Sequences & Series

MA4602
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Patterns in Numbers

$$2, 4, 6, 8, 10, \dots$$

$$1, 4, 9, 16, 25, \dots$$

$$1, -2, 4, -8, 16, -32, \dots$$

$$1, 3, 9, 27, 81, \dots$$

Sequences

A sequence is a list of real numbers:

$$u_1, u_2, u_3, u_4, \dots, u_n$$

with a rule u_n for each $n = 1, 2, 3, \dots$

Sequences

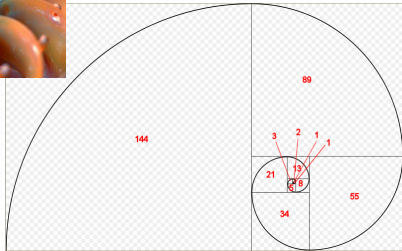|	
$2, 4, 6, 8, 10, \dots$ $u_1, u_2, u_3, u_4, u_5 \dots$ $u_n = 2n$	$1, -2, 4, -8, 16, -32, \dots$ $u_1, u_2, u_3, u_4, u_5, u_6 \dots$ $u_n = (-2)^{n-1}$
$1, 4, 9, 16, 25, \dots$ $u_1, u_2, u_3, u_4, u_5 \dots$ $u_n = n^2$	$1, 3, 9, 27, 81, \dots$ $u_1, u_2, u_3, u_4, u_5 \dots$ $u_n = (3)^{n-1}$

Fibonacci Sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

Sequence used in:

- Music to structure rhymes and rhythm
- Computer games: Metal Gear Solid 4
- Poetry
- DNA Construction
- Solar System arrangement.
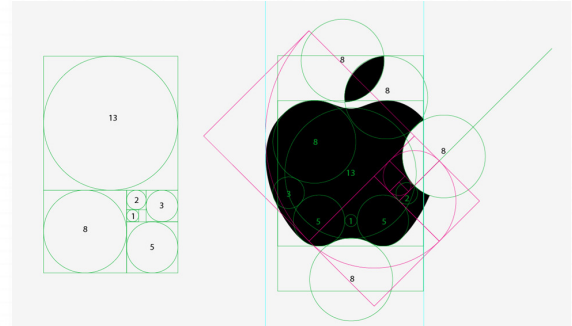- Nature
- Art: used by Marilyn Manson

Fibonacci Sequence in Nature

Milky Way and Fibonnaci Sequence

Apple Symbol and Fibonnaci Sequence

Convergence of Sequences

A sequence u_n is said to converge to a limit L if

$$\lim_{n \to \infty} u_n = L$$

If a sequence does not converge then it is said to be <u>divergent</u>.

Gauss' Genius

- Carl Friedrich Gauss' Primary school teacher wanted to keep the class busy while he completed other work so he asked the pupils to add all whole numbers between 1 and 100:

$$1 + 2 + 3 + 4 + 5 + \ldots + 99 + 100$$

- Gauss completed the task in seconds. How?

Gauss' Genius

$$1 + 2 + 3 + 4 + 5 + \ldots + 99 + 100$$

Gauss realised that addition of terms from opposite ends of the list always yielded an answer of 101:

$$1 + 2 + 3 + 4 + 5 + \ldots + 99 + 100$$
$$\underline{100 + 99 + 98 + 97 + 96 + \ldots + 2 + 1}$$
$$101 + 101 + 101 + 101 + 101 + \ldots + 101 + 101$$

$$\frac{100(101)}{2} = 5050$$

Series

Given a sequence $u_1, u_2, u_3, u_4, u_5 \ldots$ we can construct the corresponding series:

$$s_1 = u_1$$

$$s_2 = u_1 + u_2$$

$$S_3 = u_1 + u_2 + u_3$$
$$.$$
$$.$$
$$.$$
$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Series

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n$$

S_n can be written in shorthand:

$$S_n = \sum_{i=1}^{n} u_i$$

This can be interpreted as the sum starting from $i = 1$ and stopping when $i = n$

Gauss' Genius

$$1 + 2 + 3 + 4 + 5 + \ldots + 99 + 100 = 5050$$

$$\sum_{r=1}^{100} r = \frac{100(101)}{2} = 5050$$

This gives us the general formula:

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Arithmetic Series

An arithmetic sequence is one where consecutive terms have a common difference:

$$d = u_{n+1} - u_n$$

Example: Given the arithmetic sequence:

$$4, 7, 10, 13, 16, \ldots$$

Here, the common difference is 3. So $d = 3$.

Arithmetic Series

In an arithmetic sequence, the first term (u_1) is usually called a so:

$$u_1 = a$$
$$u_2 = a + d$$
$$u_3 = a + 2d$$
$$u_4 = a + 3d$$

In our sample sequence $4, 7, 10, 13, 16, \ldots$ $a = 4$ and we know $d = 3$

$$u_1 = a = 4$$
$$u_2 = a + d = 4 + 3$$
$$u_3 = a + 2d = 4 + 6$$
$$u_4 = a + 3d = 4 + 9$$

Arithmetic Series

$$u_1 = a$$
$$u_2 = a + d$$
$$u_3 = a + 2d$$
$$u_4 = a + 3d$$
$$.$$
$$.$$
$$.$$
$$u_n = a + (n-1)d$$

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots + [a + (n-1)d]$$

Arithmetic Series

$$S_n = a + (a+d) + (a+2d) + \cdots + [a + (n-2)d] + [a + (n-1)d]$$

$$S_n = [a + (n-1)d] + [a + (n-2)d] + \cdots + (a+2d) + (a+d) + a$$

$$2S_n = [2a + (n-1)d] + [2a + (n-1)d] + [2a + (n-1)d]$$
$$+ \ldots [2a + (n-1)d] + [2a + (n-1)d]$$

$$2S_n = n[2a + (n-1)d]$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Arithmetic Series – Overview

The common difference d is found by subtracting one term from the term which immediately follows it:
$$d = u_{n+1} - u_n$$

To find the nth term in an arithmetic sequence, apply the following:
$$u_n = a + (n-1)d$$

To find the sum of the first n terms in an arithmetic sequence, apply the following:
$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Arithmetic Series – Ex. 1

In the following arithmetic series:

$$3, 7, 11, 15, \dots$$

a) What is the 12th term in this sequence?

b) Find the sum of the first 12 terms in the sequence.

Solution:
We can see that $a = 3$ and $d = 4$

$$u_n = a + (n-1)d$$

$$u_{12} = 3 + (12-1)4$$

$$u_{12} = 47$$

The 12th term in the sequence is 47.

Arithmetic Series – Ex. 1

b) Find the sum of the first 12 terms in the sequence.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{12} = \frac{12}{2}[2(3) + (12-1)4]$$

$$S_{12} = 6(6 + 44) = 300$$

$$3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 + 43 + 47 = 300$$

Arithmetic Series – Ex. 2

In the following arithmetic series:

$$2, 7, 12, 17, \dots$$

a) What is the 27th term in this sequence?

b) Find the sum of the first 26 terms in the sequence.

Solution:
We can see that $a = 2$ and $d = 5$

$$u_n = a + (n-1)d$$

$$u_{27} = 2 + (27-1)5$$

$$u_{27} = 132$$

The 27th term in the sequence is 132.

Arithmetic Series – Ex. 2

b) Find the sum of the first 26 terms in the sequence.
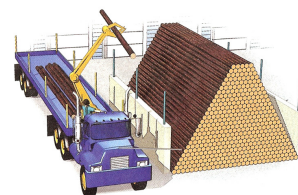
$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{26} = \frac{26}{2}[2(2) + (26-1)5]$$

$$S_{26} = 13(4 + 125) = 1677$$

$$2 + 7 + 12 + 17 \dots + 127 = 1677$$

Arithmetic Series – Ex. 3

Q. A stack of telephone poles has 30 poles in the bottom row. There are 29 poles in the second row, 28 in the next row, and so on. How many poles are in the stack if there are 5 poles in the top row?

Arithmetic Series – Ex. 3

Our arithmetic series is as follows:

$$5 + 6 + 7 + \ldots + 29 + 30$$

We know that $a = 5$ and $d = 1$

If 5 is the 1st term, 6 is the 2nd term, 7 is the 3rd term then 30 must be the 26th term.

Thus, to add up all the poles, we must find the sum of the first 26 terms i.e. S_{26}

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{26} = \frac{26}{2}[2(5) + (26-1)1]$$

$$S_{26} = 455$$

There are 455 poles in the stack.

Arithmetic Series – Ex. 4

In an arithmetic sequence, the fifth term is -18 and the tenth term is 12.

i. Find the first term and the common difference.

ii. Find the sum of the first fifteen terms of the sequence.

$$u_n = a + (n-1)d$$

$$u_5 = a + 4d = -18$$
$$u_{10} = a + 9d = 12$$

$$a + 4d = -18$$
$$\underline{-a - 9d = -12}$$
$$-5d = -30$$

$$d = 6$$

Arithmetic Series – Ex. 4

$$a + 4d = -18$$

$$a + 4(6) = -18$$

$$a = -42$$

First term is -42 and the common difference is 6

ii. Find the sum of the first fifteen terms of the sequence.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{15} = \frac{15}{2}[2(-42) + (15-1)6]$$

$$S_{15} = \frac{15}{2}(-84 + 84) = 0$$

the sum of the first fifteen terms of the sequence is zero.

Geometric Sequences & Series

A sequence is geometric if the ratio, $\frac{u_{n+1}}{u_n}$, between any two consecutive terms is a constant.

This constant is called the common ratio and is usually denoted by r.

Example:
$$2, 4, 8, 16, 32, 64, \ldots$$

The common ratio here is 2.

Geometric Sequences & Series

$$u_1 = a$$
$$u_2 = ar$$
$$u_3 = ar^2$$
$$u_4 = ar^3$$
$$.$$
$$.$$
$$.$$
$$u_n = ar^{n-1}$$

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

Geometric Sequences & Series

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

This can be shortened into the following equation:

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$

Geometric Sequences & Series - Overview

To find the common ratio r:
$$r = \frac{u_{n+1}}{u_n}$$

To find the n$^{\text{th}}$ term in the geometric sequence, apply the following:
$$u_n = ar^{n-1}$$

To find the sum of the first n terms in a geometric sequence, apply the following:
$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

Geometric Series – Ex. 1

Find the sum of the following geometric series:

$$5 + 10 + 20 + 40 + \ldots + 10240$$

Solution:
We know $a = 5$ and $r = 2$

We need to find out how many terms are in the series.

$u_n = ar^{n-1}$

$5(2^{n-1}) = 10240$

$2^{n-1} = 2048$

$\log_2 2048 = n - 1$

$11 = n - 1$

$n = 12$

Geometric Series – Ex. 1

We now know that 10240 is the 12$^{\text{th}}$ term so there are 12 terms in total in the series.

To find the sum of the geometric series we must find S_{12}

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_{12} = 5\left(\frac{1-2^{12}}{1-2}\right)$$

$$S_{12} = 5\left(\frac{4095}{-1}\right)$$

$$S_{12} = 20475$$

The sum of the geometric series is 20,475

Geometric Series – Ex. 2

Find the sum of the following geometric series:

$$3 + 12 + 48 + \ldots + 3{,}072$$

Solution:
We know $a = 3$ and $r = 4$

We need to find out how many terms are in the series.

$u_n = ar^{n-1}$

$3(4^{n-1}) = 3072$

$4^{n-1} = 1024$

$\log_4 1024 = n - 1$

$5 = n - 1$

$n = 6$

Geometric Series – Ex. 2

We now know that 3072 is the 6$^{\text{th}}$ term so there are 6 terms in total in the series.

To find the sum of the geometric series we must find S_6

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_6 = 3\left(\frac{1-4^6}{1-4}\right)$$

$$S_6 = 3\left(\frac{4095}{-3}\right)$$

$$S_6 = 4095$$

The sum of the geometric series is 4,095

Geometric Series – Ex. 3

Most lottery games in the USA allow winners of the jackpot prize to choose between two forms of the prize: an annual-payments option or a cash-value option.

In the case of the New York Lotto, there are 26 annual payments in the annual-payments option, with the first payment immediately, and the last payment in 25 years time.

The payments increase by 4% each year. The amount advertised as the jackpot prize is the total amount of these 26 payments. The cash-value option pays a smaller amount than this.

Geometric Series – Ex. 3

(a) If the amount of the first annual payment is a, write down, in terms of a, the amount of the second, third, fourth and 26th payments.

1st payment $= a$
2nd payment $= 1.04a$
3rd payment $= 1.04(1.04a) = (1.04)^2a$
4th payment $= 1.04[(1.04)^2a] = (1.04)^3a$
.
.
.
26th payment $= (1.04)^{25}a$

Geometric Series – Ex. 3

(b) The 26 payments form a geometric series. Use this fact to express the advertised jackpot prize in terms of a.

$$a + 1.04a + (1.04)^2a + (1.04)^3a + \ldots + (1.04)^{25}a$$

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$

$$S_{26} = a\left(\frac{1 - 1.04^{26}}{1 - 1.04}\right) = a\left(-\frac{1.7724}{-0.04}\right)$$

$$S_{26} = 44.312a = Jackpot\ prize$$

Geometric Series – Ex. 3

(c) Find, correct to the nearest dollar, the value of a that corresponds to an advertised jackpot prize of \$21·5 million.

$$Jackpot\ prize = 44.312a$$

$$21{,}500{,}000 = 44.312a$$

$$a = \$485{,}196$$

This means that the initial payment will be \$485,196 and will increase by 4% with each subsequent payment.

Geometric Series – Ex. 4

Q. In January 2013, HMV staff in Limerick City held a sit-in to aid negotiations for a fair severance package. The company were offering a total of €800,000 to be shared out amongst all the HMV staff in Ireland as a severance package.

The staff were not happy with this figure and instead suggested that, in the month of February 2013, HMV would pay 1 cent into the severance package for all staff on Feb. 1st, 2 cent on Feb. 2nd, 4 cent on Feb. 3rd, 8 cent on the 4th, 16 cent on the 5th and so on for the rest of that month. HMV agreed to the deal immediately. How much did HMV end up paying to the staff?

Geometric Series – Ex. 4

The payments would add up as follows:

$$€0.01 + €0.02 + €0.04 + €0.08 + €0.16 \ldots$$

We can see that this is a geometric series.

$$a = 0.01 \quad r = 2$$

There will be 28 payments as there are 28 days in February in 2013, so we need to find the sum of the first 28 terms of this series.

Geometric Series – Ex. 4

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$

$$S_{28} = 0.01\left(\frac{1 - 2^{28}}{1 - 2}\right)$$

$$S_{28} = 0.01\left(\frac{1 - 268{,}435456}{-1}\right)$$

$$S_{28} = 2{,}684{,}354.55$$

Geometric Series – Ex. 4

In the end, HMV ended up paying out €2,684,354.55 in severance fees to their Irish staff.

Dempsey's Double

The Ian Dempsey Breakfast Show on Today FM runs a daily competition called *Dempsey's Double.* Every morning one lucky listener will have the chance to try their hand at winning thousands of euro:

"We ask you ten questions starting with €5 for the first correct answer. For every answer you get right after that, we double your cash."

What's the top prize in this competition?

Dempsey's Double

The sequence is:

$$5, 10, 20, 40, \ldots$$

We can see that it is a geometric sequence with $a = 5$ and $r = 2$.

If we find the tenth term of this sequence then we will find the prize money if they answer all of the ten questions correctly.

$$u_n = ar^{n-1}$$

$$u_{10} = (5)(2)^{10-1} = 2560$$

The top prize on Dempsey's Double is €2,560

Sum to Infinity of a Geometric Series

$$S_\infty = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

$$S_\infty = \lim_{n \to \infty} S_n = \lim_{n \to \infty} a\left(\frac{1 - r^n}{1 - r}\right)$$

This limit will exist if $|r| < 1$ The series will converge giving:

$$S_\infty = a\left(\frac{1}{1 - r}\right)$$

This is because $r^n \to 0$ as $n \to \infty$ if $|r| < 1$

Sum to Infinity of a Geometric Series

$$S_\infty = \lim_{n \to \infty} S_n = \lim_{n \to \infty} a\left(\frac{1 - r^n}{1 - r}\right)$$

This limit will not exist if $|r| > 1$ Thus, the series will diverge.

This is because $r^n \to \infty$ as $n \to \infty$ if $|r| > 1$.

So, to find the sum to infinity of a geometric series, the modulus of the common ratio r must be less than 1.

Sum to Infinity – Ex. 1

Q. Find the sum to infinity of the following series:

$$1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \ldots$$

We know:

$$a = 1$$

$$r = \frac{1}{6}$$

$|r| < 1$ so the series will converge:

$$S_\infty = a\left(\frac{1}{1 - r}\right)$$

$$S_\infty = 1\left(\frac{1}{1 - \frac{1}{6}}\right)$$

$$S_\infty = \frac{6}{5}$$

Sum to Infinity – Ex. 2

Q. Write the recurring decimal $0.474747\ldots$ as an infinite geometric series and hence as a fraction.

Solution: We can write the number as a geometric series:

$$0.474747\ldots = \frac{47}{100} + \frac{47}{10,000} + \frac{47}{1,000,000} + \ldots$$

Now, we know $a = \frac{47}{100}$ and $r = \frac{1}{100}$

Sum to Infinity – Ex. 2

$|r| < 1$ so the series will converge and:

$$S_\infty = a\left(\frac{1}{1-r}\right)$$

$$S_\infty = \frac{47}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{47}{100}\left(\frac{100}{99}\right)$$

$$S_\infty = \frac{47}{99}$$

$$0.474747\ldots = \frac{47}{99}$$

Test for Convergence: Ratio Test

Compute:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = r$$

If $r < 1$, the series <u>converges</u>.

If $r > 1$, the series <u>diverges</u>.

If $r = 1$, the test is <u>inconclusive</u>.

Ratio Test – Ex. 1

Test the following series for convergence:

$$\frac{2}{1} + \frac{4}{2} + \frac{8}{6} + \ldots = \sum_{n=1}^{\infty}\frac{2^n}{n!}$$

Note: $n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$

e.g. $5! = 5 \times 4 \times 3 \times 2 \times 1$

Ratio Test – Ex. 1

$$u_n = \frac{2^n}{n!}$$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

Ratio test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2(n!)}{(n+1)!}\right|$$

Ratio Test – Ex. 1

$$= \lim_{n\to\infty}\left|\frac{2(n!)}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2}{(n+1)}\right|$$

$$= \left|\frac{2}{\infty+1}\right|$$

$$= 0$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = r$$

If $r < 1$, the series <u>converges</u>

The following series converges:

$$\frac{2}{1} + \frac{4}{2} + \frac{8}{6} + \frac{16}{12} \ldots = \sum_{n=1}^{\infty}\frac{2^n}{n!}$$

Ratio Test – Ex. 2

Test for convergence the series:

$$\frac{3}{(2)(2)} + \frac{9}{(3)(4)} + \frac{27}{(4)(8)} + \cdots = \sum_{n=1}^{\infty} \frac{3^n}{(n+1)2^n}$$

Solution:

$$u_n = \frac{3^n}{(n+1)2^n}$$

$$u_{n+1} = \frac{3^{n+1}}{(n+2)2^{n+1}}$$

Ratio Test – Ex. 2

Ratio test:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$\lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+2)2^{n+1}} \div \frac{3^n}{(n+1)2^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+2)2^{n+1}} \times \frac{(n+1)2^n}{3^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{3}{(n+2)2} \times \frac{(n+1)}{1} \right|$$

$$\lim_{n\to\infty} \left| \frac{3(n+1)}{(n+2)2} \right|$$

$$\lim_{n\to\infty} \left| \frac{3n+3}{2n+4} \right|$$

Ratio Test – Ex. 2

$$\lim_{n\to\infty} \left| \frac{3n+3}{2n+4} \right|$$

$$\lim_{n\to\infty} \left| \frac{\frac{3n}{n} + \frac{3}{n}}{\frac{2n}{n} + \frac{4}{n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{3 + \frac{3}{n}}{2 + \frac{4}{n}} \right|$$

$$= \left| \frac{3 + \frac{3}{\infty}}{2 + \frac{4}{\infty}} \right|$$

$$= \left| \frac{3}{2} \right| = \frac{3}{2}$$

$r > 1$, thus the series diverges.

Ratio Test – Ex. 3

Test for convergence the series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Solution:

$$u_n = \frac{1}{n}$$

$$u_{n+1} = \frac{1}{n+1}$$

Ratio Test – Ex. 3

Ratio test:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$\lim_{n\to\infty} \left| \frac{1}{n+1} \div \frac{1}{n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{n+1} \times \frac{n}{1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n}{n+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{1 + \frac{1}{n}} \right|$$

Ratio Test – Ex. 3

$$= \lim_{n\to\infty} \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$= \left| \frac{1}{1 + \frac{1}{\infty}} \right|$$

$$= \left| \frac{1}{1 + 0} \right|$$

$$= 1$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = r$$

If $r = 1$, the test is inconclusive.

Ratio test fails in this instance.

Maclaurin Series

How does a calculator find $\sin x, \cos x, e^x, \dots$?

Answer: by using the Maclaurin Series.
http://www.intmath.com/series-expansion/3-how-calculator-works.php

The Maclaurin Series is defined as follows:

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

Maclaurin Series – Ex. 1

Q. Find the Maclaurin Series of $f(x) = e^x$ and, hence estimate the value of e^2 using the first seven terms of the series.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$f(x) = e^x \qquad f(0) = e^0 = 1$$

$$f'(x) = e^x \qquad f'(0) = e^0 = 1$$

$$f''(x) = e^x \qquad f''(0) = e^0 = 1$$

$$f'''(x) = e^x \qquad f'''(0) = e^0 = 1$$

Maclaurin Series – Ex. 1

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$e^x = 1 + x(1) + \frac{x^2(1)}{2!} + \frac{x^3(1)}{3!} + \frac{x^4(1)}{4!} + \frac{x^5(1)}{5!} + \frac{x^6(1)}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

Maclaurin Series – Ex. 1

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

Estimate e^2:

$$e^2 = 1 + 2 + \frac{(2)^2}{2!} + \frac{(2)^3}{3!} + \frac{(2)^4}{4!} + \frac{(2)^5}{5!} + \frac{(2)^6}{6!}$$

$$e^2 = 1 + 2 + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} + \frac{64}{720} = 7.3555$$

Check e^2 on the calculator... Very close to exact value. Answer is more accurate when more terms of the series are included.

Maclaurin Series – Ex. 2

Q. Find the Maclaurin Series of $f(x) = \sin x$ and, hence estimate the value of $\sin(1)$ using the first four non-zero terms of the series.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$f(x) = \sin x \qquad f(0) = \sin(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = \cos(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = -\sin(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -\cos(0) = -1$$

Maclaurin Series – Ex. 2

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$\sin x = 0 + x(1) + \frac{x^2(0)}{2!} + \frac{x^3(-1)}{3!} + \frac{x^4(0)}{4!} + \frac{x^5(1)}{5!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

Maclaurin Series – Ex. 2

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Estimate :

$$\sin(1) = (1) - \frac{(1)^3}{3!} + \frac{(1)^5}{5!} - \frac{(1)^7}{7!}$$

$$\sin(1) = 0.84146$$

Check calculator... $\sin(1) = 0.84147$.
Note: this is the sine of one radian, not one degree.

Maclaurin Series – Ex. 3

Q. Find the Maclaurin Series of $f(x) = \cos x$ and, hence estimate the value of cos(2) using the first four non-zero terms of the series.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$f(x) = \cos x \qquad f(0) = \cos(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = -\sin(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -\cos(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = \sin(0) = 0$

Maclaurin Series – Ex. 3

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$\cos x = 1 + x(0) + \frac{x^2(-1)}{2!} + \frac{x^3(0)}{3!} + \frac{x^4(1)}{4!} + \frac{x^5(0)}{5!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots$$

Maclaurin Series – Ex. 3

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots$$

Estimate the value of cos(2) using the first five non-zero terms of the series:

$$\cos(2) = 1 - \frac{(2)^2}{2!} + \frac{(2)^4}{4!} - \frac{(2)^6}{6!} + \frac{(2)^8}{8!} = -0.4158$$

Check calculator... $\cos(2) = -0.4161$
Note: this is the cosine of two radians, not two degrees.

Maclaurin Series

The Maclaurin Series is an exact representation of a function.

But, using the first few terms of the series, we can obtain a good approximation to the function for <u>small values</u> of x.

More examples available at
http://www.studentxpress.ie/educ/maths/maths8/maths8.html