# Mathematics 1

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Undergraduate study in
**Economics, Management, Finance and the Social Sciences**

THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

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# Chapter 1

# General introduction

## Studying mathematics

The study of mathematics can be very rewarding. It is particularly satisfying to solve a problem and know that it is solved. Unlike many of the other subjects you will study, there is always a right answer in mathematics problems. Of course, part of the excitement of the social sciences arises from the fact that there may be no single 'right answer' to a problem: it is stimulating to participate in debate and discussion, to defend, or re-think (and possibly change) your position.

It would be wrong to think that, in contrast, mathematics is very dry and mechanical. It can be as much of an art as a science. Although there may be only one right (final) answer, there could be a number of different ways of obtaining that answer, some more complex than others. Thus, a given problem will have only one 'answer', but many 'solutions' (by which we mean routes to finding the answer). Generally, a mathematician likes to find the simplest solution possible to a given problem, but that does not mean that any other solution is wrong. (There may be different, equally simple, solutions.)

With mathematical questions, you first have to work out precisely what it is that the question is asking, and then try to find a method (hopefully a nice, simple one) which will solve the problem. This second step involves some degree of creativity, especially at an advanced level. You must realise that you can hardly be expected to look at every mathematics problem and write down a beautiful and concise solution, leading to the correct answer, straight away. Of course, some problems are like this (for example, 'Calculate $2 + 2$'!), but for other types of problem you should not be afraid to try various different techniques, some of which may fail. In this sense, there is a certain amount of 'trial and error' in solving some mathematical problems. This really is the way a lot of mathematics is done. For obvious reasons, teachers, lecturers, and textbooks rarely give that impression: they present the solution right there on the page or the blackboard, with no indication of the time a student might be expected to spend thinking—or of the dead-end paths he or she might understandably follow—before a solution can be found. It is a good idea to have scrap paper to work with so that you can try out various methods of solution. (It is very inhibiting only to have in front of you the crisp sheet of paper on which you want to write your final, elegant, solutions. Mathematics is **not** done that way.) You must not get frustrated if you can't solve a problem immediately. As you proceed through the subject, gathering more experience, you will develop a feel for which techniques are likely to be useful for particular problems. You should not be afraid to try different techniques, some of which may not work, if you cannot immediately recognise which technique to use.

## Mathematics in the social sciences

Many students find mathematics difficult and are tempted to ask why they have to endure the agony and anguish of learning and understanding difficult mathematical concepts and techniques. Hopefully you will not feel this way, but if you do, be assured that all the techniques you struggle to learn in this subject will be useful in the end for their applications in economics, management, and many other disciplines. Some of these applications will be illustrated in this subject guide and in the textbooks. In fact, as the textbook discussions illustrate, far from making things difficult and complicated, mathematics makes problems in economics, management and related fields 'manageable'. It's not just about working out numbers; using mathematical models, **qualitative**—and not simply **quantitative**—results can be obtained. [1]

[1] *See Anthony and Biggs, Chapter 1, for instance.*

## Aims and objectives

There is a certain amount of enjoyment to be derived from mathematics for its own sake. It is a beautiful subject, with its own concise, precise, and powerful language. To many people, however, the main attraction of mathematics is its breadth of useful applications.

The main aim of this subject is to equip you with the mathematical tools for the study of economics, management, accounting, banking and related disciplines. In conjunction with this, you will hopefully acquire what mathematicians call 'mathematical maturity'—the ability to think analytically about problems and to be able to attack them sensibly (and without fear) using mathematical techniques. This is something that will come with practice and with the increasing confidence that you will develop as you understand more deeply the key concepts and the relationships between the different parts of the subject.

## How to use the subject guide

This subject guide is **absolutely not** a substitute for the textbooks. It is only what its name suggests: a guide to the study and reading you should undertake. In each of the subsequent chapters, brief discussions of the syllabus topics are presented, together with pointers to recommended readings from the textbooks. **It is essential that you use textbooks.** Generally, it is a good idea to read the texts as you work through a chapter of the guide.

It is most useful to read what the guide says about a particular topic, then do the necessary reading, then come back and re-read what the guide says to make sure you fully understand the topic. Textbooks are also an invaluable source of examples for you to attempt.

You should not necessarily spend the same amount of time on each chapter of the guide: some chapters cover much more material than others. I have divided the guide into chapters in order to group together topics on particular central themes, rather than to create units of equal length.

The discussions of some topics in this guide are rather more thorough than others. Often, this is not because those topics are more significant, but because the textbook treatments are not as extensive as they might be.

Within each chapter of the guide you will encounter 'Activities'. You should carry out these activities as you encounter them: they are designed to help you understand the topic under discussion. Solutions to them are at the end of the chapters, but do make a serious attempt at them before consulting the solutions.

To help your time management, the chapters and topics of the subject are converted below into **approximate** percentages of total time. However, this is purely for indicative purposes. Some of you will know the basics quite well and need to spend less time on the earlier material, while others might have to work hard to comprehend the very basic topics before proceeding onto the more advanced.

| Chapter | Title | % Time |
|---|---|---|
| 2 | Basics | 20 |
| 3 | Differentiation | 20 |
| 4 | Integration | 15 |
| 5 | Functions of Several Variables | 20 |
| 6 | Matrices and Linear Equations | 15 |
| 7 | Sequences and Series | 10 |

At the end of each chapter, you will find a list of 'Learning outcomes'. This indicates what you should be able to do having studied the topics of that chapter. At the end of each chapter, there are 'Sample examination questions', which are based largely on past exam questions. Some of the sample examination questions are really only samples of **parts** of exam questions.

# Recommended books

The main recommended text is the book by Anthony and Biggs. This covers all of the required material and uses the same notations as this guide. But if you need more help with the material of the second chapter ('Basics'), you might find it useful to consult some other texts (such as the one by Booth), which treat this basic material more slowly.

## Main text

⋆ Anthony, M. and Biggs, N., *Mathematics for Economics and Finance.* (Cambridge University Press, Cambridge, UK, 1996.) [ISBN 0 521 55113 7 (hardback) and ISBN 0 521 55913 8 (paperback)].

⋆ Recommended for purchase.

## Other recommended texts

Binmore, K. and Davies, J. *Calculus.* (Cambridge University Press, Cambridge, UK, 2001) [ISBN 0521775418]

Black, J., and Bradley, J.F. *Essential Mathematics for Economists.* Second Edition (J. Wiley and Sons, Chichester, England, 1980) [ISBN 0-471-27659-6 (cloth), 0-471-27660-X (paperback)].

Booth, D.J. *Foundation Mathematics (Second Edition).* (Addison-Wesley, 1994) [ISBN 0 201 62419-2].

Bradley, T., and Patton, P. *Essential Mathematics for Economics and Business.* (Wiley, 1998) [ISBN 0 471 97511 7].

Dowling, Edward T. *Introduction to Mathematical Economics.* Second Edition. Schaum's Outline Series. (McGraw-Hill, 1992) [ISBN 0-07-017674-4]. This is a revised edition of the author's *Mathematics for Economists* (McGraw-Hill, 1980.)

Holden, K. and Pearson, A.W. *Introductory Mathematics for Economics and Business.* Second Edition. (The Macmillan Press, London, 1992) [ISBN 0-333-57649-7 (hardback), 0-333-57650-0 (paperback)]. This is a revised and expanded edition of *Introductory Mathematics for Economists* (The Macmillan Press, 1983).

Ostaszewski, A. *Mathematics in Economics: Models and Methods.* (Blackwell, Oxford, UK, 1993) [ISBN 0-631-18055-9 (hardback), 0-631-18056-7 (paperback)].

Each chapter of Anthony and Biggs has a large section of fully worked examples, and a selection of exercises for the reader to attempt.

The book by Binmore and Davies contains all the calculus you will need, and a lot more, although it is at times a bit more advanced than you will need.

Black and Bradley cover much of the necessary mathematics, explaining the economic applications.

If you find you have considerable difficulty with some of the earlier basic topics in this subject, then you should consult the book by Booth (or a similar one: there are many at that level). This book takes a slower-paced approach to these more basic topics. It would not be suitable as a main text, however, since it only covers the easier parts of the subject.

The book by Bradley and Patton covers most of the material, and has plenty of worked examples.

Dowling's book contains lots of worked examples. It is, however, less concerned with explaining the techniques. It would not be suitable as your main text, but it is a good source of additional examples.

The book by Holden and Pearson covers most of the material, and has discussions of economic applications.

Ostaszewski is at a slightly higher level than is needed for most of the subject, but it is very suitable for a number of the topics, and provides many examples.

There are many other books which cover the material of this subject, but those listed above are the ones I shall refer to explicitly.

It is important to understand how you should use the textbooks. As I mentioned above, there are no great debates in mathematics at this level: you should not, therefore, find yourself in passionate disagreement with a passage in a mathematics text! However, try not to find yourself in **passive agreement** with it either. It is so very easy to read a mathematics text and agree with it, **without engaging with it**. Always have a pen and scrap paper to hand, to make notes and to work through, for yourself, the examples an author presents. The single most important point to be made about learning mathematics is that to learn it properly, you have to do it. **Do** work through the worked examples in a textbook and **do** attempt the exercises. This is the real way to learn mathematics. In the examination, you are hardly likely to encounter a question you have seen before, so you must have practised enough examples to ensure that you know your techniques well enough to be able to cope with new problems.

## Examination advice

A sample exam paper may be found at the end of this Subject Guide. You will see that there is a section of compulsory questions and a section from which you choose questions. Any changes to exam format will be announced in examiners' reports.

It is worth making a few comments about exam technique. Perhaps the most important, though obvious, point is that you do not have to answer the questions in any particular order; choose the order that suits you best. Some students will want to do easy questions first to boost their confidence, while others will like to get the difficult ones out of the way. It is entirely up to you.

Another point, often overlooked by students, is that you should **always** include your working. This means two things.

- First, do not simply write down the answer in the exam script, but explain your method of obtaining it (that is, what I called the 'solution' earlier).
- Secondly, include your rough working. You should do this for two reasons:
  - If you have just written down the answer without explaining how you obtained it, then you have not convinced the examiner that you know the techniques, and it is the techniques that are important in this subject. (The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined.)
  - If you have not completely solved a problem, you may still be awarded marks for a partial, incomplete, or slightly wrong, solution; if you have written down a wrong answer and nothing else, no marks can be awarded. (You may have carried out a lengthy calculation somewhere on scrap paper where you made a silly arithmetical error. Had you included this calculation in the exam answer book, you would probably not have

been heavily penalised for the arithmetical error.) It is useful, also, to let the examiner know what you are thinking. For example, if you know you have obtained the wrong answer to a problem, but you can't see how to correct it, say so!

As mentioned above, you will find that, wherever appropriate, there are sample exam questions at the end of the chapters. These are based in large part on the questions appearing in past examination papers. As such, they are an indication of the types of question that might appear in future exams. But they are **just** an indication. The examiners want to test that you know and understand a number of mathematical methods and, in setting an exam paper, they are trying to test whether you do indeed know the methods, understand them, and are able to use them, and not merely whether you vaguely remember them. Because of this, you will quite possibly encounter some questions in your exam which seem unfamiliar. Of course, you will only be examined on material in the syllabus. Furthermore, you should **not** assume that your exam will be almost identical to the previous year's: for instance, just because there was a question, or a part of a question, on a certain topic last year, you should not assume there will be one on the same topic this year. For this reason, you cannot guarantee passing if you have concentrated only on a very small fraction of the topics in the subject. This may all sound a bit harsh, but it has to be emphasised.

## The use of calculators

You will not be permitted to use calculators of any type in the examination. This is not something that you should panic about: the examiners are interested in assessing that you understand the key methods and techniques, and will set questions which do not require the use of a calculator.

In this guide, I will perform some calculations for which a calculator would be needed, but you will not have to do this in the exam questions. Look carefully at the answers to the sample exam questions to see how to deal with calculations. For example, if the answer to a problem is $\sqrt{2}$, then leave the answer like that: there is no need to express this number as a decimal (for which one would need a calculator).

# Chapter 2

# Basics

## Essential reading

(For full publication details, see Chapter 1.)

Anthony, M. and Biggs, N., *Mathematics for Economics and Finance.* Chapters 1, 2, and 7.

## Further reading

Binmore, K. and Davies, J. *Calculus.* Chapter 2, Sections 2.1–2.6.

Black, J., and Bradley, J.F. *Essential Mathematics for Economists.* Chapters 1 and 4.

Booth, D.J. *Foundation Mathematics.* Chapter 1, parts of Module 1; Module 3, Units 1 and 3; Module 4; Chapter 2, Module 5, Module 6 Unit 1, Module 7 Units 1 and 2; Chapter 3, Module 11, Module 12, Module 13, Module 14, Module 15 Unit 1.

Bradley, T and Patton, P. *Essential Mathematics for Economics and Business.* Sections 1.1–1.6, 2.1, 3.1.1, 4.1–4.3.

Dowling, Edward T. *Introduction to Mathematical Economics.* Chapters 1 and 2.

Holden, K and Pearson, A.W. *Introductory Mathematics for Economics and Business.* Chapters 1 and 3.

Ostaszewski, A. *Mathematics in Economics*: Chapter 1 (though this is more advanced than is required at this stage of the subject), Chapter 5, sections 5.5 and 5.12.

Simon, C.P. and Blume, L. *Mathematics for Economists*: Sections 2.1 and 2.2.

## Introduction

This chapter discusses some of the very basic aspects of the subject, aspects on which the rest of the subject builds. It is essential to have a firm understanding of these topics before the more advanced topics can be understood.

Most things in economics and related disciplines — such as demand, sales, price, production levels, costs, and so on — are interrelated. Therefore, in order to come to rational decisions on appropriate values for many of these parameters it is of considerable benefit to form mathematical models or functional relationships between them. It should be noted at the outset that, in general, the economic models used are typically only approximations to reality, as indeed are all models. They are, nonetheless, very useful in decision making. Before we can attempt such modelling, however, we need some mathematical basics.

This chapter contains a lot of material, but much of it will be revision. If you find any of the sections difficult, please refer to the texts listed in the margin for further explanation and examples.

## Basic notations

Although there is a high degree of standardisation of notation within mathematical texts, some differences do occur. The notation given here is indicative of what is used in the rest of this guide and in most of the texts. [1] You should endeavour to familiarise yourself with as many of the common notations as possible. For example, $|a|$ means 'the absolute value of $a$', which equals $a$ if $a$ is non-negative (that is, if $a \geq 0$), and equals $-a$ otherwise. For instance, $|6| = 6$ and $|-2.5| = 2.5$. (This is sometimes termed 'the modulus of $a$'. Roughly speaking, the absolute value of a number is obtained just by ignoring any minus sign the number has.) As another example, multiplication is sometimes denoted by a dot, as in $a.b$ rather than $a \times b$. Beware of confusing multiplication and the use of a dot to indicate a decimal point. Even more commonly, one simply uses $ab$ to denote the multiplication of $a$ and $b$. Also, you should be aware of implied multiplications, as in $2(3) = 6$.

[1] *You may consult Booth, or a large number of other basic maths texts, for further information on basic notations.*

Some other useful notations are those for sums, products, and factorials. We denote the sum

$$x_1 + x_2 + \cdots + x_n$$

of the numbers $x_1, x_2, \ldots, x_n$ by

$$\sum_{i=1}^{n} x_i.$$

The '$\Sigma$' indicates that numbers are being summed, and the '$i = 1$' and $n$ below and above the $\Sigma$ show that it is the numbers $x_i$, as $i$ runs from 1 to $n$, that are being summed together. Sometimes we will be interested in adding up only some of the numbers. For example,

$$\sum_{i=2}^{n-1} x_i$$

would denote the sum $x_2 + x_3 + \cdots + x_{n-1}$, which is the sum of all the numbers except the first and last.

We denote their product $x_1 \times x_2 \times \cdots \times x_n$ (the result of multiplying all the numbers together) by

$$\prod_{i=1}^{n} x_i.$$

For a positive whole number, $n$, $n!$ ('$n$ factorial') is the product of all the numbers from 1 up to $n$. For example, $4! = 1.2.3.4 = 24$. By convention $0!$ is taken to be 1. The factorial can be expressed using the product notation:

$$n! = \prod_{i=1}^{n} i.$$

**Example:** Suppose that $x_1 = 1, x_2 = 3, x_3 = -1, x_4 = 5$. Then

$$\sum_{i=1}^{4} x_i = 1 + 3 + (-1) + 5 = 8, \quad \sum_{i=2}^{4} x_i = 3 + (-1) + 5 = 7.$$

We also have, for example,

$$\prod_{i=1}^{4} x_i = 1(3)(-1)(5) = -15, \quad \prod_{i=1}^{3} x_i = 1(3)(-1) = -3.$$

**Activity 2.1** *Suppose that* $x_1 = 3$, $x_2 = 1$, $x_3 = 4$, $x_4 = 6$. *Find* $\sum_{i=1}^{4} x_i$ *and* $\prod_{i=1}^{4} x_i$.

## Simple algebra

You should try to become confident and capable in handling simple algebraic expressions and equations. You should be proficient in:

- collecting up terms: e.g. $2a + 3b - a + 5b = a + 8b$.
- multiplication of variables: e.g.
  $(-a)(b) + (a)(-b) - 3(a)(b) + (-2a)(-4b) = -ab - ab - 3ab + 8ab = 3ab$
- expansion of bracketed terms: e.g.
  $(2x - 3y)(x + 4y) = 2x^2 - 3xy + 8xy - 12y^2 = 2x^2 + 5xy - 12y^2$.

You should also be able to factorise quadratic equations, something discussed later in this chapter.

**Activity 2.2** *Expand* $(x^2 - 1)(x + 2)$.

## Sets

A set may be thought of as a collection of objects. [2] A set is usually described by listing or describing its **members** inside curly brackets. For example, when we write $A = \{1, 2, 3\}$, we mean that the objects belonging to the set $A$ are the numbers $1, 2, 3$ (or, equivalently, the set $A$ consists of the numbers $1, 2$ and $3$). Equally (and this is what we mean by 'describing' its members), this set could have been written as

[2] *See Anthony and Biggs, Section 2.1.*

$$A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}.$$

Here, the symbol $\mid$ stands for 'such that'. Often, the symbol ':' is used instead, so that we might write

$$A = \{n : n \text{ is a whole number and } 1 \leq n \leq 3\}.$$

As another example, the set

$$B = \{x \mid x \text{ is a reader of this guide}\}$$

has as its members all of you (and nothing else). When $x$ is an object in a set $A$, we write $x \in A$ and say '$x$ belongs to $A$' or '$x$ is a member of $A$'.

The set which has no members is called the **empty set** and is denoted by $\emptyset$. The empty set may seem like a strange concept, but it has its uses.

We say that the set $S$ is a **subset** of the set $T$, and we write $S \subseteq T$, if every member of $S$ is a member of $T$. For example, $\{1, 2, 5\} \subseteq \{1, 2, 4, 5, 6, 40\}$. (Be aware that some texts use $\subset$ where we use $\subseteq$.)

Given two sets $A$ and $B$, the **union** $A \cup B$ is the set whose members belong to $A$ or $B$ (or both $A$ and $B$): that is,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

**Example:** If $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cup B = \{1, 2, 3, 4, 5, 7\}$.

Similarly, we define the **intersection**: $A \cap B$ to be the set whose members belong to both $A$ and $B$: [3]

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

[3] *See Anthony and Biggs for examples of union and intersection.*

**Activity 2.3** *Suppose $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$. Find $A \cap B$.*

## Numbers

There are some standard notations for important sets of numbers. [4] The set $\mathbb{R}$ of **real numbers**, may be thought of as the points on a line. Each such number can be described by a decimal representation.

[4] *See Anthony and Biggs, Section 2.1.*

Given two real numbers $a$ and $b$, we define the **intervals**

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$
$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$$
$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$$
$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$$
$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}\,.$$

The symbol $\infty$ means 'infinity', but it is not a real number, merely a notational convenience. You should notice that when a square bracket, '[' or ']', is used to denote an interval, the number beside the bracket is included in the interval, whereas if a round bracket, '(' or ')', is used, the adjacent number is not in the interval. For example, $[2, 3]$ contains the number 2, but $(2, 3]$ does not.

The set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ of **integers** is denoted by $\mathbb{Z}$.

The positive integers are also known as **natural numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Having defined $\mathbb{R}$, we can define the set $\mathbb{R}^2$ of **ordered pairs** $(x, y)$ of real numbers. Thus $\mathbb{R}^2$ is the set usually depicted as the set of points in a plane, $x$ and $y$ being the coordinates of a point with respect to a pair of axes. For instance, $(-1, 3/2)$ is an element of $\mathbb{R}^2$ lying to the left of and above $(0, 0)$, which is known as the **origin**.

## Functions

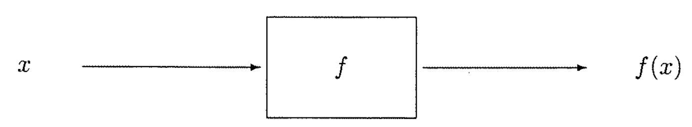
Given two sets $A$ and $B$, a **function** from $A$ to $B$ is a rule which assigns to each member of $A$ **precisely one** member of $B$. [5] For example, if $A$ and $B$ are both the set $\mathbb{Z}$, the rule which says 'add 2' is a function. Normally we express this function by a formula: if we call the function $f$, we can write the rule which defines $f$ as $f(x) = x + 2$. Two very important functions in economics are the supply and demand functions for a good. [6] These are discussed later in this chapter.

[5] *See Anthony and Biggs, Section 2.2.*

[6] *See Anthony and Biggs, Section 1.2.*

It is often helpful to think of a function as a machine which converts an input into an output, as shown in Figure 2.1.

**Figure 2.1 A function as a machine**



## Inverse functions

As the one-way arrows in Figure 2.1 indicate, a function is a one-way relationship: the function $f$ takes a number $x$ as input and it returns another number, $f(x)$. Suppose you were told that the output, $f(x)$, was a number $y$, and you wanted to know what the input was. In some cases, this is easy. For example, if $f(x) = x + 2$ and the output $f(x)$ is the number $y$, then we must have $y = f(x) = x + 2$. Solving for $x$ in terms of $y$, we find that $x = y - 2$. In other words, there is only one possible input $x$ which could have produced output $y$ for this function, namely $x = y - 2$. In a situation such as this, where for **each and every** $y$ there is **exactly one** $x$ such that $f(x) = y$, we say that the function $f$ has an inverse function. [7] The inverse function is denoted by $f^{-1}$, and it is the rule for reversing $f$. Formally, $f^{-1}(x)$ is defined by

$$x = f^{-1}(y) \iff f(x) = y.$$

(The symbol $\iff$ means 'if and only if' or 'is equivalent to'). When $f(x) = x + 2$, we have seen that

$$y = f(x) \iff x = y - 2,$$

so the inverse function (which takes as input a number $y$ and returns the number $x$ such that $f(x) = y$) is given by

$$f^{-1}(y) = y - 2.$$

(This could also be written as $f^{-1}(x) = x - 2$ or $f^{-1}(z) = z - 2$; there is nothing special about the symbol used to denote the variable (that is, the input to the function). [8])

[7] *See Anthony and Biggs, Section 2.2.*

[8] *See Anthony and Biggs, Section 2.2, for discussion of 'dummy variables'*

It should be emphasised that not every function has an inverse. For instance, the function $f(x) = x^2$, from $\mathbb{R}$ to $\mathbb{R}$, has no inverse. To see this, we can simply observe that there is not exactly one number $x$ such that $f(x) = y$, where $y = 1$; for, both when $x = 1$ and $x = -1$, $f(x) = x^2 = 1$. (Of course, this observation is true for any positive number $y$.) So, in this case, we cannot definitively answer the question 'If $f(x) = 1$, what is $x$?'.

**Activity 2.4** *If $f(x) = 3x + 2$, find a formula for $f^{-1}(x)$.*

## Composition of functions

If we are given two functions $f$ and $g$, then we can apply them consecutively to obtain what is known as the **composite** function $h$, given by the rule

$$h(x) = f(g(x)).$$

The composite function $h$ is denoted $h = fg$ and is often described in words as '$g$ followed by $f$' or as '$f$ after $g$'. [9]

[9] *See Anthony and Biggs, Section 2.3.*

**Example** Suppose that $f(x) = x + 1$ and $g(x) = x^4$. Then the composite function $h = fg$ is given by

$$fg(x) = f(g(x)) = f(x^4) = x^4 + 1.$$

On the other hand, the function $k = gf$ is given by

$$gf(x) = g(f(x)) = g(x + 1) = (x + 1)^4.$$

Note, then, that in general, the compositions $fg$ and $gf$ are different.

**Activity 2.5** *If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, find a formula for the composition $fg$.*

## Powers

When $n$ is a positive integer, the $n$th **power**[10] of the number $a$, $a^n$, is simply the product of $n$ copies of $a$, that is,

[10] *See Anthony and Biggs, Section 7.1.*

$$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ times}}.$$

The number $n$ is called the **power**, **exponent**, or **index**. We have the **power rules** (or **rules of exponents**):

$$a^x a^y = a^{x+y}, \quad (a^x)^y = a^{xy},$$

whenever $x$ and $y$ are positive integers. The power $a^0$ is defined to be 1. When $n$ is a positive integer, $a^{-n}$ means $1/a^n$. For example, $3^{-2}$ is $1/3^2 = 1/9$. The power rules hold when $x$ and $y$ are any integers, positive, negative or zero. When $n$ is a positive integer, $a^{1/n}$ is the 'positive $n$th root of $a$'; this is the number $x$ such that $x^n = a$. Formally, suppose $n$ is a positive integer and let $S$ be the set of all non-negative real numbers. Then the function $f(x) = x^n$ from $S$ to $S$ has an inverse function $f^{-1}$. We can think of $f^{-1}$ as the definition of raising a number to the power of $1/n$: explicitly, $f^{-1}(y) = y^{1/n}$. Of course, $a^{1/2}$ is usually denoted by $\sqrt{a}$, and is the **square root** of $a$. When $m$ and $n$ are integers and $n$ is positive, $a^{m/n}$ is $\left(a^{1/n}\right)^m$. So, the power rules still apply.

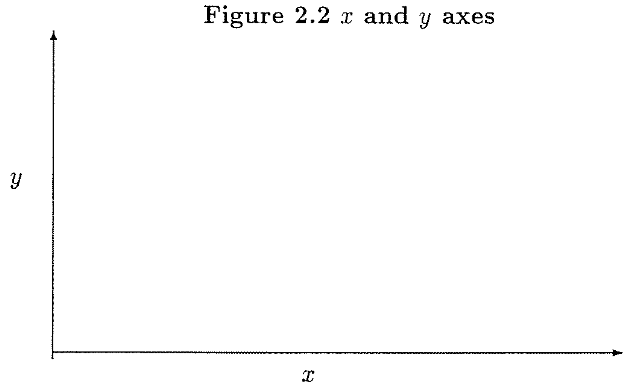
## Graphs

In this section, we consider the graphs of functions. The graphing of functions is very important in its own right, and familiarity with graphs of common functions and the ability to produce graphs systematically is a necessary and important aspect of the subject.

The **graph**[11] of a function $f(x)$ is the set of all points in the plane of the form $(x, f(x))$. Sketches of graphs can be very useful. To sketch a graph, we start with the $x$-axis and $y$-axis, as in Figure 2.2. (This figure only shows the region in which $x$ and $y$ are both non-negative, but the $x$-axis extends to the left and the $y$-axis extends downwards.

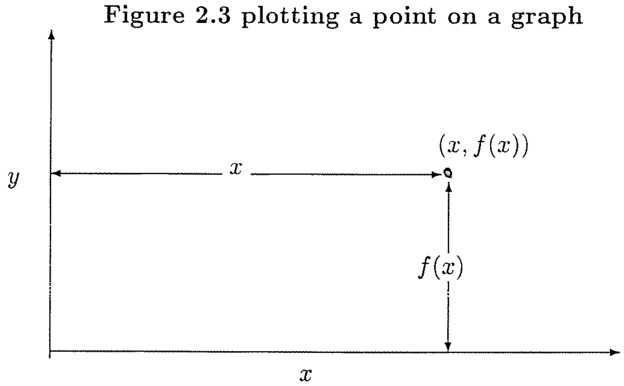
[11] *See Anthony and Biggs, Section 2.4.*

**Figure 2.2 $x$ and $y$ axes**



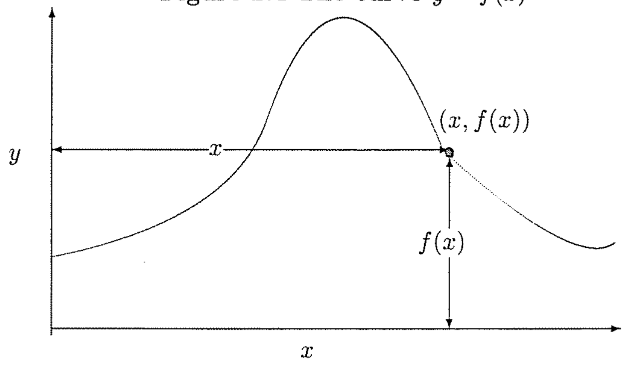

We then plot all points of the form $(x, f(x))$. Thus, at $x$ units from the origin (the point where the axes cross), we plot a point whose height above the $x$-axis (that is, whose $y$-coordinate) is $f(x)$. This is shown in Figure 2.3. The graph is sometimes described as the graph $y = f(x)$ to signify that the $y$-coordinate represents the function value $f(x)$.

**Figure 2.3 plotting a point on a graph**




Joining together all points of the form $(x, f(x))$ results in a **curve**, called the **graph** of $f(x)$. This is often described as the **curve with equation** $y = f(x)$. Figure 2.4 gives an example of what this curve might look like.

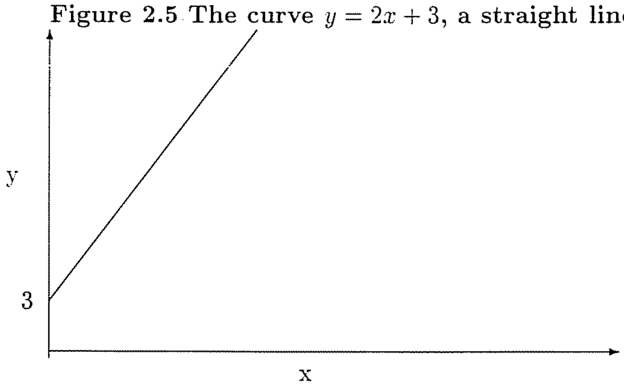
**Figure 2.4 The curve $y = f(x)$**




These figures indicate what is meant by the graph of a function, but you should not imagine that the correct way to sketch a graph is to plot a few points of the form $(x, f(x))$ and join them up; this approach rarely works well and more sophisticated techniques are needed. (Many of these will be discussed later.)

We shall discuss the graphs of some standard important functions as we progress. We start with the easiest of all: the graph of a **linear function.** In the next section we look at the graphs of quadratic functions. The linear functions are those of the form $f(x) = mx + c$ and their graphs are straight lines, with **gradient**, or **slope**, $m$, which cross the $y$-axis at the point $(0, c)$. Figure 2.5 illustrates the graph of the function $f(x) = 2x + 3$ and Figure 2.6 the graph of the function $f(x) = -x + 2$.

**Figure 2.5 The curve $y = 2x + 3$, a straight line**



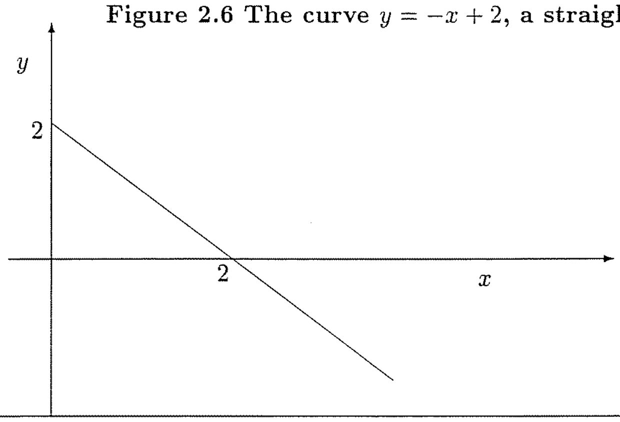


**Figure 2.6 The curve $y = -x + 2$, a straight line**

**Activity 2.6** *Sketch the curves $y = x + 3$ and $y = -3x - 2$.*

## Quadratic equations and curves

A common problem is to find the set of solutions of a **quadratic** equation[12]

$$ax^2 + bx + c = 0,$$

where we may as well assume that $a \neq 0$, because if $a = 0$ the equation reduces to a linear one. (Note that, by a solution, we mean a value of $x$ for which the equation is true.) In some cases the quadratic expression can be factorised, which means that it can be written as the product of two linear terms (of the form $x - a$ for some $a$). For example $x^2 - 6x + 5 = (x - 1)(x - 5)$, so the equation $x^2 - 6x + 5$ becomes $(x - 1)(x - 5) = 0$. Now the only way that two numbers can multiply to give 0 is if at least one of the numbers is 0, so we can conclude that $x - 1 = 0$ or $x - 5 = 0$; that is, the equation has two solutions, 1 and 5. Although factorisation may be difficult, there is a general technique for determining the solutions to a quadratic equation, as follows.[13] Suppose we have the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$. Then:

[12] *See Anthony and Biggs, Section 2.4.*

[13] *See Anthony and Biggs, Section 2.4.*

- if $b^2 - 4ac < 0$, the equation has **no** real solutions;
- if $b^2 - 4ac = 0$, the equation has **exactly one** solution, $x = -b/(2a)$;
- if $b^2 - 4ac > 0$, the equation has **two** solutions:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

For example, consider the quadratic equation $x^2 - 2x + 3 = 0$; here we have $a = 1$, $b = -2$, $c = 3$. The quantity $b^2 - 4ac$ (called the **discriminant**) is $(-2)^2 - 4(1)(3) = -8$, which is negative, so this equation has no solution. This is less mysterious than it may seem. We can write the equation as $(x - 1)^2 + 2 = 0$. Rewriting the left-hand side of the equation in this form is known as 'completing the square'. Now, the square of a number is always greater than or equal to 0, so the quantity on the left of this equation is always at least 2 and is therefore never equal to 0. The above formulae for the solutions to a quadratic equation are obtained using the technique of completing the square[14].

[14] *See Anthony and Biggs, Section 2.4, if you haven't already.*

It is instructive to look at the graphs of quadratic functions and to understand the connection between these and the solutions to quadratic equations. First, let's look at the graph of a typical quadratic function $y = ax^2 + bx + c$. Figure 2.7 shows the curves one obtains for two typical quadratics $ax^2 + bx + c$. For the first, $a$ is positive and for the second $a$ is negative. We have omitted the $x$ and $y$ axes in these figures; it is the **shape** of the graph that we want to emphasise first. Note that the first graph has a 'U'-shape and that the second is the same sort of shape, upside-down. To be more formal, the curves are **parabolas**.

**Figure 2.7 Typical quadratic curves $y = ax^2 + bx + c$**

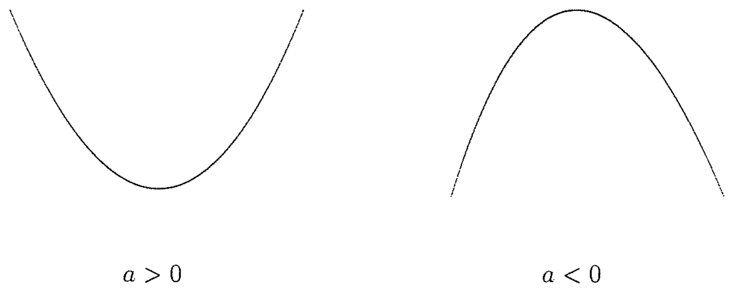
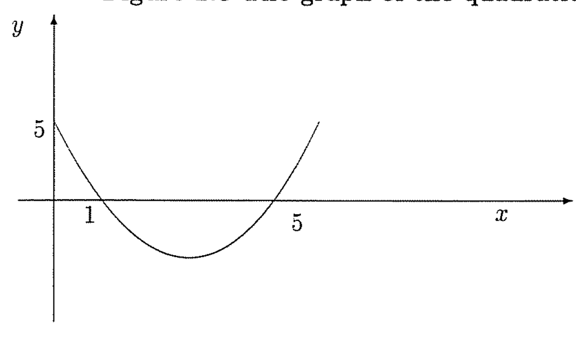



Figure 2.8 is the graph of the quadratic function $f(x) = x^2 - 6x + 5$.

**Figure 2.8 The graph of the quadratic $x^2 - 6x + 5$**

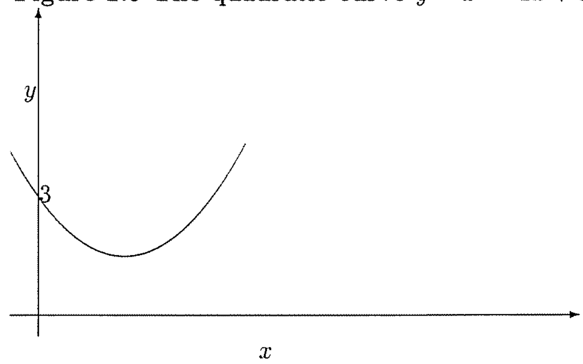



Note that, since the number in front of the $x^2$ term (what we called $a$ above) is positive, the curve is of the first type displayed in Figure 2.7. What we want to emphasise with this specific example is the positioning of the curve with respect to the axes. There is a fairly straightforward way to determine where the curve crosses the $y$-axis. Since the $y$-axis has equation $x = 0$, to find the $y$-coordinate of this crossing (or intercept), all we have to do is substitute $x = 0$ into the function. Since

$f(0) = 0^2 - 6(0) + 5 = 5$, the point where the curve crosses the $y$-axis is $(0, 5)$. (Generally, the point where the graph of a function $f(x)$ crosses the $y$-axis is $(0, f(0))$.) The other important points on the diagram are the points where the curve crosses the $x$-axis. Now, the curve has equation $y = f(x)$, and the $y$-axis has equation $y = 0$, so the curve crosses (or meets) the $x$-axis when $y = f(x) = 0$. (This argument, so far, is completely general: to find where the graph of $f(x)$ crosses the $y$-axis, we solve the equation $f(x) = 0$. In general, this may have no solution, one solution, or a number of solutions, depending on the function.) Thus, we have to solve the equation $x^2 - 5x + 6 = 0$. We did this earlier, and the solutions are $x = 1$ and $x = 5$. It follows that the curve crosses the $x$-axis at $(1, 0)$ and $(5, 0)$.

Figure 2.9 shows the graph of another quadratic, $f(x) = x^2 - 2x + 3$.

**Figure 2.9 The quadratic curve** $y = x^2 - 2x + 3$




Notice that this one does **not** cross the $x$-axis. This is because the quadratic equation $x^2 - 2x + 3 = 0$ (which we met earlier) has **no** solutions. You might ask what the coordinates of the lowest point of the 'U' are. Later, we shall encounter a general technique for answering such questions. For the moment, we can determine the point by using the observation, made earlier, that the function is $(x - 1)^2 + 2$. Now, $(x - 1)^2 \geq 0$ and is equal to 0 only when $x = 1$, so the lowest value of the function is 2, which occurs when $x = 1$; that is, the lowest point of the 'U' is the point $(1, 2)$. You can obtain quite a lot of information about quadratic curves without using very sophisticated techniques.

**Activity 2.7** *Sketch the curve* $y = x^2 + 4x + 3$. *Where does it cross the x-axis?*

# Polynomial functions

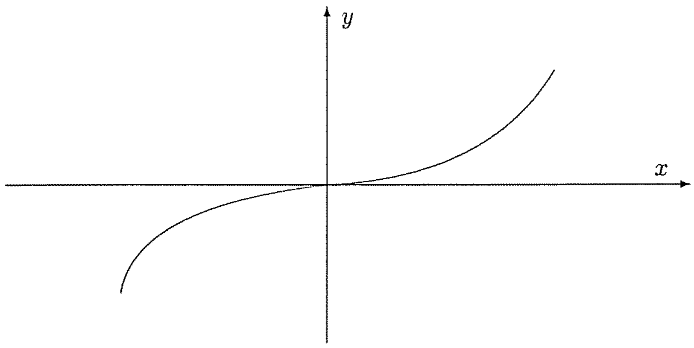
Linear and quadratic functions are examples of a more general type of function: the **polynomial functions**. A polynomial function is one of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

The right-hand side is known simply as a **polynomial**. The number $a_i$ is known as **the coefficient of** $x^i$. If $a_n \neq 0$ then $n$, the largest power of $x$ in the polynomial, is known as the **degree** of the polynomial. Thus, the linear functions are precisely the polynomials of degree 1 and the quadratics are the polynomials of degree 2.

Polynomials of degree 3 are known as cubics. The simplest is the function $f(x) = x^3$, the graph of which is shown in Figure 2.10.

**Figure 2.10 The curve $y = x^3$**




Notice that the graph of the function $f(x) = x^3$ only crosses the $x$-axis at the origin, $(0, 0)$. That is, the equation $f(x) = 0$ has just one solution. (We say that the function has just one **zero**). In general, a polynomial function of degree $n$ has at most $n$ zeroes. For example, since

$$x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3),$$

the function $f(x) = x^3 - 7x + 6$ has three zeroes; namely, $1, 2, -3$. Unfortunately, there is no general formula (as there is for quadratics) for the solutions to $f(x) = 0$ for polynomials $f$ of degree larger than 2.

**Activity 2.8** *Factorise* $f(x) = x^3 + 4x^2 + 3x$.

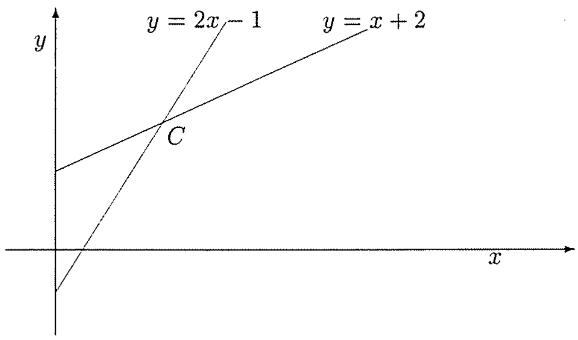
## Simultaneous equations

An important type of problem arises when we have several equations which we have to solve 'simultaneously'.[15] This means that we must find the intersection of the solution sets of the individual equations. We have already met an example of this: when we want to find the points (if any) where the curve $y = f(x)$ meets the $x$-axis, we are essentially solving two equations simultaneously. The first is $y = f(x)$ and the second (the equation of the $x$-axis) is $y = 0$. This is one way of thinking about how the equation $f(x) = 0$ arises. We shall spend a lot of time later looking at problems in which the aim is to solve simultaneously more than two equations. For the moment, we shall just illustrate with a simple example.

[15] *See Anthony and Biggs, Sections 1.3 and 2.4.*

Any two lines which are not parallel cross each other exactly once, but how do we find the crossing point? Let's consider the lines with equations $y = 2x - 1$ and $y = x + 2$. These are not parallel, since the gradient of the first is 2, whereas the gradient of the second is 1. Figure 2.11 shows the two lines. Our aim is to determine the coordinates of the crossing point $C$.

**Figure 2.11 The lines $y = 2x - 1$ and $y = x + 2$**




To find $C$, let us suppose that $C = (X, Y)$. Then, since $C$ lies on the line with equation $y = 2x - 1$, we must have $Y = 2X - 1$. But $C$ also lies on the line $y = x + 2$, so $Y = X + 2$. Therefore the coordinates $X$ and $Y$ of $C$ satisfy the following two equations, **simultaneously**:

$$\begin{aligned} Y &= 2X - 1 \\ Y &= X + 2 \end{aligned}$$

It follows that

$$Y = 2X - 1 = X + 2.$$

From $2X - 1 = X + 2$ we obtain $X = 3$. Then, to obtain $Y$, we use either the fact that $Y = 2X - 1$, obtaining $Y = 5$, or we can use the equation $Y = X + 2$, obtaining (of course) the same answer. It follows that $C = (3, 5)$.

**Activity 2.9** *Find the point of intersection of the lines with equations $y = 2x - 3$ and $y = 2 - frac12x$.*

## Supply and demand functions

Supply and demand functions[16] describe the relationship between the price of a good, the quantity supplied to the market by the manufacturer, and the amount the consumers wish to buy. The **demand function** $q^D$ of the price $p$ describes the demand quantity: $q^D(p)$ is the quantity which would be sold if the price were $p$. Similarly, the **supply function** $q^S$ is such that $q^S(p)$ is the amount supplied when the market price is $p$.

[16] *See Anthony and Biggs, Section 1.2*

In the simplest models of the market, it is assumed that the supply and demand functions are linear—in other words, their graphs are straight lines. For example, it could be that $q^D(p) = 4 - p$ and $q^S(p) = 2 + p$. Note that the graph of the demand function is a downward-sloping straight line, whereas the graph of the supply function is upward-sloping. This is to be expected, since, for example, as the price of a good increases, the consumers are prepared to buy less of the good, and so the demand function decreases as price increases.

Sometimes, the supply and demand relationships are expressed through equations. For instance, in the example just given we could equally well have described the relationship between demand quantity and price by saying that the **demand equation** is $q + p = 4$. The graphs of the demand function and supply function are known, respectively, as the **demand curve** and the **supply curve**.

There is another way to view the relationship between price and quantity demanded, where we ask how much the consumers (as a group; that is, on aggregate) would be willing to pay for each unit of a good, given that a quantity $q$ is available. From this viewpoint we are expressing $p$ in terms of $q$, instead of the other way round. We write $p^D(q)$ for the value of $p$ corresponding to a given $q$, and we call $p^D$ the **inverse demand function**. It is, as the name suggests, the inverse function to the demand function. For example, with $q^D(p) = 4 - p$, we have $q = 4 - p$ and so $p = 4 - q$; thus, $p^D(q) = 4 - q$. In a similar way, when we solve for the price in terms of the supply quantity, we obtain the **inverse supply function** $p^S(q)$.

The market is in **equilibrium**[17] when the consumers have as much of the commodity as they want and the suppliers sell as much as they want. This occurs when the quantity supplied matches the quantity demanded, or, **supply equals demand**. To find the equilibrium price $p^*$, we solve $q^D(p) = q^S(p)$ and then to determine the equilibrium quantity $q^*$ we compute $q^* = q^D(p^*)$ (or $q^* = q^S(p^*)$). (Generally there might be more than one equilibrium, but not when the supply and demand are linear.) Geometrically, the equilibrium point(s) occur where the demand curve and supply curve intersect.

[17] *See Anthony and Biggs, Section 1.3*

**Activity 2.10** *Suppose the demand function is $q^D(p) = 20 - 2p$ and that the supply function is $q^S(p) = \frac{2}{3}p - 4$. Find the equilibrium price $p^*$ and equilibrium quantity $q^*$.*

Not all supply and demand equations are linear. Consider the following example.

**Example:** Suppose that we have demand curve,

$$q = 250 - 4p - p^2$$

and supply curve

$$q = 2p^2 - 3p - 40.$$

Let us find the equilibrium price and quantity, and sketch the curves for $1 \leq p \leq 10$.

To find the equilibrium, the simplest approach is to set the demand quantity equal to the supply quantity, giving $250 - 4p - p^2 = 2p^2 - 3p - 40$. To solve this, we convert it into a quadratic equation in the standard form (that is, one of the form $ax^2 + bx + c = 0$, though clearly here we shall use $p$ rather than $x$). We have $3p^2 + p - 290 = 0$. Using the formula for the solutions of a quadratic equation, we have solutions

$$p = \frac{-1 \pm \sqrt{1 - 4(3)(-290)}}{2(3)} = \frac{-1 \pm \sqrt{3481}}{6} = \frac{-1 \pm 59}{6},$$

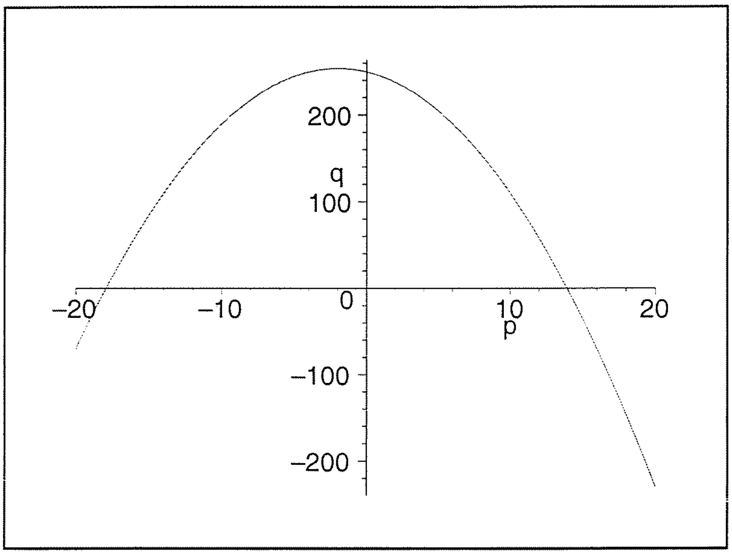
which is $p = -10$ and $p = 29/3$. (A calculator has been used here, so, given that calculators are not permitted in the exam, this precise example would not appear in an exam. This type of example, with easier arithmetic, could, however, do so: see the sample exam questions at the end of this chapter.) You could also have solved this equation using factorisation. It is not so easy in this case, but we might have been able to spot the factorisation $3p^2 + p - 290 = (3p - 29)(p + 10)$ which leads to the same answers. Clearly only the second of these two solutions is economically meaningful. So the equilibrium price is $p = 29/3$. To find the equilibrium quantity,

we can use either the supply or demand equations, and we obtain
$q = 250 - 4(29/3) - (29/3)^2 = 1061/9$.

We now turn our attention to sketching the curves. The demand curve $q = 250 - 4p - p^2$ is a quadratic with a negative squared term, and hence has an up-turned 'U' shape. It crosses the $q$-axis at $(0, 250)$. It crosses the $p$-axis where $250 - 4p - p^2 = 0$. In standard form, this quadratic equation is $-p^2 - 4p + 250 = 0$ and it has solutions

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(-1)(250)}}{2(-1)} = \frac{4 \pm \sqrt{1016}}{-2} = 13.937, -17.937.$$

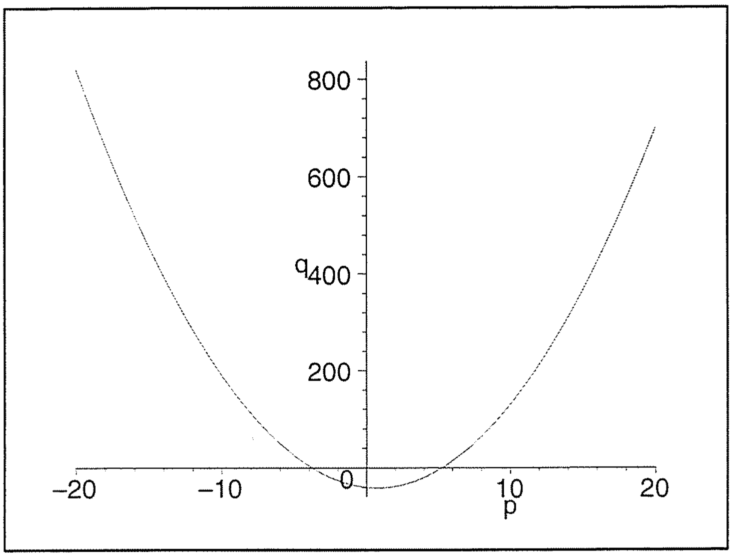
(Again, we use a calculator here, but in the exam such difficult computations would not be required.) With this information, we now know that the curve is as in the following sketch.



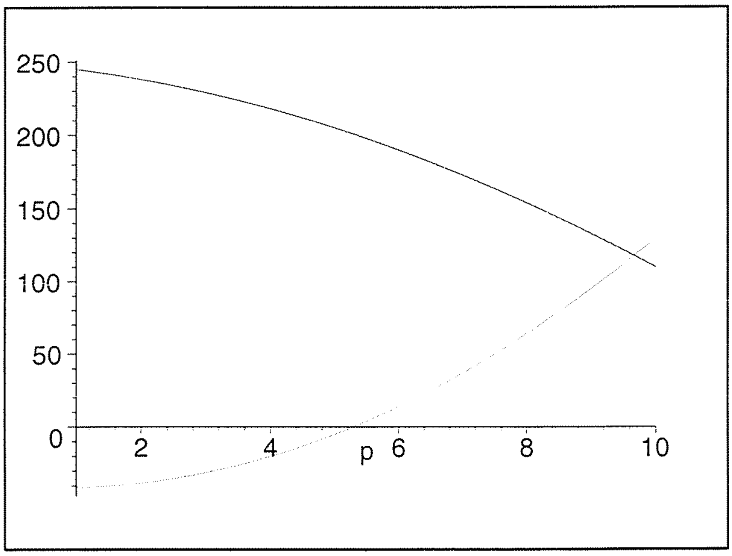

For the supply curve, we have $q = 2p^2 - 3p - 40$, which is a 'U'-shaped parabola. This curve crosses the $q$-axis at $(0, -40)$. It crosses the $P$-axis when $2p^2 - 3p - 40 = 0$. This equation has solutions

$$\frac{3 \pm \sqrt{9 - 4(2)(-40)}}{4} = \frac{3 \pm \sqrt{329}}{4} = 5.285, -3.785$$

and the curve therefore looks like the following.




The question asks us to sketch the curves for the range $1 \leq p \leq 10$. Sketching both on the same diagram we obtain:




Note that the equilibrium point $(29/3, 1061/9)$ is where the two curves intersect. (Note that the vertical line in the diagram is the line $p = 1$, not the $q$-axis.)

**Activity 2.11** *Suppose the market demand function is given by*

$$p = 4 - q - q^2$$

*and that the market supply function is*

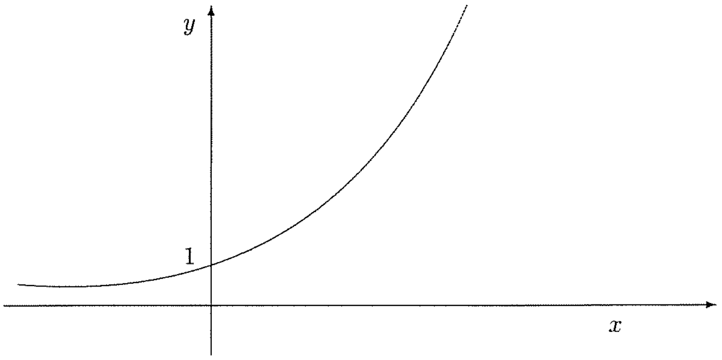
$$p = 1 + 4q + q^2.$$

*Plot both these functions on the same graph.*

## Exponentials

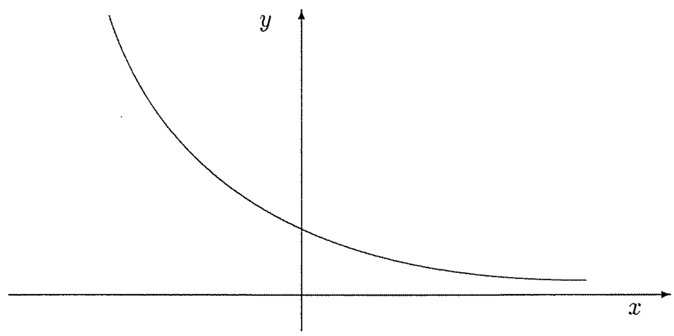
An **exponential-type** function is one of the form $f(x) = a^x$ for some number $a$. (Do not confuse it with the function which raises a number to the power $a$. An exponential-type function has the form $f(x) = a^x$, whereas the '$a$th power function' has the form $f(x) = x^a$.)

**Figure 2.12 The graph of the function $f(x) = a^x$, when $a > 1$**




There are some important points to notice about $f(x) = a^x$ and its graph, for $a > 0$. First of all, $a^x$ is always positive, for every $x$. Furthermore, if $a > 1$ then $a^x$ becomes larger and larger, without bound, as $x$ increases. We say that $a^x$ **tends to infinity** as $x$ tends to infinity. Also, for such an $a$, as $x$ becomes more and more negative, the function $a^x$ gets closer and closer to 0. In other words, $a^x$ tends to 0 as $x$ tends to 'minus infinity'. This behaviour can be seen in Figure 2.12 for the case in which $a$ is a number larger than 1. If $a < 1$ the behaviour is quite different; the resulting graph is of the form shown in Figure 2.13. (You can perhaps see why it has this shape by noting that $a^x = (1/a)^{-x}$.)

**Figure 2.13 The graph of $a^x$, when $0 < a < 1$**



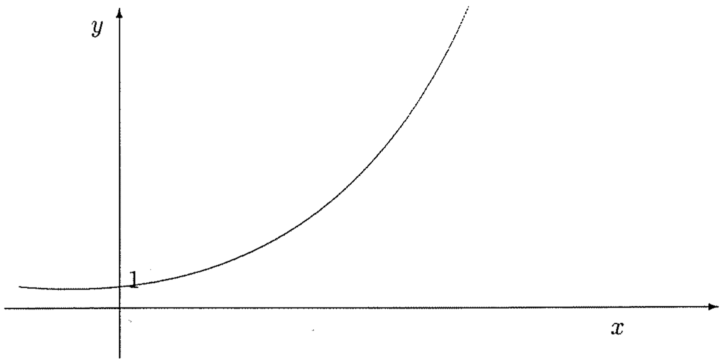

Some very important properties of exponential-type functions, exactly like the power laws, hold. In particular,

$$a^{r+s} = a^r a^s, \quad (a^r)^s = a^{rs}.$$

Another property is that, regardless of $a$, $a^0$ is equal to 1, and the point $(0, 1)$ is the only place where the graph of $a^x$ crosses the $y$-axis.

We now define the **exponential function**. This is the most important exponential-type function. It is defined to be $f(x) = e^x$, where $e$ is the special number 2.71828.... (The function $e^x$ is also sometimes written as $\exp(x)$.) The most important facts about $e^x$ to remember from this section are the shape of its graph, and its properties. The graph is shown in Figure 2.14. We shall see in the next chapter one reason why the number $e$ is so special.

**Figure 2.14 The graph of the function $f(x) = e^x$.**




## The natural logarithm

Formally, the **natural logarithm**[18] of a positive number $x$, denoted $\ln x$ (or, sometimes, $\log x$), is the number $y$ such that $e^y = x$. In other words, the natural logarithm function is the **inverse** of the exponential function $e^x$ (regarded as a function from the set of all real numbers to the set of positive real numbers). Sometimes $\ln x$ is called the **logarithm to base** $e$. The reason for this is that we can, more generally, consider the inverse of the exponential-type function $a^x$. This inverse function is called the **logarithm to base** $a$ and we use the notation $\log_a x$. Thus, $\log_a x$ is the answer to the question 'What is the number $y$ such that $a^y = x$?'.

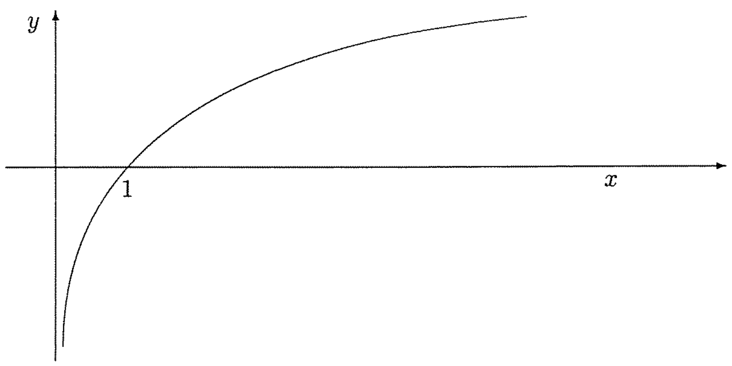
[18] *See Anthony and Biggs, Section 7.4.*

The two most common logarithms, other than the natural logarithm, are logarithms to base 2 and 10. For example, since $2^3 = 8$, we have $\log_2 8 = 3$. It may seem awkward to have to think of a logarithm as the inverse of an exponential-type function, but it is really not that strange. Confronted with the question 'What is $\log_a x$?', we simply turn it around so that it becomes, as above, 'What is the number $y$ such that $a^y = x$?

There is often some confusion caused by the notations used for logarithms. Some texts use log to mean natural logarithm, whereas others use it to mean $\log_{10}$. In this guide, we shall use ln to mean natural logarithm and we shall avoid altogether the use of 'log' without a subscript indicating its base.

Figure 2.15 shows the graph of the natural logarithm. Note that it only makes sense to define $\ln x$ for positive $x$. All the important properties of the natural logarithm follow from those of the exponential function. For example, $\ln 1 = 0$. Why? Because $\ln 1$ is, by its definition, the number $y$ such that $e^y = 1$. The only such $y$ is $y = 0$.

**Figure 2.15 The graph of the natural logarithm function,** $\ln x$




The other very important properties of $\ln x$ (which follow from properties of the exponential function[19]) are:

$$\ln(ab) = \ln a + \ln b, \ \ \ln(a/b) = \ln a - \ln b, \ \ \ln(a^b) = b \ln a.$$

[19] *See Anthony and Biggs, Section 7.4.*

These relationships are fairly simple and you will get used to them as you practise.

## Trigonometrical functions

The trigonometrical functions, $\sin x, \cos x, \tan x$ (the **sine function**, **cosine function** and **tangent function**) are very important in mathematics and they will occur later in this subject. We shall not give the definition of these functions here. If you are unfamiliar with them, consult the texts.[20]

[20] *See Dowling, Section 20.5, for example or Booth, Module 13.*

It is important to realise that, throughout this subject, angles are measured in **radians** rather than **degrees**. The conversion is as follows: 180 degrees equals $\pi$ radians, where $\pi$ is the number $3.141\ldots$. It is good practice **not** to expand $\pi$ or multiples of $\pi$ as decimals, but to leave them in terms of the symbol $\pi$. For example, since 60 degrees is one third of 180 degrees, it follows that, in radians, 60 degrees is $\pi/3$.

The graphs of the sine function, $\sin x$, and the cosine function, $\cos x$, are shown in Figures 2.16 and 2.17. Note that these functions are periodic: they repeat themselves every $2\pi$ steps. (For example, the graph of the sine function between $2\pi$ and $4\pi$ has exactly the same shape as the graph of the function between 0 and $2\pi$.)

**Figure 2.16 The graph of the function** $\sin x$

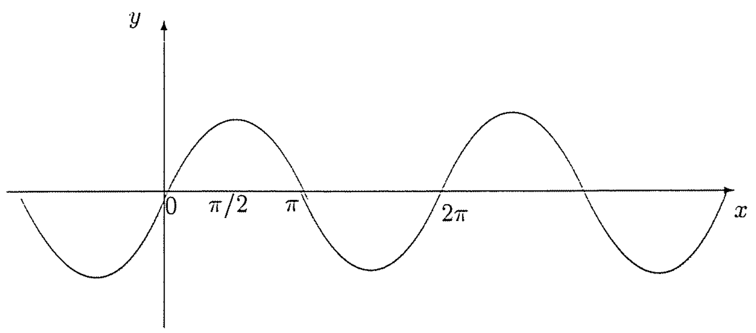
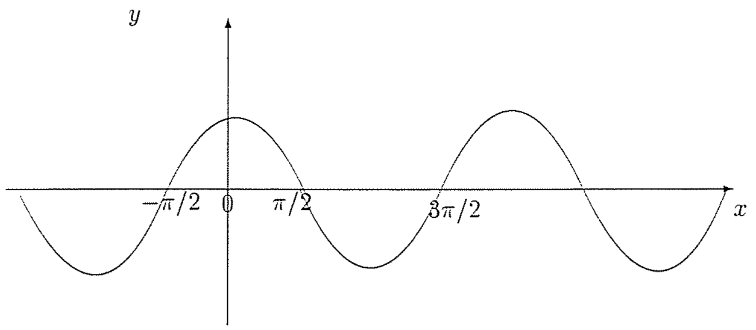



**Figure 2.17 The graph of the function** $\cos x$




Note also that the graph of $\cos x$ is a 'shift' of the graph of $\sin x$, obtained by shifting the $\sin x$ graph by $\pi/2$ to the left. Mathematically, this is equivalent to the fact that $\cos x = \sin(x + \frac{\pi}{2})$.

The tangent function, $\tan x$, is defined in terms of the sine and cosine functions, as follows:

$$\tan x = \frac{\sin x}{\cos x}.$$

Note that the sine and cosine functions always take a value between 1 and $-1$. The following table gives some important values of the trigonometrical functions.

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\pi/6$ | $1/2$ | $\sqrt{3}/2$ | $1/\sqrt{3}$ |
| $\pi/4$ | $1/\sqrt{2}$ | $1/\sqrt{2}$ | 1 |
| $\pi/3$ | $\sqrt{3}/2$ | $1/2$ | $\sqrt{3}$ |
| $\pi/2$ | 1 | 0 | undefined |

Technically, the tangent function is not defined at $\pi/2$. This means that no meaning can be given to $\tan(\pi/2)$. To see why, note that $\tan x = \sin x/\cos x$, but

$\cos(\pi/2) = 0$, and we cannot divide by 0. You might wonder what happens to $\tan x$ around $x = \pi/2$. The graph of $\tan x$ can be found in the textbooks. [21]

[21] *See, for example, Appendix A of Holden and Pearson*

There are some useful results about the trigonometrical functions, with which you should familiarise yourself. First, for all $x$,

$$(\cos x)^2 + (\sin x)^2 = 1.$$

(We use $\sin^2 x$ to mean $(\sin x)^2$, and similarly for $\cos^2 x$.) Then there are the **double-angle formulae**, which state that:

$$\sin(2x) = 2 \sin x \cos x, \quad \cos(2x) = \cos^2 x - \sin^2 x.$$

Note that, since $\cos^2 x + \sin^2 x = 1$, the double angle formula for $\cos(2x)$ may be written in another two, useful, ways:

$$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

Let $S$ be the interval $[-\pi/2, \pi/2]$. Then, regarded as a function from $S$ to the interval $[-1, 1]$, $\sin x$ has an inverse function, which we denote by $\sin^{-1}$; thus, for $-1 \leq y \leq 1$, $\sin^{-1}(y)$ is the angle $x$ (in radians) such that $-\pi/2 \leq x \leq \pi/2$ and $\sin x = y$. In a similar manner, the function $\cos x$ from the interval $[0, \pi]$ to $[-1, 1]$ has an inverse, which we denote by $\cos^{-1}$: so, for $-1 \leq y \leq 1$, $\cos^{-1} y$ is the angle $x$ (in radians) such that $0 \leq x \leq \pi$ and $\cos x = y$. Similarly, the function $\tan x$, regarded as a function from the interval $(-\pi/2, \pi/2)$ to $\mathbb{R}$ has an inverse, denoted by $\tan^{-1}$. Some texts use the notation arcsin for $\sin^{-1}$, arccos for $\cos^{-1}$, and arctan for $\tan^{-1}$.

## Further applications of functions

We have already seen that supply and demand can usefully be modelled using very simple functional relationships, such as linear functions. We now discuss a few more applications.

Suppose that the demand equation for a good is of the form $p = ax + b$ where $x$ is the quantity produced. Then, at equilibrium, the quantity $x$ is the amount supplied and sold, and hence the total revenue $TR$ at equilibrium is price times quantity, which is

$$TR = (ax + b)x = ax^2 + bx,$$

a quadratic function which may be maximised either by completing the square, or by using the techniques of calculus (discussed later).

Another very important function in applications is the total cost function of a firm. In the simplest model of a cost function, a firm has a **fixed cost**, that remain fixed independent of production or sales, and it has **variable costs** which, for the sake of simplicity, we will assume for the moment vary proportionally with production. That is, the variable cost is of the form $Vx$ for some constant $V$, where $x$ represents the production level. The Total Cost $TC$ is then the sum of these two: $TC = F + Vx$. For a limited range of $x$ this very simplistic relationship often holds well but more complicated models (for instance, involving quadratic and exponential functions) often occur.

Combining the total cost and revenue functions on one graph enables us to perform break-even analysis. The break-even output is that for which total cost equals total revenue. In simplified, linear, models the break-even point (should it exist) is unique. When non-linear relationships are used, a number of break-even points are possible.

**Example:** Let us find the break-even points in the case where the total cost function is $TC = 7 + 2x + x^2$ and the total revenue function is $TR = 10x$. To find the break-even points, we need to solve $TC = TR$; that is, $7 + 2x + x^2 = 10x$ or $x^2 - 8x + 7 = 0$. Splitting this into factors, $(x - 7)(x - 1) = 0$, so $x = 7$ or $x = 1$. (Alternatively, the formula for the solutions of a quadratic equation could be used.) Note that there are two break-even points.

**Activity 2.12** *Find the break-even points in the case where the total cost function is* $TC = 2 + 5x + x^2$ *and the total revenue function is* $TR = 12 + 8x$.

## Learning outcomes

At the end of this chapter and the relevant reading, you should be able to:

- determine inverse functions and composite functions
- sketch graphs of simple functions
- sketch quadratic curves and solve quadratic equations
- solve basic simultaneous equations.
- find equilibria from supply and demand functions, and sketch these
- find break-even points
- explain what is meant by exponential-type functions and be able to sketch their graphs
- use properties such as $a^{x+y} = a^x a^y$ and $(a^x)^y = a^{xy}$.
- explain what is meant by the exponential function $e^x$
- describe the natural logarithm ($\ln x$), logarithms to base $a$ ($\log_a x$) and their properties.
- describe the functions $\sin x, \cos x, \tan x$ and their properties, key values, and graphs.
- explain what is meant by inverse trigonometrical functions.

You do not need to know about complex (or imaginary) numbers (as you might see discussed in some texts when, in a quadratic equation, $b^2 - 4ac < 0$).

## Sample examination/practice questions

The material in this chapter of the guide is essential to what follows. Many exam questions involve this material, but additionally involve other topics, such as calculus. We give just three examples of exam-type questions which make use only of the material in this chapter.

**1.** Suppose the market demand function is given by

$$p = 4 - q - q^2$$

and that the market supply function is

$$p = 1 + 4q + q^2.$$

Determine the equilibrium price and quantity.

**2.** Suppose that the demand relationship for a product is $p = 6/(q+1)$ and that the supply relationship is $p = q + 2$. Determine the equilibrium price and quantity.

**3.** Suppose that the demand equation for a good is $q = 8 - p^2 - 2p$ and that the supply equation is $q = p^2 + 2p - 3$. Sketch the supply and demand curves on the same diagram, and determine the equilibrium price.

## Answers to activities

**2.1** $\sum_{i=1}^{4} x_i$ is $3 + 1 + 4 + 6 = 14$. The product $\Pi_{i=1}^{4} x_i$ equals $3 \times 1 \times 4 \times 6 = 72$.

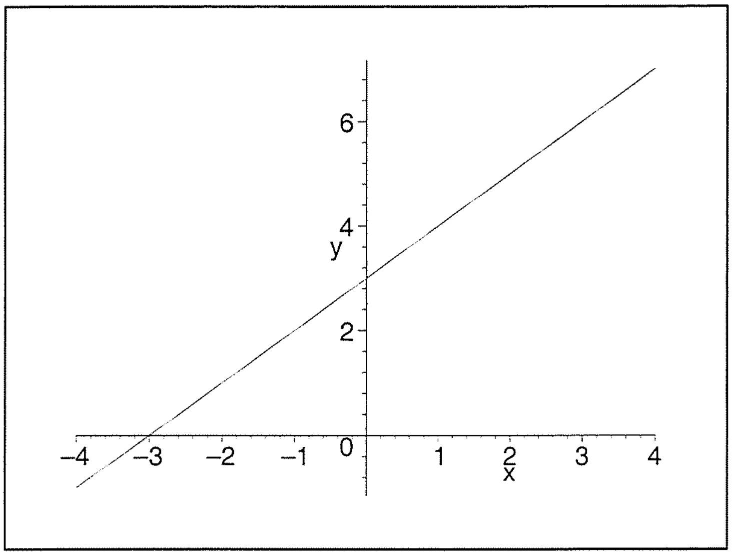
**2.2** $x^3 + 2x^2 - x - 2$.

**2.3** $A \cap B$ is the set of objects in both sets, and so it is $\{2, 5\}$.

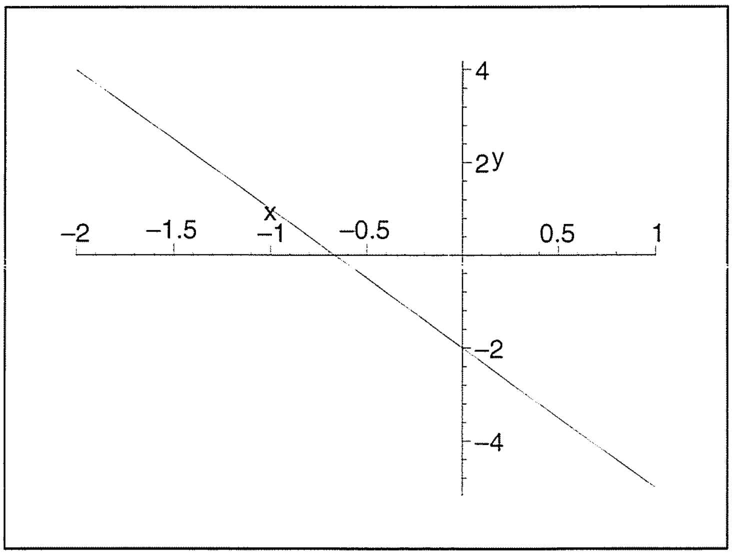
**2.4** If $y = f(x) = 3x + 2$ then we may solve this for $x$ by noting that $x = (y - 3)/2$. It follows that $f^{-1}(y) = (y - 2)/3$ or, equivalently, $f^{-1}(x) = (x - 2)/3$.

**2.5** $(fg)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^2 + 1}$.

**2.6** The curve $y = x + 3$ is a straight line with gradient 1, passing through the $y$-axis at the point $(0, 3)$. Therefore, sketching it we obtain:



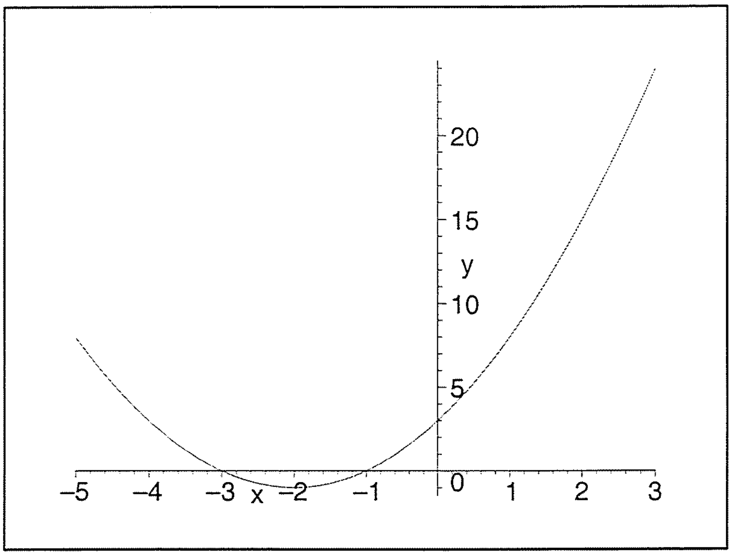

The curve $y = -3x - 2$ is a staight line with gradient $-3$ (and hence sloping downwards), passing through the $y$-axis at $(0, -2)$. Sketching the graph, we therefore obtain:




**2.7** The graph $y = x^2 + 4x + 3$ is a quadratic, with a positive $x^2$ term, and hence it has the parabolic 'U'-shape. To locate its position, we find where it crosses the axes. It crosses the $y$-axis when $x = 0$, and hence at $(0, 3)$. To find where it crosses the $x$-axis (if at all), we need to solve $y = 0$; that is, $x^2 + 4x + 3 = 0$. There are two ways we can do this. We could spot that this factorises as $(x + 3)(x + 1) = 0$, so that the solutions are $x = -3$ and $x = -1$. Alternatively, we can use the formula for the solutions of a quadratic equation, with $a = 1, b = 4, c = 3$. This gives

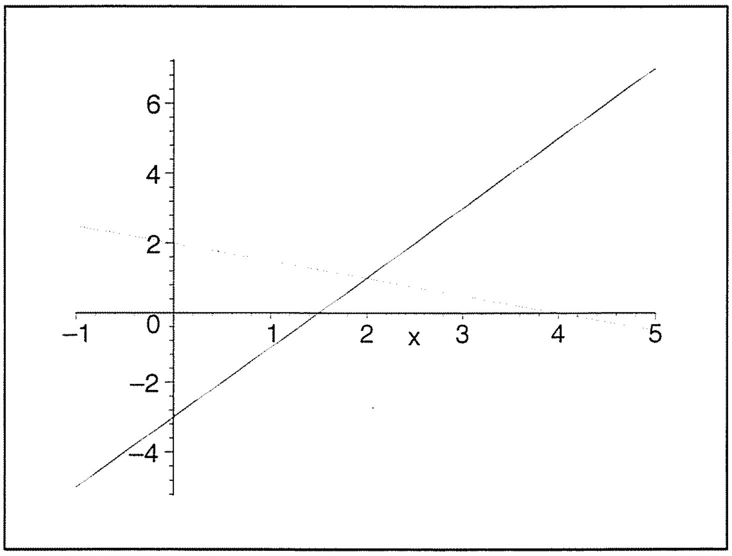
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = (-4 \pm \sqrt{4})/2 = -3, -1.$$

With this information, we can sketch the curve, as follows:

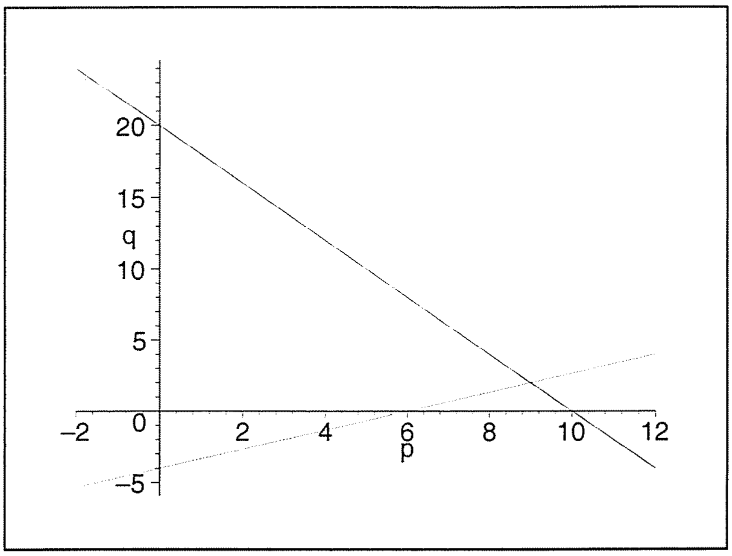



**2.8** We note first that $x$ is a factor, and so we have $x^3 + 4x^2 + 3x = x(x^2 + 4x + 3)$. To factorise the quadratic, we can simply spot the factorisation $x^2 + 4x + 3 = (x + 1)(x + 3)$ or, alternatively, we can solve the quadratic equation $x^2 + 4x + 3 = 0$, which has solutions $-1, -3$, meaning that $x^2 + 4x + 3 = (x - (-1))(x - (-3)) = (x + 1)(x + 3)$. It follows that the factorisation we require is $x(x + 1)(x + 3)$.

**2.9** To find the intersection of the two lines we solve the equations $y = 2x - 3, y = 2 - (1/2)x$ simultaneously. There is more than one way to do so, but perhaps the easiest is to write $2x - 3 = 2 - (1/2)x$, from which we obtain $(5/2)x = 5$ and hence $x = 2$. The $y$-coordinate of the intersection can then be found from either one of the initial equations: for example, $y = 2x - 3 = 2(2) - 3 = 1$. It follows that the intersection point is $(2, 1)$. The following sketch shows the two curves. (If this were an exam question, it would not be essential to include the sketch as part of your answer. I'm doing so just to help you understand what's going on.)




**2.10** To find the equilibrium, we solve simultaneously the demand and supply equations; that is, we set supply $q^D$ equal to demand $q^S$. Since $q^D = 20 - 2p$ and $q^S = (2/3)p - 4$, we set $20 - 2p = (2/3)p - 4$ and hence $(8/3)p = 24$, and $p = 9$. To find the equilibrium quantity, we can use either the supply or the demand equation. Using the demand equation gives $q = 20 - 2(9) = 2$ (and of course we will get the same answer using the supply equation). The following sketch shows the demand and supply curves (which are, of course, straight lines in this case).




**2.11** Note that, here, $p$ is given in terms of $q$, whereas in the example preceding this activity, the relationships were given the other way round, by which I mean that the quantities were expressed as functions of the prices. This is not something you should get confused about. We can think about price as a function of quantity or quantity as a function of price. In this problem, $q$ is treated as the independent variable and $p$ as the dependent variable, so we will sketch $p$ against $q$, with the vertical axis being the $P$ axis and the horizontal axis the $q$-axis. (This is in contrast to the previous question, where $q$ was the vertical coordinate and $p$ the horizontal.)

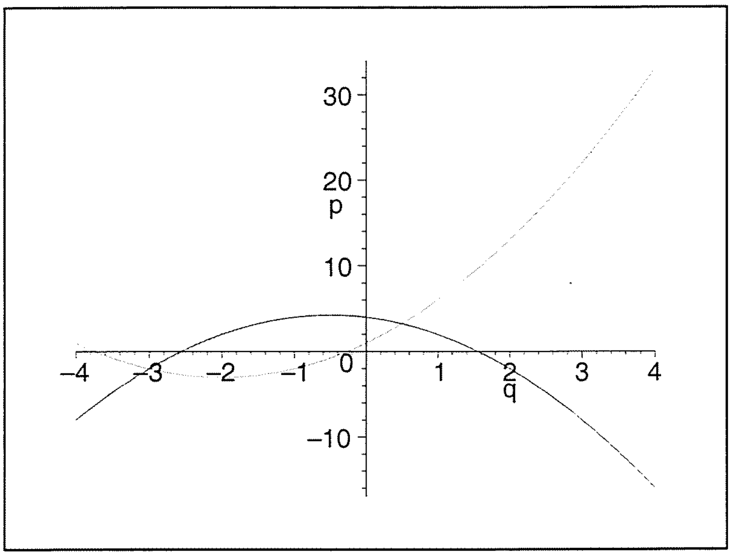
Consider the demand curve, with equation $p = 4 - q - q^2$. This is an up-turned 'U'-shape. It crosses the $p$-axis (when $q = 0$) at $(0, 4)$ and it crosses the $q$-axis when $4 - q - q^2 = 0$. Solving this quadratic in the usual way, we obtain

$$q = \frac{1 \pm \sqrt{1 - 4(-1)(4)}}{-2} = \frac{1 \pm \sqrt{17}}{-2} = -2.562, 1.562.$$

The supply curve, with equation $p = 1 + 4q + q^2$, crosses the $p$-axis at $(0, 1)$. It crosses the $q$-axis when $1 + 4q + q^2 = 0$, which is when

$$q = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2} = \frac{-4 \pm \sqrt{12}}{2} = -0.268, -3.732.$$

Sketching both curves on the same diagram, we obtain the following:




**2.12** We solve $TC = TR$, which is $2 + 5x + x^2 = 12 + 8x$. Writing this equation in the standard way, it becomes $x^2 - 3x - 10 = 0$. We can factorise this as $(x-5)(x+2) = 0$, showing that the solutions are $5, -2$. Or, we can use the formula for the solutions of a quadratic, with $a = 1, b = -3, c = -10$. Either way we see that there are two possible break-even points, $x = 5$ or $x = -2$. But the second of these has no economic significance, since it represents a negative quantity. We therefore deduce that the break-even point is $x = 5$.

## Answers to sample examination/practice questions

**1.** To find the equilibrium quantity, we solve

$$4 - q - q^2 = 1 + 4q + q^2,$$

which is

$$2q^2 + 5q - 3 = 0.$$

Using the formula for the solutions of a quadratic, we have

$$q = \frac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{4} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4} = -3, \frac{1}{2}.$$

So the equilibrium quantity is the economically meaningful solution, namely $q = 1/2$. The corresponding equilibrium price is

$$p = 4 - (1/2) - (1/2)^2 = 4 - 1/2 - 1/4 = \frac{13}{4}.$$

**2.** We solve

$$\frac{6}{q+1} = q + 2.$$

Multiplying both sides by $q + 1$, we obtain

$$6 = (q+1)(q+2) = q^2 + 3q + 2,$$

so

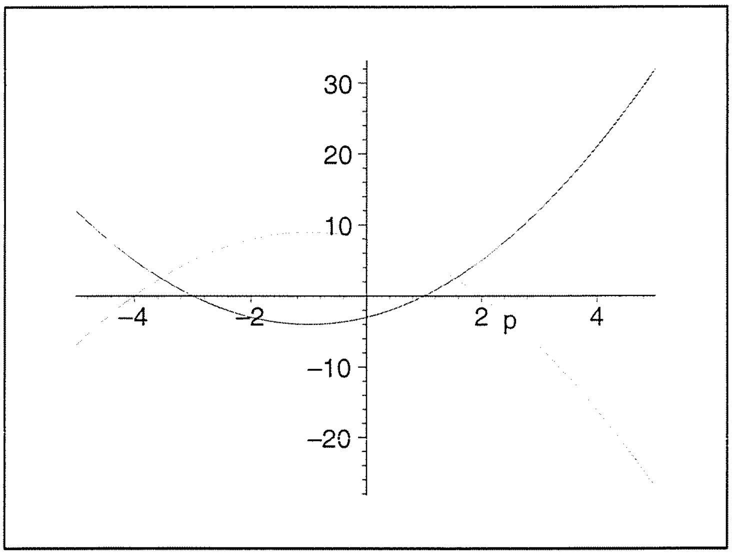
$$q^2 + 3q - 4 = 0.$$

This factorises as $(q-1)(q+4)=0$ and so has solutions 1 and $-4$. Thus the equilibrium quantity is 1, the positive solution. The equilibrium price, which can be obtained from either one of the equations, is $p = 6/(1+1) = 3$. (Here, I have used the demand equation.)

**3.** Consider first the demand curve. This is a negative quadratic and so has an upturned 'U' shape. It crosses the $p$-axis when $8 - p^2 - 2p = 0$ or, equivalently, when $p^2 + 2p - 8 = 0$. This factorises as $(p+4)(p-2) = 0$, so the solutions are $p = 2, -4$. Alternatively, we can use the formula for the solutions to a quadratic:

$$p = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2} = \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2} = -4, 2.$$

(No calculator needed!) The supply curve is $q = p^2 + 2p - 3 = (p+3)(p-1)$, which crosses the $p$-axis at $-3$ and 1, and has a 'U' shape. We notice also that the demand curve crosses the $q$-axis at $q = 8$ and the supply curve crosses the $q$-axis at $q = -3$. The curves therefore look as follows:




The equilibrium price is given by

$$8 - p^2 - 2p = p^2 + 2p - 3,$$

or $2p^2 + 4p - 11 = 0$. This has solutions

$$p = \frac{-4 \pm \sqrt{4^2 - 4(2)(-11)}}{4} = \frac{-4 \pm \sqrt{104}}{4} = -1 \pm \frac{1}{4}\sqrt{104} = -1 \pm \frac{1}{2}\sqrt{26}.$$

We know that $\sqrt{26} > 2$, so the solution $-1 + \sqrt{26}/2$ is positive and is therefore the equilibrium price. (The other solution is obviously negative.) (Note: in an exam, you should leave the answer like this, since you will not have a calculator to work the answer out as a decimal expansion.)

# Chapter 3

# Differentiation

## Essential reading

(For full publication details, see Chapter 1.)

Anthony, M. and Biggs, N., *Mathematics for Economics and Finance.* Chapters 6, 7 and 8.

## Further reading

Binmore, K. and Davies, J. *Calculus.* Chapter 2, Sections 2.7–2.10 and Chapter 4, Sections 4.2 and 4.3.

Black, J., and Bradley, J.F. *Essential Mathematics for Economists.* Chapters 5 and 6.

Booth, D.J. *Foundation Mathematics.* Chapter 5, Modules 19 and 20.

Bradley, T., and Patton, P. *Essential Mathematics for Economics and Business.* Chapter 6.

Dowling, Edward T. *Introduction to Mathematical Economics.* Chapters 3 and 4.

Holden, K and Pearson, A.W. *Introductory Mathematics for Economics and Business.* Sections 5.1, 5.2, 5.7 and 5.8.

# Introduction

In this extremely important chapter we introduce the topic of calculus, one of the most useful and powerful techniques in applied mathematics. In this chapter we focus on the process of 'differentiation' of a function. The derivative (the product of differentiation) has numerous applications in economics and related fields. It provides a rigorous mathematical way to measure how fast a quantity is changing, and it also gives us the main technique for finding the maximum or minimum value of a function.

## The definition and meaning of the derivative

The derivative is a measure of the instantaneous rate of change of a function[1] $f : \mathbb{R} \to \mathbb{R}$. The idea is to compare the value of the function at $x$ with its value at $x + h$, where $h$ is a small quantity. The change in the value of $f$ is $f(x + h) - f(x)$, and this, when divided by the change $h$ in the 'input', measures the average rate of change over the interval from $x$ to $x + h$. Informally speaking, the **instantaneous** rate of change is the quantity this average rate of change approaches as $h$ gets smaller and smaller.

[1] *See Anthony and Biggs, Section 6.1.*

A good analogy can be made with the speed of a car. Imagine that a car is driving along a straight road and that $f(x)$ represents its distance, in metres, from the starting point at time $x$, in seconds, from the start. (It would be more normal to use the symbol $t$ as the variable here rather than $x$, but you will be aware from earlier that $f(x)$ and $f(t)$ convey the same information: it does not matter which symbol is used for the variable.) Let's suppose that at time $x = 10$, the distance $f(10)$ from the start is 150 and that at time $x = 11$, the distance from the start is 170. Then the average speed between times 10 and 11 is $(170 - 150)/(11 - 10) = 20$ metres per second. However, this need not be the same as the **instantaneous** speed at 10, since the car may accelerate or decelerate in the time interval from 10 to 11. Conceivably, then, the instantaneous speed at 10 could well be higher or lower than 20. To obtain better approximations to this instantaneous speed, we should measure average speed over smaller and smaller time intervals. In other words, we should measure the average speed from 10 to $10 + h$ and see what happens as $h$ gets smaller and smaller. That is, we compute the limit as $h$ tends to 0 of $(f(10 + h) - f(10))/((10 + h) - 10) = (f(10 + h) - f(10))/h$.

We now give the definition of the derivative. The **derivative** (or instantaneous rate of change) of $f$ at a number $a$ (or 'at the point $a$') is the number which is the limit of $(f(a + h) - f(a))/h$ as $h$ tends towards 0. It is not appropriate at this level to say formally what we mean by a limit, but the idea is quite simple: we say that $g(x)$ tends to the limit $L$ as $x$ tends to $c$ if the distance between $g(x)$ and $L$ can be made as small as we like provided $x$ is sufficiently close to $c$. The derivative of $f$ at $a$ is denoted $f'(a)$. (We are assuming here that the limit exists: if it does not, then we say that the derivative does not exist at $a$. But we do not need to worry in this subject about the existence and non-existence of derivatives: these are matters for consideration in a more advanced course of study.) Now, if the derivative exists at all $a$, then for each $a$ we have a derivative $f'(a)$ and we simply call the function $f'$ the **derivative of $f$**. (Please don't be confused by this distinction between the derivative of $f$ and the derivative of $f$ at a point. For example, suppose that for each $a$, $f'(a) = 2a$. Then the derivative of $f$ is the function $f'$ given by $f'(x) = 2x$.)

Let's look at an example to make sure we understand the meaning of the derivative at a point $a$. (We will see soon that the type of numerical calculation we're about to undertake is not necessary in most cases, once we have learned some techniques for determining derivatives.)

**Example:** Suppose that $f(x) = 2^x$. Let's try to determine the derivative $f'(1)$ by working out the average rates of change $(f(1 + h) - f(1))/h$ for successively smaller values of $h$. (As just mentioned, we will later see an easier way.) The following table shows some of the values of

$$\frac{f(1+h)-f(1)}{h} = \frac{2^{1+h}-2}{h}.$$

| $h$ | $(2^{1+h}-2)/h$ |
|---|---|
| 0.5 | 1.656854 |
| 0.1 | 1.435469 |
| 0.01 | 1.39111 |
| 0.001 | 1.386775 |
| 0.0001 | 1.386342 |

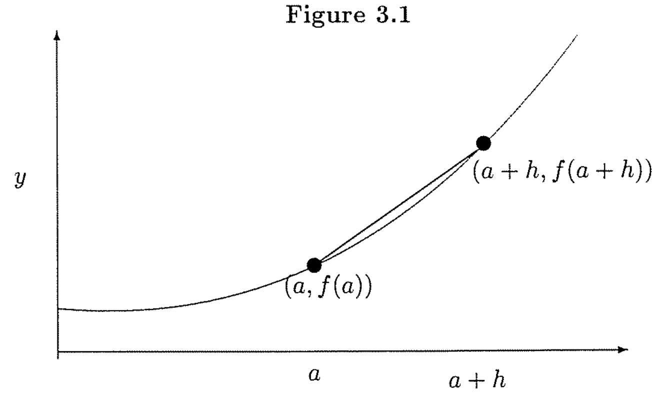
It can be seen that as $h$ approaches 0 these numbers seem to be approaching a number around 1.386. So we might guess that $f'(1)$ is around 1.386. (In fact, it turns out that the exact value of $f'(1)$ is $2\ln 2 = 1.38629436\ldots$.)

**Activity 3.1** *Calculate some more values of $(f(1+h)-f(1))/h$ for even smaller values of $h$.*

A geometrical interpretation of the derivative can be given. The ratio $(f(a+h)-f(a))/h$ is the gradient of the line joining the points $(a, f(a))$ and $(a+h, f(a+h))$; see Figure 3.1. As $h$ tends to 0, this line becomes tangent to the curve at $(a, f(a))$; that is, it just touches the curve at that point. The derivative $f'(a)$ may therefore also be thought of as the gradient of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$.

An alternative notation for $f'(x)$ is $\dfrac{df}{dx}$.

**Figure 3.1**




Derivatives can be calculated using the definition given above, in what is known as differentiation **from first principles**, but this is cumbersome and you will not need to do this in an examination. We give one example by way of illustration, but we emphasise that you are not expected to carry out such calculations.

**Example:** Suppose $f(x) = x^2$. In order to work out the derivative we calculate as follows:

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{(x^2+2xh+h^2)-x^2}{h} = 2x + h.$$

The first term is independent of $h$ and the second term approaches 0 as $h$ approaches 0, so the derivative is the function given by $f'(x) = 2x$.

## Standard derivatives

In practice, to determine derivatives (that is, to **differentiate**), we have a set of **standard derivatives** together with rules for combining these. The standard derivatives (which you should memorise) are as follows.

| $f(x)$ | $f'(x)$ |
|---|---|
| $x^k$ | $kx^{k-1}$ |
| $e^x$ | $e^x$ |
| $\ln x$ | $1/x$ |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |

We mentioned in Chapter 2 that the number $e$ is very special. We can now see one reason why. We see from above that the derivative of the power function $e^x$ is just itself, that is $e^x$. This is **not** the case for any other power function $a^x$. For example (as we shall see), the derivative of $2^x$ is **not** $2^x$, but $2^x(\ln 2)$.

These could also be stated in the '$d/dx$' notation, as $\frac{d}{dx}(e^x) = e^x$, and so on.

**Example:** The derivative of $x^5$ is $5x^4$.

**Activity 3.2** *What is the derivative of* $\frac{1}{x}$*?*

## Rules for calculating derivatives

To calculate the derivatives of functions other than the standard ones just given, it is useful to use the following rules.[2]

[2] *See Anthony and Biggs, Section 6.2.*

- **The sum rule:** If $h(x) = f(x) + g(x)$ then $h'(x) = f'(x) + g'(x)$.
- **The product rule:** If $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$.
- **The quotient rule:** If $h(x) = f(x)/g(x)$ and $g(x) \neq 0$ then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

**Example:** Let $f(x) = x^3e^x$. Then, by the product rule,
$f'(x) = (x^3)'e^x + x^3(e^x)' = 3x^2e^x + x^3e^x$.

**Activity 3.3** *Find the derivative of* $x^2 \sin x$.

**Activity 3.4** *Find the derivative of* $f(x) = (x^2 + 1)\ln x$.

**Example:** Let $f(x) = \dfrac{\ln x}{x}$. Then, by the quotient rule,

$$f'(x) = \frac{(1/x)x - (1)\ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

**Activity 3.5** *Determine the derivative of* $\sin x/x$.

Another, very important, rule is the **composite function rule**[3], or **chain rule**, which may be stated as follows:

- if $f(x) = s(r(x))$, then $f'(x) = s'(r(x))r'(x)$. If you can write a function in this way, as the composition of $s$ and $r$, then the composite function rule will tell you the derivative.

[3] *See Anthony and Biggs, Section 6.4.*

**Example:** Let $f(x) = \sqrt{x^3 + 2}$. Then $f(x) = s(r(x))$ where $s(x) = \sqrt{x} = x^{1/2}$ and $r(x) = x^3 + 2$. Now, $s'(x) = (1/2)x^{1-1/2} = (1/2)x^{-1/2}$ and $r'(x) = 3x^2$, so, by the composite function rule,

$$f'(x) = s'(r(x))r'(x) = \frac{1}{2}(x^3 + 2)^{-1/2}(3x^2) = \frac{3x^2}{2\sqrt{x^3 + 2}}.$$

**Example:** Suppose $f(x) = (ax + b)^n$. Then, by the composite function rule,

$$f'(x) = n(ax + b)^{n-1} \times (ax + b)' = an(ax + b)^{n-1}.$$

**Example:** Suppose that $f(x) = \ln(g(x))$. Then, by the composite function rule,

$$f'(x) = \frac{1}{g(x)}g'(x) = \frac{g'(x)}{g(x)}.$$

(This result is often useful in integration, something we discuss later.)

**Activity 3.6** *Find the derivative of* $(3x + 7)^{15}$.

**Activity 3.7** *Differentiate* $f(x) = \sqrt{x^2 + 1}$.

**Activity 3.8** *Differentiate* $g(x) = \ln(x^2 + 2x + 5)$.

Differentiation can sometimes be simplified by taking logarithms, as the following example demonstrates.

**Example:** We differentiate $f(x) = 2^x$ by observing that $\ln(f(x)) = \ln(2^x) = x \ln 2$. Now, the derivative of $\ln(f(x))$ is, by the composite function rule (chain rule) equal to $f'(x)/f(x)$, so, on differentiating both sides of $\ln(f(x)) = x \ln 2$, we obtain

$$\frac{f'(x)}{f(x)} = (x \ln 2)' = \ln 2,$$

and so

$$f'(x) = (\ln 2) f(x) = (\ln 2) 2^x.$$

In particular, $f'(1) = 2 \ln 2$, as alluded to in an earlier example in this chapter.

**Activity 3.9** *By taking logarithms first, find the derivative* $f'(x)$ *when* $f(x) = x^x$.

# Optimisation

### Critical points

The derivative is very useful for finding the **maximum** or **minimum** value of a function—that is, for **optimisation**.[4] Recall that the derivative $f'(x)$ may be interpreted as a measure of the rate of change of $f$ at $x$. It follows from this that we can tell whether a function is increasing or decreasing at a given point, simply by working out its derivative at that point.

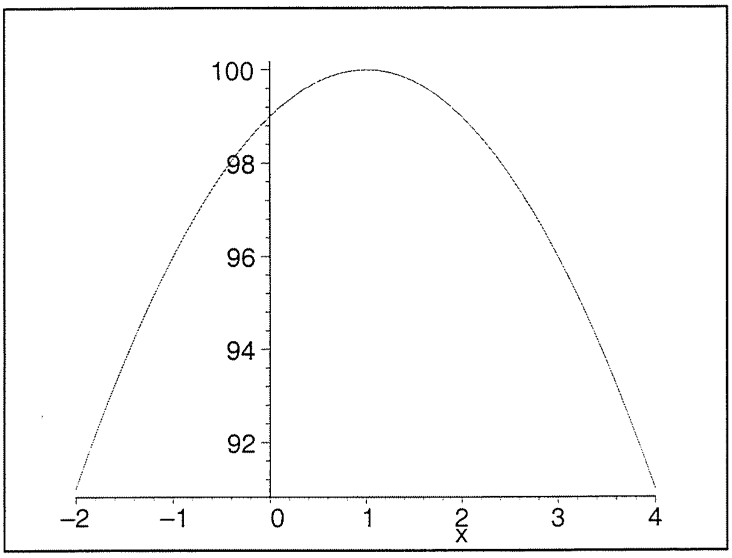
[4] *See Anthony and Biggs, Chapter 8.*

- **If $f'(x) > 0$ then $f$ is increasing at $x$.**
- **If $f'(x) < 0$ then $f$ is decreasing at $x$.**

At a point $c$ for which $f'(c) = 0$ the function $f$ is neither increasing nor decreasing: in this case we say that $c$ is a **critical point** (or **stationary point**) of $f$. It must be stressed that a function can have more than one kind of critical point. A critical point could be

- a **local maximum**, which is a $c$ such that for all $x$ close to $c$, $f(x) \leq f(c)$
- a **local minimum**, which is a $c$ such that for all $x$ close to $c$, $f(x) \geq f(c)$
- or an **inflexion point**, which is neither a local maximum nor a local minimum

In the first of the following three figures, $c = 1$ is a local maximum of the function whose graph is sketched, and in the second $c = 1$ is a local minimum of the function sketched. In the third figure, $c = 1$ is a critical point, but not a maximum or a minimum; in other words, it is an inflexion point.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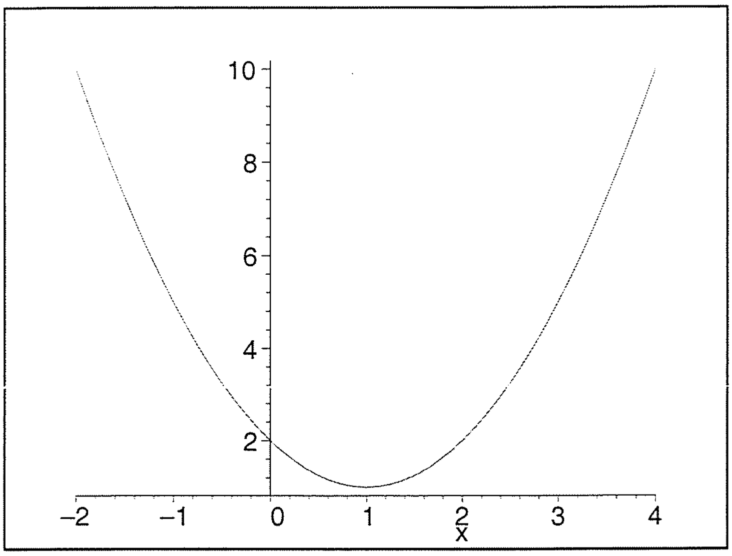
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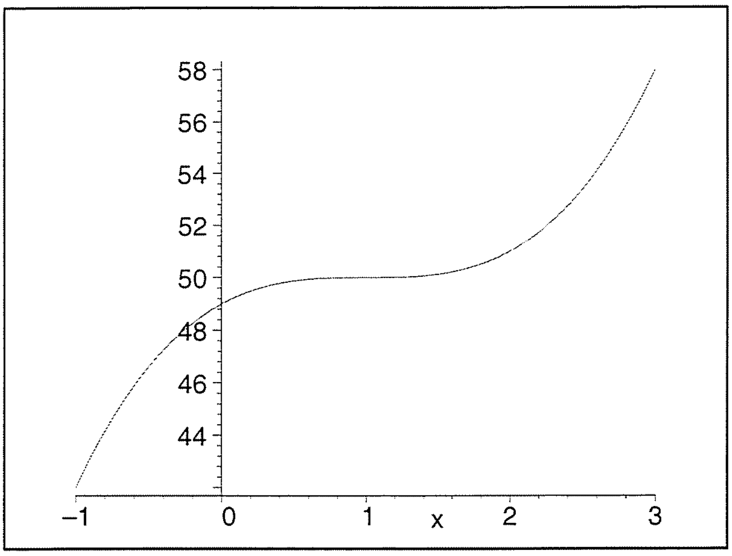



**Deciding the nature of a given critical point**

We can decide the nature of a given critical point by considering what happens to the derivative $f'$ in a region around the critical point. Suppose, for example, that $c$ is a critical point of $f$ and that $f'$ is positive for values just less than $c$, zero at $c$, and negative for values just greater than $c$. Then, $f$ is increasing just before $c$ and decreasing just after $c$, so $c$ is a local maximum. Similarly, if the derivative $f'$ changed sign from negative to positive around the point $c$ then we can deduce that $c$ is a local minimum. At an inflexion point, the derivative would not change sign: it would be either non-negative on each side of the critical point, or non-positive on each side. Thus a critical point can be **classified** by considering the sign of $f'$ on either side of the point.

There is another way of classifying critical points. Let's think about a local maximum point $c$ as described above. Note that the derivative $f'$ is decreasing at $c$ (since it goes from positive, through 0, to negative), so the derivative of the derivative $f'$ is negative at $c$. We call the derivative of $f'$ the **second derivative** of $f$, and denote it by $f''(x)$ or

$$\frac{d^2 f}{dx^2}.$$

In other words, then, $f''(c) < 0$. It can be proved that a **sufficient** condition for $f$ to have a local maximum at a critical point $c$ is $f''(c) < 0$. (By saying that this is a 'sufficient condition', we mean that if $f''(c) < 0$ then $c$ is a maximum; you should understand that even if $c$ is a maximum, it need not be the case that $f''(c) < 0$.) There is a similar condition for a minimum, in which the corresponding condition is $f''(c) > 0$. Summarising:

- **if $f'(a) = 0$ and $f''(a) < 0$ then the point $a$ is a local maximum of $f$.**

- **if $f'(b) = 0$ and $f''(b) > 0$ then the point $b$ is a local minimum of $f$.**

These observations together form the **second-order conditions** for the nature of a critical point. If a critical point $c$ is an inflexion point, then the condition $f''(c) = 0$ must hold (since the point is neither a local maximum nor a local minimum). However, as mentioned above, if $f''$ is zero at a critical point then we **cannot** conclude that the point is an inflexion point. For example, if $f(x) = x^6$ then

$f''(0) = 0$, but $f$ does not have an inflexion point at 0; it has a local minimum there.

**Example:** Let's find the critical points of the function $f(x) = 2x^3 - 9x^2 + 1$ and determine the natures of these points. The derivative is

$$f'(x) = 6x^2 - 18x = 6x(x - 3).$$

The solutions to $f(x) = 0$ are 0 and 3 and these are therefore the critical points. To determine their nature we could examine the sign of $f'$ in the vicinity of each point, or we could check the sign of $f''(x)$ at each. For completeness of exposition, we shall do both here, but in practice you only need to carry out one of these tests. First, let's examine the sign of $f'(x)$ in the vicinity of $x = 0$. We have $f'(x) = 6x(x - 3)$, which is positive for $x < 0$ (since it is then the product of two negative numbers). For $x$ just greater than 0, $x > 0$ and $x - 3 < 0$, so that $f'(x) < 0$. (Note: we are interested only in the signs of $f'(x)$ **just** to either side of the critical point, in its immediate vicinity). Thus, at $x = 0$, $f'$ changes sign from positive to negative and hence $x = 0$ is a local maximum. Now for the other critical point. When $x$ is just less than 3, $6x(x - 3) < 0$ and when $x > 3$, $6x(x - 3) > 0$; thus, since $f'$ changes from negative to positive around the point, $x = 3$ is a local minimum. Alternatively, we note that $f''(x) = 12x - 18$. Since $f''(0) < 0$, 0 is a local maximum. Since $f''(3) > 0$, 3 is a local minimum.

### Identifying local and global maxima

Now we turn to the problem of optimisation. Suppose we want to find the maximum value of a function $f(x)$. Such a wish only makes sense if the function has a maximum value; in other words, it does not take unboundedly large values. This value will occur at a local maximum point, but there may be several local maximum points. The **global maximum** is where the function attains its absolute maximum value (if such a value exists) and we can think of the local maximum points as giving the maximum value of the function in their vicinity. It should be emphasised that not all functions will have a global maximum. For instance, the function $f(x) = 2x^3 - 9x^2 + 1$ considered in the example above has no global maximum because the values $f(x)$ get increasingly large, without bound, for large positive values of $x$. Even though this function does, as we have seen, possess a local maximum, it does not have a global maximum.

If $f$ does indeed have a global maximum, then we can find it as follows. We proceed by determining all the local maximum points of $f$, using the techniques outlined above, and then we calculate the corresponding values $f(x)$ and compare these to find the largest. (Of course, if there is only one local maximum, then it is the global maximum.) The analogous procedure is carried out if we want to find the global minimum value (if the function has one): we find the minimum points and, among these, find which gives the smallest value of $f$. These techniques are, like many other things in this subject, best illustrated by examples.

**Example:** To find the maximum value of the function $f(x) = xe^{-x^2}$, we first calculate the derivative, using the product rule:

$$f'(x) = e^{-x^2} - (2x)xe^{-x^2} = e^{-x^2}(1 - 2x^2).$$

There are two solutions of $f'(x) = 0$, namely $x = 1/\sqrt{2}$ and $x = -1/\sqrt{2}$. (Note that $e^{-x^2}$ is never equal to 0.) In other words, these values of $x$ give the critical points, or stationary points. To determine their nature we could examine the sign of $f'$ in the vicinity of each point, or we could check the sign of $f''(x)$ at each. For completeness of exposition, we shall do both here, but in practice you only need to use one method. First, let's examine the sign of $f'(x)$ as $x$ goes from just less than

$-1/\sqrt{2}$ to just greater than $-1/\sqrt{2}$. For $x < -1/\sqrt{2}$, $1 - 2x^2 < 0$ and so $f'(x) < 0$, while for $x$ just greater than $-1/\sqrt{2}$, $1 - 2x^2 > 0$ and $f'(x) > 0$. It follows that $-1/\sqrt{2}$ is a local minimum. In a similar way, one can check—and you should do this—that for $x$ just less than $1/\sqrt{2}$, the derivative is positive and for $x$ just greater than $1/\sqrt{2}$, the derivative is negative, so that we may deduce $1/\sqrt{2}$ is a local maximum. Alternatively, we may calculate $f''(x)$:

$$f''(x) = -2xe^{-x^2}(1 - 2x^2) - 4xe^{-x^2}.$$

Now, $f''(-1/\sqrt{2}) > 0$, so this point is a local minimum, and $f''(1/\sqrt{2}) < 0$, so this point is a local maximum. Now, we are trying to find the maximum value of $f$. This is when $x = 1/\sqrt{2}$, and the maximum value is $f(1/\sqrt{2}) = (1/\sqrt{2})e^{-1/2} = 1/\sqrt{2e}$. (Note: this function does indeed have a global maximum and a global minimum. This might not be obvious, but it follows from that fact that for very large positive $x$ or very 'large' negative $x$, $xe^{-x^2}$ is extremely small in size.)

**Activity 3.10** *Find the critical points of $f(x) = x^3 - 6x^2 + 11x - 6$ and classify the nature each such point (that is, determine whether the point is a local maximum, local minimum, or inflexion).*

If we are trying to find the maximum value of a function $f(x)$ on an interval $[a, b]$, then it will occur either at $a$ or at $b$, or at a critical point $c$ in between $a$ and $b$. Suppose, for instance, there was just one critical point $c$ in the open interval $(a, b)$, and that this was a local maximum. To be sure that it gives the maximum value on the interval, we should compare the value of the function at $c$ with the values at $a$ and $b$. To sum up, it is possible, when maximising on an interval, that the maximum value is actually at an end-point of the interval, and we should check whether this is so. (The same argument applies to minimising.)

## Curve sketching

Another useful application of differentiation is in curve sketching. The aim in sketching the curve described by an equation $y = f(x)$ is to indicate the behaviour of the curve and the coordinates of key points. Curve sketching is a very different business from simply plotting a few points and joining them up: there's no room in this subject for such unsophisticated methods, and such 'plotting' is an inadequate substitute for proper curve sketching!

Given the equation $y = f(x)$ of a curve we wish to sketch, we have to determine key information about the curve. The main questions we should ask are as follows. Where does the curve cross the $x$-axis (if at all)? Where does it cross the $y$-axis? Where are the critical points (or stationary points, if you prefer that name)? What are the natures of the critical points? What is the behaviour of the curve for large positive values of $x$ and 'large' negative values of $x$ (where 'large' means large in absolute value)?

To outline a general technique, we take these in turn.

- **Where it crosses the $x$-axis:** The $x$-axis has equation $y = 0$ and the curve has equation $y = f(x)$, so the curve crosses the $x$-axis at the points $(x, 0)$ for which $f(x) = 0$. Thus we solve the equation $f(x) = 0$. This may have many solutions or none at all. (For instance, if $f(x) = \sin x$ there are infinitely many solutions, whereas if $f(x) = x^2 + 1$ there are none.)

- **Where it crosses the $y$-axis:** The $y$-axis has equation $x = 0$ and the curve has equation $y = f(x)$, so the curve crosses the $y$-axis at the single point $(0, f(0))$.

- **Finding the critical points:** We've seen how to do this already. We solve the equation $f'(x) = 0$.

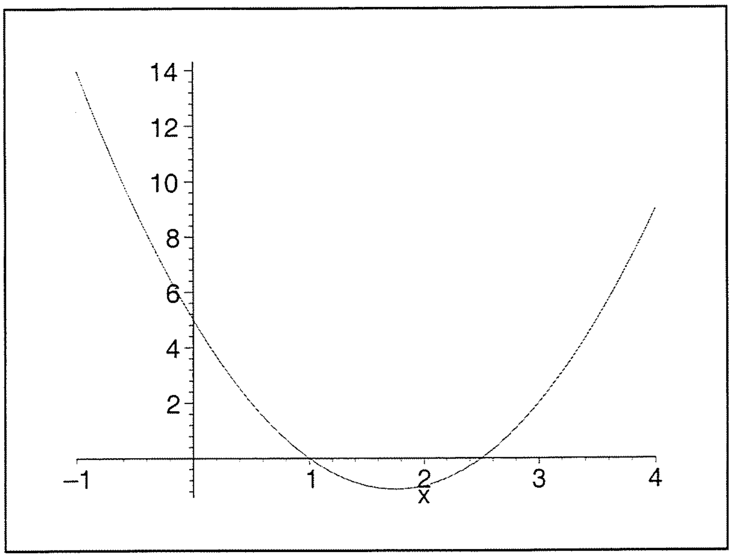
- **The natures of the critical points:** This means determining whether each one is a local maximum, local minimum, or inflexion point, and the methods for doing this have been discussed earlier in this chapter.

- **Limiting behaviour:** We have to determine what happens to $f(x)$ as $x$ tends to infinity and as $x$ tends to minus infinity; in other words, we have to ask how $f(x)$ behaves for $x$ far to the right on the $x$-axis and for $x$ far to the left on the negative side of the axis. There are two standard results here which are useful.

  First, the behaviour of a polynomial function is determined solely by its leading term, the one with the highest power of $x$. This term dominates for $x$ of large absolute value. A useful observation is that if $n$ is even then $x^n \to \infty$ as $x \to \infty$ and also as $x \to -\infty$, while if $n$ is odd, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. (To say, for example, that $f(x) \to \infty$ as $x \to \infty$ means that the values of $f(x)$ are, for $x$ large enough, greater than any value we want. For example, it means that there is some number $X$ such that for all $x > X$, $f(x) > 1000000$; and that, for some value $Y$, we have $f(x) > 100000000$ for all $x > Y$, and so on. In words, we say that '$f(x)$ tends to infinity as $x$ tends to infinity'.) Thus, for example, if $f(x) = -x^3 + 5x^2 - 7x + 2$, then we examine the leading term, $-x^3$. As $x \to \infty$, this tends to $-\infty$ and as $x \to -\infty$ it tends to $\infty$. So this is the behaviour of $f$.

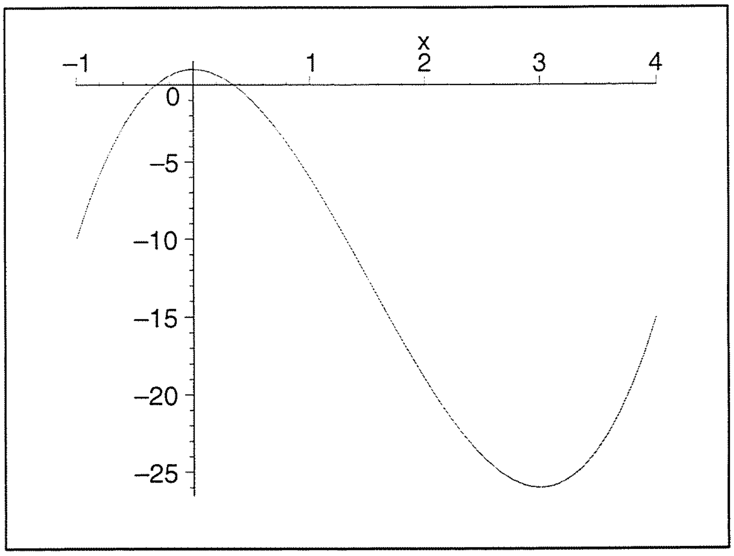
  Secondly, whenever we have a function which is the product of an exponential and a power, the exponential dominates. Thus, for example, $x^2e^{-x} \to 0$ as $x \to \infty$ (even though $x^2 \to \infty$).

**Example:** Let's do a really easy example. Consider the quadratic function $f(x) = 2x^2 - 7x + 5$. We already know a lot about sketching such curves (from the previous chapter), but let's apply the scheme suggested above. This curve crosses the $x$-axis when $2x^2 - 7x + 5 = 0$. The solutions to this equation (which can be found by using the formula or by factorising) are $x = 1$ and $x = 5/2$. The curve crosses the $y$-axis at $(0, 5)$. The derivative is $f'(x) = 4x - 7$, so there is a critical point at $x = 7/4$. The second derivative is $f''(x) = 4$, which is positive, so this critical point is a minimum. The value of $f$ at the critical point is $f(4/7) = 81/49$. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$. From this it follows that the graph of $f$ is as in the figure below.




**Example:** We considered the function $f(x) = 2x^3 - 9x^2 + 1$ earlier. We saw that it has a local maximum at $x = 0$ and a local minimum at $x = 3$. The corresponding values of $f(x)$ are $f(0) = 1$ and $f(3) = -26$. The curve crosses the $y$-axis when $y = f(0) = 1$. It crosses the $x$-axis when $2x^3 - 9x^2 + 1 = 0$. Now, this is not an easy equation to solve! However, we can get some idea of the points where it crosses the $x$-axis by considering the shape of the curve. Note that as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$. Also, we have $f(0) > 0$ and $f(3) < 0$. These observations imply that the graph must cross the $x$-axis somewhere to the left of 0 (since it must move from negative $y$-values to a positive $y$-value), it must cross again somewhere between 0 and 3 (since $f(0) > 0$ and $f(3) < 0$) and it must cross again at some point greater than 3 (because $f(x) \to \infty$ as $x \to \infty$ and hence must be positive from some point).

We therefore have the following sketch (in which I have shown the correct $x$-axis crossings):



**Activity 3.11** *Sketch the graph of* $f(x) = x^3 - 6x^2 + 11x - 6$. *(Note: this is the function considered in Activity 3.10.)*

## Marginals

We now turn our attention to economic applications of the derivative. In this section we consider 'marginals' and in the next section we approach the problem of profit maximisation using the derivative. Suppose that a firm manufactures chocolate bars and knows that in order to produce $q$ chocolate bars it will have to pay out $C(q)$ dollars in wages, materials, overheads and so on. We say that $C$ is the firm's **Cost function**. This is often called the **Total Cost**, and we shall often use the corresponding notation $TC$. The cost $TC(0)$ of producing no units (which is generally positive since a firm has certain costs in merely existing) is called the **Fixed Cost**, sometimes denoted $FC$. The difference between the cost and the fixed cost is known as the **Variable Cost**, $VC$. Other important measures are the **Average Cost**, defined by $AC = TC/q$ and the **average variable cost** $AVC = VC/q$. We have the relationship

$$TC = FC + VC.$$

An increase in production by one chocolate bar is relatively small, and may be described as 'marginal'. The corresponding increase in total cost is $TC(q+1) - TC(q)$.

Now, we know that the derivative of a function $f$ at a point $a$ is the limit of $(f(a+h) - f(a))/h$ as $h$ tends to 0. This means that if $h$ is small, then $(f(a+h) - f(a))/h$ is approximately equal to $f'(a)$. Hence, for small $h$,

$$f(a+h) - f(a) \simeq hf'(a),$$

where '$\simeq$' means 'is approximately equal to'.

If the production level $q$ is large, so that 1 unit is small compared with $q$, then we may take $f$ to be the total cost function $TC(q)$ and take $h = 1$ to see that the cost incurred in producing one extra item, namely $TC(q+1) - TC(q)$, is given approximately by

$$TC(q+1) - TC(q) \simeq 1 \times (TC)'(q) = (TC)'(q).$$

It is for this reason that we **define** the **Marginal Cost function** to be the derivative of the total cost function $TC$. This marginal cost is often denoted $MC$. The marginal cost should not be confused with the average cost, which is given by $AC(q) = TC(q)/q$. In general, the marginal cost and the average cost are different.

In the traditional language of economics, the derivative of a function $F$ is often referred to as the marginal of $F$. For example, if $TR(q)$ is the total revenue function, which describes the total revenue the firm makes when selling $q$ items, then its derivative is called the **Marginal Revenue**, denoted $MR$.

**Activity 3.12** *A firm has total cost function*

$$TC(q) = 50000 + 25q + 0.001q^2.$$

*Find the fixed cost $FC$, and the marginal cost $MC$. What is the marginal cost when the output is 100? What is the marginal cost when the output is* 10000*?*

## Profit Maximisation

The revenue or total revenue, $TR$, a firm makes is simply the amount of money generated by selling the good it manufactures. In general, this is the price of the good times the number of units sold. To calculate it for a specific firm, we need more information (as we shall see below). We have denoted the total cost function by $TC$. In this notation, the **profit** $\Pi$, is the total revenue minus the total cost, $\Pi = TR - TC$. If we consider revenue and cost as functions of $q$, then $\Pi = \Pi(q)$ is given as a function of $q$ by $\Pi(q) = TR(q) - TC(q)$. To find which value, or values, of $q$ give a maximum profit, we look for critical points of $\Pi$ by solving $\Pi'(q) = 0$. But, since $\Pi'(q) = (TR)'(q) - (TC)'(q)$, this means that the optimal value of $q$ satisfies $(TR)'(q) = (TC)'(q)$. In other words, to maximise profit, marginal revenue equals marginal cost.[5]

[5] *See Anthony and Biggs, Sections 8.1, 9.2 and 9.3 for a general discussion.*

The firm is said to be a **monopoly** if it is the only supplier of the good it manufactures. This means that if the firm manufactures $q$ units of its good, then the selling price at equilibrium is given by the inverse demand function, $p = p^D(q)$. Why is this? Well, the inverse demand function $p^D(q)$ tells us what price the consumers will be willing to pay to buy a total of $q$ units of the good. But if the firm is a monopoly and it produces $q$ units, then it is only these $q$ units that are on the market. In other words, the '$q$' in the inverse demand function (the **total** amount of the good on the market) is the same as the '$q$' that the firm produces. So the selling price is defined by the inverse demand function, as a function of the production level $q$ of the firm. The revenue is then given, as a function of $q$, by $TR(q) = qp^D(q)$.

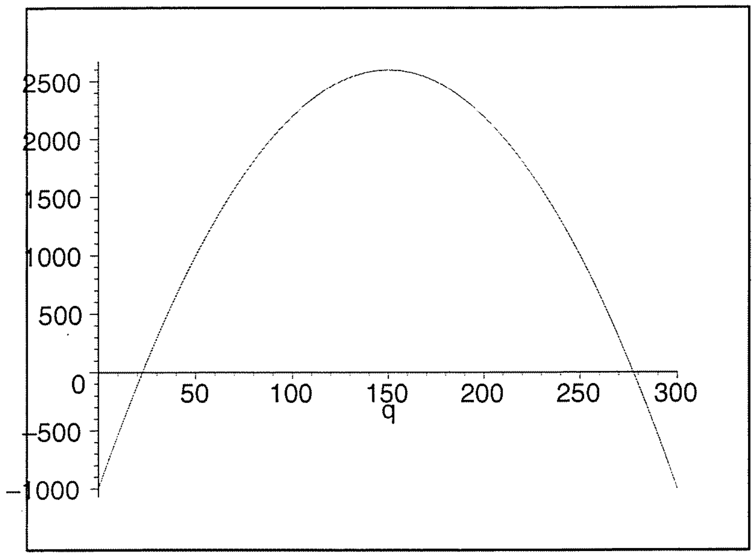
**Example:** A monopoly has cost function $TC(q) = 1000 + 2q + 0.06q^2$ and its demand curve has equation $q + 10p = 500$. What value of $q$ maximises the profit?

To answer this, we first have to determine the revenue as a function of $q$. Since the firm is a monopoly, we know that $TR(q) = qp^D(q)$. From the equation for the demand curve, $q + 10p = 500$, we obtain $p = 50 - 0.1q$, so $p^D(q) = 50 - 0.1q$ and $R(q) = q(50 - 0.1q)$. The profit is

$$\Pi(q) = TR(q) - TC(q) = q(50 - 0.1q) - (1000 + 2q + 0.06q^2) = 48q - 0.16q^2 - 1000.$$

The equation $\Pi'(q) = 0$ is $48 - 0.32q = 0$, which has solution $q = 150$. To verify that this does indeed give a maximum profit, we note that $\Pi''(q) = -0.32 < 0$.

Consider this last example a little further. The profit function $\Pi(q)$ is a quadratic function with a negative $q^2$ term, so we well know what it looks like: its graph will be as follows.




Notice that the profit is negative to start with (because there is no revenue when nothing is produced, but there is a cost of producing nothing, namely the fixed cost of 1000). As production is increased, profit starts to rise, and becomes positive. The point at which profit just starts to become positive (that is, where it first equals 0), is called the breakeven point. So, the breakeven point is the smallest positive value of $q$ such that $\Pi(q) = 0$. When the firm is producing the breakeven quantity, it is breaking even in the sense that its revenue matches its costs. In this specific example, we can calculate the breakeven point by solving the equation

$$-0.16q^2 + 48q - 1000 = 0.$$

This has the solutions 22.524 and 277.475. It is clearly the first of these that we want, the second corresponding to the higher value of $q$ at which the profit, having increased to a maximum and then decreased, becomes 0 again. So the breakeven point is 22.524.

**Activity 3.13** *A firm is a monopoly with cost function*

$$TC(q) = q + 0.02q^2.$$

*The demand equation for its product is $q + 20p = 300$. Work out (a) the inverse demand function; (b) the profit function; (c) the optimal value $q_m$ and the maximum profit; (d) the corresponding price.*

## Learning outcomes

At the end of this chapter and the relevant reading, you should be able to:

- explain what is meant by the derivative
- state the standard derivatives
- calculate derivatives using sum, product, quotient, and composite function (chain)

rules

- calculate derivatives by taking logarithms

- establish the nature of the critical/stationary points of a function.

- use the derivative to help sketch functions

- explain and use the terminology surrounding 'marginals' in economics, and be able to find fixed costs and marginal costs, given a total cost function

- explain what is meant by the breakeven point and be able to determine this

- make use of the derivative in order to minimise or maximise functions, including profit functions

You do **not** need to be able to differentiate from first principles (that is, by using the formal definition of the derivative).

## Sample examination/practice questions

**1.** Differentiate $y = (1 + 2x - e^x)$ and find when the derivative is zero.

**2.** Differentiate the following functions:

1. $y = x^3 + \exp(3x^2)$
2. $y = \dfrac{3x+5}{x^2+3x+2}$
3. $y = \dfrac{x^2+2x+7}{\sqrt{3x-1}}$
4. $y = \ln(3x^2) + 3x + \dfrac{1}{\sqrt{1+x}}$

**3.** Assume that the price/demand relationship for a particular good is given by

$$p = 10 - 0.005q$$

where $p$ is the price (\$) per unit and $q$ is the demand per unit of time. Also assume that the fixed costs are \$100 and the average variable cost per unit is $4 + 0.01q$.

(a) What is the maximum profit obtainable from this product?

(b) What level of production is required to break even?

(c) What are the marginal cost and marginal revenue functions?

**4.** The demand function relating price $p$ and quantity $x$, for a particular product, is given by

$$p = 5\exp(-x/2).$$

Find the amount of production, $x$, which will maximise revenue from selling the good, and state the value of the resulting revenue. Produce a rough sketch graph of the marginal revenue function for $0 \leq x \leq 6$.

**5.** Suppose you have a nineteenth-century painting currently worth \$2000, and that its value will increase steadily at \$500 per year, so that the amount realised by selling the painting after $t$ years will be $2000 + 500t$. An economic model shows that the optimum time to sell is the value of $t$ for which the function

$$P(t) = (2000 + 500t)e^{-(0.1)t}$$

is maximised. Given this, find the optimum time to sell, and verify that it is optimal.

**6.** A monopolist's average cost function is given by

$$AC = 10 + \frac{20}{Q} + Q.$$

Her demand equation is

$$P + 2Q = 20,$$

where $P$ and $Q$ are price and quantity, respectively. Find expressions for the total revenue and for the profit, as functions of $Q$. Determine the value of $Q$ which maximises the total revenue. Determine also the value of $Q$ maximising profit.

**7.** A firm's average cost function is given by

$$\frac{300}{Q} - 10 + Q$$

and the demand function is given by $Q + 5P = 850$, where $P$ and $Q$ are quantity and price, respectively.
Suppose the firm is a monopoly.
Find expressions for the total revenue and for the profit, as functions of $Q$.
Determine the value of $Q$ which maximises the total revenue.
Determine also the value of $Q$ maximising profit.

## Answers to activities

**3.1** Taking $h = 0.00001$, for example, $(2^{1+h} - 2)/h$ is 1.386299 and for $h = 0.000001$ it is 1.386295. These are even closer to the true value $2\ln 2 = 1.38629436\ldots$.

**3.2** The function $1/x$ can be written as $x^{-1}$. Its derivative is therefore $(-1)x^{-1-1} = -x^{-2} = -1/x^2$.

**3.3** By the product rule,

$$(x^2 \sin x)' = 2x \sin x + x^2 \cos x.$$

**3.4** By the product rule,

$$((x^2 + 1)\ln x)' = 2x \ln x + (x^2 + 1)\frac{1}{x} = 2x \ln x + x + \frac{1}{x}.$$

**3.5** By the quotient rule,

$$\left(\frac{\sin x}{x}\right)' = \frac{(\cos x(x) - (\sin x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}.$$

**3.6** By the chain rule (composite function rule),

$$\left((3x+7)^{15}\right)' = 15(3x+7)^{14}(3) = 45(3x+7)^{14}.$$

**3.7** $\sqrt{x^2+1} = (x^2+1)^{1/2}$. By the chain rule,

$$\left((x^2+1)^{1/2}\right)' = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

**3.8** By the chain rule,

$$\left(\ln(x^2+2x+5)\right)' = \frac{1}{x^2+2x+5}(x^2+2x+5)' = \frac{2x+2}{x^2+2x+5}.$$

**3.9** If $f(x) = x^x$ then $\ln f(x) = x \ln x$ and so

$$\frac{f'(x)}{f(x)} = x\left(\frac{1}{x}\right) + (1)\ln x = 1 + \ln x,$$

from which we obtain

$$f'(x) = (1+\ln x)f(x) = (1+\ln x)x^x.$$

**3.10** The derivative of $f(x) = x^3 - 6x^2 + 11x - 6$ is $f'(x) = 3x^2 - 12x + 11$. The stationary points, the solutions to $3x^2 - 12x + 11 = 0$, are $(12-\sqrt{12})/6$ and $(12+\sqrt{12})/6$. The second derivative is $6x - 12$, which is negative at the first stationary point and positive at the second. We therefore have a local maximum at $(12-\sqrt{12})/6$ and a local minimum at $(12+\sqrt{12})/6$. The corresponding values of $f$ are $2\sqrt{3}/9$ and $-2\sqrt{3}/9$. (Approximately, $2\sqrt{3}/9$ is 0.3849.)

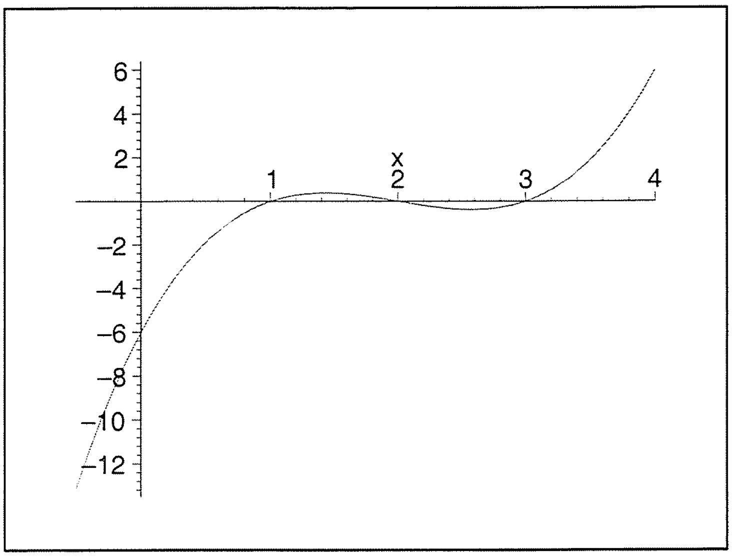
**3.11** To sketch $f(x) = x^3 - 6x^2 + 11x - 6$, we first note from the previous activity that $f$ has a local maximum at $(12-\sqrt{12})/6$ and a local minimum at $(12+\sqrt{12})/6$. The corresponding values of $f$ are $2\sqrt{3}/9$ and $-2\sqrt{3}/9$. (Approximately, $2\sqrt{3}/9$ is 0.3849.) As $x$ tends to infinity, so does $f(x)$ and as $x$ tends to $-\infty$, $f(x) \to -\infty$. The curve crosses the $y$-axis at $(0, -6)$. To find where it crosses the $x$-axis, we have so solve $x^3 - 6x^2 + 11x - 6 = 0$. One solution to this is easily found (by guesswork) to be $x = 1$, and so $(x-1)$ is a factor. Therefore, for some numbers $a, b, c$,

$$x^3 - 6x^2 + 11x - 6 = (x-1)(ax^2 + bx + c).$$

Straight away, we see that $a = 1$ and $c = 6$. To find $b$, we could notice that the number of terms in $x^2$ is $-a + b$, which should be $-6$, so that $-1 + b = -6$ and $b = -5$. Hence

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6) = (x-1)(x-2)(x-3)$$

and we see there are three solutions: $x = 1, 2, 3$. Piecing all this information together, we can sketch the curve as follows:




**3.12** The total cost function is $50000 + 25q + 0.001q^2$. The fixed cost is obtained by setting $q = 0$, giving $FC = 50000$. The marginal cost is $MC = 25 + 0.002q$. The marginal cost is \$25.2 if the output is 100, but it rises to \$45 if the output is 10000.

**3.13** (a) The inverse demand function is

$$p^D(q) = (300 - q)/20 = 15 - 0.05q.$$

(b) The profit function is

$$\Pi(q) = qp^D(q) - TC(q) = q(15 - 0.05q) - (q + 0.02q^2) = 14q - 0.07q^2.$$

(c) We have $\Pi'(q) = 14 - 0.14q$, so $q = 100$ is a critical point. The second derivative of $\Pi$ is $\Pi''(q) = -0.14$, which is negative, so the critical point is a local maximum. The value of the profit there is $\Pi(100) = 1400 - 700 = 700$, whereas $\Pi(0) = 0$ and $\Pi(200) = 0$. Since the maximum profit in the interval $[0, 200]$ must be either at a local maximum or an end-point, it follows that the maximum profit is 700, obtained when $q = 100$.

(d) The price when $q = 100$ is $p^D(100) = 15 - (0.05)(100) = 10$.

## Answers to sample examination/practice questions

**1.** The derivative is $\frac{dy}{dx} = 2 - e^x$, and this is equal to 0 when $e^x = 2$; that is, when $x = \ln 2$.

**2.** For the first, we have, using the composite function rule on the exponential term,

$$\frac{dy}{dx} = 3x^2 + 6x\exp(3x^2).$$

For the second, using the quotient rule,

$$\frac{dy}{dx} = \frac{3(x^2 + 3x + 2) - (2x + 3)(3x + 5)}{(x^2 + 3x + 2)^2} = \frac{-3x^2 - 10x - 9}{(x^2 + 3x + 2)^2}.$$

For the third problem, we take logarithms and differentiate (though it could be done directly), as follows:

$$\ln y = \ln(x^2 + 2x + 7) - \frac{1}{2}\ln(3x - 1),$$

so

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x+2}{x^2+2x+7} - \frac{1}{2}\frac{3}{3x-1},$$

and therefore

$$\frac{dy}{dx} = \frac{2x+2}{3x-1} - \frac{3}{2}(x^2+2x+7).$$

Lastly,

$$\begin{aligned}
\frac{d}{dx}\left(\ln(3x^2) + 3x + \frac{1}{\sqrt{1+x}}\right) &= \frac{d}{dx}\left(\ln(3x^2) + 3x + (1+x)^{-1/2}\right) \\
&= \frac{6x}{3x^2} + 3 - \frac{1}{2}(1+x)^{-3/2} \\
&= \frac{2}{x} + 3 - \frac{1}{2(1+x)^{3/2}}.
\end{aligned}$$

**3.** (a) The average variable cost is AVC$=4 + 0.01q$, so the variable cost is VC$=4q + 0.01q^2$. So the total cost function is $TC = 4q + 0.01q^2 + FC$, where $FC$ is the fixed cost; that is, $TC = 4q + 0.01q^2 + 100$. Given that the firm is a monopoly, the revenue is $TR = pq = (10 - 0.005q)q = 10q - 0.005q^2$. The profit function is

$$\Pi = TR - TC = 10q - 0.005q^2 - (4q + 0.01q^2 + 100) = 6q - 0.015q^2 - 100.$$

To find the maximum, we solve $\Pi' = 0$, which is $6 - 0.03q = 0$, so $q = 200$. To check it gives a maximum, we check that the second derivative is negative, which is true since $\Pi'' = -0.03$.

(b) The breakeven point is the (least) value of $q$ for which $\Pi(q) = 0$. So we solve $6q - 0.015q^2 - 100 = 0$. We can rewrite this as

$$0.015q^2 - 6q + 100 = 0,$$

which has solutions

$$\frac{6 \pm \sqrt{6^2 - 4(0.015)(100)}}{2(0.015)} = \frac{6 \pm \sqrt{36-6}}{0.03} = \frac{100}{3}(6 \pm \sqrt{30}).$$

Since $\sqrt{30} < 6$ (because $30 < 6^2 = 36$), there are two positive solutions. The breakeven point is the smaller of these, which is $100(6 - \sqrt{30})/3$.

(c) The marginal cost and marginal revenue functions are the derivatives of the total cost and total revenue. Thus,

$$MC = \frac{d}{dq}\left(4q + 0.01q^2 + 100\right) = 4 + 0.02q$$

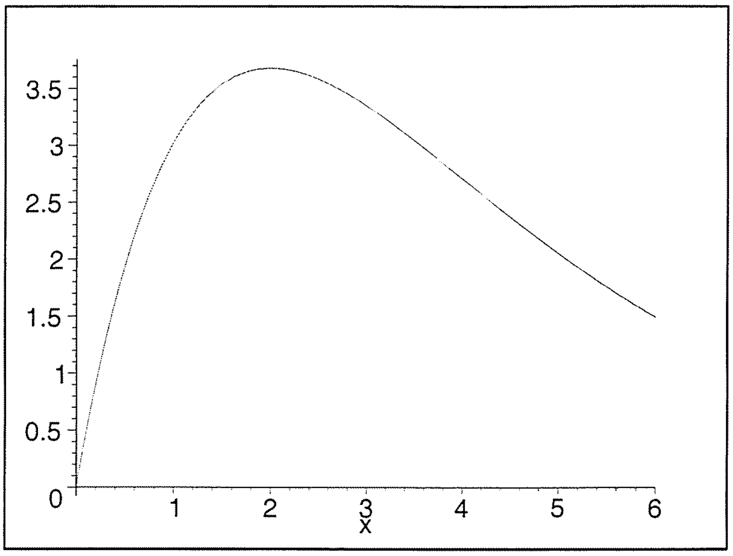
and

$$MR = \frac{d}{dq}\left(10q - 0.005q^2\right) = 10 - 0.01q.$$

**4.** The total revenue obtained from the sale of the good is simply (price times quantity) $TR = 5xe^{-x/2}$. To find the maximum, we differentiate:

$$(TR)' = 5x^{-x/2} - 5x\frac{1}{2}e^{-x/2} = \frac{5}{2}e^{-x/2}(2-x),$$

and this is zero only if $x = 2$. We can see that the derivative changes sign from positive to negative on passing through $x = 2$, and so this is a local maximum. The value of the revenue there is $5(2)e^{-2/2} = 10/e$. When $x = 0$, $TR = 0$ and there are no solutions to $TR = 0$, so the graph of $TR$ will not cross the $x$-axis. Sketching the curve between 0 and 6, we obtain the following.




**5.** By routine application of the rules for differentiation we get

$$P'(t) = 500e^{-0.1t} + (2000 + 500t)(-0.1)e^{-0.1t} = e^{-0.1t}(300 - 50t).$$

Since this is zero when $t = 6$, that is a critical point of $P$. Differentiating again we get

$$P''(t) = (-0.1)e^{-0.1t}(300 - 50t) + e^{-0.1t}(-50) = e^{-0.1t}(5t - 80).$$

It follows that $P''(6) < 0$, so the critical point $t = 6$ is a local maximum.

The fact that $t = 6$ is indeed the maximum in $[0, \infty)$ can be verified by common-sense arguments. We know that $t = 6$ is the only critical point of $P(t)$, and that it is a local maximum. It follows that $P(t)$ must decrease steadily for $t > 6$, because if at any stage it started to increase again, it would have to pass through a critical point first. (Alternatively, to see that this local maximum is a global maximum, it could be noted that $P(t) \to 0$ as $t \to \infty$. This is because the exponential part $e^{-0.01t}$ tends to 0 and exponentials 'dominate' polynomials, so that even if we multiply this by $(2000 + 500t)$, the result still tends to 0.)

**6.** First, since the total cost is $Q$ times the average cost, we have

$$TC = Q\left(10 + \frac{20}{Q} + Q\right) = 10Q + 20 + Q^2.$$

The monopolist's demand equation is $P + 2Q = 20$, so when the quantity produced is $Q$, the selling price will be $P = 20 - 2Q$ and hence the total revenue will be $TR = QP = Q(20 - 2Q) = 20Q - 2Q^2$. So the profit function is

$$\Pi(Q) = 20Q - 2Q^2 - (10Q + 20 + Q^2) = 10Q - 3Q^2 - 20.$$

To maximise $TR$, we set $(TR)' = 0$, which is $20 - 4Q = 0$, giving $Q = 5$. This does indeed maximise revenue because $(TR)'' = -4 < 0$. For the profit function, we have

$\Pi'(Q) = 10 - 6Q$, and this is 0 when $Q = 5/3$. Again, this gives a maximum because $\Pi''(Q) = -6 < 0$.

**7.** The total cost is

$$TC = Q(AC) = Q\left(\frac{300}{Q} - 10 + Q\right) = 300 - 10Q + Q^2.$$

The inverse demand function is given by $P = 170 - Q/5$, so the total revenue is

$$TR = (170 - Q/5)Q = 170Q - 0.2Q^2.$$

The derivative $(TR)'$ is $170 - 0.4Q$, which is 0 when $Q = 425$, this giving a maximum because $(TR)'' = -0.4 < 0$. The profit function is

$$\Pi(Q) = TR - TC = 170Q - 0.2Q^2 - (300 - 10Q + Q^2) = 180Q - 1.2Q^2 - 300.$$

To maximise the profit, we set $\Pi' = 0$, which is $180 - 2.4Q = 0$, so $Q = 150$. This is a maximum because $\Pi''(Q) = -2.4 < 0$.

# Chapter 4

# Integration

## Essential reading

(For full publication details, see Chapter 1.)

Anthony, M. and Biggs, N., *Mathematics for Economics and Finance.* Chapters 25 and 26.

## Further reading

Binmore, K. and Davies, J. *Calculus.* Chapter 10, Sections 10.2–10.10.

Black, J., and Bradley, J.F. *Essential Mathematics for Economists.* Chapter 9.

Booth, D.J. *Foundation Mathematics.* Chapter 6.

Bradley, T., and Patton, P. *Essential Mathematics for Economics and Business.* Chapter 8.

Holden, K and Pearson, A.W. *Introductory Mathematics for Economics and Business.* Sections 6.1–6.6, 6.14.

Dowling, Edward T. *Introduction to Mathematical Economics.* Chapters 16 and 17.

## Introduction

The next topic in calculus is integration. This is perhaps one of the most difficult topics in this subject, and I encourage you to practise on lots of examples.

# Integration

Integration is, in essence, the reverse process to differentiation and has a number of applications in economics and related subjects. (It is also essential for solving differential equations, but this topic is not part of this subject.) We start with **indefinite integration**[1].

[1] *See Anthony and Biggs, Section 25.3.*

Suppose the function $f$ is given, and the function $F$ is such that $F'(x) = f(x)$. Then we say that $F$ is an **anti-derivative** of $f$. (Sometimes the word **primitive** is used instead of 'antiderivative'.) For example, $x^4/4$ is an anti-derivative of $x^3$, and so is $x^4/4 + 5$. Any two anti-derivatives of a given function $f$ differ only by a constant. The general form of the anti-derivative of $f$ is called the **indefinite integral** of $f(x)$, and denoted by

$$\int f(x)\, dx.$$

Often we call it simply the **integral** of $f$. It is of the form $F(x) + c$, where $F$ is **any particular** anti-derivative of $f$ and $c$ denotes any constant, known as a **constant of integration**. Thus, for example, we write

$$\int x^3\, dx = \frac{x^4}{4} + c.$$

The process of finding the indefinite integral of $f$ is usually known as **integrating** $f$, and $f$ is known as the **integrand**.

Sometimes the variable of integration will be $x$; other times it will be some other symbol, but this makes no real difference. For instance,

$$\int x^3\, dx = \frac{x^4}{4} + c$$

and

$$\int t^3\, dt = \frac{t^4}{4} + c.$$

Note, however, that if we are to integrate a function of $t$, then the integral must contain a $dt$ and if we are to integrate a function of $x$, then the integral must contain a $dx$.

Just as for differentiation, we shall have a list of **standard integrals**[2] and some rules for combining these. The main standard integrals (which you should memorise) are the following.

[2] *See Anthony and Biggs, Section 25.5.*

| $f(x)$ | $\int f(x)\, dx$ |
|---|---|
| $x^n$ $(n \neq -1)$ | $\dfrac{x^{n+1}}{(n+1)} + c$ |
| $1/x$ | $\ln\lvert x\rvert + c$ |
| $e^x$ | $e^x + c$ |
| $\sin x$ | $-\cos x + c$ |
| $\cos x$ | $\sin x + c$ |

Note that the integral of $1/x$ is $\ln|x| + c$ rather than $\ln x + c$, because if $x$ is negative, then the derivative of $\ln|x|$ is the derivative of $\ln(-x)$, which is just $1/x$.

**Example:** The integral of $\sqrt{x}$, which of course can be written as $x^{1/2}$ is $2x^{3/2}/3 + c$, according to the first rule (taking $n = 1/2$).

**Activity 4.1** *Integrate the function $x^5$.*

Two important rules of integrals are easily verified:

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx,$$

for any functions $f$ and $g$, and

$$\int kf(x)\, dx = k \int f(x)\, dx,$$

for any constant $k$.

**Activity 4.2** *If a function $f$ has derivative $x^2 + 2\sin x$, and $f(0) = 1$, what is the function?*

## Definite integrals

Let $f$ be a function with an anti-derivative $F$. The **definite integral**[3] of the function $f$ over the interval $[a, b]$ is **defined to be**

[3] *See Anthony and Biggs, Section 25.4.*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Note that any anti-derivative $G(x)$ of $f$ is of the form $G(x) = F(x) + c$, for some constant $c$, so that

$$G(b) - G(a) = (F(b) + c) - (F(a) + c) = F(b) - F(a).$$

Thus, whichever anti-derivative of $f$ is chosen, the quantity on the right-hand side of the definition is the same. In calculations the notation $[F(x)]_a^b$ is often used as a shorthand for $F(b) - F(a)$.

**Example:** The definite integral of $x^4$ over $[0, 1]$ is

$$\int_0^1 x^4\, dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5} - 0 = \frac{1}{5}.$$

**Activity 4.3** *Calculate $\int_0^2 x^2\, dx$.*

**Activity 4.4** *Determine $\int_{-1}^1 e^t\, dt$.*

# Integration by substitution

## The method

We now turn our attention to a useful integration technique which may be thought of as doing for integration what the composite function rule does for differentiation. This is the technique of **integration by substitution**.[4] We illustrate with a simple example.

[4] *See Anthony and Biggs, Sections 26.1 and 26.2.*

**Example:** Suppose that we are asked to find the indefinite integral

$$\int (3x+5)^{12}\,dx.$$

We note that if we substitute $u = 3x + 5$ the integrand becomes $u^{12}$, which we know how to integrate. But we have changed the variable of integration. Originally we integrated with respect to $x$, signified by $dx$ in the integral, now we must integrate with respect to $u$. We must relate $dx$ to $du$, for the notation $\int u^{12}\,dx$ has no meaning. (Recall that a function of $u$ must be accompanied by $du$.) Making the substitution $u = 3x + 5$ is the same as saying that

$$x = \frac{1}{3}u - \frac{5}{3},$$

so $dx/du = 1/3$. Thus (and although this looks strange, it can be justified), $dx = \frac{1}{3}du$. So, when we replace $3x + 5$ by $u$ we should replace $dx$ by $(1/3)du$, giving

$$\int (3x+5)^{12}\,dx = \int u^{12}\left(\frac{1}{3}\right)du = \frac{1}{3}\frac{u^{13}}{13} + c.$$

We need the answer in terms of $x$, the original variable. Since $u = 3x + 5$ the integral is

$$\frac{1}{39}(3x+5)^{13} + c.$$

The general rule is: when we change the variable by putting $x = x(u)$ in the integral of $f(x)$ with respect to $x$, we must replace $dx$ by $(dx/du)du$. In other words, to determine

$$\int f(x)\,dx$$

we can work out

$$\int f(x(u))\,x'(u)du,$$

and then substitute back $x$ for $x(u)$.

In practice we overlook the distinction between $x$ as a function of $u$ and the inverse function, in which $u$ is regarded as a function of $x$, relying on the fact that $du/dx$ is equal to $1/(dx/du)$. This allows us to write 'shorthand' statements like

$$u = 3x+5, \quad \text{therefore} \quad du = 3\,dx,$$

which determines both $du/dx$ and $dx/du$.

As we have formally described a change of variable here, it involves expressing the variable of integration, $x$, in terms of a new variable $u$. Another way of expressing this is to say that we have made a **substitution**: we have substituted $u$ for $x$. In practice, in some problems it will be natural to express the new variable $u$ as a function of the original variable $x$, as in the example above and in some other problems it will be more natural to express $x$ as a function of $u$. The first approach is probably more common in the type of integrals studied in this subject.

## Examples

One key difficulty students have with the substitution method is not knowing which substitution to attempt. This is something you become more proficient at with practice, but it should be borne in mind that there need not be one correct substitution: for a number of problems, more than one substitution might work. In determining what substitution, if any, to try, one approach is to look at the integral and ask yourself what is complicating it. For instance, consider the example given above, where we have to integrate $(3x+5)^{12}$, we note that if we simply had $x^{12}$ rather than $(3x+5)^{12}$, then the problem would be easy. It is for this reason that we seek to transform the integral into a straightforward power, $u^{12}$, and the way to do this is to set $u = 3x+5$. Fortunately, this works, because the subsequent step of replacing $dx$ by something involving $du$ does not further complicate matters, only introducing a multiplicative factor of $1/3$. Here are a few more examples where the obvious substitution works, and one where it does not quite.

**Example:** Consider $\int x(3x+5)^7\,dx$. This is a slightly more complicated integral than the one we worked on above, but it is still the case that what makes the integral difficult is the $(3x+5)^7$. So, we try the substitution $u = 3x+5$. Then $du = 3\,dx$, so $dx = (1/3)du$ and the integral becomes $\int xu^7(1/3)\,du$. But there's something wrong here: the integrand involves both variables $x$ and $u$, whereas what we want is an integral involving only the new variable $u$. But, fear not, because, from $u = 3x+5$, we know that $x = (u-5)/3$. So, the integral is

$$\begin{aligned}
\frac{1}{3}\int \frac{(u-5)}{3}u^7\,du &= \frac{1}{9}\int \left(u^8 - 5u^7\right)\,du \\
&= \frac{1}{9}\left(\frac{u^9}{9} - 5\frac{u^8}{8}\right) + c \\
&= \frac{u^9}{81} - \frac{5}{72}u^8 + c \\
&= \frac{(3x+5)^9}{81} - \frac{5}{72}(3x+5)^8 + c,
\end{aligned}$$

not an answer you might easily have guessed(!), but which is obtained without too much difficulty using the substitution method.

**Example:** Let's think about $\int x(2x^2+7)^8\,dx$. The complicating part of the integral is $2x^2+7$, so we try the substitution $u = 2x^2+7$. With this, $du = 4x\,dx$, so that $dx = (1/(4x))du$ and the integral becomes

$$\int xu^8\left(\frac{1}{4x}\right)du = \frac{1}{4}\int u^8\,du.$$

Note that the $x$ cancels with the $1/(4x)$ factor emerging from expressing $dx$ in terms of $du$. So, here, there is no need to express $x$ in the integrand in terms of $u$. This integral is now quite straightforward. It evaluates to $u^9/36 + c$, so the answer is $(2x^2+7)^9/36 + c$.

**Example:** Consider the fairly similar-looking integral $\int x^3(2x^2+7)^8\,dx$. Again, we try the substitution $u = 2x^2+7$. We have, as before, $dx = (1/(4x))du$ and so the integral becomes

$$\int x^3 u^8 \left(\frac{1}{4x}\right) du = \frac{1}{4}\int x^2 u^8 \, du.$$

Here, the $x$ terms in the integrand do not entirely cancel. But we know that $x^2 = (u-7)/2$, so the integral simplifies as

$$\begin{aligned}
\frac{1}{4}\int \frac{(u-7)}{2} u^8 \, du &= \frac{1}{8}\int (u^9 - 7u^8)\, du \\
&= \frac{1}{8}\left(\frac{u^{10}}{10} - \frac{7u^9}{9}\right) + c \\
&= \frac{(2x^2+7)^{10}}{80} - \frac{7(2x^2+7)^9}{72} + c.
\end{aligned}$$

**Example:** The integral $\int \frac{x+1}{x^2+2x+7}\, dx$ is a different sort of integral, but there is a method that often (though not always) works. We let $u$, the new variable, be the denominator (that is, the bottom line) of the integrand, $u = x^2 + 2x + 7$. Then, $du = (2x+2)dx$, so $dx = du/(2x+2)$ and the integral is

$$\int \frac{x+1}{u}\frac{du}{2x+2} = \frac{1}{2}\int \frac{du}{u}.$$

Notice how the $x+1$ on the numerator of the integrand and the $2x+2$ factor cancel with each other to give just the constant factor $1/2$. Now the integral is

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2 + 2x + 7| + c.$$

In fact, $x^2 + 2x + 7$ is positive for all $x$, so we do not need the absolute value signs, and the answer is simply $(1/2)\ln(x^2 + 2x + 7) + c$.

**Example:** The integral $\int \frac{x}{x^2+2x+7}\, dx$ looks similar, but we shall see that the same substitution does not enable us to determine the integral. (Can you anticipate why?) Putting $u = x^2 + 2x + 7$ as before, the integral becomes

$$\int \frac{x}{u}\frac{du}{2x+2} = \int \frac{x}{2x+2}\frac{du}{u}.$$

Here, we do not get the same sort of cancellation as before. To express $x/(1+x)$ in terms of $u$ would be difficult, and would lead to a 'messy' integral which we could not easily determine. The reason that the substitution does not work here is that the numerator is not a multiple of the derivative $(2x+2)$ of the bottom line.

In the context of these last two examples, it is worth mentioning a general rule:

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c.$$

This follows on making the substitution $u = f(x)$. Noting that $du = f'(x)\, dx$, the integral is exactly $\int (1/u)\, du$, which is $\ln|u| + c$, equal to $\ln|f(x)| + c$.

**Activity 4.5** *Use the substitution $u = x^2$ to determine $\int xe^{x^2}\, dx$.*

**Activity 4.6** *Determine* $\int x(2x^2+2)^{1/2}\,dx$ *by using substitution.*

**Activity 4.7** *Determine the integral* $\int x\sqrt{x-1}\,dx$ *by using the substitution* $u = x - 1$. *Now determine it using the substitution* $u = \sqrt{x-1}$. *(You should, of course, get the same answer. The point I'm emphasising here is that there can be more than one appropriate substitution.)*

### The substitution method for definite integrals

In the case of a definite integral there is no need to revert to the original variable before evaluating the anti-derivative: we simply use the appropriate values of the new variable. If we change from the variable $x$ to the variable $u$, and the interval of integration for $x$ was $[a, b]$, the interval for $u$ will be $[\alpha, \beta]$, where $\alpha$ and $\beta$ are the values of $u$ which correspond to $x = a$ and $x = b$ respectively. Formally

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{u=\alpha}^{u=\beta} f(x(u))\,x'(u)du,$$

where $x(\alpha) = a$ and $x(\beta) = b$. This result holds provided that $u$ increases or decreases from $\alpha$ to $\beta$ as $x$ goes from $a$ to $b$.

**Example:** Making the substitution $u = 3x + 5$,

$$\int_0^1 (3x+5)^2\,dx = \int_5^8 u^2\frac{1}{3}du = \frac{1}{3}\left[\frac{u^3}{3}\right]_5^8 = \frac{1}{9}(8^3 - 5^3).$$

Here, we have used the fact that since $u = 3x + 5$, the values of $u$ corresponding to $x = 0$ and $x = 1$ are 5 and 8.

**Activity 4.8** *Determine* $\displaystyle\int_0^1 \frac{x+2}{x^2+4x+5}\,dx.$

## Integration by parts

The technique of **integration by parts**[5] may be thought of as resulting from the product rule for differentiation, which tells us that the derivative of $u(x)v(x)$ is $u'(x)v(x) + u(x)v'(x)$. Hence the anti-derivative of $u'(x)v(x) + u(x)v'(x)$ is $u(x)v(x)$, or equivalently

[5] *See Anthony and Biggs, Section 26.3.*

$$\int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx = u(x)v(x).$$

Rearranging, we get the rule for **integration by parts**:

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

Thus, we can express an integral of the form $\int u'(x)v(x)\,dx$ as a known function $(u(x)v(x))$ minus another integral. The second integral may be easier than the first. And this is the point of this rather complicated-looking rule: we might get a simpler integral as a result of replacing one 'part' $u'(x)$ by its integral $u(x)$ and the other 'part' $v(x)$ by its derivative $v'(x)$.

Often, this rule is written in the shorthand form

$$\int f\,dg = fg - \int g\,df.$$

**Example:** Consider the integral

$$\int x \ln x\,dx.$$

Taking $u'(x) = x$ and $v(x) = \ln x$ in the integration by parts rule, we have

$$\begin{aligned}
\int u'(x)v(x)\,dx &= u(x)v(x) - \int u(x)v'(x)\,dx \\
&= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \frac{1}{x}\,dx \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c.
\end{aligned}$$

**Example:** You might have wondered why, although $\ln x$ is a very important function, it does not feature in our list of standard integrals. The reason is that the integral of $\ln x$ is not particularly easy to remember. There is a rather 'sneaky' way of finding it, using integration by parts. The integral $\int \ln x\,dx$ may be thought of as the integral of $1 \times \ln x$. Taking $u'(x) = 1$ and $v(x) = \ln x$ (so $u(x) = x$ and $v'(x) = 1/x$) and using integration by parts, we have

$$\int \ln x\,dx = x \ln x - \int x\frac{1}{x}\,dx = x\ln x - \int 1\,dx = x \ln x - x + c.$$

**Activity 4.9** *Use integration by parts to find $\int xe^x\,dx$.*

## Partial fractions

This is a way of rewriting integrands of the form $p(x)/q(x)$, where $p$ and $q$ are polynomials, in a simpler form which makes them easier to integrate.[6] Here is an example.

[6] *See Anthony and Biggs, Section 26.4.*

**Example:** Consider

$$\int \frac{x}{x^2 - x - 2}\,dx.$$

The integrand is of the form $p(x)/q(x)$, where $p(x) = x$ and $q(x) = x^2 - x - 2$. Further, $q(x)$ factorises as $(x+1)(x-2)$. We claim that we can find constants $A_1$

**Activity 4.10** *Find* $\displaystyle\int \frac{dx}{x^2+4x+3}$.

## Applications of integration

We have seen that marginals are derivatives; for instance, the marginal cost $MC$ is the derivative (with respect to quantity) of the total cost function $TC$. This means that if we are given the marginal cost and we want to find the total cost function then we have to reverse the previous procedure, by integrating. However, we shall also need some additional information, since we know that when we integrate we have a constant of integration which has to be determined. Often this information is provided to us by the fixed cost, which, you will recall, is the cost $TC(0)$ of producing no units. The following example illustrates this.

**Example:** Suppose that the marginal cost function is given by $MC(q) = 2e^{0.5q}$ and that the fixed cost is 20. Then the total cost function is the integral of the marginal cost:

$$TC(q) = \int 2e^{0.5q}\,dq = 4e^{0.5q} + c,$$

for some constant $c$. To determine $c$, we use the fact that when $q = 0$, the cost $TC(0)$ must equal the fixed cost, 20; thus, $4e^0 + c = 20$, $4 + c = 20$ and so $c = 16$, and $TC(q) = 4e^{0.5q} + 16$. (An extremely common mistake in a problem of this type is to assume that the constant equals the fixed cost, which, as we see in this example, need not be the case. Beware!)

**Activity 4.11** *Find the total cost function if the marginal cost is* $q + 5q^2 + e^q$ *and the fixed cost is* 10.

## Learning outcomes

At the end of this chapter and the relevant reading, you should be able to:

- explain what is meant by an (indefinite) integral and a definite integral
- state and use the standard integrals
- use integration by substitution
- use integration by parts
- integrate using partial fractions
- calculate functions from their marginals

## Sample examination/practice questions

**1.** Determine the following integral:

$$\int \frac{x}{x^2+5x+6}\,dx$$

**2.** Evaluate $\int_1^2 x^2(x-1)^{1/2}\,dx$ using an appropriate substitution.

**3.** Determine $\int \frac{2x+3}{x^2+3x+2}\,dx$.

**4.** Determine $\int_1^e \frac{\ln x\sqrt{\ln x}}{x}\,dx$.

**5.** A company produces only product XYZ. When producing $Q$ units the marginal cost $MC$ is given by

$$MC = 1 - \frac{1}{(Q+1)^2}.$$

If the average cost per unit when producing 4 units is 3.05, what is the total cost of producing 5 units of XYZ?

**6.** A company's marginal cost function is

$$MC = 32 + 18q - 12q^2.$$

Its fixed cost is 43. Determine the firm's total cost function, average cost function, and variable cost.

**7.** The marginal revenue function for a commodity is given by

$$MR = 10 - 2x^2$$

and the total cost function for the commodity is

$$TC = x^2 + 4x + 2,$$

where $x$ is the number of units produced. Find the revenue function, and determine the maximal profit.

**8.** For a particular company, the marginal cost is a function of output as follows:

$$MC = 10 - q + q^2.$$

Determine the extra cost which is incurred when production is increased from 2 to 4.

**9.** A firm's marginal cost function is

$$\frac{20}{\sqrt{Q}}e^{\sqrt{Q}} + Q^3 + \frac{1}{Q+1}.$$

The firm's fixed costs are 20. Determine the total cost function.

## Answers to activities

**4.1** This is one of the standard derivatives, and the answer is $x^6/6 + c$.

**4.2** We know that the function is an anti-derivative of $x^2 + 2\sin x$. Integrating this, by the standard integrals and the rules just seen, we have

$$\int (x^2 + 2\sin x)\,dx = \frac{x^3}{3} - 2\cos x + c,$$

where $c$ is a constant of integration. But we know more about $f$: we know that $f(0) = 1$, so we know that

$$f(x) = \frac{x^3}{3} - 2\cos x + c,$$

where the constant $c$ is such that $f(0) = 1$. Now, substituting $x = 0$ into the expression for $f$, we have

$$f(0) = 0 - 2\cos 0 + c = -2 + c,$$

and for this to equal 1, $c$ must be 3. Therefore $f(x) = x^3/2 - 2\cos x + 3$.

**4.3** The integral $\int x^2\,dx$ is $x^3/3 + c$, so

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - 0 = \frac{8}{3}.$$

**4.4** The fact that this involves variable $t$ rather than $x$ should not confuse us. We have

$$\int_{-1}^{1} e^t\,dt = \left[e^t\right]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}.$$

**4.5** With $u = x^2$ we have $du = 2x\,dx$ and so $dx = du/(2x)$. Therefore

$$\int xe^{x^2}\,dx = \int xe^u(du/(2x)) = (1/2)\int e^u\,du = (1/2)e^u + c = (1/2)e^{x^2} + c.$$

A slightly quicker approach to making this substitution is to note that since $du = 2x\,dx$ and the integral already has $x\,dx$, we have $\int xe^{x^2}\,dx = (1/2)\int e^u\,du$. It amounts to the same thing.

**4.6** We make the substitution $u = (2x^2 + 2)$. We have $du = 4x\,dx$, so the integral reduces to

$$\int \frac{1}{4}u^{1/2}\,du = \frac{1}{4}\frac{2u^{3/2}}{3} + c = \frac{1}{6}(2x^2 + 2)^{3/2} + c.$$

**4.7** With $u = x - 1$, we have $du = dx$ and so, on noting that $x = u + 1$, the integral becomes

$$\begin{aligned}
\int x\sqrt{u}\,dx &= \int (u+1)\sqrt{u}\,du \\
&= \int (u+1)u^{1/2}\,du \\
&= \int (u^{3/2} + u^{1/2})du \\
&= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c \\
&= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c.
\end{aligned}$$

Now we try the second suggested substitution. Setting $u = \sqrt{x-1}$ we have

$$\frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{x-1}} = \frac{1}{2u},$$

and so $dx = 2u\,du$. The integral becomes $\int xu(2u)du$ and we need to replace $x$ by its expression in terms of $u$. We have $u = \sqrt{x-1}$ and so $u^2 = x - 1$, from which we obtain $x = u^2 + 1$. So the integral is

$$\begin{aligned}\int (u^2+1)u(2u)\,du &= 2\int (u^4+u^2)\,du \\ &= 2\left(\frac{u^5}{5}+\frac{u^3}{3}\right)+c \\ &= \frac{2}{5}\left(\sqrt{x-1}\right)^5 + \frac{2}{3}\left(\sqrt{x-1}\right)^3 + c.\end{aligned}$$

This is (of course!) the same as the answer obtained using the first substitution. But which is easier? Well, the details of the substitution are easier for the first, but the actual integration of the transformed integral is easier for the second (because it does not involve fractional powers). On balance, probably the first substitution is easier. But both are correct! Do not think there's necessarily only one way to solve a problem.

**4.8** We use the substitution $u = x^2 + 4x + 5$. Then $du = (2x+4)dx$ and so the integral is

$$\int \frac{(1/2)du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln(x^2+4x+5) + c.$$

**4.9** Recall that the integration by parts rule is

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

We need to integrate $xe^x$. If we were to take $u' = x$ and $v = e^x$ then $uv'$ would be $(1/2)x^2e^x$, so the integral on the right of the integration by parts equation would be even more difficult than the one we started with. There is, though, another possibility: we can take $u' = e^x$ and $v = x$, in which case $u = e^x$ and $v' = 1$. Then we have

$$\int xe^x\,dx = xe^x - \int 1.e^x\,dx = xe^x - e^x + c.$$

**4.10** The integrand is

$$\frac{1}{x^2+4x+3} = \frac{1}{(x+1)(x+3)},$$

and the partial fractions rule says that

$$\frac{1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$

for some numbers $A$ and $B$. Multiplying both sides by $(x+1)(x+3)$, we obtain

$$1 = A(x+3) + B(x+1).$$

Taking $x = -3$ gives $-2B = 1$, so $B = -1/2$. Taking $x = -1$ gives $2A = 1$, so $A = 1/2$. The integral is therefore

$$\frac{1}{2}\int\left(\frac{1}{x+1} - \frac{1}{x+3}\right)dx = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + c.$$

**4.11** We have

$$TC(q) = \int MC\,dq = \int (q + 5q^2 + e^q)dq = \frac{q^2}{2} + \frac{5q^3}{3} + e^q + c.$$

We know that $TC(0) = FC = 10$, so $0 + 0 + e^0 + c = 10$; in other words, $1 + c = 10$ and $c = 9$. Therefore the total cost function is $TC = \dfrac{q^2}{2} + \dfrac{5q^3}{3} + e^q + 9$.

## Answers to sample examination/practice questions

**1.** Noting that

$$\frac{x}{x^2 + 5x + 6} = \frac{x}{(x+2)(x+3)},$$

we use partial fractions. We have

$$\frac{x}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

for some numbers $A$ and $B$. Multiplying both sides by both factors in the usual way, we have

$$A(x+3) + B(x+1) = x.$$

Taking $x = -3$ gives $-2B = -3$, so $B = 3/2$. Taking $x = -1$ we get $2A = -1$, so $A = -1/2$. Hence

$$\int \frac{x}{x^2 + 5x + 6}\,dx = \int \left( \frac{(-1/2)}{x+2} + \frac{(3/2)}{x+3} \right) dx = -\frac{1}{2} \ln|x+2| + \frac{3}{2} \ln|x+3| + c.$$

**2.** Let us try $u = x - 1$. We have $du = dx$. When $x = 1$, $u = 0$ and when $x = 2$, $u = 1$. Furthermore, since $x = u + 1$, we may write $x^2$ as $(u+1)^2$. The integral therefore becomes

$$\begin{aligned}
\int_0^1 (u+1)^2 u^{1/2}\,du &= \int_0^1 (u^2 + 2u + 1)u^{1/2}\,du \\
&= \int_0^1 (u^{5/2} + 2u^{3/2} + u^{1/2})\,du \\
&= \left[ \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} \right]_0^1 \\
&= \frac{2}{7} + \frac{4}{5} + \frac{2}{3} \\
&= \frac{184}{105}.
\end{aligned}$$

**3.** For this integral, the substitution $u = x^2 + 3x + 2$ gives $\displaystyle\int \frac{du}{u}$, so the integral is $\ln|u| + c = \ln|x^2 + 3x + 2| + c$. An alternative approach, however, is to use partial fractions, because the denominator factorises as $(x+1)(x+2)$. Partial fractions tells us that for some numbers $A$ and $B$,

$$\frac{2x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

In the usual way, we have

$$A(x+2) + B(x+1) = 2x + 3$$

for all $x$. Taking $x = -2$ reveals that $-B = -1$, so $B = 1$; and taking $x = -1$, we obtain $A = 1$. Therefore the integral equals

$$\int \left(\frac{1}{x+1} + \frac{1}{x+2}\right) dx = \ln|x+1| + \ln|x+2| + c.$$

This is the same answer as we obtained using substitution, because

$$\ln|x+1| + \ln|x+2| = \ln|(x+1)(x+2)| = \ln|x^2+3x+2|.$$

Note that here (again), there is more than one way to solve the problem.

**4.** With $u = \ln x$, we have $du = (1/x)dx$. Also, the values $x = 1$ and $x = e$ correspond to $u = 0$ and $u = 1$. So

$$\begin{aligned} \int_1^e \frac{\ln x\sqrt{\ln x}}{x}\, dx &= \int_0^1 u\sqrt{u}\, du \\ &= \int_0^1 u^{3/2}\, du \\ &= \left[\frac{2}{5}u^{5/2}\right]_0^1 \\ &= \frac{2}{5}. \end{aligned}$$

**5.** We know that

$$TC = \int MC\, dQ = \int \left(1 - \frac{1}{(Q+1)^2}\right) dQ = \int (1-(Q+1)^{-2})dQ = Q+(Q+1)^{-1}+c.$$

So, for some constant $c$,

$$TC = Q + \frac{1}{Q+1} + c.$$

Now, we know that the average cost when $Q = 4$ is 3.05, so the total cost when $Q = 4$ is $4(3.05) = 12.2$. But $TC(4) = 4 + (1/5) + c = 4.2 + c$, so we must have $c = 8$. So $TC = Q + 1/(Q+1) + 8$. When $Q = 5$ the total cost is therefore $5 + 1/6 + 8 = 79/6$.

It's useful, perhaps, to point out how **not** to answer this question. A naive approach might be to argue as follows: the total cost at $Q = 4$ is 12.2, and the marginal cost when $Q = 4$ is $1 - (1/4^2) = 15/16$. Since the marginal cost is the cost of producing one additional item, the cost of producing 5 is therefore $12.2 + 15/16 = 13.1375$. This is incorrect. Why? The reason is that the marginal cost gives, approximately, the cost of producing one more item, but that for this approximation to be good, increasing production by one item must be a relatively small increase. But increasing from 4 to 5 is a big relative change in production. If we were increasing from 400 to 401 (say), the approximation would be better. (Recall that the formal mathematical definition of marginal cost is that it is the derivative of the total cost, and that this is **approximately** the cost of producing one additional item.)

**6.** We have

$$TC = \int MC\, dq = \int (32 + 18q - 12q^2)dq = 32q + 9q^2 - 4q^3 + c,$$

and we know that the fixed cost, which is $TC(0)$, is 43, so $0 + 0 + 0 + c = 43$ and hence $TC = 32q + 9q^2 - 4q^3 + 43$. Then, the average cost is

$$AC = \frac{MC}{q} = 32 + 9q - 4q^2 + \frac{43}{q},$$

and

$$AVC = \frac{VC}{q} = \frac{TC - FC}{q} = \frac{32q + 9q^2 - 4q^3}{q} = 32 + 9q - 4q^2.$$

**7.** The total revenue is given by

$$TR = \int MR\, dx = \int (10 - 2x^2) dx = 10x - (2/3)x^3 + c.$$

But what should $c$ be? Well, think about it: what's the revenue from selling 0 items? It's 0, of course, so $TR(0) = 0$ and hence $c = 0$. Therefore $TR = 10x - (2/3)x^3$. The profit function is

$$\Pi = TR - TC = (10x - (2/3)x^3) - (x^2 + 4x + 2) = 6x - (2/3)x^3 - x^2 - 2.$$

Setting $\Pi'(x) = 0$ we obtain $6 - 2x^2 - 2x = 0$ or, equivalently, $x^2 + x - 3 = 0$. Solving this, we obtain $x = (-1 \pm \sqrt{13})/2$ and clearly, for it to have economic significance, it is the positive solution $(-1 + \sqrt{13})/2$ that is relevant. The second derivative $\Pi''$ is $-4x - 2$, which is negative here, so this gives a maximum. The maximum value of the profit is obtained by substituting this value into the profit function. This turns out to be 2.64536.

**8.** What we need here is $TC(4) - TC(2)$. Since $TC = \int MC\, dq$, we have

$$\begin{aligned}
TC(4) - TC(2) &= \int_2^4 MC\, dq \\
&= \int_2^4 (10 - q + q^2)\, dq \\
&= \left[10q - \frac{q^2}{2} + \frac{q^3}{3}\right]_2^4 \\
&= \left(10(4) - \frac{4^2}{2} + \frac{4^3}{3}\right) - \left(10(2) - \frac{2^2}{2} + \frac{2^3}{3}\right) \\
&= \frac{98}{3}.
\end{aligned}$$

(Alternatively, you could determine $TC$ by indefinite integration to start with, and then calculate $TC(4) - TC(2)$. Of course, the constant won't be known since we aren't told the fixed costs. But this does not matter, since in working out the difference $TC(4) - TC(2)$, the constant will cancel.)

**9.** We have

$$TC = \int MC\, dQ = \int \left(\frac{20}{\sqrt{Q}} e^{\sqrt{Q}} + Q^3 + \frac{1}{Q+1}\right) dQ.$$

Now, to determine the integral of $e^{\sqrt{Q}}/\sqrt{Q}$, we use the substitution $u = \sqrt{Q}$. We have $du = (1/(2\sqrt{Q}))dQ$ and

$$\int \frac{e^{\sqrt{Q}}}{\sqrt{Q}}\, dQ = \int 2e^u\, du = 2e^u + c = 2e^{\sqrt{Q}} + c.$$

So,

$$TC = 40e^{\sqrt{Q}} + \frac{Q^4}{4} + \ln|Q+1| + c,$$

Where

$$20 = FC = TC(0) = 40e^0 + 0 + \ln(1) + c = 40 + c,$$

so $c = -20$ and

$$TC = 40e^{\sqrt{Q}} + \frac{Q^4}{4} + \ln|Q+1| - 20 = 40e^{\sqrt{Q}} + \frac{Q^4}{4} + \ln(Q+1) - 20.$$

(We've used the fact that $Q$, as quantity, is non-negative to observe that $|Q+1| = Q+1$.)