# Course notes for Science Mathematics 2 (MA4602)

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# Chapter 2

# Curve sketching

This chapter is devoted to the study of functions and their graph (curve sketching). Keeping in mind that this subject has already been studied in MA4601 (Science Mathematics 1), we will only quickly revise the main concepts of critical points of a function $f(x)$ and the study of the sign of its first derivative $f^{'}(x)$. The only new concepts introduced in this chapter are the so-called points of inflection and the study of the sign of the second derivative $f^{''}(x)$ of a function $f(x)$. The latter are used to have a better understanding of the behavior of $f(x)$ in order to be able to sketch a more precise graph of it.

## 2.1 Quick revision of MA4601: the study of the first derivative of a function

The study of the first derivative $f^{'}(x)$ of a function $f(x)$ is used to detect the critical points of $f$, including maxima and minima points. In particular, the study of the sign of $f^{'}$ gives information about where $f$ increases $\uparrow$ and where it decreases $\downarrow$, pointing out whether a critical point is a minimum, a maximum point or neither. The student can find below a quick revision of the main facts involving the study of the first derivative of a function.

### 2.1.1 Critical points: maxima and minima

It is assumed that the student is familiar with the concepts of local maxima and local minima points. We simply recall here the definition of a critical point of a function and the so-called first derivative test.

DEFINITION **2.1.1.** *A critical point for a function $f(x)$ is a point $x_0$ in the domain of $f(x)$ (i.e. where $f(x)$ is defined) such that:*

1.
$$f^{'}(x_0) = 0$$

*or*

2.
$$f^{'}(x_0) \quad \text{does not exist}$$

Remark **2.1.1.** *A local minimum or a local maximum for a function $f(x)$ is a critical point, therefore we study the critical points of $f(x)$ in order to detect its local maxima and minima points. There are three types of critical points: the local minima, the local maxima and the points of inflection, which will be new material covered in this module.*

We recall the following facts from $MA4601$.

**FIRST DERIVATIVE: INCREASING AND DECREASING FUNCTIONS**

THEOREM **2.1.2.** *If $f^{'}(x) > 0$ on an interval, then $f$ is increasing $\uparrow$ on the interval. If $f^{'}(x) < 0$ on an interval, then $f$ is decreasing $\downarrow$ on the interval.*

Therefore we have the so-called

**FIRST DERIVATIVE TEST**

Given a differentiable function

$$f : [a, b] \longrightarrow \mathbb{R},$$

suppose $x_0 \in (a, b)$ is a critical point for $f$.

1. If $f^{'}(x) > 0$, for $x \in (a, x_0)$ and $f^{'}(x) < 0$, for $x \in (x_0, b)$, then $x_0$ is a local maximum;
2. If $f^{'}(x) < 0$, for $x \in (a, x_0)$ and $f^{'}(x) > 0$, for $x \in (x_0, b)$, then $x_0$ is a local minimum.

Again we assume that the above concepts have been studied in MA4601, where the concepts we introduce next are not expected to be known by the student.

## 2.2 New concepts: the study of the second derivative of a function

The study of the second derivative $f^{''}(x)$ will give us information about the so-called concavity and convexity of the function and will also tell us what type of points are the critical points that are neither a local maximum nor a local minimum: they will be in fact called points of inflection. This is the subject of what we study next.

### 2.2.1 Concavity, convexity and points of inflection

**SECOND DERIVATIVE: THE TEST FOR CONCAVITY**

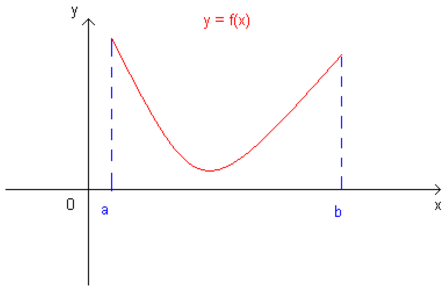


Figure 2.1: Picture showing the graph of a concave function: $f^{''}(x) > 0$ on $(a, b)$.

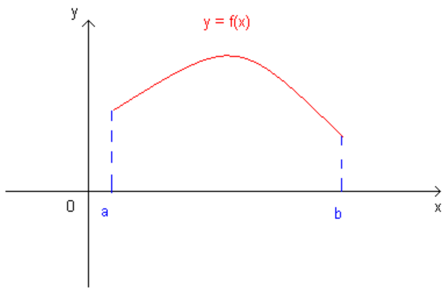


Figure 2.2: Picture showing the graph of a convex function: $f^{''}(x) < 0$ on $(a, b)$.

THEOREM **2.2.1.** *If $f^{''}(x) > 0$ on an interval, then $f$ is concave on the interval. If $f^{''}(x) < 0$ on an interval, then $f$ is convex on the interval (see above figures).*

## POINTS OF INFLECTION

DEFINITION **2.2.1.** *A point $x_0$ is a point of inflection for a function $f(x)$ if $f(x)$ changes from concave to convex of from convex to concave at $x_0$ i.e. if $f^{''}(x_0) = 0$.*

Note that there are two types of points of inflection:

1. Points of inflection $x_0$ that are critical points (see figure 2.3);

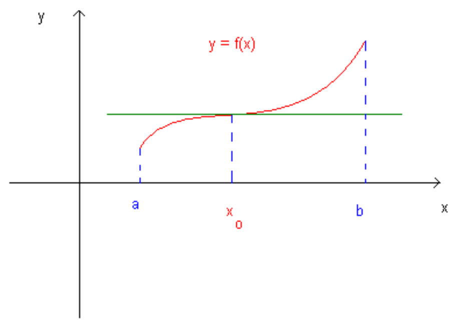


Figure 2.3: Picture showing the graph of a function $f$ with a point of inflection at $x_0$ which is a critical point: the tangent line to $y = f(x)$ at $x_0$ is horizontal ($f^{''}(x_0) = 0$ and $f^{'}(x_0) = 0$).

2. Points of inflection $x_0$ that are not critical points (see figure 2.4).

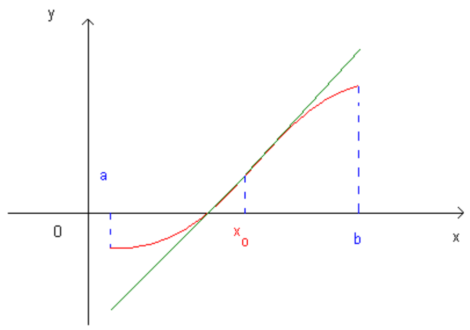


Figure 2.4: Picture showing the graph of a function $f$ with a point of inflection at $x_0$ which is not a critical point: the tangent line to $y = f(x)$ at $x_0$ is not horizontal ($f^{''}(x_0) = 0$ but $f^{'}(x_0) \neq 0$).

### 2.2.2 Critical points: minima, maxima and points of inflection

**THE SECOND DERIVATIVE TEST (for critical points)**

Given a twice differentiable function

$$f : [a, b] \longrightarrow \mathbb{R},$$

suppose $x_0 \in (a, b)$.

1. If $f^{'}(x_0) = 0$ and $f^{''}(x_0) > 0$, then $x_0$ is a local minimum;
2. If $f^{'}(x_0) = 0$ and $f^{''}(x_0) < 0$, then $x_0$ is a local maximum;
3. If $f^{'}(x_0) = 0$ and $f^{''}(x_0) = 0$, then $x_0$ is a point of inflection.

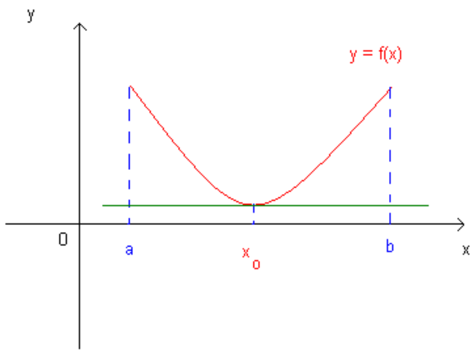


Figure 2.5: Picture showing the graph of a function $f$ with a critical point at $x_0$ which is a local minimum: $f^{'}(x_0) = 0$ (the line tangent to $y = f(x)$ at $x_0$ is horizontal) and $f^{''}(x_0) > 0$.

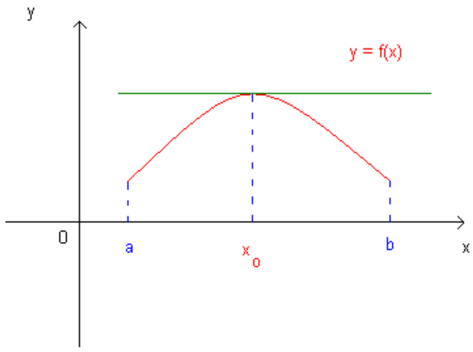


Figure 2.6: Picture showing the graph of a function $f$ with a critical point at $x_0$ which is a local maximum: $f^{'}(x_0) = 0$ (the line tangent to $y = f(x)$ at $x_0$ is horizontal) and $f^{''}(x_0) < 0$.

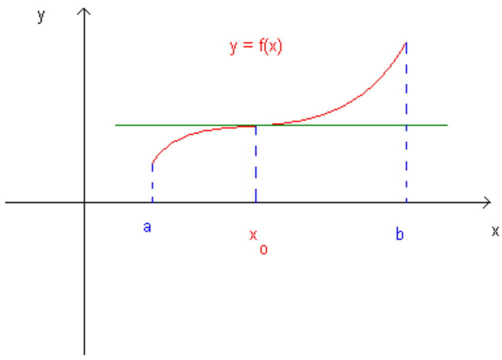


Figure 2.7: Picture showing the graph of a function $f$ with a critical point at $x_0$ which is a point of inflection: $f^{'}(x_0) = 0$ (the line tangent to $y = f(x)$ at $x_0$ is horizontal) and $f^{''}(x_0) = 0$.

## 2.3 Functions of calculus: Inverse trigonometric functions

This section is devoted to the study of the so-called Inverse Trigonometric Functions, assuming that the reader is familiar with the Trigonometric Functions. It is also assumed that the reader is familiar with concepts like $1-1$ functions and inverse of a function. The main objective of this section is to apply the concepts studied in MA4601 and so far in these notes to the study of he following functions.

### 2.3.1 The inverse of $y = \sin(x)$: $y = \arcsin(x)$

We consider the function

$$y = \sin(x)$$

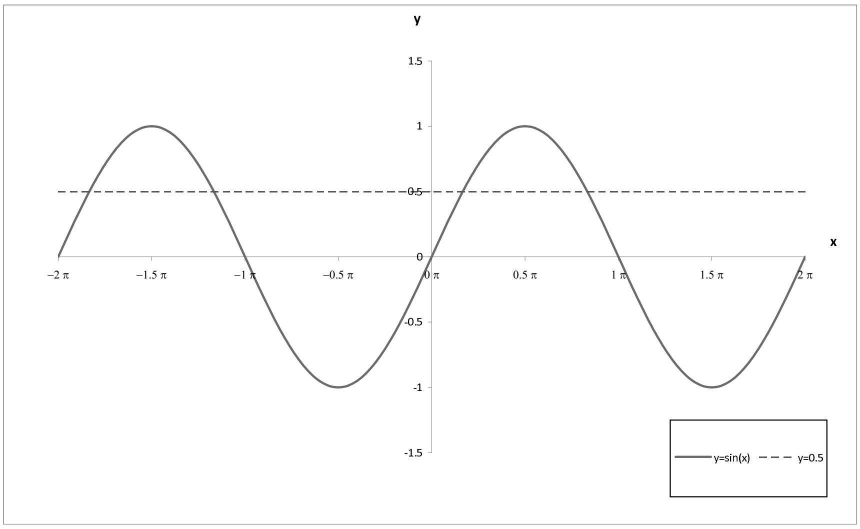


Figure 2.8: Picture showing the graph of $y = \sin(x)$ and the horizontal line $y = \frac{1}{2}$ intersecting it more than once.

and we want to invert it. This function is defined on $(-\infty, +\infty)$ but it is not $1-1$ on the entire real line: one can for example look at the graph of $y = \sin(x)$ and see that any horizontal straight line $y = k$, with $k$ any real number such that $-1 < k < 1$, intersects the graph of $y = \sin(x)$ more than once, which implies that $y = \sin(x)$ in not $1-1$ on $(-\infty, +\infty)$ (see figure 2.8).

1. Domain and Range of $y = \arcsin(x)$

   In order to invert the function $y = \sin(x)$ and determine domain and range of its inverse $y = \arcsin(x)$,we consider its restriction

   $$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1] \,.$$

   The above restriction of $y = \sin(x)$ is $1-1$, therefore invertible and the inverse is given by

   $$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] .$$

   Therefore the domain of arcsin is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

   By definition of the inverse of a function we have

   $$\begin{aligned} \arcsin(\sin(x)) &= x \\ \sin(\arcsin(x)) &= x. \end{aligned}$$

2. Derivative of $y = \arcsin(x)$

   We have

   $$\begin{aligned} \frac{d}{dy}\left(\arcsin(y)\right)\big|_{y=\sin(x)} &= \frac{1}{\frac{d}{dx}\left(\sin(x)\right)} \\ &= \frac{1}{\cos(x)} \\ &= \frac{1}{\sqrt{1-\sin^2(x)}} \\ &= \frac{1}{\sqrt{1-y^2}}. \end{aligned}$$

   Therefore, we can conclude that

   $$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}, \qquad \forall x \in (-1, 1) \,.$$

3. Graph of $y = \arcsin(x)$

   The graph of $y = \arcsin(x)$ is simply the reflection across the straight line $y = x$ of the restriction of $y = \sin(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (see figure 2.9).

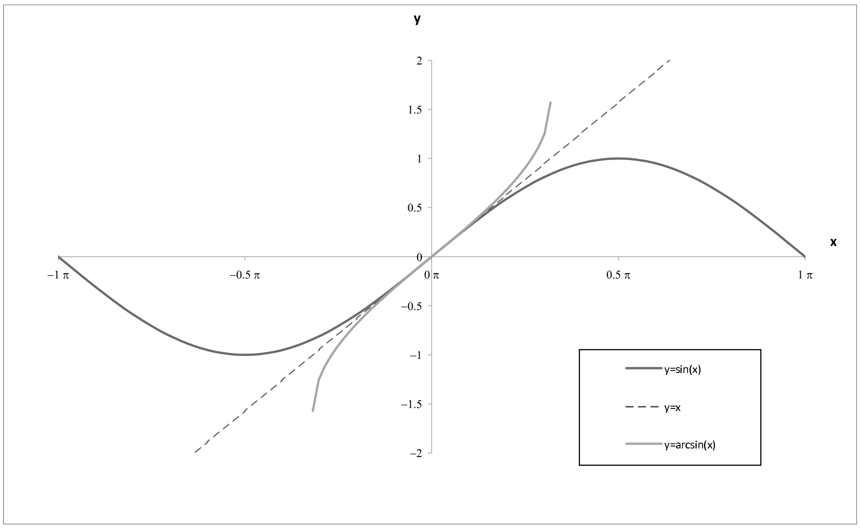


Figure 2.9: The graph of $y = \arcsin(x)$.

### 2.3.2 The inverse of $y = cos(x)$: $y = \arccos(x)$

We consider the function

$$y = \cos(x)$$

and we want to invert it. This function is defined on $(-\infty, +\infty)$ but, like $y = \sin(x)$, it is not $1 - 1$ on the entire real line: like for function $y = \sin(x)$, one can look at the graph of $y = \sin(x)$ and see that any horizontal straight line $y = k$, with $k$ any real number such that $-1 < k < 1$, intersects the graph of $y = \sin(x)$ more than once, which implies that $y = \sin(x)$ in not 1-1 on $(-\infty, +\infty)$) (see figure 2.10).

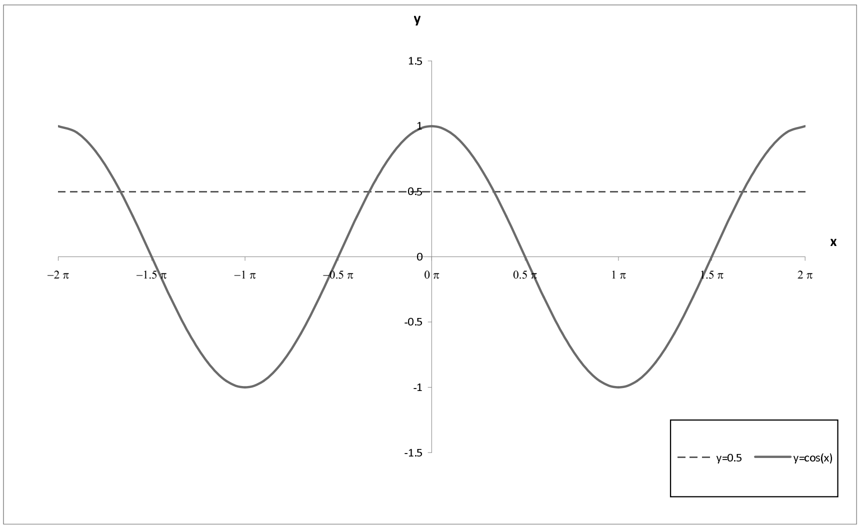


Figure 2.10: Picture showing the graph of $y = \cos(x)$ and the horizontal line $y = \frac{1}{2}$ intersecting it more than once.

1. Domain and Range of $y = \arccos(x)$

   In order to invert the function $y = \cos(x)$ and determine domain and range of its inverse $y = \arccos(x)$, we need to consider its restriction

   $$\cos : [0, \pi] \longrightarrow [-1, 1] .$$

   The above restriction of $y = \cos(x)$ is $1-1$, therefore invertible and the inverse is given by

   $$\arccos : [-1, 1] \longrightarrow [0, \pi] .$$

   Therefore the domain of arccos is $[-1, 1]$ and the range is $[0, \pi]$.

   By definition of the inverse of a function we have

   $$\begin{aligned} \arccos(\cos(x)) &= x \\ \cos(\arccos(x)) &= x. \end{aligned}$$

2. Derivative of $y = \arccos(x)$

   We have

   $$\begin{aligned} \frac{d}{dy}(\arccos(y))\,\big|_{y=\cos(x)} &= \frac{1}{\frac{d}{dx}(\cos(x))} \\ &= \frac{1}{-\sin(x)} \\ &= -\frac{1}{\sin(x)} \\ &= -\frac{1}{\sqrt{1-\cos^2(x)}} \\ &= -\frac{1}{\sqrt{1-y^2}}. \end{aligned}$$

   Therefore, we can conclude that

   $$\frac{d}{dx}(\arcsin x) = -\frac{1}{\sqrt{1-x^2}}, \qquad \forall x \in (-1, 1) .$$

3. Graph of $y = \arccos(x)$

   The graph of $y = \arccos(x)$ is simply the reflection across the straight line $y = x$ of the restriction of $y = \cos(x)$ on $[0, \pi]$ (see figure 2.11).

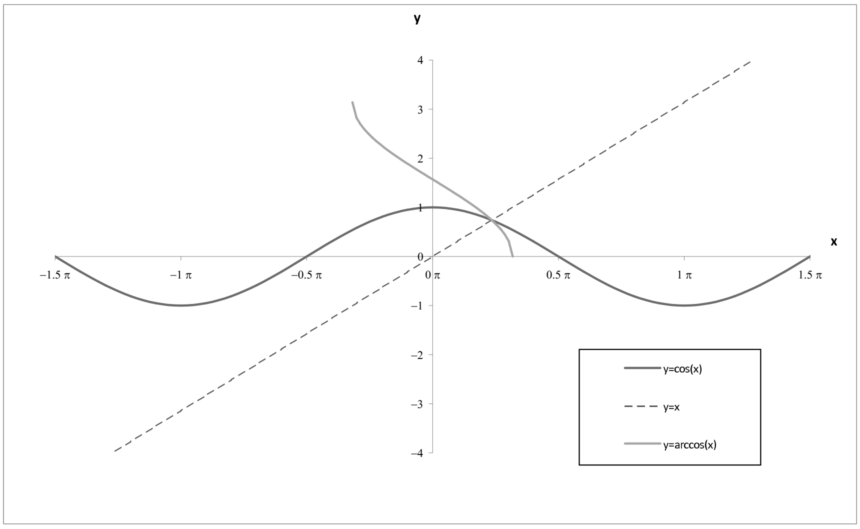


Figure 2.11: The graph of $y = \arccos(x)$.

### 2.3.3 The inverse of $y = \tan(x)$: $y = \arctan(x)$

We consider the function

$$y = \tan(x)$$

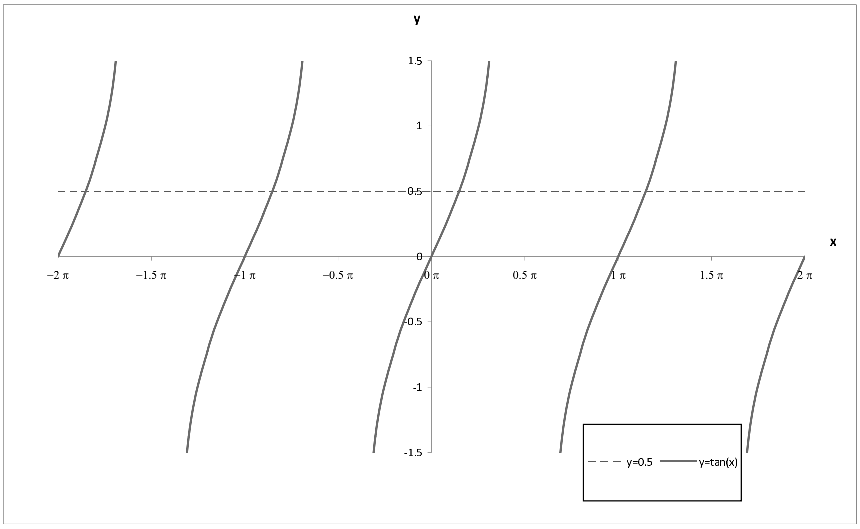


Figure 2.12: Picture showing the graph of $y = \tan(x)$ and the horizontal line $y = \frac{1}{2}$ intersecting it more than once.

and we want to invert it. This function is defined on $(-\infty, +\infty)$, with the exception of points of type $\frac{\pi}{2} + k\pi$, for $k = \pm 1, 2, 3, \ldots$ (where the function has vertical asymptotes) but it is not $1 - 1$ on its domain: once again one can look at the graph of $y = \tan(x)$ and see that any horizontal straight line $y = k$, with $k$ any real number, intersects the graph of $y = \tan(x)$ more than once, which implies that $y = \tan(x)$ in not $1 - 1$ on its domain.

1. Domain and Range of $y = \arctan(x)$

   In order to invert the function $y = \tan(x)$ and determine domain and range of its inverse $y = \arctan(x)$, we need to consider its restriction

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow (-\infty, \infty) .$$

The above restriction of $y = \tan(x)$ is $1-1$, therefore invertible and the inverse is given by

$$\arctan : (-\infty, \infty) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Therefore the domain of arctan is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

By definition of the inverse of a function we have

$$\begin{aligned} \arctan(\tan(x)) &= x \\ \tan(\arctan(x)) &= x. \end{aligned}$$

2. Derivative of $y = \arctan(x)$

   We have

$$\begin{aligned} \frac{d}{dy}(\arctan y)\ |_{y=\tan x} &= \frac{1}{\frac{d}{dx}(\tan x)} \\ &= \frac{1}{1+\tan^2 x} \\ &= \frac{1}{1+y^2}. \end{aligned}$$

   Therefore, we can conclude that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}, \qquad \forall x \in (-\infty, \infty).$$

3. Graph of $y = \arctan(x)$

   The graph of arctan is simply the reflection across the straight line $y = x$ of the restriction of $\tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (see figure 2.13).

Remark **2.3.1.** *The reader will note that we did not study the first and second derivative of the Inverse Trigonometric Functions to look for critical points, where the above functions are increasing and decreasing and where they are convex or concave : the reason for that is simply due to the fact that we did not need such information in order to plot their graph since we knew how to sketch the graph of their inverses. We next study the so-called Hyperbolic functions which require a more in-dept study in order to be able to plot their graphs.*

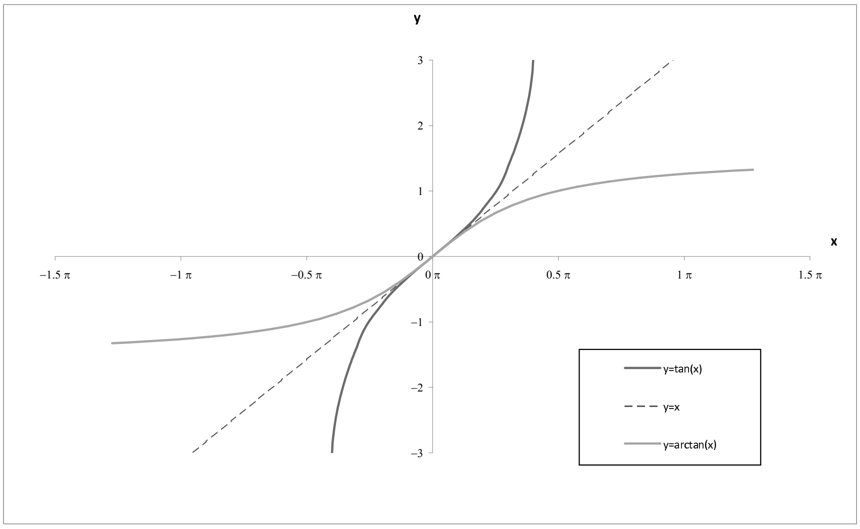


Figure 2.13: The graph of $y = \arctan(x)$.

## 2.4 Functions of calculus: Hyperbolic functions

This section is devoted to the study of the so-called Hyperbolic Functions. Here we assume that the reader is familiar with the exponential and the logarithm functions, including their derivatives. For this we refer again to the module MA4601 (Science Mathematics 1) or any calculus book as a reference.
The Hyperbolic Functions are defined in the following way

$$\begin{aligned}
\sinh(x) &= \frac{e^x - e^{-x}}{2} \\
\cosh(x) &= \frac{e^x + e^{-x}}{2} \\
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\coth(x) &= \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \\
\operatorname{sech}(x) &= \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}} \\
\operatorname{cosech}(x) &= \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}.
\end{aligned}$$

Here we will be studying functions $y = \sinh(x)$, $y = \cosh(x)$ and $y = \tanh(x)$.

### 2.4.1 The function $y = \sinh(x)$

We consider the function

$$y = \sinh(x)$$

defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

1. Domain of $y = \sinh(x)$:

   This function is defined on $(-\infty, +\infty)$, therefore the domain of $y = \sinh(x)$ is $(-\infty, \infty)$.

2. Symmetry of $y = \sinh(x)$:

   $$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh(x),$$

   therefore $y = \sinh(x)$ is an odd function which means that its graph will be symmetric across the origin of the $xy$-axes.

3. $x$ and $y$ intercepts of $y = \sinh(x)$

   $y$-intercepts:

$$\sinh(0) = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0,$$

   therefore $y = \sinh(x)$ intercepts the $y$ axis at the origin $O = (0, 0)$.

   $x$-intercepts: set $y = 0$ which gives

$$\frac{e^x - e^{-x}}{2} = 0$$

   i.e.

$$e^x - e^{-x} = 0 / \cdot e^x$$

$$e^{2x} = e^0$$

$$2x = 0$$

   i.e.

$$x = 0.$$

   Therefore $y = \sinh(x)$ intercepts the $x$ and $y$ axes at the point $O = (0, 0)$.

4. Possible asymptotes for $y = \sinh(x)$:

   The domain of $y = \sinh(x)$ is $(-\infty, +\infty)$, therefore there will not be any vertical asymptote for this function. To check whether there is any horizontal asymptote we evaluate the following limits:

$$\begin{aligned} \lim_{x\to+\infty} \frac{e^x - e^{-x}}{2} &= +\infty \\ \lim_{x\to-\infty} \frac{e^x - e^{-x}}{2} &= -\infty; \end{aligned}$$

   therefore there is no horizontal asymptote. One can check that there is no diagonal asymptote either but we will skip this.

5. First derivative of $y = \sinh(x)$:

$$\begin{aligned}
\frac{d}{dx}\sinh(x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\
&= \frac{1}{2}\left(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right) \\
&= \frac{1}{2}\left(e^x - \left(-e^{-x}\right)\right) \\
&= \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh(x).
\end{aligned}$$

(a) Critical points of $y = \sinh(x)$:

$$\frac{d}{dx}\sinh(x) = 0$$

i.e.

$$\cosh(x) = 0,$$

which gives

$$\frac{e^x + e^{-x}}{2} = 0,$$

i.e.

$$e^x + e^{-x} = 0 / \cdot e^x$$

which gives

$$e^{2x} + e^0 = 0$$

i.e.

$$e^{2x} = -1 \tag{2.4.1}$$

which is impossible since $e^{2x}$ is always positive, therefore there is no value of $x$ that satisfies equation (2.4.1), therefore there is no critical point.

(b) Sign of $\frac{d}{dx}\sinh(x)$: we study $\frac{d}{dx}\sinh(x)$ in order to understand where $y = \sinh(x)$ is increasing and where it is decreasing. Recall that the derivative of $y = \sinh(x)$ is given by

$$\frac{d}{dx}\sinh(x) = \frac{e^x + e^{-x}}{2}$$

and note that

$$\frac{e^x + e^{-x}}{2} > 0,$$

therefore

$$\frac{d}{dx}\sinh(x) > 0, \qquad \text{for all} \quad x \in (-\infty, +\infty),$$

therefore function $y = \sinh(x)$ is increasing on its entire domain $(-\infty, +\infty)$.

6. Second derivative of $y = \sinh(x)$:

$$\begin{aligned}
\frac{d^2}{dx^2}\sinh(x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{1}{2}\left(\frac{d}{dx}e^x + \frac{d}{dx}e^{-x}\right) \\
&= \frac{1}{2}\left(e^x - \left(e^{-x}\right)\right) \\
&= \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh(x).
\end{aligned}$$

(a) Points of inflection of $y = \sinh(x)$:

$$\frac{d^2}{dx^2}\sinh(x) = 0$$

i.e.

$$\frac{e^x - e^{-x}}{2} = 0,$$

i.e.

$$e^x - e^{-x} = 0 / \cdot e^x$$

which gives

$$e^{2x} - e^0 = 0$$

i.e.

$$2x = 0$$

i.e.

$$x = 0 \quad \text{is a point of inflection.}$$

(b) Sign of $\frac{d^2}{dx^2}\sinh(x)$: we study the sign of $\frac{d^2}{dx^2}\sinh(x)$ in order to see where $y = \sinh(x)$ is concave and where it is convex. Recall that the second derivative of $y = \sinh(x)$ is given by

$$\frac{d^2}{dx^2}\sinh(x) = \frac{e^x - e^{-x}}{2}$$

and note that

$$\frac{e^x - e^{-x}}{2} > 0,$$

when

$$e^x - e^{-x} > 0$$

i.e. when

$$e^x > e^{-x}$$

and the last inequality is satisfied for $x \in (0, +\infty)$. The reader can easily verify this by sketching the graphs of functions $y = e^x$ and $y = e^{-x}$ and noticing that the graph of $y = e^x$ lies above the graph of $y = e^{-x}$ for $x \in (0, +\infty)$. A function is concave where its second derivative is positive, therefore our function $y = \sinh(x)$ is concave on $(0, +\infty)$. On the other hands we have:

$$\frac{e^x - e^{-x}}{2} < 0,$$

when

$$e^x - e^{-x} < 0$$

i.e. when

$$e^x < e^{-x}$$

and the last inequality is satisfied for $x \in (-\infty, 0)$. Again the reader can easily verify this by sketching the graphs of functions $y = e^x$ and $y = e^{-x}$ and noticing that the graph of $y = e^x$ lies below the one of $y = e^{-x}$ for $x \in (-\infty, 0)$. A function is convex where its second derivative is negative, therefore our function $y = \sinh(x)$ is convex on $(0, +\infty)$.

7. Graph: Note that from the graph of $y = \sinh(x)$ we can say that the range of this function is $(-\infty, +\infty)$ (see fig. 2.14).

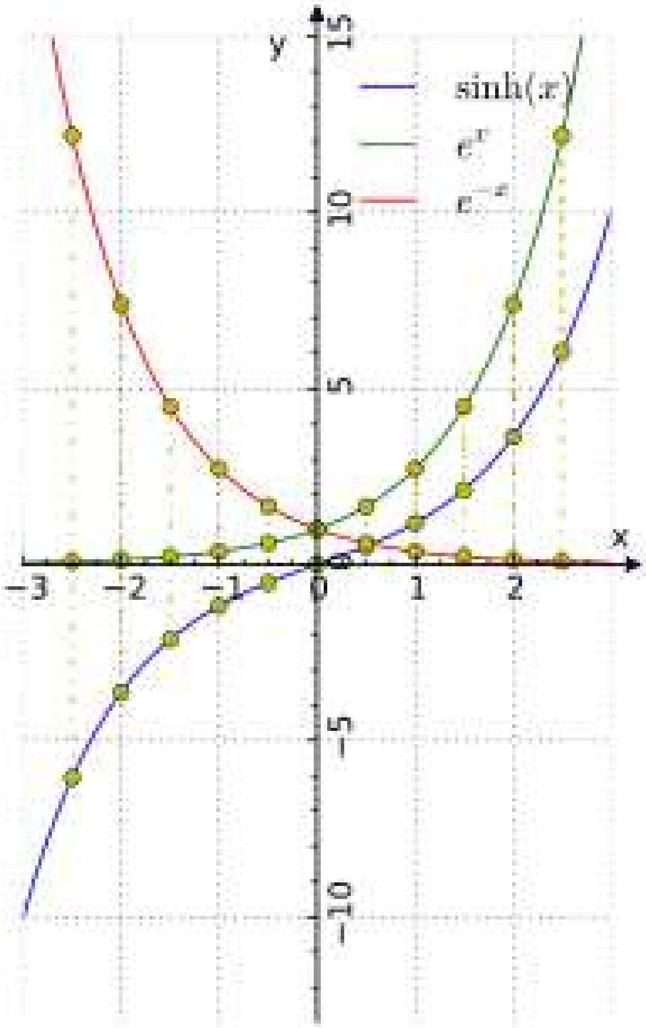


Figure 2.14: The graph of $y = \sinh(x)$ (picture published by wikipedia).

### 2.4.2 The function $y = \cosh(x)$

We consider the function

$$y = \cosh(x)$$

defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

1. Domain of $y = \cosh x$:

   This function is defined on $(-\infty, +\infty)$, therefore the domain of $y = \cosh(x)$ is $(-\infty, \infty)$.

2. Symmetry of $y = \cosh(x)$:

   $$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \frac{e^x - e^{-x}}{2} = \cosh(x),$$

   therefore $y = \cosh(x)$ is an even function which means that its graph will be symmetric across the $y$-axes.

3. $x$ and $y$ intercepts of $y = \cosh(x)$

   $y$-intercepts:

   $$\cosh(0) = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1,$$

   therefore $y = \cosh(x)$ intercepts the $y$ axis at the point $A = (0, 1)$.

   $x$-intercepts: set $y = 0$ which gives

   $$\frac{e^x + e^{-x}}{2} = 0$$

   i.e.

   $$e^x + e^{-x} = 0 / \cdot e^x$$

   $$e^{2x} = -e^0$$

   $$e^{2x} = -1 \tag{2.4.2}$$

   which is not possible since $e^{2x}$ is always positive, therefore there is no value of $x$ that satisfies equation (2.4.2), therefore function $y = \cosh(x)$ does not intercept the $x$-axis at any point.

4. Possible asymptotes for $y = \cosh(x)$:

   The domain of $y = \cosh(x)$ is $(-\infty, +\infty)$, therefore there will not be any vertical asymptote for this function. To check whether there is any horizontal asymptote we evaluate the following limits:

$$\begin{aligned}\lim_{x\to+\infty} \frac{e^x + e^{-x}}{2} &= +\infty \\ \lim_{x\to-\infty} \frac{e^x + e^{-x}}{2} &= +\infty;\end{aligned}$$

   therefore there is no horizontal asymptote. One can check that there is no diagonal asymptote either but we will skip this.

5. First derivative of $y = \cosh(x)$:

$$\begin{aligned}\frac{d}{dx}\cosh(x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{1}{2}\left(\frac{d}{dx}e^x + \frac{d}{dx}e^{-x}\right) \\ &= \frac{1}{2}\left(e^x - \left(e^{-x}\right)\right) \\ &= \sinh(x).\end{aligned}$$

   (a) Critical points of $y = \cosh(x)$:

$$\frac{d}{dx}\cosh(x) = 0$$

   i.e.

$$\sinh(x) = 0,$$

   which gives

$$\frac{e^x - e^{-x}}{2} = 0,$$

   i.e.

$$e^x - e^{-x} = 0 / \cdot e^x$$

   which gives

$$e^{2x} - e^0 = 0$$

i.e.

$$e^{2x} = e^0 \tag{2.4.3}$$

which gives

$$x = 0.$$

Therefore $x = 0$ is a critical point for our function $y = \cosh(x)$. In order to find out whether $x = 0$ is a local minimum, a local maximum or a point of inflection we study the sign of the $\frac{d}{dx}\cosh(x)$.

(b) Sign of $\frac{d}{dx}\cosh(x)$: we study $\frac{d}{dx}\cosh(x)$ in order to understand where $y = \cosh(x)$ is increasing and where it is decreasing. Recall that the derivative of $y = \cosh(x)$ is given by

$$\frac{d}{dx}\cosh(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

and from the graph of $y = \sinh(x)$ we see that this function is negative on $(-\infty, 0)$ and it is positive on $(0, +\infty)$, therefore

$$\frac{d}{dx}\cosh(x) < 0, \qquad \text{for all} \quad x \in (-\infty, 0),$$

and

$$\frac{d}{dx}\cosh(x) > 0, \qquad \text{for all} \quad x \in (0, +\infty),$$

which implies that function $y = \sinh(x)$ is decreasing on $(-\infty, 0)$ and it is increasing on $(0, +\infty)$, therefore $x = 0$ is a local minimum.

6. Second derivative of $y = \cosh(x)$:

$$\frac{d^2}{dx^2}\cosh(x) = \frac{d}{dx}\sinh(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

(a) Points of inflection of $y = \cosh(x)$:

$$\frac{d^2}{dx^2}\cosh(x) = 0$$

i.e.

$$\frac{e^x + e^{-x}}{2} = 0,$$

i.e.

$$e^x + e^{-x} = 0 / \cdot e^x$$

which gives

$$e^{2x} = -e^0$$

i.e.

$$e^{2x} = -1 \tag{2.4.4}$$

and as we noticed before there is no solution for equation (2.4.4), therefore function $y = \cosh(x)$ has no point of inflection.

(b) Sign of $\frac{d^2}{dx^2}\cosh(x)$: we study the sign of $\frac{d^2}{dx^2}\cosh(x)$ in order to understand where $y = \cosh(x)$ is concave and where it is convex. Recall that the second derivative of $y = \cosh(x)$ is given by

$$\frac{d^2}{dx^2}\cosh(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

and notice that

$$\frac{e^x + e^{-x}}{2} > 0, \qquad \text{for all} \quad x \in (-\infty, +\infty),$$

therefore function $y = \cosh(x)$ is concave on its entire domain $(-\infty, +\infty)$.

7. Graph: Note that from the graph of $y = \cosh(x)$ we can say that the range of this function is $(1, +\infty)$ (see fig. 2.15).

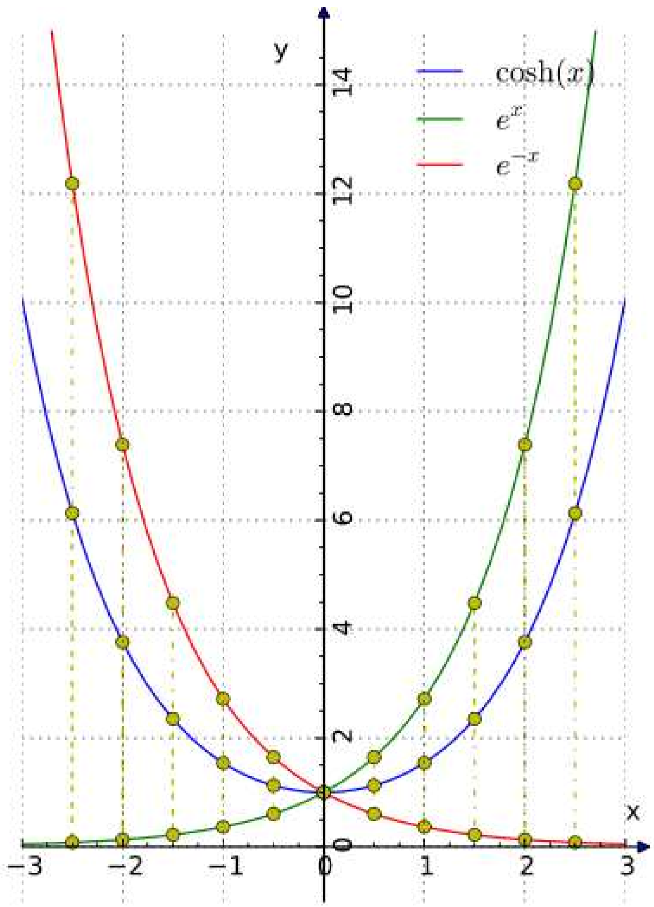


Figure 2.15: The graph of $y = \cosh(x)$ (picture published by wikipedia).

### 2.4.3 The function $y = \tanh(x)$

We consider the function

$$y = \tanh(x)$$

defined by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

1. Domain of $y = \tanh x$:
   This function is defined on $(-\infty, +\infty)$ since the denominator of $\tanh(x)$ is always positive and therefore is never zero, therefore the domain of $y = \tanh(x)$ is $(-\infty, \infty)$.
2. Symmetry of $y = \tanh(x)$:
   $$\tanh(-x) = \frac{e^{-x} - e^x}{e^{-x} + e^x} = -\frac{e^x - e^{-x}}{e^x + e^{-x}} = -\tanh(x),$$
   therefore $y = \tanh(x)$ is an odd function which means that its graph will be symmetric across the origin of the $xy$-axes.
3. $x$ and $y$ intercepts of $y = \tanh(x)$
   $y$-intercepts:
   $$\tanh(0) = \frac{e^0 - e^0}{e^0 + e^0} = 0,$$
   therefore $y = \tanh(x)$ intercepts the $y$ axis at the origin $O = (0, 0)$.

   $x$-intercepts: set $y = 0$ which gives
   $$\frac{e^x - e^{-x}}{e^x + e^{-x}} = 0$$
   i.e.
   $$e^x - e^{-x} = 0 / \cdot e^x$$
   $$e^{2x} = e^0$$
   i.e.
   $$x = 0,$$
   which gives again the above point $O(0, 0)$. We can conclude that $y = \tanh(x)$ intercepts the $x$ and $y$ axes at the origin $O(0, 0)$.

4. Possible asymptotes for $y = \tanh(x)$:

   The domain of $y = \tanh(x)$ is $(-\infty, +\infty)$, therefore there will not be any vertical asymptote for this function. To check whether there is any horizontal asymptote we evaluate the following limits:

$$\begin{aligned}
\lim_{x\to+\infty} \tanh(x) = \lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \lim_{x\to+\infty} \frac{e^x\left(1 - e^{-2x}\right)}{e^x\left(1 + e^{-2x}\right)} \\
&= \lim_{x\to+\infty} \frac{\left(1 - e^{-2x}\right)}{\left(1 + e^{-2x}\right)} = 1 \\
\lim_{x\to-\infty} \tanh(x) = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} &= \lim_{x\to-\infty} \frac{e^{-x}\left(e^{2x} - 1\right)}{e^{-x}\left(e^{2x} + 1\right)} \\
&= \lim_{x\to-\infty} \frac{\left(e^{2x} - 1\right)}{\left(e^{2x} + 1\right)} = -1;
\end{aligned}$$

   therefore there are two horizontal asymptotes: $y = 1$ and $y = -1$. One can check that there is no diagonal asymptote for this functions but we will skip this.

5. First derivative of $y = \tanh(x)$:

$$\begin{aligned}
\frac{d}{dx}\tanh(x) &= \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) \\
&= \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)} \\
&= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)}.
\end{aligned}$$

   We want to evaluate explicitly the quantity

$$\cosh^2(x) - \sinh^2(x).$$

$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - e^{2x} - e^{-2x} + 2e^x e^{-x}}{4} \\
&= \frac{4e^x e^{-x}}{4} = 1
\end{aligned}$$

   Therefore we have

$$\frac{d}{dx}\tanh(x) = \frac{1}{\cosh^2(x)}$$

(a) Critical points of $y = \tanh(x)$:

$$\frac{d}{dx}\tanh(x) = 0$$

i.e.

$$\frac{1}{\cosh^2(x)} = 0. \tag{2.4.5}$$

Equation (2.4.5) has no solution, therefore $y = \tanh(x)$ has no critical point.

(b) Sign of $\frac{d}{dx}\tanh(x)$: we study $\frac{d}{dx}\tanh(x)$ in order to see where $y = \tanh(x)$ is increasing and where it is decreasing. Recall that the derivative of $y = \tanh(x)$ is given by

$$\frac{d}{dx}\tanh(x) = \frac{1}{\cosh^2(x)}$$

and notice that

$$\frac{d}{dx}\tanh(x) = \frac{1}{\cosh^2(x)} > 0, \qquad \text{for all} \quad x \in (-\infty, +\infty),$$

which implies that function $y = \tanh(x)$ is increasing on the entire domain $(-\infty, +\infty)$.

6. Second derivative of $y = \tanh(x)$:

$$\frac{d^2}{dx^2}\tanh(x) = \frac{d}{dx}\frac{1}{\cosh^2(x)} = \frac{-2\cosh(x)\sinh(x)}{\cosh^4(x)} = -2\,\frac{\sinh(x)}{\cosh^3(x)}.$$

(a) Points of inflection of $y = \tanh(x)$:

$$-2\,\frac{\sinh(x)}{\cosh^3(x)} = 0$$

i.e.

$$-2\,\sinh(x) = 0$$

i.e.

$$x = 0,$$

therefore $x = 0$ is a point of inflection for our function $y = \tanh(x)$.

(b) Sign of $\frac{d^2}{dx^2}\tanh(x)$: we study the sign of $\frac{d^2}{dx^2}\tanh(x)$ in order to see where $y = \tanh(x)$ is concave and where it is convex. Recall that the second derivative of $y = \tanh(x)$ is given by

$$\frac{d^2}{dx^2}\tanh(x) = -2\,\frac{\sinh(x)}{\cosh^3(x)}$$

and that its sign depends on the sign of both its numerator $-2\sinh(x)$ and denominator $\cosh^3(x)$. The latter is always positive (since $\cosh(x) > 0$), therefore the sign of $-2\,\frac{\sinh(x)}{\cosh^3(x)}$ depends on the sign of the numerator $-2\ \sinh(x)$ and from our previous study about function $y = \sinh(x)$ we have

$$\begin{array}{llll} -2\ \sinh(x) > 0 & \text{when} & \sinh(x) < 0 & \text{i.e.} \quad \text{for all } x \in (-\infty, 0) \\ -2\ \sinh(x) < 0 & \text{when} & \sinh(x) > 0 & \text{i.e.} \quad \text{for all } x \in (0, +\infty). \end{array}$$

Therefore $y = \tanh(x)$ is concave on $(-\infty, 0)$ and it is convex on $(0, +\infty)$.

7. Graph: Notice that from the graph of $y = \tanh(x)$ we can say that the range of this function is $(-1, 1)$ (see fig. 2.16).

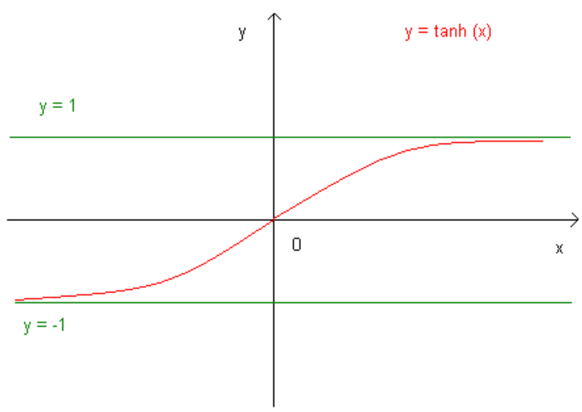


Figure 2.16: The graph of $y = \tanh(x)$

### 2.4.4 Hyperbolic Identities

1.

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - e^{2x} - e^{-2x} + 2e^x e^{-x}}{4} \\ &= \frac{4e^x e^{-x}}{4} = 1\end{aligned}$$

(see fig. 2.17).

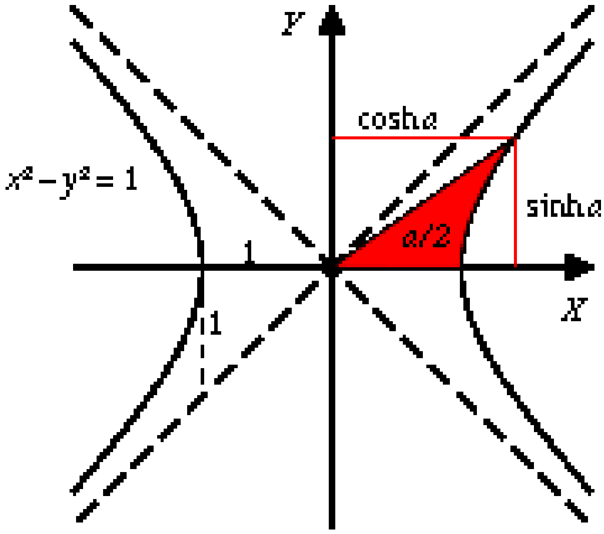


Figure 2.17: Just as the points $(cost, sint)$ form a circle with a unit radius, the points $(cosht, sinht)$ form the right half of the equilateral hyperbola (picture published by wikipedia).

By making use of the above identity we obtain

2.

$$\begin{aligned}1 - \tanh^2(x) &= 1 - \frac{\sinh^2(x)}{\cosh^2(x)} \\ &= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} \\ &= \frac{1}{\cosh^2(x)} \\ &= \operatorname{sech}^2(x)\end{aligned}$$

3.
$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

Proof: To prove the above inequality we simply start by using the definition of $\sinh(x)$ and $\cosh(x)$:

$$\begin{aligned}
& \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \\
= & \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2} \\
= & \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{(x+y)} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-(x+y)}}{4} \\
= & \frac{2e^{x+y} - 2e^{-(x+y)}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y).
\end{aligned}$$

■

4.
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

Proof: The proof of this equality is left as an exercise for the student.

## 2.5 Revision exercises on curve-sketching

Exercise **2.5.1.** *(From the Spring exam 2008/09)*

*Study the behavior of the function*

$$f(x) = \frac{x^2}{x^2 - 1},$$

*by following the steps given below:*

1. *find the domain of $f$;*
2. *find any possible symmetry in its graph;*
3. *find where the graph cuts the $x$ and the $y$ axes;*
4. *find any possible asymptote for $f$;*
5. *find where $f$ is increasing and decreasing and point out any critical point for $f$ (if there is any) by specifying when it is a maximum or a minimum point;*
6. *find where $f$ is concave and convex; point out any point of inflection (if there is any);*
7. *sketch the graph.*

Exercise **2.5.2.** *(From the Repeat Exam 2009/10)*

*Study the behavior of the function*

$$f(x) = \frac{1}{x^2 - 25},$$

*by following the steps given below:*

1. *find the domain of $f$;*
2. *find any possible symmetry in its graph;*
3. *find where the graph cuts the $x$ and the $y$ axes;*
4. *find any possible asymptote for $f$;*
5. *find where $f$ is increasing and decreasing and point out any critical point for $f$ by specifying when it is a maximum or a minimum point;*
6. *find where $f$ is concave and convex; point out any point of inflection (if there is any);*
7. *sketch the graph.*

Exercise **2.5.3.** *(From the Spring Exam 2011/12)*

*Study the behavior of the function*

$$f(x) = \frac{4 - x}{x},$$

*by following the steps given below:*

1. *find the domain of $f$;*
2. *find any possible symmetry of $f$;*
3. *find where $f$ cuts the $x$ and the $y$ axes;*
4. *find any possible asymptote for $f$;*
5. *find where $f$ is increasing and decreasing and point out any critical point for $f$ (if there is any) by specifying when it is a maximum or a minimum point;*
6. *find where $f$ is concave and convex; point out any point of inflection (if there is any);*
7. *sketch the graph of $f$.*