Course notes for Science Mathematics 2 (MA4602)

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Chapter 1

Limits

This first chapter is devoted to the study of the concept of limit. The aim here is to enhance the knowledge acquired by the student in the module MA4601 (Science Mathematics 1) where the basic concepts of limit have been already introduced. Limits are important tools employed to understand the behavior of a function "near a point x_0" or "at infinity i.e. for large values of x". This is particularly important in curve sketching, which has been partly studied in MA4601 and which will continue in Chapter 2 of these course notes. In the first section of the current chapter we also revise the concept of continuity of a function.

1.1 Limit of a function "at a point": continuity and discontinuity of a function

We start by revising the concept of the limit of a function.

1.1.1 Finite limit "at a point"

DEFINITION **1.1.1.** *The function f approaches the limit ℓ near the point x_0 if we make $f(x)$ as close as we like to ℓ by requiring that x be sufficiently close to (but not equal to) x_0 and we write*

$$\lim_{x \to x_0} f(x) = \ell, \tag{1.1.1}$$

which is read "the limit of $f(x)$ as x approaches x_0 is ℓ".

In other words we can say that in (1.1.1) the values of function $f(x)$ get closer and closer to ℓ as x gets closer and closer to x_0 (from both sides of x_0).

Example **1.1.1.**

$$\lim_{x \to 0} x^2 = 0 \tag{1.1.2}$$

1.1.2 Limits laws

We revise here the limits laws. Suppose we have two functions

$$f(x) \qquad \text{and} \qquad g(x)$$

and suppose the limits

$$\lim_{x\to x_0} f(x) = \ell_1; \qquad \lim_{x\to x_0} g(x) = \ell_2$$

exist. Then we have

1. $$\lim_{x\to x_0} [f(x) + g(x)] = \ell_1 + \ell_2$$

2. $$\lim_{x\to x_0} [f(x)g(x)] = \ell_1 \ell_2$$

3. If $c \in \mathbb{R}$ (c is a constant, i.e. a number), then:

$$\lim_{x\to x_0} [cf(x)] = c\ell_1$$

4. If $\ell_1 \neq 0$, then:

$$\lim_{x\to x_0} \frac{1}{f(x)} = \frac{1}{\ell_1}$$

5. If $\ell_2 \neq 0$, then:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\ell_1}{\ell_2}$$

6. $$\lim_{x\to x_0} |f(x)| = \left|\lim_{x\to x_0} f(x)\right| = |\ell_1|$$

Example **1.1.2.**

$$\lim_{x\to 0} \left(x^2 + 10\right) = \lim_{x\to 0} x^2 + \lim_{x\to 0} 10 = 0 + 10 = 10$$

Example **1.1.3.**

$$\lim_{x\to 0} \left(5x^2\right) = 5 \cdot \lim_{x\to 0} x^2 = 5 \cdot 0 = 0$$

Example **1.1.4.**

$$\lim_{x\to 2} \left(\frac{7}{x^2}\right) = 7 \cdot \frac{1}{\lim\limits_{x\to 2} x^2} = 7 \cdot \frac{1}{2^2} = \frac{7}{4}$$

1.2 Evaluating the limit of a function

We start with two simple examples.

Example **1.2.1.**

$$\lim_{x\to 0} x^2 = 0^2 = 0$$

Example **1.2.2.**

$$\lim_{x\to 2} x^2 = 2^2 = 4$$

Note that in the above examples the limits can be evaluated by simply substituting 0 for x and 2 for x respectively. Unfortunately this cannot always be done.
In general, trying to evaluate a limit is more complicated and it might require different techniques, some of which will be studied in this section in the following important examples.

Example **1.2.3.**

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

The first thing to do is to try to substitute the value 1 for x in

$$f(x) = \frac{x^2-1}{x-1}$$

but if we do so the result is $\frac{0}{0}$ which is meaningless. Let us rewrite $f(x)$ as

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1$$

therefore

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 1+1 = 2$$

Example **1.2.4.**

$$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$$

Again the first thing to do is to try to substitute the value 0 for x in

$$f(x) = \frac{\sqrt{4+x}-2}{x}$$

but if we do so the result is $\frac{0}{0}$ which is meaningless. Let us rewrite $f(x)$ as

$$\begin{aligned}
\frac{\sqrt{4+x}-2}{x} &= \frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}\\
&= \frac{4+x-4}{x\left(\sqrt{4+x}+2\right)}\\
&= \frac{x}{x\left(\sqrt{4+x}+2\right)}\\
&= \frac{1}{\sqrt{4+x}+2}
\end{aligned}$$

therefore

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}=\lim_{x\to 0}\frac{1}{\sqrt{4+x}+2}=\frac{1}{\sqrt{4+0}+2}=\frac{1}{4}$$

The following example illustrates how to evaluate the limit of a function given by two formulae: we are interested in evaluating the limit as x approaches the point of transition from one formula to the other.

Example **1.2.5.** *We consider the so-called Heaviside function $H(x)$ which is defined as*

$$H(x)=\begin{cases}0, & x<0,\\ 1, & x\geq 0\end{cases}$$

$H(x)$ is given by two formulae and the point of transition from one formula to the other is $x=0$ (see figure 1.1).
To evaluate the limit

$$\lim_{x\to 0}H(x)$$

we need to evaluate two limits: the so-called right-hand limit and the left-hand limit of $H(x)$ as x approaches 0. The left-hand limit is defined by

$$\lim_{x\to 0^-}H(x)=\lim_{x\to 0,\,x<0}H(x)=\lim_{x\to 0,\,x<0}0=0$$

The right-hand limit is defined by

$$\lim_{x\to 0^+}H(x)=\lim_{x\to 0,\,x>0}H(x)=\lim_{x\to 0\,x>0}1=1$$

The example above shows that the Heaviside function $H(x)$ has two different limits as x approaches 0. This is happening because $H(x)$ is discontinuous at $x=0$ i.e. there is a break or a "jump" it the graph of $H(x)$ at $x=0$ (see figure 1.1).
We give next another example of a function which is given by two formulae.

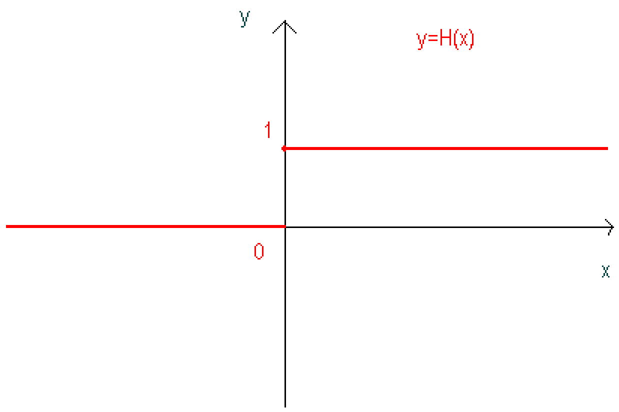

Figure 1.1: Picture showing the graph of the Heaviside function $H(x)$
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Example **1.2.6.**

$$f(x) = \begin{cases} 1, & x < 0, \\ x^2 + 1, & x \geq 0 \end{cases}$$

$f(x)$ is given by two formulae and the point of transition from one formula to the other is $x = 0$ (see figure 1.2.6).
To evaluate the limit

$$\lim_{x \to 0} f(x)$$

we need to evaluate the left- and the right-hand limits. The left-hand limit is given by

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1 = 1,$$

where the right-hand limit is given by

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 1) = 0 + 1 = 1$$

The right- and left-hand limits are equal, moreover they equal $f(0) = 1$ and this is happening because the function is continuous i.e. the function has no break in its graph (see figure 1.2).

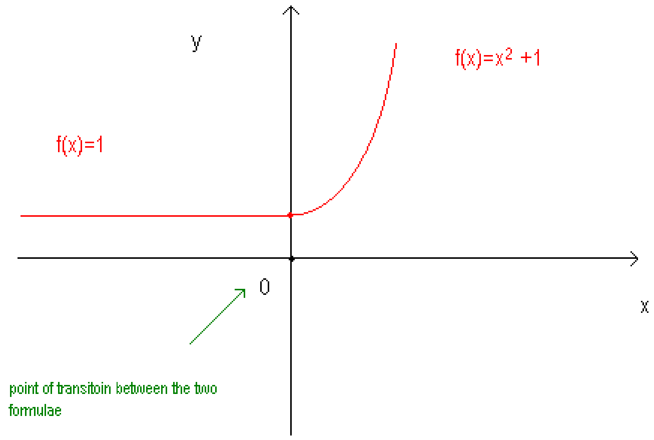

Figure 1.2: Picture showing the graph of the function $f(x)$ given in example 1.2.6

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Remark **1.2.1.** *Note that when*

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$$

we simply write

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$$

The mathematical definition of a continuous function is given below.

DEFINITION **1.2.1.** *A function $f : [a, b] \to \mathbb{R}$ is continuous at a point $x_0 \in [a, b]$ if*

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) = f(x_0)$$

1.2.1 Infinite limit "at a point": divergent functions and vertical asymptotes

DEFINITION **1.2.2.** *The function f approaches the limit $+\infty$ near the point x_0 if we can make $f(x)$ arbitrarily large by requiring that x be sufficiently close to (but not equal to) x_0 and we write*

$$\lim_{x \to x_0} f(x) = +\infty, \tag{1.2.1}$$

which is read "the limit of $f(x)$ as x approaches x_0 is $+\infty$". We also say that $f(x)$ diverges to $+\infty$ when x approaches x_0 and we call the straight vertical line $x = x_0$ a vertical asymptote for $f(x)$ (see figure 1.3).

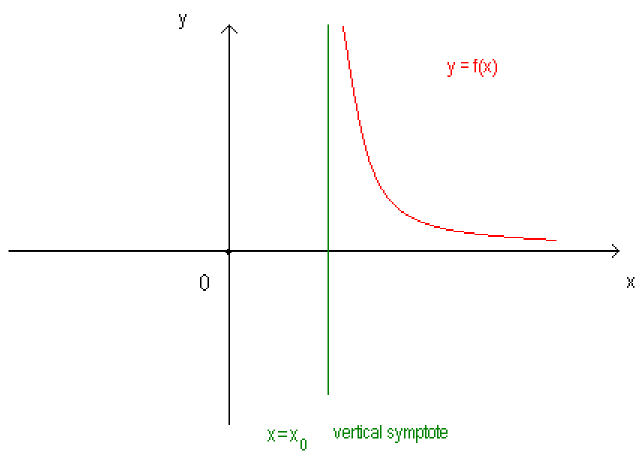

Figure 1.3: Picture showing the graph of a function with a vertical asymptote $x = x_0$

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Similarly we have

DEFINITION **1.2.3.** *The function f approaches the limit $-\infty$ near the point x_0 if we can make $f(x)$ arbitrarily large but negative by requiring that x be sufficiently close to (but not equal to) x_0 and we write*

$$\lim_{x \to x_0} f(x) = -\infty, \tag{1.2.2}$$

which is read "the limit of $f(x)$ as x approaches x_0 is $-\infty$". We also say that $f(x)$ diverges to $-\infty$ when x approaches x_0 and we call the straight vertical line $x = x_0$ a vertical asymptote for $f(x)$ (see figure 1.4).

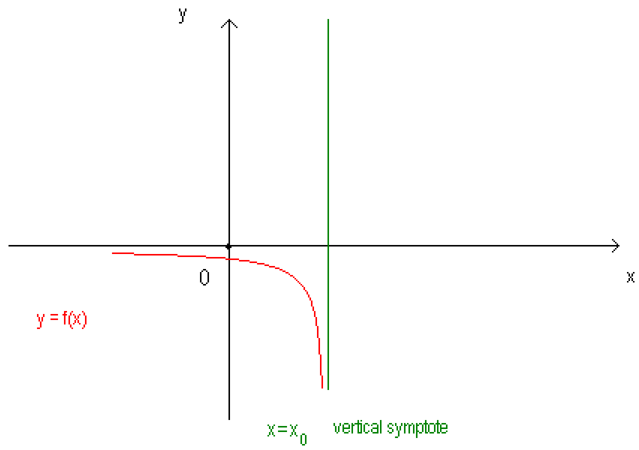

Figure 1.4: Picture showing the graph of a function with a vertical asymptote $x = x_0$

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Example **1.2.7.**

$$f(x) = \frac{1}{x}$$

We have (see picture 1.5):

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty; \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Note from the graph of $f(x) = \frac{1}{x}$ that this function is discontinuous at $x = 0$ i.e. where its vertical asymptote occurs (in fact here $f(x)$ is not even define).

Exercise **1.2.1.** *Find the vertical asymptotes of the function given by*

$$f(x) = \frac{x}{x^2 + x - 2}$$

and show explicitly what the limits of $f(x)$ are when x is approaching each vertical asymptote from the right and from the left hand side.

Answer: Vertical asymptotes are likely to occur when the denominator is 0, therefore we look for the solutions for x to the following quadratic equation

$$x^2 + x - 2 = 0$$

which gives

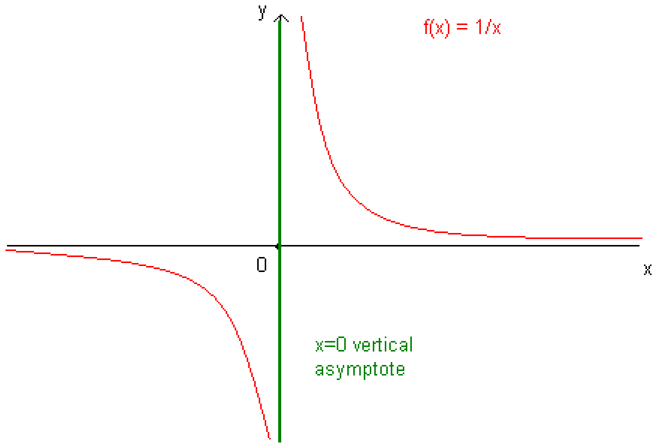

Figure 1.5: Picture showing the function $f(x) = \frac{1}{x}$ with its vertical asymptote $x = x_0$

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$$x = \frac{-1 \pm \sqrt{9}}{2} = \left\langle \begin{array}{l} 1 \\ -2 \end{array} \right.$$

therefore we can rewrite our function $f(x)$ in the following way

$$f(x) = \frac{x}{x^2 + x - 2} = \frac{x}{(x-1)(x+2)}$$

and we start by looking at the limit

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x}{(x-1)(x+2)}.$$

Since $f(x)$ could have a different behavior on the left and on the right hand side of $x = 1$, we will then evaluate the left and the right-hand limits as x approaches 1.

$$\begin{aligned} \lim_{x \to 1^-} \frac{x}{(x-1)(x+2)} &= \lim_{x \to 1,\, x<1} \frac{x}{(x-1)(x+2)} \\ &= \lim_{x \to 1,\, x-1<0} \frac{x}{(x-1)(x+2)} = -\infty; \end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+}\frac{x}{(x-1)(x+2)} &= \lim_{x\to 1,\, x>1}\frac{x}{(x-1)(x+2)}\\ &= \lim_{x\to 1,\, x-1>0}\frac{x}{(x-1)(x+2)} = +\infty;\end{aligned}$$

Similarly now for $x=-2$ we have:

$$\begin{aligned}\lim_{x\to -2^-}\frac{x}{(x-1)(x+2)} &= \lim_{x\to -2,\, x<-2}\frac{x}{(x-1)(x+2)}\\ &= \lim_{x\to 1,\, x+2<0}\frac{x}{(x-1)(x+2)} = -\infty;\end{aligned}$$

$$\begin{aligned}\lim_{x\to -2^+}\frac{x}{(x-1)(x+2)} &= \lim_{x\to -2,\, x>-2}\frac{x}{(x-1)(x+2)}\\ &= \lim_{x\to 1,\, x+2>0}\frac{x}{(x-1)(x+2)} = +\infty.\end{aligned}$$

Therefore there are two vertical asymptotes: $x=1$ and $x=-2$ and the behavior of the function on the right and on the left hand side of each asymptote is given by the above four limits.

Exercise **1.2.2.** *Find the vertical asymptotes of the following function*

$$g(x)=\frac{2x^2}{x^2-x-2}$$

and show explicitly what are the limits of $g(x)$ when x is approaching each vertical asymptote from the right and from the left hand side.

Remark **1.2.2.** *If*

$$\lim_{x\to x_0} f(x)=\pm\infty$$

then

$$\lim_{x\to x_0}\frac{1}{f(x)}=0.$$

Remark **1.2.3.** *If*

$$\lim_{x\to x_0} f(x)=0$$

then

$$\lim_{x\to x_0}\frac{1}{f(x)}=\pm\infty.$$

1.2.2 Limits law

The same limits laws we saw for the finite limits "at a point" hold here for the infinite limits at a point or a combination of finite and infinite limits "at a point" as long as the operations of addition/sunstraction and multiplication/division of numbers make sense.

We list at the end of this chapter some examples of the so-called indeterminate cases where these operations do not make sense anymore and therefore different techniques must be used to evaluate such limits.

1.3 Limit of a function "at infinity"

In the curve-sketching of a function $f(x)$ it is useful to study the behavior of $f(x)$ when x becomes larger and larger but positive or larger and larger but negative.

1.3.1 Finite limits "at infinity": convergent functions and horizontal asymptotes

DEFINITION **1.3.1.** *The function f approaches the limit ℓ at $+\infty$ if we make $f(x)$ as close as we like to ℓ by requiring that x be sufficiently large and positive and we write*

$$\lim_{x\to+\infty} f(x) = \ell, \tag{1.3.1}$$

which is read "the limit of $f(x)$ as x approaches $+\infty$ is ℓ". We also say that $f(x)$ converges to ℓ when x approaches $+\infty$ and we call the straight horizontal line $y = \ell$ an horizontal asymptote for $f(x)$ (see figure 1.6)*.*

Similarly we have the following definition.

DEFINITION **1.3.2.** *The function f approaches the limit ℓ at $-\infty$ if we make $f(x)$ as close as we like to ℓ by requiring that x be sufficiently large but negative and we write*

$$\lim_{x\to-\infty} f(x) = \ell, \tag{1.3.2}$$

which is read "the limit of $f(x)$ as x approaches $-\infty$ is ℓ". We also say that $f(x)$ converges to ℓ when x approaches $-\infty$ and we call the straight horizontal line $y = \ell$ an horizontal asymptote for $f(x)$ (see figure 1.7)*.*

Example **1.3.1.**

$$f(x) = \frac{1}{x}$$

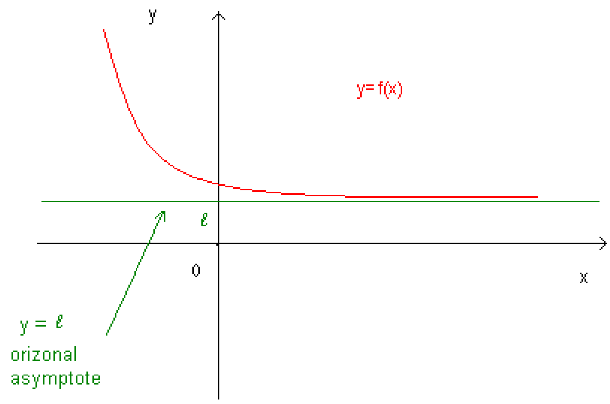

Figure 1.6: Picture showing the graph of a function with an horizontal asymptote $y = \ell$

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We have:

$$\lim_{x\to+\infty} \frac{1}{x} = 0; \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

and $y = 0$ is an horizontal asymptote for $f(x) = \frac{1}{x}$ (it is the only horizontal asymptote for this function in fact).

Exercise **1.3.1.** *Find the vertical and horizontal asymptotes of the function given by*

$$f(x) = \frac{5}{x-3},$$

hence try to sketch its graph.

Exercise **1.3.2.** *Find the horizontal asymptotes of the function given by*

$$f(x) = \frac{x}{x^2+x-2}.$$

Answer: To find the horizontal asymptotes of a function (if there is any) we evaluate the limits of the functions when x is approaching $+\infty$ and $-\infty$.

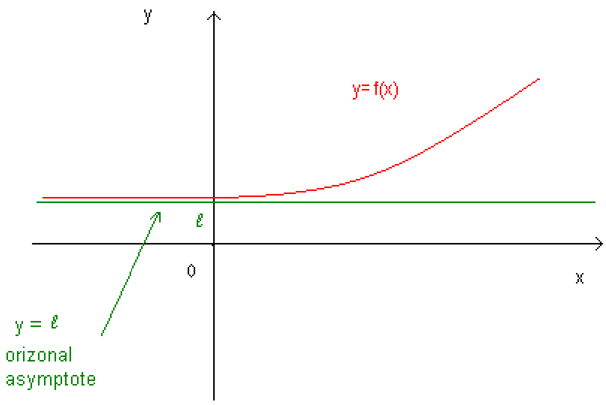

Figure 1.7: Picture showing the graph of a function with an horizontal asymptote $y = \ell$

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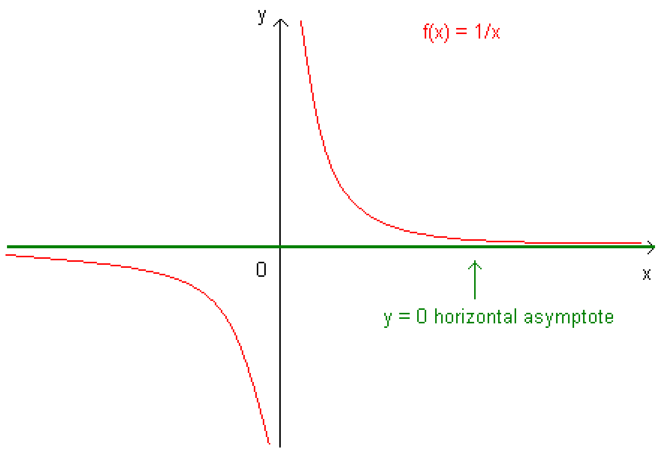

Figure 1.8: Picture showing the graph of a function $f(x) = \frac{1}{x}$ and its horizontal asymptote.

$$\lim_{x\to+\infty}\frac{x}{x^2+x-2}=\lim_{x\to+\infty}\frac{x}{x^2\left(1+\frac{1}{x}-\frac{2}{x^2}\right)}=\lim_{x\to+\infty}\frac{1}{x\left(1+\frac{1}{x}-\frac{2}{x^2}\right)}=0$$

and similarly

$$\lim_{x\to-\infty}\frac{x}{x^2+x-2}=\lim_{x\to-\infty}\frac{x}{x^2\left(1+\frac{1}{x}-\frac{2}{x^2}\right)}=\lim_{x\to-\infty}\frac{1}{x\left(1+\frac{1}{x}-\frac{2}{x^2}\right)}=0,$$

therefore $y=0$ is the only horizontal asymptote for this function when x is approaching both $+\infty$ and $-\infty$.

1.3.2 Infinite limits "at infinity": divergent functions "at infinity"

The values of a function $f(x)$ can become larger and larger (positive or negative) for very large values of x (by large values of x we mean both large positive values or large negative values of x). This property of a function $f(x)$ is explained by the following definition of "divergent function at infinity".

DIVERGENT FUNCTIONS "AT $+\infty$"

DEFINITION **1.3.3.** *The function f approaches the limit $+\infty$ at $+\infty$ if we make $f(x)$ arbitrarily large by requiring that x be sufficiently large and positive and we write*

$$\lim_{x\to+\infty} f(x)=+\infty, \tag{1.3.3}$$

which is read "the limit of $f(x)$ as x approaches $+\infty$ is $+\infty$". We also say that $f(x)$ diverges to $+\infty$ when x approaches $+\infty$ (see picture 1.9).

Note that, roughly speaking, the above definition is telling us how to "describe" the fact that the values of a function are becoming larger but positive positive for positive large values of x. Again, roughly speaking (this is not a rigorous mathematical definition), this definition is describing the fact that a function can become larger and larger (positive) as long as x is on the right hand side of the y-axis and "far away" from it!

DEFINITION **1.3.4.** *The function f approaches the limit $-\infty$ at $+\infty$ if we make $f(x)$ arbitrarily large by requiring that x be sufficiently large and negative and we write*

$$\lim_{x\to+\infty} f(x)=-\infty, \tag{1.3.4}$$

which is read "the limit of $f(x)$ as x approaches $+\infty$ is $-\infty$". We also say that $f(x)$ diverges to $-\infty$ when x approaches $+\infty$ (see picture **??**).

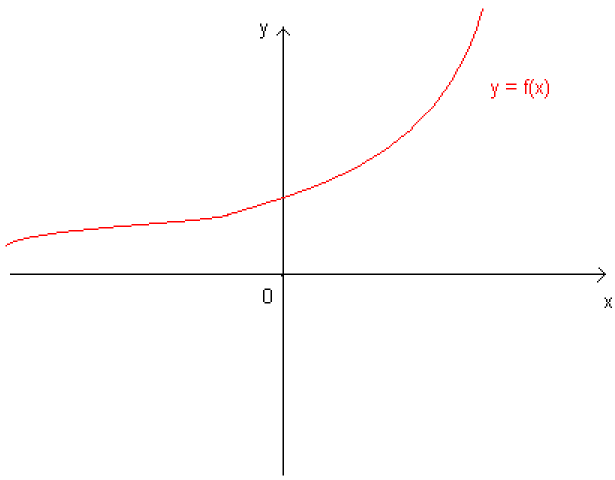

Figure 1.9: Picture showing the graph of a function with a vertical asymptote $x = x_0$

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Again, roughly speaking, the above definition is telling us how to "describe" the fact that the values of a function are becoming larger and larger but negative for negative large values of x this time. Again, roughly speaking (this is not a rigorous mathematical definition), this definition is describing the fact that a function can become larger and larger (negative) as long as x is on the right hand side of the y-axis and "far away" from it!

We can similarly give the following two definitions which describe the behavior of a function $f(x)$ that "goes to $+\infty$ and to $-\infty$" for large negative values of x.

DIVERGENT FUNCTIONS "AT $-\infty$"

In the following definition we describe a function $f(x)$ that is positive and larger and larger for negative and large values of x:

DEFINITION **1.3.5.** *The function f approaches the limit $+\infty$ at $-\infty$ if we make $f(x)$ arbitrarily large and negative by requiring that x be sufficiently large and positive and we write*

$$\lim_{x \to -\infty} f(x) = +\infty, \tag{1.3.5}$$

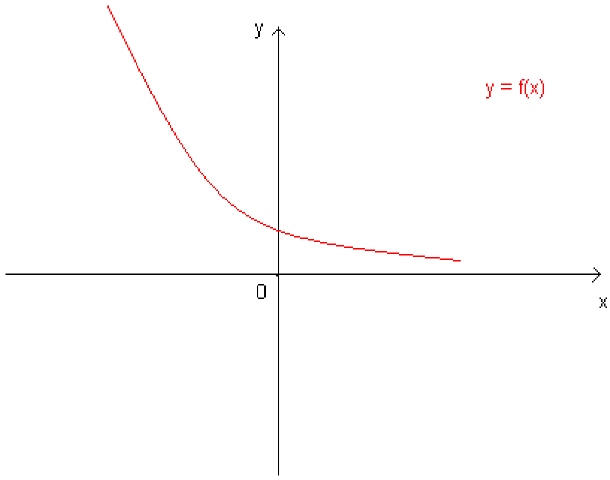

Figure 1.10: Picture showing the graph of a function with a vertical asymptote $x = x_0$

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which is read "the limit of $f(x)$ as x approaches $-\infty$ is $+\infty$". We also say that $f(x)$ diverges to $+\infty$ when x approaches $-\infty$ (see figure 1.11).

In the following definition we describe a function $f(x)$ that is negative and larger and larger negative for negative and large values of x:

DEFINITION **1.3.6.** *The function f approaches the limit $+\infty$ at $-\infty$ if we make $f(x)$ arbitrarily large and negative by requiring that x be sufficiently large and positive and we write*

$$\lim_{x\to-\infty} f(x) = +\infty, \tag{1.3.6}$$

which is read "the limit of $f(x)$ as x approaches $-\infty$ is $+\infty$". We also say that $f(x)$ diverges to $+\infty$ when x approaches $-\infty$ (see figure 1.12).

Example **1.3.2.**

$$\lim_{x\to+\infty} x = +\infty; \qquad \lim_{x\to-\infty} x = -\infty$$

Example **1.3.3.**

$$\lim_{x\to+\infty} x^2 = +\infty; \qquad \lim_{x\to-\infty} x^2 = +\infty$$

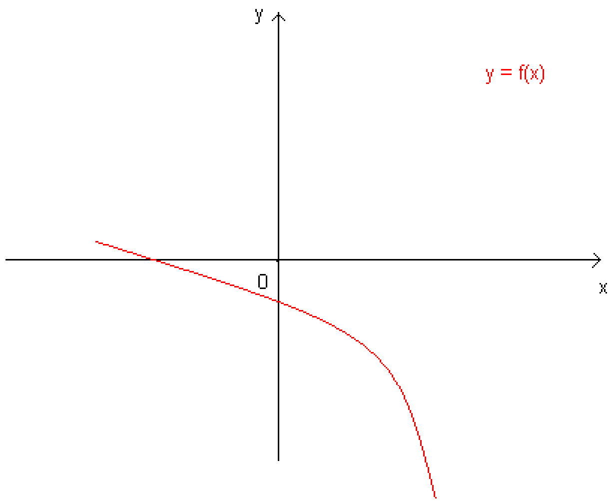

Figure 1.11: Picture showing the graph of a function with a vertical asymptote $x = x_0$

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Example **1.3.4.**

$$\lim_{x\to+\infty} x^3 = +\infty; \qquad \lim_{x\to-\infty} x^3 = -\infty$$

From the above examples we understand that he general rule is

1. When n is EVEN, then:

$$\lim_{x\to+\infty} x^n = +\infty; \qquad \lim_{x\to-\infty} x^n = +\infty$$

2. When n is ODD, then:

$$\lim_{x\to+\infty} x^n = +\infty; \qquad \lim_{x\to-\infty} x^n = -\infty$$

1.3.3 Infinite limits "at infinity": diagonal asymptotes

We conclude this section with the study of the so-called diagonal asymptotes.

DEFINITION **1.3.7.** *We say that the straight line $y = mx + c$ (where m and c are constants) is a diagonal asymptote (or slant asymptote) for $f(x)$*

$$\lim_{x\to+\infty} \Big[f(x) - (mx + c)\Big] = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} \Big[f(x) - (mx + c)\Big] = 0$$

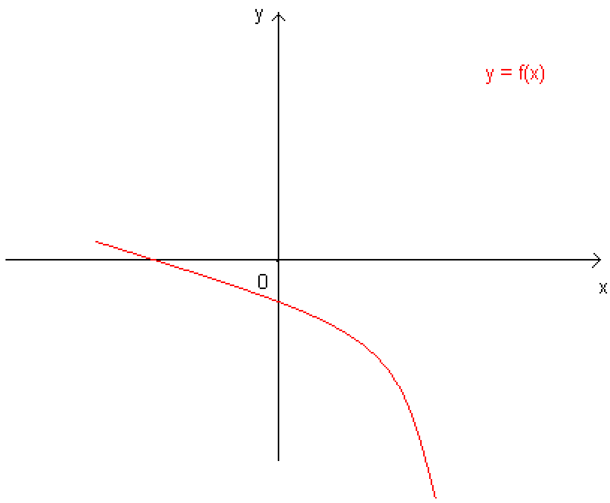

Figure 1.12: Picture showing the graph of a function with a vertical asymptote $x = x_0$

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respectively.

Example **1.3.5.** *We consider the function given by*

$$f(x) = x + \frac{1}{x}$$

and we want to see whether this function has any diagonal asymptote.

We start by imposing that the following limits be 0:

$$\lim_{x \to \pm\infty} \left[x + \frac{1}{x} - (mx + c) \right] = 0$$

and we want to see if we can find m and c so that this limit is 0.

$$\begin{aligned}
0 &= \lim_{x \to \pm\infty} \left[x + \frac{1}{x} - (mx + c) \right] \\
&= \lim_{x \to \pm\infty} \left[x + \frac{1}{x} - mx - c \right] \\
&= \lim_{x \to \pm\infty} \left[\frac{x^2 + 1 - mx^2 - cx}{x} \right] \\
&= \lim_{x \to \pm\infty} \left[\frac{(1 - m)x^2 - cx + 1}{x} \right]
\end{aligned}$$

In order for the above limit to be zero we must have

$$1 - m = 0$$

which gives

$$m = 1,$$

therefore our limit reduces to

$$0 = \lim_{x \to +\infty} \left[\frac{-cx + 1}{x} \right] = -c$$

therefore we obtain

$$-c = 0$$

i.e.

$$c = 0.$$

Therefore the function has a diagonal asymptote which equation is

$$y = x$$

and $f(x) = x + \frac{1}{x}$ approaches $y = x$ for $x \to +\infty$ and for $x \to -\infty$. See figure (1.13).

1.4 Limits: indeterminate cases

The following examples might include examples we already encountered earlier in this chapter.

Example **1.4.1.**

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

The first thing to do is to try to substitute the value 1 for x in

$$f(x) = \frac{x^2 - 1}{x - 1}$$

but if we do so the result is $\frac{0}{0}$ which is meaningless. Let us rewrite $f(x)$ as

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1$$

therefore

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

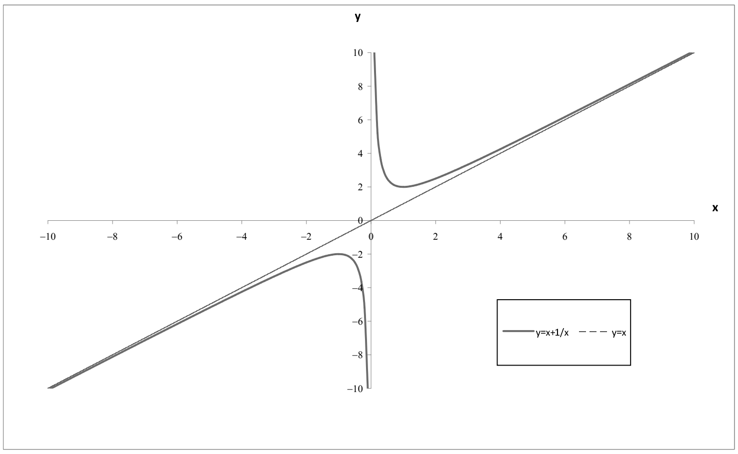

Figure 1.13: Picture showing the graph of function $f(x) = x + \frac{1}{x}$ with its diagonal asymptote $y = x$.

Example **1.4.2.**

$$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$$

Again the first thing to do is to try to substitute the value 0 for x in

$$f(x) = \frac{\sqrt{4+x}-2}{x}$$

but if we do so the result is $\frac{0}{0}$ which is meaningless. Let us rewrite $f(x)$ as

$$\begin{aligned}
\frac{\sqrt{4+x}-2}{x} &= \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} \\
&= \frac{4+x-4}{x\left(\sqrt{4+x}+2\right)} \\
&= \frac{x}{x\left(\sqrt{4+x}+2\right)} \\
&= \frac{1}{\sqrt{4+x}+2}
\end{aligned}$$

therefore

$$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{\sqrt{4+0}+2} = \frac{1}{4}$$

Example **1.4.3.** *We consider the so-called Heaviside function $H(x)$ which is defined as*

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0 \end{cases}$$

$H(x)$ is given by two formulae and the transition point between the two formulae is $x = 0$.
We need to evaluate two limits: the so-called right-hand limit and the left-hand limit of $H(x)$ as x approaches 0. The left-hand limit is defined by

$$\lim_{x \to 0^-} H(x) = \lim_{x \to 0,\, x<0} H(x) = \lim_{x \to 0} 0 = 0$$

The right-hand limit is defined by

$$\lim_{x \to 0^+} H(x) = \lim_{x \to 0,\, x>0} H(x) = \lim_{x \to 0} 1 = 1$$

Example **1.4.4.**

$$f(x) = \begin{cases} 1, & x < 0, \\ x^2 + 1, & x \geq 0 \end{cases}$$

$f(x)$ is given by two formulae and the transition point from one formula to the other is $x = 0$.
We need to evaluate the left- and the right-hand limits. The left-hand limit is given by

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1 = 1,$$

where the right-hand limit is given by

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 1) = 0 + 1 = 1$$

Therefore we simply write

$$\lim_{x \to 0} f(x) = 1.$$

The last example of this section shows how a function given by a single formula needs something to be written by making use of two different formulae in order to be written in a more explicit way. We are interested in evaluating the following limit.

Example **1.4.5.**

$$\lim_{x \to 0} \frac{|x|}{x}$$

First of all if we try to substitute the value 0 for x in

$$f(x) = \frac{|x|}{x}$$

we obtain $\frac{0}{0}$ which is meaningless as we know. In this case it is a good idea to try to write $f(x)$ in a more explicit way as follows

$$\frac{|x|}{x} = \begin{cases} \frac{-x}{x} = -1, & x < 0, \\ \frac{x}{x} = 1, & x > 0 \end{cases}$$

therefore we need to evaluate the left- and the right-hand limits as x approaches 0. We have

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0} (-1) = -1$$
$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0} 1 = 1$$

Exercise **1.4.1.** *Sketch the graph of* $f(x) = \frac{|x|}{x}$.

Example **1.4.6.**

$$\lim_{x \to +\infty} (x^2 - x) = (+\infty) - (+\infty) \qquad \text{INDETERMINATE CASE}$$

Therefore we try to evaluate the limit in the following way:

$$\lim_{x \to +\infty} (x^2 - x) = \lim_{x \to +\infty} x^2 \left(1 - \frac{1}{x}\right) = \big(+\infty(1 - 0)\big) = (+\infty) \cdot 1 = +\infty$$

Example **1.4.7.**

$$\lim_{x \to -\infty} (-x^3 - 3x^2 + 5) = (+\infty) - (+\infty) \qquad \text{INDETERMINATE CASE}$$

Therefore we try to evaluate the limit in the following way:

$$\lim_{x \to -\infty} (-x^3 - 3x^2 + 5) = \lim_{x \to -\infty} (-x^3) \left(1 - \frac{3}{x} - \frac{5}{x^3}\right) = \big(+\infty(1 + 0 - 0)\big) = (+\infty) \cdot 1 = +\infty$$

Example **1.4.8.**

$$\lim_{x \to +\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{+\infty}{+\infty} \qquad \text{INDETERMINATE CASE}$$

Therefore we try to evaluate the limit in the following way:

$$\begin{aligned}
\lim_{x\to+\infty}\frac{3x^2-x-2}{5x^2+4x+1} &= \lim_{x\to+\infty}\frac{x^2\left(3-\frac{1}{x}-\frac{2}{x^2}\right)}{x^2\left(5+\frac{4}{x}+\frac{1}{x^2}\right)}\\
&= \lim_{x\to+\infty}\frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}\\
&= \frac{3-0-0}{5+0+0}=\frac{3}{5}
\end{aligned}$$

Example **1.4.9.**

$$\lim_{x\to-\infty}\frac{6x^5+3x^2x+4}{x^3-5}=\frac{-\infty}{-\infty} \qquad \text{INDETERMINATE CASE}$$

Therefore we try to evaluate the limit in the following way:

$$\begin{aligned}
\lim_{x\to-\infty}\frac{6x^5+3x^2x+4}{x^3-5} &= \lim_{x\to-\infty}\frac{x^5\left(6+\frac{3}{x^3}+\frac{4}{x^5}\right)}{x^3\left(1-\frac{5}{x^3}\right)}\\
&= \lim_{x\to-\infty}\frac{x^2\left(6+\frac{3}{x^3}+\frac{4}{x^5}\right)}{1-\frac{5}{x^3}}\\
&= (+\infty)\,\frac{6+0+0}{1-0}=+\infty
\end{aligned}$$

Example **1.4.10.**

$$\lim_{x\to+\infty}\frac{3x^2-4x+1}{5x^4+2}=\frac{+\infty}{+\infty} \qquad \text{INDETERMINATE CASE}$$

Therefore we try to evaluate the limit in the following way:

$$\begin{aligned}
\lim_{x\to+\infty}\frac{3x^2-4x+1}{5x^4+2} &= \lim_{x\to+\infty}\frac{x^2\left(3-\frac{4}{x}+\frac{1}{x^2}\right)}{x^4\left(5+\frac{2}{x^4}\right)}\\
&= \lim_{x\to+\infty}\frac{3-\frac{4}{x}+\frac{1}{x^2}}{x^2\left(5+\frac{2}{x^4}\right)}\\
&= \frac{3-0+0}{(+\infty)\,(5+0)}=0
\end{aligned}$$

Example **1.4.11.**

$$\lim_{x\to+\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\frac{+\infty}{+\infty} \qquad \text{INDETERMINATE CASE}$$

Therefore we try to evaluate the limit in the following way:

$$\begin{aligned}
\lim_{x\to+\infty}\frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to+\infty}\frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} \\
&= \lim_{x\to+\infty}\frac{x\sqrt{\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} \\
&= \lim_{\to+\infty}\frac{\sqrt{\left(2+\frac{1}{x^2}\right)}}{\left(3-\frac{5}{x}\right)} \\
&= \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}
\end{aligned}$$

L'HOSPITAL'S RULE

Suppose we have two functions

$$f(x) \qquad \text{and} \qquad g(x)$$

that are both differentiable on an interval containing a point x_0 and that we can write

$$\frac{f(x)}{g(x)}.$$

Suppose the limit

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}$$

is an INDETERMINATE FORM $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then l'Hospital's rule says that:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

therefore the original limit can be evaluated by evaluating the limit of the quotient of the derivatives of f and g!
Note that if the limit of the derivatives

$$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

does not exist, this does not imply that the original limit

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

does not exist!

Example **1.4.12.** *(of when to use l'Hospital's rule)*

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{0}{0} \qquad \text{INDETERMINATE CASE}$$

By l'Hospital's rule we have:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\big(\sin(x)\big)'}{(x)'} = \lim_{x \to 0} \frac{\cos(x)}{1} = \frac{1}{1} = 1$$

Example **1.4.13.** *(of when to use l'Hospital's rule)*

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = \frac{0}{0} \qquad \text{INDETERMINATE CASE}$$

By l'Hospital's rule we have:

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = \lim_{x \to 0} \frac{\big(\cos(x) - 1\big)'}{(x)'} = \lim_{x \to 0} \frac{-\sin(x)}{(1} = \frac{0}{1} = 0$$

Example **1.4.14.** *(of when to use l'Hospital's rule)*

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{0}{0} \qquad \text{INDETERMINATE CASE}$$

By l'Hospital's rule we have:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{\big(e^x - 1\big)'}{(x)'} = \lim_{x \to 0} \frac{e^x}{1} = \frac{1}{1} = 1$$