UNIVERSITY *of* LIMERICK

OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

REPEAT ASSESSMENT PAPER

MODULE CODE: MS4315	SEMESTER: Repeat 2017
MODULE TITLE: Operations Research 2	DURATION OF EXAMINATION: $2\frac{1}{2}$ hours
LECTURER: Dr. M. Burke & Mr. K. O'Brien	PERCENTAGE OF TOTAL MARKS: 100%
EXTERNAL EXAMINER: Prof. J. King	

INSTRUCTIONS TO CANDIDATES:

Answer four questions correctly for full marks.

Answer two questions from Q1–Q3 and two questions from Q4–Q6.

50% of the marks are for the two questions from Q1–Q3.

50% for the two questions from Q4–Q6..

Question 1

(a) In terms of a strategic (matrix) game, define what is meant by a dominant strategy and a dominated strategy? Describe the technique of iterated elimination of dominated strategies. What is the technique used for ? 5 %

(b) By removing all strategies which are dominated by strict pure or mixed strategies, derive a reduced version of the following 2-player matrix game:

	S	T	U
A	(5,3)	(1,2)	(2,1)
B	(1,-1)	(0,0)	(3,-3)
C	(2,-2)	(3,0)	(6,3)

10 %

(c) Find all the *Nash* Equilibria of the above game. 10 %

Question 2

- In a game show, contestants Máire and Séamus start the last round with €500 and €400 respectively. Each must decide to pass or play. If a player passes, they keep their money but if they opt to play they win or lose €200 each with probability 1/2. These outcomes are independent of each other. The player with the most money at the end of the round gets a bonus of €300.

(a) If Máire goes first and Séamus sees her move, draw the game tree. 9 %

(b) Show that the strategic form of the game is

	Pass	Play
Pass	(8,4)	$\left(\frac{13}{2}, \frac{11}{2}\right)$
Play	$\left(\frac{13}{2}, \frac{11}{2}\right)$	$\left(\frac{29}{4}, \frac{23}{4}\right)$

where payoffs are expected values in 00's. 8 %

(c) Solve the game. 8 %

Question 3

(a) The costs incurred by a firm in a production period are

$$c = 50 + 2x$$

where x is the number of items produced in that period. The items sell at a price of

$$p = 14 - \frac{x}{50}$$

each. Find the level of production that maximises the firm's profits when the firm has a monopoly. 6 %

(b) If two identical firms supply the market with x_i, $i = 1, 2$ items each at a cost per period of

$$c_i = 50 + 2x_i$$

respectively and sell each item at a price of

$$p = 14 - \frac{x_1 + x_2}{50},$$

analyse the resulting one shot *Cournot* game. 9 %

(c) If this 2-firm game is to be played repeatedly, consider the following "cooperative" strategy: a firm produces half of the optimal level associated with a monopoly (see part (a) for as long as the other firm does the same, and if the other firm deviates, it reverts to the single shot *Cournot* strategy thereafter. Does it ever pay to defect from the cooperative strategy? In particular, using the discount factor ω per period, when is this cooperative strategy a *Nash* equilibrium ? 10 %

Question 4

(a) (i) Big O-notation is used to classify algorithms according to their relative complexity. Compare the complexity of algorithms of order $O(\log n)$, $O(n)$, $O(n \log n)$, $O(2^n)$and $O(n!)$. Illustrate your answer with a sketch. 2 %

(ii) What is meant by Combinatorial Explosion? Why is it relevant for Binary Integer Problems? 2 %

(b) Consider the Integer Linear Program (IP):

$$\begin{aligned} \max 5x_1 + 4x_2 & \\ x_1 + x_2 &\leq 5 \\ 10x_1 + 6x_2 &\leq 45 \\ x_1, x_2 \leq 0 \text{ and integer.} & \end{aligned}$$

The corresponding Simplex Tableau (transforming the max problem into a min problem) is:

0	-5	-4	0	0
5	1	1	1	0
45	10	6	0	1

(i) Apply one iteration of the Simplex Method and show that the Simplex Tableau now takes the form: 4 %

22.5	0	-1	0	0.5
0.5	0	0.4	1	-0.1
4.5	1	0.6	0	0.1

(ii) After a second iteration of the Simplex Method the Simplex Tableau now takes the form: (**N.B.do not perform the arithmetic!**)

23.75	0	0	2.5	0.25
1.25	0	1	2.5	-0.25
3.75	1	0	-1.5	0.25

Explain why this Tableau is optimal. 1 %

(iii) The solution to the LP Relaxation of the IP is $x_1 = 3.75, x_2 = 1.25$. Suppose that we decide to branch on x_1. The two branches are $S_0 : x_1 \leq 3$ and $S_1 : x_1 \geq 4$.

Consider the branch $S_0 : x_1 \leq 3$.

(a) First show that the basic variable x_1 may be expressed in terms of the non-basic variables x_3 & x_4 as: $x_1 = 3.75 + 1.5x_3 - 0.25x_4$. 2 %

(b) Substitute this expression for x_1 into the S_0 branch constraint and show that it takes the form $1.5x_3 - 0.25x_4 + s = -0.75$. (The variable s is the slack variable for the constraint $x_1 \leq 3$.) 1 %

(c) Show that the Simplex Tableau with the addition of this constraint takes the form: 1 %

23.75	0	0	2.5	0.25	0
1.25	0	1	2.5	-0.25	0
3.75	1	0	-1.5	0.25	0
-0.75	0	0	1.5	-0.25	1

(d) Explain why this tableau is optimal but infeasible. 1 %

(e) Apply **one** iteration of the Dual Simplex Method to this tableau and show that the Simplex Tableau now takes the form: 4 %

23	0	0	4	0	1
2	0	1	1	0	-1
3	1	0	0	0	1
3	0	0	-6	1	-4

(f) This tableau is LP optimal and integer feasible. Explain why. What is the solution to the IP? 2 %

(g) Finally, **suppose** that we had started with the branch $S_1 : x_1 \geq 4$, expressed x_1 in terms of the non-basic variables x_3 & x_4 as $x_1 = 3.75 + 1.5x_3 - 0.25x_4$ as above and applied the Dual Simplex method to the resulting tableau.
We would have found (**N.B.do not perform the arithmetic!**)

23.33	0	0	0	0.67	1.67
0.83	0	1	0	0.17	1.67
4	1	0	0	0	-1
0.17	0	0	1	-0.17	-0.67

i. This tableau is now optimal. Is the solution integer? 1 %

ii. What would be the next branch & bound step? (**N.B.do not perform the arithmetic!**) 1 %

(h) Finally, **suppose** that we had started with the branch $S_1 : x_1 \geq 4$, expressed x_1 in terms of the non-basic variables x_3 & x_4 as $x_1 = 3.75 + 1.5x_3 - 0.25x_4$ as above and applied the Dual Simplex method to the resulting tableau.

We would have found (**N.B.do not perform the arithmetic!**)

23.33	0	0	0	0.67	1.67
0.83	0	1	0	0.17	1.67
4	1	0	0	0	-1
0.17	0	0	1	-0.17	-0.67

i. Is this tableau optimal? 1%

ii. Is the solution integer? 1%

iii. What would be the next branch & bound step? (**N.B.do not perform the arithmetic!**) 1%

Question 5

(a) GAMMA Investments is considering investments into 6 projects: A, B, C, D, E and F.

Each project has an initial cost, an expected profit rate (one year from now) expressed as a percentage of the initial cost, and an associated risk of failure. These numbers are given in the table below:

	A	B	C	D	E	F
Initial Cost	1.8	1.5	1.1	1.8	2.1	3.2
Profit Rate	13%	12%	10%	12%	11%	9%
Failure Risk	6%	4%	5%	5.5%	4.5%	4.5%

(i) Provide a formulation to choose the projects that maximize total expected profit, such that GAMMA Investments does not invest more than 5M dollars and its average failure risk is not over 5%.

You may assuming equal weighting for each project when determining average risk. For example, if GAMMA Investments invests only into A,B and C, it invests only 4.2M dollars and its average failure risk is $(6\% + 4\% + 5\%)/3 = 5\%$. 4 %

(ii) Suppose that if C is chosen, D must be chosen. Modify your formulation. 2 %

(iii) Suppose that if E is chosen, F must not be chosen. Modify your formulation. 2%

(iv) Suppose that if A and C are chosen, D must be chosen. Modify your formulation. 2 %

(b) (i) Explain briefly why the following strategy for the solution of Integer Linear Programs (IPs) is not useful: "Solve the LP relaxation then round off the components of the solution to the nearest integers". 3 %

(ii) Given an LP (the *Primal* problem) we can write a closely related LP, its *Dual*:

$$\begin{aligned} z &= \max\{c^T x : Ax \le b, x \in \mathbb{R}^n, x \ge 0\} \quad \textit{Primal} \\ w &= \min\{b^T y : A^T y \ge c, y \in \mathbb{R}^m, y \ge 0\}. \quad \textit{Dual} \end{aligned}$$

Prove the Weak Duality Theorem: for *any* primal feasible point x and *any* dual feasible point y, $b^T y \ge c^T x$. 4 %

(c) (i) Provide a short description of the Dynamic Programing Paradigm. 4 %

(ii) What is a Greedy Algorithm? Support your answer with a simple example, and discuss the advantages and disadvantages of using Greedy Algorithms. 2 %

(iii) In the context of the design of algorithms, describe the Divide and Conquer paradigm. 2 %

Question 6

(a) Maximize P = 3x+2y

$$12x - 18y \leq 90$$
$$20x + 13y \leq 252$$
$$y \leq 10$$
$$x,y \geq 0$$
x,y integers

This Integer Program is to be solved using the tabular Branch and Bound method.

Use the solution grids below to solve the problem. Each node is referenced by its tree level, ordered from left to right so that the annotation **Node XY** is the node at level **X** at position **Y** where **Y** = *A* is the left-most position in level **X**, where **Y** = *B* is the 2nd from the left in level **X**, and so on.

You must draw an enumeration tree/diagram to keep track of your progress. Draw the enumeration tree on an otherwise blank page. 25%

	Node 0		Node 1A		Node 1B
(i)	$x = 7.50, y = 9.00$	(i)	$x = 6.50, y = 8.50$	(i)	$x = 7.61, y = 7.00$
(ii)	$x = 6.10, y = 10.00$	(ii)	$x = 6.00, y = 8.91$	(ii)	$x = 7.60, y = 8.00$
(iii)	$x = 7.00, y = 9.00$	(iii)	$x = 6.91, y = 7.61$	(iii)	$x = 6.70, y = 6.50$
(iv)	$x = 6.00, y = 10.50$	(iv)	$x = 7.00, y = 10.00$	(iv)	$x = 6.91, y = 8.25$
(v)	$x = 7.00, y = 8.50$	(v)	$x = 6.00, y = 11.00$	(v)	$x = 7.00, y = 8.61$
	Node 2A		Node 2B		Node 2C
(i)	$x = 7.00, y = 9.75$	(i)	$x = 6.25, y = 8.30$	(i)	$x = 7.00, y = 6.75$
(ii)	$x = 7.25, y = 8.50$	(ii)	$x = 7.50, y = 6.50$	(ii)	$x = 6.75, y = 8.00$
(iii)	$x = 6.75, y = 10.00$	(iii)	$x = 6.25, y = 5.61$	(iii)	$x = 7.40, y = 8.00$
(iv)	$x = 8.50, y = 6.61$	(iv)	$x = 8.60, y = 6.61$	(iv)	$x = 7.80, y = 8.00$
(v)	$x = 8.25, y = 8.25$	(v)	$x = 8.00, y = 9.00$	(v)	$x = 6.61, y = 9.61$

	Node 2D		Node 3A		Node 3B
(i)	$x = 6.90, y = 8.25$	(i)	$x = 7.125, y = 7.75$	(i)	$x = 6.30, y = 6.61$
(ii)	$x = 7.00, y = 8.50$	(ii)	$x = 6.50, y = 7.50$	(ii)	$x = 7.25, y = 5.50$
(iii)	$x = 7.00, y = 8.40$	(iii)	$x = 7.40, y = 6.61$	(iii)	$x = 7.25, y = 7.61$
(iv)	$x = 7.00, y = 7.90$	(iv)	$x = 6.50, y = 7.91$	(iv)	$x = 7.10, y = 8.50$
(v)	$x = 7.50, y = 7.10$	(v)	$x = 6.90, y = 7.125$	(v)	$x = 6.00, y = 7.25$
	Node 3C		Node 3D		Node 3E
(i)	$x = 9.20, y = 6.10$	(i)	$x = 8.25, y = 8.60$	(i)	$x = 7.00, y = 8.00$
(ii)	$x = 5.75, y = 8.50$	(ii)	$x = 7.40, y = 9.50$	(ii)	$x = 7.75, y = 6.80$
(iii)	$x = 8.50, y = 7.61$	(iii)	$x = 6.00, y = 8.00$	(iii)	$x = 7.00, y = 8.50$
(iv)	$x = 8.25, y = 7.61$	(iv)	$x = 7.25, y = 9.333$	(iv)	$x = 6.50, y = 8.00$
(v)	$x = 8.60, y = 9.20$	(v)	$x = 9.00, y = 8.60$	(v)	$x = 7.00, y = 7.00$
	Node 3F		Node 3G		Node 3H
(i)	$x = 9.00, y = 7.00$	(i)	$x = 7.40, y = 11.00$	(i)	$x = 7.60, y = 7.80$
(ii)	$x = 8.00, y = 7.25$	(ii)	$x = 7.50, y = 10.50$	(ii)	$x = 8.00, y = 8.00$
(iii)	$x = 7.00, y = 7.50$	(iii)	$x = 7.40, y = 8.50$	(iii)	$x = 8.30, y = 7.50$
(iv)	$x = 8.00, y = 10.00$	(iv)	$x = 7.30, y = 7.50$	(iv)	$x = 7.60, y = 6.60$
(v)	$x = 8.00, y = 7.00$	(v)	$x = 7.90, y = 6.50$	(v)	$x = 7.70, y = 7.50$