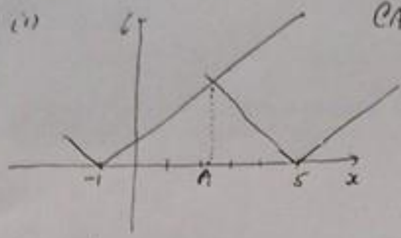
SME 2018 SOLUTIONS

Q1 (a) $x=2 \quad y=-1 \quad z=3$

Eliminate y to reduce to system in x & z
as opposed to using Gauss Elimination ab initio ?

(b) (i)



CA $x+1=-x+5$
$x=2$

Ans: $x \geq 2$

or $(x-5)^2 \leq (x+1)^2$
$\vdots$
$x \geq 2$

(ii) discriminant $=0$

$(2a)^2-4(5a-6)=0$

$a^2-5a+6=0 \rightarrow a=2$ or 3

Q2 (a) (i) $2-i=\sqrt{5}\angle \tan^{-1}(-1/2) \simeq 2.24\angle 2.68$ 2nd quadrant ~~or 5.82~~ [−0.46 in 4th quad]

(iii) The coeff of $z^2 = -$ sum of roots

$-2 = -(z_1+\bar{z}_1+z_2)$ real root; conjugate of $\bar{z}_1$

$= -(4+z_2) \Rightarrow z_2=-2$

(ii) $(z-2+i)(z-2-i)(z+2) = z^3-2z^2-3z+10$

$k=3$

(b) $\cos 3A + i\sin 3A = (\cos A + i\sin A)^3$

$= \cos^3 A + 3\cos^2 A(i\sin A) + 3\cos A(i\sin A)^2 + (i\sin A)^3$

$= \cos^3 A - 3\cos A\sin^2 A \quad + i(3\cos^2 A\sin A - \sin^3 A)$

$\therefore \cos 3A = \cos^3 A - 3\cos A(1-\cos^2 A)$

$= 4\cos^3 A - 3\cos A$

Q3 (a) (i) Ans: $2/3$

(ii) $\left(x+\frac{1}{5x}\right)^2-\left(\frac{1}{5x}-4x\right)^2 = \left(x+\frac{1}{5x}+\frac{1}{5x}-4x\right)\left(x+\frac{1}{5x}-\frac{1}{5x}+4x\right)$

$= \left(-3x+\frac{2}{5x}\right)5x$

$= -15x+2$

Ans: 2

(b) (i) $\frac{dy}{dx} = \frac{x^2}{8}-\frac{1}{x}$

@ $x=1$ $\frac{dy}{dx} = -7/8$

(ii) $\frac{dy}{dx}=0 \Rightarrow x=2$ [given $x>0$]

@ $x=2$ $y = \frac{8}{24}-\ln 4 = \frac{1}{3}-\ln 4 \approx$

Q4 (a) $x^2+y^2+2x-6y-6 = (x+1)^2+(y-3)^2-16 = 0$

Centre at $(-1,3)$ radius $= 4$

(b) x-axis cut $\Rightarrow y=0$; $x^2+2x-6=0 \Rightarrow x = -1\pm\sqrt{7}$

let $A=(-1-\sqrt{7},0)$ & $B=(-1+\sqrt{7},0)$

(c) $\frac{dy}{dx}$; $2(x+1)+2(y-3)\frac{dy}{dx}=0$ $y=0 \Rightarrow \frac{dy}{dx}=\frac{x+1}{3}$

tangent through A: $y-0 = \frac{-\sqrt{7}}{3}(x+1+\sqrt{7})$

" " B: $y-0 = \frac{\sqrt{7}}{3}(x+1-\sqrt{7})$

(d) tangents $\perp$ iff product of slopes $=-1$

but $\frac{-\sqrt{7}}{3}\times\frac{\sqrt{7}}{3} = \frac{-7}{9} \neq -1$

Q5 (a) (i) $r = \frac{x^2-x-6}{x+2} = \frac{(x+2)(x-3)}{x+2} = x-3$

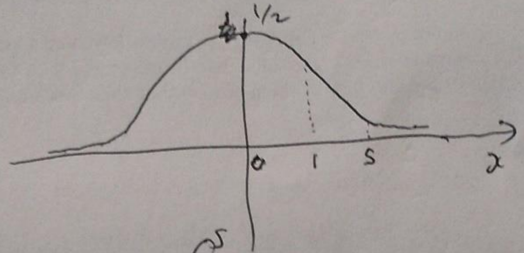
(ii) $S_\infty = T_1 \frac{1}{1-r} = \frac{x+2}{1-x+3} = \frac{x+2}{4-x} = 3$

$\Rightarrow x = 5/2$

(iii) S_∞ is well defined if $-1 < r < 1$

$-1 < x-3 < 1 \Rightarrow 2 < x < 4$

(b)



$$\int_0^1 \frac{2}{4+x^2}\,dx = \int_1^s \frac{2}{4+x^2}\,dx$$

$$\tan^{-1} x/2 \Big|_0^1 = \tan^{-1} x/2 \Big|_1^s$$

$$\tan^{-1} 1/2 - 0 = \tan^{-1} s/2 - \tan^{-1} 1/2$$

$$\Rightarrow \tan^{-1} s/2 = 2\tan^{-1} 1/2$$

$$\Rightarrow s/2 = \tan\left(2\tan^{-1} 1/2\right) = \frac{1/2 + 1/2}{1 - 1/2 \cdot 1/2} = 4/3$$

$$\Rightarrow s = 8/3$$

Q6 (a)

(i) $x=5$; (1,4) (4,1) (3,2) (2,3) $\quad p = 4/36 = 1/9$

(ii) $x<5$; (1,1)
(1,2) (2,1)
(1,3) (3,1) (2,2)

$P = 1 - \left[\frac{1}{36} + \frac{2}{36} + \frac{3}{36}\right] \frac{4}{36}$

$= \frac{26}{36} = \frac{13}{18}$ (or $P = 1 - \frac{10}{36}$)

(iii) $x=7$; (1,6) (6,1), (2,5) (5,2), (3,4) (4,3)

There are 26 ways of getting $x>5$

$P = 6/26 = \frac{3}{13}$

(b)

$$E(\text{Jo wins}) = 3P(x=5) + 1P(x>5) - qP(x<5)$$
$$= 3/9 + 1 \cdot \frac{13}{18} - q\frac{5}{36} = \frac{38-5q}{36}$$
$$= 0 \Rightarrow q = 38/5 = 7.6$$

Q7 over to you!

Q8 (i) $\frac{dy}{dx} = 0.624 - 0.026x = 0 \Rightarrow x = 24 \ [\Rightarrow y = 7.488]$

$\frac{d^2y}{dx^2} = -0.026 < 0 \Rightarrow \max @ \frac{dy}{dx} = 0$

(ii) $5 = 0.624x - 0.013x^2 \Rightarrow x = \overset{37.83}{\cancel{66.724}}, \ 10.166$

D in $(10, 5)$ E in $(38, 5)$

(iii) $\int_0^{10.16} 0.624x - 0.013x^2 \, dx + \int_{10.16}^{37.86} 5 \, dx + \int_{37.86}^{48} 0.624x - 0.013x^2 \, dx$

$= \overset{0.312}{\cancel{0.624}}x^2 - 0.0043x^3 \Big|_0^{10.16} + 5(27.72) + 0.312x^2 - 0.0043x^3 \Big|_{37.86}^{48}$

$= 27.70 + 138.60 + 243.30 - 213.70$

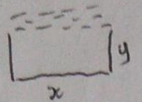
$= 195.9 \approx 196$

(iv) $y - 7.488 = -0.013(x - 24)^2 \ \cancel{+7.488}$ complete the square

(v) $y + 4 = -2(x-3)^2$

$\Rightarrow y = -2x^2 + 12x - 22$

Q9 (a)



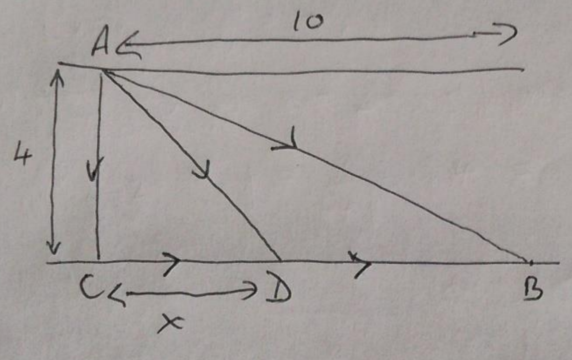
fence perimeter $P = x + 2y = 1000 \Rightarrow y = \frac{1000-x}{2}$

field area $A = xy = x\left(\frac{1000-x}{2}\right) = 500x - x^2/2$

(i) $\frac{dA}{dx} = 500 - x = 0 \Rightarrow x = 500\text{ m},\ y = 250\text{ m}$ $\left[\frac{d^2A}{dx^2} = -1 \Rightarrow \max @ \frac{dA}{dx} = 0\right]$

(ii) $A_{max} = (500)(250) = 125000\text{ m}^2$

(b)



$S_{row} = 5$

$S_{run} = 13$

$DB : 10 - x$

$AD : \sqrt{4^2 + x^2}$

$$\text{Time} = \frac{AD}{S_{row}} + \frac{DB}{S_{run}} = \frac{\sqrt{4^2+x^2}}{5} + \frac{10-x}{13}$$

$$\frac{d\,\text{Time}}{dx} = \frac{x}{5\sqrt{4^2+x^2}} - \frac{1}{13} \quad \left[\frac{d^2\,\text{Time}}{dx^2} = \frac{4^2}{(4^2+x^2)^{3/2}} > 0 \Rightarrow \min @ \frac{dT}{dx} = 0\right]$$

$$= 0 \Rightarrow x = \frac{20}{12} = 5/3$$