MA4702 - Fundamentals of Mathematics and Limits

Notation:

- $\mathbb{R}$ - All real numbers positive and negative
- $\mathbb{R}^+$ - All positive real numbers including 0
- $\mathbb{R}^-$ - All negative real numbers including 0
- $[a, b]$ - All real numbers x such that $a \leq x \leq b$
- (a, b) - All real numbers x such that $a < x < b$
- $[a, \infty)$ - All real numbers x such that $a \leq x$
- (a, ∞) - All real numbers x such that $a < x$

Question 1 : Evaluation of Functions

(i) Evaluate the following function for x = -1,0,1 and 2 respectively.

$$f(x) = \frac{e^x - e^{-x}}{2}$$

(ii) Evaluate the function for each of the following values : $0.5,\ 1,\ 1.25,\ 2.$

$$f(x) = \sqrt{1 + e^x}$$

Question 2 : Floor and Ceiling Functions (Part A)

- $\lceil x \rceil$: Ceiling function
- $\lfloor x \rfloor$: Floor Function
- $\{x\}$: Fractional Part of a number ($\{x\} = x - \lfloor x \rfloor$)

Complete the following table.

Value x	Floor $\lfloor x \rfloor$	Ceiling $\lceil x \rceil$	Fractional $\{x\}$
-1.4	-2	-1	
2.3			
7/9			
-16/3			
0			0
1		1	

Question 3 : Floor and Ceiling Functions (Part B)

Provide some values for x and y that **contradict** the following statement.

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$$

If the values of x and y were integers, would the equation be true for all values of x and y?

Question 4 : Laws for Logarithms

The following laws are very useful for working with logarithms.

1. $\log_b(X) + \log_b(Y) = \log_b(XY)$
2. $\log_b(X)$ - $\log_b(Y) = \log_b(X/Y)$
3. $\log_b(X^Y) = Y \log_b(X)$
4. $\log_b(X) = 1$ when $b = X$

Use the Laws of Logarithms to evaluate the following expressions:

(i) $\log_2(8)$

(ii) $\log_2(\sqrt{128})$

(iii) $\log_2(64)$

(iv) $\log_5(125) + \log_3(729)$

(v) $\log_2(64/4)$

(vi) $\log_3(\frac{1}{81})$

Question 5 : Cross Multiplication

Solve the following equations for A and B where $A, B \in \mathbb{R}$

(i) $\dfrac{11}{x^2 - 4x - 12} = \dfrac{A}{x-6} + \dfrac{B}{x+2}$

(ii) $\dfrac{2x+5}{x^2 - 4x - 12} = \dfrac{A}{x-6} + \dfrac{B}{x+2}$

(iii) $\dfrac{1}{(n)(n+1)} = \dfrac{A}{n} + \dfrac{B}{n+1}$

(iv) $\dfrac{2}{(n+1)(n+3)} = \dfrac{A}{n+1} + \dfrac{B}{n+3}$

Question 6 : Exponential and Logarithm Exercises

(i) Find the value of x

$$e^{2x-5} = 3.$$

(ii) Find the value of x

$$ln(e^x + 2) = 4$$

(iii) Find the value of x

$$log_3(2x-1) + log_3(5) = 3$$

(iv) Find the value of x

$$log_2(x+1) + log_2(5) = 3$$

Question 7 : Review of Differentiation

(i) $f(x) = e^{4x}$

(ii) $f(x) = \cos(4x)$

(iii) $f(x) = \sin(3x)$

(iv) $f(x) = e^{4y}\cos(4x)$

(v) $f(y) = e^{4y}\cos(4x)$

Question 8 : Expressing Repeating Decimals as Fractions

Express the following numbers as fractions. For example $0.77777... = \frac{7}{9}$

(i) 0.29292929....

(ii) 0.475475475....

(iii) 0.4545454545.....

(iv) 0.473473473.........

Question 9 : Evaluate the following limits

(i)
$$\lim_{x\to 5}(x^2)$$

(ii)
$$\lim_{x\to 2}(4x^2-3x+1)$$

(iii)
$$\lim_{x\to 4} x+5$$

(iv)
$$\lim_{x\to 2} 2x-1$$

(v)
$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

(vi)
$$\lim_{x\to 3}\frac{x^2-4x+3}{x-3}$$

(vii)
$$\lim_{x\to\infty}\frac{x+4}{x-4}$$

(viii)
$$\lim_{x\to\infty}\frac{x^3-4}{x-1}$$

(ix)
$$\lim_{x\to\infty}\frac{x^2-1}{x-1}$$

(x)
$$\lim_{x\to\infty}\frac{2x^2+8}{5x^2-7x}$$

(xi)
$$\lim_{x\to\infty}\frac{3x^2+7x^3}{x^2+5x^4}$$

(xii)
$$\lim_{x\to\infty}\frac{2x^2-8x}{4x^2-7}$$

(xiii)
$$\lim_{x\to\infty}\frac{x-3}{x^2-9}$$

Question 10: Sequences and Series

(i) Find the sum of the first 10 numbers of this arithmetic series: $1 + 11 + 21 + 31 + \ldots$

(ii) The second term u_2 of a geometric sequence is 21.
The third term u_3 is -84.

- Find the common ratio
- Find the first and fourth term

(iii) Three consecutive terms of an arithmetic series are

$$4x - 1, 2x + 11, 3x + 41.$$

Find the value of x.

(iv) Find the sum of the following geometric series:

$$3 + 6 + 12 + 24 + \ldots + 1536$$

(v) In an arithmetic sequence, three consecutive terms have a sum of - 9 and a product of 48. Find the common difference d for these terms.
Hint : Write terms as $x - d, x, x + d$

(vi) The first three terms of an arithmetic sequence are 6,-9 and x. The first three terms of a geometric sequence are 6, x, y. Find the value of x and the value of y.

(vii) Find the sum to infinity of the geometric series

$$1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \ldots$$

(viii) The first three terms of a geometric series are

$$x - 2, 2x + 3, 7x$$

Find both of the possible values of x

(ix) The $n-$th term of an arithmetic is $3n + 2$
Find S_n the sum of the first n terms, in terms of n

Question 11 : Convergence of a Sequence

A sequence u_n is said to converge to a limit L where $L \neq \infty$ if

$$\lim_{n\to\infty} u_n = L$$

If a sequence does not converge then it is said to be divergent.

Test the following sequences for convergence. see page 14 of notes for more details.

(i)

$$u_n = \frac{2n-1}{4n+3}$$

(ii)

$$u_n = \frac{n+5}{3n+4}$$

(iii)

$$u_n = \frac{n+4}{2n^3+n+3}$$

(iv)

$$u_n = \frac{n^2+5n}{n^2+2n-1}$$

(v)

$$u_n = \frac{2n^3+1}{2n^3+3n+4}$$

(vi)

$$u_n = \frac{2n^4+1}{n^3+2n^2-1}$$

Question 12 : Partitioning of Summations

For some integers m and n, with $m < n$.

$$\sum_{i=1}^{i=n} u_i = \sum_{i=1}^{i=m} u_i + \sum_{i=m+1}^{i=n} u_i$$

Suppose $n = 100$ and $m = 50$

$$\sum_{i=1}^{i=100} u_i = \sum_{i=1}^{i=50} u_i + \sum_{i=51}^{i=100} u_i$$

Compute the following

(i)
$$\sum_{i=1}^{30} i$$

(ii)
$$\sum_{i=1}^{65} i$$

(iii)
$$\sum_{i=1}^{37} i$$

(iv)
$$\sum_{i=38}^{65} i$$

Question 13 : Sum to Infinity Exercises

Compute the summations of the following infinite series

(i) $1 + 0.2 + 0.04 + \ldots$

(ii) $1 - 0.2 + 0.04 - \ldots$

(iii) $20 + 5 + 1.25 + \ldots$

(iv) $-20 + 5 - 1.25 + \ldots$

Question 14 : Summation of an Infinite Series

Find the value to which each of the following series converges.

$$\sum_{n=1}^{\infty} \frac{3}{4^{n-1}}$$

Question 15 : The Ratio test

For a series with general term u_n, if

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = r$$

then

- the series converges (absolutely) if $r < 1$
- the series diverges if $r > 1$ (or if r is infinity)
- the series could do either if $r = 1$, so the test is not conclusive.

This formula will be given in the back of the exam paper. However the indications on how to interpret r will not be given.

Example

Suppose that the following term is the general term for a series. Test this series for convergence

$$u_n = \frac{n!n!}{(2n)!}$$

then

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{4n+2}$$
$$\to \frac{1}{4}$$

so this series converges.

Use the Ratio Test to test the following series for convergence

(i)

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{4^n}$$

(iii)

$$\sum_{n=1}^{\infty} \frac{5^n}{n!}$$

(i)

$$\sum_{n=1}^{\infty} \frac{4^n}{(n+1)!}$$

(iv)

$$\sum_{n=1}^{\infty} \frac{n+1}{2^n}$$

(v) Use the Ratio test to find the values for x for which the series is convergent

$$\sum_{n=1}^{\infty} \frac{x^n}{n+2}$$

Question 16 : Evaluation of telescoping series

Find the sum of the following telescoping series

(i)

$$\sum_{n=1}^{\infty} \frac{3}{(3n+1)(3n+4)}$$

(ii)

$$\sum_{n=1}^{\infty} \frac{4}{(2n+1)(2n+3)}$$

(iii)

$$\sum_{n=1}^{\infty} \frac{5}{(5n+1)(5n+6)}$$

(iv)

$$\sum_{n=1}^{\infty} \frac{6}{(6n+1)(6n+7)}$$

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Question 17 : Evaluation of Telescoping Series

Answer the following questions

(i) Show that, where $r \neq \pm 1$.

$$\frac{2}{r^2-1} = \frac{1}{r-1} - \frac{1}{r+1}$$

(ii) Hence, evaluate the following summation

$$\sum_{r=2}^{n} \frac{2}{r^2-1}$$

(iii) Hence, evaluate the following summation

$$\sum_{r=2}^{\infty} \frac{2}{r^2-1}$$

Question 18 : Evaluation of terms in a Maclaurin Series

(i) Evaluate the following Macluarin Series for $n = \{1, 2, 3\}$.

$$\sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Question 19 : Functions

Part A - Functions

Consider the function f(x) where

$$f(x) = \frac{1}{\sqrt{3x-6}}$$

(i) Find the domain and range of f(x)

(ii) Find $f(3x^2+2)$ and simplify

Part B - Functions

Consider the functions $f(x) = \sqrt{2x-6}$ and $g(x) = log_e(2x+1)$

(i) Find $f(4–2x^2)$ and simplify answer.

(ii) Write down the domain and range of f(x).

(iii) Determine $g^{-1}(x)$, the inverse of g(x).

Part C - Composite Functions

Consider the functions $f(x) = \sqrt{2x-8}$, and $g(x) = 2x^2+4$.

(i) Find the composite function $f \circ g(x)$ and $g \circ f(x)$, simplifying your answer as much as possible.

(ii) Determine $f^{-1}(x)$, the inverse of f(x).

Question 20 : Functions

(i) Find $f^{-1}(x)$ the inverse of the function

$$f(x) = \frac{1}{2x-5}$$

(ii) Find the domain and the range of the function:

$$f(x) = 7 + 2sin(x)$$

(iii) Find $f^{-1}(x)$ the inverse of the function

$$f(x) = \sqrt{2x+3}$$

(iv) Find $f^{-1}(x)$ the inverse of the function

$$f(x) = e^{3x}$$

(v) Find the domain and the range of the function:

$$f(x) = 1 - x^2$$

(vi) Find the domain and the range of the function:

$$f(x) = ln(x)$$

Question 21 : Hyperbolic Functions - Proof of Identities

Recall

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad\qquad (e^x)^2 = e^{2x}$$

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad\qquad (e^x) \times (e^{-x}) = e^{x-x} = e^0 = 1$$

Using their definition in terms of exponentials, prove the following hyperbolic identity:

(i) Show that

$$\sinh 2x = 2\sinh(x)\cosh(x)$$

(ii) Show that

$$\sinh^2(x) = \frac{1}{2}\big[\cosh(2x) - 1\big]$$

(iii) Show that

$$\cosh^2(x) - \sinh^2(x) = 1$$

(iv) Show that

$$\cosh(x + y) = \cosh(x)\cosh(y) + sinh(x)sinh(y)$$

Question 22 : Inverse of Functions

Procedure

- To determine $f^{-1}(x)$ when given a function f, substitute $f^{-1}(x)$ for x and substitute x for f(x).
- Then solve for $f^{-1}(x)$, provided that it is also a function.

Example:
Given $f(x) = 2x - 7$, find $f^{-1}(x)$.
Substitute $f^{-1}(x)$ for x and substitute x for f(x). Then solve for $f^{-1}(x)$:

$$\begin{aligned} f(x) &= 2x - 7 \\ x &= 2[f^{-1}(x)] - 7 \\ x + 7 &= 2[f^{-1}(x)] \\ \frac{x+7}{2} &= f^{-1}(x) \end{aligned}$$

(i) Find $f^{-1}(x)$ the inverse of the function

$$f(x) = \frac{1}{2x - 5}$$

(ii) given that $g(x) = log_e(4x)$, find $g^{-1}(x)$ the inverse function of g(x).

Question 23 : Even and Odd Functions

Check whether the following functions are even, odd or neither.

(i)
$$f(x) = \frac{4}{x^2 + 1}$$

(ii)
$$f(x) = sin(4x)$$

(iii)
$$f(x) = -cos(3x)$$

(iv)
$$f(x) = \frac{3x + 2}{4x + 3}$$

(v)
$$f(x) = \frac{e^x - e^{-x}}{2}$$

(vi)
$$f(x) = \frac{x}{x^2 - 4}$$

Question 24 : Calculations for the Ratio Test

For each of the following terms, given as u_n, state u_{n+1} and hence calculate a simplified expression for r, where

$$r = \frac{u_{n+1}}{u_n}$$

(i) $u_n = 2^n$

(ii) $u_n = n!$

(iii) $u_n = \dfrac{(n+1)!}{2^n}$

(iv) $u_n = n! \times n!$

(v) $u_n = \dfrac{4^n}{2^n}$

(vi) $u_n = \dfrac{4^n}{(2n)!}$

(vii) $u_n = \dfrac{5^n}{n!}$ (2010 Exam)

(viii) $u_n = \dfrac{x^n}{n+1}$ (2005 Exam)

(ix) $u_n = \dfrac{n+1}{2^n}$ (2009 Exam)

(x) $u_n = \dfrac{4^n}{(n+1)!}$ (2007 Exam)

Question 25 : Functions - Part A

Marks

SECTION A

1 (a) Consider the functions $f(x) = \sqrt{8-2x}$, $g(x) = \log_e(2x+1)$

(i) Find $f(4-2x^2)$ and simplify answer.

(ii) Write down the domain and range of $f(x)$.

(iii) Find $g^{-1}(x)$ the inverse of $g(x)$. **10**

Question 25 : Functions - Part B

SECTION A

1(a) Consider the fimctions $f(x) = \dfrac{1}{\sqrt{x-2}}$ and $g(x) = x + \sin x$

(i) Find $f(4x^2 + 2)$ and simplify the answer.

(ii) Write down the domain and range of $f(x)$.

(iii) Prove that $g(x)$ is an odd fimction. (see tables page 9)

Question 26 : Domain and Ranges of Functions

Find the domain and the range of the functions:

(i) $f(x) = x - 2 \quad g(x) = -2x$

(ii) $f(x) = x^2 - 4 \quad g(x) = -x^2 - 4$

(iii) $f(x) = \sqrt{x} \quad g(x) = \sqrt{x-2}$

(iv) $f(x) = \sin x \quad g(x) = -2\sin x$

(v) $f(x) = \cos x \quad g(x) = \cos^2(x)$

(vi) $f(x) = e^x \quad g(x) = e^x - 2$

Question 27 : Domain and Ranges of Functions

Find the domain and the range of the functions:

(i) $f(x) = 8 - 2\sin(x)$

(ii) $f(x) = 5 - 2\cos(2x)$

(iii) $f(x) = 2\cos(x) - 6$

(iv) $f(x) = 7 + 2\sin(3x)$

(v) $f(x) = 5\sin(x) - 2\cos(x)$

(vi) $f(x) = \cos^2(x) + \sin^2(x)$

(vii) $f(x) = \cos^2(x) - \sin^2(x)$

Question 28 : Evaluation of trigonometric values

Evaluate the following values. *you may use your calculator.*

(i) $\cos(\pi/2)$

(ii) $\sin(3\pi/4)$

(i) $\tan(\pi/6)$

(ii) $\cos(1.5\pi)$

Question 29 : Composite Functions

Determine the values of $f \circ g(2)$ and $g \circ f(x)$. Henxe or otherwise, evaluate $f \circ g(2)$ and $g \circ f(2)$.

(i) $f(x) = X^2 + 1$ and $g(x) = 2x$

(ii) $f(x) = \sqrt{x+1}$ and $g(x) = x^2$

(iii) $f(x) = x^2 - 1$ and $g(x) = 2x + 1$

(iv) $f(x) = 5x$ and $g(x) = x^2 + 1$

(v) $f(x) = -4x + 9$ and $g(x) = 2x - 7$

(vi) $f(x) = e^x$ and $g(x) = \ln(x)$

(vii) $f(x) = x^2 + 1$ and $g(x) = 1 - 3x$

(viii) $f(x) = \sin(\pi x)$ and $g(x) = 2x + 1$

Question 30 : Inverse Functions

(i) $f(x) = 2x$

(ii) $f(x) = e^x$

(iii) $f(x) = x - 3$

(iv) $f(x) = \cos(x)$

(v) $f(x) = \sin^{-1}(x)$

(vi) $f(x) = \dfrac{1}{x-1}$

(vii) $f(x) = x^2 + 2$

(viii) $f(x) = \sin^{-1}(2x)$

(ix) $f(x) = 3e^{2x}$

(x) $f(x) = \sqrt{2x+3}$

(xi) $f(x) = \log_e 4x$

(xii) $f(x) = \dfrac{-2}{x-5}$

Selected Solution

(i) For the function $f(x) = 2x$ the inverse function is $f^{-1}(x) = x/2$.

(ii) For the function $f(x) = e^x$ the inverse function is $f^{-1}(x) = \log_e(x) = \ln(x)$.

(iii) For the function $f(x) = x - 3$ the inverse function is $f^{-1}(x) = x + 3$

(iv) For the function $f(x) = \cos(x)$ the inverse function is $f^{-1}(x) = \cos^{-1}(x)$.

(v) For the function $f(x) = \sin^{-1}(x)$ the inverse function is $f^{-1}(x) = \sin(x)$.

Question 31 : Revision of Product, Quotient and Chain Rule

(i)
$$f(x) = (x^4 + 4x + 2)(2x + 3)$$

(ii)
$$f(x) = (2x - 1)(3x^2 + 2)$$

(iii)
$$f(x) = (x^3 - 12x)(3x^2 + 2x)$$

(iv)
$$f(x) = (2x^5 - x)(3x + 1)$$

(v)
$$f(x) = \frac{2x + 1}{x + 5}$$

(vi)
$$f(x) = \frac{3x^4 + 2x + 2}{3x^2 + 1}$$

(vii)
$$f(x) = \frac{x^{\frac{3}{2}} + 1}{x + 2}$$

(viii)
$$f(x) = \frac{x^2 + x}{2x - 1}$$

(ix)
$$f(x) = \frac{x + 1}{2x^2 + 2x + 3}$$

(x)
$$f(x) = \frac{16x^4 + 2x^2}{x}$$

(xi)
$$f(x) = (x + 5)^2$$

(xii)
$$f(x) = \sqrt{1 - x^2}$$

(xiii)
$$f(x) = \frac{(2x + 4)^3}{4x^3 + 1}$$

(xiv)
$$f(x) = \frac{2x + 3}{(x^4 + 4x + 2)^2}$$

(xv)
$$f(x) = \frac{2x + 3}{(x^4 + 4x + 2)^2}$$

(xvi)
$$f(x) = 3e^x - 4\cos(x) - \frac{1}{4}\ln x$$

(xvii)
$$f(x) = \sin(x) + \cos(x)$$

Selected Solutions

(i)
$$10x^4 + 12x^3 + 16x + 16$$

(ii)
$$18x^2 - 6x + 4$$

(iii)
$$15x^4 + 8x^3 - 108x^2 - 48x$$

(iv)
$$36x^5 + 10x^4 - 6x - 1$$

(v)
$$30x^2 + 70x + 6$$

(vi)
$$6x(25x^2 + 1)(5x^2 + 1)^3$$

Question 32 : Introduction to Integration

Evaluate each of the following indefinite integrals:

1. $\int (5t^8 - 4t^5 + 3t + 2)\, dt$
2. $\displaystyle\int (\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3})\, dx$
3. $\displaystyle\int (\sqrt{x} + x\sqrt{x})\, dx$
4. $\displaystyle\int \left(\frac{1}{\sqrt{u}} + \frac{1}{u^2\sqrt{u}}\right)\, du$
5. $\displaystyle\int \frac{4}{\sqrt{9 - x^2}}\, dx$
6. $\displaystyle\int \frac{t^2 + 3t + 6}{t}\, dt$
7. $\displaystyle\int (3e^x + 1)\, dx$
8. $\int (5\cos x - 6\sin x)\; dx$
9. $\displaystyle\int \frac{7}{x^2 + 9}\, dx$

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Question 33 : Integration

Using appropriate substitutions, evaluate the indefinite integrals:

(i)
$$\int (x^2 - 2)^2 \, dx$$

(ii)
$$\int 8x^3 \, dx$$

(iii)
$$\int (4x^2 + 11x^3) \, dx$$

(iv)
$$\int (31x^{32} + 4x^3 - 9x^4) \, dx$$

(v)
$$\int 5x^{-2} \, dx$$

Using appropriate substitutions, evaluate the indefinite integrals:

(i)
$$\int 3x^2(x^3 + 1)^5 \, dx$$

(ii)
$$\int x^4 \sin(x^5) \, dx$$

Addition and Subtraction Rules of Integration

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

$$\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

The Power Rule for Integration

The power rule for derivatives can be reversed to give us a way to handle integrals of powers of x. Since

$$\frac{d}{dx} x^n = nx^{n-1},$$

we can conclude that

Integration

Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n, (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
$a^x \quad (a > 0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\frac{1}{\sqrt{a^2 - x^2}} \quad (a > 0)$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

$$\int nx^{n-1}\,dx = x^n + C,$$

or, a little more usefully,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Question 34 : Integration

Evaluate the following:

(i)

$$\int (x^2 - 2)^2 \, dx$$

(ii)

$$\int 8x^3 \, dx$$

(iii)

$$\int (4x^2 + 11x^3) \, dx$$

(iv)

$$\int (31x^{32} + 4x^3 - 9x^4) \, dx$$

(v)

$$\int 5x^{-2} \, dx$$

Question 35 : Introduction to Integration

Part A

Using appropriate substitutions, evaluate the indefinite integrals:

(i)

$$\int (s-4)^5 ds$$

(ii)

$$\int \frac{3}{(x+1)^4} dx$$

(iii)

$$\int (2y+3)(y^2+3y+2)^2 dy$$

Part B

Evaluate the following indefinite integrals using partial fractions:

(i)

$$\int \frac{x}{x^2-9} dx$$

(ii)

$$\int \frac{x}{x^2-4x+3} dx$$

(iii)

$$\int \frac{x}{x^2-4x+3} dx$$

Question 36 : Definite Integrals

Evaluate the following definite integrals

(i)
$$\int_1^2 (x^2-1)dx$$

(ii)
$$\int_0^{\frac{\pi}{2}} \cos x dx$$

(iii)
$$\int_0^{\pi} \cos x dx$$

(iv)
$$\int_1^2 (y^2-y^{-2})dy$$

(v)
$$\int_{-3}^1 (6x^2-5x+2)dx$$

(vi)
$$\int_4^0 \sqrt{t}(t-2)dt$$

Hint:

$$\int \sqrt{t}(t-2)dt$$

$$\sqrt{t}(t-2) = t^{1/2} \times (t-2) = t^{3/2} - 2t^{1/2}$$

Question 37 : Definite Integrals

Evaluate the following definite integrals:

(a) $\displaystyle\int_{-2}^{2} \frac{1}{x+3}\,dx$

(b) $\displaystyle\int_{0}^{2} (x^4 + 3x^2 + 2)\,dx$

(c) $\displaystyle\int_{-\pi}^{\pi} (5\sin x - 7\cos x)\,dx$

(d) $\displaystyle\int_{-3}^{2} 2x\,e^{(x^2+1)}\,dx$

(e) $\displaystyle\int_{0}^{\sqrt{\pi}} x\cos\left(x^2 - \frac{\pi}{2}\right)\,dx$

(f) $\displaystyle\int_{0}^{\pi} x\sin x\,dx$

(g) $\displaystyle\int_{0}^{1} \frac{1}{x^2-4}\,dx$

(h) $\displaystyle\int_{0}^{2} \frac{1}{x^2+4}\,dx$

Question 38 : Integration by Parts

Evaluate the following using integration by parts.

(i)
$$\int -4\ln(x)\,dx$$

(ii)
$$\int (-7x+38)\cos(x)\,dx$$

(iii)
$$\int_{0}^{\frac{\pi}{2}} (-6x+45)\cos(x)\,dx$$

(iv)
$$\int (5x+1)(x-6)^4\,dx$$

(v)
$$\int_{-1}^{1} (2x+8)^3(-x+2)\,dx$$

(vi)
$$\int \sin(x)\,e^{x}\,dx$$

Question 39 : Integration by Parts

Revision Week Exercises

Question 40 : Integration by Parts

Evaluate the following indefinite integrals by integration by parts:

(a) $\int x^2 e^x \, dx$

(b) $\int x \ln x \, dx$

(c) $\int x^2 \cos x \, dx$

(d) $\int x \sin x \, dx$

(e) $\int e^x \sin x \, dx$

(f) $\int \ln x \, dx$

Formula:

If u and v are functions of x that have continuous derivatives, then

$$\int udv = uv - \int vdu$$

The LIPET rule

It is considered a rule of thumb to remember the acronym **LIPET** when performing integration by parts. This acronym will help you to determine what to use as u.

L -logarithms,

I -inverse trigonometric functions,

P -polynomials (i.e. x, x^2) ,

E -exponentials (i.e. e^x, e^{3x}),

T -trigonometric functions.

- $\cosh(x)$ is both the derivative and integral of $\sinh(x)$
- $\sinh(x)$ is both the derivative and integral of $\cosh(x)$

Question 41 : Definite Integrals

Exercise: Evaluate the following definite integral

$$\int_1^3 \frac{x}{3}dx$$

Solution

$$\int_1^3 \frac{x}{3}dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20$$

Exercise: Evaluate the following definite integral

$$\int_1^3 \frac{x^2-4x+3}{x-3}dx$$

Factorize the numerator $x^2-4x+3=(x-1)(x-3)$

Treat it as an indefinite integral for time being.

$$\int \frac{x^2-4x+3}{x-3}dx = \int \frac{(x-1)(x-3)}{x-3}dx = \int (x-1)dx = \frac{x^2}{2} - x + c$$

$$\left[\frac{x^2}{2} - x\right]_1^3 = (4.5-3)-(0.5-1) = 2$$

Question 42 : Integration by Parts (Exam Standard)

the following questions are from previous past papers. Please be advised of the notes below.

(i) (2005) Use integration by parts to find $\int xe^x dx$

(ii) (2006) Use integration by parts to find $\int xln(x)dx$

(iii) (2007) Use integration by parts to find $\int xsinh(x)dx$

(iv) (2008) Use integration by parts to find $\int xcos(x)dx$

(v) (2009) Use integration by parts to find $\int xcosh(x)dx$

(vi) (2010) Use integration by parts to find $\int xe^x dx$

Important:

- You should expect to see hyperbolic functions (i.e. $\cosh(x)$ and $\sinh(x)$) in the end of semester exam.
- However you should expect to see terms like x^2, e^{2x} and $\ln(x)$, as well as what was in previous exams.
- **VERY Important:** Make sure you know how to integrate and differentiate expressions of the form e^{ax}, $\cos(ax)$, $\cosh(ax)$, $\sin(ax)$ and $\sinh(ax)$.

Question 43 : Definite Integrals

Evaluate the following definite integrals

(i) Find the area between $f(x) = x^2 + 4x$ and the $x-$axis between $x = -4$ and $x = 3$.

(ii) Calculate the following:

$$\int_0^1 \frac{4x^3}{x^4+1}dx$$

(iii) Evaluate

$$\int_0^{\frac{\pi}{2}} \cos^4(x)\sin(x)dx$$

Question 32 : Curve Sketching

Ex. 1 (i) Find the y intercept of the function $y = f(x)$.

(ii) Show that $(\sqrt{3}, 1)$ is a stationary point of the function. Find the other two stationary points and classify all three points as local maxima or minima.

(iii) Find the two inflection pints of $f(x)$.

(iv) Find the x values for which is $y = f(x)$ is concave up/down.

(v) Determine the behaviour of y as $x \to +\infty$ and as $x \to -\infty$

(vi) Sketch the graph of $y = f(x)$ indicating clearly the features of the curve obtained in aprts (i) to (v) of this quesiton.

Ex. 2 Find α and β so that the function

$$f(x) = \alpha x^3 + \beta x^2 + 1$$

has a point of inflection at $(-1, 2)$

Question 33 : Curve Sketching

33a - Past Paper Question - Summer 04/05

2 Consider the function $y = f(x) = x^4 - 6x^2 + 2$

(i) Find the y intercept of $f(x)$. **1**

(ii) Find and classify the critical points of $f(x)$ as local maxima or local minima or points of inflection. **6**

(iii) Find all points of inflection. **4**

(iv) Find the x values for which $y = f(x)$ is concave up/down. **2**

(v) Determine the behaviour of y as $x \to +\infty$ and as $x \to -\infty$. **2**

(vi) Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – v). **5**

33b- Past Paper Question - Summer 05/06

2 Consider the function $y = f(x) = \dfrac{1}{x-4}$ $(x \neq 4)$

(i)	Find the y intercept of $f(x)$.	**2**
(ii)	Show that the function has no local maximum or local minimum point.	**5**
(iii)	Explain why the function is decreasing for all values of x.	**2**
(iv)	Find the equation of the vertical asymptote.	**2**
(v)	Find the equation of the horizontal asymptote.	**3**
(vi)	Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – v).	**6**

Question 24 : Curve Sketching

This is Curve Sketching Example 3 from the lectures. The entirety of the material covered by this question is examinable in the End-Of-Semester exam.

Curve Sketching – Complete Ex. 3

Consider the function $f(x) = xe^{-x}$

i. Find the x and y intercepts of $f(x)$.

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

iii. Find all points of inflection.

iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$

v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – iv)

Question 25 : Curve Sketching

This is Curve Sketching Example 4 from the lectures. The entirety of the material covered by this question is examinable in the End-Of-Semester exam.

Curve Sketching – Complete Ex. 4

The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{0.17h}{h^2 + 2}$$

i. Find the axis intercepts of $A(h)$.

ii. Find and classify the critical points of $A(h)$ as local maxima or local minima.

iii. Determine the behaviour of $A(h)$ as $h \to +\infty$

iv. Sketch the graph of $y = A(h)$ for $h \geq 0$ illustrating clearly the features of the curve obtained in parts (i – iii)

Question 26 : Curve Sketching

Consider the function $y = f(x) = x^4 – 6x^2 + 2$

(i) Find the y intercept of f(x).

(ii) Find and classify the critical points of f(x) as local maxima or local minima or points of inflection.

(iii) Find all points of inflection.

(iv) Find the x values for which y = f(x) is concave up/down.

(v) Determine the behaviour of y as $x \to +\infty$ and as $x \to -\infty$.

(vi) Sketch the graph of y = f(x) illustrating clearly the features of the curve obtained in parts (i – v).

Question 34 : Simpson's Rule

(i) Use Simpson's rule with 4 equal subintervals to find an approximation for

$$\int_0^2 \sqrt{1 + 3x^2}dx$$

.

(ii) Use Simpson's rule with 4 equal subintervals to find an approximation for

$$\int_0^2 \sqrt{1+x^2}dx$$

Question 28 : Integration and Numerical Integration

1. **Integrations by Parts**
   Evaluate the following expression, using the Integration by Parts" technique

   (i)
   $$\int x\sqrt{x+1}dx$$

   (ii)
   $$\int x^5\sqrt{x^3+1}dx$$

   (ii)
   $$\int e^x cos(x)dx$$

2. **Trapezoidal Rule**
3. **Simpson's Rule**

Tutorial Sheet 5

Which of the following functions are well defined functions? If the function is not well defined, give a counterexample showing that it is not.

1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 + 1$
2. $f : \mathbb{R}^+ \to \mathbb{R}^+, \quad f(x) = x^2 + 1$
3. $f : \mathbb{R}^+ \to [1, 10], \quad f(x) = x^2 + 1$
4. $f : \mathbb{R}^+ \to [1, \infty), \quad f(x) = x^2 + 1$
5. $f : \mathbb{R}^+ \to \mathbb{R}^+, \quad f(x) = \sqrt[+]{x}$
6. $f : \mathbb{R}^- \to \mathbb{R}^-, \quad f(x) = \sqrt[+]{x}$
7. $f : \mathbb{R}^+ \to \mathbb{R}^-, \quad f(x) = \sqrt[+]{x}$
8. $f : \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = \sqrt[+]{x}$
9. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$
10. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$
11. $f : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$
12. $f : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}, \quad f(x) = \frac{1}{x-1}$
13. $f : \mathbb{R}^+ \setminus \{1\} \to \mathbb{R}^+, \quad f(x) = \frac{1}{x-1}$
14. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^x$
15. $f : \mathbb{R} \to \mathbb{R}^+, \quad f(x) = e^x - 1$
16. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \ln(x)$
17. $f : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+, \quad f(x) = \ln(x)$
18. $f : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}, \quad f(x) = \ln(x)$
19. $f : (1, \infty) \to \mathbb{R}, \quad f(x) = \ln(x + 1)$

For each of the following well defined functions, say whether the function is one-to-one, onto, or invertible. In the case of invertible functions, give the inverse function. In the case of non-invertible functions, modify the domain and codomain of the functions to make them invertible and give the corressponding inverse function.

1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x + 4$
2. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x$
3. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$
4. $f : \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2 + 4$
5. $f : \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = \sqrt[+]{x}$
6. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$
7. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^x$
8. $f : \mathbb{R}^+ \to [1, \infty), \quad f(x) = e^x$
9. $f : \mathbb{R}^+ \to \mathbb{R}^+, \quad f(x) = e^x + 1$
10. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sin(x)$
11. $f : (\text{-}\pi, \pi) \to [\text{-}1, 1], \; f(x) = \sin(x)$
12. $f : (\text{-}\frac{\pi}{2}, \frac{\pi}{2}) \to [\text{-}1, 1], \; f(x) = \sin(x)$

Question 35 : Area Between Curves and Lines

(i) (2005) Find the area bounded by the curves $y = x^2$ and $y = 2–x^2$.

(ii) (2006) Find the area bounded by the curve $y = x^2–6x + 5$ and the line $y = x–5$.

(iii) (2007) Find the area bounded by the curves defined by $y = x^2–4$ and $y = 4–x^2$.

(iv) (2008) Find the area bounded by the curve $y = x^2 - 1$ and the line $y = 4x–1$.

(v) (2009) Find the area bounded by the curve $y = 5x–x^2$ and the line $y = 2x$.

(vi) (2010) Find the area enclosed by the curve $y = 4–x^2$ and the line $y = x + 2$.

Question 26 : Integration : Partial Fraction Expansion Questions

- Suppose we have to integrate the following expression.

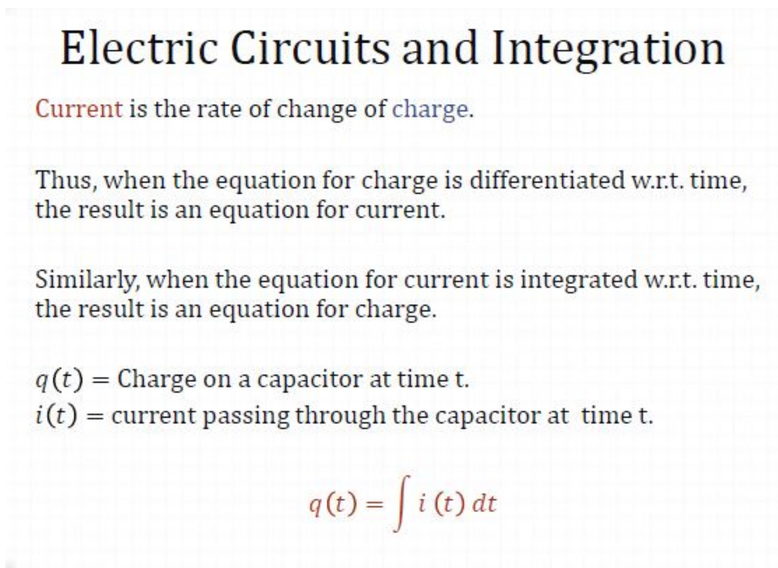
$$\int \frac{1}{x+1} + \frac{1}{x-2} dx$$

- The integral of both individual terms are $ln(x + 1)$ and $ln(x - 2)$
- The overall answer is therefore

$$\int \left(\frac{1}{x+1} + \frac{1}{x-2} \right) \, dx \; = \; \ln(x+1) + \ln(x-2) + c$$

- Further simplifcations are possible, but you will get full marks once you get to there.
- As we covered this extensively at the start of the semester, You should expect a question like this.

Question 40 : Applications of Integration - Electrical Circuits



1. A current $i(t) = 4e^{-2t}$ passes through a capacitor at time t. The capacitor is uncharged initially. Find the charge $q(t)$ at all times t.
2. A current $i(t) = 5+6\sin 3t$ passes through a capacitor at time t. The capacitor is uncharged at time $t = 0$. Find the charge $q(t)$ at all times t. ***2001 Q.3(b)***

Question 42 : Combined Integration Question (Exam Standard)

(i) Evaluate the following indefinite integral:

$$\int 3x^2 + 2e^x - 1dx$$

(ii) Evaluate the following definite integral:

$$\int_4^9 \frac{1}{\sqrt{x}}dx$$

(iii) Find

$$\int \left(e^{4x} + \cos(3) + \frac{1}{x^2} \right) dx$$

(iv) The area enclosed between the curve $y = cos(x)$ and the x-axis between $x = 0$ and $x = \frac{\pi}{3}$

Question 43 : Definite Integrals (Worked Example)

Consider the integral

$$\int_0^2 x\cos(x^2+1)\,dx$$

By using the substitution $u = x^2 + 1$, we obtain $du = 2xdx$ and

$$\begin{aligned} \int_0^2 x\cos(x^2+1)\,dx \qquad &= \frac{1}{2}\int_0^2 \cos(x^2+1)2x\,dx \\ = \frac{1}{2}\int_1^5 \cos(u)\,du \qquad &= \frac{1}{2}\Big(\sin(5) - \sin(1) \Big). \end{aligned}$$

Question 9 : Change of Base Formula

A formula that allows you to rewrite a logarithm in terms of logs written with another base. This is especially helpful when using a calculator to evaluate a log to any base other than 10 or e.

Assume that x, a, and b are all positive. Also assume that $a \neq 1$, $b \neq 1$.

Change of base formula:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example 1:

$$= \frac{\log_{32}}{\log_{16}} = \frac{\log_{32}}{\log_{16}}$$

Example 2:
(note that $2^4 = 16$ and $2^5 = 32$)

$$\log_2 3 = \frac{\log_{32}}{\log_{16}}$$

Example 3:

$$\log_a x = \frac{\log_{32}}{\log_{16}}$$

- Trigonometric Functions
- Unit Circle

MA4702 - Part 7 - Partial Differentiation

Question 41: Partial Derivatives

Exercise 41a : (Worked Example)

Solve for $\partial f/\partial x$ and $\partial f/\partial y$ if

$$f(x, y) = -x^2 + y$$

Solution :

$$\partial f/\partial x = -2x$$
$$\partial f/\partial y = 1$$

Exercise 41b :

Determine all first order and second order partial derivatives of each of the following.

(i) $f(x, y) = 3x + 4y$

(ii) $f(x, y) = xy^3 + x^2y^2$

(iii) $f(x, y) = x^3y + e^x$

(iv) $f(x, y) = xe^{2x+3y}$

(v) $f(x, y) = 2x\sin(x^2y)$.

Question 29: Partial Derivatives

Example 2 :

Solve for $\partial f/\partial x$ and $\partial f/\partial y$ if

$$f(x,y) = ln(xy) + sin(x) = ln(x) + ln(y) + sin(x)$$

.

Solution :

$$\partial f/\partial x =$$
$$\partial f/\partial y =$$

Question 30: Partial Derivatives

(i) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$ of the function $z = 2x^2y + 4xy^3 + 5x^2$.

(ii) Prove that the function $z = cos(x + 2y)$ satisfies the partial differential equation

$$\frac{\partial^2 z}{\partial^2 x} - 4\frac{\partial^2 z}{\partial^2 y} = 0$$

10

Question 29 : Maclaurin Series

- A MacLaurin series of a function f(x) for which a derivative may be taken of the function or any of its derivatives at 0 is the power series

  $$f(0) + f'(0) + \frac{f''(0)}{2!} + \frac{f''(0)}{2!}$$

  which can be written in the more compact sigma, or summation, notation as

  $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
  $$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

  where $n!$ denotes the factorial of n and $f^{(n)}(0)$ denotes the nth derivative of f evaluated at the point 0.

- The derivative of order zero f is defined to be f itself and $(x)^0$ and 0! are both defined to be 1.

- *(Relation to Taylor Series: Maclaurin series is a Taylor series expansion of a function about 0.)*

(i) Derive the Maclaurin Expansion of $sin(t)$.

Derivative	Value at t=0
$f(t) = \sin(t)$	0
$f'(t) = \cos(t)$	1
$f''(t) = -\sin(t)$	0
$f'''(t) = -\cos(t)$	-1
...	...

$$\sin(t) = f(0) + f'(0)t + \frac{f''(0)t^2}{2!} + \frac{f'''(0)t^3}{3!} + \cdots$$

$$\sin(t) = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 + \cdots$$

(i) Find the Maclaurin series of $f(x) = cos(x)$ up to and including the term containing x^4. Use your answer to estimate the value of $cos(1)$

(ii) Find the Maclaurin series of $f(x) = e^x$ up to and including the term containing x^6

(iii) Use your answer from part (ii) to estimate the value of e^3, $e^{0.5}$ and e^{-1}

Question 50

Solution :

$$\partial g/\partial x = (xy) \times (2xe^{x^2}) + ye^{x^2}$$
$$= 2ye^{x^2} + ye^{x^2} = ye^{(2+1)}$$

$$\partial g/\partial x(3,2) = 2e^(2+1)$$

$= 2(19) = 38$

$$\partial g/\partial y(3,2) = 3$$

Example 4 :
Find the rectangular box of least surface area that has volume 1000. Solution :

- Volume = 1000 = xyz
- Surface Area = 2xz + 2yz + 2xy
- z = 1000/xy
- S = 2x(1000/xy) + 2y(1000/xy) + 2xy
- = 2000/y + 2000/x + 2xy *need to minimize this*

Solve for $\partial S/\partial x = 0 and \partial S/\partial y = 0$

- $\partial S/\partial x = -2000/ + 2y = 0$
- $\partial S/\partial y = -2000/ + 2x = 0$
- $y = 1000/ and x = 1000/$
- $y() = 0$

Therefore, y = 0 or -1000 = 0 So, y = 10 and x = 10. Using the second derivative test, S does have a local minimum at (10, 10).

Appendices

Limits

- **Example 1**
  Find the limit $\lim_{x \to 2} 4x^3$.

We need to simplify the problem, since we have no rules about this expression by itself. We know from the identity rule above that $\lim_{x\to 2} x = 2$.
By the power rule,

$$\lim_{x\to 2} x^3 = \left(\lim_{x\to 2} x\right)^3 = 2^3 = 8$$

. Lastly, by the scalar multiplication rule, we get

$$\lim_{x\to 2} 4x^3 = 4 \lim_{x\to 2} x^3 = 4 \cdot 8 = 32$$

.

- **Example 2**
  Find the limit $\lim_{x\to 2}[4x^3 + 5x + 7]$.

  To do this informally, we split up the expression, once again, into its components. As above,$\lim_{x\to 2} 4x^3 = 32$.

  Also $\lim_{x\to 2} 5x = 5 \cdot \lim_{x\to 2} x = 5 \cdot 2 = 10 and \lim_{x\to 2} 7 = 7$.
  Adding these together gives

$$\lim_{x\to 2} 4x^3 + 5x + 7 = \lim_{x\to 2} 4x^3 + \lim_{x\to 2} 5x + \lim_{x\to 2} 7 = 32 + 10 + 7 = 49.$$

- **Example 3**
  Find the limit $\lim_{x\to 2} \frac{4x^3+5x+7}{(x-4)(x+10)}$.

  From the previous example the limit of the numerator is

$$\lim_{x\to 2} 4x^3 + 5x + 7 = 49.$$

  The limit of the denominator is

$$\lim_{x\to 2}(x-4)(x+10) = \lim_{x\to 2}(x-4) \cdot \lim_{x\to 2}(x+10) = (2-4) \cdot (2+10) = -24.$$

  As the limit of the denominator is not equal to zero we can divide. This gives

$$\lim_{x\to 2} \frac{4x^3 + 5x + 7}{(x-4)(x+10)} = -\frac{49}{24}.$$

- Example 4
  Find the limit $\lim_{x\to 4} \frac{x^4-16x+7}{4x-5}$.

  We apply the same process here as we did in the previous set of examples;

$$\lim_{x\to 4}\frac{x^4-16x+7}{4x-5}=\frac{\lim_{x\to 4}(x^4-16x+7)}{\lim_{x\to 4}(4x-5)}=\frac{\lim_{x\to 4}(x^4)-\lim_{x\to 4}(16x)+\lim_{x\to 4}(7)}{\lim_{x\to 4}(4x)-\lim_{x\to 4}5}.$$

We can evaluate each of these;

$$\lim_{x\to 4}(x^4)=256,$$

$$\lim_{x\to 4}(16x)=64,$$

$$\lim_{x\to 4}(7)=7,$$

$$\lim_{x\to 4}(4x)=16$$

and $\lim_{x\to 4}(5)=5$. Thus, the answer is $\frac{199}{11}$.

- **Example 5**

  Find the limit $\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$.

  In this example, evaluating the result directly will result in a division by zero. While you can determine the answer experimentally, a mathematical solution is possible as well.

  First, the numerator is a polynomial that may be factored:

  $$\lim_{x\to 2}\frac{(x-2)(x-1)}{x-2}$$

  Now, you can divide both the numerator and denominator by (x-2):

  $$\lim_{x\to 2}(x-1)=(2-1)=1$$

- **Example 6**

  Find the limit $\lim_{x\to 0}\frac{1-\cos x}{x}$.

  To evaluate this seemingly complex limit, we will need to recall some sine and cosine identities. We will also have to use two new facts. First, if f(x) is a trigonometric function (that is, one of sine, cosine, tangent, cotangent, secant or cosecant) and is defined at a, then $\lim_{x\to a}f(x)=f(a)$.

  Second, $\lim_{x\to 0}\frac{\sin x}{x}=1$. This may be determined experimentally, or by applying L'Hôpital's rule, described later in the book.

To evaluate the limit, recognize that $1 - \cos x$ can be multiplied by $1 + \cos x$ to obtain $(1 - \cos^2 x)$ which, by our trig identities, is $\sin^2 x$. So, multiply the top and bottom by $1+\cos x$. (This is allowed because it is identical to multiplying by one.) This is a standard trick for evaluating limits of fractions; multiply the numerator and the denominator by a carefully chosen expression which will make the expression simplify somehow. In this case, we should end up with:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x} &= \lim_{x\to 0}\left(\frac{1-\cos x}{x}\cdot\frac{1}{1}\right) && (1)\\
&= \lim_{x\to 0}\left(\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\right) && (2)\\
&= \lim_{x\to 0}\frac{(1-\cos x)\cdot 1 + (1-\cos x)\cdot\cos x}{x\cdot(1+\cos x)} && (3)\\
&= \lim_{x\to 0}\frac{1-\cos x+\cos x-\cos^2 x}{x\cdot(1+\cos x)} && (4)\\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x\cdot(1+\cos x)} && (5)\\
&= \lim_{x\to 0}\frac{\sin^2 x}{x\cdot(1+\cos x)} && (6)\\
&= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\right) && (7)
\end{aligned}$$

.

Our next step should be to break this up into $\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x}$ by the product rule. As mentioned above, $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Next, $\lim_{x\to 0}\frac{\sin x}{1+\cos x} = \frac{\lim_{x\to 0}\sin x}{\lim_{x\to 0}(1+\cos x)} = \frac{0}{1+\cos 0} = 0$.

Thus, by multiplying these two results, we obtain 0.