Review of complex numbers

Definition:
A complex number is a number of the form

$$z = a + ib$$

where a and b are real numbers (called the real part and imaginary part of z, respectively) and $i = \sqrt{-1}$. We sometimes use the notation:

$$a = \mathrm{Re}(z) \qquad \text{and} \qquad b = \mathrm{Im}(z)$$

The **complex conjugate** of $z = a + ib$ is $\bar{z} = a - ib$.

The **absolute value** or **modulus** of $z = a + ib$ is defined as $|z| = \sqrt{a^2 + b^2}$ and has the property that

$$z \cdot \bar{z} = a^2 + b^2 = |z|^2$$

Powers of i:

$$i^2 = -1, \qquad i^3 = -i, \qquad i^4 = 1, \qquad i^5 = i, \qquad i^6 = -1, \qquad \text{etc.}$$
$$i^{-1} = -i, \qquad i^{-2} = -1, \qquad i^{-3} = i, \qquad i^{-4} = 1, \qquad \text{etc.}$$

Operations with complex numbers:

Addition: $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

Multiplication: $z_1 z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

Inversion: $z^{-1} = \dfrac{1}{a+ib} = \dfrac{a-ib}{(a+ib)(a-ib)} = \dfrac{a-ib}{a^2+b^2} = \dfrac{a}{a^2+b^2} - i\,\dfrac{b}{a^2+b^2}$

Exercise: Find the inverse z^{-1} and complex conjugate $\bar{z}$ of each of the following complex numbers:

$$\text{(i) } z = 1 + 2i; \quad \text{(ii) } \frac{1}{2} - i\frac{1}{3}; \quad \text{(iii) } -i.$$

Exercise: Let $Z_1 = 4 - 3i$ and $Z_2 = 2 + i$. Evaluate the real and imaginary part of the complex expression

$$\frac{1}{Z_1 - Z_2} + \frac{1}{Z_1 Z_2}.$$

Polar form of complex numbers

The complex number $z = a + ib$ is represented as the point in the x, y plane which has rectangular coordinates (a, b), or polar coordinates (r, θ), where

$$r^2 = a^2 + b^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta$$

The polar representation of a complex number is

$$z = a + ib = r\left[\cos(\theta) + i\sin(\theta)\right]$$

The number r is called the **absolute value** or **modulus** and θ is called the **argument** of the complex number z, denoted by

$$r = |z| \qquad \text{and} \qquad \theta = \arg(z).$$

The polar representation of a complex number has an exponential as well as a trigonometric form. Since

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

then

$$z = r\left[\cos(\theta) + i\sin(\theta)\right] = r\,e^{i\theta}$$

Example: Express in polar form and sketch the following complex numbers

$$\text{(i) } 2+i; \quad \text{(ii) } 3-2i; \quad \text{(iii) } -1-3i; \quad \text{(iv) } (2+i)^2; \quad \text{(v) } i(4-i).$$

De Moivre's Formula

Let $z = r(\cos\theta + i\sin\theta$ be a complex number expressed in polar form. Then we have

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

Example: Calculate the following powers:

$$\text{(i) } (\sqrt{3}+i)^{20}; \qquad \text{(ii) } (4+4i)^7.$$

Quadratic and higher order equations:

Polynomial equations often have complex roots. For example, the equation $x^2 - 4x + 8 = 0$ has roots $2 \pm 2i$.

Note: If the roots of a quadratic equation with real coefficients are complex then they are of the form $a \pm ib$. In other words, if $z = a + ib$ is a root then so is the conjugate $\bar{z} = a - ib$.

If the polynomial coefficients are not real then the complex roots do not necessarily follow the pattern above.

Example: Solve the quadratic equation

$$z^2 - 4iz + 5 = 0.$$