Chapter 1

Functions of several variables

1.1 Introduction

Definition 1.2. *A real function of several real variables is a function that depends on more than one variable.*

1.2.1 Examples

i) $f_1(x, y) = 2x + 3y$ is a function of 2 variables.

ii) $f_2(x, y) = x^2 - 4y^3$ is a function of 2 variables.

iii) $f_3(x, y) = 2xy^2 - 4xe^y$ is a function of 2 variables.

iv) $f_4(x, y, z) = \frac{xy\sqrt{y+z^3}}{x^2+z^2}$ is a function of 3 variables.

1.3 Partial differentiation

Consider a function of 2 variables $f(x, y)$, then if we keep y fixed ($y = y_0$), we can define a new function of 1 variable g as follows:

$$g(x) = f(x, y_0).$$

Definition 1.4. *The derivative of g (with respect to x) is called the partial derivative of f with respect to (w.r.t.) the first variable (x) and is denoted by*

$$\frac{\partial f}{\partial x}(x, y_0) = g'(x).$$

If we keep constant x ($x = x_0$), then we can define another new function h as :

$$h(y) = f(x_0, y).$$

The derivative of h (w.r.t. y) is called the partial derivative of f w.r.t. the second variable (y) and is denoted by:

$$\frac{\partial f}{\partial y}(x_0, y) = h'(y).$$

1.4.1 Examples

i) $f_1(x, y) = 2x + 3y$, $\frac{\partial f_1}{\partial x} = 2 + 0 = 2$, $\frac{\partial f_1}{\partial y} = 0 + 3 = 3$.

ii) $f_2(x, y) = x^2 - 4y^3$, $\frac{\partial f_2}{\partial y} = 2x - 0 = 2x$, $\frac{\partial f_2}{\partial y} = 0 - 12y^2$.

iii) $f_3(x, y) = 2xy^2 - 4xe^y$, $\frac{\partial f_3}{\partial x} = 2y^2 - 4e^y$, $\frac{\partial f_3}{\partial y} = 4xy - 4xe^y$.

1.4.2 Functions of more than 2 variables

In the case of a function depending on more than 2 variables, the partial derivative w.r.t. one variable is the derivative of this function when we keep constant all other variables.

Consider function $f(x, y, z, t)$ (depending on 4 variables) defined by:

$$f(x, y, z, t) = 3xt^2 \cos(2yz^2).$$

Then we can define 4 partial derivatives:

- $\frac{\partial f}{\partial x} = 3t^2 \cos(2yz^2)$
- $\frac{\partial f}{\partial y} = -6xt^2z^2 \sin(2yz^2)$
- $\frac{\partial f}{\partial z} = -12xyzt^2 \sin(2yz^2)$
- $\frac{\partial f}{\partial t} = 6xt \cos(2yz^2)$

1.5 Higher order partial derivatives

So far we have seen the first order partial derivatives which are obtained by deriving once a function of several variables w.r.t. one variable.

Suppose we have a function $f(x, y)$, then we can define the function g as the partial derivative of f w.r.t. x:

$$g(x, y) = \frac{\partial f}{\partial x}(x, y).$$

The function g depends on the same variable as f, and we can define its partial derivatives: $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

We call the partial derivative of g w.r.t x "the second-order partial derivative" of f w.r.t to x as it was obtained by deriving f twice w.r.t. x keeping y constant and we denote it by:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}.$$

The partial derivative of g w.r.t y is called the second-order partial derivative of f with respect to x and then to y. It is denoted by:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}.$$

We can define the third-order partial derivatives of f as the partial derivatives of the second-order partial derivatives of f or equivalently as the second-order partial derivatives of the first-order partial derivatives of f.

Remark 1.6. *For a function depending on 2 variables $f(x, y)$ there exist*

- *2 first-order partial derivative :* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.
- *4 second-order partial derivatives :* $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$.
- *8 third-order partial derivatives :* $\frac{\partial^3 f}{\partial x^3}, \frac{\partial^3 f}{\partial y \partial x^2}, \frac{\partial^3 f}{\partial x \partial y \partial x}, \frac{\partial^3 f}{\partial y^2 \partial x}, \frac{\partial^3 f}{\partial x^2 \partial y}, \frac{\partial^3 f}{\partial y \partial x \partial y}, \frac{\partial^3 f}{\partial x \partial y^2}, \frac{\partial^3 f}{\partial y^3}$.
- 2^n n^{th}*-order partial derivatives.*

The reason for that is that any partial derivative is in itself a function depending on 2 variables for which we can define 2 partial derivatives $\frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial y}$.

Proposition 1.7. *Let $f(x, y)$ be a function of two variables. Suppose that the second-oder partial derivatives* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$ *are continuous. Then we have:*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

1.7.1 Example

- $f(x,y) = x^2y^3$, $\frac{\partial f}{\partial x} = 2xy^3$, $\frac{\partial f}{\partial y} = 3x^2y^2$,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(2xy^3) = 6xy^2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2.$$

1.7.2 Evaluation at a point

We can evaluate the partial derivatives at a point in the same way we do it for any function:

Suppose $f(x,y) = x^3\cos(2y)$, then $\frac{\partial^2 f}{\partial x \partial y} = -6x^2\sin(2y)$. We can evaluate these functions at $(2, \frac{\pi}{4})$:

$$f(2, \frac{\pi}{4}) = 2^3\cos(2\frac{\pi}{4}) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y}(2, \frac{\pi}{4}) = -6(2)^2\sin(2\frac{\pi}{4}) = -24.$$

1.8 Total differential

Definition 1.9. *Let $f(x,y)$ be a function of 2 variables. Then, we define the total differential of f by the following expression:*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

1.9.1 Interpretation

The previous expression is not a function or a number, it is just a formal expression relating *infinitesimal (infinitely small) changes in the variables x and y, denoted by dx and dy respectively, to* infinitesimal changes in f denoted by df.

For a function of 3 variables, $g(x,y,z)$, the total differential would be:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

1.10 Application

The partial derivatives correspond to the rate of change of a function when one variable changes.

Let $f(x, y)$ be a function of 2 variables. Suppose that we change x from its original value by δx. Then $f(x, y)$ will change by δf such that:

$$f(x + \delta x, y) = f(x, y) + \delta f,$$

which means that:

$$\delta f = f(x + \delta x, y) - f(x, y).$$

If we divide both sides by δx (the quantity by which x changed), we get:

$$\frac{\delta f}{\delta x} = \frac{f(x + \delta x, y) - f(x, y)}{\delta x}.$$

On the right-hand side, we can identify clearly the rate of change of f with respect to x which limit as δx goes to 0 is $\frac{\partial f}{\partial x}$.

It follows that the limit of $\frac{\delta f}{\delta x}$ as δx goes to 0 is $\frac{\partial f}{\partial x}$. We can deduce then, that if δx is small enough we will be close to this limit, i.e. if δx *is small enough*, then

$$\frac{\delta f}{\delta x} \simeq \frac{\partial f}{\partial x},$$

or equivalently:

$$\delta f \simeq \frac{\partial f}{\partial x} \delta x.$$

An analogous result holds when we replace x by y.

Now suppose that both x and y change, then we will have:

$$\delta f \simeq \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y,$$

which means that the value of f will be affected by both changes with the corresponding rates.

We recognize in the latter formula a similarity with the total differential. There is however a substantial difference, in the total differential we have a rigorous equality and the changes are infinitesimal (infinitely small), while in the above formula, we have an approximate equality and the changes are finite (small but finitely small).

1.10.1 Example

Consider a triangle ABC of which we measure the lengths b and c and the angle A (in radian). Suppose the measures are 120m, 85m, and $\pi/6$.

The area S of the triangle is then given by the formula:

$$S = \frac{1}{2} bc \sin(A),$$

which for our measures gives $S = 2550m^2$.

Now suppose that our measures were not exact. More precisely, we suppose that when we measure b and c we have an error that is of at most $0.1m$ and when we measure A we have an error of at most $\pi/200$.

This means that the exact value of b cannot be known, but we know from our measure that this exact value stands between $120 - 0.1 = 119.9$ and $120 + 0.1 = 120.1m$.

Since the exact value of S is not accessible (because of the inevitable errors on the measurements), we would like to have an interval to which S belongs.

This can be done using partial derivatives. In fact, S is a function of 3 variables b, c, and A. Since the errors are small, we can write:

$$\delta S = \frac{\partial S}{\partial b}\delta b + \frac{\partial S}{\partial c}\delta c + \frac{\partial S}{\partial A}\delta A,$$

or

$$\delta S = \frac{1}{2} c \sin(A) \delta b + \frac{1}{2} b \sin(A) \delta c + \frac{1}{2} bc \cos(A) \delta A.$$

The maximal error is obtained when we sum the absolute values of all maximal errors:

$$\delta S_{\text{max}} = \left|\frac{1}{2} c \sin(A)\right| \delta b_{\text{max}} + \left|\frac{1}{2} b \sin(A)\right| \delta c_{\text{max}} + \left|\frac{1}{2} bc \cos(A)\right| \delta A_{\text{max}}.$$

We obtain $\delta S_{\text{max}} = 74.5m^2$. It follows that the exact area S_{exact} satisfies:

$$2475.5m^2 = S - \delta S_{\text{max}} \leq S_{\text{exact}} \leq S - \delta S_{\text{max}} = 2624.5m^2.$$

1.11 Case of a compound function

Let f be a function depending on two variables x and y. Suppose that the variables x and y depend on a third variable t, i.e. $x = x(t)$ and $y = y(t)$.

We define the function g depending on the variable t as:

$$g(t) = f\left(x(t), y(t)\right).$$

To determine the derivative g' of g, we first write the total differential of f:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Then,

$$\begin{aligned} g'(t) &= \frac{dg}{dt} \\ &= \frac{df\big(x(t), y(t)\big)}{dt} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \end{aligned}$$

Example

Suppose that in a rectangle of sides a and b the first side is increasing at a speed of $2m/s$ and the second is decreasing at a speed of $4m/s$. How is changing the diagonal d of the triangle?

We can think of the length a, b, and d as functions of the time t. Then :

$$d(t) = \sqrt{a(t)^2 + b(t)^2}.$$

If we define f as $f(a,b) = \sqrt{a^2+b^2}$, then $\frac{\partial f}{\partial a} = \frac{a}{\sqrt{a^2+b^2}}$ and $\frac{\partial f}{\partial b} = \frac{b}{\sqrt{a^2+b^2}}$. We deduce then, that the rate of change of the length of diagonal is given by:

$$\begin{aligned} d'(t) &= \frac{\partial f}{\partial a}\frac{da}{dt} + \frac{\partial f}{\partial b}\frac{db}{dt} \\ &= \frac{a}{\sqrt{a^2+b^2}}a'(t) + \frac{b}{\sqrt{a^2+b^2}}b'(t) \\ &= \frac{2a-4b}{\sqrt{a^2+b^2}}. \end{aligned}$$

1.12 Exercises

Exercise 1

Find the first partial derivatives of :

(i) $f_1(x,y) = x^2 + y^3$

(ii) $f_2(x,y) = x^3y^5$

(iii) $f_3(x,y) = (2xy^2 - 4y)e^x$

(iv) $f_4(x,y) = sin(2x + 5y)$

(v) $f_5(x,y) = 5xy^3\cos(x+y)$

(vi) $f_6(x,y) = \frac{x}{y^2} - \frac{y}{x^2}$

(vii) $f_7(x,y) = \ln(x^2 + y)$

(viii) $f_8(x,y,z) = xy + yz + xz$

(ix) $f_9(x,y,z,w) = \frac{w^2}{xy^2z^3}$

Exercise 2

Find the second partial derivatives of the previous functions.

Exercise 3

A cylindrical hole of diameter 6 inches and height 4 inches is to be cut in a block of wood by a process in which the maximum error in diameter is 0.05 inch and in height is 0.01 inch. What is the largest possible error in the volume of the cavity?

Exercise 4

The breaking weight W of a cantilever beam is given by the formula

$$Wl = Kbd^2,$$

where b is the breadth, l the length, d the depth, and K a constant depending on the material of the beam.

If the length is increased by 1% and the breadth by 5%, by how much should the depth be altered to keep the breaking weight unchanged?

Exercise 5

A triangle ABC is being transformed so that the angle A changes at a uniform rate from $0°$ to $90°$ in 10 seconds while side AB increases by 1cm$/s$ and side AC decreases by 1cm$/s$. If at the time of observation, $A = 60°$, $AC = 16$cm, and $AB = 10$cm, find

(i) how fast is BC changing.

(ii) how fast is the area of the triangle changing.

Exercise 6

The radius of a cylinder increases at the rate of 2cm$/s$ and the height h increases at 3cm$/s$. Find the rate at which the volume is increasing when $r = 10cm$ and $h = 20cm$.

1.13 Solutions

Exercise 1

(i) $\frac{\partial f_1}{\partial x} = 2x$, $\frac{\partial f_1}{\partial y} = 3y^2$.

(ii) $\frac{\partial f_2}{\partial x} = 3x^2y^5$, $\frac{\partial f_2}{\partial y} = 5x^3y^4$.

(iii) $\frac{\partial f_3}{\partial x} = (2xy^2 + 2y^2 - 4y)e^x$, $\frac{\partial f_3}{\partial y} = 4(xy - 1)e^x$.

(iv) $\frac{\partial f_4}{\partial x} = 2\cos(2x + 5y)$, $\frac{\partial f_4}{\partial y} = 5\cos(2x + 5y)$.

(v) $\frac{\partial f_5}{\partial x} = 5y^3\cos(x + y) - 5xy^3\sin(x + y)$, $\frac{\partial f_5}{\partial y} = 15xy^2\cos(x + y) - 5xy^3\sin(x + y)$.

(vi) $\frac{\partial f_6}{\partial x} = \frac{1}{y^2} + 2\frac{y}{x^3}$, $\frac{\partial f_6}{\partial y} = -2\frac{x}{y^3} - \frac{1}{x^2}$.

(vii) $\frac{\partial f_7}{\partial x} = \frac{2x}{x^2+y}$, $\frac{\partial f_7}{\partial y} = \frac{1}{x^2+y}$.

(viii) $\frac{\partial f_8}{\partial x} = y + z$, $\frac{\partial f_8}{\partial y} = x + z$, $\frac{\partial f_8}{\partial z} = x + y$.

(ix) $\frac{\partial f_9}{\partial x} = -\frac{w^2}{x^2y^2z^3}$, $\frac{\partial f_9}{\partial y} = -2\frac{w^2}{xy^3z^3}$, $\frac{\partial f_9}{\partial z} = -3\frac{w^2}{xy^2z^4}$, $\frac{\partial f_9}{\partial w} = \frac{2w}{xy^2z^3}$

Exercise 2

(i) $\frac{\partial^2 f_1}{\partial x^2} = 2$, $\frac{\partial^2 f_1}{\partial y^2} = 6y$, $\frac{\partial^2 f_1}{\partial x \partial y} = \frac{\partial^2 f_1}{\partial y \partial x} = 0$.

(ii) $\frac{\partial^2 f_2}{\partial x^2} = 6xy^5$, $\frac{\partial^2 f_2}{\partial y^2} = 20x^3y^3$, $\frac{\partial^2 f_2}{\partial x \partial y} = \frac{\partial^2 f_2}{\partial y \partial x} = 15x^2y^4$.

(iii) $\frac{\partial^2 f_3}{\partial x^2} = (2xy^2 + 4y^2 - 4y)e^x$, $\frac{\partial^2 f_3}{\partial y^2} = 4xe^x$, $\frac{\partial^2 f_3}{\partial x \partial y} = \frac{\partial^2 f_3}{\partial y \partial x} = 4(xy + y - 1)e^x$.

(iv) $\frac{\partial^2 f_4}{\partial x^2} = -4\sin(2x + 5y)$, $\frac{\partial^2 f_4}{\partial y^2} = -25\sin(2x + 5y)$, $\frac{\partial^2 f_4}{\partial x \partial y} = \frac{\partial^2 f_4}{\partial y \partial x} = -10\sin(2x + 5y)$.

(v) $\frac{\partial^2 f_5}{\partial x^2} = -10y^3\sin(x+y) - 5xy^3\cos(x+y)$, $\frac{\partial^2 f_5}{\partial y^2} = 5xy(6+y^2)\cos(x+y)$, $\frac{\partial^2 f_5}{\partial x \partial y} = \frac{\partial^2 f_5}{\partial y \partial x} = 5y^2(3 + xy)\cos(x + y) + 5y^2(y - -3x)\sin(x + y)$.

(vi) $\frac{\partial^2 f_6}{\partial x^2} = -6\frac{y}{x^4}$, $\frac{\partial^2 f_6}{\partial y^2} = 6\frac{x}{y^4}$, $\frac{\partial^2 f_6}{\partial x \partial y} = \frac{\partial^2 f_6}{\partial y \partial x} = \frac{2}{x^3} - \frac{2}{y^3}$.

(vii) $\frac{\partial^2 f_7}{\partial x^2} = 2\frac{-x^2+y}{(x^2+y)^2}$, $\frac{\partial^2 f_7}{\partial y^2} = -\frac{1}{(x^2+y)^2}$, $\frac{\partial^2 f_7}{\partial x \partial y} = \frac{\partial^2 f_7}{\partial y \partial x} = -\frac{2x}{(x^2+y)^2}$.

(viii) $\frac{\partial^2 f_8}{\partial x^2} = \frac{\partial^2 f_8}{\partial y^2} = \frac{\partial^2 f_8}{\partial z^2} = 0$, $\frac{\partial^2 f_8}{\partial x \partial y} = \frac{\partial^2 f_8}{\partial y \partial x} = \frac{\partial^2 f_8}{\partial x \partial z} = \frac{\partial^2 f_8}{\partial z \partial x} = \frac{\partial^2 f_8}{\partial y \partial z} = \frac{\partial^2 f_8}{\partial z \partial y} = 1$.

(ix) $\frac{\partial^2 f_9}{\partial x^2} = 2\frac{w^2}{x^3y^2z^3}$, $\frac{\partial^2 f_9}{\partial y^2} = 6\frac{w^2}{xy^4z^3}$, $\frac{\partial^2 f_9}{\partial z^2} = 12\frac{w^2}{xy^2z^5}$, $\frac{\partial^2 f_9}{\partial w^2} = \frac{2}{xy^2z^3}$, $\frac{\partial^2 f_9}{\partial x \partial y} = \frac{\partial^2 f_9}{\partial y \partial x} = 2\frac{w^2}{x^2y^3z^3}$, $\frac{\partial^2 f_9}{\partial x \partial z} = \frac{\partial^2 f_9}{\partial z \partial x} = 3\frac{w^2}{x^2y^2z^4}$, $\frac{\partial^2 f_9}{\partial x \partial w} = \frac{\partial^2 f_9}{\partial w \partial x} = -2\frac{w}{x^2y^2z^3}$, $\frac{\partial^2 f_9}{\partial y \partial z} = \frac{\partial^2 f_9}{\partial z \partial y} = 6\frac{w^2}{xy^3z^4}$, $\frac{\partial^2 f_9}{\partial y \partial w} = \frac{\partial^2 f_9}{\partial w \partial y} = -4\frac{w}{xy^3z^3}$, $\frac{\partial^2 f_9}{\partial z \partial w} = \frac{\partial^2 f_9}{\partial w \partial z} = -6\frac{w}{xy^2z^4}$.

Exercise 3

Since $V = V(D, H) = \pi \left(\frac{D}{2}\right)^2 H$, we have:

$$dV = \frac{\pi}{2}DHdD + \frac{\pi}{4}D^2dH.$$

The maximum error on the volume δV is then given from the maximum errors on the diameter δD and on the height δH by:

$$\delta V = \left|\frac{\pi}{2}DH\right| \delta D + \left|\frac{\pi}{4}D^2\right| \delta H.$$

$\delta V = 2.17\text{in}^3$

Exercise 4

$$dW = \frac{Kd^2}{l}db + \frac{2Kbd}{l}d(d) - \frac{Kbd^2}{l^2}dl.$$

The modifications are : $\delta l = 0.01l$, $\delta b = 0.05b$, and $\delta d = xd$. The unknown x has to be such that the breaking weight is unchanged, which means that the corresponding $\delta W = 0$.

$$\begin{aligned}\delta W = 0 =& \frac{Kd^2}{l}\delta b + \frac{2Kbd}{l}\delta d - \frac{Kbd^2}{l^2}\delta l \\ =& \frac{Kd^2}{l}0.05b + \frac{2Kbd}{l}xd - \frac{Kbd^2}{l^2}0.01l\end{aligned}$$

It follows that:

$$x = -0.02$$

which means that the depth has to be altered by 2%.

Exercise 5

(i) The length of BC is given by the formula:

$$BC = \sqrt{AB^2 + AC^2 - 2AC\ AB\cos A}$$

Then:

$$\begin{aligned}dBC =& \frac{AB - AC\cos A}{\sqrt{AB^2 + AC^2 - 2AB\ AC\cos A}}dAB \\ &+ \frac{AC - AB\cos A}{\sqrt{AB^2 + AC^2 - 2AB\ AC\cos A}}dAC \\ &+ \frac{AB\ AC\sin A}{\sqrt{AB^2 + AC^2 - 2AB\ AC\cos A}}dA,\end{aligned}$$

and

$$\begin{aligned}\frac{dBC}{dt} =& \frac{AB - AC\cos A}{\sqrt{AB^2 + AC^2 - 2AB\ AC\cos A}}\frac{dAB}{dt} \\ &+ \frac{AC - AB\cos A}{\sqrt{AB^2 + AC^2 - 2AB\ AC\cos A}}\frac{dAC}{dt} \\ &+ \frac{AB\ AC\sin A}{\sqrt{AB^2 + AC^2 - 2AB\ AC\cos A}}\frac{dA}{dt},\end{aligned}$$

where the angle A has to be expressed in radian. ($60° = \frac{\pi}{3}$rd)

$\frac{dBC}{dt} = 0.912$cms

(ii) The area S of the triangle ABC is given by the formula:

$$S = \frac{AB.AC \sin A}{2}.$$

Then

$$\frac{dS}{dt} = \frac{AC \sin A}{2}\frac{dAB}{dt} + \frac{AB \sin A}{2}\frac{dAC}{dt} + \frac{AB.AC \cos A}{2}\frac{dA}{dt}$$

$\frac{dS}{dt} = 8.88\text{cm}^2\text{s}$

Exercise 6

The volume V of the cylinder is given by the formula:

$$V = \pi r^2 h.$$

Then:

$$\frac{dV}{dt} = 2\pi r h\frac{dr}{dt} + \pi r^2\frac{dh}{dt}.$$

$\frac{dV}{dt} = 3456\text{cm}^3\text{s}$