Limit Properties

Limit Properties The time has almost come for us to actually compute some limits. However, before we do that we will need some properties of limits that will make our life somewhat easier. So, lets take a look at those first. The proof of some of these properties can be found in the Proof of Various Limit Properties section of the Extras chapter. Properties First we will assume that and exist and that c is any constant. Then,

Limit Properties

In other words we can factor a multiplicative
constant out of a limit.
So to take the limit of a sum or difference all we
need to do is take the limit of the individual parts
and then put them back together with the
appropriate sign. This is also not limited to two
functions. This fact will work no matter how many
functions weve got separated by $+$ or -.

Limit Properties

We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

As noted in the statement we only need to worry about the limit in the denominator being zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

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In this property n can be any real number (positive, negative, integer, fraction, irrational, zero, etc.). In the case that n is an integer this rule can be thought of as an extended case of 3.

For example consider the case of n = 2.

The same can be done for any integer n.

Limit Properties

This is just a special case of the previous example. In other words, the limit of a constant is just the constant. You should be able to convince yourself of this by drawing the graph of .

Limit Properties

As with the last one you should be able to convince yourself of this by drawing the graph of .
This is really just a special case of property 5 using .

Note that all these properties also hold for the two one-sided limits as well we just didnt write them down with one sided limits to save on space.

Lets compute a limit or two using these properties. The next couple of examples will lead us to some truly useful facts about limits that we will use on a continual basis.

Example 1 Compute the value of the following limit.

Limit Properties

Solution This first time through we will use only the properties above to compute the limit.

First we will use property 2 to break up the limit into three separate limits. We will then use property 1 to bring the constants out of the first two limits. Doing this gives us,

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We can now use properties 7 through 9 to actually compute the limit.

Now, lets notice that if we had defined
then the proceeding example would have been,

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In other words, in this case we see that the limit is the same value that wed get by just evaluating the function at the point in question. This seems to violate one of the main concepts about limits that weve seen to this point.

Limit Properties

In the previous two sections we made a big deal about the fact that limits do not care about what is happening at the point in question. They only care about what is happening around the point. So how does the previous example fit into this since it appears to violate this main idea about limits?

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Despite appearances the limit still doesnt care about what the function is doing at . In this case the function that weve got is simply nice enough so that what is happening around the point is exactly the same as what is happening at the point. Eventually we will formalize up just what is meant by nice enough. At this point lets not worry too much about what nice enough is. Lets just take advantage of the fact that some functions will be nice enough, whatever that means.

Limit Properties

The function in the last example was a polynomial. It turns out that all polynomials are nice enough so that what is happening around the point is exactly the same as what is happening at the point. This leads to the following fact.

Fact If p(x) is a polynomial then,
By the end of this section we will generalize this out
considerably to most of the functions that well be
seeing throughout this course.

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Lets take a look at another example.

Example 2 Evaluate the following limit.

Solution First notice that we can use property 4) to write the limit as,

Limit Properties

Well, actually we should be a little careful. We can do that provided the limit of the denominator isnt zero. As we will see however, it isnt in this case so were okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.

Notice that the limit of the denominator wasnt zero and so our use of property 4 was legitimate.

Limit Properties

Notice in this last example that again all we really did was evaluate the function at the point in question. So it appears that there is a fairly large class of functions for which this can be done. Lets generalize the fact from above a little.

Fact Provided f(x) is nice enough we have,

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Again, we will formalize up just what we mean by nice enough eventually. At this point all we want to do is worry about which functions are nice enough. Some functions are nice enough for all x while others will only be nice enough for certain values of x. It will all depend on the function.

As noted in the statement, this fact also holds for the two one-sided limits as well as the normal limit. Here is a list of some of the more common functions that are nice enough.

Limit Properties

Polynomials are nice enough for all xs. If then f(x)
will be nice enough provided both p(x) and q(x) are
nice enough and if we dont get division by zero at
the point were evaluating at.

Limit Properties

are nice enough for all xs are nice enough provided In other words secant and tangent are nice enough everywhere cosine isnt zero. To see why recall that these are both really rational functions and that cosine is in the denominator of both then go back up and look at the second bullet above.

Limit Properties

are nice enough provided In other words cosecant and cotangent are nice enough everywhere sine isnt zero. is nice enough for all x if n is odd. is nice enough for if n is even. Here we require to avoid having to deal with complex values. are nice enough for all xs. are nice enough for x¿0. Remember we can only plug positive numbers into logarithms and not zero or negative numbers.

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Any sum, difference or product of the above functions will also be nice enough. Quotients will be nice enough provided we dont get division by zero upon evaluating the limit.

The last bullet is important. This means that for any combination of these functions all we need to do is evaluate the function at the point in question, making sure that none of the restrictions are violated. This means that we can now do a large number of limits.

Example 3 Evaluate the following limit.

Solution This is a combination of several of the functions listed above and none of the restrictions are violated so all we need to do is plug in into the function to get the limit.
Not a very pretty answer, but we can now do the limit.