Hyperbolic Functions

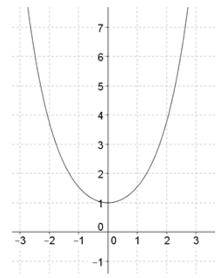
MA4702

Week 3

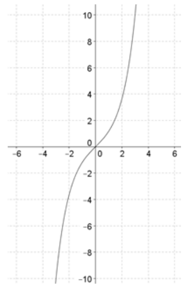
Presented by Páraic Treacy

Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



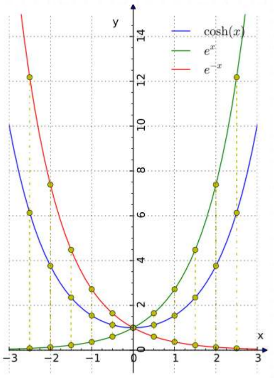
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Graphing Cosh x

This graph shows how adding e^x and e^{-x}, then dividing by 2, produces the $\cosh x$ function.

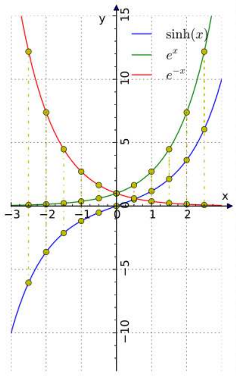
$$\cosh x = \frac{e^x + e^{-x}}{2}$$



Graphing sinh x

This graph shows how subtracting e^{-x} from e^x, then dividing by 2, produces the $\sinh x$ function.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Where are Hyperbolic Functions found in real life?

- The cables of a suspension bridge.
- A rope hanging between two posts.
- Each strand of a typical spider web.
- The Gateway Arch in St. Louis.

All of these are catenaries. Catenaries are segments from the graph of the hyperbolic cosine function $A\cosh(ax)$, where A and a are constants.

http://www.youtube.com/watch?v=AYIQYZuQNMw

Where are Hyperbolic Functions found in real life?

Catenaries occur naturally, since they minimize the gravitational potential energy of a string or rope whose location is fixed at two ends, which is equivalent to minimizing the area under the string.

They are also optimal for architects when a flexible cable (or its equivalent) is subject to a uniform force (e.g., gravity or the weight of a bridge, etc.). Example: The Golden Gate Bridge was designed to take advantage of this phenomenon.

Hyperbolic Identities – Ex. 1

Prove:

$$\sinh 2x = 2\sinh x\cosh x$$

Solution: Write the right-hand side in terms of exponentials

$$\sinh 2x = 2\sinh x\cosh x$$

$$\sinh 2x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$\sinh 2x = \frac{1}{2}(e^x - e^{-x})(e^x + e^{-x})$$

Hyperbolic Identities – Ex. 1

$$\sinh 2x = \frac{1}{2}(e^x - e^{-x})(e^x + e^{-x})$$

$$\sinh 2x = \frac{1}{2}(e^{2x} + e^0 - e^0 - e^{-2x})$$

$$\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x})$$

True: LHS = RHS

QED

(*quod erat demonstrandum* = "which had to be demonstrated")

Hyperbolic Identities – Ex. 2

Prove:

$$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x)$$

Solution: First, write the left-hand side in terms of exponentials

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{1}{2}(1 + \cosh 2x)$$

$$\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(1 + \cosh 2x)$$

Hyperbolic Identities – Ex. 2

$$\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(1 + \cosh 2x)$$

$$\frac{e^{2x} + e^0 + e^0 + e^{-2x}}{4} = \frac{1}{2}(1 + \cosh 2x)$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2}(1 + \cosh 2x)$$

Hyperbolic Identities – Ex. 2

Next, we'll adjust the right hand side of the equation:

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2}(1 + \cosh 2x)$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2}\left(1 + \frac{e^{2x} + e^{-2x}}{2}\right)$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2} + \frac{e^{2x} + e^{-2x}}{4}$$

Hyperbolic Identities – Ex. 2

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2} + \frac{e^{2x} + e^{-2x}}{4}$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{2 + e^{2x} + e^{-2x}}{4}$$

True: LHS = RHS

QED

Osbourne's Rule

Notice the similarity:

$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x)$ [Previous Proof]

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ [From log tables]

This is because of the relationship between trig functions and the complex exponential:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Osbourne's Rule

Osbourne's Rule states that:

Any algebraic relation between Cosine and Sine also applies to Cosh and Sinh with the condition that a product of Sines gets an extra minus.

Osbourne's Rule – Examples

$$\sin 2x = 2\sin x\cos x$$

$$\sinh 2x = 2\sinh x\cosh x$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

Osbourne's Rule – Key Example

$$\cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

Notice how the condition within Osbourne's Rule (that a product of Sines gets an extra minus) is applied in this example.

Next, we'll prove that

$$\cosh^2 x - \sinh^2 x = 1$$

Hyperbolic Identities – Ex. 3

Prove that:

$$\cosh^2 x - \sinh^2 x = 1$$

Proof: Replace $\cosh x$ and $\sinh x$ with their exponential equivalents and expand the equation.

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$$

Hyperbolic Identities – Ex. 3

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$$

$$\left(\frac{e^{2x} + e^0 + e^0 + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - e^0 - e^0 + e^{-2x}}{4}\right) = 1$$

$$\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right) = 1$$

$$\frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1$$

Hyperbolic Identities – Ex. 3

$$\frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1$$

$$\frac{4}{4} = 1$$

True: LHS = RHS

QED