16 April 2012

Problem Sheet 5: Fourier Transforms

1. Sketch the graph of the following triangle function

$$f(x) = \begin{cases} 1+x & \text{if } -1 < x < 0 \\ 1-x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and calculate its Fourier transform.
Hint: use integration by parts. The answer is $F(s) = 4\sin^2(\frac{s}{2})/\sqrt{2\pi}$.

2. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ -1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Sketch the graph of this function and express it in terms of the Heaviside unit step function. Then calculate its Fourier transform.
Answer:

$$f(x) = H(x-1) - 2H(x) + H(x+1); \qquad F(s) = \frac{4i}{s}\sin^2(\frac{s}{2})$$

3. Find the Fourier transform of

$$f(x) = \begin{cases} \cosh(x) & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The answer is

$$F(s) = \frac{(\mathrm{e}^2-1)\cos(s) + s(\mathrm{e}^2+1)\sin(s)}{\mathrm{e}\sqrt{2\pi}(s^2+1)}$$

4. The displacement of a damped harmonic oscillator as a function of time is given by

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ \mathrm{e}^{-t/\tau}\sin(\omega_0 t) & \text{for } t \geq 0. \end{cases}$$

Find the Fourier transform of this function.
Hint: Use the formula $\sin(\omega_0 t) = (\mathrm{e}^{i\omega_0 t} - \mathrm{e}^{-i\omega_0 t})/2i$. *The answer is*

$$F(s) = \frac{1}{2}\left[\frac{1}{s+\omega_0 - i/\tau} - \frac{1}{s-\omega_0-i/\tau}\right]$$