Mathematics for Physical Sciences III

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First topics:

1. Vector fields and vector calculus
2. Gradient, divergence and curl operators
3. Line integrals, surface integrals and Integral Theorems

Example: Maxwell's equations

Differential form	Integral form
$\nabla \cdot \mathbf{D} = \rho_f$	$\oiint_{\partial V} \mathbf{D} \cdot d\mathbf{A} = Q_f(V)$
$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0$
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} d\mathbf{l} = -\dfrac{\partial \Phi_S(\mathbf{B})}{\partial t}$
$\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_{\partial S} \mathbf{H} \cdot d\mathbf{l} = I_{f,S} + \dfrac{\partial \Phi_S(\mathbf{D})}{\partial t}$

Vector calculus

Vector calculus is concerned with differentiation and integration of vector fields, that is functions with many components, such as $\mathbf{F}(x,y) = (f(x,y), g(x,y))$ or $\mathbf{F}(t) = (f(t), g(t), h(t))$.

Review of vectors

A **vector in the plane** is a directed line segment (a quantity characterized by both magnitude and direction). Examples of vectors in physics include velocity, force, acceleration, etc.

For a general vector $\mathbf{v} = (x,y) = x\mathbf{i} + y\mathbf{j}$, the **magnitude** or **length** of $\mathbf{v}$ can be defined as

$$|\mathbf{v}| = \sqrt{x^2 + y^2}$$

Vector functions of one variable

A **vector function** or **vector field** of one variable is a rule that assigns a vector to each value of the variable. For example, when a particle moves through the plane during a time interval, each of the particle coordinates is a function of time: $x = x(t)$ and $y = y(t)$. The path of the particle is then described by the position vector

$$\mathbf{r}(t) = (x(t), y(t)) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

Example: In the case of a projectile launched from the origin (0,0), the path is described by the position vector

$$\mathbf{r}(t) = V_0 t \cos(\alpha)\,\mathbf{i} + \left[V_0 t \sin(\alpha) - \frac{1}{2} g t^2\right]\mathbf{j} \tag{1}$$

where V_0 is the initial (launch) velocity and α is the initial (launch) angle.

If $\mathbf{r}(t)$ is the position vector of a particle moving in the plane then we have

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \quad \text{is the } \textbf{velocity} \text{ of the particle;}$$

$$|\mathbf{v}(t)| \quad \text{is the } \textbf{speed} \text{ of the particle;}$$

$$\frac{\mathbf{v}}{|\mathbf{v}|} \quad \text{is the } \textbf{direction of motion}\text{;}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \quad \text{is the } \textbf{acceleration} \text{ of the particle.}$$

Note: To differentiate a vector function of one variable we simply differentiate each component with respect to that variable. For example, if $\mathbf{r}(t) = (x(t), y(t)) = x(t)\mathbf{i} + y(t)\mathbf{j}$ then

$$\frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}.$$

Exercise:

For the projectile motion defined by the position vector in Equation (1), calculate

1. The velocity, speed and direction of motion of the particle at each moment of time t;
2. The acceleration of the particle
3. The maximum height, flight time and range of the motion.

Review of some vector operations

Consider two vectors in 3-dimensional space: $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. We define

- The **dot product** or **inner product** of **A** and **B** is

$$\mathbf{A} \cdot \mathbf{B} = A_1 B_1 + A_2 B + A_3 B_3 = |\mathbf{A}| \cdot |\mathbf{B}| \cos(\theta)$$

  where θ is the angle between the vectors.

- The **cross product** of **A** and **B** is

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}| \cdot |\mathbf{B}| \sin(\theta) \cdot \mathbf{n} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}$$

  where **n** is the unit vector in the direction which is perpendicular to the plane formed by **A** and **B**.

Properties of the dot and cross products

1. The dot product of two vectors is a scalar (number) while the cross product is a vector;
2. If the dot product of two vectors is zero then the two vectors are perpendicular (the angle between them is 90°);
3. If the cross product of two vectors is zero then the vectors are parallel (the angle between them is 0°);
4. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ but $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.
5. $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$
6. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Example: Let $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Calculate $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$. Same problem for $\mathbf{A} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j}$.

Triple products

A triple product is a product of the form

$$(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}, \qquad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \quad \text{or} \quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

Scalar triple product:
If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ and $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ then the **scalar triple product** or **box product** is defined as:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \det\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

and

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \equiv [ABC]$$

The vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ has the properties

1. $$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

2. $$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$
$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$$

Exercises

1. If $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{C} = 4\mathbf{j} - 3\mathbf{k}$ find
    1. $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$
    2. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.
2. If $\mathbf{A} = x^2\mathbf{i} - y\mathbf{j} + xz\mathbf{k}$, $\mathbf{B} = y\mathbf{i} + x\mathbf{j} - xyz\mathbf{k}$ and $\mathbf{C} = \mathbf{i} - y\mathbf{j} + x^3 z\mathbf{k}$ find
    1. $$\frac{\partial^2}{\partial x \partial y}(\mathbf{A} \times \mathbf{B})$$
    2. $$\frac{\partial}{\partial x}(\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))$$

    at the point $(1, -1, 2)$.

Gradient, Divergence and Curl

First, we define the vector operator ∇ (del) as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

1. The **gradient** of a scalar function $\Phi(x, y, z)$ is

$$\text{grad}(\Phi) = \nabla\Phi = \mathbf{i}\frac{\partial\Phi}{\partial x} + \mathbf{j}\frac{\partial\Phi}{\partial y} + \mathbf{k}\frac{\partial\Phi}{\partial z}$$

2. The **divergence** of a vector field $\mathbf{A}(x, y, z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ is

$$\text{div}(\mathbf{A}) = \nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

3. The **curl** of $\mathbf{A}$ is

$$\text{curl}(\mathbf{A}) = \nabla \times \mathbf{A} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{pmatrix}$$

The curl of a vector can also be calculated with the formula

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \mathbf{k}$$

Definition: The expression

$$\nabla \cdot (\nabla \Phi) = \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

is called the **Laplacian** of Φ.

Exercise: If $\Phi = x^2 y z^3$ and $\mathbf{A} = xz\mathbf{i} - y^2\mathbf{j} + 2x^2 y\mathbf{k}$ find

(a) $\nabla \Phi$; (b) $\nabla^2 \Phi$; (c) $\nabla \cdot \mathbf{A}$; (d) $\nabla \times \mathbf{A}$; (e) $\text{div}(\Phi \mathbf{A})$; (f) $\text{curl}(\Phi \mathbf{A})$.

Properties of the ∇ operator

1. $\nabla(\Phi_1 + \Phi_2) = \nabla\Phi_1 + \nabla\Phi_2$ (gradient of sum = sum of gradients)
2. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$ (divergence of sum = sum of divergences)
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$ (curl of sum = sum of curls)
4. $\nabla \cdot (\Phi\mathbf{A}) = (\nabla\Phi) \cdot \mathbf{A} + \Phi(\nabla \cdot \mathbf{A})$ (product rule)
5. $\nabla \times (\Phi\mathbf{A}) = (\nabla\Phi) \times \mathbf{A} + \Phi(\nabla \times \mathbf{A})$ (product rule)
6. $\nabla \times (\nabla\Phi) = 0$ (the curl of the gradient is zero)
7. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ (the divergence of the curl is zero)
8. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

Properties of the gradient

If $\Phi(x, y) = c$ is a curve in the plane then the vector $\nabla\Phi$ is perpendicular to the curve at each point.

Example: If $\Phi(x, y) = x^2 + y^2$ then the vector $\nabla\Phi = (2x, 2y)$ is perpendicular to the circle $x^2 + y^2 = 1$ at each point (x, y).

Similarly, if $\Phi(x, y, z) = c$ is a 3-dimensional surface, then $\nabla\Phi$ is perpendicular to the surface at each point.

Directional derivative

Let $\mathbf{u}$ be a vector (a direction) and $\Phi(x, y, z)$ be a scalar function. The directional derivative of Φ in the direction of $\mathbf{u}$ is defined as the dot product of the gradient vector and the direction vector

$$D_{\mathbf{u}}(\Phi) = \nabla\Phi \cdot \mathbf{u}$$

Example: The directional derivative of $\Phi(x, y) = x^2y + y^3$ at the point $(1, 1)$ in the direction of $\mathbf{u} = \mathbf{i} - \mathbf{j}$ is

$$D_{\mathbf{u}}(\Phi)(1, 1) = \nabla\Phi(1, 1) \cdot \mathbf{u} = (2\mathbf{i} + 4\mathbf{j}) \cdot (\mathbf{i} - \mathbf{j}) = 2 \cdot 1 + 4 \cdot (-1) = -2.$$

Direction of maximal increase:

The gradient of a function $\Phi(x, y, z)$ at a point (x_0, y_0, z_0) defines the direction in which the function increases most rapidly at that point. The rate of maximum increase at that point is given by $|\nabla\Phi(x_0, y_0, z_0)|$.

Similarly, the function decreases most rapidly in the direction of $-\nabla\Phi$. The rate of maximum decrease at that point is given by $-|\nabla\Phi(x_0, y_0, z_0)|$. Any direction perpendicular to the gradient is a direction of zero change for the function.

Example: Find the directions of maximal increase, maximal decrease and zero change for the function $\Phi(x, y) = x^2 + y^2$ at $(1, 1)$.