FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF MATHEMATICS AND STATISTICS

END OF SEMESTER EXAMINATION PAPER 2016

MODULE CODE: MS4131

SEMESTER: Spring 2016

MODULE TITLE: Linear Algebra 1

DURATION OF EXAM: 2.5 hours

LECTURER: Mr. Kevin O'Brien

GRADING SCHEME: 80 marks
100% of module grade

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES

Scientific calculators approved by the University of Limerick can be used.
Students must attempt any 4 questions from 5.

Question 1

Part A. Matrix Multiplication (4 Marks)

Given the matrices

$$A = \begin{pmatrix} 2 & 2 & 1 & -1 & 4 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 4 \\ 0 & 5 \\ -1 & 0 \\ 7 & 1 \\ 1 & 3 \end{pmatrix}; \quad C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

calculate the products AB and CA.

Part B. Diagonal Matrices (3 Marks)

Consider the following diagonal matrix D. In terms of the values a, b and c ;

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

(i) (1 Mark) Write an expression for the trace of the matrix D.

(ii) (1 Mark) State the inverse of D, i.e. D^{-1}.

(iii) (1 Mark) State the matrix D^3.

Part C. Matrix Multiplication (4 Marks)

Suppose A is a lower triangular matrix of the form;

$$A = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

(i) (1 Mark) State the transpose of A.

(ii) (2 Marks) Compute B where B = $A \times A^T$.

(iii) (1 Mark) B is a symmetric matrix. What is meant by this?

Part B. Invertible Matrices (5 Marks)

Show that if A is an $n \times n$ invertible matrix that satisfies

$$9A^3 + A^2 - 3A = 0$$

where $A^n = \underbrace{A \dots A}_{n \text{ times}}$, I is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix, then the inverse of A is given by

$$A^{-1} = \frac{1}{3}I + 3A.$$

Prove of Transpose Identity

$$(AB)^T = B^t \times A^T$$

Question 2

Part A. Fundamental Theorem of Invertible Matrices (5 Marks)

The Fundamental Theorem of Invertible Matrices states that a set of mathematical expressions concerning a $n \times n$ matrix A are each equivalent to one another.

(i) (4×1 Mark) State any four of these expressions.

(ii) (1 Mark) What is the rank of a matrix.

Part C. Inverting a Matrix with E.R.O.s (5 Marks)

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -4 & 1 \\ -3 & 0 & -1 \end{pmatrix}.$$

using elementary row operations.

Part C. Inverting a Matrix with Co-Factor Method (10 Marks)

- (5 Marks) For each element of A, calculate the corresponding minor. Show your workings. State the matrix of minors.
- (2 Marks) Hence or otherwise, compute the determinant of A i.e. $Det(A)$.
- (1 Mark) Compute the cofactor matrix for A, cof (A).
- (2 Marks) State the Inverse Matrix of A.

Question 3

Part A. System of Linear Equations (6 Marks)

Consider the linear system

$$\begin{aligned} x_1 + x_3 &= 4 \\ 2x_1 + 4x_2 + x_3 &= -3 \\ x_2 + 3x_3 &= 7. \end{aligned}$$

(i) (1 Mark) Write down the coefficient matrix and the augmented matrix of this system.

(ii) (1 Mark) What can you say about the solution set of the system? Justify your answer.

(iii) (4 Marks) Solve the system of equations, using any appropriate method.

Part B. Row-Echelon Form of a Matrix (4 Marks)

Consider the matrices U, V, W and X presented below. For each matrix state one reason why that matrix in not in row-echelon form. Provide distinct answers for each of the four matrices.

$$U = \begin{pmatrix} 1 & 2 & 6 & 3 & 5 \\ 0 & 1 & 4 & 0 & 6 \\ 0 & 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} \qquad V = \begin{pmatrix} 1 & 1 & 6 & 8 & 5 \\ 0 & 2 & 6 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$W = \begin{pmatrix} 1 & 1 & 6 & 8 & 5 \\ 0 & 1 & 6 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 6 & 8 & 5 \\ 0 & 1 & 6 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(**Marking Scheme**: 4×1 *Marks where 1 Mark is awarded for each valid and distinct reason*)

Part C. Distance from Planes (6 Marks)

(i) (3 Marks) Give the general form of the equation of the plant π in $\mathbb{R}^3$ passing throughthe point $P_0 = (1, 0, 2)$ with the vector $n = (-5, 3, 2)$ as the normal.

(ii) (3 Marks) Show that the point $Q = (1, -1, 1)$ does not lie in the plane π and find its distance from π.

Question 4

Part A. Vector Calculations (7 Marks)

Consider the three vectors in $\mathbb{R}^3$:

$$u = (1, 2, 0), \quad v = (0, 1, 3), \quad w = (1, 2, 3).$$

(i) Evaluate $\|u\|$, $\|v\|$, $u \cdot v$, $u \times v$ and the angle between u and v.

(ii) Calculate the scalar triple product $u \cdot (v \times w)$.

Part B. Orthonormal Projections (8 Marks)

If

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix},$$

(i) find the vector component of u along a, $proj_a u$ and the vector orthogonal component to a;

(ii) calculate the norm of $proj_a u$ and the norm of $u - proj_a u$;

(iii) draw $proj_a u$ showing its direction and orientation.

Part C. Vector Proofs (5 Marks)

Prove that, for any $u,\ v\ \in \mathbb{R}^3$,

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u.$$

(It is sufficient to verify this property for one component.)

Part D. Planes (5 Marks)

1. Find the general form of the equation of the plane π in $\mathbb{R}^3$ which passes through the point $P = (3, 1, 6)$ and is orthogonal to the vector $n = (1, 7, -2)$.

2. Show that the point $Q = (1, -1, 1)$ does not lie in the plane π and find its distance from π.

Question 5

Part A. Vector Calculations (7 Marks)

Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(i) Determine the eigenvalues and corresponding eigenvectors of A.

(ii) Diagonalise A; i.e, give a matrix P and a diagonal matrix D, such that $A = PDP^{-1}$.

(iii) Hence, evaluate A^5.

Part B. Addition of Vectors

Prove that for any $u,\ v,\ \in \mathbb{R}^3$ and any $k \in \mathbb{R}$ we have

(i) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$;

(ii) $(u + v) \times w = u \times w + v \times w$;

(iii) $k(u \times v) = (ku) \times v = u \times (kv)$.

3. (i) Let u = (-1, 2, 3, 1) and v = (1, 0, 5, -2) be two vectors in R4. Evaluate ——u——, ——v——, ——u + v—— and . Check that u and v satisfy the triangle inequality. 5

Vectors

(b) Let u and v be two vectors in $\mathbb{R}^3$ and let θ be the angle between them.

i. Define the scalar product $u \cdot v$ in terms of θ and prove the so-called Cauchy- Scwarz Ineqlity

$$\|u \cdot v\| \leq \|u\| \times \|v\|$$

Prove the so-called "Triangular Inequality"

$$\|u\| + \|v| \ \leq \|u\| + \|v\|$$

Let $v = (v_1, v_2, v_3)$ be a vector in $\mathbb{R}^3$ and k be a scale ($k \in (R)$ and $k \geq 0$). Show that

$$\|kv\| = k\|v\|$$

, where $\|v\|$ denotes the euclidena norm.

Determinants

$$|A| = \begin{vmatrix} 0 & 4 & 7 \\ -1 & -1 & 7 \\ 1 & 5 & 1 \end{vmatrix}$$