Line Integrals

Physical examples of line integrals:

- The total work done by a force $\mathbf{F}$ which moves its application point along a given curve C is given by

$$W_C = \int_C \mathbf{F} \cdot d\mathbf{r}$$

  where $\mathbf{r}$ is the position vector of the application point at a given moment.

- If a loop of wire C carrying a current I is placed in a magnetic field $\mathbf{B}$ then the total force on the wire is

$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B}$$

If $\mathbf{A} = A_0\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ is a vector then its integral along a curve C, from a point P_1 to another point P_2 can be calculated as

$$\begin{aligned}\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r} &= \int_C \mathbf{A} \cdot d\mathbf{r} \\ &= \int_C (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) \\ &= \int_C A_1\,dx + A_2dy + A_3dz\end{aligned}$$

which can then be calculated in the usual manner once the equation of the curve C is known.

Example

Evaluate the line integral $\int_C \mathbf{A} \cdot d\mathbf{r}$ where $\mathbf{A} = (x+y)\mathbf{i} + (y-x)\mathbf{j}$ along each of the following 2-dimensional curves

1. the parabola $y^2 = x$ from (1,1) to (4,2);
2. the curve $x = 2t^2 + t + 1,\ y = 1 + t^2$ from (1,1) to (4,2);
3. the line $y = 1$ from (1,1) to (4,1) followed by the line $x = 4$ from (4,1) to (4,2).

Solution: Write the integral as

$$\int_C \mathbf{A} d\mathbf{r} = \int_C (x+y)dx + (y-x)dy$$

1. Along the parabola $x = y^2$ we have $dx = 2ydy$. Substitute all x in the integral in terms of y to get

$$I = \int_1^2 [(y^2 + y)2y + (y - y^2)]\, dy = \frac{34}{3}$$

2. We write x and y in terms of t so $dx = (4t + 1)dt$ and $dy = 2tdt$. The limits for t can be evaluated from the limits for x and y.

$$I = \int_0^1 [(3t^2 + t = 2)(4t + 1) - (t^2 + t)2t]\, dt = \frac{32}{3}$$

3. The line integral must be evaluated along the two line segments separately and the results added together. Note that along the line $y = 1$ we have $dy = 0$ while along the line $x = 4$ we have $dx = 0$. So

$$I = \int_1^4 (x + 1)\, dx + \int_1^2 (y - 4)\, dy = -\frac{5}{2}.$$

More examples

Example 2: Evaluate the line integral

$$I = \oint_C x\,dy$$

where C is the circle in the xy-plane defined by $x^2 + y^2 = 1$.

Example 3: If

$$\mathbf{A} = (3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k}$$

evaluate $\int_C \mathbf{A} \cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

1. $x = t$, $y = t^2$, $z = t^3$;
2. the straight lines from (0,0,0) to (0,0,1), then to (0,1,1) and then to (1,1,1);
3. the straight line joining (0,0,0) to (1,1,1)

Green's theorem in the plane

Green's theorem, also called the **divergence theorem in two dimensions** relates a line integral around a closed curve, C, to a double integral over the region R enclosed by the curve,

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Green's theorem gives a condition for a line integral to be independent of its path.

Independence of the path

Consider the line integral

$$I = \int_A^B (P\,dx + Q\,dy)$$

We say that this integral is independent of the path taken from A to B if it has the same value along any two arbitrary paths C_1 and C_2 between the points.

This means that, if we take $C = C_1 - C_2$ (the closed loop formed by C_1 and C_2, the integral around C must be zero.

It can be seen, from Green's theorem, that a necessary condition for the line integral to be zero is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

This is also a sufficient condition for path independence.

Examples

1. Show that the area bounded by a closed curve C is given by

$$\frac{1}{2}\oint_C xdy - ydx$$

   Hence calculate the area of the ellipse, $x = a\cos(t)$, $y = b\sin(t)$.

2. Show that the integral

$$\int_{(1,2)}^{(3,4)} (6xy^2 - y^3)dx + (6x^2y - 3xy^2)dy$$

   is independent of the path joining the two points. Evaluate the integral.

Conservative vector fields

A vector field $\mathbf{A}$ is called **conservative** if any of the following equivalent conditions holds

- The line integral of $\mathbf{A}$ between two points is independent of the path
- The line integral of $\mathbf{A}$ over any closed curve C is equal to zero, that is

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$$

- The curl of $\mathbf{A}$ is zero, $\nabla \times \mathbf{A} = 0$;
- There exists a scalar field $\Phi(x, y, z)$, called a **potential**, such that

$$\mathbf{A} = \nabla \Phi$$

Example: Show that $\mathbf{A} = (xy^2 + z)\mathbf{i} + (x^2 y + 2)\mathbf{j} + x\mathbf{k}$ is conservative and find its scalar potential Φ.

The divergence theorem

Let S be a closed surface bounding a region of volume V and let $\mathbf{n}$ be the unit (outward) normal to the surface.

The divergence theorem states that

$$\iint_S \mathbf{A} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{A} \, dV$$

In other words, the surface integral of the normal component of a vector **A** taken over a closed surface is equal to the integral of the divergence of **A** taken over the volume enclosed by the surface.

Calculating surface integrals

Note: The surface integral $\iint_S \mathbf{A} \cdot \mathbf{n}\, dS$ is sometimes written as

$$\iint\limits_S \mathbf{A} \cdot d\mathbf{S}$$

where dS and $d\mathbf{S}$ are referred to as the scalar and vector area elements, respectively, with $d\mathbf{S} = \mathbf{n} dS$.

If the surface S is given by the equation $z = F(x, y)$ then a surface integral over S is calculated as follows

$$\iint\limits_S \Phi\, dS = \iint\limits_R \Phi \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$$

where R is the projection of S onto the xy plane.
Similar formulas hold for projections onto the xz or yz planes.

Example 1

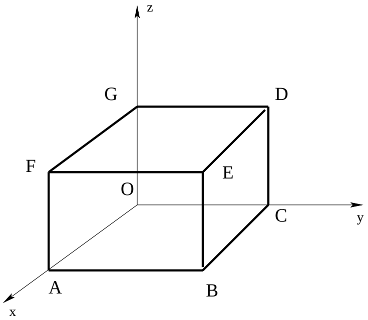
Evaluate

$$\iint\limits_S \mathbf{A} \cdot \mathbf{n} \, dS$$

where $\mathbf{A} = xy\mathbf{i} - x^2\mathbf{j} + (x+z)\mathbf{k}$, S is that portion of the plane $2x + 2y + z = 6$ included in the first octant and $\mathbf{n}$ is a unit normal to S.

Example 2

Verify the divergence theorem for $\mathbf{A} = (2x - z)\mathbf{i} + x^2y\mathbf{j} - xz^2\mathbf{k}$ taken over the region bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.



To verify the divergence theorem, calculate first the volume integral:

$$\iiint\limits_V \nabla \cdot \mathbf{A} dV = \int_0^1 \int_0^1 \int_0^1 (2 + x^2 - 2xz)\, dxdydz = \frac{11}{6}$$

Next evaluate $\iint_S \mathbf{A} \cdot \mathbf{n}$ on each face of the cube.

1. **On face AFEB** we have $\mathbf{n} = \mathbf{i}$ and $x = 1$. Then

$$\iint_{AFEB} \mathbf{A} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 \left[(2-z)\mathbf{i} + \mathbf{j} - z^2\mathbf{k}\right] \cdot \mathbf{i}\, dydz = \frac{3}{2}$$

2. **On face COGD** we have $\mathbf{n} = -\mathbf{i}$ and $x = 0$.

$$\iint_{COGD} \mathbf{A} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 (-z\mathbf{i}) \cdot (-\mathbf{i})\, dydz = \frac{1}{2}$$

3. **On face BEDC** we have $\mathbf{n} = \mathbf{j}$ and $y = 1$.

$$\iint_{BEDC} \mathbf{A} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 \left[(2x-z)\mathbf{i} + x^2\mathbf{j} - xz^2\mathbf{k}\right] \cdot \mathbf{j}\, dxdz = \frac{1}{3}$$

4. On face OAFG we have $\mathbf{n} = -\mathbf{j}$ and $y = 0$

$$\iint_{OAFG} \mathbf{A}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 \left[(2x-z)\mathbf{i} - xz^2\mathbf{k}\right]\cdot(-\mathbf{j})\, dxdz = 0$$

5. On face EFGD we have $\mathbf{n} = \mathbf{k}$ and $z = 1$

$$\iint_{EFGD} \mathbf{A}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 \left[(2x-1)\mathbf{i} + x^2y\mathbf{j} - x\mathbf{k}\right]\cdot\mathbf{k}\, dxdy = -\frac{1}{2}$$

6. On OABC we have $\mathbf{n} = -\mathbf{k}$ and $z = 0$

$$\iint_{OABC} \mathbf{A}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 \left[2x\mathbf{i} - x^2y\mathbf{j}\right]\cdot(-\mathbf{k})\, dxdy = 0$$

Adding the six faces we get $\frac{3}{2} + \frac{1}{2} + \frac{1}{3} + 0 - \frac{1}{2} + 0 = \frac{11}{6}$.

Stokes' Theorem

The line integral of a vector **A** taken around a simple closed curve (that is, a non-intersecting closed curve), C, is equal to the surface integral of the curl of **A** taken over any surface S having C as a boundary.

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Note that, if $\nabla \times \mathbf{A} = 0$ then the line integral of **A** over the closed curve C is zero and hence the vector field is conservative.

Physical examples: Surface Integrals

If **A** is a vector field, the surface integral

$$\iint_S \mathbf{A} \cdot d\mathbf{S}$$

is called the **flux** of **A** through the surface S.

Gauss' flux theorem: The electric flux through any closed surface is proportional to the enclosed electric charge

$$Q = \varepsilon_0 \oiint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is the permittivity of free space or electric constant.

Physical examples: Volume integrals

The volume of a closed region in space, D, is given by

$$V = \iiint_D dV$$

The mass of an object which occupies a region D and has density $\delta(x, y, z)$ is given by

$$M = \iiint_D \delta \, dV$$

Total electric charge enclosed within a region D is

$$Q = \iiint_V \rho \, dV$$

where $\rho(x, y, z)$ is the charge density.

Applications of the divergence theorem

The integral form of the Gauss equation is

$$Q = \varepsilon_0 \oint\!\!\!\oint_S \mathbf{E} \cdot d\mathbf{S}$$

which can be written as

$$\iint_V \rho \, dV = \varepsilon_0 \iiint_V \nabla \cdot \mathbf{E} \, dV$$

and hence

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

which is the differential form.

Applications of the Stokes theorem

The Stokes' Theorem shows the equivalence between the integral and differential forms of the Ampere-Maxwell equation.

Ampere's law for a distributed current I with current density $\mathbf{J}$ is

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}$$

for any circuit C bounding a surface S. Using Stokes' Theorem, this can be written as

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}$$

and, since this holds for any surface S, it follows that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$