**DUBLIN INSTITUTE OF TECHNOLOGY**

**KEVIN STREET, DUBLIN 8**

# B.Sc. in Mathematics (Ordinary)

**SAMPLE PAPER**

## Numerical Methods 2

Full marks for complete answers to **five** questions.

Use at least three significant digits and show all your calculations clearly.

Mathematics Tables, Graph Paper

**1** The Lagrange polynomial which interpolates the data $(x_i, f_i)$, where $f_i = f(x_i)$ for $i = 0, 1, 2 \dots n$, is given by

$$P_n(x) = \sum_{i=0}^{n} L_{n,i}(x) f_i \qquad \text{where} \qquad L_{n,i}(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}.$$

and the interpolation error is given by the formula

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0) \cdots (x - x_n)$$

Consider the function $f(x) = \sin(x)$.

(a) Construct the Lagrange form of the interpolating polynomial for $f$ passing through the points $(0, \sin(0))$, $(\pi/4, \sin(\pi/4))$ and $(\pi/2, \sin(\pi/2))$.

[8 marks]

(b) Use this polynomial to estimate $\sin(\pi/3)$. What is the error in this approximation?

[4 marks]

(c) Establish the theoretical error bound for using the polynomial found in part (a) to approximate $\sin(\pi/3)$. How does this error compare with that in part (b)?

[8 marks]

**2** The Newton form of the interpolating polynomial is

$$P(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) +$$
$$+ \cdots f[x_0, x_1, \dots x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

where

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}; \qquad f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \quad \text{etc.}$$

are the divided differences of $f(x)$ with respect to the interpolating points.

Consider the function $f(x) = \mathrm{e}^x$.

(a) Construct the Newton interpolating polynomial for $f$ passing through the points $(-1, \mathrm{e}^{-1})$, $(0, \mathrm{e}^{0})$ and $(1, \mathrm{e}^{1})$.

[8 marks]

(b) Use this polynomial to estimate $\mathrm{e}^{0.1}$. What is the error in this approximation?

[4 marks]

(c) Establish the theoretical error bound for using the polynomial found in part (a) to approximate $\mathrm{e}^{0.1}$. How does this error compare with that in part (b)?

[8 marks]

**3** (a) Consider the Cholesky factorization of a matrix, $A = LL^T$. What conditions must $A$ and $L$ satisfy? In the case of a $3\times3$ matrix, derive the equations satisfied by the entries of $L$ and indicate the order in which these should be solved.

[6 marks]

(b) Find the Cholesky factorization for the matrix

$$B = \begin{pmatrix} 6 & -2 & 3 \\ -2 & 8 & 1 \\ 3 & 1 & 7 \end{pmatrix}$$

[6 marks]

(c) Use the result of part (b) to solve the system

$$\begin{aligned} 6x_1 - 2x_2 + 3x_3 &= 4 \\ -2x_1 + 8x_2 + x_3 &= -2 \\ 3x_1 + x_2 + 7x_3 &= 8. \end{aligned}$$

[8 marks]

**4** Consider the matrix

$$A = \begin{pmatrix} 16 & 7 & -7 \\ -1 & 2 & 1 \\ 11 & 7 & -5 \end{pmatrix}$$

(a) Sketch the Gerschgorin circles for the matrix $A$. What information do they give regarding the eigenvalues of $A$?

[4 marks]

(b) Using exact arithmetic, calculate the matrix $B = (A - 3I)^{-1}$. How are the eigenvalues and eigenvectors of $B$ related to those of $A$?

[8 marks]

(c) Using 3 iterations of the inverse power method, approximate the eigenvalue of $A$ closest to the number 3, as well as an associated eigenvector. Use $\mathbf{x}^{(0)} = [1, 1, 1]^T$ as your initial approximation.

[8 marks]

**5** (a) Using the Euler method and a step size of $h = 0.25$, find an approximate solution for the initial value problem

$$\frac{dx}{dt} = 3t - \frac{x}{t}, \qquad 1 \leq t \leq 3,$$
$$x(1) = 2.$$

Given that the exact solution is $x(t) = t^2 + \frac{1}{t}$, calculate the approximation error at each step.

[10 marks]

(b) Approximate the solution of the problem in part (b) using the fourth order Runge-Kutta method with a step size of $h = 0.25$. How do the absolute errors in this case compare with those obtained from the Euler's method?

[10 marks]

*Hint: The fourth-order Runge-Kutta formula is given by* $w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ *where* $k_1 = hf(t_i, w_i)$, $k_2 = hf(t_i + \frac{h}{2}, w_i + \frac{k_1}{2})$, $k_3 = hf(t_i + \frac{h}{2}, w_i + \frac{k_2}{2})$, $k_4 = hf(t_i + h, w_i + k_3)$, *where* $w_i$ *is the approximate solution at time* $t_i = t_0 + ih$.

**6** (a) Determine the weights $w_1$, $w_2$, $w_3$, and the quadrature points $x_1$, $x_2$, $x_3$ for the three-point Gaussian quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

knowing that this approximation formula has degree of precision equal to 5.

[12 marks]

(b) Use the three-point Gaussian quadrature rule to approximate the value of the integral

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx.$$

What is the absolute error in this approximation?

[8 marks]

**7** The composite trapezoidal rule for approximating the value of the integral $I(f) = \int_a^b f(x)\,dx$ is given by

$$T_h(f) = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] - \frac{(b-a)h^2}{12} f''(\bar{\xi})$$

where $a < \bar{\xi} < b$, $h = (b-a)/n$ and $x_i = a + ih$ for all $i = 0, \dots n$.

(a) Consider the integral

$$I(f) = \int_0^{\pi} \sin(x)\,dx$$

Compute a sequence of approximations $T_h(f)$ which shows clearly the convergence to $I(f)$ and the rate of convergence corresponding to this method.

[12 marks]

(b) Determine the number of intervals needed in the composite trapezoidal rule such that, when approximating the value of the integral in part (b) the error is less than $10^{-4}$. (Do not attempt to determine this approximation.)

[8 marks]

**8** (a) Define the Chebyshev polynomials, $T_n(x)$, and prove the recurrence formula

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x),$$

for $n \geq 0$ and $-1 \leq x \leq 1$. (*Hint: Use the substitution* $\theta = \cos^{-1}(x)$.)

[5 marks]

(b) Using the definition or the recurrence formula in part (a), calculate the polynomials $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$ and $T_4(x)$.

[5 marks]

(c) Starting with the fourth-order Maclaurin polynomial and using the modified Chebyshev polynomials $\widetilde{T}_n(x) = T_n(x)/2^{n-1}$, find the polynomial of least degree which best approximates the function $f(x) = \mathrm{e}^x$ on $[-1, 1]$, while keeping the error less than 0.05.

[10 marks]