Question 2 (25 Marks)

(i) The fundamental Theorem of Invertible Matrices stats that a set of mathematical expressions concerning a $n \times n$ matrix A are each equivalent to one another. State any four of these expressions.

(i) (1 Mark) What is the trace of a square matrix

(ii) (1 Mark) What is the Rank of a matrix.

Reduce the following matrix to row echelon form
Consider the following diagonal matrix A. In terms of the values a,b and c ;

(i) Write an expression for the trace of the matrix

(ii) State the inverse of A, i.e. A^{-1}

(iii) State the matrix A^3

Part A. Inverting a Matrix using Co-Factors

Given the matrix

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix}.$$

calculate

- the determinant of A;
- the cofactor matrix of A;
- and **hence** the inverse matrix A^{-1} .
- Evaluate the minors and cofactors of A, for A given by and hence, in each case, construct the cofactor matrix Cof(A) of A.

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -4 & 1 \\ -3 & 0 & -1 \end{pmatrix}.$$

using elementary row operations.

2. Let a triangular matrix be a square matrix with either all (i, j) entries zero for either $i < j$ (in which case it is called an lower triangular matrix) or for $j < i$ (in which case it is called an upper triangular matrix). Show that any triangular matrix satisfying $AA^T = A^T A$ is a diagonal matrix.

This is also expressed by saying that Ax is a linear combination of the columns of A.

Part B. System of Linear Equations

1. Consider the linear system

$$\begin{aligned} x_1 + x_3 &= 4 \\ 2x_1 + 4x_2 + x_3 &= -3 \\ x_2 + 3x_3 &= 7. \end{aligned}$$

   (a) Write down the coefficient matrix and the augmented matrix of this system.

   (b) What can you say about the solution set of the system? Justify your answer.

   (c) Solve the system of equations, using any appropriate method.

2. Consider the homogeneous system:

$$\begin{aligned} x_1 + x_3 &= 0 \\ 2x_1 + 4x_2 + x_3 &= 0 \\ x_2 + 3x_3 &= 0. \end{aligned}$$

   What can you say about its solution set?

1. Prove that, for any $u,\ v\ \in \mathbb{R}^3$,

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u.$$

   (It is sufficient to verify this property for one component.)

2. Consider the three vectors in $\mathbb{R}^3$:

$$u = (1, 2, 0), \quad v = (0, 1, 3), \quad w = (1, 2, 3).$$

   (a) Evaluate $\|u\|$, $\|v\|$, $u \cdot v$, $u \times v$ and the angle between u and v.

   (b) Calculate the scalar triple product $u \cdot (v \times w)$.

Fundamental Theorem of Invertible Matrices Rank Trace

1. Consider the linear system

$$\begin{aligned} x_1 + x_3 &= 4 \\ 2x_1 + 4x_2 + x_3 &= -3 \\ x_2 + 3x_3 &= 7. \end{aligned}$$

   (a) Write down the coefficient matrix and the augmented matrix of this system.

   (b) What can you say about the solution set of the system? Justify your answer.

   (c) Solve the system of equations, using any appropriate method.