Graph Theory: Tutorial Sheet

Invertible Functions

1. What conditions must be satisfied for a function to have an inverse.
   (a) One-one and onto
   (b) One-to-one only
   (c) onto only
   (d) Neither onto nor One-to-One

2. If f is a function for which the rule is f(x) = 7/8 - x, where x is real, the rule for the inverse function f-1 is:
   (a) $f^{-1}(x)$ = 8/7 + x
   (b) $f^{-1}(x)$ = -8/x + 7
   (c) $f^{-1}(x)$ = 2x + 73/4
   (d) $f^{-1}(x)$ = 7/8 - x
   (e) $f^{-1}(x)$ = 8/7 + x

3. Which of the following functions is not one-to-one?
   (a) $f(x) = 9 - x2, x \geq 0$
   (b) $f(x) = 1/x^2 - 9$
   (c) $f(x) = 1 - 9x$
   (d) $f(x) = \sqrt{x}$
   (e) $f(x) = 3/x$

4. The range of the function with rule $f(x) = \|x - 4\| + 3$ is:
   (a) $(4, \infty)$
   (b) $\mathbb{R}$
   (c) $[3, \infty)$
   (d) $(4, \infty)$
   (e) $(-1, \infty)$

5. A function $f : X \to Y$, where $X = \{p, q, r, s\}$ and $Y = \{1, 2, 3, 4, 5\}$ is given by the subset of $X \times Y$, i.e. $\{(q, 3), (r, 3), (p, 5), (s, 2)\}$.
   i. Show f as an arrow diagram.
   ii. State the domain, the co-domain and range of f.
   iii. Say why f does not have the **one-to-one** property and why f does not have the **onto** property, giving a specific counter example in each case.

Solutions

i. (Done on whiteboard)

ii. Domain, Co-Domain and Range

$$\begin{aligned}\text{Domain } & \{a, b, c, d\},\\ \text{Co-Domain } & \{1, 2, 3, 4, 5\}\\ \text{Range } & \{2, 3, 5\}\end{aligned}$$

- Onto - Range is equivalent to Co-domain.
- no element of co-domain unused.
- one to one - Each element of domain has one image in the co-domain. Each image has only one ancestor.

6. Consider the functions $f : \mathcal{R} \to \mathcal{Z}$ and $g : \nabla \to \mathcal{R}$ given by

$$f(X) = \rfloor x - 1 \lfloor$$

$$g(X) = |x - 1|$$

i. Write down the domain, co-domain and range of f and g(x).

ii. For each function, say whether or not it is one to one, justifying your answer.

iii. For each function, say whether or not it is onto, justifying your answer.

7. State whether or not each of the following functions has an inverse, justifying your answer. In the cases Where there is an inverse define it fully.

(i) $f : S \to \mathcal{Z}^+$ defned in part (a) (see Section 2.9).

(ii) $g : \mathcal{R} \to \mathcal{R}$ defined by $g(x) = x^2$.

(iii) $h : \mathcal{R} \to \mathcal{R}$ defined by $h(x) = 4x - 1$.

Solutions

- No Inverse. Function is not onto, only one-to-one . Each name has only one image But each number can have more than one ancestor. Also the Co-domain and Range must be assumed to be not equal. Even very very long names do not exceed 200 letters.
- No Inverse
- Inverse Exists

$$h^{-1}(x) = \frac{x + 1}{4}$$

8. (a) Given a real number x, say how $\lfloor x \rfloor$, the floor of x, is defined.

(b) The function $f : \mathcal{R} \to \mathcal{R}$ is given by the rule

$$f(x)\lfloor x/2 \rfloor$$

i. Find $f(-3)$ and $f(3)$
ii. Justifying your answer, say whether f is one-to-one.
iii. , Justifying your answer, say whether f is onto.

Solutions

- f(3) = 0 and f(-3) = - 1
- No, Different members of Domain can take the same value.
- No, f takes on integer values only while the codomain is specified as $\mathcal{R}$.

9. For each of the following equations, give **two** different examples of a real number x which satisfies the equations:

- $\lfloor x \rfloor = 3$
- $\lceil x \rceil = -1$
- $|x - 5| = 12$

Solutions

- $\lfloor x \rfloor = 3$: two examples π and 3.5
- $\lceil x \rceil = -1$: two examples -1.2 , -1.6
- $|x - 5| = 12$ Answers: -7 and 17

10. Given any number $xin\mathcal{R}$ the floor value is denoted $\rfloor x \lfloor$ and the absolute value is denoted by $|x|$.

a Find $\rfloor\sqrt{2}\lfloor$ and $|-2|$.
b Find the set of values of a such that $\rfloor a \lfloor = 1$, and the set of values $|b| = 1$
c (Discussed Separately)

Part A:

- $\sqrt{2} = 1.414214\ldots$
- Floor function of x is the integer that precedes x.

$$\rfloor\sqrt{2}\lfloor = 1$$

- The absolute value of -2 is simply 2.
- The set of values of a for which $\rfloor a \lfloor = 1$ is all real numbers between 1 and 2. a may take the value 1, but not the value 2.

$$1 \leq a < 2, a \in \mathcal{R}$$

- The set of values of b for which $|b| = 1$ are simply the values -1 and 1.

$$b = \{-1, 1\} b \in \mathcal{Z}$$

11. i. State the condition to be satisified in order for a function to have an inverse.

    ii. Given $f : \mathcal{R} \to \mathcal{R}$ where $f(x) = 2x - 1$, define fully the inverse function f^{-1} and state the values of $f^{-1}(1)$

    ii. Given $g : \mathcal{R} \to \mathcal{R}$ where $g(x) = 3^x$, define fully the inverse function g^{-1} and state the values of $g^{-1}(1)$

    This requires giving the inverse function in algebraic terms and its domain and co-domain.

12. Evaluate the following function for $x = 1, 2$ and 5 respectively.

$$f(x) = \frac{e^x + e^{-x}}{2}$$