Eigenvalues and eigenvectors of a matrix

Definition: If A is an $n \times n$ matrix and there exists a real number λ and a non-zero column vector V such that

$$AV = \lambda V$$

then λ is called an **eigenvalue** of A and V is called an **eigenvector** corresponding to the eigenvalue λ.

Example: Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$. Note that

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 4 \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

So, the number 4 is an eigenvalue of A and the column vector $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ is an eigenvector corresponding to 4.

Finding eigenvalues and eigenvectors for a given matrix A

1. The eigenvalues of A are calculated by solving the characteristic equation of A:

$$\det(A - \lambda I) = 0$$

2. Once the eigenvalues of A have been found, the eigenvectors corresponding to each eigenvalue λ can be determined by solving the matrix equation

$$AV = \lambda V$$

Example: Find the eigenvalues of $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$.

Example:

The eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ are 1 and 4. To find the eigenvectors corresponding to 1 we write $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ and solve

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 1\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \qquad \text{so} \qquad \begin{aligned} 2V_1 + V_2 &= V_1 \\ 2V_1 + 3V_2 &= V_2 \end{aligned}$$

This gives a single independent equation $V_1 + V_2 = 0$ so any vector of the form $V = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

The power method

The power method is an iterative technique for approximating the dominant eigenvalue of a matrix together with an associated eigenvector.

Let A be an $n \times n$ matrix with eigenvalues satisfying

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$$

The largest eigenvalue, λ_1 is called the **dominant eigenvalue**.

Let $\mathbf{x}_0$ be an n-vector. Then it can be written as

$$\mathbf{x}_0 = \alpha_1 \mathbf{v}_1 + \cdots \alpha_n \mathbf{v}_n$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the eigenvectors of A.

Now construct the sequence $\mathbf{x}^{(m)} = A\mathbf{x}^{(m-1)}$, for $m \geq 1$.

$$\begin{aligned}
\mathbf{x}^{(1)} &= A\mathbf{x}^{(0)} = \alpha_1\lambda_1\mathbf{v}_1 + \cdots \alpha_n\lambda_n\mathbf{v}_n \\
\mathbf{x}^{(2)} &= A\mathbf{x}^{(1)} = \alpha_1\lambda_1^2\mathbf{v}_1 + \cdots \alpha_n\lambda_n^2\mathbf{v}_n \\
&\vdots \\
\mathbf{x}^{(m)} &= A\mathbf{x}^{(m-1)} = \alpha_1\lambda_1^m\mathbf{v}_1 + \cdots \alpha_n\lambda_n^m\mathbf{v}_n
\end{aligned}$$

and hence

$$\frac{\mathbf{x}^{(m)}}{\lambda_1^m} = \alpha_1\mathbf{v}_1 + \alpha_2\left(\frac{\lambda_2}{\lambda_1}\right)^m \mathbf{v}_2 + \cdots \alpha_n\left(\frac{\lambda_n}{\lambda_1}\right)^m \mathbf{v}_n$$

which gives

$$\lim_{n\to\infty} \frac{\mathbf{x}^{(m)}}{\lambda_1^m} = \alpha_1\mathbf{v}_1$$

Hence, the sequence $\frac{\mathbf{x}^{(m)}}{\lambda_1^m}$ converges to an eigenvector associated with the dominant eigenvalue.

The power method implementation

Choose the initial guess, $\mathbf{x}^{(0)}$ such that $\max_i |x_i^{(0)}| = 1$.

For $m \geq 1$, let

$$\mathbf{y}^{(m)} = A\mathbf{x}^{(m-1)}, \qquad \mathbf{x}^{(m)} = \frac{\mathbf{y}^{(m)}}{y_{p_m}^{(m)}}$$

where p_m is the index of the component of $\mathbf{y}^{(m)}$ which has maximum absolute value.

Note that

$$\mathbf{y}^{(m)} = A\mathbf{x}^{(m-1)} \approx \lambda_1 \mathbf{x}^{(m-1)}$$

and since $x_{p_{m-1}}^{(m-1)} = 1$ (by construction), it follows that

$$y_{p_{m-1}}^{(m)} \to \lambda_1$$

Note: Since the power method is an iterative scheme, a stopping condition could be given as

$$|\lambda^{(m)} - \lambda^{(m-1)}| < \text{required error}$$

where $\lambda^{(m)}$ is the dominant eigenvalue approximation during the m^{th} iteration, or

$$\max_i |x_i^{(m)} - x_i^{(m-1)}| < \text{required error}$$

Example Consider the matrix

$$A = \begin{pmatrix} -2 & -2 & 3 \\ -10 & -1 & 6 \\ 10 & -2 & -9 \end{pmatrix}$$

Show that its eigenvalues are $\lambda_1 = -12$, $\lambda_2 = -3$ and $\lambda_3 = 3$ and calculate the associated eigenvectors. Use the power method to approximate the dominant eigenvalue and a corresponding eigenvector.

The Gerschgorin Circle Theorem

Let A be an $n \times n$ matrix and define

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

for each $i = 1, 2, 3 \ldots n$. Also, consider the circles

$$C_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq R_i\}.$$

1. If λ is an eigenvalue of A then λ lies in one of the circles C_i.

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1. If λ is an eigenvalue of A then λ lies in one of the circles C_i.
2. If k of the circles C_i form a connected region R in the complex plane, disjoint from the remaining $n - k$ circles then the region contains exactly k eigenvalues.

Example

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 9 \end{pmatrix}$$

The radii of the Gerschgorin circles are

$$R_1 = |-1| + |0| = 1; \quad R_2 = |1| + |1| = 2; \quad R_3 = |-2| + |-1| = 3$$

and the circles are

$$C_1 = \{z \in \mathbb{C} \colon |z-1| \leq 1\}; \qquad C_2 = \{z \in \mathbb{C} \colon |z-5| \leq 2\};$$
$$C_3 = \{z \in \mathbb{C} \colon |z-9| \leq 3\}$$

Since C_1 is disjoint from the other circles, it must contain one of the eigenvalues and this eigenvalue is real. As C_2 and C_3 overlap, their union must contain the other two eigenvalues.

Matrix polynomials and inverses

Let A be an $n \times n$ matrix with eigenvalues λ_1, λ_2, ..., λ_n and associated eigenvectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$.

1. Let
$$p(x) = a_0 + a_1 x + \cdots + a_m x^m$$
Then the eigenvalues of
$$B = p(A) = a_0 + a_1 A + \cdots a_m A^m$$
are $p(\lambda_1)$, $p(\lambda_2)$, ..., $p(\lambda_m)$, with eigenvectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$.

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are $p(\lambda_1)$, $p(\lambda_2)$, ..., $p(\lambda_m)$, with eigenvectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$.
2. If $det(A) \neq 0$ then A^{-1} has eigenvalues $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, ..., $\frac{1}{\lambda_m}$, with eigenvectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_n}$.

The inverse power method

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and associated eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$.

Let q be a number for which $\det(A - qI) \neq 0$ (so q can be any number which is not an eigenvalue of A). Let $B = (A - qI)^{-1}$ so the eigenvalues of B are given by

$$\mu_1 = \frac{1}{\lambda_1 - q}, \qquad \mu_2 = \frac{1}{\lambda_2 - q}, \quad \ldots \quad \mu_m = \frac{1}{\lambda_m - q},$$

If we know that λ_k is the eigenvalue that is closest to the number q, then μ_k is the dominant eigenvalue of B and so it can be determined using the power method.

Example

Consider the matrix

$$A = \begin{pmatrix} 12 & 1 & 1 & 0 & 3 \\ -1 & 3 & 0 & 1 & 0 \\ 1 & 0 & -6 & 2 & 1 \\ 0 & 2 & 1 & 9 & 0 \\ 1 & 0 & 1 & 0 & -2 \end{pmatrix}$$

Using Gerschgorin circles obtain an estimate for one of the eigenvalues of A and then get a more accurate approximation using the inverse power method.

Determination of the smallest eigenvalue

To find the eigenvalue of A that is smallest in magnitude is equivalent to finding the dominant eigenvalue of the matrix $B = A^{-1}$. (The inverse power method with $q = 0$.)

Example: Consider the matrix

$$A = \begin{pmatrix} -2 & -2 & 3 \\ -10 & -1 & 6 \\ 10 & -2 & -9 \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = -12$, $\lambda_2 = -3$ and $\lambda_3 = 3$. Use the inverse power method to approximate the smallest eigenvalue and a corresponding eigenvector.