Approximation Theory

Let $f(x)$ be a continuous function on an interval $[a, b]$. If $P(x)$ is a polynomial, we are interested in finding

$$E[P] = \max_{a \leq x \leq b} |f(x) - P(x)|$$

the maximum possible error in the approximation of $f(x)$ by $P(x)$ on $[a, b]$.

For each degree n define

$$\rho_n(f) = \min_{deg(P) \leq n} E[P] = \min_{deg(P) \leq n} \left[\max_{a \leq x \leq b} |f(x) - P(x)| \right]$$

The **minimax error**, $\rho_n(f)$, is the smallest value for $E[P]$ that can be obtained with a polynomial of degree $\leq n$.

Minimax polynomial

It can be shown that the minimax error $\rho_n(f)$ on $[a, b]$ is achieved for a unique polynomial of degree $\leq n$ called the **minimax polynomial approximation** of order n, denoted by $M_n(x)$.

Example: Let $f(x) = \mathrm{e}^x$ on $[-1, 1]$ and consider linear polynomial approximations to f. The Taylor polynomial for this function is

$$T_1(x) = 1 + x$$

and the maximum possible error

$$E[T_1] = \max_{-1 \leq x \leq 1} |\mathrm{e}^x - T_1(x)| = 0.718$$

On the other hand, it can be shown that the linear minimax polynomial is

$$M_1(x) = 1.2643 + 1.1752x$$

for which the maximum possible error is

$$E[M_1] = \max_{-1 \leq x \leq 1} |\mathrm{e}^x - M_1(x)| = 0.279 < E[T_1]$$

Chebyshev polynomials

For any integer $n \geq 0$ define the function

$$T_n(x) = \cos(n \cos^{-1}(x)), \quad -1 \leq x \leq 1$$

We need to show that $T_n(x)$ is a polynomial of degree n. We calculate the functions $T_n(x)$ recursively.

Let $\theta = cos^{-1}(x)$ so $\cos(\theta) = x$. Then

$$T_n(x) = \cos(n\theta)$$

Easy to see that:

$$\begin{aligned}
n = 0 &\Longrightarrow T_0(x) = \cos(0) = 1 \\
n = 1 &\Longrightarrow T_1(x) = \cos(\theta) = x \\
n = 2 &\Longrightarrow T_2(x) = \cos(2\theta) = 2\cos^2(\theta) - 1 = 2x^2 - 1
\end{aligned}$$

Recurrence relations for Chebyshev polynomials

Using trigonometric formulas we can prove that

$$T_{n+m}(x) + T_{n-m}(x) = 2\,T_n(x)\,T_m(x)$$

for all $n \geq m \geq 0$ and all $x \in [-1, 1]$.

Hence, for $m = 1$ we get

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

which is then used to calculate the Chebyshev polynomials of higher order.

Example: Calculate $T_3(x)$, $T_4(x)$ and $T_5(x)$.

More properties of Chebyshev polynomials

Note that

$$|T_n(x)| \leq 1$$

and

$$T_n(x) = 2^{n-1}x^n + \text{lower degree terms}$$

for all $n \geq 0$ and all x in $[-1, 1]$.

If we define the **modified Chebyshev polynomial**:

$$\widetilde{T}_n(x) = \frac{T_n(x)}{2^{n-1}}$$

then we have

$$|\widetilde{T}_n(x)| \leq \frac{1}{2^{n-1}} \quad \text{and} \quad \widetilde{T}_n(x) = x^n + \text{lower degree terms}$$

for all $n \geq 0$ and all x in $[-1, 1]$.

Zeros of Chebyshev polynomials

We have

$$T_n(x) = \cos(n\theta), \qquad \theta = cos^{-1}(x)$$

so

$$T_n(x) = 0 \quad \Longrightarrow \quad n\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$$

which implies

$$\theta = \pm\frac{(2k+1)\pi}{2n}, \qquad k = 0, 1, 2, \dots$$

and hence the zeros of $T_n(x)$ are given by

$$x_k = \cos\left[\frac{(2k+1)\pi}{2n}\right], \qquad k = 0, 1, 2, \dots n-1.$$

The minimum size property

Let $n \geq 1$ be an integer and consider all possible monic polynomials (that is, polynomials whose highest-degree term has coefficient equal to 1) of degree n.

Then the degree n monic polynomial with the smallest maximum absolute value on $[-1, 1]$ is the modified Chebyshev polynomial $\widetilde{T}_n(x)$ and its maximum value is $1/2^{n-1}$.

A near-minimax approximation method

Let $f(x)$ be a continuous function on $[-1, 1]$. We are looking for an approximation given by an interpolating polynomial of degree 3, $C_3(x)$. Let x_0, x_1, x_2, x_3 be the interpolating nodes.

Recall the formula for the interpolation error:

$$f(x) - C_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{4!} f^{(4)}(\xi)$$

where ξ is in $[-1, 1]$.

We need to find the interpolating points so that we minimize

$$E[C_3] = \max_{-1 \leq x \leq 1} |f(x) - C_3(x)|$$

This is equivalent to minimizing

$$\max_{-1\leq x\leq 1} |(x-x_0)(x-x_1)(x-x_2)(x-x_3)|$$

But we know that the minimum value of a monic polynomial is obtained for the modified Chebyshev polynomial $\widetilde{T}_4(x)$ hence

$$(x-x_0)(x-x_1)(x-x_2)(x-x_3) = \frac{T_4(x)}{2^3} = \frac{1}{8}(8x^4 - 8x^2 + 1)$$

hence the interpolating points x_0, x_1, x_2, x_3 are the zeros of $T_4(x)$, that is

$$\cos(\frac{\pi}{8}),\ \cos(\frac{3\pi}{8}),\ \cos(\frac{5\pi}{8}),\ \cos(\frac{7\pi}{8})$$

Example

Let $f(x) = \mathrm{e}^x$ on $[-1, 1]$. In order to get the interpolating polynomial of degree 3 which approximates $f(x)$ such that the maximum error is minimized, the interpolation nodes x_0, x_1, x_2, x_3 have to be chosen as the zeros of $T_4(x)$.

We use Newton's divided difference formula for the interpolating polynomial

$$\begin{gathered} P_3(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ +(x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] \end{gathered}$$

where

$$\begin{aligned} f(x_0) &= \mathrm{e}^{\cos(\frac{\pi}{8})} \approx 2.5190 \\ f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \approx 1.94538 \\ &\vdots \end{aligned}$$

Recall that the degree n monic polynomial with the smallest maximum absolute value on $[-1, 1]$ is the modified Chebyshev polynomial $\widetilde{T}_n(x)$ and its maximum value is $1/2^{n-1}$.

Hence, the Chebyshev polynomials can be used to minimize approximation error by providing optimal interpolation points.

The Chebyshev polynomials also provide a method for reducing the degree of an approximating polynomial with minimal loss of accuracy.

Economization of power series

Consider approximating a polynomial of degree n

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

on $[-1, 1]$ with a polynomial of degree at most $n-1$. We need to choose $P_{n-1}(x)$ which minimizes

$$\max_{x \in [-1,1]} |P_n(x) - P_{n-1}(x)|$$

We know that

$$\max_{x \in [-1,1]} |\widetilde{T}_n(x)| = \frac{1}{2^{n-1}} \leq \max_{x \in [-1,1]} |\frac{1}{a_n}(P_n(x) - P_{n-1}(x))|$$

Hence we have

$$\frac{1}{a_n}(P_n(x) - P_{n-1}(x)) = \widetilde{T}_n(x)$$

so

$$P_{n-1}(x) = P_n(x) - a_n \widetilde{T}_n(x)$$

and this choice gives

$$\max_{x\in[-1,1]} |P_n(x) - P_{n-1}(x)| = \frac{|a_n|}{2^{n-1}}$$

Examples

1. Starting with the fourth-order MacLaurin polynomial, find the polynomial of least degree which best approximates the function $f(x) = \mathrm{e}^x$ on $[-1, 1]$ while keeping the error less than 0.05.
2. Find the sixth order MacLaurin polynomial for $\sin(x)$ and use Chebyshev polynomials to obtain a lesser degree polynomial approximation while keeping the error less than 0.01 on $[-1, 1]$.
3. Use the zeros of $\widetilde{T}_3$ to construct an interpolating polynomial of degree 2 for the functions (i) $\sin(x)$ and (ii) $\ln(x+2)$, on [-1,1].