12 March 2012

Problem Sheet 4: Introduction to Fourier Series

The Fourier series of a 2π-periodic function $f(x)$ on $[-\pi, \pi]$ is given by

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + \sum_{n=1}^{\infty} B_n \sin(nx)$$

and the coefficients are calculated with the formulas

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx; \quad A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx; \quad B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Exercises:

1. Consider the function $f(x) = x^2$ for $-\pi \le x \le \pi$, together with its periodic expansion, $f(x + 2\pi) = f(x)$. Sketch the graph of this function and calculate its Fourier series. Is this function even or odd?

   Answer:
   $$x^2 = \frac{\pi^2}{3} - 4 \left(\cos(x) - \frac{\cos(2x)}{2^2} + \frac{\cos(3x)}{3^2} - \cdots \right)$$

2. Consider the function $f(x) = |\sin(x)|$ for $-\infty < x < \infty$. Sketch a graph of this function and calculate its Fourier series. Is this function even or odd?

   Answer:
   $$|\sin(x)| = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos(2x)}{3} + \frac{\cos(4x)}{15} + \frac{\cos(6x)}{35} + \cdots \cdots \right)$$

3. Consider the function $f(x) = x$ for $0 \le x \le \pi$. Construct its even extension onto the interval $[-\pi, \pi]$ and then the periodic extension to the whole real axis and sketch a graph of this function. Calculate its cosine Fourier series.

   Answer:
   $$x = \pi - \frac{4}{\pi} \left(\cos(x) + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \cdots \right)$$