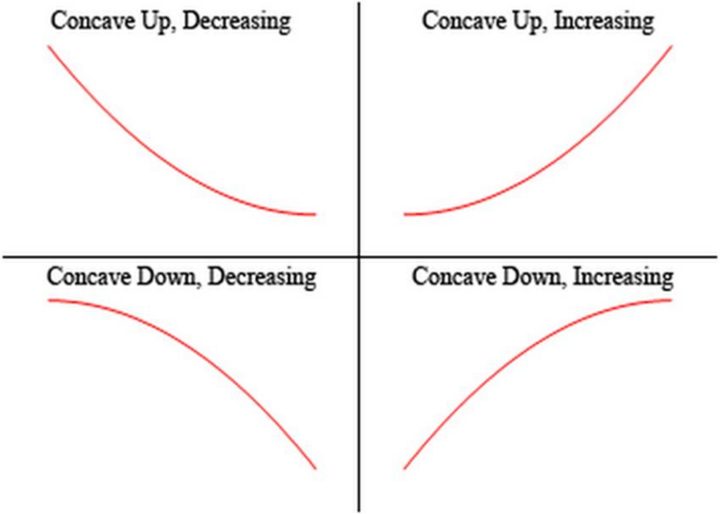
Function Concavity

In the previous section we saw how we could use the first derivative of a function to get some information about the graph of a function. In this section we are going to look at the information that the second derivative of a function can give us a about the graph of a function.

Function Concavity

- Before we do this we will need a couple of definitions out of the way.
- The main concept that well be discussing in this section is **concavity**.
- Concavity is easiest to see with a graph (well give the mathematical definition in a bit).

Function Concavity



Function Concavity

So a function is concave up if it opens up and the function is concave down if it opens down. Notice as well that concavity has nothing to do with increasing or decreasing. A function can be concave up and either increasing or decreasing. Similarly, a function can be concave down and either increasing or decreasing. Its probably not the best way to define concavity by saying which way it opens since this is a somewhat nebulous definition. Here is the mathematical definition of concavity.

Function Concavity

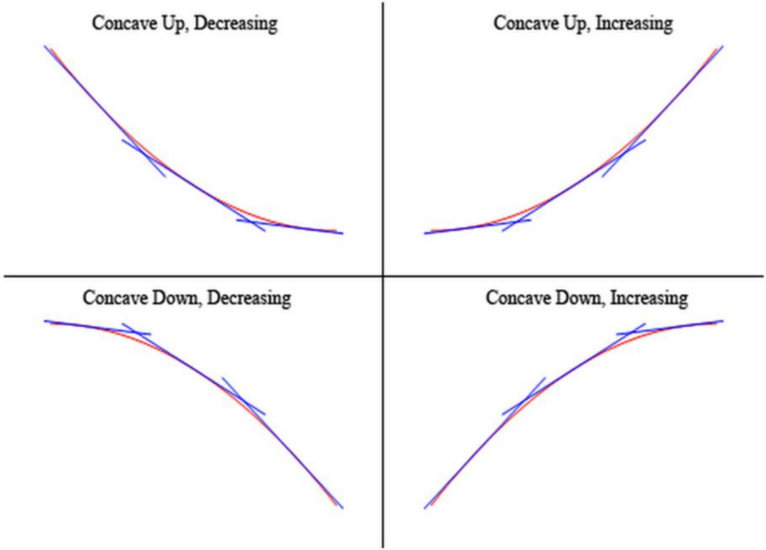
Definition 1 Given the function then

- is **concave up** on an interval **I** if all of the tangents to the curve on **I** are below the graph of .
- is **concave down** on an interval **I** if all of the tangents to the curve on **I** are above the graph of .

Function Concavity

To show that the graphs above do in fact have concavity claimed above here is the graph again (blown up a little to make things clearer).

Function Concavity



Function Concavity

- So, as you can see, in the two upper graphs all of the tangent lines sketched in are all below the graph of the function and these are concave up.
- In the lower two graphs all the tangent lines are above the graph of the function and these are concave down.

Function Concavity

- Again, notice that concavity and the increasing/decreasing aspect of the function is completely separate and do not have anything to do with the other.
- This is important to note because students often mix these two up and use information about one to get information about the other.

Shape of a Curve

Theres one more definition that we need to get out of the way.

Definition 2

- A point is called an inflection point if the function is continuous at the point and the concavity of the graph changes at that point.

Function Concavity

- Now that we have all the concavity definitions out of the way we need to bring the **second derivative** into the mix.
- We did after all start off this section saying we were going to be using the second derivative to get information about the graph.
- The following fact relates the second derivative of a function to its concavity.

Function Concavity

Fact Given the function then,

- If for all x in some interval I then is concave up on I.
- If for all x in some interval I then is concave down on I.

Function Concavity

- Notice that this fact tells us that a list of possible inflection points will be those points where the second derivative is zero or doesnt exist.
- Be careful however to not make the assumption that just because the second derivative is zero or doesnt exist that the point will be an inflection point.
- We will only know that it is an inflection point once we determine the concavity on both sides of it.
- It will only be an inflection point if the concavity is different on both sides of the point.

Function Concavity

Now that we know about concavity we can use this information as well as the increasing/decreasing information from the previous section to get a pretty good idea of what a graph should look like. Lets take a look at an example of that.

Function Concavity

Example 1

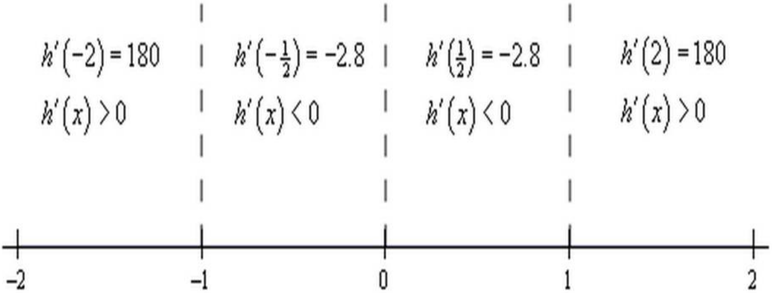
For the following function identify the intervals where the function is increasing and decreasing and the intervals where the function is concave up and concave down. Use this information to sketch the graph.

Solution Okay, we are going to need the first two derivatives so lets get those first.

Function Concavity

- Lets start with the increasing/decreasing information since we should be fairly comfortable with that after the last section.
- There are three critical points for this function : , , and .
- Below is the number line for the increasing/decreasing information.

Function Concavity



Function Concavity

So, it looks like weve got the following intervals of increasing and decreasing.
Note that from the first derivative test we can also say that is a relative maximum and that is a relative minimum. Also is neither a relative minimum or maximum.

Function Concavity

Now lets get the intervals where the function is concave up and concave down. If you think about it this process is almost identical to the process we use to identify the intervals of increasing and decreasing. This only difference is that we will be using the second derivative instead of the first derivative.

Function Concavity

The first thing that we need to do is identify the possible inflection points. These will be where the second derivative is zero or doesnt exist. The second derivative in this case is a polynomial and so will exist everywhere. It will be zero at the following points.

Function Concavity

As with the increasing and decreasing part we can draw a number line up and use these points to divide the number line into regions. In these regions we know that the second derivative will always have the same sign since these three points are the only places where the function may change sign. Therefore, all that we need to do is pick a point from each region and plug it into the second derivative.

Function Concavity

The second derivative will then have that sign in the whole region from which the point came from

Here is the number line for this second derivative.

Function Concavity

So, it looks like weve got the following intervals of concavity.

Function Concavity

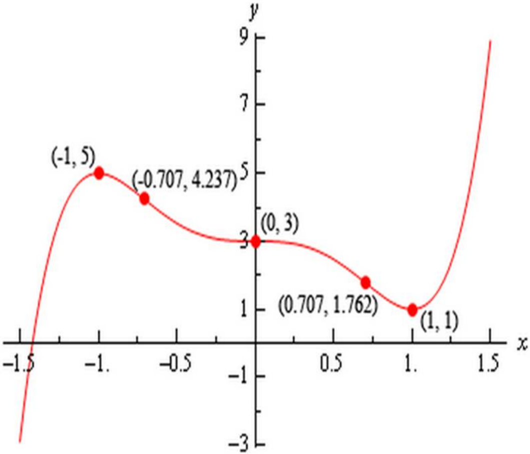
This also means that
are all inflection points.

Function Concavity

All this information can be a little overwhelming when going to sketch the graph. The first thing that we should do is get some starting points. The critical points and inflection points are good starting points. So, first graph these points. Now, start to the left and start graphing the increasing/decreasing information as we did in the previous section when all we had was the increasing/decreasing information. As we graph this we will make sure that the concavity information matches up with what were graphing.

Function Concavity

Using all this information to sketch the graph gives the following graph.



Function Concavity

We can use the previous example to illustrate another way to classify some of the critical points of a function as relative maximums or relative minimums.

Function Concavity

Notice that is a relative maximum and that the function is concave down at this point. This means that must be negative. Likewise, is a relative minimum and the function is concave up at this point. This means that must be positive.

As well see in a bit we will need to be very careful with . In this case the second derivative is zero, but that will not actually mean that is not a relative minimum or maximum. Well see some examples of this in a bit, but we need to get some other information taken care of first.

Function Concavity

It is also important to note here that all of the critical points in this example were critical points in which the first derivative was zero and this is required for this to work. We will not be able to use this test on critical points where the derivative doesnt exist.

Function Concavity

- Here is the test that can be used to classify some of the critical points of a function.
- The proof of this test is in the Proofs From Derivative Applications section of the Extras chapter.

Second Derivative Test

Second Derivative Test

Suppose that is a critical point of such that and that is continuous in a region around . Then,

- If then is a relative maximum.
- If then is a relative minimum.
- If then can be a relative maximum, relative minimum or neither.

Second Derivative Test

The third part of the second derivative test is important to notice. If the second derivative is zero then the critical point can be anything. Below are the graphs of three functions all of which have a critical point at , the second derivative of all of the functions is zero at and yet all three possibilities are exhibited.

Second Derivative Test

The first is the graph of . This graph has a relative minimum at .
Next is the graph of which has a relative maximum at .
Finally, there is the graph of and this graph had neither a relative minimum or a relative maximum at .

Second Derivative Test

So, we can see that we have to be careful if we fall into the third case. For those times when we do fall into this case we will have to resort to other methods of classifying the critical point. This is usually done with the first derivative test.

Lets go back and relook at the critical points from the first example and use the Second Derivative Test on them, if possible.