Linear Algebra Tutorial Sheet : Invertible Matrices

1. The Fundamental Theorem of Invertible Matrices states that a set of mathematical expressions concerning an $n \times n$ matrix A are each equivalent to one another.

   (i) State any four of these expressions.

   (ii) What is the trace of a square matrix

2. In this question, you are required to find the inverse of the following matrix using elementary row operations.

$$A = \begin{pmatrix} -2 & -2 & -2 \\ 2 & 3 & 2 \\ 3 & -2 & 5 \end{pmatrix}$$

   (i) Write down the augmented matrix of this system.

   (ii) Find the inverse of the matrix, using elementary row operations. Show your workings for each stage of the calculation.

3. In this question, you are required to find the inverse of the following matrix using elementary row operations.

$$A = \begin{pmatrix} 1 & -4 & -3 \\ 1 & 3 & 5 \\ -2 & 0 & -4 \end{pmatrix}$$

   (i) Write down the augmented matrix of this system.

   (ii) Find the inverse of the matrix, using elementary row operations. Show your workings for each stage of the calculation.

4. Suppose that the inverse of the following matrix M is given as M^1:

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 0 & -2 \\ -6 & -2 & 2 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ -0.25 & -1.0 & -0.75 \\ 0.50 & 0.5 & 0.50 \end{pmatrix}$$

(i) State the inverse of the matrix N where $N = 2M$.

$$N = 2M = \begin{pmatrix} 4 & 4 & 4 \\ 8 & 0 & -4 \\ -12 & -4 & 4 \end{pmatrix}$$

5. Consider the matrix B specified as

$$B = \begin{pmatrix} -4 & 3 & -2 \\ -2 & 3 & 4 \\ -1 & 1 & 0 \end{pmatrix}.$$

(i) For each element of B, calculate the corresponding minor. Show your workings for each calculation. State the matrix of minors.

(ii) Hence or otherwise, compute the determinant of B i.e. $\det(B)$.

(iii) Compute the cofactor matrix for B i.e. $\text{cof}(B)$.

(iv) State the inverse matrix of B, given by

$$B^{-1} = \frac{1}{\det(B)} \text{cof}(B)^T.$$

6. Show that if A is an $n \times n$ invertible matrix that satisfies

$$9A^3 + A^2 - 3A = 0$$

where $A^n = \underbrace{A \dots A}_{n \text{ times}}$, I is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix, then the inverse of A is given by

$$A^{-1} = \frac{1}{3} I + 3A.$$

7. Consider the following diagonal matrix D. Provide answers for the following questions in terms of the values a, b and c.

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

(i) (1 Mark) Write an expression for the trace of the matrix D.

(ii) (1 Mark) State the inverse of D, i.e. D^{-1}.

(iii) (1 Mark) State the matrix D^3.