Mathematics for Physical Sciences II
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# Problem Sheet 8: Differential equations and applications

1. Verify that each of the following functions is a solution of the given differential equation
   (a) $f(x) = \mathrm{e}^{-x} + \mathrm{e}^{-\frac{3}{2}x}$ is a solution of $2\frac{df}{dx} + 3f(x) = \mathrm{e}^{-x}$;
   (b) $f(x) = -\frac{1}{x+1}$ is a solution of $\frac{df}{dx} = f^2(x)$.

2. Solve the following separable differential equations

$$\text{(i) } \frac{dy}{dx} = \frac{x}{y^2}; \quad \text{(ii) } \frac{dy}{dx} = y^2x^3;$$
$$\text{(iii) } \frac{dy}{dx} = \frac{x^2+2}{y}; \quad \text{(iv) } \frac{dy}{dx} = y^2 - 3y + 2.$$

3. If 16 grams of a radioactive substance were present at time $t = 1$ year and 2 grams were present at time $t = 4$ years, how much was present initially ($t = 0$) and what is the half-life of the substance?

4. The **carbon-dating technique** is based on the fact that, in a living organism, there is a constant amount of $C^{14}$, balanced by decay and absorbtion. This amount is denoted by $N_0$. When the organism dies, absorbtion stops and the amount of carbon $N(t)$ decays, according to the equation $\frac{dN}{dt} = -kN(t)$.

   The half-life of $C^{14}$ (carbon-14) has been measured to be 5730 years. What is the age of a human bone for which only 10% of its original carbon-14 is left?

5. A cup of tea is at 70°C but after 10 minutes has cooled to 50°. If the ambient temperature is 25°C find the total time required for the temperature to drop to 30°C.

6. A detective called to the scene of a murder measures the temperature of the victim to be 29°C at 3 p.m. and 27°C at 3.30 p.m. Assuming that the normal body temperature is 37°C and that the temperature of the surrounding medium is a constant 21°C, use Newton's law of cooling to estimate the time of death.

7. Solve the following first-order linear differential equations.

$$\text{(i) } x\frac{df}{dx} + f(x) = \mathrm{e}^x; \qquad \text{(ii) } \frac{df}{dx} + 2f(x) = 3,\ f(0) = 1; \qquad \text{(iii) } 2\frac{df}{dx} = \mathrm{e}^{\frac{x}{2}} + f(x);$$
$$\text{(iv) } \frac{df}{dx} + xf(x) = x,\ f(0) = -6; \qquad \text{(v) } \frac{dy}{dx} = 1 + x + y + xy;$$
$$\text{(vi) } \frac{df}{dx} + \frac{f(x)}{x+5} = 4,\ f(0) = 0.$$