Mathematics for Physical Sciences III: Sample Paper

1 (a) The position vector of a particle moving in the plane is given by

$$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}.$$

Calculate the velocity, speed, direction of motion and acceleration of the particle.

[10 marks]

(b) If $\Phi = xy + yz + zx$ and $\mathbf{A} = x^2y\mathbf{i} + y^2z\mathbf{j} + z^2x\mathbf{k}$, find (a) $\mathbf{A} \cdot \nabla\Phi$; (b) $\Phi\nabla \cdot \mathbf{A}$ and (c) $(\nabla\Phi) \times \mathbf{A}$, at the point (3,-1,2).

[15 marks]

2 (a) A vector field $\mathbf{A}$ is defined as

$$\mathbf{A} = (z^2 + 2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}$$

Show that $\mathbf{A}$ is conservative and calculate the scalar potential Φ.

[12 marks]

(b) Evaluate the line integral $\int_C (x + y)\,dx + (y - x)\,dy$ along each of the following 2-dimensional curves:

i. the parabola $y^2 = x$ from (1,1) to (4,2);

ii. the curve $x = 2t^2 + t + 1$, $y = 1 + t^2$ from (1,1) to (4,2);

[13 marks]

3 (a) Evaluate the line integral

$$\int_C xydy - y^2dx$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

[10 marks]

(b) Using Green's theorem in the plane, calculate the integral in (i) by transforming it into a double integral first. [15 marks]

Note: Green's theorem in the plane relates a line integral around a closed curve, C, to a double integral over the region R enclosed by the curve,

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

4 Consider the function $f(x) = x^2$ for $-\pi \leq x \leq \pi$, together with its periodic expansion, $f(x + 2\pi) = f(x)$. Sketch the graph of this function and calculate its Fourier series. Is this function even or odd?

[25 marks]

Note: The Fourier series of a 2π-periodic function $f(x)$ on $[-\pi, \pi]$ is given by

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + \sum_{n=1}^{\infty} B_n \sin(nx)$$

and the coefficients are calculated with the formulas

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx; \quad A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx; \quad B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

5 Consider the function

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ -1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the graph of this function and express it in terms of the Heaviside unit step function.

[10 marks]

(b) Calculate the Fourier transform of the function $f(x)$.

[15 marks]

Hint: The Fourier transform of $f(t)$ is given by $F(s) = \int_{-\infty}^{\infty} \mathrm{e}^{-ist} f(t)\, dt$

6 (a) A component manufacturer knows that 2% of components produced are defective, but guarantees that there are no more than two defective items in a box of 20. What is the probability that a box satisfies the guarantee?

[10 marks]

(b) The height of male students at a particular university is normally distributed with mean $\mu = 70$in and standard deviation $\sigma = 4$in. Calculate: (i) the probability that a randomly selected male is between 67 and 76 inches; (ii) the probability that a randomly selected male is taller than 80 inches.

[15 marks]