CONTINUOUS ASSESSMENT TEST

Full marks for complete answers to any FOUR questions.
All questions carry equal marks.

1. Consider the nonlinear equation

$$x^4 - 18x^2 + 45 = 0$$

which is known to have a root in the interval $(1, 2)$.

(a) Use Newton's method to evaluate this root to six exact digits. [*13 marks*]

(b) Use the secant method to evaluate this root to six exact digits. [*12 marks*]

(*Hint: The exact root is* $\sqrt{3} = 1.73205080$.)

2. Consider the nonlinear equation

$$e^x - x - 2 = 0.$$

(a) Show there is a root α in the interval $(1, 2)$. [*2 marks*]

(b) Estimate how many iterations will be needed in order to approximate this root with an accuracy of $\epsilon = 0.001$ using the bisection method. [*8 marks*]

(c) Then approximate α with an accuracy of $\epsilon = 0.001$ using the bisection method. [*15 marks*]

(*Note: The root is* $\alpha = 1.14619322062$.)

3. Consider the nonlinear equation

$$f(x) = x^3 - 2x^2 - 3 = 0$$

which has a root α between 2 and 3.

(a) Rewrite the equation $f(x) = 0$ as the fixed-point problem $g_1(x) = x$ where

$$g_1(x) = 2 + \frac{3}{x^2}$$

and, using the convergence criterion, show that the iteration algorithm associated with this problem converges to the root α. [*7 marks*]

(b) Create two other fixed-point iteration schemes with functions $g_2(x)$ and $g_3(x)$. [*8 marks*]

(c) Perform ten iterations with six exact digits for each of the three schemes $g_1(x)$, $g_2(x)$ and $g_3(x)$. Knowing that the true value of the root is $\alpha = 2.485583998$, compare the approximations obtained from the three algorithms after the 10th iteration. (Use the same starting point for all algorithms.) *[10 marks]*

4. Consider the following system

$$\begin{aligned} 0.002x_1 + 1.231x_2 + 2.471x_3 &= 3.704 \\ 1.196x_1 + 3.165x_2 + 2.543x_3 &= 6.904 \\ 1.475x_1 + 4.271x_2 + 2.142x_3 &= 7.888. \end{aligned}$$

(a) By direct substitution, show that the exact solution is $x_1 = 1$, $x_2 = 1$ and $x_3 = 1$. *[2 marks]*

(b) Use Gaussian elimination with and without pivoting to solve this system. Use 4-digit rounding arithmetic. *[15 marks]*

(c) Calculate the relative errors for each of the methods used in part (b). Can you explain why the results are so different? *[8 marks]*

5. Let

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 4 \\ 4 & -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 5 & 0 \\ 2 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.$$

(a) Find a 3×3 matrix X such that

$$AXB = A + B + C.$$

[15 marks]

(b) Solve the system of equations

$$AY = D.$$

[10 marks]

6. Consider the following system of equations

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= -3 \\ 2x_1 + 5x_2 - 3x_3 &= 0 \\ -4x_1 + x_2 + 10x_3 &= 15 \end{aligned}$$

Write the system in matrix form and use the LU decomposition method to solve it. If A is the matrix of coefficients, use the following factorisations for A:

(a) Write $A = L{\cdot}U$, where L is a lower triangular matrix with 1's along the main diagonal and U is a general upper triangular matrix. *[12.5 marks]*

(b) Write $A = L \cdot U$ where L is a general lower triangular matrix and U is an arbitrary upper triangular matrix with 1's along the main diagonal. *[12.5 marks]*

(Obviously, the two decompositions should yield the same solution in the end!)