Question 1 part A i

$$(u_2(t) - u_\infty(t)) \times [t-2]$$

$u_\infty(t)$=0

$$g(t) \text{ in the form } u_a(t) \times f(t-a) \text{i.e. in the form } u_2(t) \times f(t-2)$$

- $f(t-2) = t-2$ therefore $f(t) = t$.
- Also $a = 2$
- We will use Table Entry 17.

$$F(s) = \frac{1}{s^2}$$

$$G(s) = e^{-as}F(s) = \frac{e^{-2s}}{s^2}$$

Question 1 part A ii

$$g(t) = [u_o(t) - u_3(t)]\, sin(\pi t) = [1 - u_3(t)]sin(\pi t)$$

- We need to re-express $sin(\pi t)$ in form $f(t-3)$
- Remark: t = (t-3)+3
- $sin(\pi t) = sin(\pi(t-3) + 3\pi)$
- We can use the trigonometric idenity sin(A+B) =sinA cos B + cosA sin B
- We will specifically look at sin B and cos B, i.e. sin (3π) and cos (3π).
- sin (3π) = 0 and cos (3π) = -1.
- Therefore
- $sin(\pi t) = sin(\pi(t-3) + 3\pi) = -sin(\pi(t-3)$

$$g(t) = sin(\pi t) + u_3(t)sin(\pi(t-3)$$

- The first term is straightforward
- The second term is the first term shifted by 3 time units.

Question 1 part A iii

The base function B(s)

$$\int_0^p f(t)e^{-st}dt$$

Can be re-expressed as:

$$\int_0^1 5e^{-st}dt + \int_1^2 4e^{-st}dt$$

$$5\int_0^1 e^{-st}dt + 4\int_1^2 e^{-st}dt$$

$$\int e^{-st}dt = \frac{e^{-st}}{-s}$$

This will give us the Laplace transform of the Base function.Another way of solving this:

$$(u_0(t) - u_1(t)) \times 5 + (u_1(t) - u_2(t)) \times 4 = 5u_0(t) - 5u_1(t) + 4u_1(t) - 4u_2(t)$$

$$= 5u_o(t) - u_1(t) - 4u_2(t) = 5 - u_1(t) - 4u_2(t)$$

The Laplace transform of the Base function is

$$B(s) = \frac{5}{s} - \frac{e^{-s}}{s} - \frac{4e^{-2s}}{s}$$

N.B. This is Quadratic in nature

$$B(s) = \frac{(4e^{-s} - 5)(e^{-s} + 1)}{s}$$

The Laplace transform of the given function is

$$G(s) = \frac{B(s)}{1 - e^{-sp}} = \frac{B(s)}{1 - e^{-2s}}$$

The denominator is quadratic in nature too. (difference of squares)

$$G(s) = \frac{(4e^{-s} - 5)(e^{-s} + 1)}{(s)(1 - e^{-s})(1 + e^{-s})} = \frac{4e^{-s} - 5}{(s)(1 - e^{-s})}$$

Partial Fraction Expansion may help simplify further

Question 1 part b

Inverse Laplace Transforms :
Find the inverse Laplace transforms of the following 2,3,3

Question 1 part b i

$$\frac{2s-2}{s^2-s-6} = \frac{2s-2}{(s-3)(s+2)} = \frac{2s}{(s-3)(s+2)} - \frac{2}{(s-3)(s+2)}$$

Using Table Entries 8 and 9, with a=3 and b=-2. solve on the board.

Question 1 part b ii

$$\frac{e^{-s}}{s^2+4s+4} = \frac{e^{-s}}{(s+2)^2}$$

- Laplace transform is in form $e{-}as \times F(S)$, with $a = 1$
- See Table Entry 17 (between Heaviside and Ramp Functions)
- First we find the inverse Laplace Transform of F(S).

$$F(S) = \frac{1}{(s+2)^2}$$

- $F(s)$ is in form

  $$F(S) = k\frac{n!}{(s-a)^{n+1}}$$

  Necessarily $n = 1$, $a = -2$ and $k = 1$ k is not necessary and we will drop it.
- The inverse laplace transform of F(S) is therefore

$$\mathcal{L}^{-1}[F(s)] = t^n e^{at}$$

$$f(t)\mathcal{L}^{-1}[\frac{1}{(s+2)^2}] = te^{-2t}$$

- Now to find $g(t)$

$$g(t) = u_a(t) \times f(t-a) = u_1(t) \times f(t-1) = u_1(t) \times (t-1)e^{-2t-2}$$

Question 1 part b iii

$$G(s) = tan^1(s+2)$$

- Q1 part b iii usually employs table entry 18

Convolution

Notation:

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

We can compute $(f * g)(t)$, the convolution of two functions $f(t)$ and $g(t)$, by following these steps:

- Get the Laplace transforms of the two component functions : $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$
- Multiply these two Laplace transforms: $F(s) \times G(s)$
- Find the inverse Laplace transform of the product: $\mathcal{L}^{-1}[F(s) \times G(s)]$

Question 2

- Differential Equations
- Integral Equations (using Convolution)

Question 2 Part A

- Table Entry 12: $\mathcal{L}[y'(t)] = sY(s) - y(0)$
- Table Entry 13: $\mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0)$
- Be Mindful of Boundary Conditions. In most past papers, they are zero, but not this year.
- $y(0) = 1$ $y'(0) = 0$

$$y'' + 4y = 5e^{-t}$$

- Finding the Laplace Transform of everything

$$(s^2Y(s) - s) - (4Y(s)) = \frac{5}{s+1}$$

- Simplifying the RHS

$$(s^2 - 4)Y(s) = \frac{5}{s+1} + s = \frac{5}{s+1} + \frac{s^2+s}{s+1} = \frac{s^2+s+5}{s+1}$$

$$(s^2 - 4)Y(s) = \frac{s^2+s+5}{s+1}$$

$$Y(s) = \frac{s^2+s+5}{(s+1)(s-2)(s+2)}$$

Partial Fraction Expansion to solve

Question 2 Part B

$$y(t) = sin3t - 2\int_0^t cos3(t-u)y(u)du$$

$$y(t) = sin3t - 2\left[cos(3t) * y(t)\right]$$

Find the Laplace transform for both sides

$$Y(s) = \frac{3}{s^2+9} - 2\left[\frac{s}{s^2+9} \times Y(s)\right]$$

$$Y(s) = \frac{3 - 2sY(s)}{s^2 + 9}$$

Cross Multiplication

$$Y(s)(s^2 + 9) = 3 - 2sY(s)$$

$$Y(s)(s^2 + 2s + 9) = 3$$

$$Y(s) = \frac{3}{s^2 + 2s + 9} = \frac{3}{(s + 1)^2 + 8}$$

Let us look at this matric is form (see Table Entry 17)
$F(s - a)$i.e. F(s+1)$\frac{3}{(s+1)^2+8}$
Necessarily

$$F(s) = \frac{3}{s^2 + 8}$$

$f(t) = t - 2$

$$u_2(t) - u_\infty(t) \times [t - 2]$$

$$u_o(t) - u_3(t)sin(\pi t) = 1 - u_3(t)sin(\pi t)$$

$$u_0(t) - u_1(t) \times 5 + u_1(t) - u_2(t) \times 4 = 5u_0(t) - 5u_1(t) + 4u_1(t) - 4u_2(t)$$

$$= 5u_o(t) - u_1(t) - 4u_2(t)$$

Question 1 part ii

Inverse Laplace Transforms :
Find the inverse Laplace transforms of the following 2,3,3
$\frac{2s-2}{s^2-s+6}$ $\frac{e^{-s}}{s^2+4s+4}$ $tan^1(s+2)$
Notation:

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

We can compute $(f * g)(t)$, the convolution of two functions $f(t)$ and $g(t)$, by following these steps:

- Get the Laplace transforms of the two component functions : $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$
- Multiply these two Laplace transforms: $F(s) \times G(s)$
- Find the inverse Laplace transform of the product: $\mathcal{L}^{-1}[F(s) \times G(s)]$

Question 2

- Differential Equations
- Integral Equations (using Convolution)
- Table Entry 12: $y''(t)$
- Table Entry 13: $y''(t)$

$$y(t) = sin3t - 2\int_0^t cos3(t-u)y(u)du$$
$$y(t) = sin3t - 2\left[cos3 * y(t)\right]$$

Find the Laplace transform for both sides

$$Y(s) = \frac{3}{s^2+9} - 2\left[\frac{s}{s^2+9} \times Y(s)\right]$$