Tutorial Sheet 6

1. For the vectors given below, evaluate the following expressions where it is possible.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -5 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

i) $2\mathbf{u} + 3\mathbf{v}$
ii) $3\mathbf{u} - \mathbf{v}$
iii) $\mathbf{x} + 3\mathbf{v}$
iv) $2\mathbf{z} - \mathbf{w}$
v) $\mathbf{u} + \mathbf{x}$
vi) $\mathbf{v} + \mathbf{w}$
vii) $\mathbf{u} \cdot \mathbf{v}$
viii) $(2\mathbf{u}) \cdot (3\mathbf{v})$
ix) $\mathbf{x} \cdot \mathbf{y}$
x) $\mathbf{w} \cdot \mathbf{z}$
xi) $\mathbf{w} \cdot (\mathbf{z} + \mathbf{w})$
xii) $|\mathbf{x}|$
xiii) $|\mathbf{w}|$
xiv) $|\mathbf{y}| + |\mathbf{w}|$

2. Calculate the angles between the pairs $\mathbf{u}, \mathbf{v}$, $\mathbf{x}, \mathbf{y}$, and $\mathbf{w}, \mathbf{z}$ from the previous question. Give your answers in both radians and degrees.

3. Calculate the area of the parallelogram spanned by the vectors $\mathbf{x}$ and $\mathbf{y}$.

4. Show that the volume of the parallelopiped spanned by the vectors $\mathbf{u}, \mathbf{v}$ and $2\mathbf{u} + 3\mathbf{v}$ is zero.

5. For the matrices below, evaluate the following expressions where it is possible.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 1 & -7 \end{bmatrix}, C = \begin{bmatrix} 3 & 2 & -2 \\ 4 & 8 & 2 \end{bmatrix}, D = \begin{bmatrix} 3 & 2 & -2 \\ 4 & 8 & 2 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, F = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 4 & 1 \\ 3 & 1 & 0 \end{bmatrix},$$

i) $2A + 3B$
ii) $3C - D$
iii) $8A + 4C$
iv) $2000A + 3000B$
v) $E - F$
vi) $\det(A) + \det(B)$
vii) $\det(A + B)$
viii) $\det(C)$
ix) $\det(E)$
x) $A\mathbf{x}$
xi) $B\mathbf{x}$
xii) $A\mathbf{y} + B\mathbf{x}$
xiii) $A\mathbf{u}$
xiv) $C\mathbf{x}$
xv) $C\mathbf{w}$
xvi) $E\mathbf{u}$
xvii) $E\mathbf{w} - \mathbf{F}\mathbf{w}$

Tutorial Sheet 7

1. For each of the following systems of linear equations, write down the corressponding coefficient matrix A, vector of unknowns $\mathbf{x}$, and vector of right hand sides $\mathbf{b}$ so that the system can be expressed in the form $A\mathbf{x} = \mathbf{b}$

   i)
   $$\begin{aligned} 2x + 3y &= 1 \\ 5x + 7y &= 3 \end{aligned}$$

   ii)
   $$\begin{aligned} 2x + 3y + 4z &= 1 \\ x - 2y + 2z &= 7 \\ 3x + 2y + z &= 0.2 \end{aligned}$$

   iii)
   $$\begin{aligned} 3x + y + z &= 1 \\ y + 4z &= -4 \\ x - y &= 2 \end{aligned}$$

2.

Let the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be given by

$$\mathbf{a} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

These vectors span a three dimensional parallelopiped as shown in the figure to the right.

i) Find the area of the parallelogram $S_{\mathbf{ab}}$ which is spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$. Hence state the area of the parallelogram $S'_{\mathbf{ab}}$ on the opposite side of the parrallelopiped.

ii) Find the areas of the parallelograms $S_{\mathbf{bc}}$ and $S_{\mathbf{ac}}$ spanned by the relevant pairs of vectors and hence find the total surface area of the parrallelopiped.

iii) Find the signed volume of the parrallelopiped.

3. Rotate the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ anti-clockwise $\frac{\pi}{4}$ radians about the origin.

4. Rotate the point $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ anti-clockwise $\frac{\pi}{4}$ radians about the point $\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

5. Rotate the line segment with endpoints $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ anti-clockwise $\frac{\pi}{2}$ radians about the point $\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Give the new endpoints $\mathbf{x}'$ and $\mathbf{y}'$ of the rotated line segment.