Curve Sketching

MA4602

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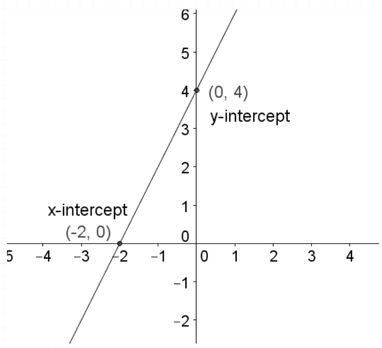
What is needed to sketch a curve?

To plot $y = f(x)$ we typically consider the following:

1. The domain and range of $f(x)$.
2. The x and y intercepts.
3. Vertical asymptotes (if any).
4. Behaviour as $x \to \pm\infty$ i.e. horizontal asymptotes.
5. Maximum and Minimum points.
6. Points of Inflection.

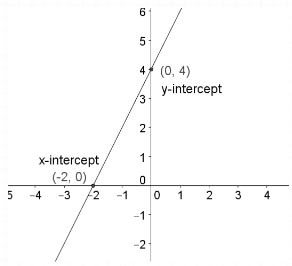
x and y Intercepts



x and y Intercepts

At the point where a curve (or a line) crosses the x-axis, the y value is always zero.

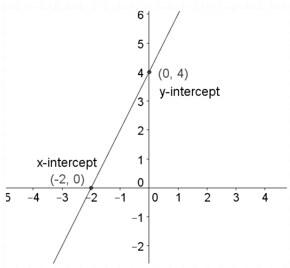
Thus, to find the x-intercept, we let $y = 0$ in our function and find the x value.



x and y Intercepts

At the point where a curve (or a line) crosses the y-axis, the x value is always zero.

Thus, to find the y-intercept, we let $x = 0$ in our function and find the y value.



x and y Intercepts – Ex. 1

Q. Find the x-intercept and the y-intercept of the function

$$f(x) = 2x^2 + 7x + 3$$

Solution: To find the x-intercept, let $y = 0$ and find x.

$$f(x) = 2x^2 + 7x + 3$$

$$0 = 2x^2 + 7x + 3$$

$$(2x + 1)(x + 3) = 0$$

x and y Intercepts – Ex. 1

$$(2x + 1)(x + 3) = 0$$

$$2x + 1 = 0 \qquad x + 3 = 0$$

$$x = -\frac{1}{2} \qquad x = -3$$

The curve $f(x) = 2x^2 + 7x + 3$ crosses the x axis at $-\frac{1}{2}$ and -3.

This gives the points $(-\frac{1}{2}, 0)$ and $(-3, 0)$

x and y Intercepts – Ex. 1

To find the y-intercept, let $x = 0$ and find y.

$$y = 2x^2 + 7x + 3$$

$$y = 2(0)^2 + 7(0) + 3$$

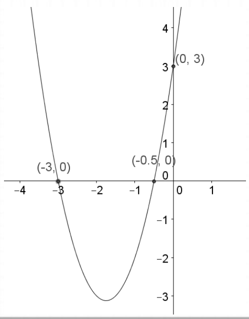
$$y = 3$$

The curve $f(x) = 2x^2 + 7x + 3$ crosses the y axis at 3.

This gives the points $(0, 3)$

x and y Intercepts – Ex. 1

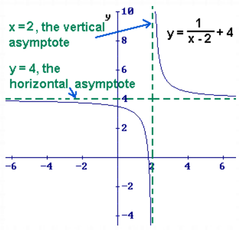
Graph of the curve $f(x) = 2x^2 + 7x + 3$



Asymptotes

An Asymptote is a line that continually approaches a given curve but does not meet it at any finite distance.

Typically, there are three types of asymptote: vertical, horizontal, and diagonal.



Vertical Asymptotes

How to find a vertical asymptote:

Let the Denominator of the given function equal zero and find value(s) for x.

Vertical Asymptote - Example

Identify the vertical asymptote of the following function:

$$f(x) = \frac{x^2 + 4}{x^2 - 4}$$

Solution: Let Denominator equal zero (note: no denominator, no vertical asymptote) and find value(s) for x

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

Vertical Asymptotes at $x = 2$ and $x = -2$

Why?

Why are asymptotes located at $x = 2$ and $x = -2$ for this particular function?

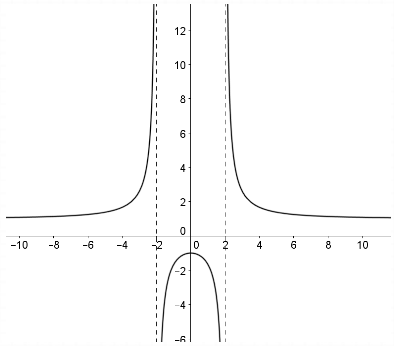
Reason:

$$\lim_{x\to 2}\frac{x^2+4}{x^2-4}=\frac{(2)^2+4}{(2)^2-4}$$

$$=\frac{4+4}{4-4}=\frac{8}{0}=\infty$$

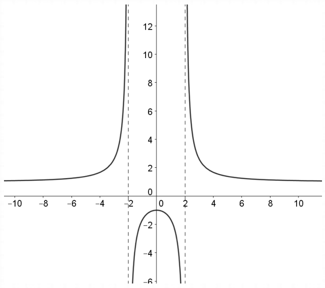
As the x value approaches 2 in this particular function, the corresponding y value approaches ∞

How does this affect the graph of the function?



How does this affect the graph of the function?

Notice how the curve never crosses at $x = 2$ and $x = -2$. At $x = 2$ and $x = -2$ the curve tends towards ∞ or $-\infty$



Horizontal Asymptotes

To determine the horizontal asymptote when graphing a function, find $\lim_{x\to\infty} f(x)$

Example: Identify the horizontal asymptote of the following function:

$$f(x)=\frac{x^2+4}{x^2-4}$$

$$\lim_{x\to\infty}\frac{x^2+4}{x^2-4}$$

Horizontal Asymptotes

$$\lim_{x\to\infty}\frac{x^2+4}{x^2-4}$$

$$=\lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{4}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}}$$

$$=\lim_{x\to\infty}\frac{1+\frac{4}{x^2}}{1-\frac{4}{x^2}}=\frac{1+\frac{4}{\infty^2}}{1-\frac{4}{\infty^2}}$$

$$=\frac{1+0}{1-0}=1$$

Horizontal Asymptote at

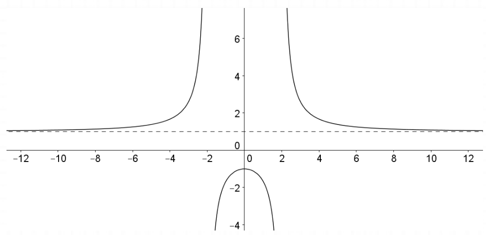
$$y = 1$$

Note: if the answer turns out to be ∞ then there is no horizontal asymptote.

Horizontal Asymptotes

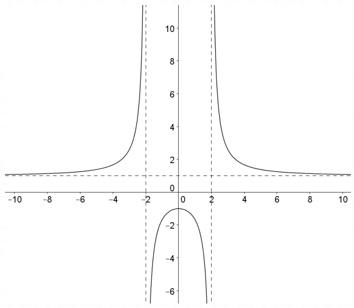
$$\lim_{x\to\infty}\frac{x^2+4}{x^2-4}=1$$

This indicates that as x approaches infinity, y approaches 1.



Asymptotes - Overview

$$f(x) = \frac{x^2 + 4}{x^2 - 4}$$



Asymptotes Overview

To determine the Horizontal Asymptote when graphing a function, find $\lim_{x\to\infty} f(x)$

To determine the Vertical Asymptote: let the Denominator of the given function equal zero and find value(s) for x.

Asymptotes Example

Q. Calculate the horizontal and vertical asymptotes of the following function:

$$f(x) = \frac{x}{x^2 - 9}$$

Solution: vertical asymptote – let denominator equal zero and find value for x.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

Two vertical asymptotes – at $x = 3$ and $x = -3$

Asymptotes Example

Horizontal asymptote: To determine the horizontal asymptote, find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{x}{x^2 - 9}$$

$$= \frac{\infty}{\infty}$$

Not defined…

$$\lim_{x\to\infty} \frac{x}{x^2 - 9}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2} - \frac{9}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{9}{x^2}}$$

Asymptotes Example

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{9}{x^2}}$$

$$= \frac{\frac{1}{\infty}}{1 - \frac{9}{\infty}}$$

$$= \frac{0}{1 - 0} = 0$$

Horizontal asymptote at $y = 0$

Asymptotes:

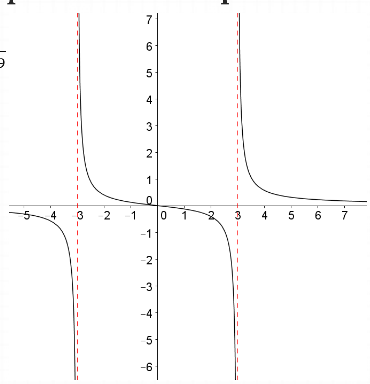
$$x = 3$$

$$x = -3$$

$$y = 0$$

Asymptotes Example

Graph of $f(x) = \frac{x}{x^2 - 9}$



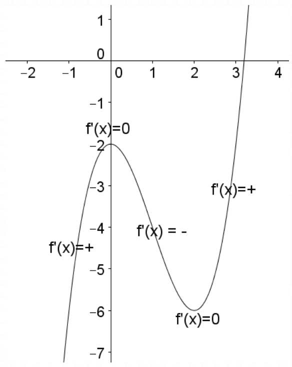
Differentiation

Differentiation is the process of finding the derivative, or rate of change, of a function.

The derivative of a function can indicate whether a function is increasing or decreasing at certain points.

$f'(x) = +$	Function is increasing
$f'(x) = -$	Function is decreasing
$f'(x) = 0$	Function is neither increasing nor decreasing

Differentiation



Maximum and Minimum Points

To find and classify critical points (max./min. points) of a function, complete the following steps:

1. Find $f'(x)$ and let $f'(x) = 0$
2. Calculate value(s) for x from this equation
3. Find the corresponding y values
4. Use $f'(x)$ to check the nature of the curve around these points to determine whether these points are max. or min.

Max. and Min. Points – Ex. 1

Example: Find the maximum and minimum points of the curve:

$$f(x) = x^3 - 3x^2 - 2$$

Solution:

$$f'(x) = 3x^2 - 6x$$

Let $f'(x) = 0$

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$

$3x = 0$	$x - 2 = 0$
$x = 0$	$x = 2$

Max. and Min. Points – Ex. 1

Find Corresponding y values:

$$y = x^3 - 3x^2 - 2$$

$x = 0$

$$y = (0)^3 - 3(0)^2 - 2$$

$$y = -2$$

Critical Point: $(0, -2)$

$x = 2$

$$y = (2)^3 - 3(2)^2 - 2$$

$$y = -6$$

Critical Point: $(2, -6)$

Max. and Min. Points – Ex. 1

How do we know whether these points $(0, -2)$ $(2, -6)$ are max. or min. points?

Analyse whether the function is increasing/decreasing before and after each of the turning points. $f'(x) = 3x^2 - 6x$

$x = 0$ $\qquad$ $x = 2$

$x = -1$	$x = 1$	$x = 3$
$f'(-1) = 9$	$f'(1) = -3$	$f'(3) = 9$
Function is increasing	Function is decreasing	Function is increasing

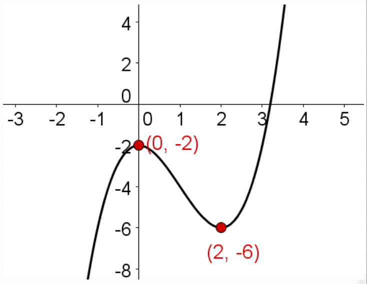
This indicates that there is a maximum point at $x = 0$ and a minimum point at $x = 2$

Max. and Min. Points – Ex. 1

Graph of $f(x) = x^3 - 3x^2 - 2$

Max. point: $(0, -2)$

Minimum point: $(2, -6)$



Max and Min Points – Ex. 2

Find the maximum and minimum points of the curve:

$$f(x) = \frac{x-1}{x-2} \qquad \frac{u}{v}$$

$u = x - 1$	$v = x - 2$
$\frac{du}{dx} = 1$	$\frac{dv}{dx} = 1$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Max and Min Points – Ex. 2

$$f'(x) = \frac{(x-2)(1) - (x-1)(1)}{(x-2)^2}$$

$$f'(x) = \frac{x - 2 - x + 1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

Let $f'(x) = 0$ and find value(s) for x

$$\frac{-1}{(x-2)^2} = 0$$

Not possible, so no max. or min. points.

Max. and Min. Points – Ex. 3

Example: Find the maximum and minimum points of the curve:

$$f(x) = 2x^3 + x^2 - 4x - 2$$

Solution:

$$f'(x) = 6x^2 + 2x - 4$$

Let $f'(x) = 0$

$$6x^2 + 2x - 4 = 0$$
$$(6x - 4)(x + 1) = 0$$

$6x - 4 = 0$	$x + 1 = 0$
$x = \frac{2}{3}$	$x = -1$

Max. and Min. Points – Ex. 3

Find Corresponding y values:

$$y = 2x^3 + x^2 - 4x - 2$$

$x = \frac{2}{3}$

$$y = 2\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) - 2$$

$$y = -3.63$$

Critical Point: $\left(\frac{2}{3}, -3.63\right)$

$x = -1$

$$y = 2(-1)^3 + (-1)^2 - 4(-1) - 2$$

$$y = 1$$

Critical Point: $(-1, 1)$

Max. and Min. Points – Ex. 3

How do we know whether these points $\left(\frac{2}{3}, -3.63\right)$ $(-1, 1)$ are max or min points?

Analyse whether the function is increasing/decreasing before and after each of the turning points. $f'(x) = 6x^2 + 2x - 4$

$x = -1$ $\qquad x = \frac{2}{3}$

$x = -2$	$x = 0$	$x = 1$
$f'(-2) = 16$	$f'(0) = -4$	$f'(1) = 4$
Function is increasing	Function is decreasing	Function is increasing

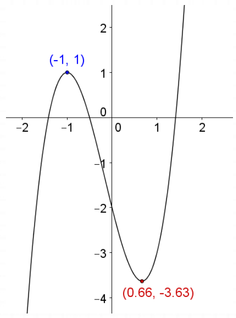
This indicates that there is a maximum point at $x = -1$ and a minimum point at $x = \frac{2}{3}$

Max. and Min. Points – Ex. 3

Graph of $f(x) = 2x^3 + x^2 - 4x - 2$

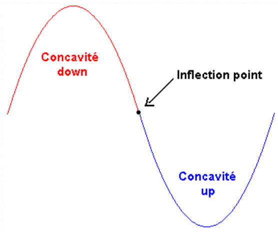
Max. point: $(-1, 1)$

Minimum point: $\left(\frac{2}{3}, -3.63\right)$



Points of Inflection

At a point of inflection, the curve changes from being concave upwards (positive curvature) to concave downwards (negative curvature), or vice versa.



Method for finding Points of Inflection

- Find $f''(x)$ and let $f''(x) = 0$
- Calculate value(s) for x from this equation
- Find the corresponding y values

Points of Inflection – Ex. 1

Example: Find the point of inflection of the curve:

$$f(x) = x^3 - 3x^2 - 2$$

Solution:

Step 1: Find $f''(x)$ and let $f''(x) = 0$

$$f(x) = x^3 - 3x^2 - 2$$

$$f'(x) = 3x^2 - 6x$$

$$f''(x) = 6x - 6$$

Points of Inflection – Ex. 1

Let $f''(x) = 0$

$$f''(x) = 6x - 6$$

$$6x - 6 = 0$$

$$x = 1$$

Find the corresponding y values:

$$y = x^3 - 3x^2 - 2$$

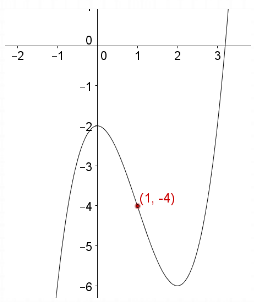
$$= (1)^3 - 3(1)^2 - 2 = -4$$

Point of inflection at $(1, -4)$

Points of Inflection – Ex. 1

Graph of $f(x) = x^3 - 3x^2 - 2$

Point of inflection at $(1, -4)$



Curve Sketching – Complete Ex. 1

Consider the function $f(x) = 2x^4 - 4x^2 + 1$

i. Find the y intercept of $f(x)$.

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

iii. Find all points of inflection.

iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$

v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – iv)

Curve Sketching – Complete Ex. 1

i. Find the y intercept of $f(x)$.

Let $x = 0$

$$f(x) = 2x^4 - 4x^2 + 1$$

$$f(0) = 2(0)^4 - 4(0)^2 + 1$$

$$f(0) = 1$$

y intercept is at the point (0,1)

Curve Sketching – Complete Ex. 1

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

$$f(x) = 2x^4 - 4x^2 + 1$$

$$f'(x) = 8x^3 - 8x$$

Let $f'(x) = 0$

$$8x^3 - 8x = 0$$

$$8x(x^2 - 1) = 0$$

Curve Sketching – Complete Ex. 1

$$8x(x^2 - 1) = 0$$

$8x = 0$ | $x^2 - 1 = 0$

$x = 0$ | $x = \pm 1$

Find corresponding y-values

$$f(x) = 2x^4 - 4x^2 + 1$$

$$f(0) = 2(0)^4 - 4(0)^2 + 1$$

$f(0) = 1$ Point: $(0, 1)$

$$f(x) = 2x^4 - 4x^2 + 1$$

$$f(1) = 2(1)^4 - 4(1)^2 + 1$$

$$f(1) = -1$$

Point: $(1, -1)$

$$f(-1) = 2(-1)^4 - 4(-1)^2 + 1$$

$$f(-1) = -1$$

Point: $(-1, -1)$

Curve Sketching – Complete Ex. 1

Critical points: $(0,1), (1,-1), (-1,-1)$

Classify these points: maximum or minimum points?

Analyse whether the function is increasing/decreasing before and after each of the turning points. $f'(x) = 8x^3 - 8x$

$x = -1$ $x = 0$ $x = 1$

$x = -2$	$x = -0.5$	$x = 0.5$	$x = 2$
$f'(-2) = -48$	$f'(-0.5) = 3$	$f'(0.5) = -3$	$f'(2) = 48$
Function is decreasing	Function is increasing	Function is decreasing	Function is increasing

Curve Sketching – Complete Ex. 1

Minimum point at $(-1, -1)$

Maximum point at $(0, 1)$

Minimum point at $(1, -1)$

Curve Sketching – Complete Ex. 1

iii. Find all points of inflection.

Find $f''(x)$ and let $f''(x) = 0$

$f'(x) = 8x^3 - 8x$

$f''(x) = 24x^2 - 8$

$24x^2 - 8 = 0$

$3x^2 - 1 = 0$

$3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}}$

$x = \pm 0.577$

Curve Sketching – Complete Ex. 1

$x = 0.577$

$f(x) = 2x^4 - 4x^2 + 1$

$f(0.577)$
$= 2(0.577)^4 - 4(0.577)^2 + 1$

$f(0.577) = -0.11$

Point of Inflection:
$(0.577, -0.11)$

$x = -0.577$

$f(x) = 2x^4 - 4x^2 + 1$

$f(-0.577)$
$= 2(-0.577)^4 - 4(-0.577)^2 + 1$

$f(-0.577) = -0.11$

Point of Inflection:
$(-0.577, -0.11)$

Curve Sketching – Complete Ex. 1

iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$

$f(x) = 2x^4 - 4x^2 + 1$

$\lim_{x\to+\infty} 2x^4 - 4x^2 + 1$

Take highest power of x outside brackets:

$\lim_{x\to+\infty} x^4\left(\frac{2x^4}{x^4} - \frac{4x^2}{x^4} + \frac{1}{x^4}\right)$

$\lim_{x\to+\infty} x^4\left(2 - \frac{4}{x^2} + \frac{1}{x^4}\right)$

$= \infty^4\left(2 - \frac{4}{\infty^2} + \frac{1}{\infty^4}\right)$

$= \infty(2 - 0 + 0)$

$= +\infty$

As the x value approaches $+\infty$ the y value will approach $+\infty$

Curve Sketching – Complete Ex. 1

Now, we'll check the function as $x \to -\infty$

$f(x) = 2x^4 - 4x^2 + 1$

$\lim_{x\to-\infty} 2x^4 - 4x^2 + 1$

Take highest power of x outside brackets:

$\lim_{x\to-\infty} x^4\left(\frac{2x^4}{x^4} - \frac{4x^2}{x^4} + \frac{1}{x^4}\right)$

$\lim_{x\to-\infty} x^4\left(2 - \frac{4}{x^2} + \frac{1}{x^4}\right)$

$= (-\infty)^4\left(2 - \frac{4}{(-\infty)^2} + \frac{1}{(-\infty)^4}\right)$

$= \infty(2 - 0 + 0)$

$= +\infty$

As the x value approaches $-\infty$ the y value will approach $+\infty$

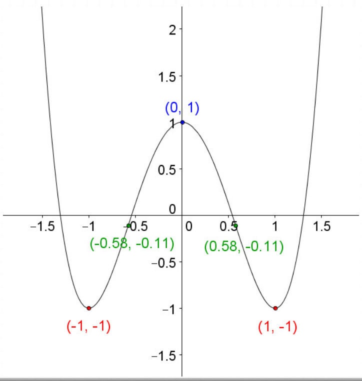
Curve Sketching – Complete Ex. 1

v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – v)

What we know:

- y intercept is at the point $(0, 1)$
- Minimum point at $(-1, -1)$
- Maximum point at $(0, 1)$
- Minimum point at $(1, -1)$
- Points of Inflection: $(0.577, -0.11)$ and $(-0.577, -0.11)$
- As the x value approaches $+\infty$ the y value will approach $+\infty$
- As the x value approaches $-\infty$ the y value will approach $+\infty$

Curve Sketching – Complete Ex. 1



Curve Sketching – Complete Ex. 2

Consider the function $f(x) = \frac{x}{x-1}$

i. Find the x and y intercepts of $f(x)$.
ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.
iii. Find all points of inflection.
iv. Find the vertical and horizontal asymptotes
v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – iv)

Curve Sketching – Complete Ex. 2

x-intercept: let $y = 0$

$$f(x) = \frac{x}{x-1}$$

$$0 = \frac{x}{x-1}$$

$$x = 0$$

x-intercept: (0,0)

y-intercept: let $x = 0$

$$f(x) = \frac{x}{x-1}$$

$$f(0) = \frac{0}{0-1}$$

$$f(0) = 0$$

y-intercept: (0,0)

Curve Sketching – Complete Ex. 2

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

Critical points: Find $f'(x)$ and let $f'(x) = 0$

$$f(x) = \frac{x}{x-1} \quad \frac{u}{v}$$

$$u = x \qquad v = x - 1$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 1$$

$$f'(x) = \frac{(x-1)(1) - (x)(1)}{(x-1)^2} = -\frac{1}{(x-1)^2}$$

Curve Sketching – Complete Ex. 2

Let $f'(x) = 0$

$$-\frac{1}{(x-1)^2} = 0$$

Not possible: -1 divided by a number cannot equal zero.

Result: no critical points.

Curve Sketching – Complete Ex. 2

iii. Find all points of inflection.

Points of inflection: find $f''(x)$ and let $f''(x) = 0$

$$f'(x) = -\frac{1}{(x-1)^2} = -1(x-1)^{-2}$$

$$f''(x) = -1(-2)(x-1)^{-3}(1)$$

$$f''(x) = \frac{2}{(x-1)^3}$$

Curve Sketching – Complete Ex. 2

Let $f''(x) = 0$

$$f''(x) = \frac{2}{(x-1)^3}$$

$$\frac{2}{(x-1)^3} = 0$$

Not possible: 2 divided by a number cannot equal zero.

Result: no points of inflection.

Curve Sketching – Complete Ex. 2

iv. Find the vertical and horizontal asymptotes

Vertical asymptote: let denominator = 0 and find a value for x

$$f(x) = \frac{x}{x-1}$$

$$x - 1 = 0$$

$$x = 1$$

Vertical asymptote at $x = 1$

Horizontal Asymptotes: find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{x}{x-1}$$

$$\lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{x}{x} - \frac{1}{x}} = \lim_{x\to\infty} \frac{1}{1 - \frac{1}{x}}$$

$$\frac{1}{1 - \frac{1}{\infty}} = \frac{1}{1-0} = 1$$

Horizontal asymptote at $y = 1$

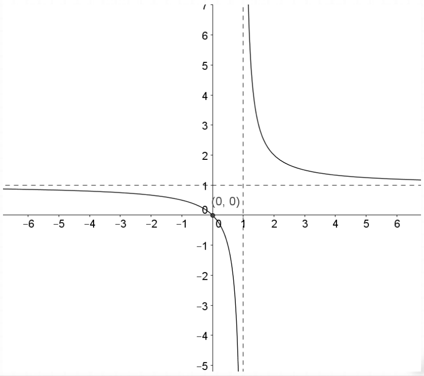
Curve Sketching – Complete Ex. 2

What we know:

- x- and y-intercept: (0,0)
- No Critical points.
- No Points of Inflection.
- Vertical asymptote at $x = 1$
- Horizontal asymptote at $y = 1$

Curve Sketching – Complete Ex. 2

$$f(x) = \frac{x}{x-1}$$



Curve Sketching – Complete Ex. 3

Consider the function $f(x) = xe^{-x}$

i. Find the x and y intercepts of $f(x)$.

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

iii. Find all points of inflection.

iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$

v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – iv)

Curve Sketching – Complete Ex. 3

i. Find the x and y intercepts of $f(x)$.

$$f(x) = xe^{-x}$$

Let $x = 0$

$$f(0) = (0)e^{-(0)} = 0$$

x-Intercept at (0,0)

x intercept will also turn out to be (0,0). Let $y = 0$

$$0 = xe^{-x}$$

$x = 0$ | $e^{-x} = 0$ Not possible

Thus y intercept at $x = 0$

y-intercept: (0,0)

Curve Sketching – Complete Ex. 3

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

$$f(x) = xe^{-x}$$

Use product rule to find $f'(x)$

$$u = x \qquad v = e^{-x}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -e^{-x}$$

$$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$f'(x) = x(-e^{-x}) + e^{-x}(1)$$

$$f'(x) = e^{-x}(1-x)$$

Let $f'(x) = 0$

$$e^{-x}(1-x) = 0$$

Curve Sketching – Complete Ex. 3

$e^{-x}(1-x) = 0$

$e^{-x} = 0$ | $1 - x = 0$

Not possible | $x = 1$

Critical point at $x = 1$

Corresponding y–value

$f(x) = xe^{-x}$

$f(1) = (1)e^{-1}$

$f(1) = e^{-1} = 0.37$

Critical point: $(1, 0.37)$

Curve Sketching – Complete Ex. 3

Critical point: $(1, 0.37)$. Is it a max. or a min. point?

Analyse whether the function is increasing/decreasing before and after each of the turning points. $f'(x) = e^{-x}(1-x)$

$x = 1$

$x = 0$	$x = 2$
$f'(0) = 1$	$f'(2) = -0.135$
Function is increasing	Function is decreasing

It is clear that the point $(1, 0.37)$ is a maximum point.

Curve Sketching – Complete Ex. 3

iii. Find all points of inflection.

$f'(x) = e^{-x}(1-x)$

Use product rule to find $f''(x)$

$u = e^{-x}$ $\quad v = 1 - x$

$\frac{du}{dx} = -e^{-x}$ $\quad \frac{dv}{dx} = -1$

$$f''(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$f''(x) = e^{-x}(-1) + (1-x)(-e^{-x})$$

$$f''(x) = e^{-x}(x-2)$$

Let $f''(x) = 0$

$$e^{-x}(x-2) = 0$$

Curve Sketching – Complete Ex. 3

$e^{-x}(x-2) = 0$

$e^{-x} = 0$ | $x - 2 = 0$

Not possible | $x = 2$

Point of Inflection at $x = 2$

Find corresponding y-value

$f(x) = xe^{-x}$

$f(2) = (2)e^{-2}$

$f(2) = 0.27$

Point of Inflection at $(2, 0.27)$

Curve Sketching – Complete Ex. 3

iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$

$$\lim_{x \to +\infty} xe^{-x}$$

$$\frac{\infty}{e^{\infty}} = 0$$

Reason: the exponential will grow much quicker.

For example:

At 10:

$$\frac{10}{e^{10}} = 0.00045$$

At 100:

$$\frac{100}{e^{100}} = 3.72 \times 10^{-42}$$

As x approaches $+\infty$, the y value approaches 0

Curve Sketching – Complete Ex. 3

Behaviour as $x \to -\infty$

$$\lim_{x \to -\infty} xe^{-x}$$

$$= (-\infty)e^{-(-\infty)}$$

$$= (-\infty)e^{\infty}$$

$$= (-\infty)(\infty)$$

$$= -\infty$$

As x approaches $-\infty$, the y value approaches $-\infty$

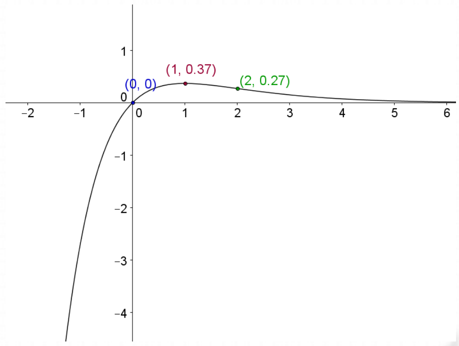
Curve Sketching – Complete Ex. 3

What we know:

- x- and y-Intercept at (0,0)
- Maximum point at $(1, 0.37)$
- Point of Inflection at $(2, 0.27)$
- As x approaches $+\infty$, the y value approaches 0
- As x approaches $-\infty$, the y value approaches $-\infty$

Curve Sketching – Complete Ex. 3

$f(x) = xe^{-x}$



Curve Sketching – Complete Ex. 4

The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{0.17h}{h^2+2}$$

i. Find the axis intercepts of $A(h)$.

ii. Find and classify the critical points of $A(h)$ as local maxima or local minima.

iii. Determine the behaviour of $A(h)$ as $h \to +\infty$

iv. Sketch the graph of $y = A(h)$ for $h \geq 0$ illustrating clearly the features of the curve obtained in parts (i – iii)

Curve Sketching – Complete Ex. 4

i. Find the axis intercepts of $A(h)$.

$$A(h) = \frac{0.17h}{h^2+2}$$

Let $h = 0$

$$A(0) = \frac{0.17(0)}{(0)^2+2} = 0$$

Intercept at (0,0)

The other intercept will also turn out to be (0,0). Let $A(h) = 0$

$$0 = \frac{0.17h}{h^2+2}$$

$$0 = 0.17h$$

$$h = 0$$

intercept: (0,0)

Curve Sketching – Complete Ex. 4

ii. Find and classify the critical points of $A(h)$ as local maxima or local minima.

$$A(h) = \frac{0.17h}{h^2+2}$$

Use quotient rule to find $A'(h)$

$u = 0.17h \qquad v = h^2 + 2$

$\frac{du}{dh} = 0.17 \qquad \frac{dv}{dh} = 2h$

$$A'(h) = \frac{v\frac{du}{dh} - u\frac{dv}{dh}}{v^2}$$

$$= \frac{(h^2+2)(0.17) - (0.17h)(2h)}{(h^2+2)^2}$$

$$= \frac{0.17h^2 + 0.34 - 0.34h^2}{(h^2+2)^2}$$

$$= \frac{-0.17h^2 + 0.34}{(h^2+2)^2}$$

Curve Sketching – Complete Ex. 4

Let $A'(h) = 0$

$$\frac{-0.17h^2 + 0.34}{(h^2+2)^2} = 0$$

$$-0.17h^2 + 0.34 = 0$$

$$0.17h^2 = 0.34$$

$$h^2 = 2$$

$$h = \pm\sqrt{2}$$

Corresponding *A(h)* value

$$A(h) = \frac{0.17h}{h^2+2}$$

$$A(\sqrt{2}) = \frac{0.17(\sqrt{2})}{(\sqrt{2})^2+2} = 0.06$$

Critical point: $(\sqrt{2}, 0.06)$

$$A(-\sqrt{2}) = \frac{0.17(-\sqrt{2})}{(-\sqrt{2})^2+2} = -0.06$$

Curve Sketching – Complete Ex. 4

Critical points: $(\sqrt{2}, 0.06)$ and $(-\sqrt{2}, -0.06)$. Max. or a min. point?
Only need to check the first point as we are graphing for $h \geq 0$

Analyse whether the function is increasing/decreasing before and after the turning point. $A'(h) = \frac{-0.17h^2+0.34}{(h^2+2)^2}$

$x = \sqrt{2}$

$x = 1$	$x = 2$
$A'(1) = 0.0188$	$A'(2) = -0.00944$
Function is increasing	Function is decreasing

It is clear that the point $(\sqrt{2}, 0.06)$ is a maximum point.

Curve Sketching – Complete Ex. 4

iii. Determine the behaviour of $A(h)$ as $h \to +\infty$

$$\lim_{h\to\infty} \frac{0.17h}{h^2+2}$$

$$= \lim_{h\to\infty} \frac{0.17(\infty)}{(\infty)^2+2}$$

$$= \frac{\infty}{\infty}$$

Not defined…

$$\lim_{h\to\infty} \frac{0.17h}{h^2+2}$$

$$= \lim_{h\to\infty} \frac{\frac{0.17}{h}}{1+\frac{2}{h^2}}$$

$$= \frac{\frac{0.17}{\infty}}{1+\frac{2}{(\infty)^2}}$$

$$\frac{0}{1+0} = 0$$

Curve Sketching – Complete Ex. 4

$$\lim_{h\to\infty} \frac{0.17h}{h^2+2} = 0$$

So, as $h \to \infty$, $A(h) \to 0$.

This means that, as number of hours after the drug was injected h increases toward infinity, the concentration of the drug $A(h)$ will approach zero.

Curve Sketching – Complete Ex. 4

What we know:

- Axis Intercept at $(0,0)$
- Maximum point at $(\sqrt{2}, 0.06)$
- As h approaches $+\infty$, the $A(h)$ value approaches 0

Curve Sketching – Complete Ex. 4

$$A(h) = \frac{0.17h}{h^2+2}$$