13 February 2012

Problem Sheet 2: More vector calculus

1. Show that div(curl($\mathbf{A}$))=0 for every vector field $\mathbf{A}$.
2. If $\Phi = 2x^2y - xz^3$ find $\nabla\Phi$ and $\nabla^2\Phi$.
3. Find a unit normal to the surface $\Phi(x, y, z) = x^2y - 2xz + 2y^2z^4 = 10$ at the point $(2, 1, -1)$.
4. The temperature in degrees Celsius on the surface of a metal plate is

   $$T(x, y) = 20 - 4x^2 - y^2$$

   where x and y are measured in centimetres. In what direction from (2,-3) does the temperature increase most rapidly? What is this rate of increase?
5. The surface of a mountain is modeled by the equation

   $$h(x, y) = 5000 - 0.001x^2 - 0.004y^2.$$

   A mountain climber is at the point (500,300,4300). In what direction should he move in order to ascend at the greatest rate?
6. Newton's law of gravitation, where a small particle of mass m orbits a much larger mass M is

   $$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^3}\,\mathbf{r}$$

   where $\mathbf{r}$ is the position vector of the small mass and $r = |\mathbf{r}|$.

   Show that the vector $\mathbf{r} \times \frac{d\mathbf{r}}{dt}$ is a constant of the motion.
7. The general equation of motion for a particle of mass m and charge q under the influence of a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ is

   $$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

   where $\mathbf{r}$ is the position vector of the particle at time t and $\dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial t}$, etc.

   (a) Write this vector equation as 3 separate equations in terms of the components of the vectors involved.

   (b) (*Optional*) Rewrite the equations of motion for the simple case $\mathbf{E} = E\mathbf{i}$, $\mathbf{B} = B\mathbf{j}$. If the particle starts from the origin at $t = 0$ with $\mathbf{r} = v_0\mathbf{k}$ and $v_0 = \frac{E}{B}$ show that the particle continues its original motion for all time t.