Introduction to Fourier Series

A function $f(x)$ is called **periodic** with period T if

$$f(x+T) = f(x)$$

for all numbers x. The most familiar examples of periodic functions are the trigonometric functions sin and cos which are periodic with period 2π.

Under certain conditions, a periodic function $f(x)$ can be represented as an infinite sum of sine and cosine terms (called a **Fourier series expansion**) as follows

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + \sum_{n=1}^{\infty} B_n \sin(nx)$$

Example

The **sawtooth wave** function has the form

$$f(x) = x \qquad -\pi \leq x \leq \pi$$
$$f(x+2k\pi) = f(x) \quad \text{for} \quad k = \pm 1, \pm 2, \pm 3, \ldots$$

The Fourier series expansion for this function is

$$x = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots\right)$$

for $-\pi < x < \pi$.

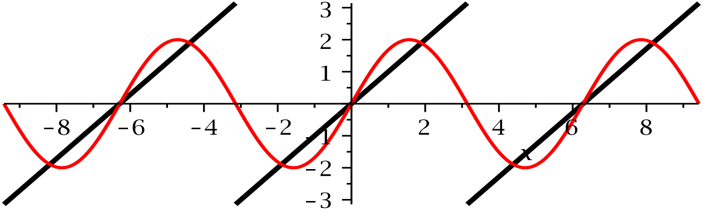


Figure: The first term in the Fourier series expansion, $x \approx 2\sin(x)$

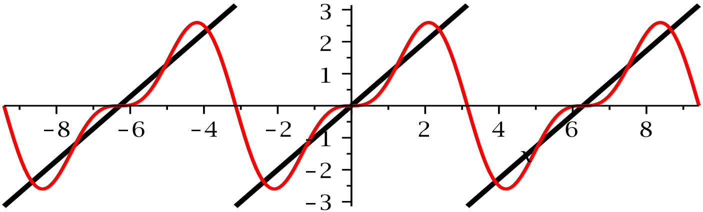


Figure: The first 2 terms in the Fourier series expansion, $x \approx 2\sin(x) - \sin(2x)$

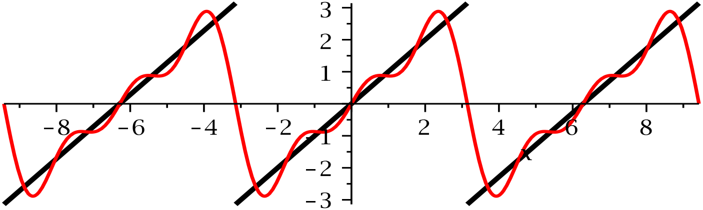


Figure: The first 3 terms, $x \approx 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x)$

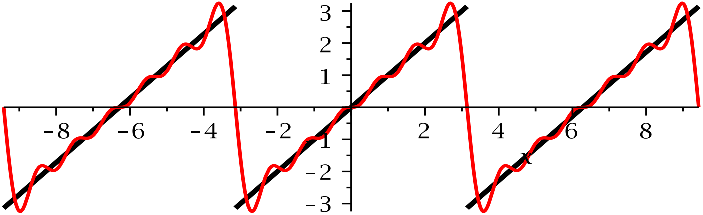


Figure: The first 4 terms, $x \approx 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x)$

Remarks

1. The Fourier series expansion provides an approximation method: the more terms are selected in the series, the better the approximation
2. The Fourier series allows a periodic signal to be expressed in terms of simple sinusoidal oscillations
3. We have approximated the function $f(x) = x$ for $-\pi < x < \pi$ in terms of sinusoidal terms so even non-periodic functions can have a Fourier series expansion!

Even and odd functions

A function $f(x)$ is called **even** if $f(-x) = f(x)$ for all x (so its graph is symmetric with respect to the y-axis).

A function $f(x)$ is called **odd** if $f(-x) = -f(x)$ for all x (so its graph is symmetric with respect to the origin).

For example, $\cos(x)$ and even powers of x are even functions while $\sin(x)$ and odd powers of x are odd functions.

The Fourier series for an even function contains only cos terms ($B_n = 0$ for all n), while the Fourier series of an odd function contains only sin terms ($A_n = 0$ for all n).

Examples

The sawtooth wave function is an **odd** function which has the expansion

$$x \approx 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \cdots\right)$$

for $-\pi < x < \pi$.

The function $f(x) = x^2$, $-\pi \leq x \leq \pi$, together with its periodic extension, is an **even** function which has the expansion

$$x^2 \approx \frac{\pi^2}{3} - 4\left(\cos(x) - \frac{\cos(2x)}{2^2} + \frac{\cos(3x)}{3^2} - \cdots\right)$$

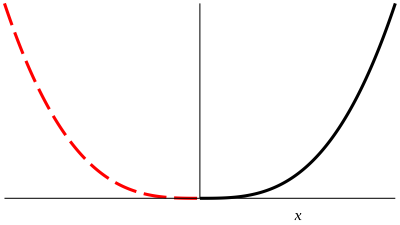
Even and odd extensions

Given a function $f(x)$ defined on $[0, \pi]$ we often need an expansion in sine terms (or cosine terms) only.

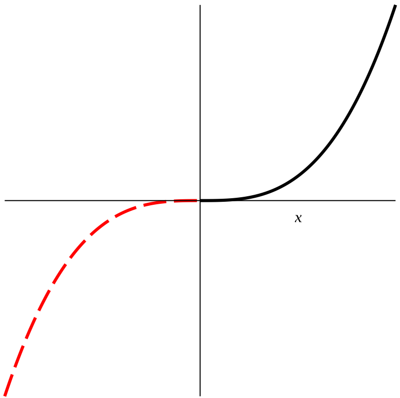
To expand $f(x)$ in cosine terms we make an **even extension** of the function from the interval $[0, \pi]$ onto the interval $[-\pi, 0]$. This will give us a function which is even on the interval $[-\pi, \pi]$.

To expand $f(x)$ in sine terms we make an **odd extension** of the function from the interval $[0, \pi]$ onto the interval $[-\pi, 0]$. This will give us a function which is odd on the interval $[-\pi, \pi]$.

Even and odd extensions



(a) Even extension



(b) Odd extension

Example

Expand $f(x) = 1$, $0 \leq x \leq \pi$ in **sine series**.

Answer:

$$1 \approx \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right)$$

Functions of period $2L$

It is often necessary to work with periodic functions with period other than 2π. The Fourier series for such functions is easily modified as

$$f(x) \approx \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\frac{\pi n x}{L}) + \sum_{n=1}^{\infty} B_n \sin(\frac{\pi n x}{L})$$

where the coefficients are given as

$$A_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx; \quad A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{\pi n x}{L})\,dx;$$
$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{\pi n x}{L})\,dx$$

The complex form of a Fourier series

Let $f(x)$ be an integrable function on the interval $[-\pi, \pi]$. Its Fourier series can be written in complex form as

$$f(x) \approx \sum_{n=-\infty}^{\infty} C_n \mathrm{e}^{\mathrm{i}nx}$$

where

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-\mathrm{i}nx}\, dx$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Also we have

$$C_0 = \frac{A_0}{2}; \qquad C_n = \frac{A_n - \mathrm{i}B_n}{2}; \qquad C_{-n} = \frac{A_n + \mathrm{i}B_n}{2}; \qquad n > 0.$$

Note: The simplification from trigonometric form to exponential form makes use of the formula

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

Note: If $f(x)$ is a real function then the coefficient C_{-n} is the complex conjugate of C_n, for all $n > 0$.

Exercise: Calculate the complex Fourier series for the function $f(x) = x$ in the interval $-2 \leq x \leq 2$.

We find that

$$x \approx \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2i(-1)^n}{n\pi} \exp\left(\frac{\pi i n x}{2}\right)$$

The convergence of the Fourier series

The Fourier series of a function $f(x)$ of period 2π converges for all values of x. The sum of the series is equal to

- $f(x)$, if x is a point of continuity for $f(x)$;
- $\frac{1}{2}[f(x+0)+f(x-0)]$ (the mean of left-hand and right-hand limits) if x is a point of discontinuity.

The Fourier series is therefore a good approximation for the values of a function at all points where the function is continuous.

Close to a point of discontinuity, the Fourier series approximation will overshoot the value of the function. This behaviour is known as the **Gibbs phenomenon**.

Example

Consider the square-wave function (black) represented below together with its Fourier series approximation (red):

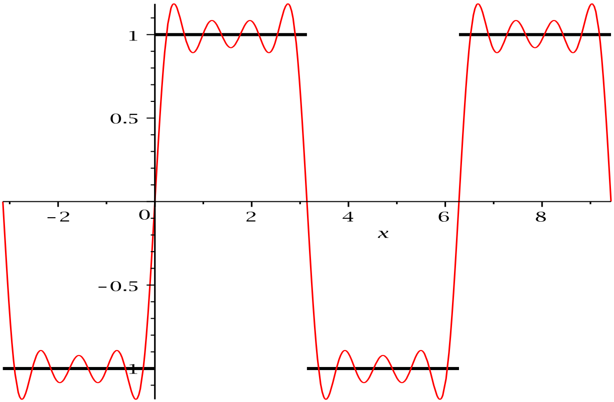


Figure: The first 3 terms, $1 \approx \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right)$

Parseval's identity

This formula relates a function $f(x)$ to the coefficients of its Fourier series. If f is periodic with period 2π then we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \sum_{n=-\infty}^{\infty} |C_n|^2 = \frac{A_0^2}{2} + \sum_{n=1}^{\infty}(A_n^2 + B_n^2)$$

where C_n are the coefficients of the exponential Fourier series and A_n, B_n are the coefficients of the trigonometric Fourier series.

Example: Using Parseval's identity and the Fourier series for the function $f(x) = x^2$ on $[-2,2]$, calculate the sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

The Fourier transform

Given a function $f(x)$ (a time signal), we can define a new function

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-isx} \, dx$$

which is called the **Fourier transform** of $f(x)$.

The signal $f(x)$ can be recovered from $F(s)$ using the **inverse Fourier transform**:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \mathrm{e}^{isx} \, dx$$

The Fourier transform

In electrical engineering, a slightly different notation is used:

$$F(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-isx} \, dx$$

In this context, the Fourier transform $F(s)$ is also called the **spectrum** of the time signal $f(x)$ and the new variable s denotes a frequency.

The **inverse Fourier transform** is:

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{isx} \, dx$$

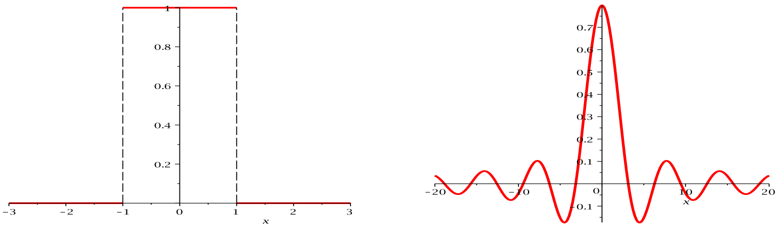
Example

The Fourier transform of the “top-hat” or “gate” function

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

is equal to

$$F(s) = \int_{-1}^{1} \mathrm{e}^{-isx}\, dx = \sqrt{\frac{2}{\pi}}\, \frac{\sin(s)}{s}$$



(a) Gate function (b) Fourier transform

The Fourier transform can be regarded as a generalization of the Fourier series representation of a function. Recall the complex (exponential) Fourier series

$$f(x) \approx \sum_{n=-\infty}^{\infty} C_n \mathrm{e}^{\mathrm{i}nx} \quad \text{where} \quad C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-\mathrm{i}nx}\, dx$$

The Fourier transform replaces the infinite sum by an integral and the integer n becomes a continuous variable, s:

$$f(x) = \int_{-\infty}^{\infty} F(s) \mathrm{e}^{isx}\, dx \quad \text{and} \quad F(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-isx}\, dx$$

Sine and cosine Fourier transform

We can also define the sine and cosine Fourier transforms (which represent continuous forms of the usual trigonometric series) using the formulas:

Sine transform:

$$F_S(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(sx)\, dx$$

Inverse sine transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S(s) \sin(sx)\, ds$$

Sine and cosine Fourier transform

Cosine transform:

$$F_C(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(sx)\,dx$$

Inverse cosine transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(s)\cos(sx)\,ds$$

Existence of the Fourier transform

In order to calculate a Fourier transform, the function $f(x)$ must be defined on the whole real axis $(-\infty < x < \infty)$ and it must satisfy

$$\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$$

This means that the function $f(x)$ must be zero at $\pm\infty$.

Example: Calculate the Fourier transform of the one-sided exponential function:

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The unit step function

The unit step function (or Heaviside function) is defined by

$$H(x-a) = \begin{cases} 1 & \text{if } x > a \\ 0 & \text{if } x < a \end{cases}$$

• A combination of step functions can be used to denote a function which has a constant value over a limited range. For example, the function

$$F(x) = 3\,[H(x-a) - H(x-b)]$$

has the value 3 for $a < x < b$ and 0 outside this interval.

• The step function can be used to denote a change from 0 to a different type of behaviour. For example, the one-sided exponential function,

$$f(x) = \begin{cases} \mathrm{e}^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad = \quad H(x) \cdot \mathrm{e}^{-x}$$

The Dirac delta function

The Dirac delta function, $\delta(x)$, is defined by the properties

$$\delta(x) = 0 \text{ for } x \neq 0$$
$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Similarly, we can define a delta function centred at a point $x = a$ (different from zero) and denote this by $\delta(x - a)$.

The delta function is not a function in the classical sense so it is often referred to as a **generalized function** or a **distribution**.

The delta function can also be defined as the limit of a sequence of functions which get progressively "thinner" and "taller", for example, the gate functions:

$$\Delta_n(x) = \begin{cases} n & \text{if } -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

$$= n\left[H(x+\frac{1}{2n}) - H(x-\frac{1}{2n})\right]$$

Properties of the delta function

The fundamental property of the delta function is

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

where $f(x)$ is any continuous function.

The delta function can therefore be thought of as transforming a function $f(x)$ into a number $f(a)$ (where a is the "centre" of the delta function).

We also have

$$\frac{d}{dx}\,H(x) = \delta(x)$$

(the delta function is the derivative of the unit step function).

Fourier transform

The Fourier transform of the delta function is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-isx} \delta(x)\, dx = \frac{1}{\sqrt{2\pi}}.$$

Hence, the delta function can be recovered from $F(s)$ using the inverse Fourier transform:

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{isx} F(s)\, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{isx}\, ds$$

which is an alternative representation for the delta function.

Interpretation

The Dirac delta function is used in physics to represent a **point source**. For example, the total electric charge in a finite volume V is given by $Q = \iiint_V \rho(x)\,dx$ where $\rho(x)$ is the charge density.

Now suppose a charge, Q, is concentrated at one point $x = a$. The charge density should be zero everywhere except at $x = a$ where it should be infinite, seeing we have a finite charge in an infinitely small volume. Therefore

$$Q = \iiint\limits_V \rho(x)\,dx \qquad \text{where} \qquad \rho(x) = Q\delta(x - a)$$