Differential Equations

A *differential equation (DE)* is an equation which involves an unknown function $f(x)$ as well as some of its derivatives.

To solve a differential equation means to find the unknown function $f(x)$.

Example: Solve the differential equation

$$\frac{df}{dx} - f(x) = 0 \tag{1}$$

The function $f(x) = \mathrm{e}^x$ is a *particular solution* of the differential equation. However, the function $f(x) = C\mathrm{e}^x$, where C is an arbitrary constant, is called the *general solution*. This means that any solution of the equation will take this form.

An *initial value problem (IVP)* consists of a differential equation involving an unknown function $f(x)$ as well as an initial condition (that is, an equation giving the value of $f(x)$ at a particular point).

Example: Solve the IVP:

$$\frac{df}{dx} + f(x) = 0, \quad f(0) = 2. \tag{2}$$

The general solution is $f(x) = Ce^{-x}$. Using the initial condition, we get $C = 2$ and so the solution is $f(x) = 2e^{-x}$.

Note: The differential equation (1) has an infinity of solutions, while the IVP (2) has only one solution.

Classification of DE's:

There are many types of differential equations. We have *ordinary* DE's (the unknown function is a function of one variable only) or *partial* DE's (the unknown function depends on many variables). A DE can also be classified by order (the order of the highest derivative which appears in the equation). Moreover, DE's can be linear/nonlinear, with constant/variable coefficients, homogeneous/non-homogenous, etc.

Examples:

1. Equations (1) and (2) above are first-order linear differential equations.

2. $\dfrac{df}{dx} + f^2(x) = 0$ is a first-order nonlinear equation (it contains f^2).

3. $\dfrac{d^2 f}{dx^2} + f(x)\dfrac{df}{dx} + x + 1 = 0$ is a second-order nonlinear equation (it contains $f\frac{df}{dx}$).

Verifying that a function is a solution

Example 1: Show that the function $f(x) = \mathrm{e}^{-x} + \mathrm{e}^{-\frac{3}{2}x}$ is a solution of the differential equation

$$2\frac{df}{dx} + 3f(x) = \mathrm{e}^{-x}$$

Example 2: Show that the function $f(x) = \frac{1}{x} + \frac{x}{2}$ is a solution of the initial value problem

$$\frac{df}{dx} = 1 - \frac{1}{x}f(x), \qquad f(2) = \frac{3}{2}.$$

Physical models involving first-order differential equations

1. Law of radioactive decay
The rate at which a radioactive element decays is proportional to the amount present. So, if $N(t)$ is the amount present at time t, we can write the differential equation

$$\frac{dN}{dt} = -kN(t), \qquad N(0) = N_0 \text{ (known)}.$$

where k is a proportionality constant (usually determined experimentally).

2. Newton's law of cooling

The rate at which the temperature of an object is changing is proportional to the difference between its current temperature and the temperature of the surrounding medium. So, if $T(t)$ is the temperature of the object at time t and T_S is the temperature of the surrounding medium, we have

$$\frac{dT}{dt} = k\left(T(t) - T_S\right), \qquad T(0) = T_0.$$

where k is a proportionality constant (usually determined experimentally).

Separable differential equations

A *separable differential equation* is a first order differential equation of the form

$$\frac{df}{dx} = A(f)B(x)$$

(the derivative can be expressed as the product of one factor involving only f and another involving only x).

To solve a separable equation, take all f factors on one side and all x factors on the other

$$\frac{df}{A(f)} = B(x)\,dx$$

and then integrate both sides

$$\int \frac{df}{A(f)} = \int B(x)\,dx$$

Example 1: Solve the separable equation

$$\frac{df}{dx} = e^x f(x)$$

Example 2: Sometimes differential equations are given in the x and y notation, so $y = y(x)$ is the unknown function. Solve the initial value problem:

$$\frac{dy}{dx} = x^2(1+y), \qquad y(0) = 0.$$

Example 3: The law of radioactive decay

$$\frac{dN}{dt} = -kN(t), \qquad N(0) = N_0$$

is a separable differential equation. The solution is

$$N(t) = N_0 \mathrm{e}^{-kt} \qquad \text{or} \qquad N(t) = N_0 \mathrm{e}^{-\frac{t\ln(2)}{t_H}}$$

The constant k is usually determined by knowing the **half-life** of the given radioactive material, that is, the time t_H at which exactly half the original amount is left.

Example 4: Newton's law of cooling

$$\frac{dT}{dt} = -k\left(T(t) - T_S\right), \qquad T(0) = T_0.$$

is a separable differential equation. The solution is

$$T(t) = T_S + (T_0 - T_S)\,\mathrm{e}^{-kt}$$

Linear first-order differential equations

A *linear first-order differential equation* has the following standard form

$$\frac{df}{dx} + A(x)f(x) = B(x) \tag{3}$$

where $A(x)$ and $B(x)$ are any functions of x.

Example:

$$\frac{df}{dx} + f(x) = \mathrm{e}^{2x}, \qquad f(0) = 0.$$

Solution procedure

1. Make sure the equation is written in the standard form (3) above. (If it's not, bring it to the standard form by moving terms around, dividing, etc.)
2. Calculate the *integrating factor*

$$I(x) = \mathrm{e}^{\int A(x)dx}$$

3. Multiply both sides of the equation by the integrating factor $I(x)$. The left-hand side will then become

$$I(x)\frac{df}{dx} + f(x)\frac{dI}{dx} = \frac{d}{dx}\left[I(x)f(x)\right]$$

4. Integrate both sides of the equation with respect to x.
5. The solution is obtained as

$$f(x) = \frac{\int I(x)B(x)\,dx}{I(x)}$$

More examples

Solve the following linear first-order equations:

1. $$x\frac{df}{dx} = x^2 + 3f(x), \qquad f(1) = 2;$$
2. $$\frac{df}{dx} + xf(x) = x, \qquad f(0) = -6.$$

Linear second-order differential equations

A *linear second-order differential equation* has the following standard form

$$a\frac{d^2 f}{dx^2} + b\frac{df}{dx} + cf(x) = 0 \tag{4}$$

where a, b and c are numbers (or constants).

To solve a second-order linear equation of the form (4), first construct the **auxiliary quadratic equation**:

$$am^2 + bm + c = 0$$

and find its two roots, m_1 and m_2.

We have the following possibilities:

1. If the roots are real and different, the solution is written in the form

$$f(x) = C_1\,\mathrm{e}^{m_1 x} + C_2\,\mathrm{e}^{m_2 x}$$

2. If the roots are real and equal $m_1 = m_2 = m$ (repeated roots) the solution is written in the form

$$f(x) = (C_1 + C_2 x)\,\mathrm{e}^{mx}$$

3. If the roots are complex, of the form $m_1 = a + \mathrm{i}b$, $m_2 = a - \mathrm{i}b$ then the solution can be written in the form

$$f(x) = \mathrm{e}^{ax}\left[C_1 \cos(bx) + C_2 \sin(bx)\right].$$

where C_1 and C_2 are constants of integration.

Examples: Solve the following linear second-order equations:

1. $$\frac{d^2f}{dx^2} + 5\frac{df}{dx} + 6f(x) = 0, \qquad f(0) = 0;\ \frac{df}{dx}(0) = 1;$$

2. $$\frac{d^2f}{dx^2} + 4\frac{df}{dx} + 4f(x) = 0, \qquad f(0) = 0;\ \frac{df}{dx}(0) = 1;$$

3. $$\frac{d^2f}{dx^2} + 4\frac{df}{dx} + 13f(x) = 0, \qquad f(0) = 0;\ \frac{df}{dx}(0) = 1.$$

Example: A mass-spring system

Consider a suspended spring with a mass m attached to one end. We make the following assumptions about this mechanical system:

- The elastic (restoring) force acting on the spring is proportional to the elongation (Hooke's law)

  $$F(x) = -kx$$

  where k is the stiffness of the spring.
- Air-resistance is proportional to the velocity of the moving object

We want to find the further displacement of the spring as a function of time, $x(t)$.

Writing Newton's second law, we obtain the differential equation

$$\frac{d^2x}{dt^2} + 2\delta \frac{dx}{dt} + \omega_0^2 x(t) = 0$$

where δ is a damping parameter (related to the air resistance) and $\omega_0^2 = k/m$ is called the natural frequency of the spring.

I. Simple harmonic motion

If $\delta = 0$ (no damping) then the previous equation becomes

$$\frac{d^2x}{dt^2} + \omega_0^2 x(t) = 0$$

which is said to describe **simple harmonic motion.** The solution of this equation is

$$x(t) = C_1 \sin(\omega_0 t) + C_2 \cos(\omega_0 t)$$

which is a periodic function. The model predicts that, in the absence of air resistance, the system will oscillate indefinitely with frequency ω_0.

II. Damped harmonic motion

Case 1: If $\delta > \omega_0$, the system is called overdamped. The solution in this case is

$$x(t) = C_1 \, \mathrm{e}^{m_1 t} + C_2 \, \mathrm{e}^{m_2 t}$$

since the solutions of the quadratic equation are given by

$$m_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

and they are real. Also note that m_1 and m_2 are both negative and so the solution $x(t)$ decays exponentially to zero.

II. Damped harmonic motion

Case 2: If $\delta < \omega_0$, the system is called underdamped. The solution in this case is of the form

$$x(t) = \mathrm{e}^{-\delta t}\left[C_1 \cos(t\sqrt{\omega_0^2 - \delta^2}) + C_2 \sin(t\sqrt{\omega_0^2 - \delta^2})\right].$$

since the solutions of the quadratic equation are given by

$$m_{1,2} = -\delta \pm \mathrm{i}\sqrt{\omega_0^2 - \delta^2}$$

and they are complex. The motion in this case is described by an oscillatory function with an exponentially decaying amplitude.

III. Forced harmonic oscillator

Suppose the mass-spring system is now subjected to external periodic vibrations. The differential equation becomes

$$\frac{d^2x}{dt^2} + 2\delta \frac{dx}{dt} + \omega_0^2 x(t) = A\sin(\omega t) \tag{5}$$

where A and ω are the amplitude and frequency of this imposed oscillation.

Exercise: Show that if $\delta = 0$ (no damping) the following is a solution for the forced harmonic oscillator (5),

$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) - \frac{A\sin(\omega t)}{\omega^2 - \omega_0^2} \tag{6}$$

Note that if ω is close to ω_0, the solution (6) describes an oscillation with a very large amplitude.

Resonance

Exercise: If $\omega = \omega_0$ then the following expression is a solution for the forced harmonic oscillator (5) with $\delta = 0$,

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) - \frac{At}{2\omega} \cos(\omega t)$$

This effect is called **resonance** and occurs when the frequency of external vibrations is equal (or close) to the natural frequency of a system. Resonance oscillations are characterised by amplitudes which increase linearly with time.

This phenomenon is of great importance in engineering applications and can occur in any system which is subjected to periodic forces.