



FACULTY OF SCIENCE AND ENGINEERING
DEPARTMENT OF MATHEMATICS AND STATISTICS

END OF SEMESTER EXAMINATION PAPER 2016

MODULE CODE: MS4131

SEMESTER: Spring 2016

MODULE TITLE: Linear Algebra 1

DURATION OF EXAM: 2.5 hours

LECTURER: Mr. Kevin O'Brien

GRADING SCHEME: 80 marks

100% of module grade

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES

Scientific calculators approved by the University of Limerick can be used.
Students must attempt any 4 questions from 5.

Question 1

Part A. Matrix Multiplication (4 Marks)

Given the matrices

$$A = \begin{pmatrix} 2 & 2 & 1 & -1 & 4 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 4 \\ 0 & 5 \\ -1 & 0 \\ 7 & 1 \\ 1 & 3 \end{pmatrix}; \quad C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

calculate the products AB and CA .

Part B. Diagonal Matrices (3 Marks)

Consider the following diagonal matrix D . In terms of the values a , b and c ;

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

- (i) (1 Mark) Write an expression for the trace of the matrix D .
- (ii) (1 Mark) State the inverse of D , i.e. D^{-1} .
- (iii) (1 Mark) State the matrix D^3 .

Part C. Matrix Multiplication (4 Marks)

Suppose A is a lower triangular matrix of the form;

$$A = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

- (i) (1 Mark) State the transpose of A .
- (ii) (2 Marks) Compute B where $B = A \times A^T$.
- (iii) (1 Mark) B is a symmetric matrix. What is meant by this?

Question 2

Part A. Fundamental Theorem of Invertible Matrices (5 Marks)

The Fundamental Theorem of Invertible Matrices states that a set of mathematical expressions concerning a $n \times n$ matrix A are each equivalent to one another.

- (i) (4×1 Mark) State any four of these expressions.
- (ii) (1 Mark) What is the rank of a matrix.

Part B. Invertible Matrices (5 Marks)

Show that if A is an $n \times n$ invertible matrix that satisfies

$$9A^3 + A^2 - 3A = 0$$

where $A^n = \underbrace{A \dots A}_{n \text{ times}}$, I is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix, then the inverse of A is given by

$$A^{-1} = \frac{1}{3}I + 3A.$$

Part C. Inverting a Matrix with E.R.O.s (5 Marks)

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -4 & 1 \\ -3 & 0 & -1 \end{pmatrix}.$$

using elementary row operations.

Question 3

Part A. System of Linear Equations (6 Marks)

Consider the linear system

$$\begin{aligned}x_1 + x_3 &= 4 \\2x_1 + 4x_2 + x_3 &= -3 \\x_2 + 3x_3 &= 7.\end{aligned}$$

- (i) (1 Mark) Write down the coefficient matrix and the augmented matrix of this system.
- (ii) (1 Mark) What can you say about the solution set of the system? Justify your answer.
- (iii) (4 Marks) Solve the system of equations, using any appropriate method.

Part B. Row-Echelon Form of a Matrix (4 Marks)

Consider the matrices U, V, W and X presented below. For each matrix state one reason why that matrix is not in row-echelon form. Provide distinct answers for each of the four matrices.

$$\begin{aligned}U &= \begin{pmatrix} 1 & 2 & 6 & 3 & 5 \\ 0 & 1 & 4 & 0 & 6 \\ 0 & 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} & V &= \begin{pmatrix} 1 & 1 & 6 & 8 & 5 \\ 0 & 2 & 6 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix} \\ W &= \begin{pmatrix} 1 & 1 & 6 & 8 & 5 \\ 0 & 1 & 6 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} & X &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 6 & 8 & 5 \\ 0 & 1 & 6 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

(Marking Scheme: 4×1 Marks where 1 Mark is awarded for each valid and distinct reason)

Part C. Distance from Planes (6 Marks)

- (i) (3 Marks) Give the general form of the equation of the plane π in \mathbb{R}^3 passing through the point $P_0 = (1, 0, 2)$ with the vector $n = (-5, 3, 2)$ as the normal.
- (ii) (3 Marks) Show that the point $Q = (1, -1, 1)$ does not lie in the plane π and find its distance from π .

Question 4

Part A. Vector Calculations (7 Marks)

Consider the three vectors in \mathbb{R}^3 :

$$u = (1, 2, 0), \quad v = (0, 1, 3), \quad w = (1, 2, 3).$$

- (i) Evaluate $\|u\|$, $\|v\|$, $u \cdot v$, $u \times v$ and the angle between u and v .
- (ii) Calculate the scalar triple product $u \cdot (v \times w)$.

Part B. Orthonormal Projections (8 Marks)

If

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix},$$

- (i) find the vector component of u along a , $proj_a u$ and the vector orthogonal component to a ;
- (ii) calculate the norm of $proj_a u$ and the norm of $u - proj_a u$;
- (iii) draw $proj_a u$ showing its direction and orientation.

Part C. Vector Calculations (7 Marks)

Prove that, for any $u, v \in \mathbb{R}^3$,

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u.$$

(It is sufficient to verify this property for one component.)

Part D. Vector Calculations (7 Marks)

1. Find the general form of the equation of the plane π in \mathbb{R}^3 which passes through the point $P = (3, 1, 6)$ and is orthogonal to the vector $n = (1, 7, -2)$.
2. Show that the point $Q = (1, -1, 1)$ does not lie in the plane π and find its distance from π .

Question 5

Part A. Vector Calculations (7 Marks)

Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- (i) Determine the eigenvalues and corresponding eigenvectors of A .
- (ii) Diagonalise A ; i.e, give a matrix P and a diagonal matrix D , such that $A = PDP^{-1}$.
- (iii) Hence, evaluate A^5 .

Part B. Addition of Vectors

Prove that for any $u, v, \in \mathbb{R}^3$ and any $k \in \mathbb{R}$ we have

- (i) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$;
- (ii) $(u + v) \times w = u \times w + v \times w$;
- (iii) $k(u \times v) = (ku) \times v = u \times (kv)$.