

FOURIER SERIES

1 – Periodic Functions

Recall that f has period T if $f(x + T) = f(x)$ for all x .

If f and g are periodic with period T , then if $h(x) = af(x) + bg(x)$, where a and b are constants,

$$\begin{aligned}h(x + T) &= af(x + T) + bg(x + T) \\ &= af(x) + bg(x) = h(x)\end{aligned}$$

and thus h also has period T .

1.1 – Functions of period 2π

We want to represent general functions of period 2π in terms of cosine and sine functions which also have period 2π ,

i.e., by the **trigonometric series**:

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

where a_0, a_1, a_2, \dots and b_1, b_2, b_3, \dots are real constants, the **coefficients** of the series.

Clearly, since each term of this infinite series has period 2π , if it converges its sum will also have period 2π .

Assume then that we can write $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We have to determine the coefficients:

To determine a_0 :

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) dx &= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx dx \\ &= 2\pi a_0 + 0 + 0\end{aligned}$$

$$\text{Thus, } a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

To determine a_n :

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx \, dx &= a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \\ &\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos mx \cos nx \, dx \\ &+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos mx \sin nx \, dx \end{aligned}$$

Remark

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \frac{1}{2} \left[\int_{-\pi}^{\pi} \cos(m+n)x \, dx + \int_{-\pi}^{\pi} \cos(m-n)x \, dx \right] \\ &= \frac{1}{2} \left\{ \left[\frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} + \left[\frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi} \right\} \\ &= 0\end{aligned}$$

unless $n = m$,

in which case

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \pi$$

Similarly

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \sin nx \, dx &= \frac{1}{2} \left[\int_{-\pi}^{\pi} \sin(m+n)x \, dx + \int_{-\pi}^{\pi} \sin(n-m)x \, dx \right] \\ &= -\frac{1}{2} \left\{ \left[\frac{\cos(m+n)x}{m+n} \right]_{-\pi}^{\pi} + \left[\frac{\cos(n-m)x}{n-m} \right]_{-\pi}^{\pi} \right\} \\ &= 0\end{aligned}$$

(even when $m = n$.)

Thus

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \pi a_m$$

and so

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

and similarly

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

These formulae for the coefficients a_0 , a_n and b_n are called the **Euler formulae**:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Note If $f(x)$ is periodic with period 2π then $\int_{-\pi}^{\pi} f(x) dx$ can be integrated over any interval of length 2π , e.g., $\int_0^{2\pi} f(x) dx$

The expression

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called the **Fourier series** for $f(x)$ with **Fourier coefficients** a_0 , a_n and b_n .

Example. Square wave

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$
$$f(x + 2\pi) = f(x)$$



Figure 1: $f(x)$

$$\begin{aligned}a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\&= \frac{1}{2\pi} \left[\int_{-\pi}^0 (-k) dx + \int_0^{\pi} k dx \right] \\&= \frac{1}{2\pi} [-k\pi + k\pi] = 0\end{aligned}$$

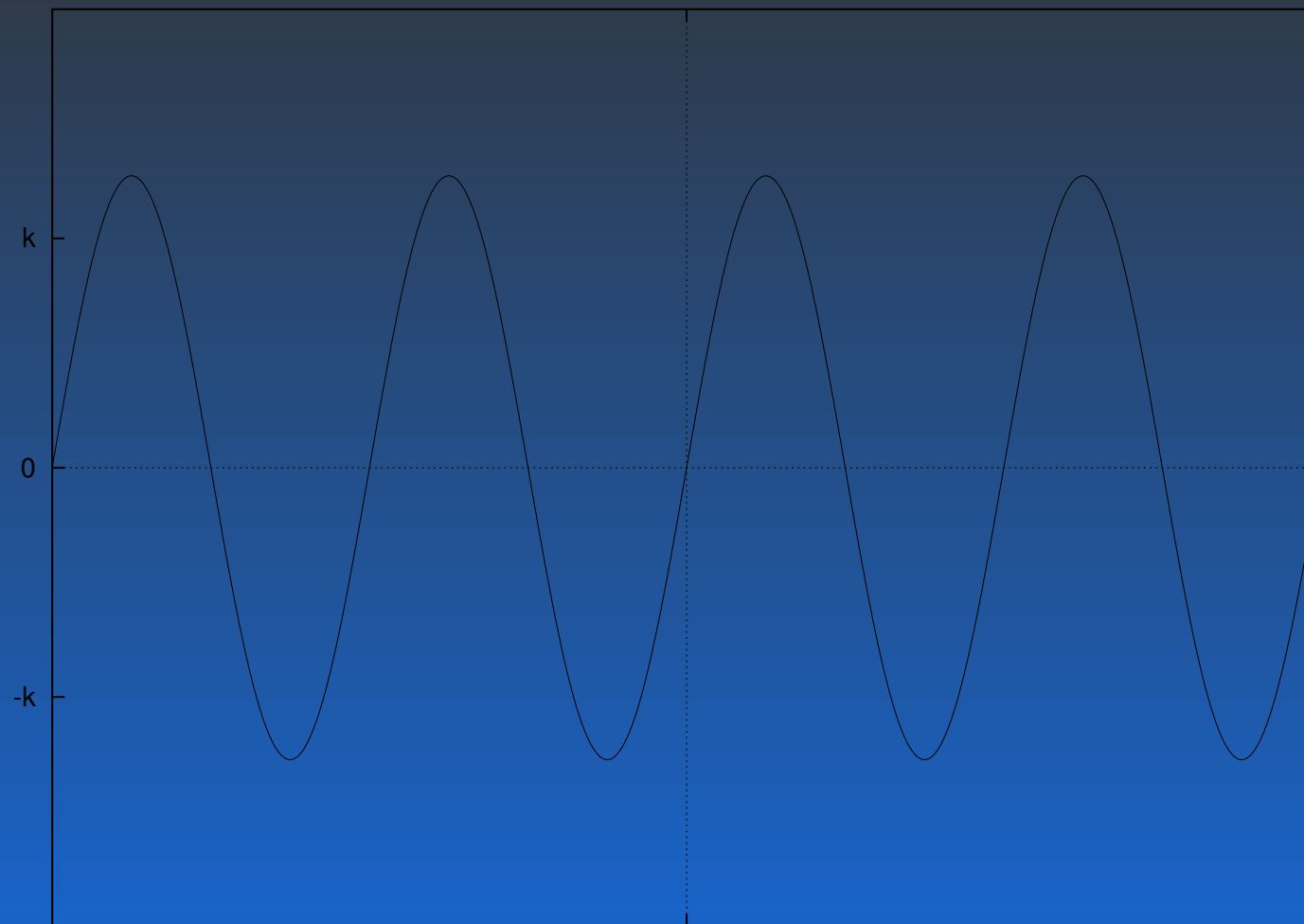
$$\begin{aligned}a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\&= \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) \cos nx \, dx + \int_0^{\pi} k \cos nx \, dx \right] \\&= \frac{1}{\pi} \left\{ \left[-\frac{k}{n} \sin nx \right]_{-\pi}^0 + \left[\frac{k}{n} \sin nx \right]_0^{\pi} \right\} \\&= 0\end{aligned}$$

$$\begin{aligned}b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\&= \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) \sin nx \, dx + \int_0^{\pi} k \sin nx \, dx \right] \\&= \frac{1}{\pi} \left\{ \left[\frac{k}{n} \cos nx \right]_{-\pi}^0 - \left[\frac{k}{n} \cos nx \right]_0^{\pi} \right\} \\&= \frac{k}{n\pi} \{ \cos 0 - \cos(-n\pi) - \cos(n\pi) + \cos 0 \} \\&= \frac{2k}{n\pi} \{ 1 - \cos n\pi \} \\&= \frac{2k}{n\pi} [1 - (-1)^n] = \begin{cases} 0, & n \text{ even} \\ \frac{4k}{n\pi}, & n \text{ odd} \end{cases}\end{aligned}$$

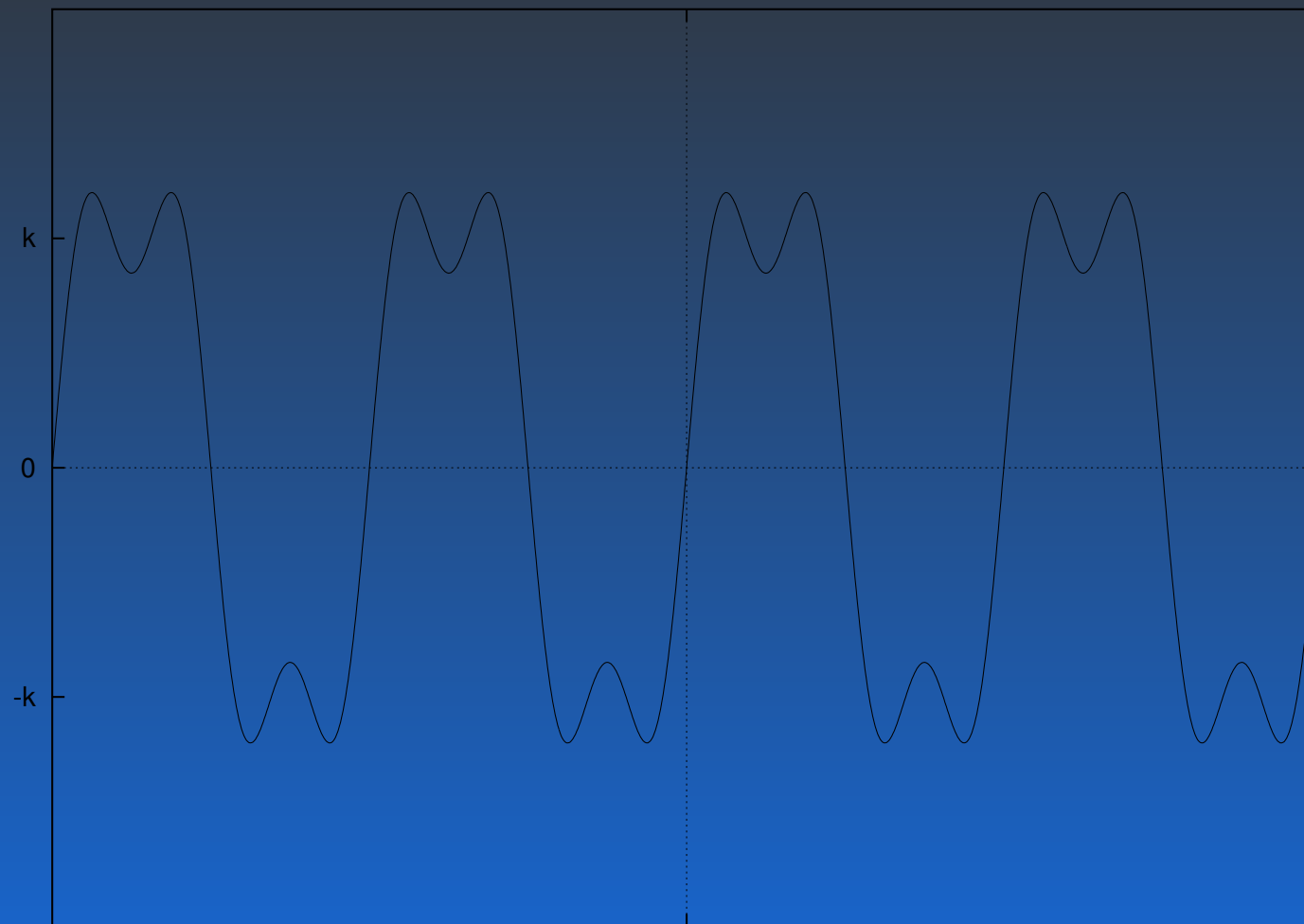
Thus, $b_1 = \frac{4k}{\pi}$, $b_2 = 0$, $b_3 = \frac{4k}{3\pi}$, $b_4 = 0$, etc.

and the Fourier series is

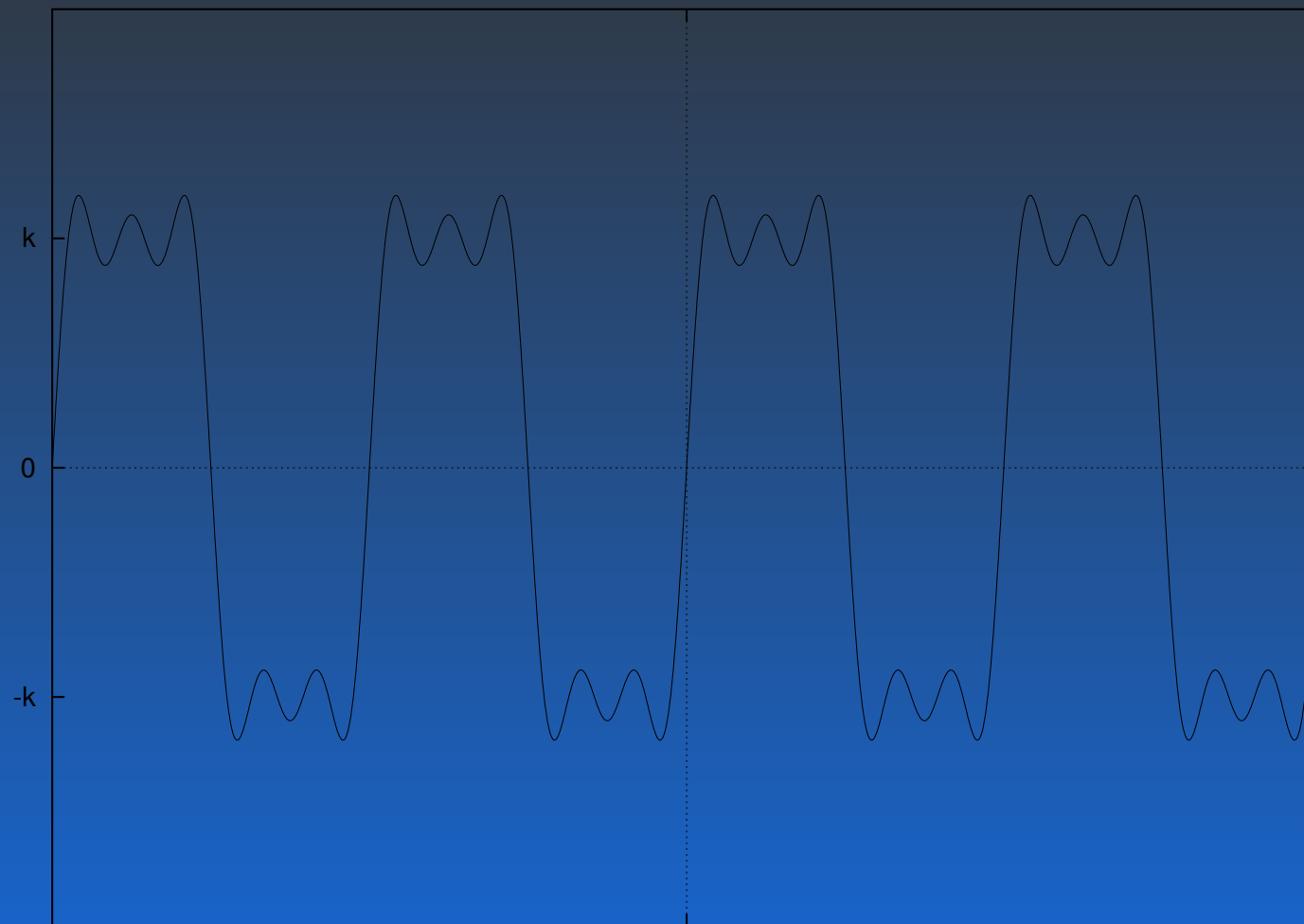
$$\frac{4k}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right]$$



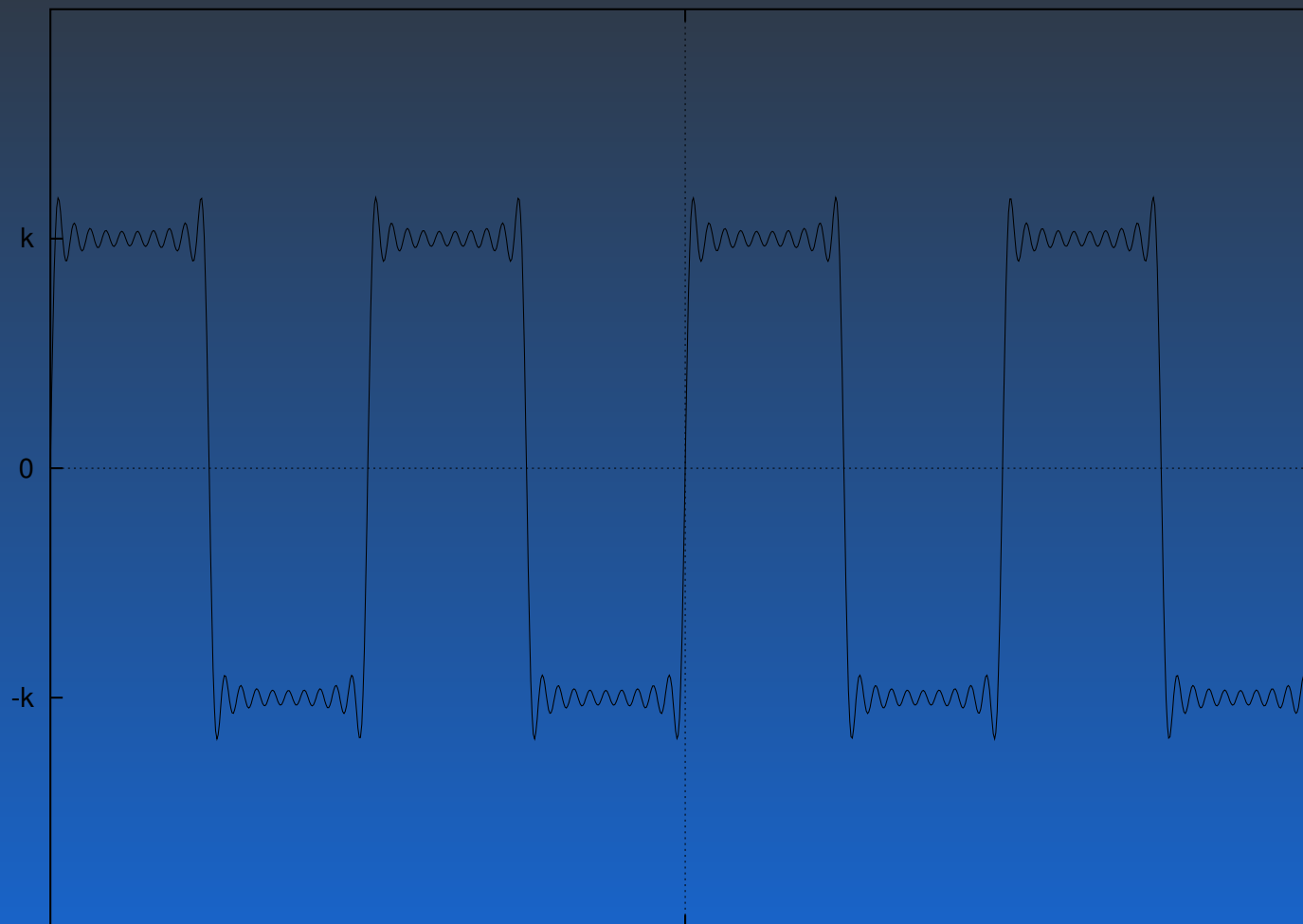
1 term of series



2 terms of series



3 terms of series



10 terms of series

Formal definition.

If $f(x)$ is periodic, with period 2π , piecewise continuous in $[-\pi, \pi]$, has a left and right-hand derivative at each $x_0 \in [-\pi, \pi]$, then its Fourier series is convergent to $f(x)$, except at a discontinuity x_0 , where the sum is $\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)]$.

Remark. The left hand derivative of f at x_0 is

$$\lim_{h \rightarrow 0_-} \frac{f(x_0 - 0) - f(x_0 - h)}{h}$$

where $f(x_0 - 0) = \lim_{x \rightarrow x_0_-} f(x)$

and similarly the right hand derivative of f at x_0 is

$$\lim_{h \rightarrow 0_+} \frac{f(x_0 + h) - f(x_0 + 0)}{h}$$

We can in these circumstances write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Example. Saw-tooth wave.

$$f(x) = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \end{cases}$$

with period 2π .

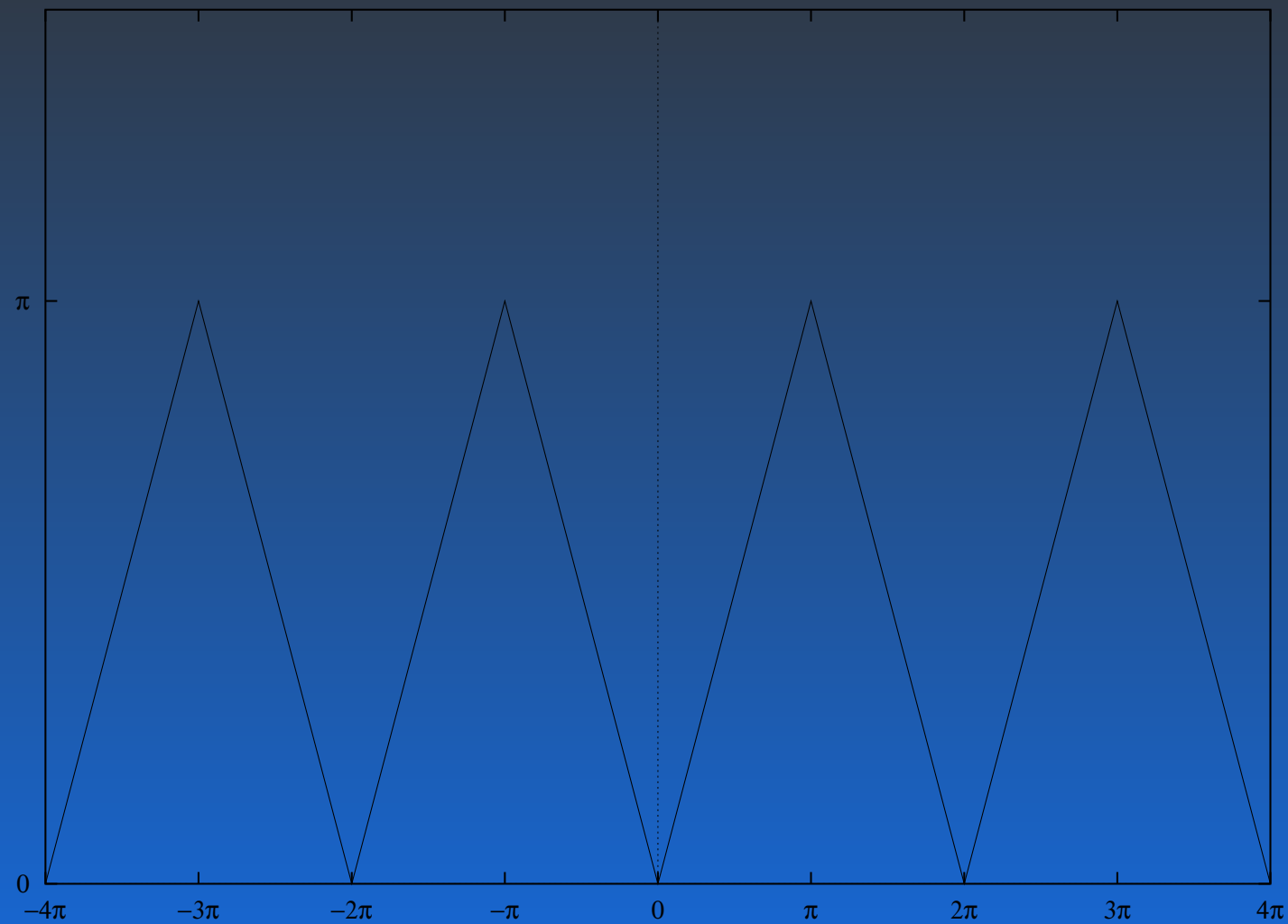


Figure 2: $f(x)$

$$\begin{aligned}a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\&= \frac{1}{2\pi} \left[\int_{-\pi}^0 (-x) dx + \int_0^{\pi} x dx \right] \\&= \frac{1}{\pi} \int_0^{\pi} x dx \\&= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\&= \frac{1}{\pi} \left[\int_{-\pi}^0 (-x) \cos nx \, dx + \int_0^{\pi} x \cos nx \, dx \right] \\&= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx \\&= \frac{2}{\pi} \left\{ \left[\frac{x}{n} \sin nx \right]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx \, dx \right\} \\&= \frac{2}{\pi} \left\{ 0 + \frac{1}{n^2} [\cos nx]_0^{\pi} \right\} \\&= \frac{2}{n^2 \pi} [\cos n\pi - 1] = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{n^2 \pi}, & n \text{ odd} \end{cases}\end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 (-x) \sin nx \, dx + \int_0^{\pi} x \sin nx \, dx \right] \\ &= 0 \end{aligned}$$

Thus

$$\begin{aligned} f(x) &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos nx}{n^2} \\ &= \frac{\pi}{2} - \frac{4}{\pi} \left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \dots \right] \end{aligned}$$

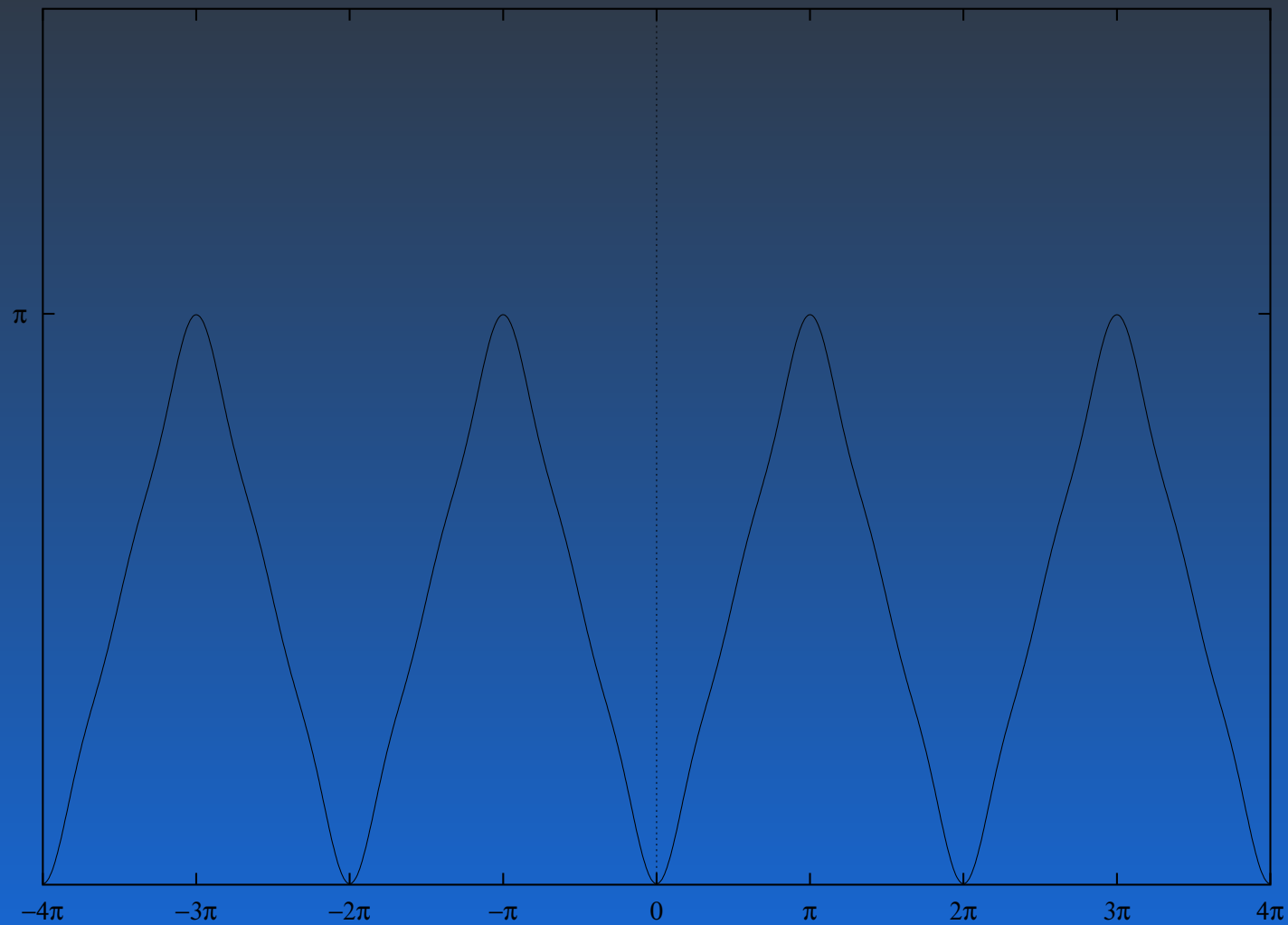


Figure 3: First 4 terms of Fourier series for $f(x)$

Remark. If $f(x)$ and $g(x)$ each have Fourier series expansions, then the Fourier series for the sum $f(x) + g(x)$ is just the sum of the two expansions.

For example, the Fourier series of the square wave

$f(x) = k, 0 < x < \pi \quad -k, -\pi < x < 0$ with period 2π was shown to be

$$\frac{4k}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right]$$

Thus the Fourier series for $g(x) = 2k, 0 < x < \pi \quad 0, -\pi < x < 0$ with period 2π is

$$k + \frac{4k}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right]$$

1.2 – Functions of arbitrary period.

Let $f(x)$ have period 2ℓ .

If we make the substitution $u = \frac{\pi}{\ell}x$, then,

when $x = \pm\ell$, $u = \pm\pi$

and so f is periodic in the new variable u with period 2π

and thus if f has a Fourier series, then

$$f(x) = f\left(\frac{\ell}{\pi}u\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nu + b_n \sin nu)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{\ell}{\pi}u\right) du$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{\ell}{\pi}u\right) \cos nu \, du$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{\ell}{\pi}u\right) \sin nu \, du$$

or, in terms of x , where $u = \frac{\pi}{\ell}x \Rightarrow du = \frac{\pi}{\ell} dx$

$$a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) dx$$

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx$$

and

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

Example. Rectified sine wave.

$$f(x) = \sin \pi x \quad 0 < x < 1$$

$$f(x+1) = f(x)$$

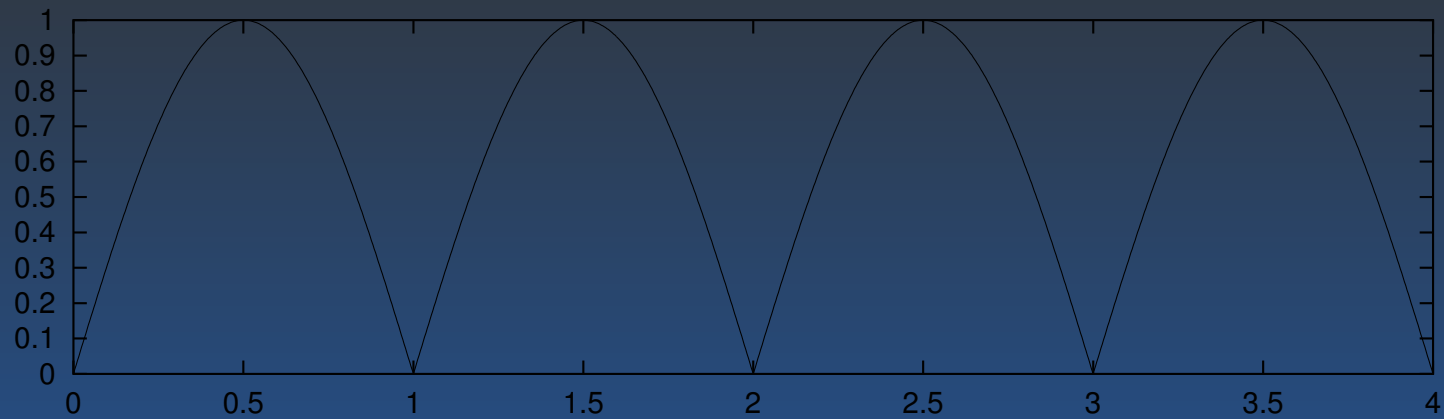


Figure 4: $f(x) = |\sin \pi x|$

$$\begin{aligned}a_0 &= \int_{-1/2}^{1/2} f(x) dx = \int_0^1 f(x) dx \\&= \int_0^1 \sin \pi x dx = -\frac{1}{\pi} [\cos \pi x]_0^1 \\&= \frac{2}{\pi}\end{aligned}$$

$$\begin{aligned}a_n &= 2 \int_0^1 \sin \pi x \cos 2n\pi x \, dx \\&= \int_0^1 \sin(2n+1)\pi x \, dx + \int_0^1 \sin(2n-1)\pi x \, dx \\&= -\frac{1}{(2n+1)\pi} [\cos(2n+1)\pi x]_0^1 - \frac{1}{(2n-1)\pi} [\cos(2n-1)\pi x]_0^1 \\&= -\frac{1}{(2n+1)\pi} (-2) - \frac{1}{(2n-1)\pi} (-2) \\&= \frac{2}{(2n+1)\pi} + \frac{2}{(2n-1)\pi} = \frac{8n}{(4n^2-1)\pi}\end{aligned}$$

$$\begin{aligned} &= \int_0^1 \cos(2n-1)\pi x \, dx - \int_0^1 \cos(2n+1)\pi x \, dx \\ &= \frac{1}{(2n-1)\pi} [\sin(2n-1)\pi x]_0^1 - \frac{1}{(2n+1)\pi} [\sin(2n+1)\pi x]_0^1 \\ &= 0 \end{aligned}$$

So,

$$\begin{aligned} f(x) &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{8n}{(4n^2 - 1)\pi} \cos 2n\pi x \\ &= \frac{2}{\pi} + \frac{8}{\pi} \left[\frac{\cos 2\pi x}{3} + \frac{2 \cos 4\pi x}{15} + \dots \right] \end{aligned}$$

Remark. In each of the last two examples $f(x) = f(-x)$ for all x and in each case the coefficients b_n turned out to be zero.

Was this a coincidence or can some work be saved by making use of this property?

1.3 – Even and odd functions.

The function f is said to be **even** if $f(-x) = f(x)$ for all x .

For example, the functions $\cos x$, 1 , x^2 , x^4 are even,

Similarly, f is said to be **odd** if $f(-x) = -f(x)$ for all x .

For example, the functions $\sin x$, x , x^3 are odd.

If $f(x)$ is even then

$$\int_{-k}^k f(x) dx = \int_{-k}^0 f(x) dx + \int_0^k f(x) dx$$

$$(u = -x, f(x) = f(-u) = f(u)) = \int_0^k f(u) du + \int_0^k f(x) dx$$

$$(du = -dx) = 2 \int_0^k f(x) dx$$

Similarly, if $f(x)$ is odd, then $\int_{-k}^k f(x) dx = 0$.

Note that if $f(x)$ is **even** and $g(x)$ is **odd** then if $h(x) = f(x)g(x)$,
 $h(-x) = f(x)[-g(x)] = -h(x)$ and so h is **odd**.

Thus, if $f(x)$ is even, the product $f(x) \sin \frac{n\pi x}{\ell}$ is odd and hence $b_n = 0$.

Similarly, if $f(x)$ is odd, the product $f(x) \cos \frac{n\pi x}{\ell}$ is odd and hence $a_n = 0$, and obviously $a_0 = 0$ also.

Thus, we have the following:

The Fourier series of an even function $f(x)$ of period 2ℓ is a **Fourier cosine series**:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{\ell} x$$

where

$$a_0 = \frac{1}{\ell} \int_0^{\ell} f(x) dx \quad a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx$$

The Fourier series of an odd function $f(x)$ of period 2ℓ is a **Fourier sine series**:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{\ell} x$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx$$

2 – Half-range expansions.

Even though Fourier series were introduced for periodic functions on $(-\infty, \infty)$, we can also find Fourier series for functions defined on an interval $[0, \ell]$, by finding either even or odd periodic extensions of period 2ℓ as follows:

2.1 – Periodic extensions

Definition 1. The *even periodic extension* $f_1(x)$ of period 2ℓ of the function $f(x)$ defined on $[0, \ell]$ is

$$f_1(x) = \begin{cases} f(x), & 0 \leq x \leq \ell \\ f(-x), & -\ell \leq x \leq 0 \end{cases}$$

with $f_1(x + 2\ell) = f_1(x)$

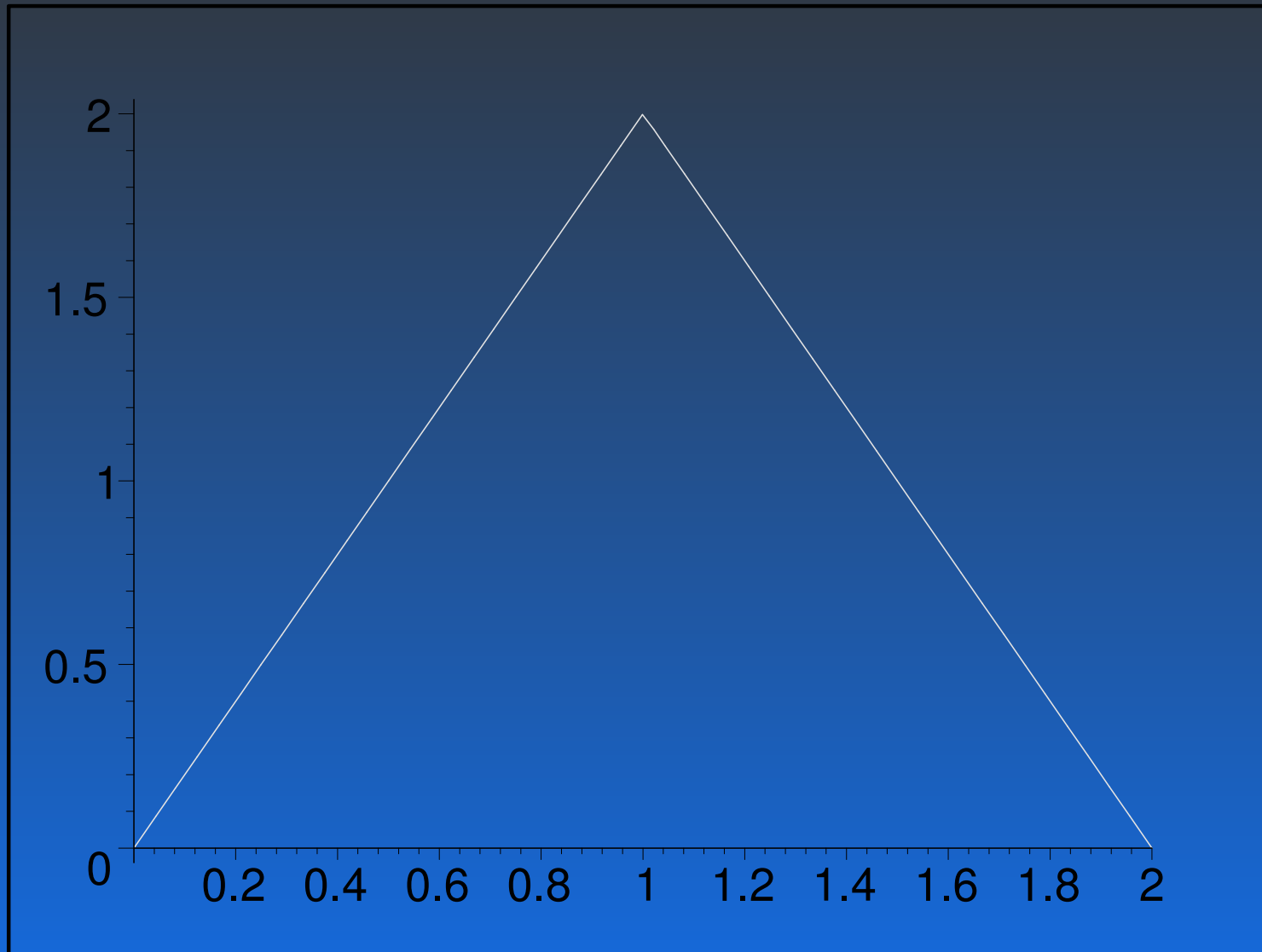


Figure 5: $f(x)$

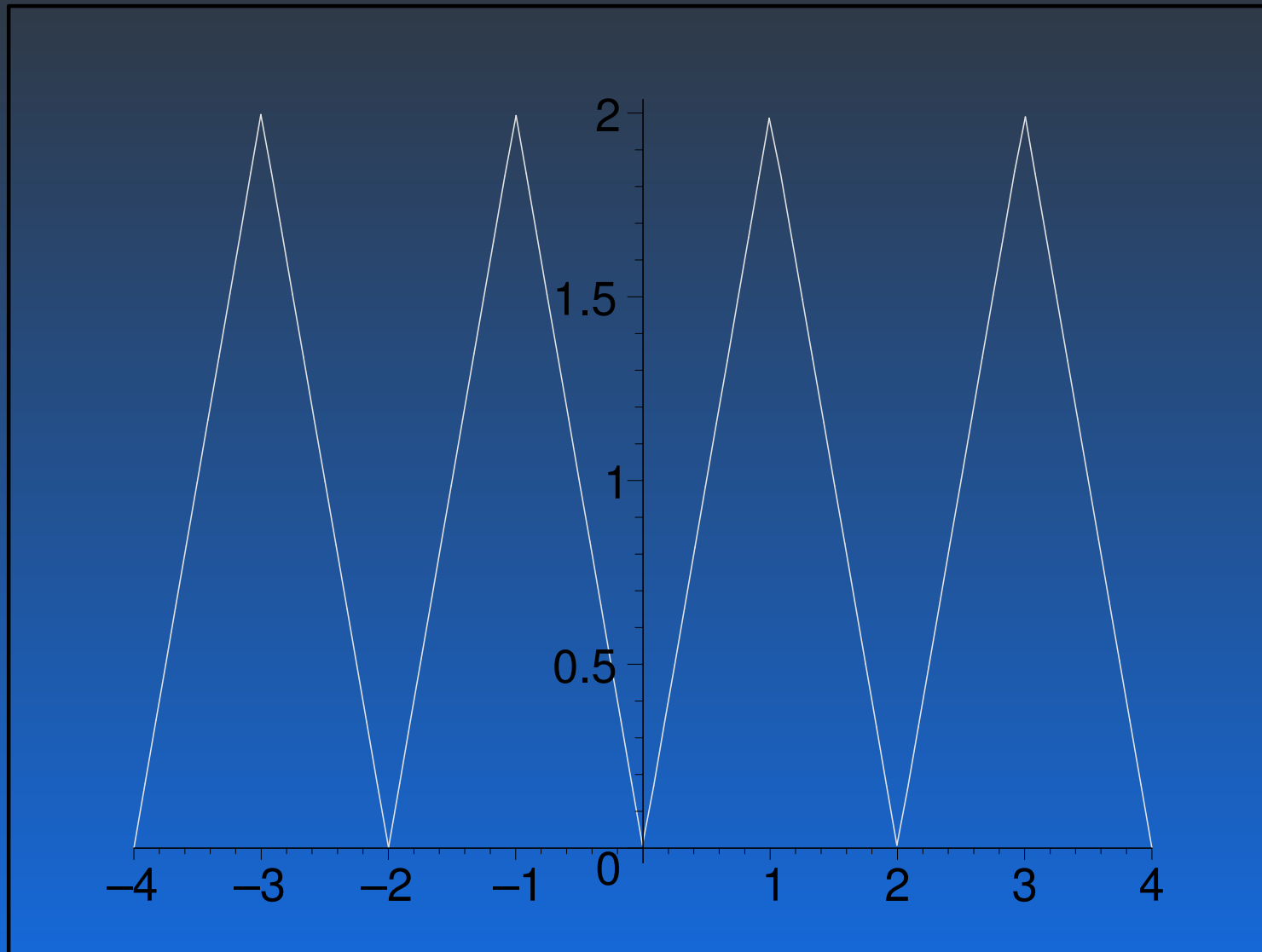


Figure 6: $f_1(x)$

$f_1(x)$ has a Fourier cosine series:

$$f_1(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{\ell} x$$

where

$$a_0 = \frac{1}{\ell} \int_0^{\ell} f(x) dx \quad a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx$$

This can now be regarded as a Fourier cosine series for $f(x)$ on the interval $0 \leq x \leq \ell$ if we restrict x to that interval, i.e.,

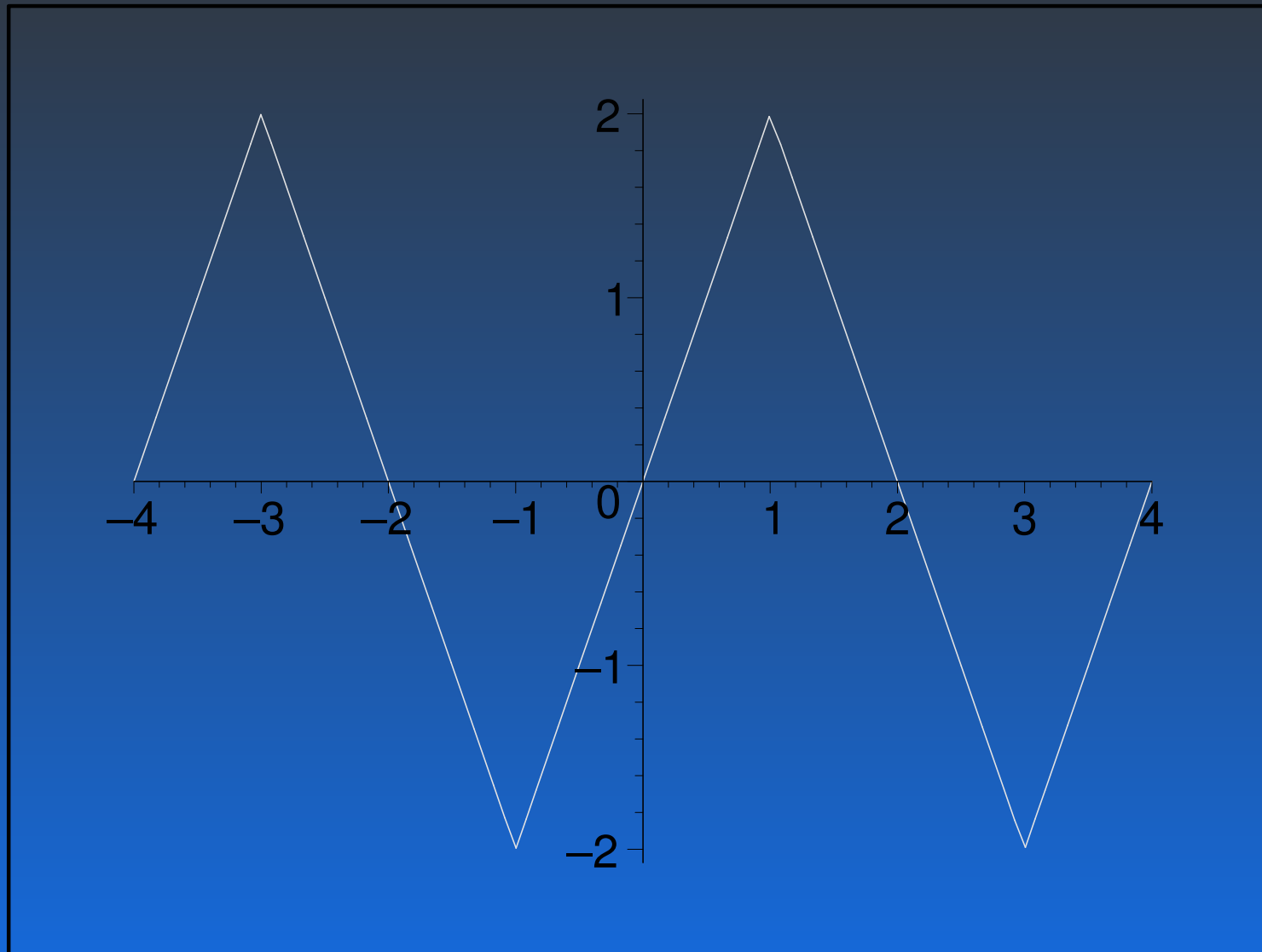
$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

for $0 \leq x \leq \ell$

Definition 2. The *odd periodic extension* $f_2(x)$ of period 2ℓ of the function $f(x)$ defined on $[0, \ell]$ is

$$f_2(x) = \begin{cases} f(x), & 0 < x < \ell \\ -f(-x), & -\ell < x < 0 \end{cases}$$

with $f_2(x + 2\ell) = f_2(x)$

Figure 7: $f_2(x)$

$f_2(x)$ has a Fourier sine series:

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{\ell} x$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx$$

This can now be regarded as a Fourier sine series for $f(x)$ on the interval $0 \leq x \leq \ell$ if we restrict x to that interval,

i.e.,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

for $0 < x < \ell$

Example.

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$$

Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} \quad 0 \leq x \leq 2$$

where

$$\begin{aligned}a_0 &= \frac{1}{2} \int_0^2 f(x) dx \\&= \frac{1}{2} \left\{ \int_0^1 x dx + \int_1^2 (2-x) dx \right\} \\&= \frac{1}{2} \left\{ \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2 \right\} \\&= \frac{1}{2} \left\{ \frac{1}{2} + (4-2) - \left(2 - \frac{1}{2} \right) \right\} \\&= \frac{1}{2} \left\{ \frac{1}{2} + \frac{1}{2} \right\} = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}a_n &= \frac{1}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx \\&= \int_0^1 x \cos \frac{n\pi x}{2} dx + \int_1^2 (2-x) \cos \frac{n\pi x}{2} dx\end{aligned}$$

$$u = x \quad dv = \cos \frac{n\pi x}{2} dx$$

$$du = dx \quad v = \frac{2}{n\pi} \sin \frac{n\pi x}{2}$$

$$u = (2-x) \quad dv = \cos \frac{n\pi x}{2} dx$$

$$du = -dx \quad v = \frac{2}{n\pi} \sin \frac{n\pi x}{2}$$

$$\begin{aligned}
\Rightarrow a_n &= \frac{2}{n\pi} \left[x \sin \frac{n\pi x}{2} \right]_0^1 - \frac{2}{n\pi} \int_0^1 \sin \frac{n\pi x}{2} dx \\
&\quad + \frac{2}{n\pi} \left[(2-x) \sin \frac{n\pi x}{2} \right]_1^2 + \frac{2}{n\pi} \int_1^2 \sin \frac{n\pi x}{2} dx \\
&= \frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \left[\cos \frac{n\pi x}{2} \right]_0^1 \\
&\quad - \frac{2}{n\pi} \sin \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \left[\cos \frac{n\pi x}{2} \right]_1^2 \\
&= \frac{4}{n^2\pi^2} \left(\cos \frac{n\pi}{2} - 1 - \cos n\pi + \cos \frac{n\pi}{2} \right) \\
&= \frac{4}{n^2\pi^2} \left(2 \cos \frac{n\pi}{2} - 1 - \cos n\pi \right)
\end{aligned}$$

Note that

$$\cos \frac{n\pi}{2} = 0, \quad n \text{ odd}$$

$$\cos \frac{n\pi}{2} = 1, \quad n/2 \text{ even}$$

$$\cos \frac{n\pi}{2} = -1, \quad n/2 \text{ odd}$$

Thus, $n \text{ odd} \Rightarrow a_n = 0$

$$n \text{ even and } n/2 \text{ even} \Rightarrow a_n = \frac{4}{n^2\pi^2}(2 - 2) = 0$$

$$n \text{ even and } n/2 \text{ odd} \Rightarrow a_n = \frac{4}{n^2\pi^2}(-2 - 2) = -\frac{16}{n^2\pi^2}$$

Alternatively Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \quad 0 \leq x \leq 2$$

where

$$\begin{aligned} b_n &= \frac{1}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} dx \\ &= \int_0^1 x \sin \frac{n\pi x}{2} dx + \int_1^2 (2-x) \sin \frac{n\pi x}{2} dx \end{aligned}$$

$$u = x \quad dv = \sin \frac{n\pi x}{2} dx$$

$$du = dx \quad v = -\frac{2}{n\pi} \cos \frac{n\pi x}{2}$$

$$u = (2 - x) \quad dv = \sin \frac{n\pi x}{2} dx$$

$$du = -dx \quad v = -\frac{2}{n\pi} \cos \frac{n\pi x}{2}$$

$$\begin{aligned}
\Rightarrow b_n &= -\frac{2}{n\pi} \left[x \cos \frac{n\pi x}{2} \right]_0^1 + \frac{2}{n\pi} \int_0^1 \cos \frac{n\pi x}{2} dx \\
&\quad - \frac{2}{n\pi} \left[(2-x) \cos \frac{n\pi x}{2} \right]_1^2 - \frac{2}{n\pi} \int_1^2 \cos \frac{n\pi x}{2} dx \\
&= -\frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \left[\sin \frac{n\pi x}{2} \right]_0^1 \\
&\quad + \frac{2}{n\pi} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \left[\sin \frac{n\pi x}{2} \right]_1^2 \\
&= \frac{4}{n^2\pi^2} \left(\sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right) \\
&= \frac{8}{n^2\pi^2} \left(\sin \frac{n\pi}{2} \right)
\end{aligned}$$

Note that

$$\sin \frac{n\pi}{2} = 0, \quad n \text{ even}$$

$$\sin \frac{n\pi}{2} = 1, \quad (n+1)/2 \text{ odd}$$

$$\cos \frac{n\pi}{2} = -1, \quad (n+1)/2 \text{ even}$$

Thus, $n \text{ even} \Rightarrow b_n = 0$

$$n \text{ odd and } (n+1)/2 \text{ odd} \Rightarrow b_n = \frac{8}{n^2\pi^2}$$

$$n \text{ odd and } (n+1)/2 \text{ even} \Rightarrow b_n = -\frac{8}{n^2\pi^2}$$

3 – Fourier series solutions of differential equations

There is an immediate application of Fourier series in the solution of constant coefficient second order linear differential equations,

i.e., equations of the form

$$y'' + ay' + by = R(x)$$

where a and b are constant.

If $R(x)$ is a constant or a cosine or sine term, this equation can be solved by the method of undetermined coefficients.

Thus, if $R(x)$ can be expanded in a Fourier series, the solution can also be found in this way.

Example

$$y'' + y' = r(x)$$

$$\text{where } r(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$$

To find y_h the solution of the homogeneous equation

$$y_h'' + y_h' = 0$$

let $y_h = e^{\lambda x}$; substituting we obtain $(\lambda^2 + \lambda)e^{\lambda x} = 0$ and thus the characteristic equation $\lambda(\lambda + 1) = 0$ which has roots 0 and -1 .

Thus $y_h = A + Be^{-x}$ where A and B are arbitrary constants.

To find now the particular integral y_p corresponding to the RHS $r(x)$, note that we have already found a Fourier cosine series expansion for $r(x)$, i.e.,

$$r(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} \quad 0 \leq x \leq 2$$

where

$$a_0 = \frac{1}{2}$$
$$a_n = \frac{4}{n^2 \pi^2} \left(2 \cos \frac{n\pi}{2} - 1 - \cos n\pi \right)$$

The particular integral will take the form

$$y_p = y_0 + \sum_{n=1}^{\infty} y_n$$

where y_0 is the particular integral corresponding to the constant term a_0 and y_n is the particular integral corresponding to $a_n \cos \frac{n\pi x}{2}$ for each n .

Note that y_0 would usually be a constant (as a_0 is) but, as one of the solutions of the homogeneous equations is constant, in this case

$$y_0 = c_0 x$$

for some constant c_0 .

Substituting into the equation

$$y_0'' + y_0' = a_0$$

we obtain

$$c_0 = a_0 = \frac{1}{2}$$

Similarly y_n satisfies the equation

$$y_n'' + y_n' = a_n \cos \frac{n\pi x}{2}$$

y_n takes the form

$$y_n = A_n \cos \frac{n\pi x}{2} + B_n \sin \frac{n\pi x}{2}$$

and has derivatives

$$\begin{aligned}y'_n &= -\frac{n\pi}{2}A_n \sin \frac{n\pi x}{2} + \frac{n\pi}{2}B_n \cos \frac{n\pi x}{2} \\y''_n &= -\left(\frac{n\pi}{2}\right)^2 A_n \cos \frac{n\pi x}{2} - \left(\frac{n\pi}{2}\right)^2 B_n \sin \frac{n\pi x}{2}\end{aligned}$$

Substituting for y''_n and y'_n then gives

$$\begin{aligned}\left[-\left(\frac{n\pi}{2}\right)^2 A_n + \frac{n\pi}{2}B_n\right] \cos \frac{n\pi x}{2} &- \left[\left(\frac{n\pi}{2}\right)^2 B_n + \frac{n\pi}{2}A_n\right] \sin \frac{n\pi x}{2} \\&= a_n \cos \frac{n\pi x}{2}\end{aligned}$$

Equating the coefficients of $\cos \frac{n\pi x}{2}$ and $\sin \frac{n\pi x}{2}$ on each side of the equation we obtain

$$\begin{aligned} -\left(\frac{n\pi}{2}\right)^2 A_n + \frac{n\pi}{2} B_n &= a_n \\ \left(\frac{n\pi}{2}\right)^2 B_n + \frac{n\pi}{2} A_n &= 0 \end{aligned}$$

The second equation gives

$$A_n = -\frac{n\pi}{2} B_n$$

Substituting in the first equation we obtain

$$\left[\left(\frac{n\pi}{2} \right)^3 + \frac{n\pi}{2} \right] B_n = a_n$$

or

$$B_n = \frac{a_n}{\left[\left(\frac{n\pi}{2} \right)^2 + 1 \right] \frac{n\pi}{2}}$$

and consequently

$$A_n = - \frac{a_n}{\left[\left(\frac{n\pi}{2} \right)^2 + 1 \right]}$$

Thus, the general solution of the differential equation is

$$y = A + Be^{-x} + \frac{x}{2} + \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi x}{2} + B_n \sin \frac{n\pi x}{2} \right]$$

with A_n and B_n as above.

Example. RLC circuit.

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{dv(t)}{dt}$$

where $L = 10$ henrys, $R = 100$ ohms, $C = 0.01$ farads; i.e.,

$$10 \frac{d^2 i}{dt^2} + 100 \frac{di}{dt} + 100i = \frac{dv(t)}{dt}$$

We first find the solution i_h of the homogeneous equation

$$10 \frac{d^2 i}{dt^2} + 100 \frac{di}{dt} + 100i = 0$$

by setting $i = e^{\lambda t}$ and substituting to obtain the characteristic equation

$$10\lambda^2 + 100\lambda + 100 = 0$$

which has two (negative) roots $\lambda_{1,2} = -5 \pm \sqrt{15}$

Thus $i_h = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \rightarrow 0$ as $t \rightarrow \infty$.

Now, let $v(t) = 100t(\pi^2 - t^2)$ volts for $-\pi < t < \pi$ and let v have period 2π .

We now can find a particular integral i_p of the differential equation by expanding $v'(t)$ in a Fourier series and then finding a particular integral corresponding to each term of the series.

$$v'(t) = 100(\pi^2 - 3t^2)$$

and since $v'(-t) = v'(t)$, $v'(t)$ is **even** and so has a Fourier cosine series:

$$v'(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

where

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_0^{\pi} v'(t) dt \\&= \frac{1}{\pi} \int_0^{\pi} 100(\pi^2 - 3t^2) dt \\&= \frac{100}{\pi} [\pi^2 t - t^3]_0^{\pi} \\&= \frac{100}{\pi} (\pi^3 - \pi^3) \\&= 0\end{aligned}$$

Or we could simply note that

$$\int_0^{\pi} v'(t) dt = [v(t)]_0^{\pi} = v(\pi) - v(0) = 0$$

$$\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^\pi v'(t) dt \\
&= \frac{2}{\pi} \left\{ \int_0^\pi 100\pi^2 \cos nt dt - \int_0^\pi 300t^2 \cos nt dt \right\} \\
&= \frac{2}{\pi} \left\{ \frac{100\pi^2}{n} [\sin nt]_0^\pi - 300 \left(\left[\frac{t^2}{n} \sin nt \right]_0^\pi - \frac{2}{n} \int_0^\pi t \sin nt dt \right) \right\} \\
&= \frac{1200}{n\pi} \int_0^\pi t \sin nt dt \\
&= \frac{1200}{n\pi} \left\{ - \left[\frac{t}{n} \cos nt \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos nt dt \right\} \\
&= -\frac{1200}{n^2} \cos n\pi + \frac{1200}{n^3\pi} [\sin nt]_0^\pi = (-1)^{n+1} \frac{1200}{n^2}
\end{aligned}$$

That is

$$v'(t) = 1200 \left(\cos t - \frac{1}{4} \cos 2t + \frac{1}{9} \cos 3t - \dots \right)$$

Let i_n be the particular integral corresponding to $\frac{\cos nt}{n^2}$.

This takes the form

$$i_n = A_n \cos nt + B_n \sin nt$$

where A_n and B_n are constants (which depend on n)

$$i'_n = -nA_n \sin nt + nB_n \cos nt$$

and

$$i''_n = -n^2 A_n \cos nt - n^2 B_n \sin nt$$

Substituting into the equation for i_n , i.e.,

$$10 \frac{d^2 i_n}{dt^2} + 100 \frac{di_n}{dt} + 100 i_n = \frac{\cos nt}{n^2}$$

gives

$$\begin{aligned} & (-10n^2 A_n + 100n B_n + 100 A_n) \cos nt \\ & + (-10n^2 B_n - 100n A_n + 100 B_n) \sin nt = \frac{\cos nt}{n^2} \end{aligned}$$

Equating the coefficients of $\cos nt$ and $\sin nt$ on each side of this equation (and dividing by 10) gives

$$\begin{aligned}(10 - n^2)B_n - 10nA_n &= 0 \\ (10 - n^2)A_n + 10nB_n &= \frac{1}{10n^2}\end{aligned}$$

Thus the first equation \Rightarrow

$$A_n = \frac{10 - n^2}{10n} B_n$$

Substituting this for A_n in the second equation gives

$$\left[\frac{(10 - n^2)^2}{10n} + 10n \right] B_n = \frac{1}{10n^2}$$

or

$$\left[\frac{(10 - n^2)^2 + 100n^2}{10n} \right] B_n = \frac{1}{10n^2}$$

So

$$B_n = \frac{1}{n [(10 - n^2)^2 + 100n^2]}$$

and consequently

$$A_n = \frac{10 - n^2}{10n^2 [(10 - n^2)^2 + 100n^2]}$$

Therefore the particular integral for $v'(t)$ is

$$i_p = 1200 \sum_{n=1}^{\infty} (-1)^{n+1} [A_n \cos nt + B_n \sin nt]$$

with A_n and B_n defined as above.