## 2.4 The definite integral

### 2.4.1 Definition

**Definition 2.5.** *Let $a, b \in \mathbb{R}$ such that $a \leq b$ and let $f$ be an integrable function over $[a, b]$. Suppose that $F = \int f(x)dx$, i.e. $\frac{dF(x)}{dx} = f(x)$.*

*The definite integral of $f$ between $a$ and $b$ is denoted by $\int_a^b f(x)dx$ and is given by:*

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

In the following example we compute the integral of $f(x) = x^2 + 3x - 5$ between $-1$ and 4.

$$\begin{aligned}
\int_{-1}^4 f(x)dx &= \int_{-1}^4 (x^2 + 3x - 5)dx \\
&= \left[\frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x\right]_{-1}^4 \\
&= \left(\frac{1}{3}(4)^3 + \frac{3}{2}(4)^2 - 5(4)\right) - \left(\frac{1}{3}(-1)^3 + \frac{3}{2}(-1)^2 - 5(-1)\right) \\
&= \frac{115}{6}
\end{aligned}$$

### 2.5.1 Properties

(i) Suppose that for any $x \in [a, b]$, $f(x) \geq 0$, then:

$$\int_a^b f(x)dx \geq 0.$$

(ii) For any $a, b \in \mathbb{R}$, we have:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

(iii) for any $a, b, c \in \mathbb{R}$, we have:

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

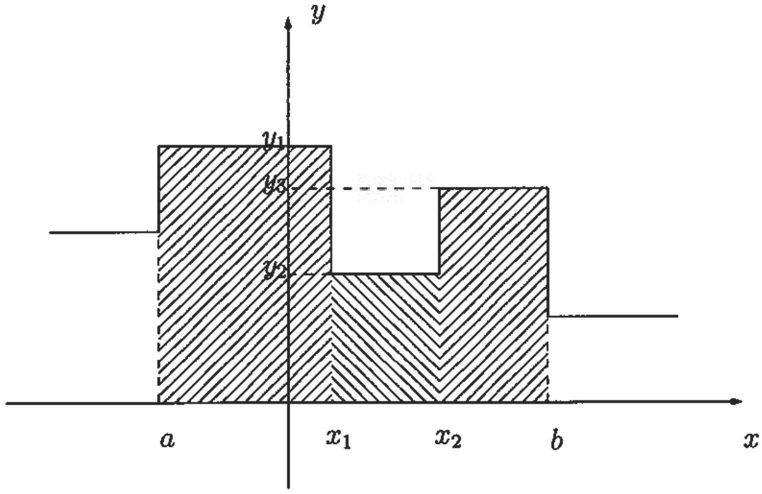


Figure 2.1: Integration of a piecewise constant function.

### 2.5.2 Interpretation

Suppose we have to integrate a piecewise constant function between $a$ and $b$ as shown on Figure 2.1.

Since:

$$f(x) = \begin{cases} y_1 & \forall x \in (a, x_1) \\ y_2 & \forall x \in (x_1, x_2) \\ y_3 & \forall x \in (x_2, x_3) \end{cases}$$

we can deduce from property (iii) that:

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx \\ &= \int_a^{x_1} y_1 dx + \int_{x_1}^{x_2} y_2 dx + \int_{x_2}^{x_3} y_3 dx \\ &= y_1(x_1 - a) + y_2(x_2 - x_1) + y_3(b - x_2)\end{aligned}$$

We see clearly that in the case of a piecewise constant function $f$, the integral $\int_a^b f(x)dx$ corresponds to the area of the surface located between the curve of $f$, the $x$ axis and the vertical lines passing at $a$ and $b$.

Now suppose we want to determine the area of the surface located between the curve of $f$, the $x$ axis and the vertical lines passing at $a$ and $b$ for a more general positive function on $[a, b]$.

If we divide the interval $[a, b]$ into 10 subintervals $[x_n, x_{n+1}]$, $0 \leq n \leq 9$, of equal lengths $\frac{b-a}{10}$, where $x_0 = a$ and $x_{10} = b$, we can approximate the area of the surface by the area of the rectangles of sides $[x_n, x_{n+1}]$ and $[0, f(x_n)]$ as shown on Figure 2.2.

This approximation can be improved if we divide $[a, b]$ into $N > 10$ intervals of length $\frac{b-a}{N}$. The higher is $N$, the better the approximation will be.

If $N$ increases indefinitely, we will converge to the area of the original surface which can be interpreted as the sum of areas of an infinity of rectangles of infinitesimal (infinitely small) width and of height $f(x)$, having an infinitesimal surface $f(x)dx$.

The integral $\int_a^b f(x)dx$ corresponds then to the area of the surface located between the curve of $f$, the $x$ axis and the vertical lines passing at $a$ and $b$.

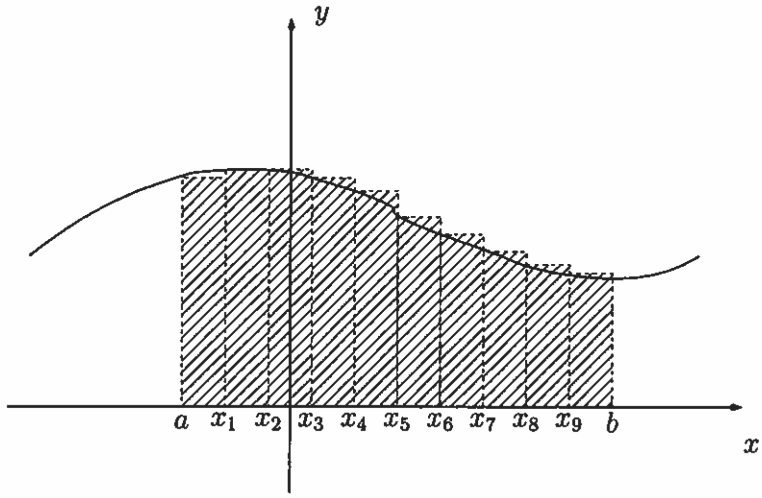


Figure 2.2: Integration of a more general positive function.

Suppose now that the function $f$ satisfies the following condition:

$$\begin{cases} f(x) \leq 0 & \forall x \in (a, c) \\ f(x) \geq 0 & \forall x \in (c, b) \end{cases}$$

as shown on Figure 2.3.

Then using property (iii), we have:

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= -\int_a^c -f(x)dx + \int_c^b f(x)dx \end{aligned}$$

Since the curve of $-f$ is the symmetric of the one of $f$ about the $x$ axis, $\int_a^c -f(x)dx$ is equal to the area of the surface located between the curve of $f$, the $x$ axis and the vertical lines passing at $a$ and $c$. (because $-f$ is positive)

We can now generalize the interpretation of the integral to general functions as follows : $\int_a^b f(x)dx$ corresponds to the algebraic area of the surface located between the curve of $f$, the $x$ axis and the vertical lines passing at $a$ and $b$. The adjective algebraic means that the surface will be counted positively if the function is positive and negatively if the function is negative.

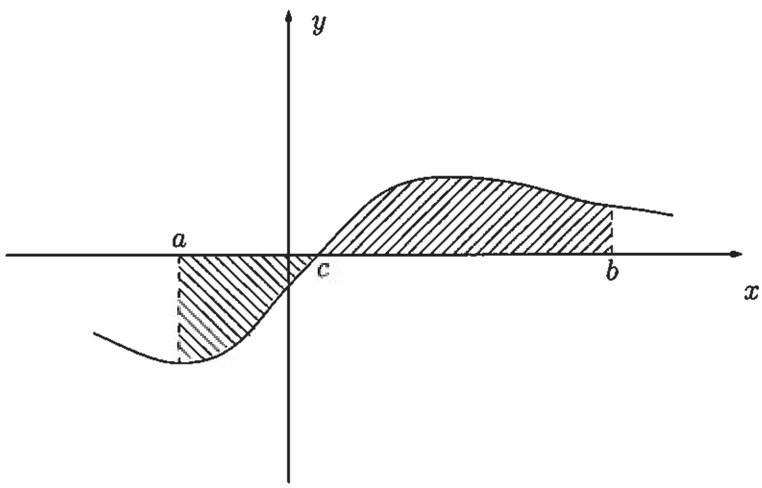


Figure 2.3: Integration of a general function.

### 2.5.3 Applications

#### Find the area between two curves

Let $A$ be the area of the surface between the curves of two functions $f$ and $g$ for $x \in [a, b]$, then:

$$A = \int_a^b |f(x) - g(x)|dx.$$

In fact, we can see in the example of Figure 2.4 that:

$$\begin{aligned} A =& A_1 + A_2 \\ =& \int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx \\ =& \int_a^c |f(x) - g(x)|dx + \int_c^b |g(x) - f(x)|dx \\ =& \int_a^b |f(x) - g(x)|dx \end{aligned}$$

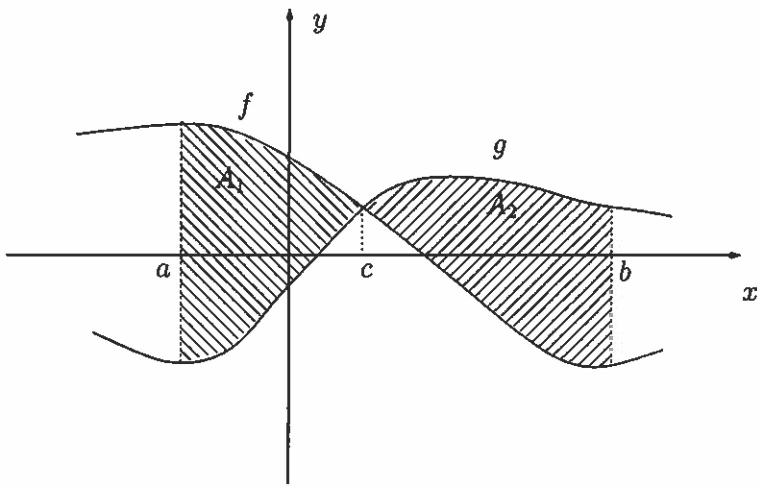


Figure 2.4: Area between 2 curves.

### Volumes of revolution

Suppose we want to determine the volume delimited by the surface obtained when rotating the curve of a function $f$ about the $x$ axis for $x \in [a, b]$. Then we can divide this volume into infinitesimal cylinders around $(x, 0)$ of radius $f(x)$ and height $dx$.

The volume $V$ is then obtained by summing the infinitesimal volumes $dv = \pi f(x)^2 dx$:

$$V = \int_a^b \pi f(x)^2 dx.$$

This is shown diagrammatically on Figure 2.5.

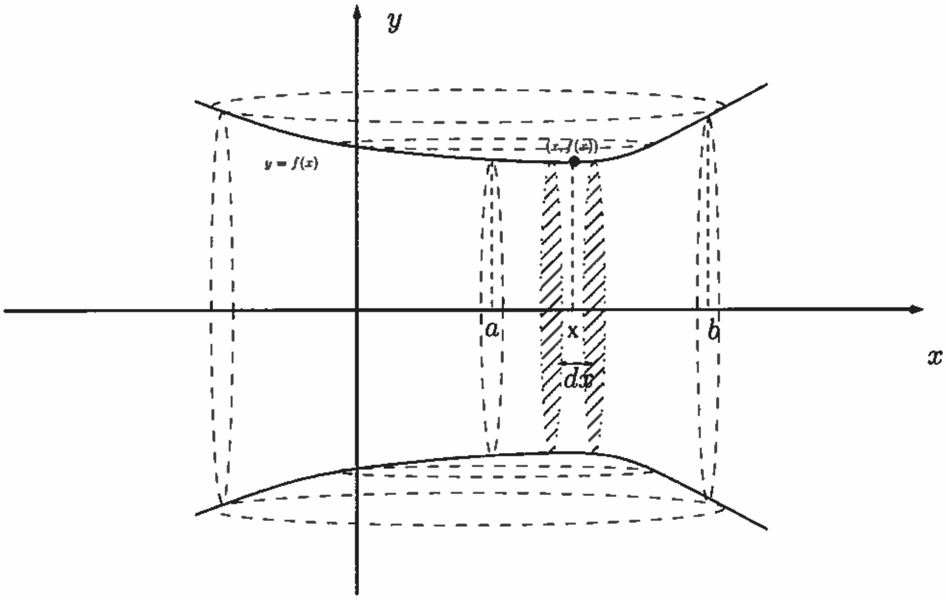


Figure 2.5: Volume of revolution.

Example: to find the volume to a cone of height $h$ and radius $R$, we can consider it as the product of the revolution of the curve $y = f(x) = \frac{R}{h}x$ about the $x$ axis, the basis being at $x = h$. Then following the previous formula, we have:

$$\begin{aligned}
V &= \int_0^h \pi f(x)^2 dx \\
&= \int_0^h \pi \left(\frac{R}{h}x\right)^2 dx \\
&= \pi \frac{R^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\
&= \frac{\pi R^2 h}{3}
\end{aligned}$$

### Length of a curve

Consider a function $f$ and suppose we have to determine the length of the curve of $f$ between $x = a$ and $x = b$. We can approach this curve by $N$ straight lines $[P_0, P_1]$, $[P_1, P_2]$,..., $[P_N, P_{N+1}]$ as shown on figure 2.6.

Suppose that the points $(x_j)_{0 \le j \le N+1}$ are equidistant, i.e. $x_{j+1} - x_j = \frac{b-a}{N} = \Delta x$. Then, since the point $P_j$ has coordinates $(x_j, f(x_j))$, the length

$\delta l_j$ of $[P_j, P_{j+1}]$ is given by:

$$\begin{aligned}\delta l_j^2 =&(x_{j+1} - x_j)^2 + (f(x_{j+1}) - f(x_j))^2 \\ =&(\Delta x)^2 + \Delta f(x_j)^2\end{aligned}$$

Now suppose that the number of subdivisions $N$ goes to infinity, then the sum of the lengths of $[P_j, P_{j+1}]$ will converge to the length of the curve.

But if the $N$ goes to infinity, we will obtain an infinity of straight lines of infinitesimal length $dl$ given by:

$$\begin{aligned}dl^2 =&(dx)^2 + df(x)^2 \\ =&(dx)^2 + \left(\frac{df}{dx}\right)^2 dx^2 \\ =&\left(1 + f'(x)^2\right) dx^2\end{aligned}$$

and

$$ds = \sqrt{1 + f'(x)^2} dx$$

Since the length of the curve is given by $L = \int_a^b dl$, we have:

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

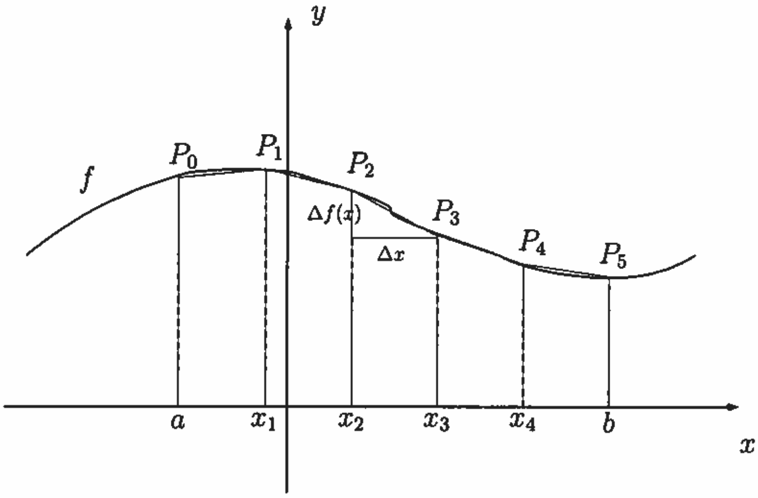


Figure 2.6: Length of a curve.

We can use this formula for finding the circumference of a circle of radius R. The equation of the circle of center $O$ and radius $R$ is:

$$x^2 + y^2 = R^2.$$

The circle can be considered as made of two halfs (one in the half-plane $y \geq 0$ and one in the half-plane $y \leq 0$). The circumference is then twice the length of a half-circle.

The half-circle lying in the half-plane $y \geq 0$ is the curve of the function $f(x) = \sqrt{R^2 - x^2}$. To find the derivative of $f$ we can derive the following equations:

$$x^2 + f(x)^2 = R^2,$$

which gives:

$$2x + 2f(x)f'(x) = 0,$$

and then

$$f'(x) = -\frac{x}{f(x)} = -\frac{x}{\sqrt{R^2 - x^2}}.$$

It follows that the circumference of the circle is :

$$\begin{aligned}
S =& 2\int_{-R}^{+R} \sqrt{1 + f'(x)^2}dx \\
=& 2\int_{-R}^{+R} \sqrt{1 + \frac{x^2}{R^2 - x^2}}dx \\
=& 2R\int_{-R}^{+R} \frac{1}{\sqrt{R^2 - x^2}}dx \\
=& 2R\left[\sin^{-1}\left(\frac{x}{R}\right)\right]_{-R}^{+R} \\
=& 2R\left(\sin^{-1}(1) - \sin^{-1}(-1)\right) \\
=& 2R\left(\frac{\pi}{2} - (-\frac{\pi}{2})\right) \\
=& 2\pi R.
\end{aligned}$$

### Centroid of area under a curve

For a set of point masses $(m_i)_{1\leq i\leq N}$ located at points $(x_i, y_i)_{1\leq i\leq N}$, the centroid has coordinates $(\bar{x}, \bar{y})$ given by:

$$\begin{cases} \bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i} \\ \bar{y} = \frac{\sum_i m_i y_i}{\sum_i m_i} \end{cases}$$

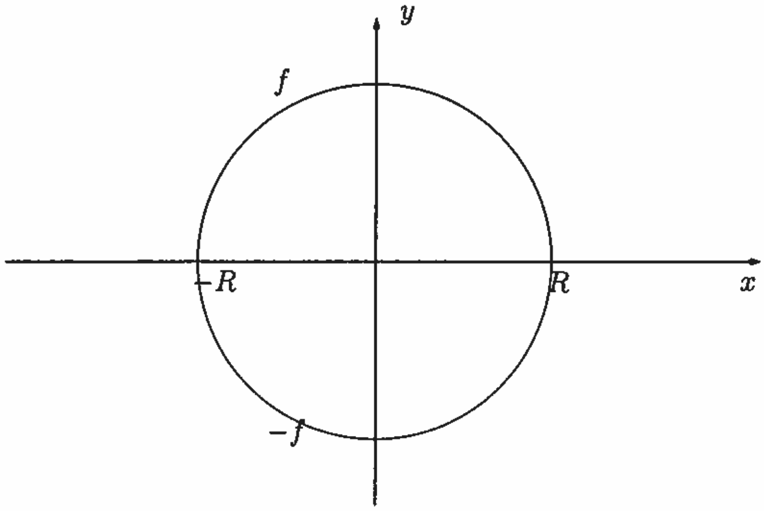


Figure 2.7: Circumference of a circle.

Consider now the area under the curve of a function $f$ for $x \in [a, b]$. The coordinates of the centroid of this area can be obtained by considering this area as an infinite sum of point masses $dxdy$ at points $(x, y)$, $a \leq x \leq b$, $0 \leq y \leq f(x)$.

In order to find the centroid we can even divide the area into infinitesimal rectangles of width $dx$ and height $f(x)$ centered around $x$ which has an infinitesimal mass $dM = |f(x)|dx$ and which centroid is located at $\left(x, \frac{f(x)}{2}\right)$.

Then, the coordinates of the centroid of the area are given by:

$$\begin{cases} \bar{x} = \dfrac{1}{A} \displaystyle\int_a^b x dM \\ \bar{y} = \dfrac{1}{A} \displaystyle\int_a^b \dfrac{f(x)}{2} dM \end{cases}$$

where $A$ is the total area:

$$A = \int_a^b f(x)dx.$$

Replacing $dM$ by its expression in terms of $x$, we get:

$$\begin{cases} \bar{x} = \dfrac{1}{A} \displaystyle\int_a^b x|f(x)|dx \\ \bar{y} = \dfrac{1}{A} \displaystyle\int_a^b \dfrac{f(x)}{2} |f(x)|dx \end{cases}$$

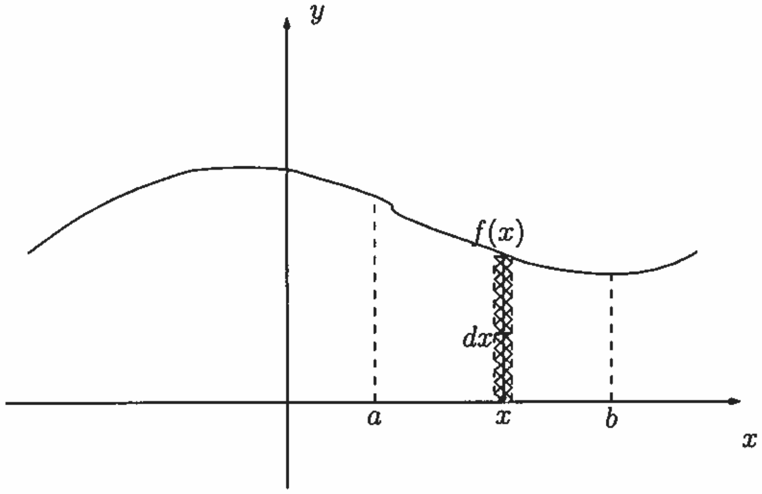


Figure 2.8: Centroid of an area under a curve.

This formula can be generalized to the area between two curves $f$ and $g$, where $f(x) \leq g(x)$ for any $x \in [a, b]$. In this case the infinitesimal rectangles will have a width of $dx$ and extend between $f(x)$ and $g(x)$ (instead of 0 and $f(x)$), then their infinitesimal mass is given by $dM = |g(x) - f(x)|dx$ and their centroid is located at $(x, \frac{f(x)+g(x)}{2})$.

Then:

$$\begin{aligned}\bar{x} =& \frac{1}{A}\int_a^b x dM \\ =& \frac{1}{A}\int_a^b x|g(x) - f(x)|dx\end{aligned}$$

and

$$\begin{aligned}\bar{y} =& \frac{1}{A}\int_a^b \frac{f(x)+g(x)}{2} dM \\ =& \frac{1}{A}\int_a^b \frac{f(x)+g(x)}{2} |g(x) - f(x)|dx\end{aligned}$$

### Moments of inertia

For a set of point masses $(m_i)_{1 \leq i \leq N}$ located at points $(x_i, y_i)_{1 \leq i \leq N}$, the moments of inertia $I_x$ about the $x$ axis, $I_y$ about the $y$, and $I_0$ about the origin

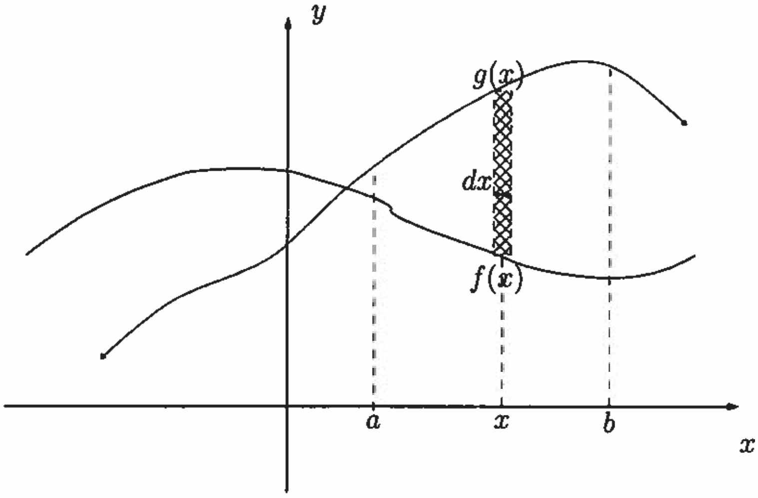


Figure 2.9: Centroid of the area between 2 curves.

are given by:

$$I_x = \sum_i y_i^2 m_i,$$

$$I_y = \sum_i x_i^2 m_i,$$

$$I_0 = I_x + I_y = \sum_i (y_i^2 + x_i^2) m_i.$$

Now, suppose we have to determine the moment of inertia $I_y$ of an area extending between the curves of two functions $y = f(x)$ and $y = g(x)$, $f(x) \leq g(x)$, as shown in Figure 2.10. We can divide this area into infinitesimal rectangles of width $dx$ extending between $y = f(x)$ and y=$g(x)$. All points in such a rectangle are at a distance $x$ from the $y$ axis. The total mass of the rectangle is $dM = |g(x) - f(x)|dx$. It follows that:

$$\begin{aligned} I_y &= \int_a^b x^2 dM \\ &= \int_a^b x^2 |g(x) - f(x)| dx \end{aligned}$$

Let us now determine the moment of inertia $I_x$ of the same surface about the $x$ axis. To do so, we can consider an infinitesimal element of surface of width $dx$ and height $dy$ located at a point $(x, y)$ of the surface as shown on Figure 2.11. (this means that $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$)

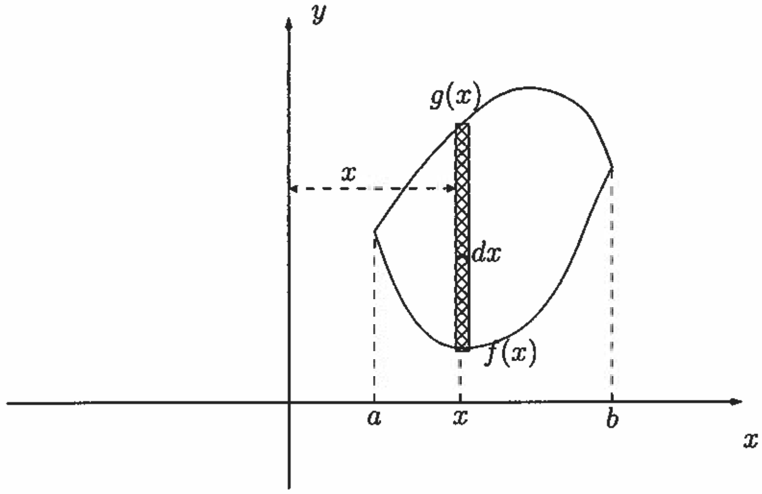


Figure 2.10: Moment of inertial about $y$ axis.

This element has a mass equal to $d^2M = dxdy$ and is located at a distance $y$ from the $x$ axis. Its moment $d^2I_x$ is then given by:

$$d^2I_x = y^2d^2M = y^2dxdy.$$

Now consider the infinitesimal strip of width $dx$ located at $x$ and extending from $f(x)$ to $g(x)$. The moment of inertial of this strip about the $x$ axis is given by summing the moments of inertia of all elements $dxdy$ located at points $(x, y)$ for $y$ satisfying $f(x) \leq y \leq g(x)$. This moment $dI_x$ is given by:

$$\begin{aligned}
dI_x &= \int_{y=f(x)}^{g(x)} d^2I_x \\
&= \int_{y=f(x)}^{g(x)} y^2d^2M \\
&= \int_{y=f(x)}^{g(x)} y^2dxdy \\
&= dx\int_{y=f(x)}^{g(x)} y^2dy \\
&= dx\left[\frac{y^3}{3}\right]_{y=f(x)}^{g(x)} \\
&= \frac{g^3(x) - f^3(x)}{3}dx
\end{aligned}$$

Finally to get the moment $I_x$ of the total surface about the $x$ axis, we have to sum the moments $dI_X$ for $x \in [a, b]$:

$$\begin{aligned} I_x &= \int_{x=a}^{b} dI_x \\ &= \int_{x=a}^{b} \frac{g^3(x) - f^3(x)}{3} dx \end{aligned}$$

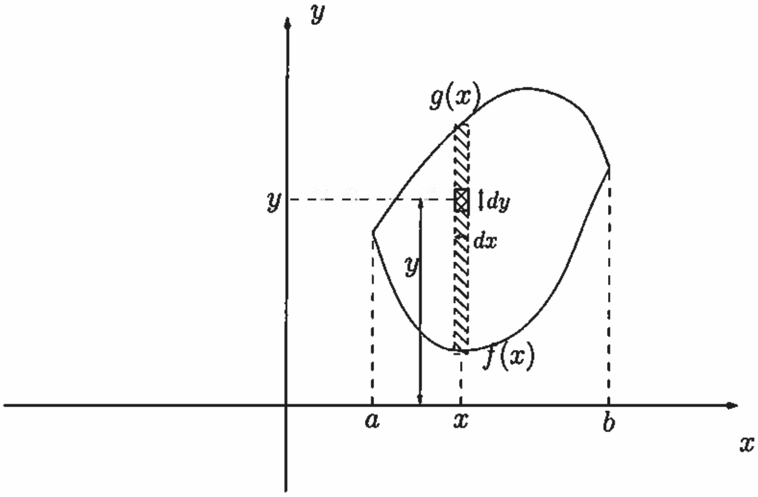


Figure 2.11: Moment of inertial about $x$ axis.

Suppose now that we consider the moment of inertial $I_x$ about the $x$ axis of an area extending between the curves of two functions $x = h(y)$ and $x = k(y)$, $h(y) \leq k(y)$, as shown in Figure 2.12. This moment can be found dividing the area into infinitesimal rectangles of height $dy$ extending between $x = h(y)$ and $x = k(y)$. All points in such a rectangle are at a distance $y$ from the $x$ axis. The total mass of the rectangle is $dM = |k(y) - h(y)|dy$. It follows that:

$$I_x = \int_a^b y^2 |k(y) - h(y)| dy$$

We can apply these formula to find the moments of inertial for the area defined by:

$$x^2 + y^2 \leq 1, \qquad x \geq 0, \quad y \geq 0.$$

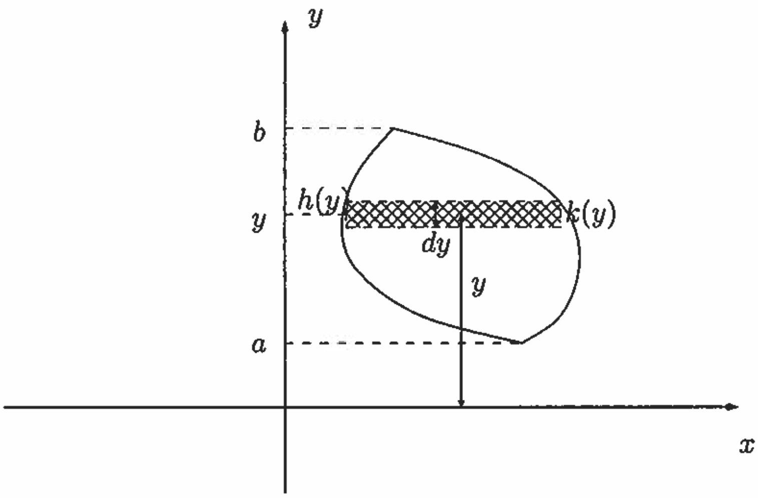


Figure 2.12: Moment of inertial about $x$ axis.

To find $I_y$, we set $a = 0$, $b = 1$, $f(x) = 0$, and $g(x) = \sqrt{1 - x^2}$. Then

$$I_y = \int_0^1 x^2\sqrt{1 - x^2}.$$

Define $\theta$ as $x = \sin\theta$, then $dx = \cos\theta d\theta$, and

$$\begin{aligned}
I_y =& \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1 - \sin^2\theta}\cos\theta d\theta \\
=& \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta \\
=& \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\sin(2\theta)\right)^2 d\theta \\
=& \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1 - \cos(4\theta)}{2} d\theta \\
=& \frac{1}{8}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\frac{\pi}{2}} \\
=& \frac{1}{8}\left[\frac{\pi}{2}\right] \\
=& \frac{\pi}{16}
\end{aligned}$$

Because of the symmetry of the shape, $I_x = I_y$.

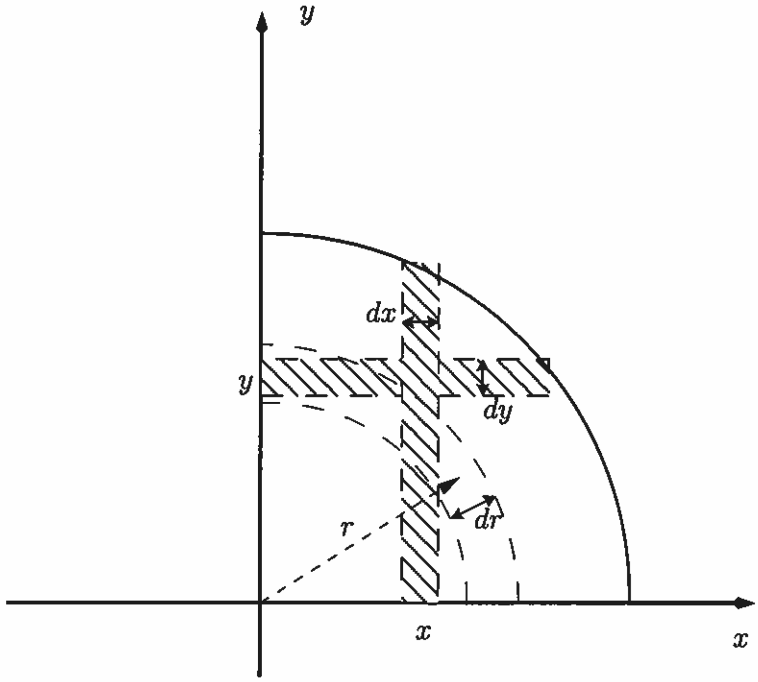


Figure 2.13: Moment of inertial of a quarter circle.

**Definition 2.6.** *If the total mass of an area is $M$ and its moment of inertial is $I$, then we define the* radius of gyration $k$ *about the corresponding axis by:*

$$k^2 = \frac{I}{M}.$$

**Proposition 2.7.** *Let $k_x$, $ky$, and $k_o$ be the radius of gyration about the $x$ axis, the $y$ axis, and the origin respectively. Then:*

$$k_o^2 = k_x^2 + k_y^2.$$

This result is a direct consequence of the fact that:

$$I_o = I_x + I_y,$$

which is a consequence of the obvious formula:

$$x^2 + y^2 = r^2.$$

In fact, in the previous example, we can solve directly $I_o$ by dividing the are into rings of infinitesimal width $dr$ having a mass $dM = \frac{2\pi r}{4} dr = \frac{\pi r}{2} dr$, and were each point is at a distance $r$ from the origin. It follows that;

$$\begin{aligned} I_0 &= \int_0^r r^2 dM \\ &= \int_0^r r^2 \frac{\pi r}{2} dr \\ &= \int_0^r \frac{\pi r^3}{2} dr \\ &= \frac{\pi}{2} \left[\frac{r^4}{4}\right]_0^1 \\ &= \frac{\pi}{8} \end{aligned}$$

We can then check that

$$I_x + I_y = \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8} = I_o.$$

This relation can be useful to find one moment when the two other moments are known.

## 2.8 Numerical methods of integration

In many situations, we deal with integrals of complicated functions for which it is impossible to find an explicit expression. To solve such integrals, we use some numerical methods which allow us to find an approximate value of the integral. These methods can be implemented on computers and their precision depends on the discretization parameters. As the discretization increases, the precision is improved, but at the same time, the need of time and memory increases.

### 2.8.1 The trapezoidal rule

Suppose we have to find the integral between $a$ and $b$ of some function $f$. Then we divide the interval $[a, b]$ into $N$ subintervals of equal length $h = \frac{b-a}{N}$: $[x_n, x_{n+1}]$, $0 \leq n \leq N-1$, where $x_n = a + n\frac{b-a}{N} = a + nh$, $0 \leq n \leq N$.

The integral $I = \int_a^b f(x)dx$ can then be approximated by $I_N$ given by:

$$I_N = \frac{h}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{N-2}) + 2f(x_{N-1}) + f(x_N)\Big).$$

This formula comes from the fact that we approximate the area under the curve by the sum of the areas of right trapezia of height $[x_n, x_{n+1}]$ and for which the parallel sides have lengths $f(x_n)$ and $f(x_{n+1})$. Its area is then equal to $\frac{h}{2}(f(x_{n+1}) + f(x_n))$.

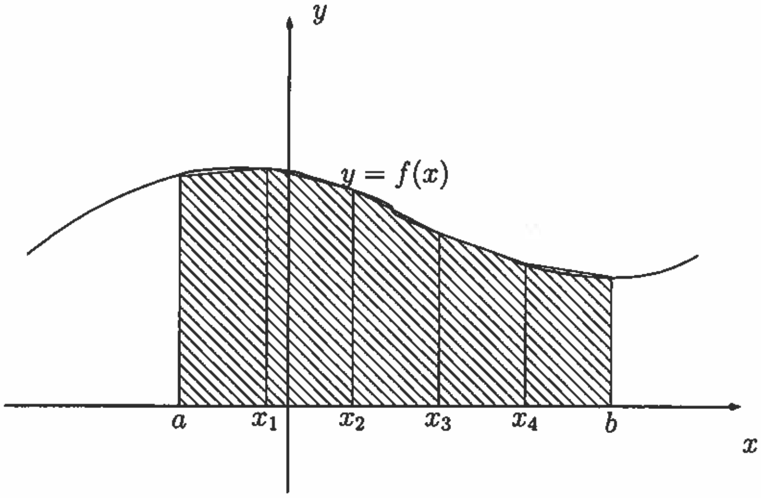


Figure 2.14: The trapezoidal rule.

**Example**

We can give an approximation of $I = \int_1^5 \frac{dx}{\sqrt{1+x^3}}$ using 5 strips. We first define $x_0 = 1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, $x_5 = 5$. Then we compute $I_5$ as follows:

$$I_5 = \frac{1}{2}\Big(f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\Big).$$

We find $I_5 \simeq 1.8975$.

## 2.8.2 The Simpson's rule

The Simpson's rule is another method for numerical integration. Its justification is more complicated. However, for the same number of subdivisions, the Simpson's rule give a more accurate result than the trapezoidal rule.

The Simpson's rule needs an even number of subdivisions $2N$ of the interval of integration $[a, b]$, defining intervals of width $h = \frac{b-a}{2N}$ which edges are the points $(x_n)_{0\leq n\leq 2N}$, $x_n = a + \frac{b-a}{2N}n$. We have then $x_0 = a$ and $x_{2N} = b$.

The approximation of $\int_a^b f(x)dx$ given by the Simpson's rule $I_N$ is defined by:

$$\begin{aligned} I_{2N} =& \frac{h}{3}\Big( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{2n-1}) \\ & + 4f(x_{2n}) + 2f(x_{2n+1}) + \ldots + 2f(x_{2N-2}) + 4f(x_{2N-1}) + f(x_{2N})\Big). \end{aligned}$$

**Example**

Using the Simpson's rule for finding an approximation of $I = \int_1^5 \frac{dx}{\sqrt{1+x^3}}$ with 10 subdivisions, we get:

$$\begin{aligned} I_{10} =& \frac{1}{3}\Big( f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) \\ & + 4f(4.5) + 2f(5) + 4f(5.5) + 2f(6) + 4f(6.5) + 2f(7) + 4f(7.5) \\ & + 2f(8) + 4f(8.5) + 2f(9) + 4f(9.5) + f(10)\Big) \\ \simeq& 1.1913 \end{aligned}$$