Limits

MA4602

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Limits Notation

$$\lim_{x \to c} f(x)$$

"The limit of $f(x)$ as x approaches c"

A limit looks at the behaviour of $f(x)$ as x gets closer to c.

Dividing by Zero and Infinity

Any real number divided by ∞ will be equal to zero

e.g. $\frac{10}{\infty} = 0$ $\quad -\frac{597}{\infty} = 0$

Any real number divided by zero will be equal to ∞

e.g. $\frac{17}{0} = \infty$ $\quad -\frac{24}{0} = \infty$

Solving Limits – Ex. 1

Solve the following limit:

$$\lim_{x \to 0} \frac{x^2 + 1}{x}$$

Solution:

$$= \frac{(0)^2 + 1}{0}$$

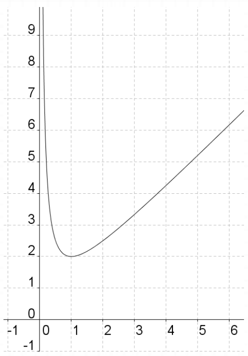
$$= \frac{1}{0} = \infty$$

$$\lim_{x \to 0} \frac{x^2 + 1}{x} = \infty$$

Solving Limits – Ex. 1

$$\lim_{x \to 0} \frac{x^2 + 1}{x} = \infty$$

As the x value approaches 0, the corresponding $f(x)$ value (or y value) approaches ∞.



Factorising Formulae

Difference of two squares:

$$x^2 - a^2 = (x - a)(x + a)$$

Difference of two cubes:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

Sum of two cubes:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Undefined Answers

Answers such as $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are not defined

If the answer obtained for a limit is one of these then a different approach needs to be applied…

Solving Limits – Ex. 2

Solve the following limit:

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \frac{(2)^3-8}{2-2} = \frac{0}{0}$$

This answer is not defined… Need to figure out a way to get a defined answer.

Solving Limits – Ex. 2

Solution: Factorise the numerator

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}$$

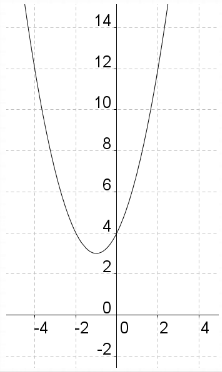
$$= \lim_{x\to 2} x^2+2x+4$$

$$= (2)^2+2(2)+4 = 12$$

Solving Limits – Ex. 2

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = 12$$

As the x value approaches 2, the corresponding $f(x)$ value (or y value) approaches 12.



Steps for when an answer of $\frac{0}{0}$ is obtained from a limit

1. Start problem again
2. Factorise the top and/or bottom line of the function (numerator and denominator) if possible.
3. Cancel any common factors i.e. divide above and below by the common factor.
4. Find the limit of the function i.e. sub for x in the function with the value it is approaching.

Solving Limits – Ex. 3

Solve the following limit:

$$\lim_{x\to 5}\frac{x^2-3x-10}{x-5} = \frac{(5)^2-3(5)-10}{5-5} = \frac{0}{0}$$

This answer is not defined…

Solving Limits – Ex. 3

Solution: Factorise the numerator

$$\lim_{x\to 5}\frac{x^2-3x-10}{x-5}=\lim_{x\to 5}\frac{(x-5)(x+2)}{x-5}$$

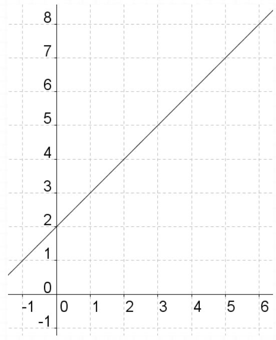
$$=\lim_{x\to 5} x+2$$

$$=5+2=7$$

Solving Limits – Ex. 3

$$\lim_{x\to 5}\frac{x^2-3x-10}{x-5}=7$$

As the x value approaches 5, the corresponding $f(x)$ value (or y value) approaches 7.



Solving Limits – Ex. 4

Solve the following limit:

$$\lim_{n\to\infty}\frac{3}{\sqrt{n^2+3}}$$

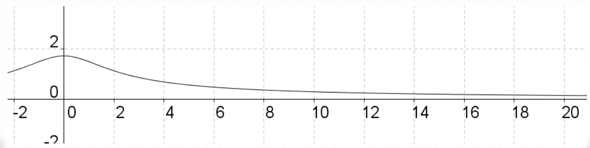
$$=\frac{3}{\sqrt{(\infty)^2+3}}=\frac{3}{\infty}=0$$

$$\lim_{n\to\infty}\frac{3}{\sqrt{n^2+3}}=0$$

Solving Limits – Ex. 4

$$\lim_{n\to\infty}\frac{3}{\sqrt{n^2+3}}=0$$

As the n value approaches infinity, the corresponding $f(n)$ value approaches 0.



Solving Limits – Ex. 5

Solve the following limit:

$$\lim_{x\to\infty}\frac{x^2+1}{x^3-3}$$

$$=\frac{(\infty)^2+1}{(\infty)^3-3}=\frac{\infty}{\infty}$$

Not defined…

Solving Limits – Ex. 5

Solution: divide above and below by the highest power of x in the function.

$$\lim_{x\to\infty}\frac{x^2+1}{x^3-3}=\lim_{x\to\infty}\frac{\frac{x^2}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}-\frac{3}{x^3}}$$

$$\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^3}}{1-\frac{3}{x^3}}$$

Solving Limits – Ex. 5

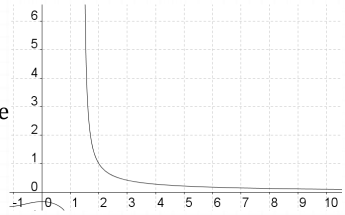
$$\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^3}}{1-\frac{3}{x^3}} = \frac{\frac{1}{\infty}+\frac{1}{(\infty)^3}}{1-\frac{3}{(\infty)^3}}$$

$$\frac{0+0}{1-0} = \frac{0}{1} = 0$$

Solving Limits – Ex. 5

$$\lim_{x\to\infty}\frac{x^2+1}{x^3-3} = 0$$

As the x value approaches infinity, the corresponding $f(x)$ value (or y value) approaches 0.



Steps for evaluating limits as x approaches infinity

1. Sub for x in the function with ∞ . Usually (but not always) this will give an answer of $\frac{\infty}{\infty}$ which is not defined.
2. If this occurs, start the problem again, this time divide above and below by the highest power of x in the function.
3. Sub for x with ∞ and complete your calculations – this should give a defined answer. Remember ∞ is a defined answer, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are not defined answers.

Summary

1. Answer of $\frac{0}{0}$ is not defined. Solution: factorise the function.
2. Answer of $\frac{\infty}{\infty}$ is not defined. Solution: divide above and below by highest power of x in the function.
3. $\frac{\textit{Real Number}}{0} = \infty$
4. $\frac{\textit{Real Number}}{\infty} = 0$

Drug Concentration

The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{0.17h}{h^2+2}$$

Find and interpret:

$$\lim_{h\to\infty} A(h)$$

Drug Concentration

$$\lim_{h\to\infty} A(h) = \lim_{h\to\infty}\frac{0.17h}{h^2+2}$$

$$\lim_{h\to\infty}\frac{0.17h}{h^2+2} = \lim_{h\to\infty}\frac{0.17(\infty)}{(\infty)^2+2} = \frac{\infty}{\infty}$$

Not defined…

$$\lim_{h\to\infty}\frac{0.17h}{h^2+2} = \lim_{h\to\infty}\frac{\frac{0.17}{h}}{1+\frac{2}{h^2}}$$

Drug Concentration

$$\lim_{h\to\infty} \frac{\frac{0.17}{h}}{1+\frac{2}{h^2}} = \frac{\frac{0.17}{\infty}}{1+\frac{2}{(\infty)^2}}$$

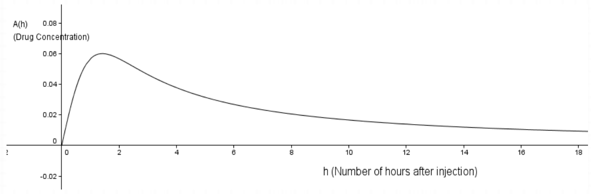
$$\frac{0}{1+0} = 0$$

So, as $h \to \infty$, $A(h) \to 0$.

This means that, as number of hours after the drug was injected h increases toward infinity, the concentration of the drug $A(h)$ will approach zero.

Drug Concentration

As number of hours after the drug was injected h increases, the concentration of the drug $A(h)$ will approach zero.



Piecewise Functions

A Piecewise Function is a function which is defined by multiple subfunctions, each subfunction applying to a certain interval of the main functions domain (a subdomain).

Example:

$$f(x) = \begin{cases} (x-2)^2, & x < 3 \\ x-1, & x \geq 3 \end{cases}$$

Example of a Real Life Piecewise Function

The state charges companies an annual fee of €20 per ton for each ton of pollution emitted by that company into the atmosphere, up to a maximum of 4,000 tons.

No additional fees are charged for emissions beyond the 4,000 ton limit.

(a) Write a piecewise definition of the fees $f(x)$ charged for the emission of x tons of pollution in a year.

(b) What is the limit of $f(x)$ as x approaches 4,000 tons?

(c) What is the limit of $f(x)$ as x approaches 8,000 tons?

Example of a Real Life Piecewise Function

(a)

$$f(x) = \begin{cases} 20x & x \leq 4{,}000 \\ 80{,}000 & x > 4{,}000 \end{cases}$$

(b) What is the limit of $f(x)$ as x approaches 4,000 tons?

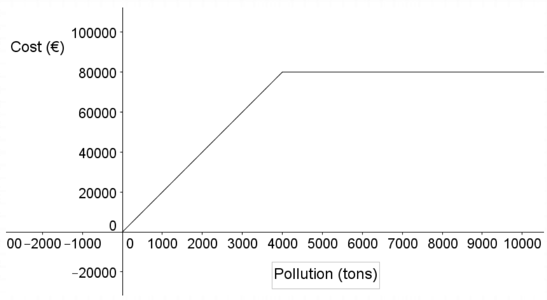
$$\lim_{x\to 4{,}000} f(x)$$

$$= \lim_{x\to 4{,}000} 20x$$

$$= 20(4{,}000) \quad = \quad €80{,}000$$

As the amount of pollution approaches 4,000 tons, the cost to the company will approach €80,000.

Example of a Real Life Piecewise Function



Example of a Real Life Piecewise Function

(c) What is the limit of $f(x)$ as x approaches 8,000 tons?

$$\lim_{x\to 8{,}000} f(x)$$

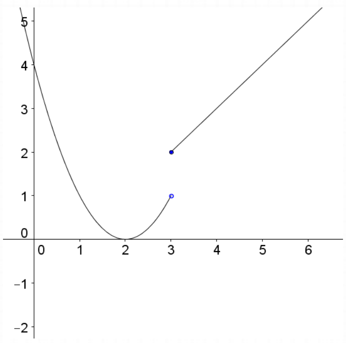
$$\lim_{x\to 8{,}000} 80{,}000$$

$$= €80{,}000$$

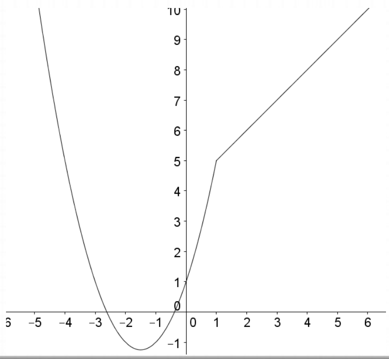
As the amount of pollution approaches 8,000 tons, the cost to the company will approach €80,000.

More Examples of Piecewise Functions

$$f(x) = \begin{cases} (x-2)^2, & x < 3 \\ x-1, & x \geq 3 \end{cases}$$



$$g(x) = \begin{cases} x^2+3x+1, & x < 1 \\ x+4, & x \geq 1 \end{cases}$$



Continuity

What's the difference between the previous two examples of piecewise functions?

In the first example, there is a gap in the graph between the two subfunctions. This indicates that $f(x)$ is not continuous at $x = 3$.

In the second example, there is no gap in the graph between the two subfunctions. This indicates that $g(x)$ is continuous at $x = 1$ and, indeed, everywhere else.

How can we check continuity without drawing out the graph?

A function $f(x)$ is continuous at a point c if:

i. $f(c)$ exists
ii. $\lim_{x\to c} f(x)$ exists
iii. $f(c) = \lim_{x\to c} f(x)$

The first aspect we check (i) is relatively straightforward, the second aspect (ii) requires a little more work…

How to check if $\lim_{x \to c} f(x)$ exists

$\lim_{x \to c} f(x)$ exists if:

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

We find the limit of the function $f(x)$ as x approaches c from the right hand side (c^+) and we find the limit of the function $f(x)$ as x approaches c from the left hand side (c^-).

If these two are equal then $\lim_{x \to c} f(x)$ exists and is equal to the value found for the $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$.

Continuous Functions Ex. 1

In the following piecewise function:

$$f(x) = \begin{cases} (x-2)^2, & x < 3 \\ x - 1, & x \geq 3 \end{cases}$$

Is $f(x)$ continuous at $x = 3$?

Solution: First, find $f(3)$:

$$f(3) = 3 - 1 = 2$$

Next, find if $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x)$

Continuous Functions Ex. 1

$$f(x) = \begin{cases} (x-2)^2, & x < 3 \\ x - 1, & x \geq 3 \end{cases}$$

Find: $\lim_{x \to 3^+} f(x)$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x - 1$$

$$\lim_{x \to 3^+} x - 1 = 3 - 1$$

$$\lim_{x \to 3^+} x - 1 = 2$$

Find: $\lim_{x \to 3^-} f(x)$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x-2)^2$$

$$\lim_{x \to 3^-} (x-2)^2 = (3-2)^2$$

$$\lim_{x \to 3^-} (x-2)^2 = 1$$

Continuous Functions Ex. 1

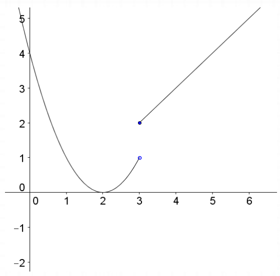
It is clear from the answers we obtained that:

$$\lim_{x \to 3^+} f(x) \neq \lim_{x \to 3^-} f(x)$$

Thus $\lim_{x \to 3} f(x)$ does not exist.

So: $f(3) \neq \lim_{x \to 3} f(x)$

Thus the function $f(x)$ is not continuous at $x = 3$



Continuous Functions Ex. 2

In the following piecewise function:

$$g(x) = \begin{cases} x^2 + 3x + 1, & x < 1 \\ x + 4, & x \geq 1 \end{cases}$$

Is $g(x)$ continuous at $x = 1$?

Solution: First, find $g(1)$:

$$g(1) = 1 + 4 = 5$$

Next, find if $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^-} g(x)$

Continuous Functions Ex. 2

$$g(x) = \begin{cases} x^2 + 3x + 1, & x < 1 \\ x + 4, & x \geq 1 \end{cases}$$

Find: $\lim_{x \to 1^+} g(x)$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} x + 4$$

$$\lim_{x \to 1^+} x + 4 = 1 + 4$$

$$\lim_{x \to 1^+} x + 4 = 5$$

Find: $\lim_{x \to 1^-} g(x)$

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x^2 + 3x + 1$$

$$\lim_{x \to 1^-} x^2 + 3x + 1 = (1)^2 + 3(1) + 1$$

$$\lim_{x \to 1^-} x^2 + 3x + 1 = 5$$

Continuous Functions Ex. 2

It is clear from the answers we obtained that:

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x)$$

Thus $\lim_{x\to 1} g(x)$ exists and

$$\lim_{x\to 1} g(x) = 5$$

Earlier, we found that

$$g(1) = 5$$

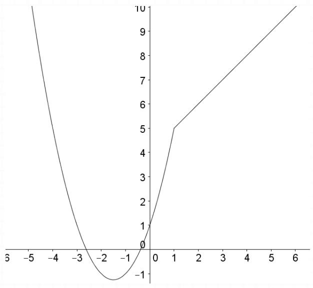
So:

$$g(1) = \lim_{x\to 1} g(x)$$

Thus the function $g(x)$ is continuous at $x = 1$

Continuous Functions Ex. 2

Conclusion: The function $g(x)$ is continuous at $x = 1$



Continuous Functions Ex. 3

In the following piecewise function:

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Is $f(x)$continuous everywhere?

Solution: The function changes at $x = 2$ so we will check the continuity at that point. First, find $f(2)$ and eventually check if it equals $\lim_{x\to 2} f(x)$

$$f(2) = 1$$

Next, find if $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$

Continuous Functions Ex. 3

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Find: $\lim_{x\to 2^+} f(x)$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2 - x - 2}{x - 2}$$

$$= \frac{(2)^2 - (2) - 2}{2 - 2} = \frac{0}{0}$$

Not defined.

$$\lim_{x\to 2^+} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x\to 2^+} \frac{(x-2)(x+1)}{x - 2}$$

$$= \lim_{x\to 2^+} x + 1$$

$$= 3$$

Continuous Functions Ex. 3

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Find: $\lim_{x\to 2^-} f(x)$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2 - x - 2}{x - 2}$$

$$= \frac{(2)^2 - (2) - 2}{2 - 2} = \frac{0}{0}$$

Not defined.

$$\lim_{x\to 2^-} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x\to 2^-} \frac{(x-2)(x+1)}{x - 2}$$

$$= \lim_{x\to 2^-} x + 1$$

$$= 3$$

Continuous Functions Ex. 3

It is clear from the answers we obtained that:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$

Thus $\lim_{x\to 2} f(x)$ exists:

$$\lim_{x\to 2} f(x) = 3$$

We found earlier that

$$f(2) = 1$$

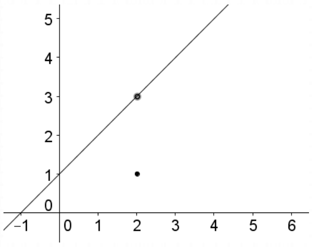
So:

$$\lim_{x\to 2} f(x) \neq f(2)$$

Thus, the piecewise function is not continuous at $x = 2$

Continuous Functions Ex. 3

Conclusion: The function is not continuous everywhere as it is not continuous at $x = 2$



Summary of checking Continuity of a piecewise function

A piecewise function is continuous at the point $x = c$ if

$$f(c) = \lim_{x\to c} f(x)$$

These are typically the steps taken:

1. Find $f(c)$
2. Check whether $\lim_{x\to c} f(x)$ exists. If $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$ then $\lim_{x\to c} f(x)$ exists and is equal to the answer obtained for both $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.
3. Check whether $f(c) = \lim_{x\to c} f(x)$

Organic Fertilizer

A company charges €1.20 per kg for organic fertilizer on orders not over 100 kg and €1.10 per kg for orders over 100 kg. Let $f(x)$ represent the cost for buying x kg of organic fertilizer.

(a) Find the cost of buying the following:

i. 80 kg
ii. 150 kg
iii. 100 kg

(b) Is $f(x)$ continuous everywhere? What can you conclude from the answer to this question?

Organic Fertilizer

$$f(x) = \begin{cases} 1.2x, & x \le 100 \\ 1.1x & x > 100 \end{cases}$$

(i) Cost of buying 80 kg of organic fertilizer:

$$f(80) = 1.2(80)$$

$$f(80) = 96$$

The cost of 80 kg of organic fertilizer will be €96.

(ii) Cost of buying 150 kg of organic fertilizer:

$$f(150) = 1.1(150)$$

$$f(150) = 165$$

The cost of 150 kg of organic fertilizer will be €165.

Organic Fertilizer

(iii) Cost of buying 100 kg of organic fertilizer:

$$f(100) = 1.2(100)$$

$$f(100) = 120$$

The cost of 100 kg of organic fertilizer will be €120.

Organic Fertilizer

Is $f(x)$continuous everywhere?

Solution: The function changes at $x = 100$ so we will check the continuity at that point. First, find $f(100)$ and eventually check if it equals $\lim_{x\to 100} f(x)$

$$f(100) = 120 \qquad \text{[Solved this previously]}$$

Next, find if $\lim_{x\to 100^+} f(x) = \lim_{x\to 100^-} f(x)$

Organic Fertilizer

$$\lim_{x \to 100^+} f(x)$$

$$\lim_{x \to 100^+} 1.1x$$

$$= 1.1(100)$$

$$= 110$$

$$\lim_{x \to 100^-} f(x)$$

$$\lim_{x \to 100^-} 1.2x$$

$$= 1.2(100)$$

$$= 120$$

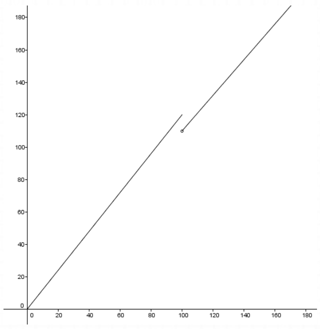
$$\lim_{x \to 100^+} f(x) \neq \lim_{x \to 100^-} f(x)$$

Organic Fertilizer

$$\lim_{x \to 100^+} f(x) \neq \lim_{x \to 100^-} f(x)$$

Thus, $\lim_{x \to 100} f(x)$ does <u>not</u> exist and we can conclude that $f(x)$ is not continuous at $x = 100$

This would indicate that the price of this fertilizer changes irregularly when the weight of the order is around 100 kg e.g. 98 kg will cost the same as 106.9 kg.



L'Hôpital's rule

If

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \quad or \quad \pm\infty$$

And there exists

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

And

$$g'(x) \neq 0$$

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

L'Hôpital's rule - Example

Q. Evaluate

$$\lim_{x \to 0} \frac{\sin x}{x}$$

If we sub in our limit:

$$\frac{\sin(0)}{0} = \frac{0}{0}$$

This answer is not defined so we will use L'Hôpital's rule to help solve this particular limit.

L'Hôpital's rule - Example

$$\lim_{x \to 0} \frac{\sin x}{x}$$

$$= \lim_{x \to 0} \frac{\cos x}{1} \quad \text{[Differentiate above and below]}$$

$$= \frac{\cos(0)}{1} = \frac{1}{1} = 1 \qquad \text{[Sub in limit]}$$

Thus

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$