3	68	72	10	81	76	17	104	94
4	69	81	11	85	85	18	105	98
5	70	60	12	87	82	19	112	108
6	70	67	13	87	90	20	120	131
7	73	72	14	87	96	21	132	131

Table 5.19.5: Transmitral volumetric flow(MF) and left ventricular stroke volume (SV) in 21 patients. (Zhang et al 1986)

5.20 Deming Regression

As stated previously, the fundamental flaw of simple linear regression is that it allows for measurement error in one variable only. This causes a downward biased slope estimate.

Deming regression is a regression fitting approach that assumes error in both variables. Deming regression is recommended by **?** as the preferred Model II regression for use in method comparison studies. The sum of squared distances from measured sets of values to the regression line is minimized at an angles specified by the ratio λ of the residual variance of both variables. I When λ is one, the angle is 45 degrees. In ordinary linear regression, the distances are minimized in the vertical directions (**?**). In cases involving only single measurements by each method, λ may be unknown and is therefore assumes a value of one. While this will bias the estimates, it is less biased than ordinary linear regression.

The Bland Altman Plot is uninformative about the comparative influence of proportional bias and fixed bias. Model II approaches, such as Deming regression, can provide independent tests for both types of bias.

Comparison of Deming and Linear Regression
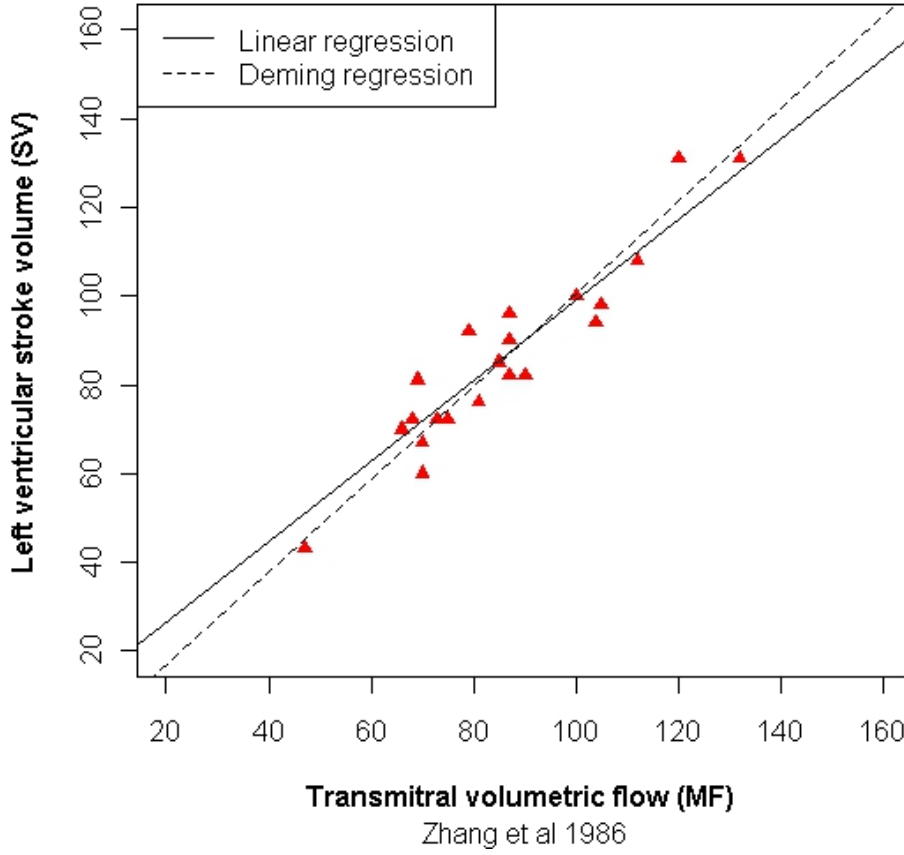
Figure 5.19.4: Deming Regression For Zhang's Data

For a given λ, **?** derived the following estimate that would later be used for the Deming regression slope parameter. The intercept estimate α is simply estimated in the same way as in conventional linear regression, by using the identity $\bar{Y} - \hat{\beta}\bar{X}$;

$$\hat{\beta} = \frac{S_{yy} - \lambda S_{xx} + [(S_{yy} - \lambda S_{xx})^2 + 4\lambda S_{xy}^2]^{1/2}}{2S_{xy}} \quad (5.22)$$

, with λ as the variance ratio. As stated previously λ is often unknown, and therefore must be assumed to equal one. **?** states that Deming regression is acceptable only when the precision ratio (λ,in their paper as η) is correctly specified, but in practice this is often not the case, with the λ being underestimated. Several candidate models, with varying variance ratios may be fitted, and estimates of the slope and intercept are

produced. However no model selection information is available to determine the best fitting model.

As with conventional regression methodologies, Deming regression calculates an estimate for both the slope and intercept for the fitted line, and standard errors thereof. Therefore there is sufficient information to carry out hypothesis tests on both estimates, that are informative about presence of fixed and proportional bias.

A 95% confidence interval for the intercept estimate can be used to test the intercept, and hence fixed bias, is equal to zero. This hypothesis is accepted if the confidence interval for the estimate contains the value 0 in its range. Should this be, it can be concluded that fixed bias is not present. Conversely, if the hypothesis is rejected, then it is concluded that the intercept is non zero, and that fixed bias is present.

Testing for proportional bias is a very similar procedure. The 95% confidence interval for the slope estimate can be used to test the hypothesis that the slope is equal to 1. This hypothesis is accepted if the confidence interval for the estimate contains the value 1 in its range. If the hypothesis is rejected, then it is concluded that the slope is significant different from 1 and that a proportional bias exists.

For convenience, a new data set shall be introduced to demonstrate Deming regression. Measurements of transmitral volumetric flow (MF) by doppler echocardiography, and left ventricular stroke volume (SV) by cross sectional echocardiography in 21 patients with aortic valve disease are tabulated in **?**. This data set features in the discussion of method comparison studies in **?**, p.398 .

? states that Deming's regression is acceptable only when the precision ratio (λ, in their paper as η) is correctly specified, but in practice this is often not the case, with the λ being underestimated.

Patient	MF (cm^3)	SV (cm^3)	Patient	MF (cm^3)	SV (cm^3)	Patient	MF (cm^3)	SV (cm^3)
1	47	43	8	75	72	15	90	82
2	66	70	9	79	92	16	100	100
3	68	72	10	81	76	17	104	94
4	69	81	11	85	85	18	105	98
5	70	60	12	87	82	19	112	108
6	70	67	13	87	90	20	120	131
7	73	72	14	87	96	21	132	131

Table 5.20.6: Transmitral volumetric flow(MF) and left ventricular stroke volume (SV) in 21 patients. (Zhang et al 1986)

5.21 Other Types of Studies

? categorize method comparison studies into three different types. The key difference between the first two is whether or not a 'gold standard' method is used. In situations where one instrument or method is known to be 'accurate and precise', it is considered as the'gold standard' (**?**). A method that is not considered to be a gold standard is referred to as an 'approximate method'. In calibration studies they are referred to a criterion methods and test methods respectively.

1. **Calibration problems**. The purpose is to establish a relationship between methods, one of which is an approximate method, the other a gold standard. The results of the approximate method can be mapped to a known probability distribution of the results of the gold standard (**?**). (In such studies, the gold standard method and corresponding approximate method are generally referred to a criterion method and test method respectively.) **?** make clear that their methodology is not intended for calibration problems.

2. **Comparison problems**. When two approximate methods, that use the same

units of measurement, are to be compared. This is the case which the Bland-Altman methodology is specfically intended for, and therefore it is the most relevant of the three.

3. Conversion problems. When two approximate methods, that use different units of measurement, are to be compared. This situation would arise when the measurement methods use 'different proxies', i.e different mechanisms of measurement. **?** deals specifically with this issue. In the context of this study, it is the least relevant of the three.

?, p.47 cautions that 'gold standards' should not be assumed to be error free. 'It is of necessity a subjective decision when we come to decide that a particular method or instrument can be treated as if it was a gold standard'. The clinician gold standard , the sphygmomanometer, is used as an example thereof. The sphygmomanometer 'leaves considerable room for improvement' (**?**). **?** similarly addresses the issue of glod standards, 'well-established gold standard may itself be imprecise or even unreliable'.

The NIST F1 Caesium fountain atomic clock is considered to be the gold standard when measuring time, and is the primary time and frequency standard for the United States. The NIST F1 is accurate to within one second per 60 million years (**?**).

Measurements of the interior of the human body are, by definition, invasive medical procedures. The design of method must balance the need for accuracy of measurement with the well-being of the patient. This will inevitably lead to the measurement error as described by **?**. The magnetic resonance angiogram, used to measure internal anatomy, is considered to the gold standard for measuring aortic dissection. Medical test based upon the angiogram is reported to have a false positive reporting rate of 5% and a false negative reporting rate of 8%. This is reported as sensitivity of 95% and a specificity of 92% (**?**).

In literature they are, perhaps more accurately, referred to as 'fuzzy gold standards' (**?**). Consequently when one of the methods is essentially a fuzzy gold standard, as opposed to a 'true' gold standard, the comparison of the criterion and test methods

should be consider in the context of a comparison study, as well as of a calibration study.

5.22 Methods of assessing agreement

1. Pearson's Correlation Coefficient

2. Intraclass correlation coefficient

3. Bland Altman Plot

4. Bartko's Ellipse (1994)

5. Blackwood Bradley Test

6. Lin's Reproducibility Index

7. Luiz Step function

Bland and Altman attend to the issue of repeated measures in 1996. Repeated measurements on several subjects can be used to quantify measurement error, the variation between measurements of the same quantity on the same individual. Bland and Altman discuss two metrics for measurement error; the within-subject standard deviation ,and the correlation coefficient.

The above plot incorporates both the conventional limits of agreement (the inner pair of dashed lines), the 't' limits of agreement (the outer pair of dashed lines) centred around the inter-method bias (indicated by the full line). This plot is intended for expository purposes only, as the sample size is small.

5.23 Bland Altman Plots In Literature

? contains a study the use of Bland Altman plots of 44 articles in several named journals over a two year period. 42 articles used Bland Altman's limits of agreement, wit the other two used correlation and regression analyses. ? remarks that 3 papers, from 42 mention predefined maximum width for limits of agreement which would not impair medical care.

The conclusion of **?** is that there are several inadequacies and inconsistencies in the reporting of results ,and that more standardization in the use of Bland Altman plots is required. The authors recommend the prior determination of limits of agreement before the study is carried out. This contention is endorsed by **?**, which makes a similar recommendation for the sample size, noting that *sample sizes required either was not mentioned or no rationale for its choice was given.*

> In order to avoid the appearance of "data dredging", both the sample size and the (limits of agreement) should be specified and justified before the actual conduct of the trial. (**?**)

? remarks that the limits of agreement should be compared to a clinically acceptable difference in measurements.

5.24   Gold Standard

This is considered to be the most accurate measurement of a particular parameter.

5.25   Agreement

Bland and Altman (1986) define Perfect agreement as 'The case where all of the pairs of rater data lie along the line of equality'. The Line of Equality is defined as the 45 degree line passing through the origin, or X=Y on a XY plane.

5.26   Lack Of Agreement

1. Constant Bias

2. Proportional Bias

Constant Bias

This is a form of systematic deviations estimated as the average difference between the test and the reference method

Proportional Bias

Two methods may agree on average, but they may exhibit differences over a range of

5.27 Bland Altman plots using 'Gold Standard' raters

According to Bland and Altman, one should use the methodology previous outlined, even when one of the raters is a Gold Standard.

5.28 Bias Detection

further to this method, the presence of constant bias may be indicated if the average value differences is not equal to zero. Bland and Altman does, however, indicate the indication of absence of bias does not provide sufficient information to allow a judgement as to whether or not one method can be substituted for another.

Chapter 6

Repeated Measurements and Repeatability

6.1 Definition of Replicate measurements

Further to **?**, a formal definition is required of what exactly replicate measurements are

By replicates we mean two or more measurements on the same individual taken in identical conditions. In general this requirement means that the measurements are taken in quick succession.

Roy accords with Bland and Altman?s definition of a replicate, as being two or more measurements on the same individual under identical conditions. Roy allows the assumption that replicated measurements are equi-correlated. Roy allows unequal numbers of replicates.

Replicate measurements are linked over time. However the method can be easily extended to cover situations where they are not linked over time.

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Replicate measurements are linked over time. However the method can be easily extended to cover situations where they are not linked over time.

6.3 Model for replicate measurements

We generalize the single measurement model for the replicate measurement case, by additionally specifying replicate values. Let y_{mir} be the $r-$th replicate measurement for subject "i" made by method "m". Further to **?** fixed effect can be expressed with a single term α_{mi}, which incorporate the true value μ_i.

$$y_{mir} = \mu_i + \alpha_m + e_{mir}$$

Combining fixed effects (**?**), we write,

$$y_{mir} = \alpha_{mi} + e_{mir}.$$

The following assumptions are required

- e_{mir} is independent of the fixed effects with mean $\mathrm{E}(e_{mir}) = 0$.

- Further to **?** between-item and within-item variances $\text{Var}(\alpha_{mi}) = \sigma_{Bm}^2$ and $\text{Var}(e_{mir}) = \sigma_{Wm}^2$

6.4 Carstensen's Model

? presents a model to describe the relationship between a value of measurement and its real value. The non-replicate case is considered first, as it is the context of the Bland Altman plots. This model assumes that inter-method bias is the only difference between the two methods.

A measurement y_{mi} by method m on individual i is formulated as follows;

$$y_{mi} = \alpha_m + \mu_i + e_{mi} \qquad e_{mi} \sim \mathcal{N}(0, \sigma_m^2) \tag{6.1}$$

The differences are expressed as $d_i = y_{1i} - y_{2i}$. For the replicate case, an interaction term c is added to the model, with an associated variance component. All the random effects are assumed independent, and that all replicate measurements are assumed to be exchangeable within each method.

$$y_{mir} = \alpha_m + \mu_i + c_{mi} + e_{mir}, \qquad e_{mi} \sim \mathcal{N}(0, \sigma_m^2), \quad c_{mi} \sim \mathcal{N}(0, \tau_m^2). \tag{6.2}$$

Of particular importance is terms of the model, a true value for item i (μ_i). The fixed effect of Roy's model comprise of an intercept term and fixed effect terms for both methods, with no reference to the true value of any individual item. A distinction can be made between the two models: Roy's model is a standard LME model, whereas Carstensen's model is a more complex additive model.

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Of particular importance is terms of the model, a true value for item i (μ_i). The fixed effect of Roy's model comprise of an intercept term and fixed effect terms for both methods, with no reference to the true value of any individual item. A distinction can be made between the two models: Roy's model is a standard LME model, whereas Carstensen's model is a more complex additive model.

6.6 Two Way ANOVA

? develop their model from a standard two-way analysis of variance model, reformulated for the case of replicate measurements, with random effects terms specified as appropriate. Their model describing y_{mir}, again the rth replicate measurement on the ith item by the mth method ($m = 1, 2$, $i = 1, \ldots, N$, and $r = 1, \ldots, n$), can be written as

$$y_{mir} = \alpha_m + \mu_i + a_{ir} + c_{mi} + \epsilon_{mir}. \tag{6.5}$$

The fixed effects α_m and μ_i represent the intercept for method m and the 'true value' for item i respectively. The random-effect terms comprise an item-by-replicate interaction

127

term $a_{ir} \sim \mathcal{N}(0, \varsigma^2)$, a method-by-item interaction term $c_{mi} \sim \mathcal{N}(0, \tau_m^2)$, and model error terms $\varepsilon \sim \mathcal{N}(0, \varphi_m^2)$. All random-effect terms are assumed to be independent. For the case when replicate measurements are assumed to be exchangeable for item i, a_{ir} can be removed.

The model expressed in (2) describes measurements by m methods, where $m = \{1, 2, 3 \ldots\}$. Based on the model expressed in (2), **?** compute the limits of agreement as

$$\alpha_1 - \alpha_2 \pm 2\sqrt{\tau_1^2 + \tau_2^2 + \varphi_1^2 + \varphi_2^2}$$

? notes that, for $m = 2$, separate estimates of τ_m^2 can not be obtained. To overcome this, the assumption of equality, i.e. $\tau_1^2 = \tau_2^2$ is required.

6.7   Two Way ANOVA

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The fixed effects α_m and μ_i represent the intercept for method m and the 'true value' for item i respectively. The random-effect terms comprise an item-by-replicate interaction term $a_{ir} \sim \mathcal{N}(0, \varsigma^2)$, a method-by-item interaction term $c_{mi} \sim \mathcal{N}(0, \tau_m^2)$, and model error terms $\varepsilon \sim \mathcal{N}(0, \varphi_m^2)$. All random-effect terms are assumed to be independent. For the case when replicate measurements are assumed to be exchangeable for item i, a_{ir} can be removed.

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? notes that, for $m = 2$, separate estimates of τ_m^2 can not be obtained. To overcome this, the assumption of equality, i.e. $\tau_1^2 = \tau_2^2$ is required.

6.8 Statistical Model For Replicate Measurements

Let y_{Aij} and y_{Bij} be the jth repeated observations of the variables of interest A and B taken on the ith subject. The number of repeated measurements for each variable may differ for each individual. Both variables are measured on each time points. Let n_i be the number of observations for each variable, hence $2 \times n_i$ observations in total.

It is assumed that the pair y_{Aij} and y_{Bij} follow a bivariate normal distribution.

$$\begin{pmatrix} y_{Aij} \\ y_{Bij} \end{pmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \text{ where } \boldsymbol{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix} \tag{6.7}$$

The matrix $\boldsymbol{\Sigma}$ represents the variance component matrix between response variables at a given time point j.

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{pmatrix} \tag{6.8}$$

σ_A^2 is the variance of variable A, σ_B^2 is the variance of variable B and σ_{AB} is the covariance of the two variable. It is assumed that $\boldsymbol{\Sigma}$ does not depend on a particular time point, and is the same over all time points.

6.9 Replicate measurements

? accords with Bland and Altman?s definition of a replicate, as being two or more measurements on the same individual under identical conditions. Roy allows the assumption that replicated measurements are equi-correlated. Roy allows unequal numbers of replicates.

Replicate measurements are linked over time. However the method can be easily extended to cover situations where they are not linked over time.

In this model , the variances of the random effects must depend on m, since the different methods do not necessarily measure on the same scale, and different methods naturally must be assumed to have different variances. **?** attends to the issue of comparative variances.

? also remark that an important feature of replicate observations is that they should be independent of each other. This issue is addressed by **?**, in terms of exchangeability and linkage. Carstenen advises that repeated measurements come in two *substantially different* forms, depending on the circumstances of their measurement: exchangable and linked.

Repeated measurements are said to be exchangeable if no relationship exists between successive measurements across measurements. If the condition of exchangeability exists, a group of measurement of the same item determined by the same method can be re-arranged in any permutation without prejudice to proper analysis. There is no reason to believe that the true value of the underlying variable has changed over the course of the measurements.

Exchangeable repeated measurements can be treated as true replicates. For the purposes of method comparison studies the following remarks can be made. The $r-$th measurement made by method 1 has no special correspondence to the $r-$th measurement made by method 2, and consequently any pairing of repeated measurements are as good as each other.

Repeated measurements are said to be linked if a direct correspondence exists between successive measurements across measurements, i.e. pairing. Such measurements are commonly made with a time interval between them, but simultaneously for both methods. Paired measurements are exchangeable, but individual measurements are not.

If the paired measurements are taken in a short period of time so that no real systemic changes can take place on each item, they can be considered true replicates. Should enough time elapse for systemic changes, linked repeated measurements can not be treated as true replicates.

6.10 Replicate Measurements

Thus far, the formulation for comparison of two measurement methods is one where one measurement by each method is taken on each subject. Should there be two or more measurements by each methods, these measurement are known as 'replicate measurements'. **?** recommends the use of replicate measurements, but acknowledges the additional computational complexity.

? address this problem by offering two different approaches. The premise of the first approach is that replicate measurements can be treated as independent measurements. The second approach is based upon using the mean of the each group of replicates as a representative value of that group. Using either of these approaches will allow an analyst to estimate the inter method bias.

However, because of the removal of the effects of the replicate measurements error, this would cause the estimation of the standard deviation of the differences to be unduly small. **?** propose a correction for this.

? takes issue with the limits of agreement based on mean values of replicate measurements, in that they can only be interpreted as prediction limits for difference between means of repeated measurements by both methods, as opposed to the difference of all measurements. Incorrect conclusions would be caused by such a misinterpretation. **?** demonstrates how the limits of agreement calculated using the mean of replicates are 'much too narrow as prediction limits for differences between future single measurements'. This paper also comments that, while treating the replicate measurements as independent will cause a downward bias on the limits of agreement calculation, this method is preferable to the 'mean of replicates' approach.

6.11 Replicate Measurements

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? address this situation via two different approaches. The premise of the first approach is that replicate measurements can be treated as independent measurements. The second approach is based upon using the mean of the each group of replicates as one single representative value.

Although either approach may be used to estimate the inter-method bias, removal of the effects of replicate measurements error leads to the underestimation of the standard deviation of the differences. **?** propose a correction for this.

? take issue with the limits of agreement based on mean values of replicate measurements, since these must be interpreted as prediction limits for the difference between means of repeated measurements by both methods, rather than the difference of individual measurements. **?** demonstrates how the limits of agreement calculated using the mean of replicates are 'much too narrow as prediction limits for differences between future single measurements'. This paper also comments that, while treating the replicate measurements as independent will cause a downward bias on the limits of agreement calculation, this method is preferable to the 'mean of replicates' approach.

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Exchangeable repeated measurements can be treated as true replicates. For the purposes of method comparison studies the following remarks can be made. The r−th measurement made by method 1 has no special correspondence to the r−th measurement made by method 2, and consequently any pairing of repeated measurements are as good as each other.

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If the paired measurements are taken in a short period of time so that no real systemic changes can take place on each item, they can be considered true replicates. Should enough time elapse for systemic changes, linked repeated measurements can not be treated as true replicates.

6.13   Repeated Measurements in LME models

In many statistical analyzes, the need to determine parameter estimates where multiple measurements are available on each of a set of variables often arises. Further to **?**, **?** performs an analysis of the correlation of replicate measurements, for two variables of interest, using LME models.

Let y_{Aij} and y_{Bij} be the jth repeated observations of the variables of interest A and B taken on the ith subject. The number of repeated measurements for each variable may differ for each individual. Both variables are measured on each time points. Let n_i be the number of observations for each variable, hence $2 \times n_i$ observations in total.

It is assumed that the pair y_{Aij} and y_{Bij} follow a bivariate normal distribution.

$$\begin{pmatrix} y_{Aij} \\ y_{Bij} \end{pmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \text{ where } \boldsymbol{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix}$$

The matrix Σ represents the variance component matrix between response variables at a given time point j.

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{pmatrix}$$

σ_A^2 is the variance of variable A, σ_B^2 is the variance of variable B and σ_{AB} is the covariance of the two variable. It is assumed that $\boldsymbol{\Sigma}$ does not depend on a particular time point, and is the same over all time points.

6.14   Linked replicates

? proposes the addition of an random effects term to their model when the replicates are linked. This term is used to describe the 'item by replicate' interaction, which is independent of the methods. This interaction is a source of variability independent of the methods. Therefore failure to account for it will result in variability being wrongly attributed to the methods.

? introduces a second data set; the oximetry study. This study done at the Royal Childrens Hospital in Melbourne to assess the agreement between co-oximetry and pulse oximetry in small babies.

In most cases, measurements were taken by both method at three different times. In some cases there are either one or two pairs of measurements, hence the data is unbalanced. **?** describes many of the children as being very sick, and with very low oxygen saturations levels. Therefore it must be assumed that a biological change can occur in interim periods, and measurements are not true replicates.

? demonstrate the necessity of accounting for linked replicated by comparing the limits of agreement from the 'oximetry' data set using a model with the additional term, and one without. When the interaction is accounted for the limits of agreement are (-9.62,14.56). When the interaction is not accounted for, the limits of agreement are (-11.88,16.83). It is shown that the failure to include this additional term results in an over-estimation of the standard deviations of differences.

Limits of agreement are determined using Roy's methodology, without adding any additional terms, are found to be consistent with the 'interaction' model; $(-9.562, 14.504)$. Roy's methodology assumes that replicates are linked. However, following Carstensen's example, an addition interaction term is added to the implementation of Roy's model to assess the effect, the limits of agreement estimates do not change. However there is a conspicuous difference in within-subject matrices of Roy's model and the modified model (denoted 1 and 2 respectively);

$$\hat{\mathbf{\Lambda}}_1 = \left(\begin{array}{cc} 16.61 & 11.67 \\ 11.67 & 27.65 \end{array} \right) \qquad \hat{\mathbf{\Lambda}}_2 = \left(\begin{array}{cc} 7.55 & 2.60 \\ 2.60 & 18.59 \end{array} \right). \tag{6.9}$$

(The variance of the additional random effect in model 2 is 3.01.)

? introduces the Akaike information criterion (AIC), a model selection tool based on the likelihood function. Given a data set, candidate models are ranked according to their AIC values, with the model having the lowest AIC being considered the best fit.Two candidate models can said to be equally good if there is a difference of less than 2 in their AIC values.

137

The Akaike information criterion (AIC) for both models are $AIC_1 = 2304.226$ and $AIC_2 = 2306.226$, indicating little difference in models. The AIC values for the Carstensen 'unlinked' and 'linked' models are 1994.66 and 1955.48 respectively, indicating an improvement by adding the interaction term.

The $\hat{\mathbf{\Lambda}}$ matrices are informative as to the difference between Carstensen's unlinked and linked models. For the oximetry data, the covariance terms (given above as 11.67 and 2.6 respectively) are of similar magnitudes to the variance terms. Conversely for the 'fat' data the covariance term (-0.00032) is negligible. When the interaction term is added to the model, the covariance term remains negligible. (For the 'fat' data, the difference in AIC values is also approximately 2).

To conclude, Carstensen's models provided a rigorous way to determine limits of agreement, but don't provide for the computation of $\hat{\mathbf{D}}$ and $\hat{\mathbf{\Lambda}}$. Therefore the test's proposed by **?** can not be implemented. Conversely, accurate limits of agreement as determined by Carstensen's model may also be found using Roy's method. Addition of the interaction term erodes the capability of Roy's methodology to compare candidate models, and therefore shall not be adopted.

Finally, to complement the blood pressure (i.e.'J vs S') method comparison from the previous section (i.e.'J vs S'), the limits of agreement are $15.62 \pm 1.96 \times 20.33 = (-24.22, 55.46)$.)

6.15   Repeatability

As mentioned previously, **?** emphasizes the importance of repeatability as part of an overall method comparison study. The coefficient of repeatability was proposed by **?**, and is referenced in subsequent papers, such as **?**. The coefficient of repeatability is a measure of how well a measurement method agrees with itself over replicate measurements (**?**). The coefficient of repeatability is a measure of how well a measurement method agrees with itself over replicate measurements (**?**). Once the the standard deviations of the differences between the two measurements (in some texts called the residual standard deviation or within-item variability) $sigma_m$ is determined, the computation of the coefficients of repeatability for both methods is straightforward. The coefficient is calculated from the (in some texts called the residual standard deviation) as $1.96 \times \sqrt{2} \times \sigma_m = 2.83\sigma_m$).

6.16   What is Repeatability

The quality of repeatability is the ability of a measurement method to give consistent results for a particular subject. That is to say that a measurement will agree with prior and subsequent measurements of the same subject.

6.17   Coefficient of Repeatability

The coefficient of repeatability is a measure of how well a measurement method agrees with itself over replicate measurements (**?**). Once the within-item variability is known, the computation of the coefficients of repeatability for both methods is straightforward.

The British standards Insitute [1979] define a coefficient of repeatability as *the value below which the difference between two single test results....may be expected to lie within a specified probability.* Unless otherwise instructed, the probability is assumed to be 95%.

The Bland Altman method offers a measurement on the repeatability of the methods. The *Coefficient of Repeatability* (CR) can be calculated as 1.96 (or 2) times the standard deviations of the differences between the two measurements (d_2 and d_1).

6.18 Repeatability coefficient

? introduces the repeatability coefficient for a method, which is defined as the upper limits of a prediction interval for the absolute difference between two measurements by the same method on the same item under identical circumstances (**?**).

σ_x^2 is the within-subject variance of method x. The repeatability coefficient is $2.77\sigma_x$ (i.e. $1.96 \times \sqrt{2}\sigma_x$). For 95% of subjects, two replicated measurement by the same method will be within this repeatability coefficient.

6.19 Repeatability

Repeatability is the ability of a measurement method to give consistent results for a particular subject, i.e. a measurement will agree with prior and subsequent measurements of the same subject. **?** emphasizes the importance of repeatability as part of an overall method comparison study, a view endorsed by **?**. Before there can be good agreement between two methods, a method must have good agreement with itself. If one method has poor repeatability in the sense of considerable variability, then agreement between two methods is bound to be poor (**?**). **?** remarks that it is important to report repeatability when assessing measurement, because it measures the purest form of random error not influenced by other factors, while further remarking '*curiously replicate measurements are rarely made in method comparison studies, so that an important aspect of comparability is often overlooked.* **?** strongly recommends the simultaneous estimation of repeatability and agreement be collecting replicated data. However **?** notes the lack of convenience in such calculations. Repeatability is defined by the **?** as '*the closeness of agreement between independent results obtained with the*

same method on identical test material, under the same conditions (same operator, same apparatus, same laboratory and after short intervals of time)' and is determined by taking multiple measurements on a series of subjects.

A measurement is said to be repeatable when this variation is smaller than some pre-specified limit. In these situations, there is often a predetermined "critical difference", and for differences in monitored values that are smaller than this critical difference, the possibility of pre-test variability as a sole cause of the difference may be considered in addition to, for examples, changes in diseases or treatments.

The British Standards Institute (1979) defines a coefficient of repeatability as *the value below which the difference between two single test results may be expected to lie within a specified probability.* In the absence of other indications, the probability is 95%.

6.20   Repeatability and Gold Standards

Currently the phrase 'gold standard' describes the most accurate method of measurement available. No other criteria are set out. Further to **?**, various gold standards have a varying levels of repeatability. Dunn cites the example of the sphygmomanometer, which is prone to measurement error. Consequently it can be said that a measurement method can be the 'gold standard', yet have poor repeatability.

? recognizes this problem. Hence, if the most accurate method is considered to have poor repeatability, it is referred to as a "bronze standard". Again, no formal definition of a 'bronze standard' exists.

The coefficient of repeatability may provide the basis of formulation a formal definition of a 'gold standard'. For example, by determining the ratio of CR to the sample mean $\bar{X}$. Advisably the sample size should specified in advance. A gold standard may be defined as the method with the lowest value of $\lambda = CR/\bar{X}$ with $\lambda < 0.1\%$. Similarly, a silver standard may be defined as the method with the lowest value of λ with $0.1\% \leq \lambda < 1\%$. Such thresholds are solely for expository purposes.

6.21 Repeatability

Repeatability (or *test-retest reliability*) describes the variation in measurements taken by a single method of measurement on the same item and under the same conditions. A less-than-perfect test-retest reliability causes test-retest variability. Such variability can be caused by, for example, intra-individual variability and intra-observer variability. A measurement may be said to be repeatable when this variation is smaller than some agreed limit.

Test-retest variability is practically used, for example, in medical monitoring of conditions. In these situations, there is often a predetermined "critical difference", and for differences in monitored values that are smaller than this critical difference, the possibility of pre-test variability as a sole cause of the difference may be considered in addition to, for examples, changes in diseases or treatments.

According to the *Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results*, the following conditions need to be fulfilled in the establishment of repeatability:

- the same measurement procedure

- the same observer

- the same measuring instrument, used under the same conditions

- the same location

- repetition over a short period of time.

- same objectives

Repeatability is defined by the **IUPAC** as '*the closeness of agreement between independent results obtained with the same method on identical test material, under the same conditions (same operator, same apparatus, same laboratory and after short*

intervals of time)' and is determined by taking multiple measurements on a series of subjects.

A measurement method can be said to have a good level of repeatability if there is consistency in repeated measurements on the same subject using that method. Conversely, a method has poor repeatability if there is considerable variation in repeated measurements.

6.22 Importance of Repeatability in MCS

Barnhart emphasizes the importance of repeatability as part of an overall method comparison study. Before there can be good agreement between two methods, a method must have good agreement with itself. The coefficient of repeatability , as proposed by Bland & Altman (1999) is an important feature of both Carstensen's and Roy's methodologies. The coefficient is calculated from the residual standard deviation (i.e. $1.96 \times \sqrt{2} \times \sigma_m = 2.83\sigma_m$).

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? strongly recommends the simultaneous estimation of repeatability and agreement be collecting replicated data. **?** notes the lack of convenience in such calculations. It is important to report repeatability when assessing measurement, because it measures the purest form of random error not influenced by other factors (**?**).

importance of repeatability' curiously replicate measurements are rarely made in method comparison studies, so that an important aspect of comparability is often overlooked.

Repeatability is important in the context of method comparison because the repeatability of two methods influence the amount of agreement which is possible between those methods. If one method has poor repeatability, the agreement is bound to be poor. If both methods have poor repeatability, agreement is even worse. If one method has poor repeatability in the sense of considerable variability, then agreement between two methods is bound to be poor (**?**).

? and **?** highlight the importance of reporting repeatability in method comparison, because it measures the purest random error not influenced by any external factors. Statistical procedures on within-subject variances of two methods are equivalent to

tests on their respective repeatability coefficients. A formal test is introduced by **?**, which will discussed in due course.

As noted by Bland and Altman 1999, the repeatability of two methods of measurement can potentially limit Repeatability (using Bland-Altman plot) The Bland-Altman plot may also be used to assess a method?s repeatability by comparing repeated measurements using one single measurement method on a sample of items. The plot can then also be used to check whether the variability or precision of a method is related to the size of the characteristic being measured. Since for the repeated measurements the same method is used, the mean difference should be zero. Therefore the Coefficient of Repeatability (CR) can be calculated as 1.96 (often rounded to 2) times the standard deviation of the case-wise differences.

6.23 Repeatability coefficient from LME Models

? introduces the repeatability coefficient for a method, which is defined as the upper limits of a prediction interval for the absolute difference between two measurements by the same method on the same item under identical circumstances (**?**).

σ_x^2 is the within-subject variance of method x. The repeatability coefficient is $2.77\sigma_x$ (i.e. $1.96 \times \sqrt{2}\sigma_x$). For 95% of subjects, two replicated measurement by the same method will be within this repeatability coefficient.

6.24 Repeatability in Bland-Altman Blood Data Analysis

- Two readings by the same method will be within $1.96\sqrt{2}\sigma_w$ or $2.77\sigma_w$ for 95% of subjects. Thisvalue is called the repeatability coefficient.

- For observer J using the sphygmomanometer $\sigma_w = \sqrt{37.408} = 6.116$ and so the repeatability coefficient is $2 : 77 \times 6.116 = 16 : 95$ mmHg.

- For the machine S,$\sigma_w = \sqrt{83.141} = 9.118$ and the repeatability coefficient is $2:77 \times 9.118 = 25.27$ mmHg.

- Thus, the repeatability of the machine is 50% greater than that of the observer.

6.25 Carstensen

- The limits of agreement are not always the only issue of interest the assessment of method specific repeatability and reproducibility are of interest in their own right.

- Repeatability can only be assessed when replicate measurements by each method are available.

- The repeatability coefficient for a method is defined as the upper limits of a prediction interval for the absolute difference between two measurements by the same method on the same item under identical circumstances.

- If the standard deviation of a measurement is σ the repeatability coefficient is $2 \times \sqrt{2}\sigma = 2.83 \times \sigma \approx 2.8\sigma$.

- The repeatability of measurement methods is calculated differently under the two models

- Under the model assuming exchangeable replicates (1), the repeatability is based only on the residual standard deviation, i.e. $2.8\sigma_m$

- Under the model for linked replicates (2) there are two possibilities depending on the circumstances.

- If the variation between replicates within item can be considered a part of the repeatability it will be $2.8\sqrt{\omega^2 + \sigma_m^2}$.

146

- However, if replicates are taken under substantially different circumstances, the variance component ω^2 may be considered irrelevant in the repeatability and one would therefore base the repeatability on the measurement errors alone, i.e. use $2.8\sigma_m$.

6.26 Notes from BXC Book (chapter 9)

The assessment of method-specific repeatability and reproducibility is of interest in its own right. Repeatability and reproducibility can only be assessed when replicate measurements by each method are available. If replicate measurements by a method are available, it is simple to estimate the measurement error for a method, using a model with fixed effects for item, then taking the residual standard deviation as measurement error standard deviation. However, if replicates are linked, this may produce an estimate that biased upwards. The repeatability coefficient (or simply repeatability) for a method is defined as the upper limit of a prediction interval for the absolute difference between two measurements by the same method on the same item under identical circumstances (see above conditions)

$$y_{mir} = \alpha_m + \beta_m(\mu_i + a_{ir} + c_{mi}) + e_{mir}$$

The variation between measurements under identical circumstances.

6.27 Model for replicate measurements

We generalize the single measurement model for the replicate measurement case, by additionally specifying replicate values. Let y_{mir} be the $r-$th replicate measurement for subject "i" made by method "m". Further to **?** fixed effect can be expressed with a single term α_{mi}, which incorporate the true value μ_i.

$$y_{mir} = \mu_i + \alpha_m + e_{mir}$$

Combining fixed effects (**?**), we write,

$$y_{mir} = \alpha_{mi} + e_{mir}.$$

The following assumptions are required

- e_{mir} is independent of the fixed effects with mean $\mathrm{E}(e_{mir}) = 0$.

- Further to **?** between-item and within-item variances $\mathrm{Var}(\alpha_{mi}) = \sigma^2_{Bm}$ and $\mathrm{Var}(e_{mir}) = \sigma^2_{Wm}$

6.28 Exchangeable and Linked measurements

6.29 Sampling Scheme : Linked and Unlinked Replicates

Measurements taken in quick succession by the same observer using the same instrument on the same subject can be considered true replicates. **?** notes that some measurements may not be 'true' replicates.

Roy's methodology assumes the use of 'true replicates'. However data may not be collected in this way. In such cases, the correlation matrix on the replicates may require a different structure, such as the autoregressive order one $AR(1)$ structure. However determining MLEs with such a structure would be computational intense, if possible at all.

One important feature of replicate observations is that they should be independent of each other. In essence, this is achieved by ensuring that the observer makes each measurement independent of knowledge of the previous value(s). This may be difficult to achieve in practice. (Check who said this)

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6.31   The Technology Acceptance Model

Davis (1989) proposes the TAM model, which suggests an hypothesis as to why users may adopt particular technologies, and not others. According to this theory, when users are presented with a new technology, two important factors will influence their decision about how and when they will adopt it.

Perceived usefulness (PU) - This was defined by Fred Davis as "the degree to which a person believes that using a particular system would enhance his or her job performance".

Perceived ease-of-use (PEOU) - Davis defined this as "the degree to which a person believes that using a particular system would be free from effort"

Davis's explanations of these term can be rephrased for application to statistical analysis. Perceived Use could refer to the degree to which an user would deem a particular statistical method would properly establish the results of an analaysis. In the case of method comparison studies, proper indication of agreement, or lack thereof.

Perceived ease-of-use requires only applying the context of a satistical problem. A very modest statistical skill set is the only prerequistive for constructing a Bland-Altman plot, and computing limits of agreement. The main building blocks are simple

descriptive, statistics and a knowledge of the normal distribution. These are topics that feature in almost every undergraduate statistics courses. Furthermore **?** recommends including the Bland-Altman method itself in undergraduate teaching.

In short, the user perceives the Bland-Altman methodology to be an easy-to-implement technique, that will properly address the question of agreement.

Conversely the Survival plot is a derivative of the Kaplan-Meier Curve, a non-parametric graphical technique that features in Survival Analysis. This subject area is a well known domain of statistics, but would be encountered on curriculums of specialist courses.

The Mountain Plot is formally called the empirical folder cumulative distribution plot. While not particularly hard to render, the procedure is not straight-forward for the casual user. Currently there is only one software implementation, ***medcalc.be*** toolkit.