Chapter 2

Integration

2.1 The indefinite integral

2.1.1 Definition

Let f be a function defined from $\mathbb{R}$ into $\mathbb{R}$. Then we can define the function g as the derivative of f:

$$g(x) = f'(x), \quad \forall x \in \mathbb{R}.$$

This defines a transform which to any function associates another function as its derivative.

The reciprocal transform would associate to a function f a function F such that the derivative of the latter function is f, i.e.

$$F'(x) = f(x), \quad \forall x \in \mathbb{R}.$$

This transform is called the anti derivative transform and F is called the anti derivative of f and is usually denoted by:

$$F = \int f(x)dx.$$

2.1.2 Examples

(i) $F(x) \int x^n dx$:

The function F has to satisfy $F'(x) = x^n$. If we derive ax^m, we get amx^{m-1}. Then, to get the answer, one has to find a and m such that $m - 1 = n$ and $am = 1$.

Then, $m = n + 1$ and $a(n + 1) = 1$. Here we have to face 2 cases: if $n \neq -1$, then $a = \frac{1}{n+1}$ and $F(x) = \frac{1}{n+1}x^{n+1}$ is a solution. If $n = -1$, then we cannot find a such that $a(n+1) = 1$ because $n+1 = 0$. In that case the solution is given by the ln function:

$$\int \frac{1}{x} dx = \ln(x),$$

because $\frac{d\ln(x)}{dx} = \frac{1}{x}$.

We can remark that if F is an anti derivative of f, then for any constant $c \in \mathbb{R}$, $F + c$ is an anti-derivative of f. In fact, if $\frac{dF(x)}{dx} = f(x)$, then $\frac{dF(x)+c}{dx} = f(x)$.

The anti derivative, or indefinite integral, is defined up to an additive constant.

Therefore, we can write:

$$\frac{d\ln(x)}{dx} = \frac{1}{x} + c, \qquad c \in \mathbb{R}.$$

(ii) $\int \sin(x)dx$:
Since $\frac{d(-\cos(x))}{dx} = \sin(x)$, we have:

$$\int \sin(x)dx = -cos(x) + c, \qquad c \in \mathbb{R}.$$

(iii) $\int e^x dx$:
Since $\frac{de^x}{dx} = e^x$, we have:

$$\int e^x dx = e^x + c, \qquad c \in \mathbb{R}.$$

(iv) $\int e^{ax} dx$:
From $\frac{d\alpha e^{\beta x}}{dx} = \alpha\beta e^{\beta x}$, we deduce that in order to get e^{ax} on the right-hand side, we have to satisfy $\alpha\beta = 1$ and $\beta = a$, i.e. $\alpha = \frac{1}{a}$. Then we have:

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c, \qquad c \in \mathbb{R}.$$

(v) $\int \cos(ax)dx$:
From the previous examples, we can deduce that:

$$\int \cos(ax)dx = \frac{\sin(ax)}{a} + c, \qquad c \in \mathbb{R}.$$

In general, all indefinite integrals of elementary functions can be treated in a similar way and can be found in the tables.

2.1.3 Properties

The indefinite integral is a linear transform, which means that for any functions f and g and any real constant α, the following properties hold:

(i)
$$\int \alpha f(x)dx = \alpha \int f(x)dx.$$

(ii)
$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx.$$

This property is very useful, since it allows us to determine easily indefinite integrals of linear combinations of elementary functions.

$$\begin{aligned}\int \left(3x^5 - 7x^2 + e^{2x} - \frac{3}{x}\right)dx =& 3\int x^5 dx - 7\int x^2 dx + \int e^{2x}dx - 3\int \frac{1}{x}dx\\ =& \frac{3}{6}x^6 - \frac{7}{3}x^3 + \frac{1}{2}e^{2x} - 3\ln(x)\end{aligned}$$

Remark 2.2. *The above properties apply only for linear combinations of functions (sums and multiplication by constants). In any case the following integral:*

$$\int e^x \sin(x)dx$$

can be transformed into

$$e^x \int \sin(x)dx, \quad \text{or} \quad \int e^x dx \int \sin(x)dx.$$

This is COMPLETELY FALSE!!

2.3 Usual methods of integration

2.3.1 Integration by substitution

Many integrals can be solved using an appropriate substitution. There is no universal rule to determine whether we should use a substitution and which substitution should be used.

We can see how the substitution method works with the following examples:

(i)

$$I = \int \frac{dx}{\sqrt{a^2 - x^2}},$$

where a is some real constant.

Here we introduce the variable u as:

$$x = a\sin(u).$$

Then

$$\frac{dx}{du} = a\cos(u),$$

and

$$dx = a\cos(u)du.$$

We also have:

$$\begin{aligned}\sqrt{a^2 - x^2} =& \sqrt{a^2 - a^2\sin(u)}\\ =& \sqrt{a^2\cos^2(u)}\\ =& |a\cos(u)|.\end{aligned}$$

If, in addition, we suppose that $a > 0$, we can prove that $\cos(u) > 0$, and then:

$$\sqrt{a^2 - x^2} = a\cos(u).$$

Finally, we get:

$$\begin{aligned}I =& \int \frac{a\cos(u)du}{a\cos(u)}\\ =& \int du\\ =& u\\ =& \sin^{-1}\left(\frac{x}{a}\right) + c\end{aligned}$$

(ii)

$$I = \int \frac{dx}{\sqrt{2 + 3x}}.$$

Here we introduce the variable u as:

$$u = 2 + 3x.$$

Then:

$$du = 3dx,$$

or

$$dx = \frac{1}{3}du.$$

So,

$$\begin{aligned} I = &\int \frac{\frac{1}{3}du}{\sqrt{u}} \\ =&\frac{1}{3}\int u^{-\frac{1}{2}}du + c \\ =&\frac{1}{3}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ =&\frac{2}{3}\sqrt{2+3x} + c. \end{aligned}$$

(iii)

$$I = \int \frac{\cos(\theta)}{\sin^3(\theta)}d\theta.$$

Here we introduce the variable u as $u = \sin(\theta)$, then we have:

$$du = \cos(theta)d\theta,$$

and

$$\begin{aligned} I = &\int \frac{du}{u^3} \\ =&\frac{1}{-2}u^{-2} + c \\ =&-\frac{1}{2\sin^2(\theta)} + c. \end{aligned}$$

A shortened version of this is:

$$\begin{aligned} I = &\int \frac{d\big(sin(\theta)\big)}{\sin^3(\theta)} \\ =&\int \sin^{-3}(\theta)d\big(\sin(\theta)\big) \\ =&\frac{1}{-2}\sin^{-2}(\theta) + c \\ =&-\frac{1}{2\sin^2(\theta)}. \end{aligned}$$

2.3.2 Standard integrals

Integrals of standard functions can be found on tables, but in many cases, one has to transform the integral into a standard form before applying the table results.

A standard result is the following:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$

The following example shows how we transform the integral $I = \int \frac{dx}{5x^2+7}$ into a standard form in order to apply the previous result.

$$\begin{aligned} I &= \int \frac{dx}{5x^2+7} \\ &= \frac{1}{5}\int \frac{1}{x^2+\frac{7}{5}} \\ &= \frac{1}{5}\int \frac{1}{x^2+\left(\sqrt{\frac{7}{5}}\right)^2} \\ &= \frac{1}{5}\frac{1}{\sqrt{\frac{7}{5}}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{7}{5}}}\right) \\ &= \frac{1}{\sqrt{35}}\tan^{-1}\left(\sqrt{\frac{5}{7}}x\right) \end{aligned}$$

2.3.3 Trigonometric fractions

In the case of integrals of polynomials and fractions of trigonometric functions of the variable x, the substitution by

$$t = \tan\left(\frac{x}{2}\right)$$

can sometimes give the solution.

With this substitution, we have:

$$\tan(x) = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1-t^2},$$

$$\sin(x) = \frac{2t}{1+t^2},$$

$$\cos(x) = \frac{1-t^2}{1+t^2},$$

and

$$\frac{dt}{dx} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right),$$

which means that

$$dx = \frac{2dt}{1+t^2}.$$

We can use this method to solve $I = \int \frac{dx}{1+2\cos(x)}$. If we define t as

$$t = \tan\frac{x}{2},$$

then, using the previous formulas, we get

$$\begin{aligned}
I =& \int \frac{2dt}{(1+t^2)\left(1+2\frac{1-t^2}{1+t^2}\right)} \\
=& \int \frac{2dt}{3-t^2} \\
=& 2\int \frac{dt}{(\sqrt{3})^2 - t^2} && \text{this is in standard form, see table} \\
=& 2\frac{1}{\sqrt{3}} \tan\cancel{h}^{-1}\left(\frac{t}{\sqrt{3}}\right) + c \\
=& \frac{2}{\sqrt{3}} \tan\cancel{h}^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + c
\end{aligned}$$

Alternatively, we can write:

$$\begin{aligned}
\frac{2}{(\sqrt{3})^2 - t^2} =& \frac{2}{(\sqrt{3}-t)(\sqrt{3}-t)} \\
=& \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}-t} + \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}+t}.
\end{aligned}$$

Then,

$$\begin{aligned}
I =& \int \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}-t}dt + \int \frac{\frac{1}{\sqrt{3}}}{\sqrt{3}+t}dt \\
=& -\frac{1}{\sqrt{3}}\ln(\sqrt{3}-t) + \frac{1}{\sqrt{3}}\ln(\sqrt{3}+t) \\
=& \frac{1}{\sqrt{3}}\ln\left(\frac{\sqrt{3}+t}{\sqrt{3}-t}\right)
\end{aligned}$$

2.3.4 Completing the square

This method consists in completing $ax^2 + bx$ by the right constant in order to obtain the square formula

$$a\left(x + \frac{b}{2a}\right)^2.$$

Suppose we have to solve the following integral:

$$I = \int \frac{dx}{2x^2 + 2x + 3},$$

then

$$\begin{aligned} I =& \frac{1}{2}\int \frac{dx}{x^2 + x + \frac{3}{2}} \\ =& \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{3}{4}} \\ =& \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{5}{4}} \end{aligned}$$

Now, if we introduce u by $u = x + \frac{1}{2}$, then

$$\begin{aligned} I =& \frac{1}{2}\int \frac{du}{u^2 + \left(\frac{\sqrt{5}}{2}\right)^2} && \text{this is a standard form} \\ =& \frac{1}{2}\frac{1}{\frac{\sqrt{5}}{2}}\tan^{-1}\frac{u}{\frac{\sqrt{5}}{2}} \\ =& \frac{1}{\sqrt{5}}\tan^{-1}\frac{2x+1}{\sqrt{5}} \end{aligned}$$

2.3.5 Integration by parts

If u and v are two functions, then the derivative of the product uv is given by the following formula:

$$(uv)' = u'v + uv'.$$

Now, we integrate this identity, because of the linearity of the integral, we get:

$$\int (uv)'dx = \int u'vdx + \int uv'dx.$$

Suppose now that we have to solve the integral of a particular function f that can be written as a product of a function u and the derivative of a function v, i.e.

$$f = uv'.$$

Then, we can easily deduce from the previous formula that:

$$\begin{aligned}\int f dx = & \int uv' dx \\ = & \int (uv)' dx - \int u'v dx \\ = & uv - \int u'v dx\end{aligned}$$

This formula can be summarized as follows:

$$\int u dv = uv - \int v du.$$

Example 1

To solve

$$I = \int x \cos x dx,$$

we define $u = x$ and $\cos x dx = dv$. Then, we have $du = dx$ and $v = \int \cos x dx = \sin x$. Now if we substitute these values in the previous formula, we get:

$$\begin{aligned}I = & x \sin x - \int \sin x dx \\ = & x \sin x + \cos x\end{aligned}$$

Example 2

Sometimes, we need integrate by parts the new integral as in the following example:

$$J = \int x^2 \sin x dx$$

Define $u = x^2$ and $\sin x dx = dv$. Then, we have $du = 2xdx$ and $v = \int \sin x dx = -\cos x$. Then:

$$\begin{aligned} J =& -x^2 \cos x - \int 2x(-\cos x)dx \\ =& -x^2 \cos x + 2\int x \cos x dx \\ =& -x^2 \cos x + 2I \\ =& -x^2 \cos x + 2(x \sin x + \cos x) \\ =& (-x^2 + 2)\cos x + 2x \sin x \end{aligned}$$

After the first integration by parts, we arrived to the result $J = x^2 \cos x + 2\int x \cos x dx$, then we needed to do another integration by parts to solve $\int x \cos x dx$ (which has been done in the previous example).

Example 3

In other cases, integration by parts leads back to the original integral. This can allow to solve the integral by solving a simple algebraic equation.

$$K = \int e^x \sin x dx.$$

Define $u = e^x$ and $dv = \sin x dx$. It follows that $du = e^x dx$ and $v = -\cos x$. Then:

$$\begin{aligned} K =& -e^x \cos x - \int (-\cos x)e^x dx \\ =& -e^x \cos x + \int \cos x e^x dx. \end{aligned}$$

Now define again $u = e^x$ and $dv = \cos x dx$. It follows that $du = e^x dx$ and $v = \sin x$. Then:

$$\begin{aligned} \int \cos x e^x dx =& e^x \sin x - \int e^x \sin x dx \\ =& e^x \sin x - K \end{aligned}$$

Now substituting this in the previous equations we get the following algebraic equation:

$$K = -e^x \cos x + e^x \sin x - K,$$

or equivalently

$$2K = -e^x \cos x + e^x \sin x,$$

which means that:

$$K = \frac{1}{2}e^x(\sin x - \cos x).$$