The Power Rule of Differentiation

The power rule is one of the most important differentiation rules in calculus. Since differentiation is linear, polynomials can be differentiated using this rule.

$$\frac{d}{dx}x^n = nx^{n-1}, \qquad n \neq 0.$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \qquad n \neq -1.$$

$$\int x^{-1} \, dx = \ln|x| + c$$

Trigonometric Functions

$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$

Integration by parts

$$\int u\,dv = uv - \int v\,du.$$

In order to calculate

$$I = \int x \cos(x)\, dx$$

let:

$$u = x \Rightarrow du = dx$$

$$dv = \cos(x)\, dx \Rightarrow v = \int \cos(x)\, dx = \sin x$$

then:

$$\begin{aligned}\int x\cos(x)\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= x\sin(x) - \int \sin(x)\,dx \\ &= x\sin(x) + \cos(x) + C,\end{aligned}$$

where C is an arbitrary constant of integration.

Integration by Substitution: Example

$$\begin{aligned}\int x\cos(x^2+1)\,dx &= \tfrac{1}{2}\int 2x\cos(x^2+1)\,dx\\ &= \tfrac{1}{2}\int \cos u\,du\\ &= \tfrac{1}{2}\sin u + C = \tfrac{1}{2}\sin(x^2+1)+C\end{aligned}$$

Additivity of integration on intervals

$$\begin{aligned} \int_a^c f(x)\,dx &= \int_a^b f(x)\,dx - \int_c^b f(x)\,dx \\ &= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \end{aligned}$$

Hyperbolic Function

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$(\sinh x)' = \cosh x = \frac{e^x + e^{-x}}{2}$$

Jacobian

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}.$$

Logarithmic Transformation

$$(\ln f)' = \frac{f'}{f}$$

Quotient Rule

$$f(x) = \frac{u(x)}{v(x)}$$

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Chain Rule

h(x) = sin(2x) is the composition of the functions f(g) = sin g and g(x) = 2x: As such we have $f'(g)$ = cos g and $g'(x) = 2$
; and so the chain rule tells us that

$$h'(x) = (cosg)(2) = 2cos(2x)$$

Maclaurin Series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots .$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(x-a)^n$$

Fundamental Theorem of Calculus

The fundamental theorem of calculus states that the integral of a function f over the interval $[a, b]$ can be calculated by finding an antiderivative F of f:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

ODEs: Integrating factor

The integrating factor is a function that is chosen to facilitate the solving of a given equation involving differentials. It is commonly used to solve ordinary differential equations.

$$y' + P(x)y = Q(x)$$

the integration factor is

$$M(x) = e^{\int P(x')dx'}$$

ODEs: Example

Solve the differential equation

$$y' - \frac{2y}{x} = 0.$$

We can see that in this case

$$P(x) = \frac{-2}{x}$$

$$M(x) = e^{\int P(x)\,dx}$$

$$M(x) = e^{\int \frac{-2}{x}\,dx} = e^{-2\ln x} = \left(e^{\ln x}\right)^{-2} = x^{-2}$$

(Note we do not need to include the integrating constant - we need only a solution, not the general solution)

$$M(x) = \frac{1}{x^2}.$$

Multiplying both sides by

$$M(x)$$

we obtain

$$\frac{y'}{x^2} - \frac{2y}{x^3} = 0$$

$$\frac{y'x^3 - 2x^2y}{x^5} = 0$$

$$\frac{x(y'x^2 - 2xy)}{x^5} = 0$$

$$\frac{y'x^2 - 2xy}{x^4} = 0.$$

Partial Derivatives: Volume of a Cone

The volume "V" of a cone depends on the cone's height "h" and its radius 'r' according to the formula

$$V(r,h) = \frac{\pi r^2 h}{3}.$$

The partial derivative of "V" with respect to 'r' is

$$\frac{\partial V}{\partial r} = \frac{2\pi r h}{3},$$

which represents the rate with which a cone's volume changes if its radius is varied and its height is kept constant. The partial derivative with respect to "h" is

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Numerical Integration: Simpson's Rule

Numerical integration constitutes a broad family of algorithms for calculating the numerical value of a definite integral, and by extension, the term is also sometimes used to describe the numerical solution of differential equations.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

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Integration by Parts

$$\int xe^x dx$$

Let $u = x$ therefore $\frac{du}{dx} = 1$ and hence $du = dx$
Let $dv = e^x dx$.

$$v = intdv = inte^x dx = e^x$$

$$I = uv - intvdu$$

$$I = xe^x - inte^x dx = xe^x - e^x + c$$

First Order Ordinary Differential Equations

Solve the first order differential Equation

$$\frac{dy}{dx} + y = x$$

subject to the boundary condition $y(0) = 1$.

PFE: Examples

$$\int \frac{dx}{x^2 - 9} = \frac{1}{6} ln \left| \frac{x-3}{x+3} \right| + c$$

$$\int \frac{dx}{x^2 + 7x + 6} = \frac{1}{5} ln \left| \frac{x+1}{x+6} \right| + c$$

$$\int \frac{xdx}{(x-2)^2} = ln \left| x - 2 \right| - \frac{2}{x-2} + c$$

Absolute Value Function

The absolute Value Function: $|x|$
Integrating Absolute Value Function:

Integration of Hyberolic Functions

$$\int sinh(x)dx = cosh(x) + c$$

$$\int cosh(x)dx = sinh(x) + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} dx = sinh^{-1}(x/a) + c \qquad x^2 < a^2$$

$$\int sinh(x/2)dx = 2cosh(x/2) + c$$

Chain Rule : Example

$$y = -cos(2x)$$

$$\frac{dy}{dx} = 2 \times sin(2x)$$

Partial Fraction Expansion

$$\frac{s+1}{s^2(s+2)} = \frac{As+B}{s^2} + \frac{C}{s+2}$$

Cross-multiply the RHS terms

$$\frac{(As+B)(s+2)}{s^2(s+2)} + \frac{(C)(s^2)}{s^2(s+2)}$$

$$= \frac{As^2+Bs+2As+2B}{s^2(s+2)} + \frac{Cs^2}{s^2(s+2)}$$

$$= \frac{(A+C)s^2+(2A+B)s+2B}{s^2(s+2)}$$

- A+C=0
- 2B = 1 B=1/2
- A=1/4
- C=-1/4

$$\int \frac{dx}{x^2 - 4}$$

$x^2 - 4 = (x - 2) \times (x + 2)$

$$\frac{1}{x^2 - 4} = \frac{A}{x - 2} + \frac{B}{x + 2}$$

$$\frac{1}{x^2 - 4} = \frac{A(x + 2)}{(x - 2)(x + 2)} + \frac{B(x - 2)}{(x - 2)(x + 2)}$$

$$1 = A(X + 2) + B(X - 2)$$

$$1 = (A + B)x + (2A - 2B)$$

- A+B=0
- 2A-2B=1
- Solving $A = 1/4$ and B= $-1/4$

$$\frac{1}{x^2-4} = \frac{1/4}{x-2} + \frac{-1/4}{x+2}$$

$$\int \frac{dx}{x^2-4} = \frac{1}{4}\int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+2}$$

$$\int \frac{dx}{x^2-4} = \frac{1}{4} ln|x-2| - \frac{1}{4} ln|x+2|$$

Integrals: Worked Example

$$\int x(x+2)^{7/2}dx = \frac{2}{9}x(x+2)^{9/2} - \int \frac{2}{9}(x+2)^{9/2}dx$$

$$\frac{2}{9}x(x+2)^{9/2} - \frac{4}{99}(x+2)^{11/2} + c$$

Integrals: Worked Example 2

$$\int \frac{e^x}{e^{2x} - 1} dx$$

Letting $e^x = u$ we get $du = e^x dx$

$$\int \frac{du}{u^2 - 1} = \int du(u+1)(u-1)$$

Determine the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Verify that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y)$$

Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & 1 & 3 \end{pmatrix}$$

(i) Find the rank of A.

(ii) Find a basis for the column space of A.

Substitution

$$I = \int \frac{sin(2x^2)}{3x} dx$$

- Let $u = 2x^2$
- $du/dx = 4x$
- $du = 4xdx$

Integration by Parts

$$I = \int u dv = uv - \int v du$$

Example

$$I = \int x^2 e^{2x} dx$$

- Let $u = x^2$. We can say that $\frac{du}{dx} = 2x$. Furthermore $du = 2xdx$
- $dv = e^{2x} dx$

Partial Derivatives

$$f(x, y) = y^3 - x^3 - 2xy + 5$$

$$\begin{aligned} f_x &= -3x^2 - 2y \\ f_y &= 3y^2 - 2x \end{aligned} \tag{1}$$

Partial Derivatives

$$f(x,y) = y^3 - x^3 - 2xy + 5$$

$$\begin{aligned} f_x &= -3x^2 - 2y \\ f_y &= 3y^2 - 2x \\ f_{xx} &= -6x \\ f_{yy} &= 6y \\ f_{xy} &= -2 \end{aligned} \tag{2}$$

Optimization

- Local Maxima
- Local Minima
- Saddle Point
- Concavity
- Convexity