MA4702 2014 Solutions.

Q1 (a)

$$F(x) = \sqrt{2x+3}$$

$$y = \sqrt{2x+3}$$

$$y^2 = 2x+3$$

$$2x = y^2 - 3$$

$$x = \frac{y^2-3}{2}$$

$$\Rightarrow \quad F^{-1}(x) = \frac{x^2-3}{2}$$

Q1 (b)

$$F(x) = \ln(x)$$

Domain: $(0, \infty)$

Reason: Cannot find log of zero or a negative value.

Range: $(-\infty, \infty)$

Q1(c) $F(x) = \dfrac{x^2 + x + 9}{2x^2 - 18}$

Vertical Asymptote:

$$2x^2 - 18 = 0$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

Vertical Asymptotes at $x = 3$ and $x = -3$

Horizontal Asymptotes:

$$\lim_{x \to \infty} \frac{x^2 + x + 9}{2x^2 - 18} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + \frac{x}{x^2} + \frac{9}{x^2}}{\frac{2x^2}{x^2} - \frac{18}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{1}{x} + \frac{9}{x^2}}{2 - \frac{18}{x^2}} = \frac{1 + \frac{1}{\infty} + \frac{9}{\infty^2}}{2 - \frac{18}{\infty}}$$

$$= \frac{1 + 0 + 0}{2 - 0} = \frac{1}{2}$$

Horizontal asymptote at $y = \frac{1}{2}$.

Q1(d) $\lim_{x \to 3} \frac{x-3}{x^2-9} = \frac{0}{0} \Rightarrow$ not defined

$$\Rightarrow \lim_{x \to 3} \frac{x-3}{x^2-9} = \lim_{x \to 3} \frac{(x-3)}{(x-3)(x+3)}$$

$$= \lim_{x \to 3} \frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}.$$

Answer $= \frac{1}{6}$.

Q1 (e) $\int x^2 - 4e^x + 2 \, dx$

$$= \frac{x^3}{3} - 4e^x + 2x + C.$$

Q1 (F) $\int_1^3 \frac{1}{x} \, dx = \left[\ln x\right]_1^3$

$$= \ln(3) - \ln(1)$$

$$= 1.0986.$$

Q1(g)

$$z = -x^2 y + x^4 e^y$$

$$\frac{\partial z}{\partial x} = -2xy + 4x^3 e^y$$

$$\frac{\partial z}{\partial y} = -x^2 + x^4 e^y$$

Q1(h)

$$F(x) = \frac{e^x + e^{-x}}{2}$$

$$F(-x) = \frac{e^{-x} + e^{-(-x)}}{2}$$

$$F(-x) = \frac{e^{-x} + e^x}{2}$$

This is an even Function

because $F(x)$ $F(-x)$

Q1 (i) $a = 4$ $\quad r = 3$

How many terms?

$$U_n = ar^{n-1}$$

$$U_n = 8748$$

$$\Rightarrow ar^{n-1} = 8748$$

$$4(3^{n-1}) = 8748$$

$$3^{n-1} = 2187$$

$$\log_3 2187 = n-1$$

$$7 = n-1$$

$$n = 8.$$

$$S_8 = 4\left(\frac{1-3^8}{1-3}\right) = 4(3280)$$

$$= 13,120$$

Sum of series = 13,120.

Q2 (a) $$F(x) = \frac{25x}{x^2+4}$$

y-Int : let $x = 0$

$$F(0) = \frac{25(0)}{(0)^2+4} = \frac{0}{4} = 0$$

y-Intercept : $(0, 0)$.

(b)

$u = 25x$	$v = x^2+4$
$\frac{du}{dx} = 25$	$\frac{dv}{dx} = 2x$

$$F'(x) = \frac{(x^2+4)(25) - (25x)(2x)}{(x^2+4)^2}$$

$$F'(x) = \frac{25x^2 + 100 - 50x^2}{(x^2+4)^2}$$

$$F'(x) = \frac{-25x^2 + 100}{(x^2+4)^2}$$

Let $f'(x) = 0$:

$$\frac{-25x^2+100}{(x^2+4)^2} = 0$$

$$-25x^2 + 100 = 0$$

$$25x^2 = 100$$

$$\Rightarrow \quad x = \pm 2$$

Find corresponding y-values :

$x = 2$: $\quad F(x) = \dfrac{25x}{x^2+4}$

$$F(2) = \frac{25(2)}{(2)^2+4} = 6.25$$

$x = -2$: $\quad F(-2) = \dfrac{25(-2)}{(-2)^2+4} = -6.25$

Turning points $(2, 6.25)$ $(-2, -6.25)$

Max or min?

$$F'(x) = \frac{-25x^2 + 100}{(x^2+4)^2}$$

$x = -2$ $\qquad\qquad$ $x = 2$

$F'(-3) = -$ $\qquad$ $F'(0) = +$ $\qquad$ $F'(3) = -$

Function decreasing $\qquad$ Function increasing $\qquad$ Function decreas

$\searrow$ $\qquad\qquad$ $\nearrow$ $\qquad\qquad$ $\searrow$

$\Rightarrow$ Min point at $(-2, -6.25)$

Max point at $(2, 6.25)$

Q2 (c) $\qquad F(x) = \dfrac{25x}{x^2+4}$

$$F(-x) = \frac{25(-x)}{(-x)^2 + 4} = \frac{-25x}{x^2+4}$$

$$-F(x) = -\left(\frac{25x}{x^2+4}\right) = \frac{-25x}{x^2+4}$$

$F(-x) = -F(x) \Rightarrow$ Odd Function.

Q2 (d) $F(x) = \frac{25x}{x^2+4}$

$$\lim_{x \to +\infty} \frac{25x}{x^2+4} = \lim_{x \to +\infty} \frac{\frac{25x}{x^2}}{\frac{x^2}{x^2} + \frac{4}{x^2}}$$

$$= \lim_{x \to +\infty} \frac{\frac{25}{x}}{1 + \frac{4}{x^2}} = \frac{\frac{25}{\infty}}{1 + \frac{4}{\infty^2}} = \frac{0}{1}$$

$$= 0$$

$$\lim_{x \to -\infty} \frac{25x}{x^2+4} = 0$$

Conclusion: As $x \to +\infty$ $y \to 0$

As $x \to -\infty$ $y \to 0$.

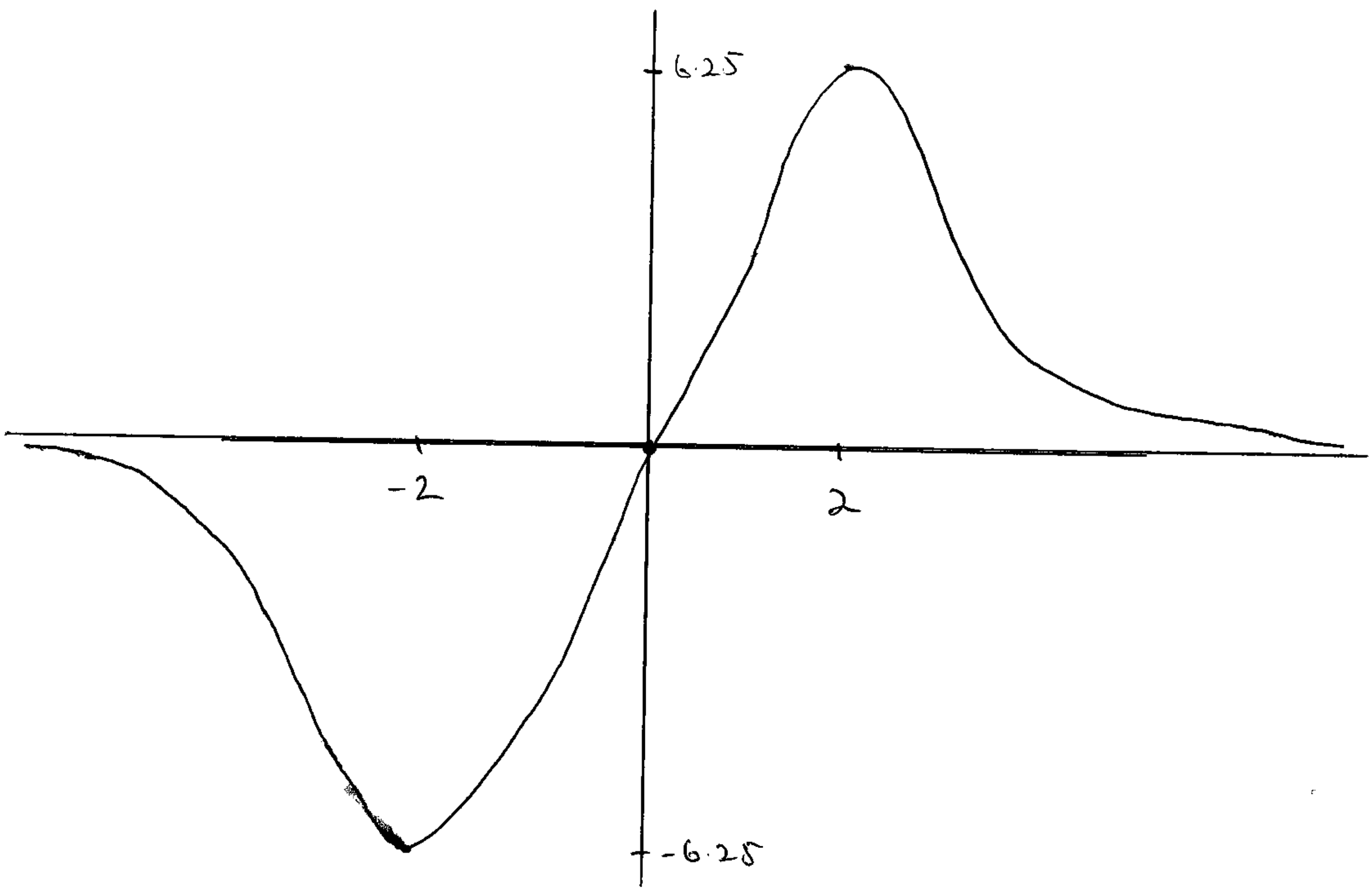
Summary: Intercept : (0,0)

Max point : (2, 6·25)

Min point : (−2, −6·25)

Symmetry : Odd function

As $x \to \pm\infty$ $y \to 0$.



Q3 (a) $a(t) = 3t^2 + 2$

$$V(t) = \int 3t^2 + 2 \, dt$$

$$V(t) = t^3 + 2t + C$$

$$V(0) = (0)^3 + 2(0) + C$$

$$V(0) = C$$

Also: $V(0) = 2$

$$\Rightarrow C = 2.$$

$$\Rightarrow V(t) = t^3 + 2t + 2.$$

Q3(b)

$$\int_0^{12} 5.45t^3 - 105t^2 + 391t + 1798$$

$$= \left[\frac{5.45t^4}{4} - \frac{105t^3}{3} + \frac{391t^2}{2} + 1798t\right]_0^{12}$$

$$= \left[1.3625t^4 - 35t^3 + 195.5t^2 + 1798t\right]_0^{12}$$

$$= 1.3625(12)^4 - 35(12)^3 + 195.5(12)^2 + 1798(12)$$

$$= 28252.8 - 60480 + 28152 + 21576$$

$$= 17500.8$$

There was 17,500 homicides in New York City between 1998 and 2009.

Q4.(a) $$\int \frac{4x-5}{2x^2-5x+3}\,dx$$

Let $u = 2x^2-5x+3$

$$\frac{du}{dx} = 4x-5$$

$$\frac{du}{4x-5} = dx$$

$$= \int \frac{4x-5}{u}\,\frac{du}{4x-5}$$

$$= \int \frac{1}{u}\,du$$

$$= \ln u + c$$

$$= \ln(2x^2-5x+3)$$

Q4 (b) $x^2 - 7x + 10 = 0$

$$(x-5)(x-2) = 0$$

$$x = 5 \qquad x = 2$$

$$\int_2^5 x^2 - 7x + 10 \, dx$$

$$= \left[\frac{x^3}{3} - \frac{7x^2}{2} + 10x\right]_2^5$$

$$= \left(\frac{(5)^3}{3} - \frac{7(5)^2}{2} + 10(5)\right) - \left(\frac{(2)^3}{3} - \frac{7(2)^2}{2} + 10(2)\right)$$

$$= \left(\frac{125}{3} - \frac{175}{2} + 50\right) - \left(\frac{8}{3} - \frac{28}{2} + 20\right)$$

$$\frac{25}{6} - \frac{26}{3} = \frac{-9}{2} = -4.5$$

Area $= 4.5$ units^2

Q4(c)

$$\int_0^5 (3t+4)^{\frac{1}{3}}\,dt$$

let $u = 3t+4$

$\frac{du}{dt} = 3$

$\frac{du}{3} = dt$

$t=5 \quad u=19$

$t=0 \quad u=4$

$$= \int_4^{19} u^{\frac{1}{3}} \frac{du}{3}$$

$$= \frac{1}{3}\int_4^{19} u^{\frac{1}{3}}\,du$$

$$= \left[\frac{1}{3}\left(\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\right)\right]_4^{19}$$

$$= \left[\frac{u^{\frac{4}{3}}}{4}\right]_4^{19}$$

$$= \frac{(19)^{\frac{4}{3}}}{4} - \frac{(4)^{\frac{4}{3}}}{4} = 11.087$$

Weight will increase by 11.087 grams.

Q5(a)

$F(x) = e^x \qquad F(0) = e^0 = 1$

$F'(x) = e^x \qquad F'(0) = 1$

$F''(x) = e^x \qquad F''(0) = 1$

$\vdots$

$$F(x) = F(0) + xF'(0) + \frac{x^2 F''(0)}{2!} \cdots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}$$

$$e^3 = 1 + 3 + \frac{(3)^2}{2} + \frac{(3)^3}{6} + \frac{(3)^4}{24} + \frac{(3)^5}{120} + \frac{(3)^6}{720}$$

$e^3 = 19.4125$

Q2(b)(i)

$$Z = 2x^2y + 4x^2y^3 - 7x^2$$

$$\frac{\partial z}{\partial x} = 4xy + 8xy^3 - 14x$$

$$\frac{\partial^2 z}{\partial x^2} = 4y + 8y^3 - 14$$

$$\frac{\partial^2 z}{\partial y \partial x} = 4x + 24xy^2$$

(ii) $$Z = \cos(x + 3y)$$

$$\frac{\partial z}{\partial x} = -\sin(x+3y)$$

$$\frac{\partial z}{\partial y} = -3\sin(x+3y)$$

$$\frac{\partial^2 z}{\partial x^2} = -\cos(x+3y)$$

$$\frac{\partial^2 z}{\partial y^2} = -9\cos(x+3y)$$

$$\frac{\partial^2 z}{\partial y^2} - 9\frac{\partial^2 z}{\partial x^2} = 0$$

$$-9\cos(x+3y) - 9(-\cos(x+3y)) = 0$$

$$-9\cos(x+3y) + 9\cos(x+3y) = 0$$

$\Rightarrow$ True.

QED.

Q6 (a)

35,000, 38,000, 41,000, 44,000 .....

$a = 35,000 \qquad d = 3,000$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_{24} = \frac{24}{2}\left[2(35,000) + 23(3,000)\right]$$

$$S_{24} = 12(70,000 + 69,000)$$

$$= €1,668,000 = €1.668\ m$$

If Joe Cole plays in 24 games in the next season, he will earn €1.668 million

Q 6(b)

$$S_{44} = \frac{44}{2}\left[2(35{,}000) + 43(3{,}000)\right]$$

$$= 22(70{,}000 + 129{,}000)$$

$$= €4{,}378{,}000$$

The maximum amount he can earn is €4·378 million.

Q6(c) $$1{,}400{,}000 = \frac{n}{2}\left[2(35{,}000) + (n-1)(3{,}000)\right]$$

$$1{,}400{,}000 = \frac{n}{2}(70{,}000 + 3{,}000n - 3{,}000)$$

$$2{,}800{,}000 = n(67{,}000 + 3{,}000n)$$

$$2{,}800{,}000 = 67{,}000n + 3{,}000n^2$$

$$2{,}800 = 67n + 3n^2$$

$$3n^2 + 67n - 2{,}800 = 0$$

$$3n^2 + 67n - 2,800 = 0$$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$n = \frac{-67 \pm \sqrt{(67)^2 - 4(3)(-2,800)}}{6}$$

$$n = \frac{-67 \pm \sqrt{38089}}{6} = \frac{-67 \pm 195.16}{6}$$

$$n = 21.36 \text{ or } -43.69$$

$\Rightarrow$ n must be positive (number of games)

$\Rightarrow$ $n = 21.36$

Conclusion: Joe Cole must play 22 games or more to earn more than €1,400,000