Arithmetic Series

An arithmetic sequence is one where consecutive terms have a common difference:

$$d = u_{n+1} - u_n$$

Example: Given the arithmetic sequence:

$$4, 7, 10, 13, 16, \ldots$$

Here, the common difference is 3. So $d = 3$.

Arithmetic Series

In an arithmetic sequence, the first term (u_1) is usually called a so:

$$u_1 = a$$
$$u_2 = a + d$$
$$u_3 = a + 2d$$
$$u_4 = a + 3d$$

In our sample sequence 4, 7, 10, 13, 16, ... $a = 4$ and we know $d = 3$

$$u_1 = a = 4$$
$$u_2 = a + d = 4 + 3$$
$$u_3 = a + 2d = 4 + 6$$
$$u_4 = a + 3d = 4 + 9$$

Arithmetic Series

$$u_1 = a$$
$$u_2 = a + d$$
$$u_3 = a + 2d$$
$$u_4 = a + 3d$$
$$\vdots$$
$$u_n = a + (n-1)d$$

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots + [a + (n-1)d]$$

Arithmetic Series

$$S_n = a + (a+d) + (a+2d) + \cdots + [a+(n-2)d] + [a+(n-1)d]$$

$$\underline{S_n = [a+(n-1)d] + [a+(n-2)d] + \cdots + (a+2d) + (a+d) + a}$$

$$2S_n = [2a+(n-1)d] + [2a+(n-1)d] + [2a+(n-1)d] + \ldots [2a+(n-1)d] + [2a+(n-1)d]$$

$$2S_n = n[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Arithmetic Series – Overview

The common difference d is found by subtracting one term from the term which immediately follows it:

$$d = u_{n+1} - u_n$$

To find the nth term in an arithmetic sequence, apply the following:

$$u_n = a + (n-1)d$$

To find the sum of the first n terms in an arithmetic sequence, apply the following:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Arithmetic Series – Ex. 1

In the following arithmetic series:

$$3, 7, 11, 15, \ldots$$

a) What is the 12^{th} term in this sequence?

b) Find the sum of the first 12 terms in the sequence.

Solution:

We can see that $a = 3$ and $d = 4$

$$u_n = a + (n-1)d$$

$$u_{12} = 3 + (12-1)4$$

$$u_{12} = 47$$

The 12^{th} term in the sequence is 47.

Arithmetic Series – Ex. 1

b) Find the sum of the first 12 terms in the sequence.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{12} = \frac{12}{2}[2(3) + (12-1)4]$$

$$S_{12} = 6(6+44) = 300$$

$$3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 + 43 + 47 = 300$$

Arithmetic Series – Ex. 2

In the following arithmetic series:

$$2, 7, 12, 17, \ldots$$

a) What is the 27th term in this sequence?

b) Find the sum of the first 26 terms in the sequence.

Solution:

We can see that $a = 2$ and $d = 5$

$$u_n = a + (n-1)d$$

$$u_{27} = 2 + (27-1)5$$

$$u_{27} = 132$$

The 27th term in the sequence is 132.

Arithmetic Series – Ex. 2

b) Find the sum of the first 26 terms in the sequence.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{26} = \frac{26}{2}[2(2) + (26-1)5]$$

$$S_{26} = 13(4 + 125) = 1677$$

$$2 + 7 + 12 + 17 \ldots + 127 = 1677$$

Arithmetic Series – Ex. 3

Q. A stack of telephone poles has 30 poles in the bottom row. There are 29 poles in the second row, 28 in the next row, and so on. How many poles are in the stack if there are 5 poles in the top row?

Arithmetic Series – Ex. 3

Our arithmetic series is as follows:

$$5 + 6 + 7 + \ \dots + 29 + 30$$

We know that $a = 5$ and $d = 1$

If 5 is the 1st term, 6 is the 2nd term, 7 is the 3rd term then 30 must be the 26th term.

Thus, to add up all the poles, we must find the sum of the first 26 terms i.e. S_{26}

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{26} = \frac{26}{2}[2(5) + (26-1)1]$$

$$S_{26} = 455$$

There are 455 poles in the stack.

Arithmetic Series – Ex. 4

In an arithmetic sequence, the fifth term is -18 and the tenth term is 12.

i. Find the first term and the common difference.

ii. Find the sum of the first fifteen terms of the sequence.

$$u_n = a + (n-1)d$$

$$u_5 = a + 4d = -18$$
$$u_{10} = a + 9d = 12$$

$$a + 4d = -18$$
$$\underline{-a - 9d = -12}$$
$$-5d = -30$$

$$d = 6$$

Arithmetic Series – Ex. 4

$$a + 4d = -18$$

$$a + 4(6) = -18$$

$$a = -42$$

First term is -42 and the common difference is 6

ii. Find the sum of the first fifteen terms of the sequence.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{15} = \frac{15}{2}[2(-42) + (15-1)6]$$

$$S_{15} = \frac{15}{2}(-84 + 84) = 0$$

the sum of the first fifteen terms of the sequence is zero.

Geometric Sequences & Series

A sequence is geometric if the ratio, $\frac{u_{n+1}}{u_n}$, between any two consecutive terms is a constant.

This constant is called the common ratio and is usually denoted by r.

Example:

$$2, 4, 8, 16, 32, 64, \ldots$$

The common ratio here is 2.

Geometric Sequences & Series

$$u_1 = a$$
$$u_2 = ar$$
$$u_3 = ar^2$$
$$u_4 = ar^3$$
$$\vdots$$
$$u_n = ar^{n-1}$$

$$S_n = u_1 + u_2 + u_3 + \dots + u_n$$

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

Geometric Sequences & Series

$$S_n = a + ar + ar^2 + ar^3 + \ \ldots + ar^{n-1}$$

This can be shortened into the following equation:

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$

Geometric Sequences & Series - Overview

To find the common ratio r:

$$r = \frac{u_{n+1}}{u_n}$$

To find the n^{th} term in the geometric sequence, apply the following:

$$u_n = ar^{n-1}$$

To find the sum of the first n terms in a geometric sequence, apply the following:

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

Geometric Series – Ex. 1

Find the sum of the following geometric series:

$5 + 10 + 20 + 40 + \ \ldots + 10240$

Solution:

We know $a = 5$ and $r = 2$

We need to find out how many terms are in the series.

$$u_n = ar^{n-1}$$

$$5(2^{n-1}) = 10240$$

$$2^{n-1} = 2048$$

$$\log_2 2048 = n - 1$$

$$11 = n - 1$$

$$n = 12$$

Geometric Series – Ex. 1

We now know that 10240 is the 12th term so there are 12 terms in total in the series.

To find the sum of the geometric series we must find S_{12}

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_{12} = 5\left(\frac{1-2^{12}}{1-2}\right)$$

$$S_{12} = 5\left(-\frac{4095}{-1}\right)$$

$$S_{12} = 20475$$

The sum of the geometric series is 20,475

Geometric Series – Ex. 2

Find the sum of the following geometric series:

$$3 + 12 + 48 + \ \ldots + 3{,}072$$

Solution:

We know $a = 3$ and $r = 4$

We need to find out how many terms are in the series.

$$u_n = ar^{n-1}$$

$$3(4^{n-1}) = 3072$$

$$4^{n-1} = 1024$$

$$\log_4 1024 = n - 1$$

$$5 = n - 1$$

$$n = 6$$

Geometric Series – Ex. 2

We now know that 3072 is the 6th term so there are 6 terms in total in the series.

To find the sum of the geometric series we must find S_6

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_6 = 3\left(\frac{1-4^6}{1-4}\right)$$

$$S_6 = 3\left(-\frac{4095}{-3}\right)$$

$$S_6 = 4095$$

The sum of the geometric series is 4,095

Geometric Series – Ex. 3

Most lottery games in the USA allow winners of the jackpot prize to choose between two forms of the prize: an annual-payments option or a cash-value option.

In the case of the New York Lotto, there are 26 annual payments in the annual-payments option, with the first payment immediately, and the last payment in 25 years time.

The payments increase by 4% each year. The amount advertised as the jackpot prize is the total amount of these 26 payments. The cash-value option pays a smaller amount than this.

Geometric Series – Ex. 3

(a) If the amount of the first annual payment is a, write down, in terms of a, the amount of the second, third, fourth and 26th payments.

1^{st} payment = a

2^{nd} payment = $1.04a$

3^{rd} payment = $1.04(1.04a) = (1.04)^2 a$

4^{th} payment = $1.04[(1.04)^2 a] = (1.04)^3 a$

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26^{th} payment = $(1.04)^{25} a$

Geometric Series – Ex. 3

(b) The 26 payments form a geometric series. Use this fact to express the advertised jackpot prize in terms of a.

$$a + 1.04a + (1.04)^2 a + (1.04)^3 a + \ \dots + (1.04)^{25} a$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_{26} = a\left(\frac{1-1.04^{26}}{1-1.04}\right) = a\left(-\frac{1.7724}{-0.04}\right)$$

$$S_{26} = 44.312a = Jackpot\ prize$$

Geometric Series – Ex. 3

(c) Find, correct to the nearest dollar, the value of a that corresponds to an advertised jackpot prize of \$21·5 million.

$$\text{Jackpot prize} = 44.312a$$

$$21{,}500{,}000 = 44.312a$$

$$a = \$485{,}196$$

This means that the initial payment will be \$485,196 and will increase by 4% with each subsequent payment.

Geometric Series – Ex. 4

Q. In January 2013, HMV staff in Limerick City held a sit-in to aid negotiations for a fair severance package. The company were offering a total of €800,000 to be shared out amongst all the HMV staff in Ireland as a severance package.

The staff were not happy with this figure and instead suggested that, in the month of February 2013, HMV would pay 1 cent into the severance package for all staff on Feb. 1st, 2 cent on Feb. 2nd, 4 cent on Feb. 3rd, 8 cent on the 4th, 16 cent on the 5th and so on for the rest of that month. HMV agreed to the deal immediately. How much did HMV end up paying to the staff?

Geometric Series – Ex. 4

The payments would add up as follows:

$$€0.01 + €0.02 + €0.04 + €0.08 + €0.16 \ldots$$

We can see that this is a geometric series.

$$a = 0.01 \quad r = 2$$

There will be 28 payments as there are 28 days in February in 2013, so we need to find the sum of the first 28 terms of this series.

Geometric Series – Ex. 4

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$

$$S_{28} = 0.01\left(\frac{1 - 2^{28}}{1 - 2}\right)$$

$$S_{28} = 0.01\left(\frac{1 - 268{,}435456}{-1}\right)$$

$$S_{28} = 2{,}684{,}354.55$$

Geometric Series – Ex. 4

In the end, HMV ended up paying out €2,684,354.55 in severance fees to their Irish staff.

Sum to Infinity of a Geometric Series

$$S_\infty = u_1 + u_2 + u_3 + \ \dots + u_n + \ \dots$$

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right)$$

This limit will exist if $|r| < 1$ The series will converge giving:

$$S_\infty = a\left(\frac{1}{1-r}\right)$$

This is because $r^n \to 0$ as $n \to \infty$ if $|r| < 1$

Sum to Infinity of a Geometric Series

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right)$$

This limit will not exist if $|r| > 1$ Thus, the series will diverge.

This is because $r^n \to \infty$ as $n \to \infty$ if $|r| > 1$.

So, to find the sum to infinity of a geometric series, the modulus of the common ratio r must be less than 1.

Sum to Infinity – Ex. 1

Q. Find the sum to infinity of the following series:

$$1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \ \ldots$$

We know:

$$a = 1$$

$$r = \frac{1}{6}$$

$|r| < 1$ so the series will converge:

$$S_\infty = a\left(\frac{1}{1-r}\right)$$

$$S_\infty = 1\left(\frac{1}{1-\frac{1}{6}}\right)$$

$$S_\infty = \frac{6}{5}$$

Sum to Infinity – Ex. 2

Q. Write the recurring decimal 0.474747... as an infinite geometric series and hence as a fraction.

Solution: We can write the number as a geometric series:

$$0.474747\ldots = \frac{47}{100} + \frac{47}{10{,}000} + \frac{47}{1{,}000{,}000} + \ldots$$

Now, we know $a = \frac{47}{100}$ and $r = \frac{1}{100}$

Sum to Infinity – Ex. 2

$|r| < 1$ so the series will converge and:

$$S_\infty = a\left(\frac{1}{1-r}\right)$$

$$S_\infty = \frac{47}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{47}{100}\left(\frac{100}{99}\right)$$

$$S_\infty = \frac{47}{99}$$

$$0.474747\ldots = \frac{47}{99}$$

Telescoping Series

A Telescoping Series is a series whose partial sums eventually only have a fixed number of terms after cancellation.

This approach is commonly used for sums of certain infinite series.

Telescoping Series – Ex. 1

Q. Find

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution: Normally we would outline the series and check if it is arithmetic or geometric and solve accordingly:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ \dots$$

But we can see that it is neither arithmetic nor geometric so we use a different method to solve this type of series.

Telescoping Series – Ex. 1

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

First, we separate the function into two distinct functions:

$$\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}$$

Now we need to figure out what a and b are equal to…

Telescoping Series – Ex. 1

$$\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{a(n+1) + b(n)}{n(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{an + a + bn}{n(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{(a+b)n + a}{n(n+1)}$$

$$1 = (a+b)n + a$$

Equate coefficients:

n-coefficient on the left of the equation equals n-coefficient on the right of the equation:

$$0 = a + b$$

Constant value on the left of the equation equals constant value on the right of the equation:

$$1 = a$$