Chapter 1 - Functions

Topic: Limits
Week 1
Presented by Páraic Treacy

Limits Notation

$$\lim_{x \to c} f(x)$$

"The limit of $f(x)$ as x approaches c"

A limit looks at the behaviour of $f(x)$ as x gets closer to c.

Dividing by Zero and Infinity

Any real number divided by ∞ will be equal to zero

e.g. $\frac{10}{\infty} = 0$ $\quad -\frac{597}{\infty} = 0$

Any real number divided by zero will be equal to ∞

e.g. $\frac{17}{0} = \infty$ $\quad -\frac{24}{0} = \infty$

Solving Limits – Ex. 1

Solve the following limit:

$$\lim_{x \to 0} \frac{x^2 + 1}{x}$$

Solution:

$$= \frac{(0)^2 + 1}{0}$$

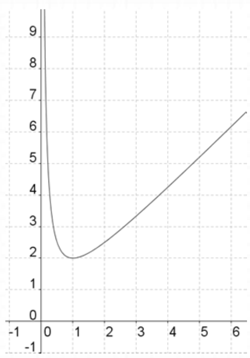
$$= \frac{1}{0} = \infty$$

$$\lim_{x \to 0} \frac{x^2 + 1}{x} = \infty$$

Solving Limits – Ex. 1

$$\lim_{x \to 0} \frac{x^2 + 1}{x} = \infty$$

As the x value approaches 0, the corresponding $f(x)$ value (or y value) approaches ∞.



Factorising Formulae

Difference of two squares:

$$x^2 - a^2 = (x - a)(x + a)$$

Difference of two cubes:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

Sum of two cubes:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Undefined Answers

Answers such as $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are not defined

If the answer obtained for a limit is one of these then a different approach needs to be applied…

Solving Limits – Ex. 2

Solve the following limit:

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \frac{(2)^3-8}{2-2} = \frac{0}{0}$$

This answer is not defined… Need to figure out a way to get a defined answer.

Solving Limits – Ex. 2

Solution: Factorise the numerator

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}$$

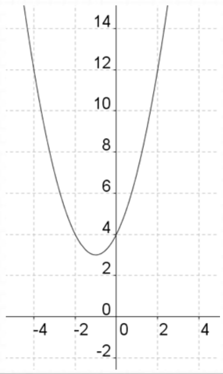
$$= \lim_{x\to 2} x^2+2x+4$$

$$= (2)^2+2(2)+4 = 12$$

Solving Limits – Ex. 2

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = 12$$

As the x value approaches 2, the corresponding $f(x)$ value (or y value) approaches 12.



Steps for when an answer of $\frac{0}{0}$ is obtained from a limit

1. Start problem again
2. Factorise the top and/or bottom line of the function (numerator and denominator) if possible.
3. Cancel any common factors i.e. divide above and below by the common factor.
4. Find the limit of the function i.e. sub for x in the function with the value it is approaching.

Solving Limits – Ex. 3

Solve the following limit:

$$\lim_{x\to 5}\frac{x^2-3x-10}{x-5} = \frac{(5)^2-3(5)-10}{5-5} = \frac{0}{0}$$

This answer is not defined…

Solving Limits – Ex. 3

Solution: Factorise the numerator

$$\lim_{x\to 5}\frac{x^2-3x-10}{x-5}=\lim_{x\to 5}\frac{(x-5)(x+2)}{x-5}$$

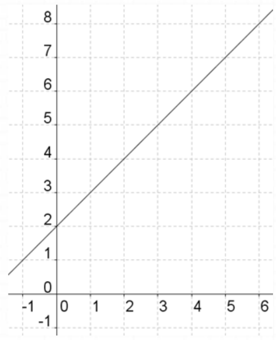
$$=\lim_{x\to 5} x+2$$

$$=5+2=7$$

Solving Limits – Ex. 3

$$\lim_{x\to 5}\frac{x^2-3x-10}{x-5}=7$$

As the x value approaches 5, the corresponding $f(x)$ value (or y value) approaches 7.



Solving Limits – Ex. 4

Solve the following limit:

$$\lim_{n\to\infty}\frac{3}{\sqrt{n^2+3}}$$

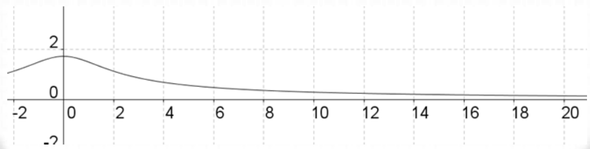
$$=\frac{3}{\sqrt{(\infty)^2+3}}=\frac{3}{\infty}=0$$

$$\lim_{n\to\infty}\frac{3}{\sqrt{n^2+3}}=0$$

Solving Limits – Ex. 4

$$\lim_{n\to\infty}\frac{3}{\sqrt{n^2+3}}=0$$

As the n value approaches infinity, the corresponding $f(n)$ value approaches 0.



Solving Limits – Ex. 5

Solve the following limit:

$$\lim_{x\to\infty}\frac{x^2+1}{x^3-3}$$

$$=\frac{(\infty)^2+1}{(\infty)^3-3}=\frac{\infty}{\infty}$$

Not defined…

Solving Limits – Ex. 5

Solution: divide above and below by the highest power of x in the function.

$$\lim_{x\to\infty}\frac{x^2+1}{x^3-3}=\lim_{x\to\infty}\frac{\frac{x^2}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}-\frac{3}{x^3}}$$

$$\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^3}}{1-\frac{3}{x^3}}$$

Solving Limits – Ex. 5

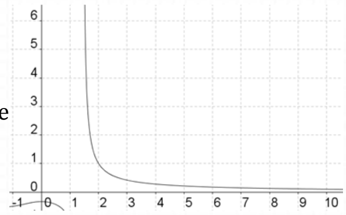
$$\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^3}}{1-\frac{3}{x^3}} = \frac{\frac{1}{\infty}+\frac{1}{(\infty)^3}}{1-\frac{3}{(\infty)^3}}$$

$$\frac{0+0}{1-0} = \frac{0}{1} = 0$$

Solving Limits – Ex. 5

$$\lim_{x\to\infty} \frac{x^2+1}{x^3-3} = 0$$

As the x value approaches infinity, the corresponding $f(x)$ value (or y value) approaches 0.



Steps for evaluating limits as x approaches infinity

1. Sub for x in the function with ∞. Usually (but not always) this will give an answer of $\frac{\infty}{\infty}$ which is not defined.
2. Start the problem again, this time divide above and below by the highest power of x in the function.
3. Sub for x with ∞ and complete your calculations – this should give a defined answer. Remember ∞ is a defined answer, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are not defined answers.

Summary

1. Answer of $\frac{0}{0}$ is not defined. Solution: factorise the function.
2. Answer of $\frac{\infty}{\infty}$ is not defined. Solution: divide above and below by highest power of x in the function.
3. $\frac{Real\ Number}{0} = \infty$
4. $\frac{Real\ Number}{\infty} = 0$

Drug Concentration

The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{0.17h}{h^2+2}$$

Find and interpret:

$$\lim_{h\to\infty} A(h)$$

Drug Concentration

$$\lim_{h\to\infty} A(h) = \lim_{h\to\infty} \frac{0.17h}{h^2+2}$$

$$\lim_{h\to\infty} \frac{0.17h}{h^2+2} = \lim_{h\to\infty} \frac{0.17(\infty)}{(\infty)^2+2} = \frac{\infty}{\infty}$$

Not defined...

$$\lim_{h\to\infty} \frac{0.17h}{h^2+2} = \lim_{h\to\infty} \frac{\frac{0.17}{h}}{1+\frac{2}{h^2}}$$

4

Drug Concentration

$$\lim_{h\to\infty} \frac{\frac{0.17}{h}}{1+\frac{2}{h^2}} = \frac{\frac{0.17}{\infty}}{1+\frac{2}{(\infty)^2}}$$

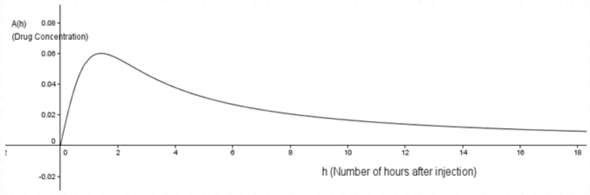
$$\frac{0}{1+0} = 0$$

So, as $h \to \infty$, $A(h) \to 0$.

This means that, as number of hours after the drug was injected h increases toward infinity, the concentration of the drug $A(h)$ will approach zero.

Drug Concentration

As number of hours after the drug was injected h increases, the concentration of the drug $A(h)$ will approach zero.



Piecewise Functions

A Piecewise Function is a function which is defined by multiple subfunctions, each subfunction applying to a certain interval of the main functions domain (a subdomain).

Example:

$$f(x) = \begin{cases} (x-2)^2, & x < 3 \\ x-1, & x \geq 3 \end{cases}$$

Example of a Real Life Piecewise Function

The state charges companies an annual fee of €20 per ton for each ton of pollution emitted by that company into the atmosphere, up to a maximum of 4,000 tons.

No additional fees are charged for emissions beyond the 4,000 ton limit.

(a) Write a piecewise definition of the fees $f(x)$ charged for the emission of x tons of pollution in a year.

(b) What is the limit of $f(x)$ as x approaches 4,000 tons?

(c) What is the limit of $f(x)$ as x approaches 8,000 tons?

Example of a Real Life Piecewise Function

(a)

$$f(x) = \begin{cases} 20x & x \leq 4{,}000 \\ 80{,}000 & x > 4{,}000 \end{cases}$$

(b) What is the limit of $f(x)$ as x approaches 4,000 tons?

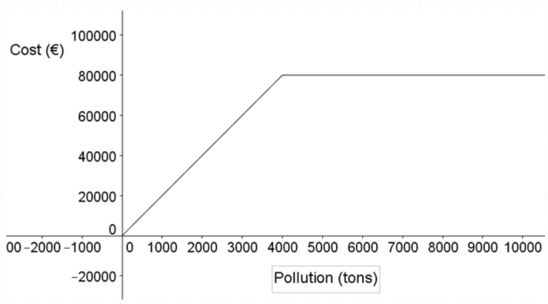
$$\lim_{x\to 4{,}000} f(x)$$

$$= \lim_{x\to 4{,}000} 20x$$

$$= 20(4{,}000) \quad = \quad €80{,}000$$

As the amount of pollution approaches 4,000 tons, the cost to the company will approach €80,000.

Example of a Real Life Piecewise Function



5

Example of a Real Life Piecewise Function

(c) What is the limit of $f(x)$ as x approaches 8,000 tons?

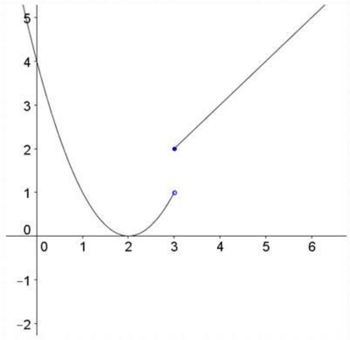
$$\lim_{x \to 8{,}000} f(x)$$

$$\lim_{x \to 8{,}000} 80{,}000$$

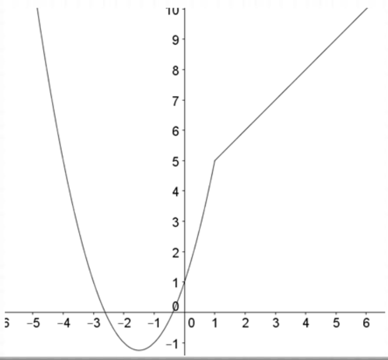
$$= €80{,}000$$

As the amount of pollution approaches 8,000 tons, the cost to the company will approach €80,000.

$$f(x) = \begin{cases} (x-2)^2, & x < 3 \\ x - 1, & x \geq 3 \end{cases}$$



$$g(x) = \begin{cases} x^2 + 3x + 1, & x < 1 \\ x + 4, & x \geq 1 \end{cases}$$



Continuity

What's the difference between the previous two examples of piecewise functions?

In the first example, there is a gap in the graph between the two subfunctions. This indicates that $f(x)$ is not continuous at $x = 3$.

In the second example, there is no gap in the graph between the two subfunctions. This indicates that $g(x)$ is continuous at $x = 1$ and, indeed, everywhere else.

How can we check continuity without drawing out the graph?

A function $f(x)$ is continuous at a point c if:

i. $f(c)$ exists

ii. $\lim_{x \to c} f(x)$ exists

iii. $f(c) = \lim_{x \to c} f(x)$

The first aspect we check (i) is relatively straightforward, the second aspect (ii) requires a little more work…

How to check if $\lim_{x \to c} f(x)$ exists

$\lim_{x \to c} f(x)$ exists if:

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

We find the limit of the function $f(x)$ as x approaches c from the right hand side (c^+) and we find the limit of the function $f(x)$ as x approaches c from the left hand side (c^-).

If these two are equal then $\lim_{x \to c} f(x)$ exists and is equal to the value found for the $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$.

Continuous Functions Ex. 1

In the following piecewise function:

$$f(x) = \begin{cases} (x-2)^2, & x < 3 \\ x - 1, & x \geq 3 \end{cases}$$

Is $f(x)$ continuous at $x = 3$?

<u>Solution</u>: First, find $f(3)$:

$$f(3) = 3 - 1 = 2$$

Next, find if $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

Continuous Functions Ex. 1

$$f(x) = \begin{cases} (x-2)^2, & x < 3 \\ x - 1, & x \geq 3 \end{cases}$$

Find: $\lim_{x\to 3^+} f(x)$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x - 1$$

$$\lim_{x\to 3^+} x - 1 = 3 - 1$$

$$\lim_{x\to 3^+} x - 1 = 2$$

Find: $\lim_{x\to 3^-} f(x)$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x-2)^2$$

$$\lim_{x\to 3^-} (x-2)^2 = (3-2)^2$$

$$\lim_{x\to 3^-} (x-2)^2 = 1$$

Continuous Functions Ex. 1

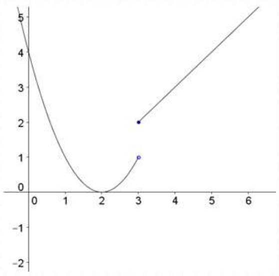
It is clear from the answers we obtained that:

$$\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$$

Thus $\lim_{x\to 3} f(x)$ does not exist.

So: $f(3) \neq \lim_{x\to 3} f(x)$

Thus the function $f(x)$ is <u>not continuous</u> at $x = 3$



Continuous Functions Ex. 2

In the following piecewise function:

$$g(x) = \begin{cases} x^2 + 3x + 1, & x < 1 \\ x + 4, & x \geq 1 \end{cases}$$

Is $g(x)$ continuous at $x = 1$?

Solution: First, find $g(1)$:

$$g(1) = 1 + 4 = 5$$

Next, find if $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x)$

Continuous Functions Ex. 2

$$g(x) = \begin{cases} x^2 + 3x + 1, & x < 1 \\ x + 4, & x \geq 1 \end{cases}$$

Find: $\lim_{x\to 1^+} g(x)$

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} x + 4$$

$$\lim_{x\to 1^+} x + 4 = 1 + 4$$

$$\lim_{x\to 1^+} x + 4 = 5$$

Find: $\lim_{x\to 1^-} g(x)$

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x^2 + 3x + 1$$

$$\lim_{x\to 1^-} x^2 + 3x + 1 = (1)^2 + 3(1) + 1$$

$$\lim_{x\to 1^-} x^2 + 3x + 1 = 5$$

Continuous Functions Ex. 2

It is clear from the answers we obtained that:

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x)$$

Thus $\lim_{x\to 1} g(x)$ exists and

$$\lim_{x\to 1} g(x) = 5$$

Earlier, we found that $g(1) = 5$

So:

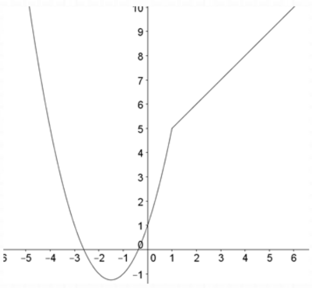
$$g(1) = \lim_{x\to 1} g(x)$$

Thus the function $g(x)$ is <u>continuous</u> at $x = 1$

7

Continuous Functions Ex. 2

Conclusion: The function $g(x)$ is continuous at $x = 1$



Continuous Functions Ex. 3

In the following piecewise function:

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Is $f(x)$continuous everywhere?

Solution: The function changes at $x = 2$ so we will check the continuity at that point. First, find $f(2)$ and eventually check if it equals $\lim_{x \to 2} f(x)$

$$f(2) = 1$$

Next, find if $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

Continuous Functions Ex. 3

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Find: $\lim_{x \to 2^+} f(x)$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x^2 - x - 2}{x - 2}$$

$$= \frac{(2)^2 - (2) - 2}{2 - 2} = \frac{0}{0}$$

Not defined.

$$\lim_{x \to 2^+} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{(x - 2)(x + 1)}{x - 2}$$

$$= \lim_{x \to 2^+} x + 1$$

$$= 3$$

Continuous Functions Ex. 3

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Find: $\lim_{x \to 2^-} f(x)$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - x - 2}{x - 2}$$

$$= \frac{(2)^2 - (2) - 2}{2 - 2} = \frac{0}{0}$$

Not defined.

$$\lim_{x \to 2^-} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{(x - 2)(x + 1)}{x - 2}$$

$$= \lim_{x \to 2^-} x + 1$$

$$= 3$$

Continuous Functions Ex. 3

It is clear from the answers we obtained that:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$$

Thus $\lim_{x \to 2} f(x)$ exists:

$$\lim_{x \to 2} f(x) = 3$$

We found earlier that

$$f(2) = 1$$

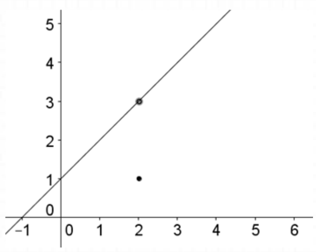
So:

$$\lim_{x \to 2} f(x) \neq f(2)$$

Thus, the piecewise function is not continuous at $x = 2$

Continuous Functions Ex. 3

Conclusion: The function is not continuous everywhere as it is not continuous at $x = 2$



8

Summary of checking Continuity of a piecewise function

A piecewise function is continuous at the point $x = c$ if

$$f(c) = \lim_{x \to c} f(x)$$

These are typically the steps taken:

1. Find $f(c)$
2. Check whether $\lim_{x \to c} f(x)$ exists. If $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$ then $\lim_{x \to c} f(x)$ exists and is equal to the answer obtained for both $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$.
3. Check whether $f(c) = \lim_{x \to c} f(x)$

Organic Fertilizer

A company charges €1.20 per kg for organic fertilizer on orders not over 100 kg and €1.10 per kg for orders over 100 kg. Let $f(x)$ represent the cost for buying x kg of organic fertilizer.

(a) Find the cost of buying the following:

i. 80 kg
ii. 150 kg
iii. 100 kg

(b) Is $f(x)$ continuous everywhere? What can you conclude from the answer to this question?

Organic Fertilizer

$$f(x) = \begin{cases} 1.2x, & x \leq 100 \\ 1.1x & x > 100 \end{cases}$$

(i) Cost of buying 80 kg of organic fertilizer:

$$f(80) = 1.2(80)$$

$$f(80) = 96$$

The cost of 80 kg of organic fertilizer will be €96.

(ii) Cost of buying 150 kg of organic fertilizer:

$$f(150) = 1.1(150)$$

$$f(150) = 165$$

The cost of 150 kg of organic fertilizer will be €165.

Organic Fertilizer

(iii) Cost of buying 100 kg of organic fertilizer:

$$f(100) = 1.2(100)$$

$$f(100) = 120$$

The cost of 100 kg of organic fertilizer will be €120.

Organic Fertilizer

Is $f(x)$continuous everywhere?

<u>Solution</u>: The function changes at $x = 100$ so we will check the continuity at that point. First, find $f(100)$ and eventually check if it equals $\lim_{x \to 100} f(x)$

$f(100) = 120$ [Solved this previously]

Next, find if $\lim_{x \to 100^+} f(x) = \lim_{x \to 100^-} f(x)$

Organic Fertilizer

$$\lim_{x \to 100^+} f(x)$$

$$\lim_{x \to 100^+} 1.1x$$

$$= 1.1(100)$$

$$= 110$$

$$\lim_{x \to 100^-} f(x)$$

$$\lim_{x \to 100^-} 1.2x$$

$$= 1.2(100)$$

$$= 120$$

$$\lim_{x \to 100^+} f(x) \neq \lim_{x \to 100^-} f(x)$$

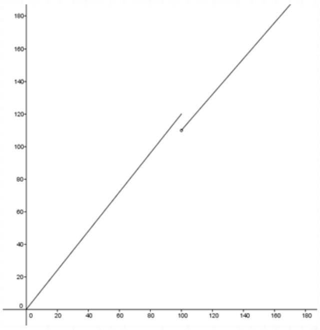
9

Organic Fertilizer

$\lim_{x \to 100^+} f(x) \neq \lim_{x \to 100^-} f(x)$

Thus, $\lim_{x \to 100} f(x)$ does <u>not</u> exist and we can conclude that $f(x)$ is not continuous at $x = 100$

This would indicate that the price of this fertilizer changes irregularly when the weight of the order is around 100 kg e.g. 98 kg will cost the same as 106.9 kg.



10

Trigonometric Functions and the Unit Circle

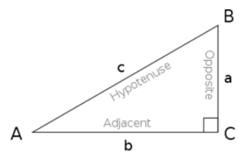
Presented by Páraic Treacy
MA4702 – Week 1

Where do Sine and Cosine come from?

We know:

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}.$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}.$$

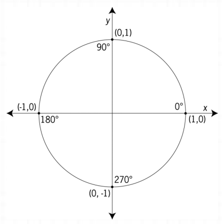


This can be explained using the unit circle.

Interactive Unit Circle

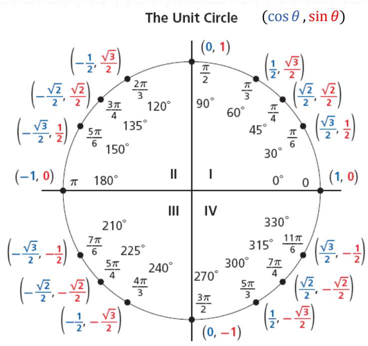
The Unit Circle is a circle with a radius of 1.

The centre of the Unit Circle is at the point (0, 0) on the Cartesian plane.



How it works:
http://www.mathsisfun.com/algebra/trig-interactive-unit-circle.html

The Unit Circle



This explains why…

In the log tables, it is stated that:

$$\cos(-A) = \cos A$$

Analyse the unit circle to see why that is

For example: $\cos(-30°) = \cos 30°$

This explains why…

In the log tables, it is stated that:

$$\sin(-A) = -\sin A$$

Analyse the unit circle to see why that is

For example: $\sin(-60°) = -\sin 60°$

This explains why…

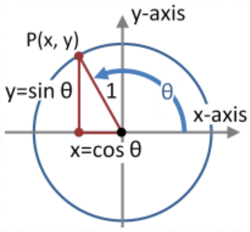
In the log tables, it is also stated that:

$$cos^2\theta + sin^2\theta = 1$$

This is derived from Pythagoras' theorem:

$$(\cos\theta)^2+(\sin\theta)^2=(1)^2$$

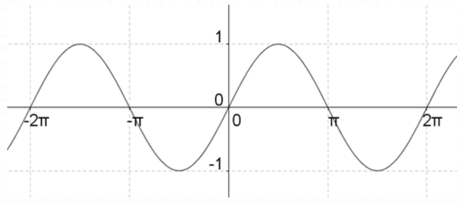
$$cos^2\theta + sin^2\theta = 1$$



Sine Curve

$$f(x) = \sin x$$

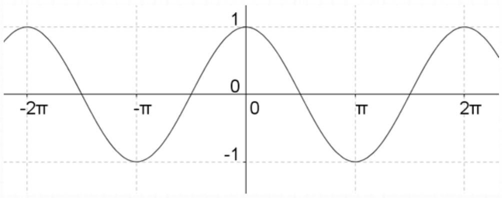
$x =$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(x) =$	0	1	0	−1	0



Cosine Curve

$$f(x) = \cos x$$

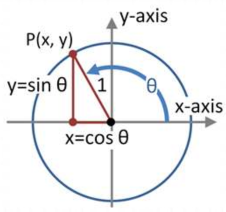
$x =$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(x) =$	1	0	-1	0	1



Sine and Cosine Curve

Notice how the output of the sine function and the cosine function is always between – 1 and 1

This can also be explained by the unit circle – if the radius of the unit circle is 1 then the maximum that the sine or cosine of an angle can be is also 1, similarly the minimum that the sine or cosine of an angle can be is -1.



Chapter 5:
Sequences & Series

MA4702
Week 11 & 12
Presented by Dr. Páraic Treacy

Patterns in Numbers

$2, 4, 6, 8, 10, \ldots$

$1, 4, 9, 16, 25, \ldots$

$1, -2, 4, -8, 16, -32, \ldots$

$1, 3, 9, 27, 81, \ldots$

Sequences

A sequence is a list of real numbers:

$$u_1, u_2, u_3, u_4, \ldots, u_n$$

with a rule u_n for each $n = 1, 2, 3, \ldots$

Sequences

$2, 4, 6, 8, 10, \ldots$
$u_1, u_2, u_3, u_4, u_5 \ldots$

$u_n = 2n$

$1, -2, 4, -8, 16, -32, \ldots$
$u_1, u_2, u_3, u_4, u_5, u_6 \ldots$

$u_n = (-2)^{n-1}$

$1, 4, 9, 16, 25, \ldots$
$u_1, u_2, u_3, u_4, u_5 \ldots$

$u_n = n^2$

$1, 3, 9, 27, 81, \ldots$
$u_1, u_2, u_3, u_4, u_5 \ldots$

$u_n = (3)^{n-1}$

Convergence of Sequences

A sequence u_n is said to converge to a limit L if

$$\lim_{n\to\infty} u_n = L$$

If a sequence does not converge then it is said to be divergent.

Convergence – Ex. 1

Q. What does the sequence with terms $u_n = \frac{1}{2^n}$ converge to?

Solution: Find

$$\lim_{n\to\infty} u_n$$

$$\lim_{n\to\infty} \frac{1}{2^n}$$

$$= \frac{1}{2^\infty}$$

$$= \frac{1}{\infty}$$

$$= 0$$

This sequence converges to zero

Convergence – Ex. 1

$$u_n = \frac{1}{2^n}$$

This means the series is:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

Or in decimal terms:

$$0.5, 0.25, 0.125, \dots$$

The number will keep getting smaller and smaller and will eventually approach zero. That's why it converges to zero.

Convergence – Ex. 2

Q. What does the sequence with terms $u_n = \frac{n+4}{3n+1}$ converge to?

Solution: Find

$$\lim_{n\to\infty} u_n$$

$$\lim_{n\to\infty} \frac{n+4}{3n+1}$$

Divide above and below by highest power of n:

$$\lim_{n\to\infty} \frac{\frac{n}{n}+\frac{4}{n}}{\frac{3n}{n}+\frac{1}{n}}$$

Convergence – Ex. 2

$$\lim_{n\to\infty} \frac{\frac{n}{n}+\frac{4}{n}}{\frac{3n}{n}+\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{4}{n}}{3+\frac{1}{n}}$$

$$\frac{1+\frac{4}{\infty}}{3+\frac{1}{\infty}}$$

$$= \frac{1+0}{3+0}$$

$$= \frac{1}{3}$$

The series converges to $\frac{1}{3}$

Convergence – Ex. 2

Check this sequence:

$$u_n = \frac{n+4}{3n+1}$$

$$\frac{5}{4}, \frac{6}{7}, \frac{7}{10}, \dots, \frac{1004}{3001}, \dots, \frac{10004}{30001}, \dots$$

In decimal terms:

$$1.25, 0.857, 0.7 \dots, 0.3345, \dots, 0.3334$$

We can see that the terms are getting closer and closer to 0.3333 or $\frac{1}{3}$

Convergence – Ex. 3

Q. What does the sequence with terms $u_n = \frac{3^n}{3^n+1}$ converge to?

Solution: Find

$$\lim_{n\to\infty} u_n$$

$$\lim_{n\to\infty} \frac{3^n}{3^n+1} = \frac{\infty}{\infty}$$

Divide above and below by 3^n

$$\lim_{n\to\infty} \frac{\frac{3^n}{3^n}}{\frac{3^n}{3^n}+\frac{1}{3^n}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{3^n}}$$

Convergence – Ex. 3

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{3^n}}$$

$$= \frac{1}{1+\frac{1}{3^\infty}}$$

$$= \frac{1}{1+\frac{1}{\infty}}$$

$$= \frac{1}{1+0}$$

$$= 1$$

This sequence converges to 1.

Convergence – Ex. 3

Check this sequence:

$$u_n = \frac{3^n}{3^n + 1}$$

$$\frac{3}{4}, \frac{9}{10}, \frac{27}{28}, \frac{81}{82} \dots$$

In decimal terms:

0.75, 0.9, 0.96, 0.987 ...

We can see that the terms are getting closer and closer to 1.

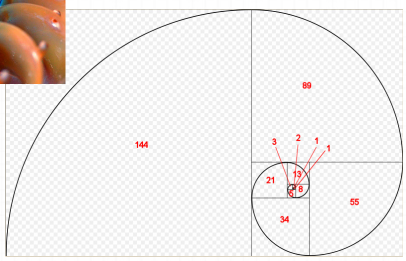
Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Sequence used in:

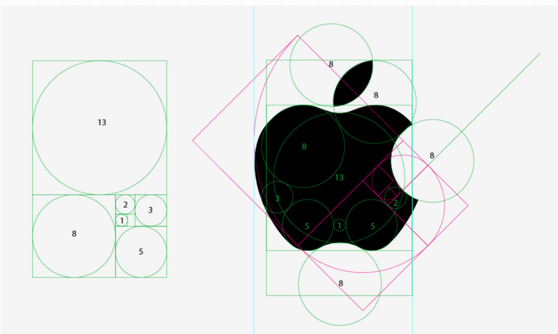
- Music to structure rhymes and rhythm
- Computer games: Metal Gear Solid 4
- Poetry
- DNA Construction
- Solar System arrangement.
- Nature
- Art: used by Marilyn Manson

Fibonacci Sequence in Nature



Milky Way and Fibonnaci Sequence

Apple Symbol and Fibonnaci Sequence



Gauss' Genius

- Carl Friedrich Gauss' Primary school teacher wanted to keep the class busy while he completed other work so he asked the pupils to add all whole numbers between 1 and 100:

$$1 + 2 + 3 + 4 + 5 + \ \dots + 99 + 100$$

- Gauss completed the task in seconds. How?

Gauss' Genius

$1 + 2 + 3 + 4 + 5 + \ \ldots + 99 + 100$

Gauss realised that addition of terms from opposite ends of the list always yielded an answer of 101:

$$\begin{array}{c} 1 + 2 + \ 3 + \ \ 4 + \ \ 5 + \ \ldots + 99 + 100 \\ \underline{100 + 99 + 98 + 97 + 96 + \ \ldots + 2 \ + 1} \\ 101 + 101 + 101 + 101 + 101 + \ \ldots + 101 + 101 \end{array}$$

$$\frac{100(101)}{2} = 5050$$

Series

Given a sequence $u_1, u_2, u_3, u_4, u_5 \ldots$ we can construct the corresponding series:

$$s_1 = u_1$$

$$s_2 = u_1 + u_2$$

$$S_3 = u_1 + u_2 + u_3$$

.

.

.

$$S_n = u_1 + u_2 + u_3 + \ \ldots + u_n$$

Series

$$s_n = u_1 + u_2 + u_3 + \ \ldots + u_n$$

S_n can be written in shorthand:

$$S_n = \sum_{i=1}^{n} u_i$$

This can be interpreted as the sum starting from $i = 1$ and stopping when $i = n$

Gauss' Genius

$$1 + 2 + 3 + 4 + 5 + \ \ldots + 99 + 100 = 5050$$

$$\sum_{r=1}^{100} r = \frac{100(101)}{2} = 5050$$

This gives us the general formula:

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Arithmetic Series

An arithmetic sequence is one where consecutive terms have a common difference:

$$d = u_{n+1} - u_n$$

Example: Given the arithmetic sequence:

$$4, 7, 10, 13, 16, \ldots$$

Here, the common difference is 3. So $d = 3$.

Arithmetic Series

In an arithmetic sequence, the first term (u_1) is usually called a so:

$$\begin{aligned} u_1 &= a \\ u_2 &= a + d \\ u_3 &= a + 2d \\ u_4 &= a + 3d \end{aligned}$$

In our sample sequence $4, 7, 10, 13, 16, \ldots$ $a = 4$ and we know $d = 3$

$$\begin{aligned} u_1 &= a = 4 \\ u_2 &= a + d = 4 + 3 \\ u_3 &= a + 2d = 4 + 6 \\ u_4 &= a + 3d = 4 + 9 \end{aligned}$$

Arithmetic Series

$u_1 = a$
$u_2 = a + d$
$u_3 = a + 2d$
$u_4 = a + 3d$
.
.
.
$u_n = a + (n-1)d$

$S_n = u_1 + u_2 + u_3 + \ ... + u_n$

$S_n = a + (a+d) + (a+2d) + (a+3d) + \ ... + [a + (n-1)d]$

Arithmetic Series

$S_n = a + (a+d) + (a+2d) + \cdots + [a + (n-2)d] + [a + (n-1)d]$

$\underline{S_n = [a + (n-1)d] + [a + (n-2)d] + \cdots + (a+2d) + (a+d) + a}$

$2S_n = [2a + (n-1)d] + [2a + (n-1)d] + [2a + (n-1)d] + \ ... [2a + (n-1)d] + [2a + (n-1)d]$

$2S_n = n[2a + (n-1)d]$

$S_n = \frac{n}{2}[2a + (n-1)d]$

Arithmetic Series – Overview

The common difference d is found by subtracting one term from the term which immediately follows it:

$d = u_{n+1} - u_n$

To find the nth term in an arithmetic sequence, apply the following:

$u_n = a + (n-1)d$

To find the sum of the first n terms in an arithmetic sequence, apply the following:

$S_n = \frac{n}{2}[2a + (n-1)d]$

Arithmetic Series – Ex. 1

In the following arithmetic series:

$3, 7, 11, 15, \ldots$

a) What is the 12th term in this sequence?

b) Find the sum of the first 12 terms in the sequence.

Solution:
We can see that $a = 3$ and $d = 4$

$u_n = a + (n-1)d$

$u_{12} = 3 + (12-1)4$

$u_{12} = 47$

The 12th term in the sequence is 47.

Arithmetic Series – Ex. 1

b) Find the sum of the first 12 terms in the sequence.

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{12} = \frac{12}{2}[2(3) + (12-1)4]$

$S_{12} = 6(6 + 44) = 300$

$3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39 + 43 + 47 = 300$

Arithmetic Series – Ex. 2

In the following arithmetic series:

$2, 7, 12, 17, \ldots$

a) What is the 27th term in this sequence?

b) Find the sum of the first 26 terms in the sequence.

Solution:
We can see that $a = 2$ and $d = 5$

$u_n = a + (n-1)d$

$u_{27} = 2 + (27-1)5$

$u_{27} = 132$

The 27th term in the sequence is 132.

Arithmetic Series – Ex. 2

b) Find the sum of the first 26 terms in the sequence.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

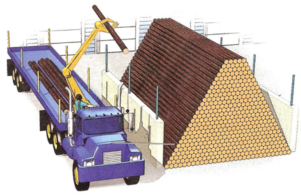
$$S_{26} = \frac{26}{2}[2(2) + (26-1)5]$$

$$S_{26} = 13(4 + 125) = 1677$$

$$2 + 7 + 12 + 17 \ldots + 127 = 1677$$

Arithmetic Series – Ex. 3

Q. A stack of telephone poles has 30 poles in the bottom row. There are 29 poles in the second row, 28 in the next row, and so on. How many poles are in the stack if there are 5 poles in the top row?



Arithmetic Series – Ex. 3

Our arithmetic series is as follows:

$$5 + 6 + 7 + \ldots + 29 + 30$$

We know that $a = 5$ and $d = 1$

If 5 is the 1st term, 6 is the 2nd term, 7 is the 3rd term then 30 must be the 26th term.

Thus, to add up all the poles, we must find the sum of the first 26 terms i.e. S_{26}

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{26} = \frac{26}{2}[2(5) + (26-1)1]$$

$$S_{26} = 455$$

There are 455 poles in the stack.

Arithmetic Series – Ex. 4

In an arithmetic sequence, the fifth term is -18 and the tenth term is 12.

i. Find the first term and the common difference.

ii. Find the sum of the first fifteen terms of the sequence.

$$u_n = a + (n-1)d$$

$$u_5 = a + 4d = -18$$
$$u_{10} = a + 9d = 12$$

$$a + 4d = -18$$
$$\underline{-a - 9d = -12}$$
$$-5d = -30$$

$$d = 6$$

Arithmetic Series – Ex. 4

$$a + 4d = -18$$

$$a + 4(6) = -18$$

$$a = -42$$

First term is -42 and the common difference is 6

ii. Find the sum of the first fifteen terms of the sequence.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{15} = \frac{15}{2}[2(-42) + (15-1)6]$$

$$S_{15} = \frac{15}{2}(-84 + 84) = 0$$

the sum of the first fifteen terms of the sequence is zero.

Geometric Sequences & Series

A sequence is geometric if the ratio, $\frac{u_{n+1}}{u_n}$, between any two consecutive terms is a constant.

This constant is called the common ratio and is usually denoted by r.

Example:

$$2, 4, 8, 16, 32, 64, \ldots$$

The common ratio here is 2.

Geometric Sequences & Series

$$u_1 = a$$
$$u_2 = ar$$
$$u_3 = ar^2$$
$$u_4 = ar^3$$
.
.
.
$$u_n = ar^{n-1}$$

$$S_n = u_1 + u_2 + u_3 + \ ... + u_n$$

$$S_n = a + ar + ar^2 + ar^3 + \ ... + ar^{n-1}$$

Geometric Sequences & Series

$$S_n = a + ar + ar^2 + ar^3 + \ ... + ar^{n-1}$$

This can be shortened into the following equation:

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

Geometric Sequences & Series - Overview

To find the common ratio r:

$$r = \frac{u_{n+1}}{u_n}$$

To find the n[th] term in the geometric sequence, apply the following:

$$u_n = ar^{n-1}$$

To find the sum of the first n terms in a geometric sequence, apply the following:

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

Geometric Series – Ex. 1

Find the sum of the following geometric series:

$$5 + 10 + 20 + 40 + \ ... + 10240$$

Solution:
We know $a = 5$ and $r = 2$

We need to find out how many terms are in the series.

$$u_n = ar^{n-1}$$

$$5(2^{n-1}) = 10240$$

$$2^{n-1} = 2048$$

$$\log_2 2048 = n - 1$$

$$11 = n - 1$$

$$n = 12$$

Geometric Series – Ex. 1

We now know that 10240 is the 12[th] term so there are 12 terms in total in the series.

To find the sum of the geometric series we must find S_{12}

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_{12} = 5\left(\frac{1-2^{12}}{1-2}\right)$$

$$S_{12} = 5\left(-\frac{4095}{-1}\right)$$

$$S_{12} = 20475$$

The sum of the geometric series is 20,475

Geometric Series – Ex. 2

Find the sum of the following geometric series:

$$3 + 12 + 48 + \ ... + 3,072$$

Solution:
We know $a = 3$ and $r = 4$

We need to find out how many terms are in the series.

$$u_n = ar^{n-1}$$

$$3(4^{n-1}) = 3072$$

$$4^{n-1} = 1024$$

$$\log_4 1024 = n - 1$$

$$5 = n - 1$$

$$n = 6$$

Geometric Series – Ex. 2

We now know that 3072 is the 6th term so there are 6 terms in total in the series.

To find the sum of the geometric series we must find S_6

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_6 = 3\left(\frac{1-4^6}{1-4}\right)$$

$$S_6 = 3\left(-\frac{4095}{-3}\right)$$

$$S_6 = 4095$$

The sum of the geometric series is 4,095

Geometric Series – Ex. 3

Most lottery games in the USA allow winners of the jackpot prize to choose between two forms of the prize: an annual-payments option or a cash-value option.

In the case of the New York Lotto, there are 26 annual payments in the annual-payments option, with the first payment immediately, and the last payment in 25 years time.

The payments increase by 4% each year. The amount advertised as the jackpot prize is the total amount of these 26 payments. The cash-value option pays a smaller amount than this.

Geometric Series – Ex. 3

(a) If the amount of the first annual payment is a, write down, in terms of a, the amount of the second, third, fourth and 26th payments.

1st payment = a
2nd payment = $1.04a$
3rd payment = $1.04(1.04a) = (1.04)^2a$
4th payment = $1.04[(1.04)^2a] = (1.04)^3a$
.
.
.
26th payment = $(1.04)^{25}a$

Geometric Series – Ex. 3

(b) The 26 payments form a geometric series. Use this fact to express the advertised jackpot prize in terms of a.

$$a + 1.04a + (1.04)^2a + (1.04)^3a + \ldots + (1.04)^{25}a$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_{26} = a\left(\frac{1-1.04^{26}}{1-1.04}\right) = a\left(-\frac{1.7724}{-0.04}\right)$$

$$S_{26} = 44.312a = Jackpot\ prize$$

Geometric Series – Ex. 3

(c) Find, correct to the nearest dollar, the value of a that corresponds to an advertised jackpot prize of $21·5 million.

$$\text{Jackpot prize} = 44.312a$$

$$21{,}500{,}000 = 44.312a$$

$$a = \$485{,}196$$

This means that the initial payment will be $485,196 and will increase by 4% with each subsequent payment.

Geometric Series – Ex. 4

Q. In January 2013, HMV staff in Limerick City held a sit-in to aid negotiations for a fair severance package. The company were offering a total of €800,000 to be shared out amongst all the HMV staff in Ireland as a severance package.

The staff were not happy with this figure and instead suggested that, in the month of February 2013, HMV would pay 1 cent into the severance package for all staff on Feb. 1st, 2 cent on Feb. 2nd, 4 cent on Feb. 3rd, 8 cent on the 4th, 16 cent on the 5th and so on for the rest of that month. HMV agreed to the deal immediately. How much did HMV end up paying to the staff?

Geometric Series – Ex. 4

The payments would add up as follows:

€0.01 + €0.02 + €0.04 + €0.08 + €0.16 ...

We can see that this is a geometric series.

$$a = 0.01 \quad r = 2$$

There will be 28 payments as there are 28 days in February in 2013, so we need to find the sum of the first 28 terms of this series.

Geometric Series – Ex. 4

$$S_n = a\left(\frac{1-r^n}{1-r}\right)$$

$$S_{28} = 0.01\left(\frac{1-2^{28}}{1-2}\right)$$

$$S_{28} = 0.01\left(\frac{1-268{,}435456}{-1}\right)$$

$$S_{28} = 2{,}684{,}354.55$$

Geometric Series – Ex. 4

In the end, HMV ended up paying out €2,684,354.55 in severance fees to their Irish staff.

Sum to Infinity of a Geometric Series

$$S_\infty = u_1 + u_2 + u_3 + \ ... + u_n + \ ...$$

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right)$$

This limit will exist if $|r| < 1$ The series will converge giving:

$$S_\infty = a\left(\frac{1}{1-r}\right)$$

This is because $r^n \to 0$ as $n \to \infty$ if $|r| < 1$

Sum to Infinity of a Geometric Series

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right)$$

This limit will not exist if $|r| > 1$ Thus, the series will diverge.

This is because $r^n \to \infty$ as $n \to \infty$ if $|r| > 1$.

So, to find the sum to infinity of a geometric series, the modulus of the common ratio r must be less than 1.

Sum to Infinity – Ex. 1

Q. Find the sum to infinity of the following series:

$$1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \ ...$$

We know:

$$a = 1$$

$$r = \frac{1}{6}$$

$|r| < 1$ so the series will converge:

$$S_\infty = a\left(\frac{1}{1-r}\right)$$

$$S_\infty = 1\left(\frac{1}{1-\frac{1}{6}}\right)$$

$$S_\infty = \frac{6}{5}$$

Sum to Infinity – Ex. 2

Q. Write the recurring decimal 0.474747… as an infinite geometric series and hence as a fraction.

Solution: We can write the number as a geometric series:

$$0.474747\ldots = \frac{47}{100} + \frac{47}{10{,}000} + \frac{47}{1{,}000{,}000} + \ldots$$

Now, we know $a = \frac{47}{100}$ and $r = \frac{1}{100}$

Sum to Infinity – Ex. 2

$|r| < 1$ so the series will converge and:

$$S_\infty = a\left(\frac{1}{1-r}\right)$$

$$S_\infty = \frac{47}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{47}{100}\left(\frac{100}{99}\right)$$

$$S_\infty = \frac{47}{99}$$

$$0.474747\ldots = \frac{47}{99}$$

Telescoping Series

A Telescoping Series is a series whose partial sums eventually only have a fixed number of terms after cancellation.

This approach is commonly used for sums of certain infinite series.

Telescoping Series – Ex. 1

Q. Find

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution: Normally we would outline the series and check if it is arithmetic or geometric and solve accordingly:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$$

But we can see that it is neither arithmetic nor geometric so we use a different method to solve this type of series.

Telescoping Series – Ex. 1

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

First, we separate the function into two distinct functions:

$$\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}$$

Now we need to figure out what a and b are equal to…

Telescoping Series – Ex. 1

$$\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{a(n+1) + b(n)}{n(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{an + a + bn}{n(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{(a+b)n + a}{n(n+1)}$$

$$1 = (a+b)n + a$$

Equate coefficients:

n-coefficient on the left of the equation equals n-coefficient on the right of the equation:

$$0 = a + b$$

Constant value on the left of the equation equals constant value on the right of the equation:

$$1 = a$$

Telescoping Series – Ex. 1

We now know:

$$a = 1$$
$$a + b = 0$$

So:

$$1 + b = 0$$
$$b = -1$$

We started with:

$$\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}$$

We can sub for a and b:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

This means that:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Can be re-written as:

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

Telescoping Series – Ex. 1

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cancel{\ldots}$$

$$= 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Telescoping Series – Ex. 2

Q. Find

$$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$$

Solution: Normally we would outline the series and check if it is arithmetic or geometric and solve accordingly:

$$\frac{1}{4} + \frac{2}{15} + \frac{1}{12} + \frac{2}{35} + \frac{1}{24} + \ldots$$

But we can see that it is neither arithmetic nor geometric so we use a different method to solve this type of series.

Telescoping Series – Ex. 2

$$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$$

First, we separate the function into two distinct functions:

$$\frac{2}{(n+1)(n+3)} = \frac{a}{n+1} + \frac{b}{n+3}$$

Now we need to figure out what a and b are equal to…

Telescoping Series – Ex. 2

$$\frac{2}{(n+1)(n+3)} = \frac{a}{n+1} + \frac{b}{n+3}$$

$$\frac{2}{(n+1)(n+3)} = \frac{a(n+3) + b(n+1)}{(n+1)(n+3)}$$

$$\frac{2}{(n+1)(n+3)} = \frac{an + 3a + bn + b}{(n+1)(n+3)}$$

$$\frac{2}{(n+1)(n+3)} = \frac{(a+b)n + 3a + b}{(n+1)(n+3)}$$

Telescoping Series – Ex. 2

$$\frac{2}{(n+1)(n+3)} = \frac{(a+b)n + 3a + b}{(n+1)(n+3)}$$

$$2 = (a+b)n + 3a + b$$

Equate co-efficients:

$$2 = 3a + b$$
$$0 = a + b$$

Telescoping Series – Ex. 2

$$2 = 3a + b$$
$$0 = a + b$$

So:

$$a = -b$$

Sub into first equation:

$$2 = 3(-b) + b$$

$$2 = -2b$$

$$b = -1$$

$$\frac{2}{(n+1)(n+3)} = \frac{a}{n+1} + \frac{b}{n+3}$$

$$\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$$

$$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$$

Can now be re-written as:

$$\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$$

Telescoping Series – Ex. 2

$$\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}\right) + \cancel{\cdots}$$

$$= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)} = \frac{5}{6}$$

Telescoping Series

How do we know when we can use this method to evaluate a series?

If the function is in the form:

$$\frac{b-a}{(k+a)(k+b)}$$

It can then be re-written in the form:

$$\frac{1}{k+a} - \frac{1}{k+b}$$

Test for Convergence: Ratio Test

Compute:

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = r$$

If $r < 1$, the series converges.

If $r > 1$, the series diverges.

If $r = 1$, the test is inconclusive.

Ratio Test – Ex. 1

Test the following series for convergence:

$$\frac{2}{1} + \frac{4}{2} + \frac{8}{6} + \cdots = \sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Note: $n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$

e.g. $5! = 5 \times 4 \times 3 \times 2 \times 1$

Ratio Test – Ex. 1

$$u_n = \frac{2^n}{n!}$$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

Ratio test:

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$$

$$\lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{2^{\cancel{n}+1}}{(n+1)!} \times \frac{n!}{\cancel{2^n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{2(n!)}{(n+1)!}\right|$$

Ratio Test – Ex. 1

$$= \lim_{n\to\infty}\left|\frac{2(n!)}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2}{(n+1)}\right|$$

$$= \left|\frac{2}{\infty+1}\right|$$

$$= 0$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = r$$

If $r < 1$, the series converges

The following series converges:

$$\frac{2}{1}+\frac{4}{2}+\frac{8}{6}+\frac{16}{12}\ldots = \sum_{n=1}^{\infty}\frac{2^n}{n!}$$

Ratio Test – Ex. 2

Test for convergence the series:

$$\frac{3}{(2)(2)}+\frac{9}{(3)(4)}+\frac{27}{(4)(8)}+\cdots = \sum_{n=1}^{\infty}\frac{3^n}{(n+1)2^n}$$

Solution:

$$u_n = \frac{3^n}{(n+1)2^n}$$

$$u_{n+1} = \frac{3^{n+1}}{(n+2)2^{n+1}}$$

Ratio Test – Ex. 2

Ratio test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$\lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+2)2^{n+1}} \div \frac{3^n}{(n+1)2^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{\cancel{3^{n}}3^{1}}{(n+2)\cancel{2^{n}}2^{1}} \times \frac{(n+1)\cancel{2^n}}{\cancel{3^n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{3}{(n+2)2} \times \frac{(n+1)}{1}\right|$$

$$\lim_{n\to\infty}\left|\frac{3(n+1)}{(n+2)2}\right|$$

$$\lim_{n\to\infty}\left|\frac{3n+3}{2n+4}\right|$$

Ratio Test – Ex. 2

$$\lim_{n\to\infty}\left|\frac{3n+3}{2n+4}\right|$$

$$\lim_{n\to\infty}\left|\frac{\frac{3n}{n}+\frac{3}{n}}{\frac{2n}{n}+\frac{4}{n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{3+\frac{3}{n}}{2+\frac{4}{n}}\right|$$

$$= \left|\frac{3+\frac{3}{\infty}}{2+\frac{4}{\infty}}\right|$$

$$= \left|\frac{3}{2}\right| = \frac{3}{2}$$

$r > 1$, thus the series diverges.

Ratio Test – Ex. 3

Test for convergence the series:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

Solution:

$$u_n = \frac{1}{n}$$

$$u_{n+1} = \frac{1}{n+1}$$

Ratio Test – Ex. 3

Ratio test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$\lim_{n\to\infty}\left|\frac{1}{n+1} \div \frac{1}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{n+1} \times \frac{n}{1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right|$$

Ratio Test – Ex. 3

$$= \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right|$$

$$= \left|\frac{1}{1+\frac{1}{\infty}}\right|$$

$$= \left|\frac{1}{1+0}\right|$$

$$= 1$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = r$$

If $r = 1$, the test is inconclusive.

Ratio test fails in this instance.

Maclaurin Series

How does a calculator find $\sin x, \cos x, e^x, \ldots$?

Answer: by using the Maclaurin Series.
http://www.intmath.com/series-expansion/3-how-calculator-works.php

The Maclaurin Series is defined as follows:

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

Maclaurin Series – Ex. 1

Q. Find the Maclaurin Series of $f(x) = e^x$ and, hence estimate the value of e^2 using the first seven terms of the series.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$f(x) = e^x \qquad f(0) = e^0 = 1$$

$$f'(x) = e^x \qquad f'(0) = e^0 = 1$$

$$f''(x) = e^x \qquad f''(0) = e^0 = 1$$

$$f'''(x) = e^x \qquad f'''(0) = e^0 = 1$$

Maclaurin Series – Ex. 1

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$e^x = 1 + x(1) + \frac{x^2(1)}{2!} + \frac{x^3(1)}{3!} + \frac{x^4(1)}{4!} + \frac{x^5(1)}{5!} + \frac{x^6(1)}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

Maclaurin Series – Ex. 1

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

Estimate e^2:

$$e^2 = 1 + 2 + \frac{(2)^2}{2!} + \frac{(2)^3}{3!} + \frac{(2)^4}{4!} + \frac{(2)^5}{5!} + \frac{(2)^6}{6!}$$

$$e^2 = 1 + 2 + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} + \frac{64}{720} = 7.3555$$

Check e^2 on the calculator… Very close to exact value. Answer is more accurate when more terms of the series are included.

Maclaurin Series – Ex. 2

Q. Find the Maclaurin Series of $f(x) = \sin x$ and, hence estimate the value of $\sin(1)$ using the first four non-zero terms of the series.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$f(x) = \sin x \qquad f(0) = \sin(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = \cos(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = -\sin(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -\cos(0) = -1$$

Maclaurin Series – Ex. 2

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$\sin x = 0 + x(1) + \frac{x^2(0)}{2!} + \frac{x^3(-1)}{3!} + \frac{x^4(0)}{4!} + \frac{x^5(1)}{5!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

Maclaurin Series – Ex. 2

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Estimate :

$$\sin(1) = (1) - \frac{(1)^3}{3!} + \frac{(1)^5}{5!} - \frac{(1)^7}{7!}$$

$$\sin(1) = 0.84146$$

Check calculator... $\sin(1) = 0.84147$.

Note: this is the sine of one radian, not one degree.

Maclaurin Series – Ex. 3

Q. Find the Maclaurin Series of $f(x) = \cos x$ and, hence estimate the value of $\cos(2)$ using the first four non-zero terms of the series.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$f(x) = \cos x$ $\qquad f(0) = \cos(0) = 1$

$f'(x) = -\sin x$ $\qquad f'(0) = -\sin(0) = 0$

$f''(x) = -\cos x$ $\qquad f''(0) = -\cos(0) = -1$

$f'''(x) = \sin x$ $\qquad f'''(0) = \sin(0) = 0$

Maclaurin Series – Ex. 3

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$\cos x = 1 + x(0) + \frac{x^2(-1)}{2!} + \frac{x^3(0)}{3!} + \frac{x^4(1)}{4!} + \frac{x^5(0)}{5!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots$$

Maclaurin Series – Ex. 3

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots$$

Estimate the value of $\cos(2)$ using the first five non-zero terms of the series:

$$\cos(2) = 1 - \frac{(2)^2}{2!} + \frac{(2)^4}{4!} - \frac{(2)^6}{6!} + \frac{(2)^8}{8!} = -0.4158$$

Check calculator... $\cos(2) = -0.4161$

Note: this is the cosine of two radians, not two degrees.

Maclaurin Series

The Maclaurin Series is an exact representation of a function.

But, using the first few terms of the series, we can obtain a good approximation to the function for <u>small values</u> of x.

More examples available at
http://www.studentxpress.ie/educ/maths/maths8/maths8.html

Chapter 1: Functions

Presented by Páraic Treacy
Week 2
Domain and Range; Even/Odd Functions; Composite Functions; Inverse Functions

Mathematical Functions

Functions are used to describe the rules which define the ways in which a *change* occurs.

This rule operates on an input to produce an output.

Converting Degrees Fahrenheit to Degrees Celsius

$$C = \frac{5}{9}(F - 32)$$

Input: 95°F

Function: $C = \frac{5}{9}(95 - 32) = \frac{5}{9}(63) = 35°\text{C}$

Output: 35°C

The Domain

The Domain of a function is the set of all possible input values.

Basically, this is the set of input values or x values which will produce a defined answer or output.

Rules for the Domain of a Function

To determine the Domain of a function, we use the following two conditions:

1. The denominator of a function cannot equal zero.
2. The Radicand (value inside a square root) must be positive or equal zero.

Domain Example 1

Find the domain of the function $f(x) = \sqrt{x - 10}$

Solution: The Radicand cannot be negative.

$$x - 10 \geq 0$$

$$x \geq 10$$

Domain: $[10, \infty)$

Domain Example 2

Find the domain of the function $f(x) = \sqrt{3x+5}$

Solution: The Radicand cannot be negative.

$$3x + 5 \geq 0$$

$$3x \geq -5$$

$$x \geq -\frac{5}{3}$$

Domain: $[-\frac{5}{3}, \infty)$

Domain Example 3

Find the domain of the function $f(x) = \frac{3}{2x+6}$

Solution: The denominator $\neq 0$

$$2x + 6 \neq 0$$

$$2x \neq -6$$

$$x \neq -3$$

Domain: $(-\infty, -3) \cup (-3, \infty)$

Domain Example 4

Find the domain of the function $f(x) = \frac{-7}{2-3x}$

Solution: The denominator $\neq 0$

$$2 - 3x \neq 0$$

$$-3x \neq -2$$

$$x \neq \frac{2}{3}$$

Domain: $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Domain Example 5

Find the domain of the function $f(x) = \frac{3}{\sqrt{3x-12}}$

Solution: Here we have both a denominator and a square root. Firstly, the Radicand cannot be negative:

$$3x - 12 \geq 0$$

$$x \geq 4$$

Domain Example 5 Continued

Secondly, the denominator $\neq 0$

$$\sqrt{3x-12} \neq 0$$

$$3x - 12 \neq 0 \qquad \text{[Square both sides]}$$

$$3x \neq 12$$

$$x \neq 4$$

Domain Example 5 Continued

Now we have <u>two</u> conditions:

$$x \geq 4 \quad \text{and} \quad x \neq 4$$

These two *combine* to produce the overall condition:

$$x > 4$$

Domain: $(4, \infty)$

Domain Example 6

Find the domain of the function $f(x) = 7 + \sqrt{x^2 - x - 2}$

Solution: The Radicand cannot be negative.

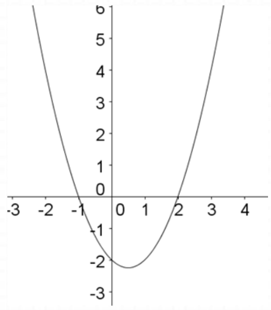
$$x^2 - x - 2 \geq 0$$

$$(x-2)(x+1) \geq 0$$

Roots are $x = 2$ and $x = -1$

Sketch the graph of the function to check where it is ≥ 0

Domain Example 6



When is $x^2 - x - 2 \geq 0$?

Domain Example 6

It is clear from the graph that $x^2 - x - 2 \geq 0$ when:

$$x \leq -1 \text{ and } x \geq 2$$

This produces the following domain:

$$(-\infty, -1] \cup [2, \infty)$$

The Domain – why can't the radicand be negative?

Example: $f(x) = \sqrt{x}$

If we let $x = -1$ then:

$$f(-1) = \sqrt{-1} = i$$

Thus, the output value is an *imaginary number* so it is not defined on the Cartesian plane.

Therefore, our x values for this particular function cannot be negative. Domain: $[0, \infty)$

The Domain – why can't the denominator equal zero?

Example: $f(x) = \frac{1}{x}$

If we let $x = 0$ then:

$$f(x) = \frac{1}{0} = \infty$$

Again, this is not a defined point on the Cartesian plane, thus we cannot include 0 in our domain for this function.

Domain: $(-\infty, 0) \cup (0, \infty)$

The Range

The Range of a function is the set of all possible output values.

In other words, it is the range of possible answers you can obtain from a particular function.

How can we determine the Range?

1. First analyse whether it's possible to obtain answers that are positive, negative, and/or equal to zero. $+, -, 0$

2. Analyse the function for any specific features which will affect the type of answer obtained.

Range Example 1

Find the range of the function $f(x) = \sqrt{x - 10}$

Domain: $[10, \infty)$

Solution: $+$ answers: Yes
$-$ answers: No
Can it equal zero: Yes
Anything else?: No.

Range: $[0, \infty)$

Range Example 2

Find the range of the function $f(x) = \sqrt{3x + 5}$

Domain: $[-\frac{5}{3}, \infty)$

Solution: $+$ answers: Yes
$-$ answers: No
Can it equal zero: Yes
Anything else?: No.

Range: $[0, \infty)$

Range Example 3

Find the range of the function $f(x) = \frac{3}{2x+6}$

Domain: $(-\infty, -3) \cup (-3, \infty)$

Solution: $+$ answers: Yes
$-$ answers: Yes
Can it equal zero: No
Anything else?: No.

Range: $(-\infty, 0) \cup (0, \infty)$

Range Example 4

Find the range of the function $f(x) = \frac{-7}{2-3x}$

Domain: $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Solution: $+$ answers: Yes
$-$ answers: Yes
Can it equal zero: No
Anything else?: No.

Range: $(-\infty, 0) \cup (0, \infty)$

Range Example 5

Find the range of the function $f(x) = \frac{3}{\sqrt{3x-12}}$

Domain: $(4, \infty)$

Solution: $+$ answers: Yes
$-$ answers: No
Can it equal zero: No
Anything else?: No.

Range: $(0, \infty)$

Range Example 6

Find the range of the function $f(x) = 7 + \sqrt{x^2 - x - 2}$

Domain: $(-\infty, -1] \cup [2, \infty)$

Solution: $+$ answers: Yes
$-$ answers: No
Can it equal zero: No
Anything else?: All answers will be ≥ 7

Range: $[7, \infty)$

Domain and Range – Special Cases

There are certain functions that must be considered a little more carefully.

These include log, sine, cosine, and exponential functions.

Domain and Range – Special Cases – Ex. 1

Q. Find the domain and range of the following function:

$$f(x) = e^{3x}$$

Domain:
Notice that there is no denominator or radicand, so any value for x can be used within this function

Domain: $(-\infty, \infty)$

Domain and Range – Special Cases – Ex. 1

$$f(x) = e^{3x}$$

Range:

$+$ answers: Yes
$-$ answers: No
Can it equal zero: No
Anything else?: No.

Range: $(0, \infty)$

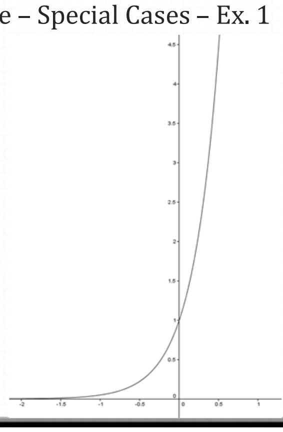
Domain and Range – Special Cases – Ex. 1

$$f(x) = e^{3x}$$

- There will be no negative answers as e raised to any power (either positive or negative) will produce a positive value.
- The answer will never be zero as e raised to any defined power will never equal zero. Remember $e^0 = 1$ so even if the power is zero this still will not produce the answer zero.

Domain and Range – Special Cases – Ex. 1

Graph of $f(x) = e^{3x}$



Domain and Range – Special Cases – Ex. 2

Q. Find the domain and range of the following function:

$$f(x) = e^{-7x}$$

Domain:

Notice that there is no <u>denominator</u> or <u>radicand</u>, so any value for x can be used within this function

Domain: $(-\infty, \infty)$

Domain and Range – Special Cases – Ex. 2

$$f(x) = e^{-7x}$$

Range:

$+$ answers: Yes

$-$ answers: No

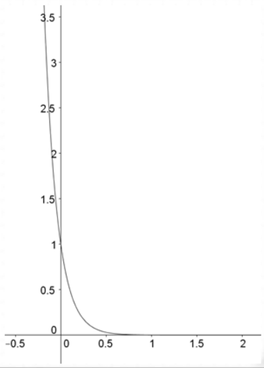
Can it equal zero: No

Anything else?: No.

Range: $(0, \infty)$

Domain and Range – Special Cases – Ex. 2

Graph of $f(x) = e^{-7x}$



Domain and Range – Special Cases – Ex. 3

Q. Find the domain and range of the following function:

$$f(x) = \log_e 3x$$

Domain:

Notice that there is no <u>denominator</u> or <u>radicand</u> BUT you cannot calculate the log of a negative number or zero therefore $3x > 0$ and thus $x > 0$

Domain: $(0, \infty)$

Why can't you find the log of a negative value?

We know:

$$\log_a c = b$$

Converts to:

$$a^b = c$$

Lets take the log of a negative number and consider what type of answer we will find:

$$\log_{10}(-100) = x$$

If we convert that to its index form:

$$\log_{10}(-100) = x$$

Converts to:

$$10^x = -100$$

So, 10 raised to what power will give us an answer of -100?

Answer: there is no such power, which means the original equation is invalid. This is why we can never find the log of a negative value.

Domain and Range – Special Cases – Ex. 3

$$f(x) = \log_e 3x$$

Range:

$+$ answers: Yes

$-$ answers: Yes

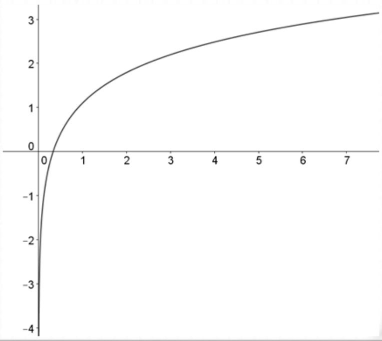
Can it equal zero: Yes

Anything else?: No.

Range: $(-\infty, \infty)$

Domain and Range – Special Cases – Ex. 3

Graph of
$f(x) = \log_e 3x$



Domain and Range – Special Cases – Ex. 4

Q. Find the domain and range of the following function:

$$f(x) = 4\sin x$$

Domain:

Notice that there is no <u>denominator</u> or <u>radicand</u> and there are no restrictions on the input value for the sine function.

Domain: $(-\infty, \infty)$

Domain and Range – Special Cases – Ex. 4

$$f(x) = 4\sin x$$

Range:

$+$ answers: Yes
$-$ answers: Yes
Can it equal zero: Yes
Anything else?: Must be between -4 and 4

Range: $[-4, 4]$

Note: The sine of any value will produce an answer between -1 and 1.

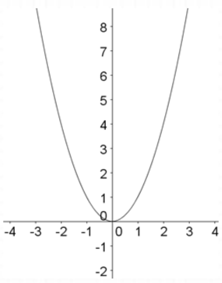
Odd and Even Functions

Determining whether a function is odd, even, or neither, will give an indication of what the graph of the function will look like.

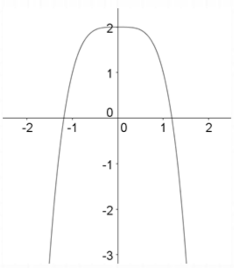
When a function is odd or even, it has a certain type of symmetry.

Even Functions

$f(x) = x^2$

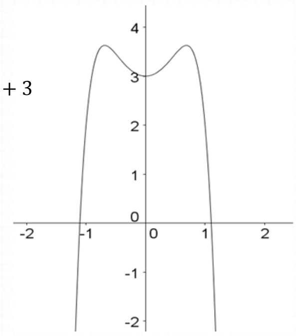


$f(x) = -x^4 + 2$



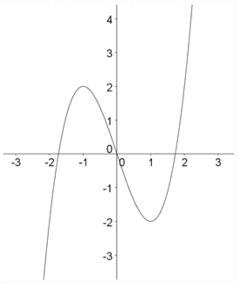
Even Function

$f(x) = -3x^6 + 2x^2 + 3$

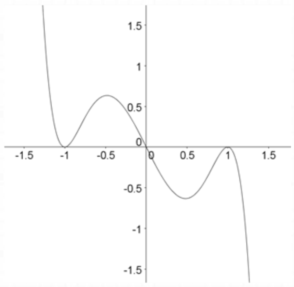


Odd Functions

$f(x) = x^3 - 3x$



$f(x) = -x^7 + 3x^3 - 2x$



When is a function even, odd, or neither?

Function is even if $f(x) = f(-x)$

Function is odd if $-f(x) = f(-x)$

Even/Odd Functions Example 1

Determine whether the following function is odd, even, or neither:

$$f(x) = 3x^2 + 7$$

Step 1: Find $f(-x)$

$$f(-x) = 3(-x)^2 + 7$$

$$f(-x) = 3x^2 + 7$$

Even/Odd Functions Example 1

$$f(x) = f(-x)$$

Thus, the function is even. If it is even then it can't be odd so no need to check.

Answer: $f(x) = 3x^2 + 7$ is an <u>even</u> function

Even/Odd Functions Example 2

Determine whether the following function is odd, even, or neither:

$$f(x) = x^3 + 5x$$

Step 1: Find $f(-x)$

$$f(-x) = (-x)^3 + 5(-x)$$

$$f(-x) = -x^3 - 5x$$

Even/Odd Functions Example 2

$$f(x) \neq f(-x)$$

Thus, the function is **not** even.

Step 2: Check if it is odd by finding $-f(x)$

$$f(x) = x^3 + 5x$$

$$-f(x) = -(x^3 + 5x)$$

Even/Odd Functions Example 2

$$-f(x) = -(x^3 + 5x)$$

$$-f(x) = -x^3 - 5x$$

We obtained the same answer for both $f(-x)$ *and* $-f(x)$

Thus $f(-x) = -f(x)$

This indicates that $f(x) = x^3 + 5x$ is an odd function

Even/Odd Functions Example 3

Determine whether the following function is odd, even, or neither:

$$f(x) = -4x^3 + 7$$

Step 1: Find $f(-x)$

$$f(-x) = -4(-x)^3 + 7$$

$$f(-x) = 4x^3 + 7$$

Even/Odd Functions Example 3

$$f(x) \neq f(-x)$$

Thus, the function is **not** even.

Step 2: Check if it is odd by finding $-f(x)$

$$f(x) = -4x^3 + 7$$

$$-f(x) = -(-4x^3 + 7)$$

Even/Odd Functions Example 3

$$-f(x) = -(-4x^3 + 7)$$

$$-f(x) = 4x^3 - 7$$

We obtained different answers for $f(-x)$ *and* $-f(x)$

Thus $f(-x) \neq -f(x)$

This indicates that $f(x) = -4x^3 + 7$ is neither odd nor even.

Even/Odd Functions - Example 4

Determine whether the following function is odd, even, or neither:

$$f(x) = x^2 \cos x$$

Step 1: Find $f(-x)$

$$f(-x) = (-x)^2 \cos(-x)$$

$$f(-x) = x^2 \cos x$$

Even/Odd Functions Example 4

$$f(x) = f(-x)$$

Thus, the function is even.

Even/Odd Functions – Example 5

Determine whether the following function is odd, even, or neither:

$$f(x) = \sin x$$

Step 1: Find $f(-x)$

$$f(-x) = \sin(-x)$$

$$f(-x) = -\sin x$$

Thus, the function is not even as $f(x) \neq f(-x)$

Even/Odd Functions – Example 5

Step 2: Check if the function is odd by finding $-f(x)$

$$f(x) = \sin x$$

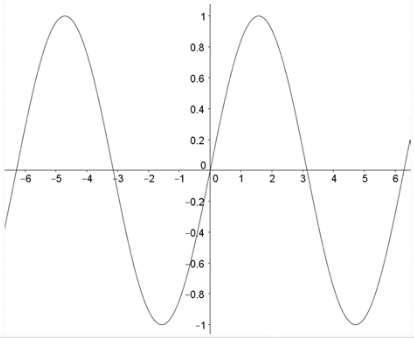
$$-f(x) = -\sin x$$

The function is odd as $-f(x) = f(-x)$

Answer: $f(x) = \sin x$ is an odd function

Even/Odd Functions – Example 5

$f(x) = \sin x$



Composite Functions

Composition of Functions is the process of combining two functions where one function is substituted in place of each x in the other function.

$f{\circ}g(x)$ is a composition of the individual functions $f(x)$ and $g(x)$

Applications of Composite Functions:
http://www.youtube.com/watch?v=97P6p9Gxyw8

Composite Functions Example

Given $f(x) = x^2 + 2$ and $g(x) = \sqrt{x}$ Find $f{\circ}g(x)$

$$\boxed{f{\circ}g(x) = f\big(g(x)\big) = f(\sqrt{x})}$$

$$f(x) = x^2 + 2$$

$$f(\sqrt{x}) = (\sqrt{x})^2 + 2$$

$$f(\sqrt{x}) = x + 2$$

Answer: $f{\circ}g(x) = x + 2$

Composite Functions Example

Given $f(x) = x^2 + 2$ and $g(x) = \sqrt{x}$ Find $g{\circ}f(x)$

$$\boxed{g{\circ}f(x) = g\big(f(x)\big) = g(x^2 + 2)}$$

$$g(x) = \sqrt{x}$$

$$g(x^2 + 2) = \sqrt{x^2 + 2}$$

Answer: $g{\circ}f(x) = \sqrt{x^2 + 2}$

The Inverse of a Function

Input: Bruce Banner

Function: Gets angry

Output: The Incredible Hulk

Inverse Function: Calms Down

What does the Inverse do?

Function: $f(x) = \frac{2x+3}{5}$

It's Inverse: $f^{-1}(x) = \frac{5x-3}{2}$

If we let $x = 6$

$$f(6) = \frac{2(6)+3}{5} = 3$$

Input of 6 produces an output of 3.

What does the Inverse do?

It's Inverse: $f^{-1}(x) = \frac{5x-3}{2}$

If we take the output and use it as the input of the inverse then…

$$f^{-1}(3) = \frac{5(3)-3}{2} = 6$$

We get back to where we started. The inverse reverses the process.

Finding the Inverse: Ex. 1

Find the inverse of the function: $f(x) = 2x + 1$

$$y = 2x + 1$$

$$y - 1 = 2x$$

$$\frac{y-1}{2} = x$$

$$f^{-1}(x) = \frac{x-1}{2}$$

Steps for Finding the Inverse

1. Replace $f(x)$ with y.
2. Find x in terms of y.
3. Replace y with x. This will result in the inverse.

Finding the Inverse: Ex. 2

Find the inverse of the function: $f(x) = \sqrt{x-7}$

$$y = \sqrt{x-7}$$

$$y^2 = x - 7 \quad \text{[Square both sides]}$$

$$y^2 + 7 = x$$

$$x = y^2 + 7$$

$$f^{-1}(x) = x^2 + 7$$

Finding the Inverse: Ex. 3

Find the inverse of the function: $f(x) = \sqrt{x^2 + 1}$

$$y = \sqrt{x^2 + 1}$$

$$y^2 = x^2 + 1$$

$$y^2 - 1 = x^2$$

$$x = \pm\sqrt{y^2 - 1}$$

Conclusion: **No Inverse**. There can only be one solution to the inverse.

Finding the Inverse: Ex. 4

Find the inverse of the function: $f(x) = e^{3x}$

Solution: $y = e^{3x}$

$3x = log_e y$ [Convert to log form]

$$x = \frac{1}{3} log_e y$$

$$f^{-1}(x) = \frac{1}{3}\log_e x$$

Finding the Inverse: Ex. 5

Find the inverse of the function:

$$f(x) = 3\sin(2x + 1)$$

Solution:

$$y = 3\sin(2x + 1)$$

$$\frac{y}{3} = \sin(2x + 1)$$

$$\sin^{-1}\left(\frac{y}{3}\right) = 2x + 1$$

$$\sin^{-1}\left(\frac{y}{3}\right) - 1 = 2x$$

$$x = \frac{1}{2}\left[\sin^{-1}\left(\frac{y}{3}\right) - 1\right]$$

$$f^{-1}(x) = \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - 1\right]$$

Blood Alcohol Percent

The number of drinks and the resulting blood alcohol percent for a man weighing 13 stone (182 lbs) is given by the table below:

No. of Drinks	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Blood Alcohol %	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.21

Note: One drink is equal to 1.25 oz of 80-proof liqour, 12 oz of regular beer (just over half a pint), or 5 oz of wine.

Write the equation of the function that models the blood alcohol percent as a funtion of the number of drinks.

Blood Alcohol Percent

No. of Drinks	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Blood Alcohol %	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.21

We can see that the percentage increases by 0.02 for every drink so the function is linear:

$$f(x) = 0.02x + 0.01$$

Where x is the number of drinks consumed.

This function is determined by finding the slope using two of the above points, then using the slope and a point on the line to find the equation of the line.

Blood Alcohol Percent

- The drink driving limit in Ireland is 0.05 percent of alcohol in the blood for drivers with a full license and 0.02 percent for learner, novice, and professional drivers.
- If a 13-stone male driver with a full license wished to find out how many drinks he would be allowed consume before he reaches the limit of 0.05 percent of alcohol in the blood, how would he do this? What about a 13-stone male learner driver?

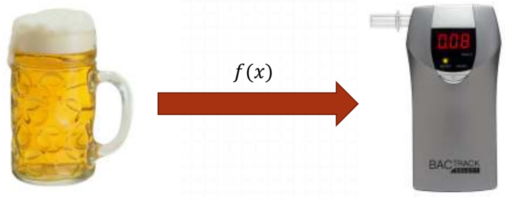
Blood Alcohol Percent

- We know we can calculate the percent of alcohol in the blood of a 13-stone man when given the number of drinks consumed by using:

$$f(x) = 0.02x + 0.01$$

- But we want to figure out the number of drinks that can be consumed when given the set percent of alcohol in the blood.

Blood Alcohol Percent



Blood Alcohol Percent

Solution: Find the inverse of the function.

$$f(x) = 0.02x + 0.01$$

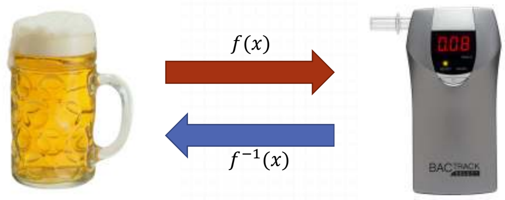
$$y = 0.02x + 0.01$$

$$0.02x = y - 0.01$$

$$x = 50y - 0.5$$

$$f^{-1}(x) = 50x - 0.5$$

Blood Alcohol Percent



Blood Alcohol Percent

$$f^{-1}(x) = 50x - 0.5$$

Now our input will be the percent of alcohol in the blood and the output will be the number of drinks.

0.05 percent of alcohol in the blood:

$$f^{-1}(0.05) = 50(0.05) - 0.5$$

$$f^{-1}(0.05) = 2.5 - 0.5 = 2$$

A 13 stone male driver with a full license could consume up to two drinks before reaching the limit of 0.05 percent of alcohol in the blood.

Blood Alcohol Percent

$$f^{-1}(x) = 50x - 0.5$$

0.02 percent of alcohol in the blood:

$$f^{-1}(0.02) = 50(0.02) - 0.5$$

$$f^{-1}(0.02) = 1 - 0.5 = 0.5$$

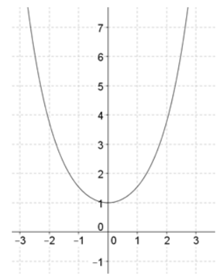
A 13 stone male driver with a learner license could consume up to half a standard drink before reaching the limit of 0.02 percent of alcohol in the blood.

Hyperbolic Functions

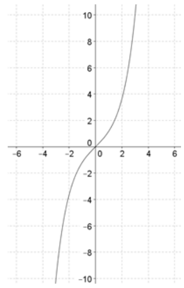
MA4702
Week 3
Presented by Páraic Treacy

Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



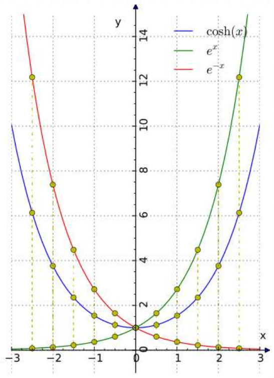
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Graphing Cosh x

This graph shows how adding e^x and e^{-x}, then dividing by 2, produces the $\cosh x$ function.

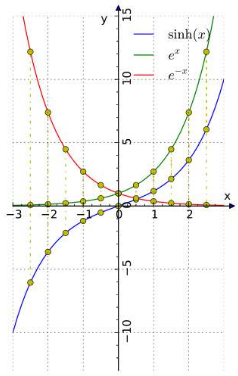
$$\cosh x = \frac{e^x + e^{-x}}{2}$$



Graphing sinh x

This graph shows how subtracting e^{-x} from e^x, then dividing by 2, produces the $\sinh x$ function.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Where are Hyperbolic Functions found in real life?

- The cables of a suspension bridge.
- A rope hanging between two posts.
- Each strand of a typical spider web.
- The Gateway Arch in St. Louis.

All of these are catenaries. Catenaries are segments from the graph of the hyperbolic cosine function $A\cosh(ax)$, where A and a are constants.

http://www.youtube.com/watch?v=AYIQYZuQNMw

Where are Hyperbolic Functions found in real life?

Catenaries occur naturally, since they minimize the gravitational potential energy of a string or rope whose location is fixed at two ends, which is equivalent to minimizing the area under the string.

They are also optimal for architects when a flexible cable (or its equivalent) is subject to a uniform force (e.g., gravity or the weight of a bridge, etc.). Example: The Golden Gate Bridge was designed to take advantage of this phenomenon.

Hyperbolic Identities – Ex. 1

Prove:

$$\sinh 2x = 2\sinh x \cosh x$$

Solution: Write the right-hand side in terms of exponentials

$$\sinh 2x = 2\sinh x \cosh x$$

$$\sinh 2x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$\sinh 2x = \frac{1}{2}(e^x - e^{-x})(e^x + e^{-x})$$

Hyperbolic Identities – Ex. 1

$$\sinh 2x = \frac{1}{2}(e^x - e^{-x})(e^x + e^{-x})$$

$$\sinh 2x = \frac{1}{2}(e^{2x} + e^0 - e^0 - e^{-2x})$$

$$\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x})$$

True: LHS = RHS

QED

(*quod erat demonstrandum* = "which had to be demonstrated")

Hyperbolic Identities – Ex. 2

Prove:

$$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x)$$

Solution: First, write the left-hand side in terms of exponentials

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{1}{2}(1 + \cosh 2x)$$

$$\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(1 + \cosh 2x)$$

Hyperbolic Identities – Ex. 2

$$\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}(1 + \cosh 2x)$$

$$\frac{e^{2x} + e^0 + e^0 + e^{-2x}}{4} = \frac{1}{2}(1 + \cosh 2x)$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2}(1 + \cosh 2x)$$

Hyperbolic Identities – Ex. 2

Next, we'll adjust the right hand side of the equation:

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2}(1 + \cosh 2x)$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2}\left(1 + \frac{e^{2x} + e^{-2x}}{2}\right)$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2} + \frac{e^{2x} + e^{-2x}}{4}$$

Hyperbolic Identities – Ex. 2

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1}{2} + \frac{e^{2x} + e^{-2x}}{4}$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{2 + e^{2x} + e^{-2x}}{4}$$

True: LHS = RHS

QED

Osbourne's Rule

Notice the similarity:

$$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x) \qquad \text{[Previous Proof]}$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \text{[From log tables]}$$

This is because of the relationship between trig functions and the complex exponential:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Osbourne's Rule

Osbourne's Rule states that:

Any algebraic relation between Cosine and Sine also applies to Cosh and Sinh with the condition that a product of Sines gets an extra minus.

Osbourne's Rule – Examples

$$\sin 2x = 2\sin x\cos x$$

$$\sinh 2x = 2\sinh x\cosh x$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

Osbourne's Rule – Key Example

$$\cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

Notice how the condition within Osbourne's Rule (that a product of Sines gets an extra minus) is applied in this example.

Next, we'll prove that

$$\cosh^2 x - \sinh^2 x = 1$$

Hyperbolic Identities – Ex. 3

Prove that:

$$\cosh^2 x - \sinh^2 x = 1$$

Proof: Replace $\cosh x$ and $\sinh x$ with their exponential equivalents and expand the equation.

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$$

Hyperbolic Identities – Ex. 3

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1$$

$$\left(\frac{e^{2x} + e^0 + e^0 + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - e^0 - e^0 + e^{-2x}}{4}\right) = 1$$

$$\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right) = 1$$

$$\frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1$$

Hyperbolic Identities – Ex. 3

$$\frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1$$

$$\frac{4}{4} = 1$$

True: LHS = RHS

QED

Chapter 2 – Curve Sketching

Week 3, 4, & 5

MA4702

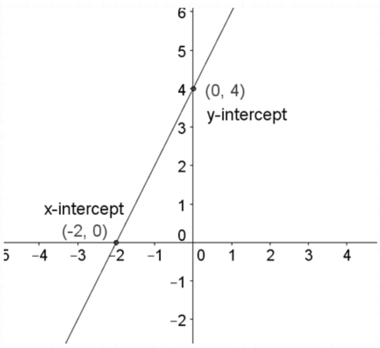
Presented by Páraic Treacy

What is needed to sketch a curve?

To plot $y = f(x)$ we typically consider the following:

1. The domain and range of $f(x)$.
2. The x and y intercepts.
3. Vertical asymptotes (if any).
4. Behaviour as $x \to \pm\infty$ i.e. horizontal asymptotes.
5. Maximum and Minimum points.
6. Points of Inflection.

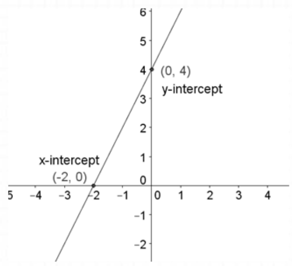
x and y Intercepts



x and y Intercepts

At the point where a curve (or a line) crosses the x-axis, the y value is always zero.

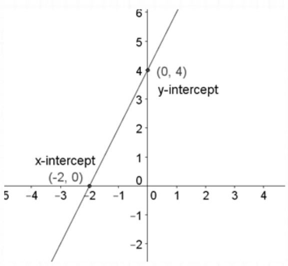
Thus, to find the x-intercept, we let $y = 0$ in our function and find the x value.



x and y Intercepts

At the point where a curve (or a line) crosses the y-axis, the x value is always zero.

Thus, to find the y-intercept, we let $x = 0$ in our function and find the y value.



x and y Intercepts – Ex. 1

Q. Find the x-intercept and the y-intercept of the function

$$f(x) = 2x^2 + 7x + 3$$

Solution: To find the x-intercept, let $y = 0$ and find x.

$$f(x) = 2x^2 + 7x + 3$$

$$0 = 2x^2 + 7x + 3$$

$$(2x + 1)(x + 3) = 0$$

x and y Intercepts – Ex. 1

$$(2x + 1)(x + 3) = 0$$

$$2x + 1 = 0 \qquad x + 3 = 0$$

$$x = -\tfrac{1}{2} \qquad x = -3$$

The curve $f(x) = 2x^2 + 7x + 3$ crosses the x axis at $-\frac{1}{2}$ and -3.

This gives the points $(-\frac{1}{2}, 0)$ and $(-3, 0)$

x and y Intercepts – Ex. 1

To find the y-intercept, let $x = 0$ and find y.

$$y = 2x^2 + 7x + 3$$

$$y = 2(0)^2 + 7(0) + 3$$

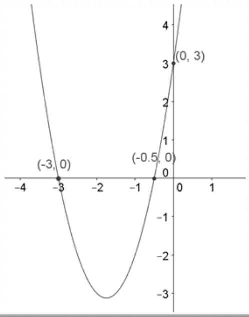
$$y = 3$$

The curve $f(x) = 2x^2 + 7x + 3$ crosses the y axis at 3.

This gives the points $(0, 3)$

x and y Intercepts – Ex. 1

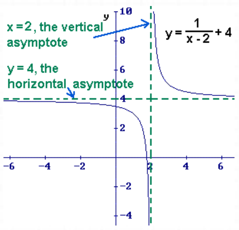
Graph of the curve $f(x) = 2x^2 + 7x + 3$



Asymptotes

An Asymptote is a line that continually approaches a given curve but does not meet it at any finite distance.

Typically, there are three types of asymptote: vertical, horizontal, and diagonal.



Vertical Asymptotes

How to find a vertical asymptote:

Let the Denominator of the given function equal zero and find value(s) for x.

Vertical Asymptote - Example

Identify the vertical asymptote of the following function:

$$f(x) = \frac{x^2 + 4}{x^2 - 4}$$

Solution: Let Denominator equal zero (note: no denominator, no vertical asymptote) and find value(s) for x

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

Vertical Asymptotes at $x = 2$ and $x = -2$

Why?

Why are asymptotes located at $x = 2$ and $x = -2$ for this particular function?

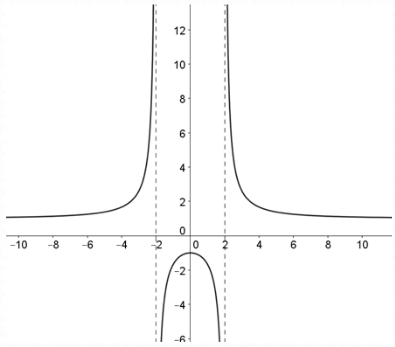
Reason:

$$\lim_{x\to 2}\frac{x^2+4}{x^2-4} = \frac{(2)^2+4}{(2)^2-4}$$

$$= \frac{4+4}{4-4} = \frac{8}{0} = \infty$$

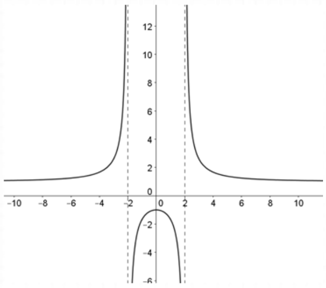
As the x value approaches 2 in this particular function, the corresponding y value approaches ∞

How does this affect the graph of the function?



How does this affect the graph of the function?

Notice how the curve never crosses at $x = 2$ and $x = -2$. At $x = 2$ and $x = -2$ the curve tends towards ∞ or $-\infty$



Horizontal Asymptotes

To determine the horizontal asymptote when graphing a function, find $\lim_{x\to\infty} f(x)$

Example: Identify the horizontal asymptote of the following function:

$$f(x) = \frac{x^2+4}{x^2-4}$$

$$\lim_{x\to\infty}\frac{x^2+4}{x^2-4}$$

Horizontal Asymptotes

$$\lim_{x\to\infty}\frac{x^2+4}{x^2-4}$$

$$= \lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{4}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}}$$

$$= \lim_{x\to\infty}\frac{1+\frac{4}{x^2}}{1-\frac{4}{x^2}} = \frac{1+\frac{1}{\infty^2}}{1-\frac{4}{\infty^2}}$$

$$= \frac{1+0}{1-0} = 1$$

Horizontal Asymptote at

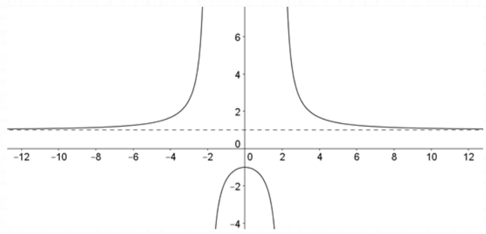
$y = 1$

Note: if the answer turns out to be ∞ then there is no horizontal asymptote.

Horizontal Asymptotes

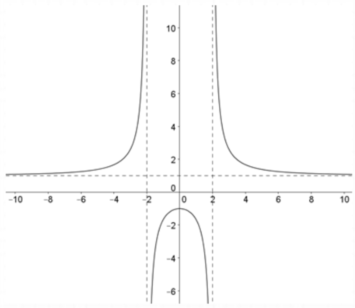
$$\lim_{x\to\infty}\frac{x^2+4}{x^2-4} = 1$$

This indicates that as x approaches infinity, y approaches 1.



Asymptotes - Overview

$$f(x) = \frac{x^2+4}{x^2-4}$$



Asymptotes Overview

To determine the <u>Horizontal Asymptote</u> when graphing a function, find $\lim_{x\to\infty} f(x)$

To determine the <u>Vertical Asymptote</u>: let the Denominator of the given function equal zero and find value(s) for x.

Asymptotes Example

Q. Calculate the horizontal and vertical asymptotes of the following function:

$$f(x) = \frac{x}{x^2-9}$$

Solution: vertical asymptote – let denominator equal zero and find value for x.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

Two vertical asymptotes – at $x = 3$ and $x = -3$

Asymptotes Example

Horizontal asymptote: To determine the horizontal asymptote, find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{x}{x^2-9}$$

$$= \frac{\infty}{\infty}$$

Not defined…

$$\lim_{x\to\infty} \frac{x}{x^2-9}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2} - \frac{9}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{9}{x^2}}$$

Asymptotes Example

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{9}{x^2}}$$

$$= \frac{\frac{1}{\infty}}{1 - \frac{9}{\infty}}$$

$$= \frac{0}{1-0} = 0$$

Horizontal asymptote at $y = 0$

Asymptotes:

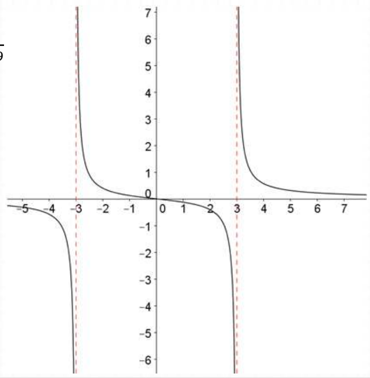
$$x = 3$$

$$x = -3$$

$$y = 0$$

Asymptotes Example

Graph of $f(x) = \frac{x}{x^2-9}$



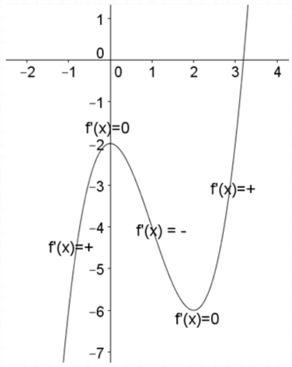
Differentiation

Differentiation is the process of finding the derivative, or rate of change, of a function.

The derivative of a function can indicate whether a function is increasing or decreasing at certain points.

$f'(x) = +$	Function is increasing
$f'(x) = -$	Function is decreasing
$f'(x) = 0$	Function is neither increasing nor decreasing

Differentiation



Maximum and Minimum Points

To find and classify critical points (max./min. points) of a function, complete the following steps:

1. Find $f'(x)$ and let $f'(x) = 0$
2. Calculate value(s) for x from this equation
3. Find the corresponding y values
4. Use $f'(x)$ to check the nature of the curve around these points to determine whether these points are max. or min.

Max. and Min. Points – Ex. 1

Example: Find the maximum and minimum points of the curve:

$$f(x) = x^3 - 3x^2 - 2$$

Solution:

$$f'(x) = 3x^2 - 6x$$

Let $f'(x) = 0$

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$

$3x = 0$	$x - 2 = 0$
$x = 0$	$x = 2$

Max. and Min. Points – Ex. 1

Find Corresponding y values:

$$y = x^3 - 3x^2 - 2$$

$x = 0$

$$y = (0)^3 - 3(0)^2 - 2$$

$$y = -2$$

Critical Point: $(0, -2)$

$x = 2$

$$y = (2)^3 - 3(2)^2 - 2$$

$$y = -6$$

Critical Point: $(2, -6)$

Max. and Min. Points – Ex. 1

How do we know whether these points $(0, -2)$ $(2, -6)$ are max. or min. points?

Analyse whether the function is increasing/decreasing before and after each of the turning points. $f'(x) = 3x^2 - 6x$

$x = 0$ $\quad$ $x = 2$

$x = -1$	$x = 1$	$x = 3$
$f'(-1) = 9$	$f'(1) = -3$	$f'(3) = 9$
Function is increasing	Function is decreasing	Function is increasing

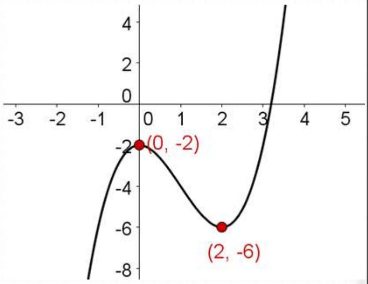
This indicates that there is a maximum point at $x = 0$ and a minimum point at $x = 2$

Max. and Min. Points – Ex. 1

Graph of $f(x) = x^3 - 3x^2 - 2$

Max. point: $(0, -2)$

Minimum point: $(2, -6)$



Max and Min Points – Ex. 2

Find the maximum and minimum points of the curve:

$$f(x) = \frac{x-1}{x-2} \qquad \frac{u}{v}$$

$u = x - 1$	$v = x - 2$
$\frac{du}{dx} = 1$	$\frac{dv}{dx} = 1$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Max and Min Points – Ex. 2

$$f'(x) = \frac{(x-2)(1) - (x-1)(1)}{(x-2)^2}$$

$$f'(x) = \frac{x - 2 - x + 1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

Let $f'(x) = 0$ and find value(s) for x

$$\frac{-1}{(x-2)^2} = 0$$

Not possible, so no max. or min. points.

Max. and Min. Points – Ex. 3

Example: Find the maximum and minimum points of the curve:

$$f(x) = 2x^3 + x^2 - 4x - 2$$

Solution:

$$f'(x) = 6x^2 + 2x - 4$$

Let $f'(x) = 0$

$$6x^2 + 2x - 4 = 0$$
$$(6x - 4)(x + 1) = 0$$

$6x - 4 = 0$	$x + 1 = 0$
$x = \frac{2}{3}$	$x = -1$

Max. and Min. Points – Ex. 3

Find Corresponding y values:

$$y = 2x^3 + x^2 - 4x - 2$$

$x = \frac{2}{3}$

$$y = 2\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) - 2$$

$$y = -3.63$$

Critical Point: $\left(\frac{2}{3}, -3.63\right)$

$x = -1$

$$y = 2(-1)^3 + (-1)^2 - 4(-1) - 2$$

$$y = 1$$

Critical Point: $(-1, 1)$

Max. and Min. Points – Ex. 3

How do we know whether these points $\left(\frac{2}{3}, -3.63\right)$ $(-1, 1)$ are max or min points?

Analyse whether the function is increasing/decreasing before and after each of the turning points. $f'(x) = 6x^2 + 2x - 4$

$x = -1$ $\qquad$ $x = \frac{2}{3}$

$x = -2$	$x = 0$	$x = 1$
$f'(-2) = 16$	$f'(0) = -4$	$f'(1) = 4$
Function is increasing	Function is decreasing	Function is increasing

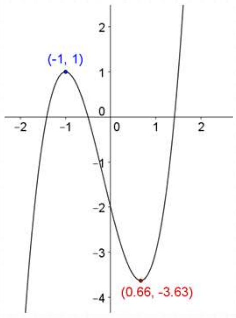
This indicates that there is a maximum point at $x = -1$ and a minimum point at $x = \frac{2}{3}$

Max. and Min. Points – Ex. 3

Graph of $f(x) = 2x^3 + x^2 - 4x - 2$

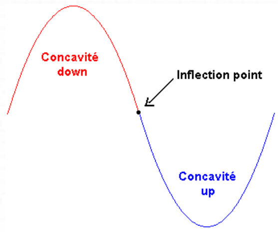
Max. point: $(-1, 1)$

Minimum point: $\left(\frac{2}{3}, -3.63\right)$



Points of Inflection

At a point of inflection, the curve changes from being concave upwards (positive curvature) to concave downwards (negative curvature), or vice versa.



Method for finding Points of Inflection

- Find $f''(x)$ and let $f''(x) = 0$
- Calculate value(s) for x from this equation
- Find the corresponding y values

Points of Inflection – Ex. 1

Example: Find the point of inflection of the curve:

$$f(x) = x^3 - 3x^2 - 2$$

Solution:

Step 1: Find $f''(x)$ and let $f''(x) = 0$

$$f(x) = x^3 - 3x^2 - 2$$

$$f'(x) = 3x^2 - 6x$$

$$f''(x) = 6x - 6$$

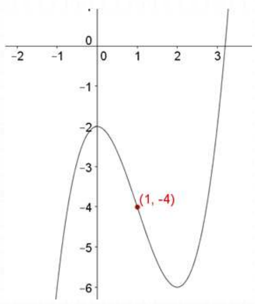
Points of Inflection – Ex. 1

Let $f''(x) = 0$	Find the corresponding y values:
$f''(x) = 6x - 6$	$y = x^3 - 3x^2 - 2$
$6x - 6 = 0$	$= (1)^3 - 3(1)^2 - 2 = -4$
$x = 1$	Point of inflection at $(1, -4)$

Points of Inflection – Ex. 1

Graph of $f(x) = x^3 - 3x^2 - 2$

Point of inflection at $(1, -4)$



Curve Sketching – Complete Ex. 1

Consider the function $f(x) = 2x^4 - 4x^2 + 1$

i. Find the y intercept of $f(x)$.

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

iii. Find all points of inflection.

iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$

v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – iv)

Curve Sketching – Complete Ex. 1

i. Find the y intercept of $f(x)$.

Let $x = 0$

$$f(x) = 2x^4 - 4x^2 + 1$$

$$f(0) = 2(0)^4 - 4(0)^2 + 1$$

$$f(0) = 1$$

y intercept is at the point (0,1)

Curve Sketching – Complete Ex. 1

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

$$f(x) = 2x^4 - 4x^2 + 1$$

$$f'(x) = 8x^3 - 8x$$

Let $f'(x) = 0$

$$8x^3 - 8x = 0$$

$$8x(x^2 - 1) = 0$$

Curve Sketching – Complete Ex. 1

$$8x(x^2 - 1) = 0$$

$8x = 0$ | $x^2 - 1 = 0$

$x = 0$ | $x = \pm 1$

Find corresponding y-values

$$f(x) = 2x^4 - 4x^2 + 1$$

$$f(0) = 2(0)^4 - 4(0)^2 + 1$$

$f(0) = 1$ Point: $(0, 1)$

$$f(x) = 2x^4 - 4x^2 + 1$$

$$f(1) = 2(1)^4 - 4(1)^2 + 1$$

$$f(1) = -1$$

Point: $(1, -1)$

$$f(-1) = 2(-1)^4 - 4(-1)^2 + 1$$

$$f(-1) = -1$$

Point: $(-1, -1)$

Curve Sketching – Complete Ex. 1

Critical points: $(0,1), (1,-1), (-1,-1)$

Classify these points: maximum or minimum points?

Analyse whether the function is increasing/decreasing before and after each of the turning points. $f'(x) = 8x^3 - 8x$

$x = -1$ $x = 0$ $x = 1$

$x = -2$	$x = -0.5$	$x = 0.5$	$x = 2$
$f'(-2) = -48$	$f'(-0.5) = 3$	$f'(0.5) = -3$	$f'(2) = 48$
Function is decreasing	Function is increasing	Function is decreasing	Function is increasing

Curve Sketching – Complete Ex. 1

Minimum point at $(-1, -1)$

Maximum point at $(0, 1)$

Minimum point at $(1, -1)$

Curve Sketching – Complete Ex. 1

iii. Find all points of inflection.

Find $f''(x)$ and let $f''(x) = 0$

$f'(x) = 8x^3 - 8x$

$f''(x) = 24x^2 - 8$

$24x^2 - 8 = 0$

$3x^2 - 1 = 0$

$3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}}$

$x = \pm 0.577$

Curve Sketching – Complete Ex. 1

$x = 0.577$

$f(x) = 2x^4 - 4x^2 + 1$

$f(0.577)$
$= 2(0.577)^4 - 4(0.577)^2 + 1$

$f(0.577) = -0.11$

Point of Inflection:
$(0.577, -0.11)$

$x = -0.577$

$f(x) = 2x^4 - 4x^2 + 1$

$f(-0.577)$
$= 2(-0.577)^4 - 4(-0.577)^2 + 1$

$f(-0.577) = -0.11$

Point of Inflection:
$(-0.577, -0.11)$

Curve Sketching – Complete Ex. 1

iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$

$f(x) = 2x^4 - 4x^2 + 1$

$\lim_{x\to+\infty} 2x^4 - 4x^2 + 1$

Take highest power of x outside brackets:

$\lim_{x\to+\infty} x^4\left(\frac{2x^4}{x^4} - \frac{4x^2}{x^4} + \frac{1}{x^4}\right)$

$\lim_{x\to+\infty} x^4\left(2 - \frac{4}{x^2} + \frac{1}{x^4}\right)$

$= \infty^4\left(2 - \frac{4}{\infty^2} + \frac{1}{\infty^4}\right)$

$= \infty(2 - 0 + 0)$

$= +\infty$

As the x value approaches $+\infty$ the y value will approach $+\infty$

Curve Sketching – Complete Ex. 1

Now, we'll check the function as $x \to -\infty$

$f(x) = 2x^4 - 4x^2 + 1$

$\lim_{x\to-\infty} 2x^4 - 4x^2 + 1$

Take highest power of x outside brackets:

$\lim_{x\to-\infty} x^4\left(\frac{2x^4}{x^4} - \frac{4x^2}{x^4} + \frac{1}{x^4}\right)$

$\lim_{x\to-\infty} x^4\left(2 - \frac{4}{x^2} + \frac{1}{x^4}\right)$

$= (-\infty)^4\left(2 - \frac{4}{(-\infty)^2} + \frac{1}{(-\infty)^4}\right)$

$= \infty(2 - 0 + 0)$

$= +\infty$

As the x value approaches $-\infty$ the y value will approach $+\infty$

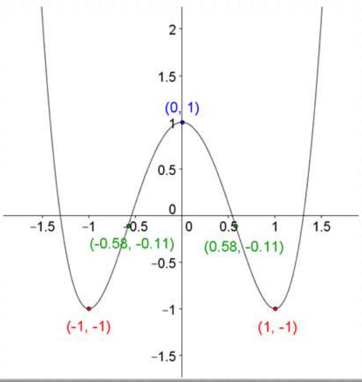
Curve Sketching – Complete Ex. 1

v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – v)

What we know:

- y intercept is at the point $(0, 1)$
- Minimum point at $(-1, -1)$
- Maximum point at $(0, 1)$
- Minimum point at $(1, -1)$
- Points of Inflection: $(0.577, -0.11)$ and $(-0.577, -0.11)$
- As the x value approaches $+\infty$ the y value will approach $+\infty$
- As the x value approaches $-\infty$ the y value will approach $+\infty$

Curve Sketching – Complete Ex. 1



Curve Sketching – Complete Ex. 2

Consider the function $f(x) = \frac{x}{x-1}$

i. Find the x and y intercepts of $f(x)$.
ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.
iii. Find all points of inflection.
iv. Find the vertical and horizontal asymptotes
v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – iv)

Curve Sketching – Complete Ex. 2

x-intercept: let $y = 0$

$$f(x) = \frac{x}{x-1}$$

$$0 = \frac{x}{x-1}$$

$$x = 0$$

x-intercept: (0,0)

y-intercept: let $x = 0$

$$f(x) = \frac{x}{x-1}$$

$$f(0) = \frac{0}{0-1}$$

$$f(0) = 0$$

y-intercept: (0,0)

Curve Sketching – Complete Ex. 2

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

Critical points: Find $f'(x)$ and let $f'(x) = 0$

$$f(x) = \frac{x}{x-1} \quad \frac{u}{v}$$

$$u = x \quad \Big| \quad v = x - 1$$

$$\frac{du}{dx} = 1 \quad \Big| \quad \frac{dv}{dx} = 1$$

$$f'(x) = \frac{(x-1)(1) - (x)(1)}{(x-1)^2} = -\frac{1}{(x-1)^2}$$

Curve Sketching – Complete Ex. 2

Let $f'(x) = 0$

$$-\frac{1}{(x-1)^2} = 0$$

Not possible: -1 divided by a number cannot equal zero.

Result: no critical points.

Curve Sketching – Complete Ex. 2

iii. Find all points of inflection.

Points of inflection: find $f''(x)$ and let $f''(x) = 0$

$$f'(x) = -\frac{1}{(x-1)^2} = -1(x-1)^{-2}$$

$$f''(x) = -1(-2)(x-1)^{-3}(1)$$

$$f''(x) = \frac{2}{(x-1)^3}$$

Curve Sketching – Complete Ex. 2

Let $f''(x) = 0$

$$f''(x) = \frac{2}{(x-1)^3}$$

$$\frac{2}{(x-1)^3} = 0$$

Not possible: 2 divided by a number cannot equal zero.

Result: no points of inflection.

Curve Sketching – Complete Ex. 2

iv. Find the vertical and horizontal asymptotes

Vertical asymptote: let denominator = 0 and find a value for x

$$f(x) = \frac{x}{x-1}$$

$$x - 1 = 0$$

$$x = 1$$

Vertical asymptote at $x = 1$

Horizontal Asymptotes: find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{x}{x-1}$$

$$\lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{x}{x} - \frac{1}{x}} = \lim_{x\to\infty} \frac{1}{1 - \frac{1}{x}}$$

$$\frac{1}{1 - \frac{1}{\infty}} = \frac{1}{1-0} = 1$$

Horizontal asymptote at $y = 1$

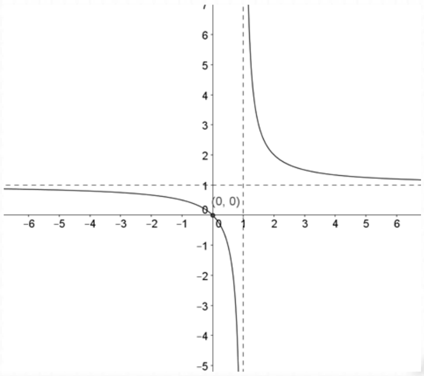
Curve Sketching – Complete Ex. 2

What we know:

- x- and y-intercept: (0,0)
- No Critical points.
- No Points of Inflection.
- Vertical asymptote at $x = 1$
- Horizontal asymptote at $y = 1$

Curve Sketching – Complete Ex. 2

$$f(x) = \frac{x}{x-1}$$



Curve Sketching – Complete Ex. 3

Consider the function $f(x) = xe^{-x}$

i. Find the x and y intercepts of $f(x)$.
ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.
iii. Find all points of inflection.
iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$
v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – iv)

Curve Sketching – Complete Ex. 3

i. Find the x and y intercepts of $f(x)$.

$$f(x) = xe^{-x}$$

Let $x = 0$

$$f(0) = (0)e^{-(0)} = 0$$

x-Interecept at (0,0)

x intercept will also turn out to be (0,0). Let $y = 0$

$$0 = xe^{-x}$$

$x = 0$	$e^{-x} = 0$ Not possible

Thus y intercept at $x = 0$

y-intercept: (0,0)

Curve Sketching – Complete Ex. 3

ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.

$$f(x) = xe^{-x}$$

Use product rule to find $f'(x)$

$u = x \qquad v = e^{-x}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -e^{-x}$

$$f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$f'(x) = x(-e^{-x}) + e^{-x}(1)$$

$$f'(x) = e^{-x}(1-x)$$

Let $f'(x) = 0$

$$e^{-x}(1-x) = 0$$

Curve Sketching – Complete Ex. 3

$e^{-x}(1-x)=0$

$e^{-x}=0$ | $1-x=0$

Not possible | $x=1$

Critical point at $x=1$

Corresponding y–value

$f(x)=xe^{-x}$

$f(1)\ =(1)e^{-1}$

$f(1)=e^{-1}=0.37$

Critical point: $(1,0.37)$

Curve Sketching – Complete Ex. 3

Critical point: $(1,0.37)$. Is it a max. or a min. point?

Analyse whether the function is increasing/decreasing before and after each of the turning points. $f'(x)=e^{-x}(1-x)$

$x=1$

$x=0$	$x=2$
$f'(0)=1$	$f'(2)=-0.135$
Function is increasing	Function is decreasing

It is clear that the point $(1,0.37)$ is a maximum point.

Curve Sketching – Complete Ex. 3

iii. Find all points of inflection.

$f'(x)=e^{-x}(1-x)$

Use product rule to find $f''(x)$

$u=e^{-x}$ $\quad v=1-x$

$\frac{du}{dx}=-e^{-x}$ $\quad \frac{dv}{dx}=-1$

$$f''(x)=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$f''(x)=e^{-x}(-1)+(1-x)(-e^{-x})$$

$$f''(x)=e^{-x}(x-2)$$

Let $f''(x)=0$

$$e^{-x}(x-2)=0$$

Curve Sketching – Complete Ex. 3

$e^{-x}(x-2)=0$

$e^{-x}=0$ | $x-2=0$

Not possible | $x=2$

Point of Inflection at $x=2$

Find corresponding y-value

$f(x)=xe^{-x}$

$f(2)=(2)e^{-2}$

$f(2)=0.27$

Point of Inflection at $(2,0.27)$

Curve Sketching – Complete Ex. 3

iv. Determine the behaviour of y as $x\to-\infty$ and as $x\to+\infty$

$$\lim_{x\to+\infty} xe^{-x}$$

$$\frac{\infty}{e^{\infty}}=0$$

Reason: the exponential will grow much quicker.

For example:

At 10:

$$\frac{10}{e^{10}}=0.00045$$

At 100:

$$\frac{100}{e^{100}}=3.72\times10^{-42}$$

As x approaches $+\infty$, the y value approaches 0

Curve Sketching – Complete Ex. 3

Behaviour as $x\to-\infty$

$$\lim_{x\to-\infty} xe^{-x}$$

$$=(-\infty)e^{-(-\infty)}$$

$$=(-\infty)e^{\infty}$$

$$=(-\infty)(\infty)$$

$$=-\infty$$

As x approaches $-\infty$, the y value approaches $-\infty$

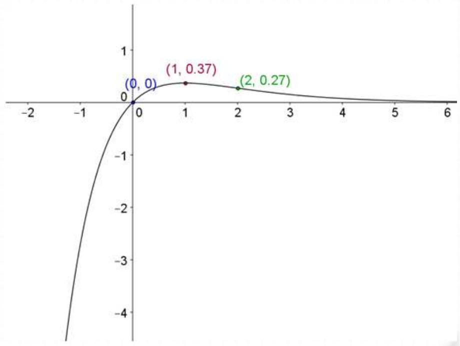
Curve Sketching – Complete Ex. 3

What we know:

- x- and y-Intercept at (0,0)
- Maximum point at $(1, 0.37)$
- Point of Inflection at $(2, 0.27)$
- As x approaches $+\infty$, the y value approaches 0
- As x approaches $-\infty$, the y value approaches $-\infty$

Curve Sketching – Complete Ex. 3

$f(x) = xe^{-x}$



Curve Sketching – Complete Ex. 4

The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{0.17h}{h^2 + 2}$$

i. Find the axis intercepts of $A(h)$.

ii. Find and classify the critical points of $A(h)$ as local maxima or local minima.

iii. Determine the behaviour of $A(h)$ as $h \to +\infty$

iv. Sketch the graph of $y = A(h)$ for $h \geq 0$ illustrating clearly the features of the curve obtained in parts (i – iii)

Curve Sketching – Complete Ex. 4

i. Find the axis intercepts of $A(h)$.

$$A(h) = \frac{0.17h}{h^2 + 2}$$

Let $h = 0$

$$A(0) = \frac{0.17(0)}{(0)^2 + 2} = 0$$

Intercept at (0,0)

The other intercept will also turn out to be (0,0). Let $A(h) = 0$

$$0 = \frac{0.17h}{h^2 + 2}$$

$$0 = 0.17h$$

$$h = 0$$

intercept: (0,0)

Curve Sketching – Complete Ex. 4

ii. Find and classify the critical points of $A(h)$ as local maxima or local minima.

$$A(h) = \frac{0.17h}{h^2 + 2}$$

Use quotient rule to find $A'(h)$

$u = 0.17h \qquad v = h^2 + 2$

$\frac{du}{dh} = 0.17 \qquad \frac{dv}{dh} = 2h$

$$A'(h) = \frac{v\frac{du}{dh} - u\frac{dv}{dh}}{v^2}$$

$$= \frac{(h^2 + 2)(0.17) - (0.17h)(2h)}{(h^2 + 2)^2}$$

$$= \frac{0.17h^2 + 0.34 - 0.34h^2}{(h^2 + 2)^2}$$

$$= \frac{-0.17h^2 + 0.34}{(h^2 + 2)^2}$$

Curve Sketching – Complete Ex. 4

Let $A'(h) = 0$

$$\frac{-0.17h^2 + 0.34}{(h^2 + 2)^2} = 0$$

$$-0.17h^2 + 0.34 = 0$$

$$0.17h^2 = 0.34$$

$$h^2 = 2$$

$$h = \pm\sqrt{2}$$

Corresponding $A(h)$ value

$$A(h) = \frac{0.17h}{h^2 + 2}$$

$$A(\sqrt{2}) = \frac{0.17(\sqrt{2})}{(\sqrt{2})^2 + 2} = 0.06$$

Critical point: $(\sqrt{2}, 0.06)$

$$A(-\sqrt{2}) = \frac{0.17(-\sqrt{2})}{(-\sqrt{2})^2 + 2} = -0.06$$

Curve Sketching – Complete Ex. 4

Critical points: $(\sqrt{2}, 0.06)$ and $(-\sqrt{2}, -0.06)$. Max. or a min. point?
Only need to check the first point as we are graphing for $h \geq 0$

Analyse whether the function is increasing/decreasing before and after the turning point. $A'(h) = \frac{-0.17h^2+0.34}{(h^2+2)^2}$

$$x = \sqrt{2}$$

$x = 1$	$x = 2$
$A'(1) = 0.0188$	$A'(2) = -0.00944$
Function is increasing	Function is decreasing

It is clear that the point $(\sqrt{2}, 0.06)$ is a maximum point.

Curve Sketching – Complete Ex. 4

iii. Determine the behaviour of $A(h)$ as $h \to +\infty$

$$\lim_{h\to\infty} \frac{0.17h}{h^2+2}$$

$$= \lim_{h\to\infty} \frac{0.17(\infty)}{(\infty)^2+2}$$

$$= \frac{\infty}{\infty}$$

Not defined…

$$\lim_{h\to\infty} \frac{0.17h}{h^2+2}$$

$$= \lim_{h\to\infty} \frac{\frac{0.17}{h}}{1+\frac{2}{h^2}}$$

$$= \frac{\frac{0.17}{\infty}}{1+\frac{2}{(\infty)^2}}$$

$$\frac{0}{1+0} = 0$$

Curve Sketching – Complete Ex. 4

$$\lim_{h\to\infty} \frac{0.17h}{h^2+2} = 0$$

So, as $h \to \infty$, $A(h) \to 0$.

This means that, as number of hours after the drug was injected h increases toward infinity, the concentration of the drug $A(h)$ will approach zero.

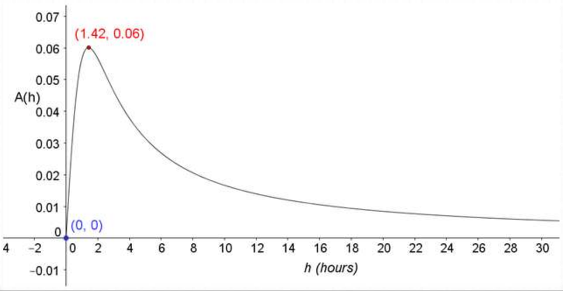
Curve Sketching – Complete Ex. 4

What we know:

- Axis Intercept at (0,0)
- Maximum point at $(\sqrt{2}, 0.06)$
- As h approaches $+\infty$, the $A(h)$ value approaches 0

Curve Sketching – Complete Ex. 4

$$A(h) = \frac{0.17h}{h^2+2}$$



Chapter 3: Integration

MA4702

Week 5, 6 & 7

Presented by Páraic Treacy

What is Integration?

Integration is the reverse process of differentiation.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

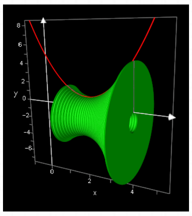
Add one to the power   Divide by new power

"c" represents the constant that may be present (to be explained later).

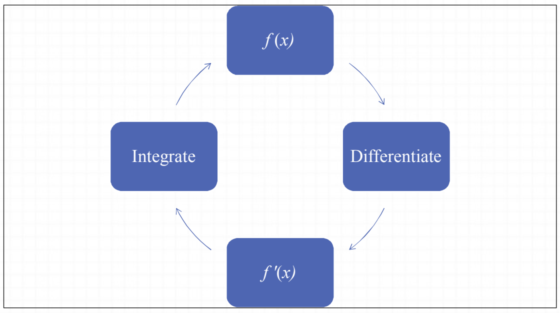
Applications of Integration

Integration is used to calculate the following:

- Moments of Inertia.
- Volume of a solid revolution.
- Electric Charges.
- Force by a liquid pressure.
- Area under a curve.
- Area between two curves.
- Work by a variable force.
- displacement, Velocity, Acceleration.



Integration and Differentiation



Integration and Differentiation

If we start with the function $f(x) = 3x^2 + 4x - 10$

Differentiate $f(x)$

$$\frac{df}{dx} = 6x + 4$$

Integrate to reverse this process:

$$\int df = \int 6x + 4\, dx$$

$$f(x) = \frac{6x^2}{2} + 4x + c$$

$$f(x) = 3x^2 + 4x + c$$

Integration and Differentiation

We started with

$$f(x) = 3x^2 + 4x - 10$$

Integration reversed the process of differentiation and we ended up with

$$f(x) = 3x^2 + 4x + c$$

The constant (-10) was eliminated during differentiation thus the reason for including "c" when integrating.

Rules for Integration

$\int k\ dx = kx + c$ [k is a constant]

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ [$n \neq -1$]

$\int k\ f(x) = k \int f(x)$ [k is a constant]

$\int [f(x) \pm g(x)]dx = \int f(x)\ dx \ \pm \ \int g(x)\ dx$

COMMON INTEGRALS	
$\int k\ dx = kx + C$	$\int \sec^2 x\ dx = \tan x + C$
$\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$	$\int \sec x \tan x\ dx = \sec x + C$
$\int x^{-1}\ dx = \int \frac{1}{x}\ dx = \ln\|x\| + C$	$\int \csc x \cot x\ dx = -\csc x + C$
$\int \frac{1}{ax+b}\ dx = \frac{1}{a}\ln\|ax+b\| + C$	$\int \csc^2 x\ dx = -\cot x + C$
$\int \ln(x)\ dx = x\ln(x) - x + C$	$\int \tan x\ dx = \ln\|\sec x\| + C$
$\int e^x\ dx = e^x + C$	$\int \sec x\ dx = \ln\|\sec x + \tan x\| + C$
$\int \cos x\ dx = \sin x + C$	$\int \frac{1}{a^2+u^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \sin x\ dx = -\cos x + C$	$\int \frac{1}{\sqrt{a^2-u^2}}\ dx = \sin^{-1}\left(\frac{u}{a}\right) + C$

Basic Examples

$$\int x^2 + 2\cos x\ dx$$

$$= \frac{x^3}{3} + 2\sin x + c$$

$$\int 4e^x + \frac{1}{x} - 20$$

$$= 4e^x + lnx - 20x + c$$

Remember...

When differentiating the product of two functions we used the product rule:

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

When differentiating the quotient of two functions we used the quotient rule:

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Remember...

When finding the derivative of the composition of two functions, we use the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Substitution and Integration by Parts

Integration requires a similar type of approach.

When integrating the product, quotient, or composition of two functions, we can integrate using one of the following:

1. Substitution.
2. Integration by parts.

We'll explore substitution first.

Substitution – Ex. 1

Evaluate:

$$\int \frac{2x}{(x^2+4)^{10}}\,dx$$

Solution:

Let $u = x^2 + 4$

From here, we will replace all "x parts" with "u parts"

$$u = x^2 + 4$$

$$\frac{du}{dx} = 2x$$

$$\frac{du}{2x} = dx$$

Now we can replace dx with what it is in terms of u.

Substitution – Ex. 1

$$\int \frac{2x}{(x^2+4)^{10}}\,dx$$

Replace with "u parts":

$$\int \frac{2x}{u^{10}}\frac{du}{2x}$$

$$\int \frac{1}{u^{10}}\,du$$

Notice how it's simplified into one function – now we can integrate.

$$\int u^{-10}\,du$$

$$= \frac{u^{-9}}{-9} + c$$

$$= -\frac{1}{9u^9} + c$$

$$= -\frac{1}{9(x^2+4)^9} + c$$

Substitution – Ex. 2

Evaluate:

$$\int x \sin x^2\ dx$$

We have the product of two functions so we must use substitution to integrate.

Let $u = x^2$

Note: knowing what to let u equal may take some trial and error.

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{du}{2x} = dx$$

Now we can replace dx with what it is in terms of u.

Substitution – Ex. 2

$$\int x \sin x^2\ dx$$

$$= \int x \sin u\,\frac{du}{2x}$$

$$= \int \frac{\sin u}{2}\,du$$

$$= \frac{1}{2}\int \sin u\,du$$

Notice how it's simplified into one function – now we can integrate.

$$= \frac{1}{2}(-\cos u) + c$$

$$= -\frac{1}{2}\cos u + c$$

$$= -\frac{1}{2}\cos x^2 + c$$

Steps for Integrating by Substitution

1. Decide what to let u equal (this may take some trial and error).
2. Find $\frac{du}{dx}$ and adjust it to get a value for dx
3. Replace dx and substitute in u wherever possible.
4. At this point, there should be no x values in the integral.
5. Integrate the function then, at the end, replace u with what it was originally.

Substitution – Ex. 3

Evaluate:

$$\int (x+2)^{20}\,dx$$

Let $u = x + 2$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Now we can replace dx with what it is in terms of u.

$$\int (x+2)^{20}\,dx$$

$$= \int u^{20}\,du$$

$$= \frac{u^{21}}{21} + c$$

$$= \frac{(x+2)^{21}}{21} + c$$

Substitution – Ex. 4

Evaluate:

$$\int e^{3x+5}\,dx$$

Let $u = 3x+5$

$$\frac{du}{dx} = 3$$

$$\frac{du}{3} = dx$$

$$\int e^{3x+5}\,dx$$

$$= \int e^u \frac{du}{3}$$

$$= \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + c$$

$$= \frac{1}{3}e^{3x+5} + c$$

Substitution – Ex. 5

Evaluate:

$$\int \sin^4 x \cos x\;dx$$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$\frac{du}{\cos x} = dx$$

Now we can replace dx with what it is in terms of u.

$$\int \sin^4 x \cos x\;dx$$

$$= \int u^4 \cos x \frac{du}{\cos x}$$

$$= \int u^4\,du$$

Substitution – Ex. 5

$$\int u^4\,du$$

$$= \frac{u^5}{5} + c$$

$$= \frac{(\sin x)^5}{5} + c$$

$$= \frac{\sin^5 x}{5} + c$$

$$\int \sin^4 x \cos x\;dx = \frac{\sin^5 x}{5} + c$$

Applying Integration to Dynamics

When we differentiate w.r.t. time a function representing the displacement of an object, we obtain an equation for the velocity of that object. If we differentiate again, we get an equation for the acceleration of that object.

Displacement → Velocity → Acceleration

As integration is the opposite of differentiation, the same process can be applied through integration but in reverse:

Acceleration → Velocity → Displacement

Applying Integration to Dynamics

Acceleration: $a(t)$

Velocity: $v(t)$

Displacement: $s(t)$

$$\int a(t)dt = v(t)$$

$$\int v(t)dt = s(t)$$

Dynamics – Ex. 1

Q. A car has an acceleration:

$$a(t) = 2 + 6t$$

The car starts from rest at time $t = 0$ from position $s = 10$. Find its position and velocity at all times t.

Find velocity first:

$$v(t) = \int a(t)\,dt$$

$$v(t) = \int (2+6t)dt$$

$$v(t) = 2t + 3t^2 + c$$

Dynamics – Ex. 1

We know that velocity was zero at $t = 0$. This info will help us find out what c is:

$$v(t) = 2t + 3t^2 + c$$

$$v(0) = 2(0) + 3(0)^2 + c$$

$$v(0) = c$$

Velocity was zero at $t = 0$

$$v(0) = 0$$

Thus, $c = 0$

$$v(t) = 2t + 3t^2$$

This is the velocity of the car at all times t.

Next, find the displacement of the car $s(t)$

$$s(t) = \int v(t)\,dt$$

$$s(t) = \int (2t + 3t^2)dt$$

Dynamics – Ex. 1

$$s(t) = \int 2t + 3t^2$$

$$s(t) = t^2 + t^3 + c$$

We know that the car started from rest at time $t = 0$ from position $s = 10$. This info will help us find c.

$$s(0) = (0)^2 + (0)^3 + c$$

$$s(0) = c$$

We know $s(0) = 10$

Thus, $c = 10$

$$s(t) = t^2 + t^3 + 10$$

This is the position of the car at all times t.

Dynamics – Ex. 2

Q. A boat has a velocity:

$$v(t) = 3t^2 + 4t$$

The car starts from rest at time $t = 0$ from position $s = 4$. Find its position at all times t.

$$s(t) = \int v(t)dt$$

$$s(t) = \int (3t^2 + 4t)dt$$

$$s(t) = \frac{3t^3}{3} + \frac{4t^2}{2} + c$$

$$s(t) = t^3 + 2t^2 + c$$

Dynamics – Ex. 2

$$s(t) = t^3 + 2t^2 + c$$

Remember: the car starts from rest at time $t = 0$ from position $s = 4$.

Thus:

$$s(0) = 4$$

$$s(t) = t^3 + 2t^2 + c$$

$$s(0) = (0)^3 + 2(0)^2 + c$$

$$s(0) = c$$

We know: $s(0) = 4$

$$c = 4$$

$$s(t) = t^3 + 2t^2 + 4$$

Dynamics – Ex. 3

Q. When an object is dropped, its acceleration (ignoring air resistance) is constant at 9.8 m/s^2

$$a(t) = -9.8$$

The negative sign is used because the object is falling.

Suppose an object is thrown from the 381 metre high rooftop of the Empire State Building in New York. If the initial velocity of the object is $-6\ m/s$, find out how long it takes the object to hit the ground and the velocity at which the object is travelling just before it hits the ground.

Dynamics – Ex. 3

$$a(t) = -9.8$$

To find the velocity of the object we must integrate the acceleration:

$$v(t) = \int a(t)\,dt$$

$$v(t) = \int (-9.8)\,dt$$

$$v(t) = -9.8t + c$$

We know the initial velocity was $-6\ m/s$ so we can find "c" using this information:

$$v(0) = -9.8(0) + c$$

$$v(0) = c$$

But we know:

$$v(0) = -6$$

Thus

$$c = -6$$

$$v(t) = -9.8t - 6$$

Dynamics – Ex. 3

$v(t) = -9.8t - 6$

To find the distance of the object from the ground, we must integrate the velocity:

$$s(t) = \int v(t)\,dt$$

$$s(t) = \int (-9.8t - 6)\,dt$$

$$s(t) = -4.9t^2 - 6t + c$$

We know the initial distance from the ground was 381 metres so we can find "c" using this information:

$$s(0) = -9.8(0)^2 - 6(0) + c$$

$$s(0) = c$$

$$s(0) = 381$$

Thus

$$c = 381$$

$$s(t) = -4.9t^2 - 6t + 381$$

Dynamics – Ex. 3

When the object hits the ground, its distance from the ground will be zero:

$$s(t) = -4.9t^2 - 6t + 381$$

$$-4.9t^2 - 6t + 381 = 0$$

Solve for t using quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(-4.9)(381)}}{2(-4.9)}$$

$$= \frac{6 \pm \sqrt{7503.6}}{-9.8}$$

$$t = -9.45 \text{ or } 8.23$$

We take our value for time as 8.23 seconds, because a negative value would indicate going backwards in time.

The object takes 8.23 seconds to reach the ground.

Dynamics – Ex. 3

To find the velocity of the object just before it hits the ground, substitute 8.23 instead of t in the equation for the velocity of the object as it falls:

$$v(t) = -9.8t - 6$$

$$v(8.23) = -9.8(8.23) - 6$$

$$v(8.23) = -86.654$$

This indicates that the object was travelling at a velocity of 86.654 m/s just as it hit the ground.

This is equivalent to a velocity of 311.95 km/h

The minus indicates that it was moving in a downward direction i.e. towards the ground.

Note: air resistance would alter this velocity considerably.

Electric Circuits and Integration

Current is the rate of change of charge.

Thus, when the equation for charge is differentiated w.r.t. time, the result is an equation for current.

Similarly, when the equation for current is integrated w.r.t. time, the result is an equation for charge.

$q(t)$ = Charge on a capacitor at time t.
$i(t)$ = current passing through the capacitor at time t.

$$q(t) = \int i(t)\,dt$$

Electric Circuits – Ex. 1

A current $i(t) = 4e^{-2t}$ passes through a capacitor at time t. The capacitor is uncharged initially. Find the charge $q(t)$ at all times t.

Solution:

$$q(t) = \int i(t)\,dt$$

$$q(t) = \int 4e^{-2t}dt$$

$$q(t) = \frac{4e^{-2t}}{-2} + c$$

$$q(t) = -2e^{-2t} + c$$

The capacitor is uncharged initially thus $q(0) = 0$

$$q(0) = -2e^{-2(0)} + c$$

$$q(0) = -2e^{0} + c$$

$$q(0) = -2 + c$$

Electric Circuits – Ex. 1

$$q(0) = -2 + c$$

Also $\quad q(0) = 0$

So:

$$-2 + c = 0$$

$$c = 2$$

Thus:

$$q(t) = -2e^{-2t} + 2$$

The charge of the capacitor $q(t)$ at all times t is:

$$q(t) = -2e^{-2t} + 2$$

Electric Circuits – Ex. 2

2001 Q.3(b)

A current $i(t) = 5 + 6\sin 3t$ passes through a capacitor at time t.

The capacitor is uncharged at time $t = 0$. Find the charge $q(t)$ at all times t.

Solution:

$$q(t) = \int i(t)\,dt$$

$$q(t) = \int (5 + 6\sin 3t)dt$$

$$q(t) = 5t - \frac{6\cos 3t}{3} + c$$

$$q(t) = 5t - 2\cos 3t + c$$

Electric Circuits – Ex. 2

$$q(t) = 5t - 2\cos 3t + c$$

Capacitor is uncharged at $t = 0$, so:

$$q(0) = 0$$

$$q(0) = 5(0) - 2\cos 3(0) + c$$

$$q(0) = 0 - 2(1) + c$$

$$q(0) = c - 2$$

If:

$$q(0) = 0$$

and

$$q(0) = c - 2$$

then

$$c - 2 = 0$$

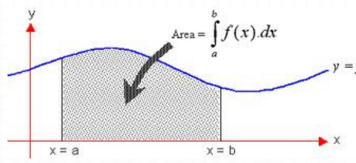
$$c = 2$$

$$q(t) = 5t - 2\cos 3t + 2$$

The Definite Integral

To determine the area under a curve $f(x)$ between $x = a$ and $x = b$ we use the definite integral:

$$\int_a^b f(x)\,dx$$



Definite Integral – Ex. 1

Evaluate:

$$\int_0^2 x^3 + 1$$

$$= \left[\frac{x^4}{4} + x\right]_0^2$$

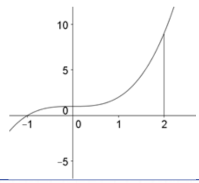
Sub in the limits 2 and 0

$$= \left(\frac{(2)^4}{4} + 2\right) - \left(\frac{(0)^4}{4} + 0\right)$$

$$= 6 - 0$$

$$= 6$$

The area under the curve $f(x) = x^3 + 1$ between $x = 0$ and $x = 2$ is 6 units^2



Definite Integral – Ex. 1

Note: when the function was integrated, the "c" that is normally added on was left out.

The reason for this is that when calculating the definite integral, "c" will be cancelled out every time:

$$\int_0^2 (x^3+1)dx$$

$$= \left[\frac{x^4}{4} + x + c\right]_0^2$$

Sub in the limits 2 and 0

$$= \left(\frac{(2)^4}{4} + 2 + c\right) - \left(\frac{(0)^4}{4} + 0 + c\right)$$

$$= 6 + c - 0 - c = 6$$

"c" always cancels so we just leave it out at the start.

Definite Integral – Ex. 2

Find the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = \pi$

Solution:

$$\int_0^\pi \sin x\,dx$$

$$= [-\cos x]_0^\pi$$

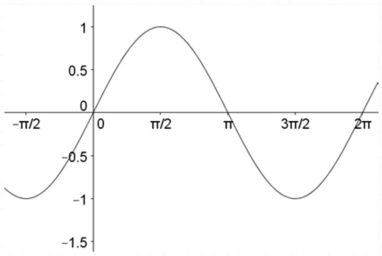
$$= (-\cos\pi) - (-\cos 0)$$

$$= (1) - (-1)$$

$$= 2$$

The area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = \pi$ is 2 units^2

Definite Integral – Ex. 2



Definite Integral – Ex. 3

Find the area between $f(x) = \sin x$ and the x axis between $x = \pi$ and $x = 2\pi$

Solution:

$$\int_{\pi}^{2\pi} \sin x \, dx$$

$$= [-\cos x]_{\pi}^{2\pi}$$

$$= (-\cos 2\pi) - (-\cos \pi)$$

$$= (-1) - (1)$$

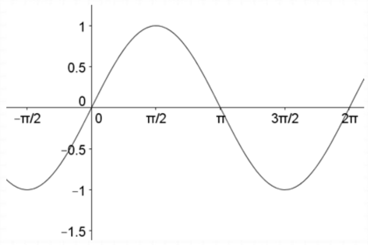
$$= -2$$

Because it is an area that we are calculating, we ignore the minus.

The area between $f(x) = \sin x$ and the x axis between $x = \pi$ and $x = 2\pi$ is 2 units^2

Definite Integral – Ex. 3

When the area is below the x axis, the calculation of the area between the curve and the x axis will produce a negative value as it did in this example.



Definite Integral – Ex. 4

Find the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$

Solution:

$$\int_{0}^{2\pi} \sin x \, dx$$

$$= [-\cos x]_{0}^{2\pi}$$

$$= (-\cos 2\pi) - (-\cos 0)$$

$$= (-1) - (-1)$$

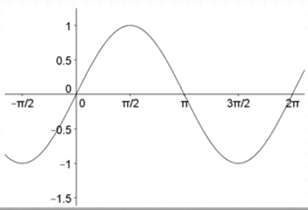
$$= 0$$

This suggest that there is no area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$

But if we look at the graph of $f(x) = \sin x$ we can see that this is obviously not true…

Definite Integral – Ex. 4

The positive area between $x = 0$ and $x = \pi$ was cancelled out by the negative area between $x = \pi$ and $x = 2\pi$. As such, the area between the curve and the x axis above the x axis needs to be calculated separately to the area between the curve and the x axis below the x axis, then combined to find the full area.



Definite Integral – Ex. 4

In Example 2, we found:

$$\int_{0}^{\pi} \sin x \, dx = 2 \text{ units}^2$$

In Example 3, we found:

$$\int_{\pi}^{2\pi} \sin x \, dx = 2 \text{ units}^2$$

This means that the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$ is:

$$2 + 2 = 4 \text{ units}^2$$

Definite Integral – Ex. 5

Find the area between $f(x) = x^2 + 4x$ and the x axis between $x = -4$ and $x = 3$

Solution:

We must check if the curve crosses the x axis and how that might affect our calculations.

$$f(x) = x^2 + 4x$$

Crosses x axis when $y = 0$

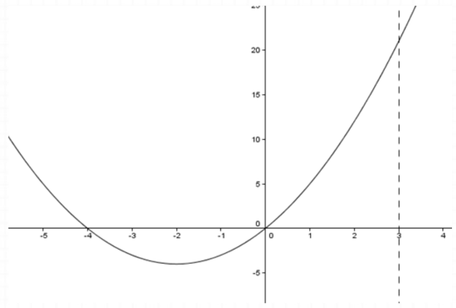
$$0 = x^2 + 4x$$

$$x(x+4) = 0$$

$$x = 0 \quad or \quad x = -4$$

Crosses x axis at 0 and -4

Definite Integral – Ex. 5



Definite Integral – Ex. 5

As such, we need to find the area below the x axis and the area above the x axis separately.

We'll start with the area below the x axis:

$$\int_{-4}^{0} (x^2 + 4x)dx$$

$$= \left[\frac{x^3}{3} + 2x^2\right]_{-4}^{0}$$

$$= \left(\frac{(0)^3}{3} + 2(0)^2\right) - \left(\frac{(-4)^3}{3} + 2(-4)^2\right)$$

$$= 0 - \left(-\frac{64}{3} + 32\right)$$

$$= -\frac{32}{3}$$

The area is $\frac{32}{3}$ units^2

Definite Integral – Ex. 5

Next we need to find the area above the x axis:

$$\int_{0}^{3} (x^2 + 4x)dx$$

$$= \left[\frac{x^3}{3} + 2x^2\right]_{0}^{3}$$

$$= \left(\frac{(3)^3}{3} + 2(3)^2\right) - \left(\frac{(0)^3}{3} + 2(0)^2\right)$$

$$= (9 + 18) - (0)$$

$$= 27 \text{ units}^2$$

The area above the x axis is 27 units^2

Definite Integral – Ex. 5

To find the area between $f(x) = x^2 + 4x$ and the x axis between $x = -4$ and $x = 3$

We add the two areas we found:

$$Total\ Area = \frac{32}{3} + 27 = 37\frac{2}{3}$$

Answer: Area is $37\frac{2}{3}$ units^2

Definite Integrals – Ex. 6

Q. Calculate the following:

$$\int_{0}^{1} \frac{4x^3}{x^4 + 1}\, dx$$

Integrating the quotient of two functions requires us to use substitution.

Let $u = x^4 + 1$

$$\frac{du}{dx} = 4x^3$$

$$\frac{du}{4x^3} = dx$$

Now, we must also change the limits:

$x = 1 \qquad u = (1)^4 + 1 = 2$

$x = 0 \qquad u = (0)^4 + 1 = 1$

Definite Integrals – Ex. 6

$$\int_0^1 \frac{4x^3}{x^4+1}\,dx$$

$$= \int_1^2 \frac{4x^3}{u}\frac{du}{4x^3}$$

$$= \int_1^2 \frac{1}{u}\,du$$

$$= [ln|u|]_1^2$$

$$= ln|2| - ln|1|$$

$$= 0.693 - 0$$

$$= 0.693$$

Definite Integrals – Ex. 7

Q. Evaluate:

$$\int_0^{\frac{\pi}{2}} \cos^4 x \sin x\,dx$$

Solution:

Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$\frac{du}{-\sin x} = dx$$

Now, we must also change the limits:

$x = \frac{\pi}{2}$ $\quad u = \cos\left(\frac{\pi}{2}\right) = 0$

$x = 0$ $\quad u = \cos(0) = 1$

Definite Integrals – Ex. 7

$$\int_0^{\frac{\pi}{2}} \cos^4 x \sin x\,dx$$

$$= \int_1^0 u^4 \sin x \frac{du}{-\sin x}$$

$$= \int_1^0 -u^4\,du$$

$$= \left[-\frac{u^5}{5}\right]_1^0$$

$$= \left(-\frac{(0)^5}{5}\right) - \left(-\frac{(1)^5}{5}\right)$$

$$= 0 - \left(-\frac{1}{5}\right)$$

$$= \frac{1}{5}$$

Explaining Integration

If we know the rate at which a quantity is changing, we can find the total change over a period of time by integrating.

For example, if we know the equation for the velocity of an object, which is the rate of change in displacement over time, then integrating it will give us the displacement over a certain period of time.

Another example: If we know the increase in length per year of the horn of a bighorn ram then integrating the function that represents this growth will allow us to calculate the total growth of the horn over a number of years.

Rams

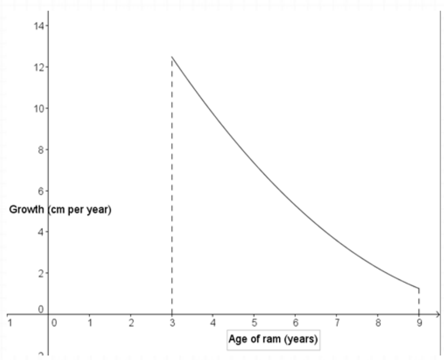
The average annual increase in the horn length (in cm) of bighorn rams can be approximated by

$$y = 0.1762x^2 - 3.986x + 22.68$$

Where x is the ram's age in years for $3 \le x \le 9$

Integrate this function to find the total increase in the length of a ram's horn during this time.

Rams



Rams

$$y = 0.1762x^2 - 3.986x + 22.68$$

Total growth between the ages of 3 and 9:

$$\int_3^9 0.1762x^2 - 3.986x + 22.68$$

$$= \frac{0.1762x^3}{3} - \frac{3.986x^2}{2} + 22.68x \Big|_3^9$$

$$= 0.0587x^3 - 1.993x^2 + 22.68x|_3^9$$

Rams

$$= 0.0587x^3 - 1.993x^2 + 22.68x|_3^9$$

$$= [0.0587(9)^3 - 1.993(9)^2 + 22.68(9)]$$
$$- [0.0587(3)^3 - 1.993(3)^2 + 22.68(3)]$$

$$= 85.4793 - 51.6879$$

$$= 33.79$$

The total increase in the length of a ram's horn between the ages of 3 and 9 was 33.79 cm

Dynamics & Def. Integral – Example

Q. Find the distance travelled $s(t)$ by a car in metres with velocity $v(t) = t^3 + 2$ in the first 4 seconds from take off.

Solution:

$$s(t) = \int v(t)dt$$

$$s(t) = \int_0^4 (t^3 + 2)dt$$

$$s(t) = \left[\frac{t^4}{4} + 2t\right]_0^4$$

$$s(t) = \left(\frac{(4)^4}{4} + 2(4)\right) - \left(\frac{(0)^4}{4} + 2(0)\right)$$

$$s(t) = (64 + 8) - 0 = 72$$

The car travelled 72 metres in the first 4 seconds.

Area Problems – Ex. 1

Q. Find the area enclosed by the curve $f(x) = x^2 - x$ and the x axis.

Solution:

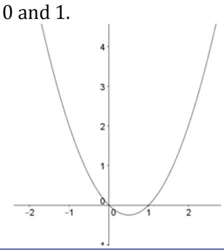
First, find where the curve cuts the x axis:

$$x^2 - x = 0$$

$$x(x - 1) = 0$$

$$x = 0 \quad or \quad x = 1$$

$f(x) = x^2 - x$ cuts the x axis at 0 and 1.



Area Problems – Ex. 1

This gives us the limits of our definite integral. To find the area between the curve and the x axis, we must calculate the following:

$$\int_0^1 (x^2 - x)\,dx$$

$$\int_0^1 (x^2 - x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

$$= \left(\frac{(1)^3}{3} - \frac{(1)^2}{2}\right) - \left(\frac{(0)^3}{3} - \frac{(0)^2}{2}\right)$$

$$= -\frac{1}{6}$$

Area Problems – Ex. 1

$$\int_0^1 (x^2 - x)\,dx = -\frac{1}{6}$$

This means that the area enclosed by the curve $f(x) = x^2 - x$ and the x axis is $\frac{1}{6}$ $units^2$

Area Problems – Ex. 2

Q. Find the area enclosed by the curve $f(x) = x^3 - 4x$ and the x axis.

Solution:

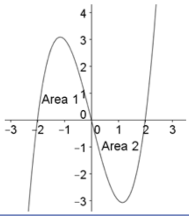
First, find where the curve cuts the x axis:

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$x = 0$ | $x^2 = 4$, $x = \pm 2$

Curve cuts the x axis at -2, 0 and 2



Area Problems – Ex. 2

We must find Area 1 and Area 2 separately.

Area 1:

$$\int_{-2}^{0} (x^3 - 4x)\, dx$$

$$\left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}$$

$$= \left(\frac{(0)^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right)$$

$$= 0 - (-4)$$

$$= 4\ units^2$$

Area Problems – Ex. 2

Area 2:

$$\int_{0}^{2} (x^3 - 4x)\, dx$$

$$\left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= \left(\frac{(2)^4}{4} - 2(2)^2\right) - \left(\frac{(0)^4}{4} - 2(0)^2\right)$$

$$= -4$$

Area 1: 4 $units^2$
Area 2: 4 $units^2$

Total Area = 8 $units^2$

The area enclosed by the curve $f(x) = x^3 - 4x$ and the x axis is 8 $units^2$

Improper Integrals

Improper Integrals are integrals involving infinity.

Example: The current flowing into a capacitor at time t is e^{-t} amps.

Find the total charge (as $t \to \infty$) which builds up on the capacitor which is initially uncharged.

Improper Integrals – Ex. 1

Solution:
Find the total charge (q) by integrating the function representing current ($i(t)$)

$$q = \int_{0}^{\infty} e^{-t}\, dt$$

$$q = \left[-e^{-t}\right]_{0}^{\infty}$$

$$q = (-e^{-\infty}) - (-e^{0})$$

$$q = \left(-\frac{1}{e^{\infty}}\right) - (-1)$$

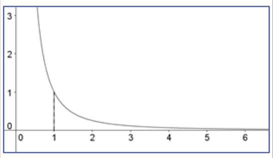
$$q = \left(-\frac{1}{\infty}\right) + 1$$

$$q = 0 + 1 = 1$$

The total charge is 1 coulomb.

Improper Integrals – Ex. 2

Q. Find the area enclosed between $y = \frac{1}{x^2}$ and the x axis for $x \geq 1$



Solution: Evaluate

$$\int_{1}^{\infty} \frac{1}{x^2}\, dx$$

$$= \int_{1}^{\infty} x^{-2}\, dx$$

$$= \left[\frac{x^{-1}}{-1}\right]_{1}^{\infty}$$

Improper Integrals – Ex. 2

$$= \left[\frac{x^{-1}}{-1}\right]_1^{\infty}$$

$$= \left[\frac{-1}{x}\right]_1^{\infty}$$

$$= \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{1}\right)$$

$$= 0 + 1 = 1$$

The area enclosed between $y = \frac{1}{x^2}$ and the x axis for $x \geq 1$ is $1\ unit^2$

Finding the Area between two curves

To find the area between two curves $g(x)$ and $f(x)$ with points of intersection at $x = a$ and $x = b$, we use the following approach:

$$\int_a^b [g(x) - f(x)]\, dx$$

Area between two curves – Ex. 1

Find the area of the region bounded by the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$

Solution:

1. Sketch the curve and the line then find the points of intersection of the curve and the line.

2. Use integration methods to find the area between the line and curve:

$$\int_a^b [g(x) - f(x)]\, dx$$

Area between two curves – Ex. 1

$y = -x^2 + 5x - 4$

x intercepts:

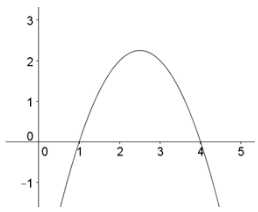
$$-x^2 + 5x - 4 = 0$$

$$x^2 - 5x + 4 = 0$$

$$(x - 4)(x - 1) = 0$$

$$x = 4 \qquad x = 1$$

Curve cuts x axis at 1 and 4.



Area between two curves – Ex. 1

Find the points of intersection of the line and the curve using simultaneous equations:

$$y = -x^2 + 5x - 4$$
$$y = x - 1$$

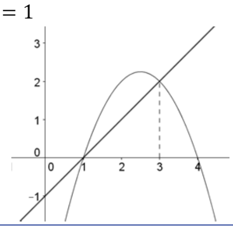
$$-x^2 + 5x - 4 = x - 1$$

$$x^2 - 4x + 3 = 0$$

$$(x - 3)(x - 1) = 0$$

$$x = 3 \qquad x = 1$$

The line and the curve intersect at $x = 3$ and at $x = 1$



Area between two curves – Ex. 1

We now know the equation for the two curves:

$$y = -x^2 + 5x - 4$$
$$y = x - 1$$

And the x values at which they intersect: 1 and 3.

So we can apply:

$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_1^3 [(-x^2 + 5x - 4) - (x - 1)]dx$$

$$\int_1^3 (-x^2 + 4x - 3)dx$$

$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$

Area between two curves – Ex. 1

$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$

$$= \left(-\frac{(3)^3}{3} + 2(3)^2 - 3(3)\right) - \left(-\frac{(1)^3}{3} + 2(1)^2 - 3(1)\right)$$

$$= (-9 + 18 - 9) - \left(-\frac{1}{3} - 1\right)$$

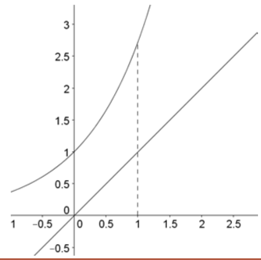
$$= 0 - \left(-\frac{4}{3}\right)$$

$$= \frac{4}{3}\ units^2$$

The area of the region bounded by the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$ is $\frac{4}{3}\ units^2$

Area between two curves – Ex. 2

Q. Find the area between $y = e^x$ and $y = x$ for $0 \le x \le 1$



$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_0^1 (e^x - x)\, dx$$

$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$

Area between two curves – Ex. 2

$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$

$$= \left(e^1 - \frac{(1)^2}{2}\right) - \left(e^0 - \frac{(0)^2}{2}\right)$$

$$= e - \frac{1}{2} - 1$$

$$= 1.218\ units^2$$

The area between $y = e^x$ and $y = x$ for $0 \le x \le 1$ is

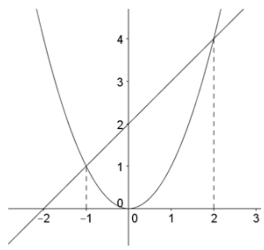
$1.218\ units^2$

Area between two curves – Ex. 3

Find the area between $y = x^2$ and $y = x + 2$

Solution:

Sketch the curves and find where they intersect.



Area between two curves – Ex. 3

Use simultaneous equations to find where they intersect:

$$y = x^2$$
$$y = x + 2$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2 \qquad x = -1$$

The curve and the line intersect at $x = 2$ and $x = -1$

Now we can apply:

$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_{-1}^2 [(x^2) - (x + 2)\ dx$$

Area between two curves – Ex. 3

$$\int_{-1}^2 [(x^2) - (x + 2)\ dx$$

$$\int_{-1}^2 (x^2 - x - 2)\ dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^2$$

$$= \left(\frac{(2)^3}{3} - \frac{(2)^2}{2} - 2(2)\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2} - 2(-1)\right)$$

$$= \left(\frac{8}{3} - 6\right) - \left(-\frac{5}{6} + 2\right)$$

$$= -4\frac{1}{2}$$

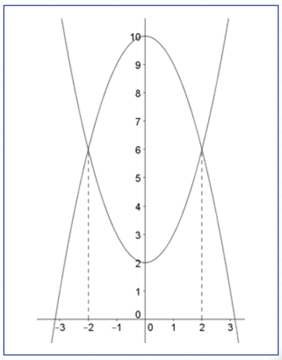
Area $= 4\frac{1}{2}$ units^2

Area between two curves – Ex. 4

Find the area enclosed by the curves $y = x^2 + 2$ and $y = 10 - x^2$

Solution:

Find the points at which the curves intersect.



Area between two curves – Ex. 4

Point of Intersection:

$$y = x^2 + 2$$
$$y = 10 - x^2$$

$$x^2 + 2 = 10 - x^2$$

$$2x^2 = 8$$

$$x = \pm 2$$

Curves intersect at $x = 2$ and at $x = -2$

Now we can find the area between the curves using:

$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_{-2}^{2} [(x^2 + 2) - (10 - x^2)]\, dx$$

Area between two curves – Ex. 4

$$\int_{-2}^{2} [(x^2 + 2) - (10 - x^2)]\, dx$$

$$= \int_{-2}^{2} (2x^2 - 8) dx$$

$$= \left[\frac{2x^3}{3} - 8x\right]_{-2}^{2}$$

$$= \left(\frac{2(2)^3}{3} - 8(2)\right) - \left(\frac{2(-2)^3}{3} - 8(-2)\right)$$

$$= \left(\frac{16}{3} - 16\right) - \left(-\frac{16}{3} + 16\right)$$

$$= -21\frac{1}{3}$$

$$\text{Area} = 21\tfrac{1}{3}\, units^2$$

Integration by Parts

- Used if the substitution method does not work.
- Typically used for the following types of integrals:

$$\int x \cos x\, dx$$

$$\int x e^x dx$$

$$\int x \ln x\, dx$$

Integration by Parts

The formula (can be found in log tables):

$$\int u\, dv = uv - \int v\, du$$

Integration by Parts – Ex. 1

Q. Evaluate:

$$\int x \cos x\, dx$$

$$\int u\, dv = uv - \int v\, du$$

First, identify u and dv, then use them to find v and du

We can see that u corresponds to x and dv corresponds to $\cos x\, dx$

Now, find du and v

$$u = x$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Integration by Parts – Ex. 1

$dv = \cos x\, dx$

$\int dv = \int \cos x\, dx$

$v = \sin x$

Now we know:

$u = x$
$du = dx$
$dv = \cos x\, dx$
$v = \sin x$

$\int u\, dv = uv - \int v\, du$

$\int x \cos x\, dx$

$= (x)(sinx) - \int \sin x\, dx$

$= x \sin x - (-\cos x) + c$

$= x \sin x + \cos x\ + c$

Integration by Parts – Ex. 2

Q. Evaluate:

$$\int_0^1 xe^x dx$$

$\int u\, dv = uv - \int v\, du$

$u = x$	$dv = e^x dx$
$\frac{du}{dx} = 1$	$v = e^x$
$du = dx$	

$\int u\, dv = uv - \int v\, du$

$\int xe^x dx = x(e^x) - \int e^x dx$

$= xe^x - e^x + c$

$\int_0^1 xe^x dx = [xe^x - e^x]_0^1$

Integration by Parts – Ex. 2

$= [xe^x - e^x]_0^1$

$= (1(e^1) - e^1) - (0(e^0) - e^0)$

$= (0) - (-1)$

$= 1$

$\int_0^1 xe^x dx = 1$

Integration by Parts – Ex. 3

Q. Evaluate:

$$\int_{-\pi}^{\pi} x \sin x\, dx$$

$\int u\, dv = uv - \int v\, du$

$u = x$	$dv = \sin x\, dx$
$\frac{du}{dx} = 1$	$v = -\cos x$
$du = dx$	

$\int u\, dv = uv - \int v\, du$

$\int x \sin x\, dx$

$= x(-\cos x) - \int (-\cos x) dx$

$= -x \cos x + \sin x + c$

Integration by Parts – Ex. 3

$\int_{-\pi}^{\pi} x \sin x\, dx$

$= [-x \cos x + \sin x]_{-\pi}^{\pi}$

$= (-\pi \cos \pi + \sin \pi) - (-(-\pi)\cos(-\pi) + \sin(-\pi))$

$= (-\pi(-1) + 0) - (\pi(-1) + 0)$

$= 2\pi$

$\int_{-\pi}^{\pi} x \sin x\, dx = 2\pi$

Steps for Integrating by Parts

1. Identify u and dv
2. Differentiate u to get a value for du
3. Integrate dv to get a value for v
4. Apply the values found to the equation:

$$\int u\, dv = uv - \int v\, du$$

5. Solve the equation and simplify your answer.

Chapter 4: Partial Differentiation

MA4702

Week 9 & 10

Presented by Dr. Páraic Treacy

Basic Differentiation – Ex. 1

Differentiate $f(x) = x^3 - 2x^2 - 7$

Solution:

$$f'(x) = (3)(x^2) + (2)(-2x^1)$$

$$f'(x) = 3x^2 - 4x$$

Basic Differentiation – Ex. 2

Differentiate $y = -3x^4 + 2x^3 - 7$

Solution:

$$\frac{dy}{dx} = (4)(-3x^3) + (3)(2x^2)$$

$$\frac{dy}{dx} = -12x^3 + 6x^2$$

Basic Differentiation – Ex. 3

Differentiate $f(x) = \frac{1}{x^3} + \frac{1}{x}$

Solution: $f(x) = x^{-3} + x^{-1}$

$$f'(x) = (-3)(x^{-4}) + (-1)(x^{-2})$$

$$f'(x) = (-3)\left(\frac{1}{x^4}\right) + (-1)(\frac{1}{x^2})$$

Basic Differentiation – Ex. 3

$$f'(x) = (-3)\left(\frac{1}{x^4}\right) + (-1)(\frac{1}{x^2})$$

$$f'(x) = \frac{-3}{x^4} - \frac{1}{x^2}$$

Basic Differentiation – Trig. & Exponential Examples

(a) $y = 4\sin x$

$$\frac{dy}{dx} = 4\cos x$$

(b) $y = 3\cos 4x$

$$\frac{dy}{dx} = 3(-\sin 4x)(4)$$

$$\frac{dy}{dx} = -12\sin 4x$$

(c) $y = -4\sin x^2$

$$\frac{dy}{dx} = -4\,(\cos x^2)(2x)$$

$$\frac{dy}{dx} = -8x\cos x^2$$

(d) $y = 3e^{4x}$

$$\frac{dy}{dx} = 12e^{4x}$$

Functions of two variables

A function of two variables is a rule f that assigns a unique value $f(x, y)$ to each pair (x, y).

Example:

$$f(x, y) = x^2 + 4xy - y$$

$x = 1$
$y = 2$

$$f(1,2) = (1)^2 + 4(1)(2) - (2)$$

$$f(1,2) = 7$$

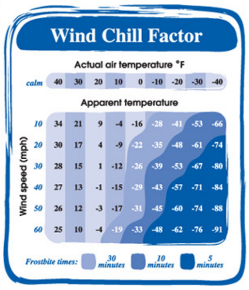
Wind Chill Index

An example of a function of two variables is the Wind Chill index developed in 1990.

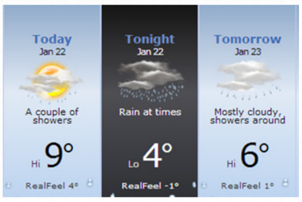
The Wind Chill Index is a means of quantifying the threat of heat loss from the human body during windy and cold conditions.

This depends on two factors (or two variables), the temperature of the air and velocity of the wind.

Wind Chill Index



calm	40	30	20	10	0	-10	-20	-30	-40
10	34	21	9	-4	-16	-28	-41	-53	-66
20	30	17	4	-9	-22	-35	-48	-61	-74
30	28	15	1	-12	-26	-39	-53	-67	-80
40	27	13	-1	-15	-29	-43	-57	-71	-84
50	26	12	-3	-17	-31	-45	-60	-74	-88
60	25	10	-4	-19	-33	-48	-62	-76	-91



Wind Chill Index

The Wind Chill Index can be calculated using the following function of two variables:

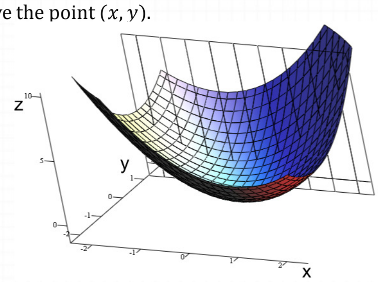
$$W(V, T) = 91.4 - \frac{(10.45 + 6.69\sqrt{V} - 0.447V)(91.4 - T)}{22}$$

Where V is the wind velocity in miles per hour $(4 \leq V \leq 45)$ and T is the temperature in Fahrenheit.

This is an example of how two inputs produce one output.

Functions in 3D

A function of two variables, $f(x, y)$ can be represented as a surface in 3 dimensions by defining $z = f(x, y)$ to be the height above the point (x, y).



Graphing 3D

Graphing 3D

Partial Differentiation

The rate of change of z in the x-direction at (x, y) is the partial derivative of z with respect to x.

$$\frac{\partial z}{\partial x}$$

To determine this partial derivative we differentiate z with respect to x, treating y as a constant.

Partial Differentiation

The rate of change of z in the y-direction at (x, y) is the partial derivative of z with respect to y.

$$\frac{\partial z}{\partial y}$$

To determine this partial derivative we differentiate z with respect to y, treating x as a constant.

Partial Differentiation – Ex. 1

Q. Find the rate of change of $z = x^2 + 5xy - 3y^2$ w.r.t. x and then w.r.t. y.

Solution: to find $\frac{\partial z}{\partial x}$ we differentiate z with respect to x, treating y as a constant.

$$z = x^2 + 5xy - 3y^2$$

$$\frac{\partial z}{\partial x} = 2x + 5y$$

To find $\frac{\partial z}{\partial y}$ we differentiate z with respect to y, treating x as a constant.

$$z = x^2 + 5xy - 3y^2$$

$$\frac{\partial z}{\partial y} = 5x - 6y$$

Partial Differentiation – Ex. 2

Q. Find the first partial derivatives and the second partial derivatives of

$$z = x^2 \sin y + y^2 \cos x$$

Solution:

$$\frac{\partial z}{\partial x} = 2x \sin y + y^2(-\sin x)$$

$$\frac{\partial z}{\partial x} = 2x \sin y - y^2 \sin x$$

$$z = x^2 \sin y + y^2 \cos x$$

$$\frac{\partial z}{\partial y} = x^2 \cos y + 2y \cos x$$

First Partial derivatives:

$$\frac{\partial z}{\partial x} = 2x \sin y - y^2 \sin x$$

$$\frac{\partial z}{\partial y} = x^2 \cos y + 2y \cos x$$

Partial Differentiation – Ex. 2

Second partial derivatives: differentiate again.

$$\frac{\partial z}{\partial x} = 2x \sin y - y^2 \sin x$$

$$\frac{\partial^2 z}{\partial x^2} = 2 \sin y - y^2 \cos x$$

$$\frac{\partial z}{\partial y} = x^2 \cos y + 2y \cos x$$

$$\frac{\partial^2 z}{\partial y^2} = x^2(-\sin y) + 2 \cos x$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2 \sin y + 2 \cos x$$

Partial Differentiation – Ex. 3

Q. Find the rate of change of $z = 3x^2 + 5x^2y + 4y$ w.r.t. x and then w.r.t. y.

Solution: to find $\frac{\partial z}{\partial x}$ we differentiate z with respect to x, treating y as a constant.

$$z = 3x^2 + 5x^2y + 4y$$

$$\frac{\partial z}{\partial x} = 6x + 10xy$$

To find $\frac{\partial z}{\partial y}$ we differentiate z with respect to y, treating x as a constant.

$$z = 3x^2 + 5x^2y + 4y$$

$$\frac{\partial z}{\partial y} = 5x^2 + 4$$

Second Partial Derivative Notation

Differentiating z w.r.t. x:

$$\frac{\partial}{\partial x}(z) = \frac{\partial z}{\partial x}$$

Differentiating again w.r.t. x:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

Mixed Partial Derivatives

Mixed partial derivatives occur when z is differentiated w.r.t. x, then the answer is differentiated w.r.t. y, or vice versa.

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Note:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Partial Differentiation – Ex. 3

Q. Find all the first, second, and mixed partial derivatives of

$$z = x^4 + 3x^2y^2 + 7$$

Solution:

$$z = x^4 + 3x^2y^2 + 7$$

$$\frac{\partial z}{\partial x} = 4x^3 + 6xy^2$$

$$z = x^4 + 3x^2y^2 + 7$$

$$\frac{\partial z}{\partial y} = 6x^2y$$

First partial derivatives:

$$\frac{\partial z}{\partial x} = 4x^3 + 6xy^2$$

$$\frac{\partial z}{\partial y} = 6x^2y$$

Partial Differentiation – Ex. 3

Second partial derivatives:

$$\frac{\partial z}{\partial x} = 4x^3 + 6xy^2$$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 + 6y^2$$

$$\frac{\partial z}{\partial y} = 6x^2y$$

$$\frac{\partial^2 z}{\partial y^2} = 6x^2$$

Mixed partial derivatives:

$$\frac{\partial z}{\partial x} = 4x^3 + 6xy^2$$

$$\frac{\partial z}{\partial x \partial y} = 12xy$$

$$\frac{\partial z}{\partial y} = 6x^2y$$

$$\frac{\partial z}{\partial y \partial x} = 12xy$$

Partial Differentiation – Ex. 4

Q. Prove that $z = e^x \sin y$ satisfies the partial differential equation known as Laplace's equation i.e.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Solution:

$$z = e^x \sin y$$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

Partial Differentiation – Ex. 4

$$z = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = e^x\,(-\sin y)$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

Now we can check:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

$$= e^x \sin y - e^x \sin y$$

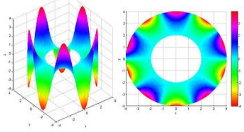
$$= 0$$

$$QED$$

Laplace's Equation

The solutions of Laplace's equation are the harmonic functions, which are important in many fields of science, notably the fields of:

- Electromagnetism.
- Astronomy.
- Fluid dynamics.



They can be used to accurately describe the behaviour of electric, gravitational, and fluid potentials. In the study of heat conduction, the Laplace equation is the steady-state heat equation.

Partial Differentiation – Ex. 5

Evaluate the second partial derivatives for

$$z = \cosh(2x - 5y)$$

Solution:

$$\frac{\partial z}{\partial x} = 2\sinh(2x - 5y)$$

$$\frac{\partial^2 z}{\partial x^2} = 4\cosh(2x - 5y)$$

$$z = \cosh(2x - 5y)$$

$$\frac{\partial z}{\partial y} = -5\sinh(2x - 5y)$$

$$\frac{\partial^2 z}{\partial y^2} = 25\cosh(2x - 5y)$$

Partial Differentiation – Ex. 6

Q. Show that $z = e^{-2x} \sin y$ satisfies the partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$$

Solution: Find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ first.

$$z = e^{-2x} \sin y$$

$$\frac{\partial z}{\partial x} = -2e^{-2x} \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = 4e^{-2x} \sin y$$

Partial Differentiation – Ex. 6

$$z = e^{-2x} \sin y$$

$$\frac{\partial z}{\partial y} = e^{-2x} \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = e^{-2x}(-\sin y)$$

$$\frac{\partial^2 z}{\partial y^2} = -e^{-2x} \sin y$$

Now we can check:

$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$$

$$4e^{-2x} \sin y + 4(-e^{-2x} \sin y) = 0$$

$$4e^{-2x} \sin y - 4e^{-2x} \sin y = 0$$

True

QED

Why is Partial Differentiation Useful?

The partial derivatives correspond to the rate of change of a function when one variable changes.

When a function is defined by two or more variables then we can assess how a change in one of those variables affects the overall function.

For example: a person's Body Mass Index (BMI) depends on their height and their weight – if one of those changes (while the other remains the same) then it will have an effect on the person's BMI reading. The partial derivative will tell us what kind of effect this change will have.

Partial Derivatives Application – Ex. 1

A person's Body Mass Index (BMI) can be calculated by dividing their weight (kg) by their height squared (m^2), so:

$$BMI = \frac{w}{h^2}$$

According to the NHS in Britain:
Normal weight: $18.5 \leq BMI \leq 25$
Overweight: $25 < BMI \leq 30$
Obese: $BMI > 30$

(a) Calculate a person's body mass index when they are 1.83 m tall and weigh 86 kg. Interpret your answer.
(b) Calculate $\frac{\partial BMI}{\partial w}$ and $\frac{\partial BMI}{\partial h}$ and interpret your answers.

Partial Derivatives Application – Ex. 1

(a) Calculate a person's body mass index when they are 1.83 metres tall and weigh 86 kg. Interpret your answer.

$$BMI = \frac{w}{h^2}$$

$$BMI = \frac{86}{(1.83)^2} = 25.7$$

This would indicate that the person in question is "overweight" but only marginally so.

Partial Derivatives Application – Ex. 1

(b) Calculate $\frac{\partial BMI}{\partial w}$ and $\frac{\partial BMI}{\partial h}$ and interpret your answers.

$$BMI = \frac{w}{h^2}$$

$$\frac{\partial BMI}{\partial w} = \frac{1}{h^2}$$

This indicates that the rate of change of BMI with respect to weight is positive. In other words an increase in weight would lead to an increase in BMI (if height remains constant).

Partial Derivatives Application – Ex. 1

$$BMI = \frac{w}{h^2} = wh^{-2}$$

$$\frac{\partial BMI}{\partial h} = -2wh^{-3} = -\frac{2w}{h^3}$$

This indicates that the rate of change of BMI with respect to height is negative. In other words, an increase in height would lead to a decrease in BMI (if weight remains constant) and vice versa.

Partial Derivatives Application – Ex. 2

Example: Your demand for a mobile phone plan depends on two important factors: the price of the plan and your income. This demand could be modelled by the following:

$$d = \frac{1}{p^2} + I$$

Where d is your level of demand for the plan, p is the monthly price of the plan, and I in your monthly income

Partial Derivatives Application – Ex. 2

If we differentiate the equation for demand w.r.t. price, this will indicate how a change in price (with income remaining the same) affects your level of demand for the phone plan:

$$d = \frac{1}{p^2} + I = p^{-2} + I$$

$$\frac{\partial d}{\partial p} = p^{-3}(-2) = -\frac{2}{p^3}$$

This indicates that an increase in price will negatively affect your level of demand for the mobile phone plan.

Partial Derivatives Application – Ex. 2

If we differentiate the equation for demand w.r.t. income, this will indicate how a change in income (with price remaining the same) affects your level of demand for the phone plan:

$$d = \frac{1}{p^2} + I$$

$$\frac{\partial d}{\partial I} = 1$$

This indicates that an increase in income will positively affect your level of demand for the mobile phone plan.

Partial Derivatives Application – Ex. 3

We can calculate the surface gravitational force of a planet using the following formula:

$$g = \frac{GM}{r^2}$$

Where:

$g = surface\ gravitational\ force$
$G = 6.67 \times 10^{-11}\ (Gravitational\ Constant)$
$M = mass\ of\ the\ planet$
$r = radius\ of\ the\ planet$

(a) What effect does the mass of a planet have on it's gravitational force?
(b) What effect does the radius of a planet have on it's gravitational force?

Partial Derivatives Application – Ex. 3

$$g = \frac{GM}{r^2}$$

$$\frac{\partial g}{\partial M} = \frac{G}{r^2}$$

This indicates that a change in mass (M) positively affects the gravitational force.

In other words, the greater the mass, the greater the gravitational force of a planet (if the radius remains constant). So if two planets had the same radius but one had a greater mass, then that planet would have a greater gravitational force.

Partial Derivatives Application – Ex. 3

$$g = \frac{GM}{r^2} = GMr^{-2}$$

$$\frac{\partial g}{\partial r} = -2GMr^{-3} = -\frac{2GM}{r^3}$$

This indicates that a change in radius (r) negatively affects the gravitational force.

In other words, the greater the radius, the less the gravitational force of a planet (if the mass remains constant). So if two planets had the same mass but one had a larger radius, then that planet would have a lesser gravitational force.