Problem Sheet 4: The LU decomposition method

1. For each of the following pairs A, B, find an LU decomposition for the matrix A and use it to solve the system $AX = B$.

$$\text{(i) } A = \begin{pmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}; \qquad \text{(ii) } A = \begin{pmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{pmatrix}, \quad B = \begin{pmatrix} -3 \\ 0 \\ 15 \end{pmatrix};$$

$$\text{(iii) } A = \begin{pmatrix} 1 & 3 & 1 & -2 \\ 2 & 4 & -1 & 2 \\ 3 & 1 & 1 & 5 \\ 4 & 2 & -1 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 7 \\ 10 \\ 11 \end{pmatrix}.$$

2. Determine whether each of the following matrices is invertible. In the case that a matrix is invertible, find its inverse.

$$\begin{pmatrix} 2 & 7 \\ 1 & -9 \end{pmatrix}; \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ -2 & 0 & 0 \end{pmatrix}; \qquad \begin{pmatrix} -4 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & -1 & 3 \end{pmatrix}$$

3. Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ -1 & -2 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix}.$$

Calculate the inverse of the matrix A and then find a 3×3 matrix X such that $AX = A+B$.

4. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & -1 \end{pmatrix}$$

   (a) Find a LU-decomposition such that L has 1's along its diagonal.

   (b) Find a LU-decomposition such that U has 1's along its diagonal.

   (c) Find matrices L, D and U such that L is lower triangular with 1's along its diagonal, U is upper triangular and D is diagonal.

5. Show that the matrix below has no LU decomposition.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix}$$

Rearrange the rows of A so that the resulting matrix does have an LU decomposition.