20 February 2012

Problem Sheet 3: Line integrals and conservative fields

1. Evaluate
$$\oint (2x - y + 4)\,dx + (5y + 3x - 6)\,dy$$
   (a) around a triangle in the xy plane with vertices at (0,0), (3,0) and (3,2), traversed in a counterclockwise direction. *(Answer:=12)*
   (b) around a circle of radius 4 with centre at (0,0). *Answer:*=64π

2. A vector field $\mathbf{A}$ is defined as
$$\mathbf{A} = 2xz\mathbf{i} + 2yz^2\mathbf{j} + (x^2 + 2y^2z - 1)\mathbf{k}$$
   Show that $\mathbf{A}$ is conservative and calculate the scalar potential Φ. *(Answer:=$x^2z + y^2z^2 - z$)*

3. A vector field $\mathbf{A}$ is defined as
$$\mathbf{A} = (z^2 + 2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}$$
   Show that $\mathbf{A}$ is conservative and calculate the line integral of $\mathbf{A}$ between (1,1,1) and (1,2,2). (Choose any path you wish!) *(Answer:=11)*

4. A vector field $\mathbf{A}$ is defined as
$$\mathbf{A} = -\frac{zx}{r^3}\mathbf{i} - \frac{zy}{r^3}\mathbf{j} + \frac{x^2 + y^2}{r^3}\mathbf{k}$$
   where $r^2 = x^2 + y^2 + z^2$. Show that $\mathbf{A}$ is conservative and construct the scalar potential Φ. *(Answer:=$\frac{z}{r} + c$)*

5. Using Green's theorem in the plane show that
$$\iint_R (x - y)\,dxdy = -\frac{2}{3}$$
   where R is the upper half of the unit circle centred at (0,0). Calculate the same integral using direct integration.