MA4702 - Fundamentals of Mathematics and Limits

Notation:

- $\mathbb{R}$ - All real numbers positive and negative
- $\mathbb{R}^+$ - All positive real numbers including 0
- $\mathbb{R}^-$ - All negative real numbers including 0
- $[a,b]$ - All real numbers x such that $a \leq x \leq b$
- (a,b) - All real numbers x such that $a < x < b$
- $[a,\infty)$ - All real numbers x such that $a \leq x$
- (a,∞) - All real numbers x such that $a < x$

Question 1 : Evaluation of Functions

(i) Evaluate the following function for x = -1,0,1 and 2 respectively.

$$f(x) = \frac{e^x - e^{-x}}{2}$$

(ii) Evaluate the function for each of the following values : $0.5,\ 1,\ 1.25,\ 2.$

$$f(x) = \sqrt{1 + e^x}$$

Question 2 : Floor and Ceiling Functions (Part A)

- $\lceil x \rceil$: Ceiling function
- $\lfloor x \rfloor$: Floor Function
- $\{x\}$: Fractional Part of a number ($\{x\} = x - \lfloor x \rfloor$)

Complete the following table.

Value x	Floor $\lfloor x \rfloor$	Ceiling $\lceil x \rceil$	Fractional $\{x\}$
-1.4	-2	-1	
2.3			
7/9			
-16/3			
0			0
1		1	

Question 3 : Floor and Ceiling Functions (Part B)

Provide some values for x and y that **contradict** the following statement.

$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$$

If the values of x and y were integers, would the equation be true for all values of x and y?

Question 4 : Laws for Logarithms

The following laws are very useful for working with logarithms.

1. $\log_b(X) + \log_b(Y) = \log_b(XY)$
2. $\log_b(X)$ - $\log_b(Y) = \log_b(X/Y)$
3. $\log_b(X^Y) = Y\log_b(X)$
4. $\log_b(X) = 1$ when $b = X$

Use the Laws of Logarithms to evaluate the following expressions:

(i) $\log_2(8)$

(ii) $\log_2(\sqrt{128})$

(iii) $\log_2(64)$

(iv) $\log_5(125) + \log_3(729)$

(v) $\log_2(64/4)$

(vi) $\log_3(\frac{1}{81})$

Question 5 : Cross Multiplication

Solve the following equations for A and B where $A, B \in \mathbb{R}$

(i) $$\frac{11}{x^2-4x-12} = \frac{A}{x-6} + \frac{B}{x+2}$$

(ii) $$\frac{2x+5}{x^2-4x-12} = \frac{A}{x-6} + \frac{B}{x+2}$$

(iii) $$\frac{1}{(n)(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

(iv) $$\frac{2}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$

Question 6 : Exponential and Logarithm Exercises

(i) Find the value of x

$$e^{2x-5} = 3.$$

(ii) Find the value of x

$$ln(e^x + 2) = 4$$

(iii) Find the value of x

$$log_3(2x-1) + log_3(5) = 3$$

(iv) Find the value of x

$$log_2(x+1) + log_2(5) = 3$$

Question 7 : Review of Differentiation

(i) $f(x) = e^{4x}$

(ii) $f(x) = \cos(4x)$

(iii) $f(x) = \sin(3x)$

(iv) $f(x) = e^{4y}\cos(4x)$

(v) $f(y) = e^{4y}\cos(4x)$

Question 8 : Expressing Repeating Decimals as Fractions

Express the following numbers as fractions. For example $0.77777... = \frac{7}{9}$

(i) 0.29292929....

(ii) 0.475475475....

(iii) 0.4545454545.....

(iv) 0.473473473.........

Question 9 : Evaluate the following limits

(i)
$$\lim_{x \to 5} (x^2)$$

(ii)
$$\lim_{x \to 2} (4x^2 - 3x + 1)$$

(iii)
$$\lim_{x \to 4} x + 5$$

(iv)
$$\lim_{x \to 2} 2x - 1$$

(v)
$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

(vi)
$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3}$$

(vii)
$$\lim_{x \to \infty} \frac{x + 4}{x - 4}$$

(viii)
$$\lim_{x \to \infty} \frac{x^3 - 4}{x - 1}$$

(ix)
$$\lim_{x \to \infty} \frac{x^2 - 1}{x - 1}$$

(x)
$$\lim_{x \to \infty} \frac{2x^2 + 8}{5x^2 - 7x}$$

(xi)
$$\lim_{x \to \infty} \frac{3x^2 + 7x^3}{x^2 + 5x^4}$$

(xii)
$$\lim_{x \to \infty} \frac{2x^2 - 8x}{4x^2 - 7}$$

(xiii)
$$\lim_{x \to \infty} \frac{x - 3}{x^2 - 9}$$

Question 10: Sequences and Series

(i) Find the sum of the first 10 numbers of this arithmetic series: $1 + 11 + 21 + 31 + \ldots$

(ii) The second term u_2 of a geometric sequence is 21.
The third term u_3 is -84.

- Find the common ratio
- Find the first and fourth term

(iii) Three consecutive terms of an arithmetic series are

$$4x - 1, 2x + 11, 3x + 41.$$

Find the value of x.

(iv) Find the sum of the following geometric series:

$$3 + 6 + 12 + 24 + \ldots + 1536$$

(v) In an arithmetic sequence, three consecutive terms have a sum of - 9 and a product of 48. Find the common difference d for these terms.
Hint : Write terms as $x - d, x, x + d$

(vi) The first three terms of an arithmetic sequence are 6,-9 and x. The first three terms of a geometric sequence are 6, x, y. Find the value of x and the value of y.

(vii) Find the sum to infinity of the geometric series

$$1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \ldots$$

(viii) The first three terms of a geometric series are

$$x - 2, 2x + 3, 7x$$

Find both of the possible values of x

(ix) The $n-$th term of an arithmetic is $3n + 2$
Find S_n the sum of the first n terms, in terms of n

Question 11 : Convergence of a Sequence

A sequence u_n is said to converge to a limit L where $L \neq \infty$ if

$$\lim_{n\to\infty} u_n = L$$

If a sequence does not converge then it is said to be divergent.

Test the following sequences for convergence. see page 14 of notes for more details.

(i)
$$u_n = \frac{2n-1}{4n+3}$$

(ii)
$$u_n = \frac{n+5}{3n+4}$$

(iii)
$$u_n = \frac{n+4}{2n^3+n+3}$$

(iv)
$$u_n = \frac{n^2+5n}{n^2+2n-1}$$

(v)
$$u_n = \frac{2n^3+1}{2n^3+3n+4}$$

(vi)
$$u_n = \frac{2n^4+1}{n^3+2n^2-1}$$

Question 12 : Partitioning of Summations

For some integers m and n, with $m < n$.

$$\sum_{i=1}^{i=n} u_i = \sum_{i=1}^{i=m} u_i + \sum_{i=m+1}^{i=n} u_i$$

Suppose $n = 100$ and $m = 50$

$$\sum_{i=1}^{i=100} u_i = \sum_{i=1}^{i=50} u_i + \sum_{i=51}^{i=100} u_i$$

Compute the following

(i)
$$\sum_{i=1}^{30} i$$

(ii)
$$\sum_{i=1}^{65} i$$

(iii)
$$\sum_{i=1}^{37} i$$

(iv)
$$\sum_{i=38}^{65} i$$

Question 13 : Sum to Infinity Exercises

Compute the summations of the following infinite series

(i) $1 + 0.2 + 0.04 + \dots$

(ii) $1 - 0.2 + 0.04 - \dots$

(iii) $20 + 5 + 1.25 + \dots$

(iv) $-20 + 5 - 1.25 + \dots$

Question 14 : Summation of an Infinite Series

Find the value to which each of the following series converges.

$$\sum_{n=1}^{\infty} \frac{3}{4^{n-1}}$$

Question 15 : The Ratio test

For a series with general term u_n, if

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = r$$

then

- the series converges (absolutely) if $r < 1$
- the series diverges if $r > 1$ (or if r is infinity)
- the series could do either if $r = 1$, so the test is not conclusive.

This formula will be given in the back of the exam paper. However the indications on how to interpret r will not be given.

Example

Suppose that the following term is the general term for a series. Test this series for convergence

$$u_n = \frac{n!n!}{(2n)!}$$

then

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{4n+2}$$
$$\to \frac{1}{4}$$

so this series converges.

Use the Ratio Test to test the following series for convergence

(i)

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{4^n}$$

(iii)

$$\sum_{n=1}^{\infty} \frac{5^n}{n!}$$

(i)

$$\sum_{n=1}^{\infty} \frac{4^n}{(n+1)!}$$

(iv)

$$\sum_{n=1}^{\infty} \frac{n+1}{2^n}$$

(v) Use the Ratio test to find the values for x for which the series is convergent

$$\sum_{n=1}^{\infty} \frac{x^n}{n+2}$$

Question 16 : Evaluation of telescoping series

Find the sum of the following telescoping series

(i)
$$\sum_{n=1}^{\infty} \frac{3}{(3n+1)(3n+4)}$$

(ii)
$$\sum_{n=1}^{\infty} \frac{4}{(2n+1)(2n+3)}$$

(iii)
$$\sum_{n=1}^{\infty} \frac{5}{(5n+1)(5n+6)}$$

(iv)
$$\sum_{n=1}^{\infty} \frac{6}{(6n+1)(6n+7)}$$

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Question 17 : Evaluation of Telescoping Series

Answer the following questions

(i) Show that, where $r \neq \pm 1$.
$$\frac{2}{r^2-1} = \frac{1}{r-1} - \frac{1}{r+1}$$

(ii) Hence, evaluate the following summation
$$\sum_{r=2}^{n} \frac{2}{r^2-1}$$

(iii) Hence, evaluate the following summation
$$\sum_{r=2}^{\infty} \frac{2}{r^2-1}$$

Question 18 : Evaluation of terms in a Maclaurin Series

(i) Evaluate the following Macluarin Series for $n = \{1, 2, 3\}$.

$$\sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Question 19 : Functions

Part A - Functions

Consider the function f(x) where

$$f(x) = \frac{1}{\sqrt{3x-6}}$$

(i) Find the domain and range of f(x)

(ii) Find $f(3x^2+2)$ and simplify

Part B - Functions

Consider the functions $f(x) = \sqrt{2x-6}$ and $g(x) = log_e(2x+1)$

(i) Find $f(4–2x^2)$ and simplify answer.

(ii) Write down the domain and range of f(x).

(iii) Determine $g^{-1}(x)$, the inverse of g(x).

Part C - Composite Functions

Consider the functions $f(x) = \sqrt{2x-8}$, and $g(x) = 2x^2+4$.

(i) Find the composite function $f \circ g(x)$ and $g \circ f(x)$, simplifying your answer as much as possible.

(ii) Determine $f^{-1}(x)$, the inverse of f(x).

Question 20 : Functions

(i) Find $f^{-1}(x)$ the inverse of the function

$$f(x) = \frac{1}{2x-5}$$

(ii) Find the domain and the range of the function:

$$f(x) = 7 + 2sin(x)$$

(iii) Find $f^{-1}(x)$ the inverse of the function

$$f(x) = \sqrt{2x+3}$$

(iv) Find $f^{-1}(x)$ the inverse of the function

$$f(x) = e^{3x}$$

(v) Find the domain and the range of the function:

$$f(x) = 1 - x^2$$

(vi) Find the domain and the range of the function:

$$f(x) = ln(x)$$

Question 21 : Hyperbolic Functions - Proof of Identities

Recall

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad (e^x)^2 = e^{2x}$$

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (e^x) \times (e^{-x}) = e^{x-x} = e^0 = 1$$

Using their definition in terms of exponentials, prove the following hyperbolic identity:

(i) Show that

$$\cosh^2(x) - \sinh^2(x) = 1$$

(ii) Show that

$$\cosh^2 x = \cosh 2x + \sinh 2x$$

(iii) Show that

$$\cosh(x+y) = \cosh(x)\cosh(y) + sinh(x)sinh(y)$$

(iv) Show that

$$\sinh 2x = 2\sinh(x)\cosh(x)$$

Question 22 : Inverse of Functions

Procedure

- To determine $f^{-1}(x)$ when given a function f, substitute $f^{-1}(x)$ for x and substitute x for f(x).
- Then solve for $f^{-1}(x)$, provided that it is also a function.

Example:
Given $f(x) = 2x - 7$, find $f^{-1}(x)$.
Substitute $f^{-1}(x)$ for x and substitute x for f(x). Then solve for $f^{-1}(x)$:

$$f(x) = 2x - 7$$
$$x = 2[f^{-1}(x)] - 7$$
$$x + 7 = 2[f^{-1}(x)]$$
$$\frac{x+7}{2} = f^{-1}(x)$$

(i) Find $f^{-1}(x)$ the inverse of the function

$$f(x) = \frac{1}{2x-5}$$

(ii) given that $g(x) = log_e(4x)$, find $g^{-1}(x)$ the inverse function of g(x).

Question 23 : Even and Odd Functions

Check whether the following functions are even, odd or neither.

(i)
$$f(x) = \frac{4}{x^2+1}$$

(ii)
$$f(x) = sin(4x)$$

(iii)
$$f(x) = -cos(3x)$$

(iv)
$$f(x) = \frac{3x+2}{4x+3}$$

(v)
$$f(x) = \frac{e^x - e^{-x}}{2}$$

(vi)
$$f(x) = \frac{x}{x^2-4}$$

Question 24 : Calculations for the Ratio Test

For each of the following terms, given as u_n, state u_{n+1} and hence calculate a simplified expression for r, where

$$r = \frac{u_{n+1}}{u_n}$$

(i) $u_n = n^2$

(ii) $u_n = 2^n$

(iii) $u_n = \dfrac{2^n}{n^2}$

(iv) $u_n = (2n)!$

(v) $u_n = \dfrac{2^n}{n!}$

(vi) $u_n = \dfrac{2^n \times n}{n!}$

(vii) $u_n = n! \times n!$

(viii) $u_n = \dfrac{2^{n+1} \times n^2}{4^n \times n!}$

Question 25 : Functions - Part A

Marks

SECTION A

1 (a) Consider the functions $f(x) = \sqrt{8-2x}$, $g(x) = \log_e(2x+1)$

(i) Find $f(4-2x^2)$ and simplify answer.

(ii) Write down the domain and range of $f(x)$.

(iii) Find $g^{-1}(x)$ the inverse of $g(x)$. **10**

Question 25 : Functions - Part B

SECTION A

1(a) Consider the fnctions $f(x) = \dfrac{1}{\sqrt{x-2}}$ and $g(x) = x + \sin x$

(i) Find $f(4x^2 + 2)$ and simplify the answer.
(ii) Write down the domain and range of $f(x)$.
(iii) Prove that $g(x)$ is an odd fnction. (see tables page 9)

Question 26 : Domain and Ranges of Functions

Find the domain and the range of the functions:

(i) $f(x) = x - 2 \quad g(x) = -2x$

(ii) $f(x) = x^2 - 4 \quad g(x) = -x^2 - 4$

(iii) $f(x) = \sqrt{x} \quad g(x) = \sqrt{x-2}$

(iv) $f(x) = \sin x \quad g(x) = -2\sin x$

(v) $f(x) = \cos x \quad g(x) = \cos^2(x)$

(vi) $f(x) = e^x \quad g(x) = e^x - 2$

Question 27 : Domain and Ranges of Functions

Find the domain and the range of the functions:

(i) $f(x) = 8 - 2\sin(x)$

(ii) $f(x) = 5 - 2\cos(2x)$

(iii) $f(x) = 2\cos(x) - 6$

(iv) $f(x) = 7 + 2\sin(3x)$

(v) $f(x) = |5\sin(x)|$

(vi) $f(x) = \cos^2(x) + \sin^2(x)$

(vii) $f(x) = \cos^2(x) - \sin^2(x)$

Question 28 : Evaluation of trigonometric values

Evaluate the following values. *you may use your calculator.*

(i) $\cos(\pi/2)$

(ii) $\sin(3\pi/4)$

(i) $\tan(\pi/6)$

(ii) $\cos(1.5\pi)$

Question 29 : Composite Functions

Determine the values of $f \circ g(2)$ and $g \circ f(x)$. Henxe or otherwise, evaluate $f \circ g(2)$ and $g \circ f(2)$.

(i) $f(x) = X^2 + 1$ and $g(x) = 2x$

(ii) $f(x) = \sqrt{x+1}$ and $g(x) = x^2$

(iii) $f(x) = x^2 - 1$ and $g(x) = 2x + 1$

(iv) $f(x) = 5x$ and $g(x) = x^2 + 1$

(v) $f(x) = -4x + 9$ and $g(x) = 2x - 7$

(vi) $f(x) = e^x$ and $g(x) = \ln(x)$

(vii) $f(x) = x^2 + 1$ and $g(x) = 1 - 3x$

(viii) $f(x) = \sin(\pi x)$ and $g(x) = 2x + 1$

Question 30 : Inverse Functions

(i) $f(x) = 2x$

(ii) $f(x) = e^x$

(iii) $f(x) = x - 3$

(iv) $f(x) = \cos(x)$

(v) $f(x) = \sin^{-1}(x)$

(vi) $f(x) = \dfrac{1}{x-1}$

(vii) $f(x) = x^2 + 2$

(viii) $f(x) = \sin^{-1}(2x)$

(ix) $f(x) = 3e^{2x}$

(x) $f(x) = \sqrt{2x+3}$

(xi) $f(x) = \log_e 4x$

(xii) $f(x) = \dfrac{-2}{x-5}$

Selected Solution

(i) For the function $f(x) = 2x$ the inverse function is $f^{-1}(x) = x/2$.

(ii) For the function $f(x) = e^x$ the inverse function is $f^{-1}(x) = \log_e(x) = \ln(x)$.

(iii) For the function $f(x) = x - 3$ the inverse function is $f^{-1}(x) = x + 3$

(iv) For the function $f(x) = \cos(x)$ the inverse function is $f^{-1}(x) = \cos^{-1}(x)$.

(v) For the function $f(x) = \sin^{-1}(x)$ the inverse function is $f^{-1}(x) = \sin(x)$.

Question 31 : Revision of Product, Quotient and Chain Rule

(i)
$$f(x) = (x^4 + 4x + 2)(2x + 3)$$

(ii)
$$f(x) = (2x - 1)(3x^2 + 2)$$

(iii)
$$f(x) = (x^3 - 12x)(3x^2 + 2x)$$

(iv)
$$f(x) = (2x^5 - x)(3x + 1)$$

(v)
$$f(x) = \frac{2x + 1}{x + 5}$$

(vi)
$$f(x) = \frac{3x^4 + 2x + 2}{3x^2 + 1}$$

(vii)
$$f(x) = \frac{x^{\frac{3}{2}} + 1}{x + 2}$$

(viii)
$$f(x) = \frac{x^2 + x}{2x - 1}$$

(ix)
$$f(x) = \frac{x + 1}{2x^2 + 2x + 3}$$

(x)
$$f(x) = \frac{16x^4 + 2x^2}{x}$$

(xi)
$$f(x) = (x + 5)^2$$

(xii)
$$f(x) = \sqrt{1 - x^2}$$

(xiii)
$$f(x) = \frac{(2x + 4)^3}{4x^3 + 1}$$

(xiv)
$$f(x) = \frac{2x + 3}{(x^4 + 4x + 2)^2}$$

(xv)
$$f(x) = \frac{2x + 3}{(x^4 + 4x + 2)^2}$$

(xvi)
$$f(x) = 3e^x - 4\cos(x) - \frac{1}{4}\ln x$$

(xvii)
$$f(x) = \sin(x) + \cos(x)$$

Selected Solutions

(i)
$$10x^4 + 12x^3 + 16x + 16$$

(ii)
$$18x^2 - 6x + 4$$

(iii)
$$15x^4 + 8x^3 - 108x^2 - 48x$$

(iv)
$$36x^5 + 10x^4 - 6x - 1$$

(v)
$$30x^2 + 70x + 6$$

(vi)
$$6x(25x^2 + 1)(5x^2 + 1)^3$$

Question 32 : Introduction to Integration

Evaluate each of the following indefinite integrals:

1. $\int (5t^8 - 4t^5 + 3t + 2)\, dt$
2. $\displaystyle\int (\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3})\, dx$
3. $\displaystyle\int (\sqrt{x} + x\sqrt{x})\, dx$
4. $\displaystyle\int \left(\frac{1}{\sqrt{u}} + \frac{1}{u^2\sqrt{u}}\right)\, du$
5. $\displaystyle\int \frac{4}{\sqrt{9 - x^2}}\, dx$
6. $\displaystyle\int \frac{t^2 + 3t + 6}{t}\, dt$
7. $\displaystyle\int (3e^x + 1)\, dx$
8. $\int (5\cos x - 6\sin x)\ dx$
9. $\displaystyle\int \frac{7}{x^2 + 9}\, dx$

MA4702 Integration

Integration

Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n, (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\|x\|$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
$a^x \quad (a > 0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\|\sec x\|$
$\frac{1}{\sqrt{a^2 - x^2}} \quad (a > 0)$	$\sin^{-1} \frac{x}{a}$
$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$

Addition and Subtraction Rules of Integration

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

$$\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

The Power Rule for Integration

The power rule for derivatives can be reversed to give us a way to handle integrals of powers of x. Since

$$\frac{d}{dx}x^n = nx^{n-1},$$

we can conclude that

$$\int nx^{n-1}\,dx = x^n + C,$$

or, a little more usefully,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Question 33 : Introduction to Integration

Part A

Using appropriate substitutions, evaluate the indefinite integrals:

(i)

$$\int (s-4)^5 ds$$

(ii)

$$\int \frac{3}{(x+1)^4} dx$$

(iii)

$$\int (2y+3)(y^2+3y+2)^2 dy$$

Part B

Using appropriate substitutions, evaluate the indefinite integrals:

(i)

$$\int 3x^2(x^3+1)^5\,dx$$

(ii)

$$\int x^4 \sin(x^5)\,dx$$

Question 34 : Integration

Evaluate the following indefinite integrals using partial fractions:

(i)

$$\int \frac{x}{x^2 - 9} dx$$

(ii)

$$\int \frac{x - 2}{x^2 - 4x + 3} dx$$

(iii)

$$\int \frac{2x - 4}{x^2 - 4x + 8} dx$$

Question 35 : Integration by Parts

Evaluate the following using integration by parts.

(i)

$$\int -4 \ln(x)\, dx$$

(ii)

$$\int (-7x + 38) \cos(x)\, dx$$

(iii)

$$\int_0^{\frac{\pi}{2}} (-6x + 45) \cos(x)\, dx$$

(iv)

$$\int (5x + 1)(x - 6)^4\, dx$$

(v)

$$\int_{-1}^{1} (2x + 8)^3 (-x + 2)\, dx$$

(vi)

$$\int \sin(x)\, e^x\, dx$$

Question 36 : Integration by Parts

Formula:
If u and v are functions of x that have continuous derivatives, then

$$\int u dv = uv - \int v du$$

The LIPET rule

It is considered a rule of thumb to remember the acronym **LIPET** when performing integration by parts. This acronym will help you to determine what to use as u.

L -logarithms,

I -inverse trigonometric functions,

P -polynomials (i.e. x, x^2) ,

E -exponentials (i.e. e^x, e^{3x}),

T -trigonometric functions.

- $\cosh(x)$ is both the derivative and integral of $\sinh(x)$
- $\sinh(x)$ is both the derivative and integral of $\cosh(x)$

Question 37 : Integration by Parts

Evaluate the following indefinite integrals by integration by parts:

(a) $\int x^2 e^x \, dx$

(b) $\int x \ln x \, dx$

(c) $\int x^2 \cos x \, dx$

(d) $\int x \sin x \, dx$

(e) $\int e^x \sin x \, dx$

(f) $\int \ln x \, dx$

Question 38 : Integration (Video)

Evaluate the following:

(i)

$$\int x^2 - (2x)^2 \, dx$$

(ii)

$$\int 8x^3 \, dx$$

(iii)
$$\int (4x^2 + 11x^3)\, dx$$

(iv)
$$\int (31x^{32} + 4x^3 - 9x^4)\, dx$$

(v)
$$\int 5x^{-2}\, dx$$

Question 39 : Definite Integrals (Video)

Evaluate the following definite integrals

(i)
$$\int_1^2 (x^2 - 1)dx$$

(ii)
$$\int_0^{\frac{\pi}{2}} \cos x dx$$

(iii)
$$\int_0^{\pi} \cos x dx$$

(iv)
$$\int_1^2 (y^2 - y^{-2})dy$$

(v)
$$\int_{-3}^1 (6x^2 - 5x + 2)dx$$

(vi)
$$\int_4^0 \sqrt{t}(t-2)dt$$

Hint:
$$\int \sqrt{t}(t-2)dt$$
$$\sqrt{t}(t-2) = t^{1/2} \times (t-2) = t^{3/2} - 2t^{1/2}$$

Question 40 : Definite Integrals (Video)

Evaluate the following definite integrals:

(a) $\displaystyle\int_{-2}^{2} \frac{1}{x+3}\, dx$

(b) $\displaystyle\int_{0}^{2} (x^4 + 3x^2 + 2)\, dx$

(c) $\displaystyle\int_{-\pi}^{\pi} (5\sin x - 7\cos x)\, dx$

(d) $\displaystyle\int_{-3}^{2} 2x\, e^{(x^2+1)}\, dx$

(e) $\displaystyle\int_{0}^{\sqrt{\pi}} x\cos\left(x^2 - \frac{\pi}{2}\right)\, dx$

(f) $\displaystyle\int_{0}^{\pi} x\sin x\, dx$

(g) $\displaystyle\int_{0}^{1} \frac{1}{x^2-4}\, dx$

(h) $\displaystyle\int_{0}^{2} \frac{1}{x^2+4}\, dx$

Question 41 : Definite Integrals

Exercise: Evaluate the following definite integral

$$\int_1^3 \frac{x}{3} dx$$

Solution

$$\int_1^3 \frac{x}{3} dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20$$

Exercise: Evaluate the following definite integral

$$\int_1^3 \frac{x^2-4x+3}{x-3}dx$$

Factorize the numerator $x^2-4x+3=(x-1)(x-3)$

Treat it as an indefinite integral for time being.

$$\int \frac{x^2-4x+3}{x-3}dx = \int \frac{(x-1)(x-3)}{x-3}dx = \int (x-1)dx = \frac{x^2}{2}-x+c$$

$$\left[\frac{x^2}{2}-x\right]_1^3 = (4.5-3)-(0.5-1)=2$$

Question 42 : Integration by Parts (Exam Standard)

the following questions are from previous past papers. Please be advised of the notes below.

(i) (2005) Use integration by parts to find $\int xe^x dx$

(ii) (2006) Use integration by parts to find $\int xln(x)dx$

(iii) (2007) Use integration by parts to find $\int xsinh(x)dx$

(iv) (2008) Use integration by parts to find $\int xcos(x)dx$

(v) (2009) Use integration by parts to find $\int xcosh(x)dx$

(vi) (2010) Use integration by parts to find $\int xe^x dx$

Important:

- You should expect to see hyperbolic functions (i.e. $\cosh(x)$ and $\sinh(x)$) in the end of semester exam.
- However you should expect to see terms like x^2, e^{2x} and $\ln(x)$, as well as what was in previous exams.
- **VERY Important:** Make sure you know how to integrate and differentiate expressions of the form e^{ax}, $\cos(ax)$, $\cosh(ax)$, $\sin(ax)$ and $\sinh(ax)$.

Question 43 : Definite Integrals

Evaluate the following definite integrals

(i) Find the area between $f(x) = x^2 + 4x$ and the $x-$axis between $x = -4$ and $x = 3$.

(ii) Calculate the following:

$$\int_0^1 \frac{4x^3}{x^4+1} dx$$

(iii) Evaluate

$$\int_0^{\frac{\pi}{2}} \cos^4(x)\sin(x) dx$$

Question 44 : Integration : Partial Fraction Expansion Questions

- Suppose we have to integrate the following expression.

$$\int \frac{1}{x+1} + \frac{1}{x-2} dx$$

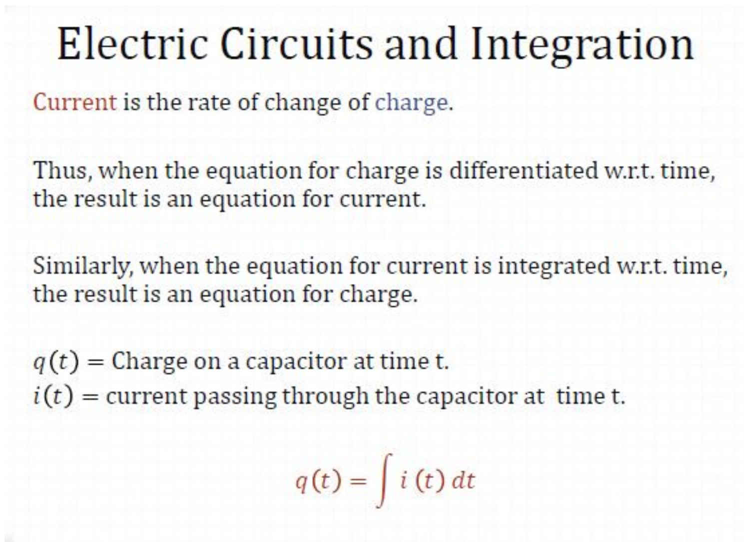
- The integral of both individual terms are $ln(x+1)$ and $ln(x-2)$
- The overall answer is therefore

$$\int \left(\frac{1}{x+1} + \frac{1}{x-2} \right) dx = \ln(x+1) + \ln(x-2) + c$$

- Further simplifcations are possible, but you will get full marks once you get to there.

- As we covered this extensively at the start of the semester, You should expect a question like this.

Question 45 : Applications of Integration - Electrical Circuits



1. A current $i(t) = 4e^{-2t}$ passes through a capacitor at time t. The capacitor is uncharged initially. Find the charge $q(t)$ at all times t.
2. A current $i(t) = 5 + 6\sin 3t$ passes through a capacitor at time t. The capacitor is uncharged at time $t = 0$. Find the charge $q(t)$ at all times t. ***2001 Q.3(b)***

Question 46 : Combined Integration Question (Exam Standard)

(i) Evaluate the following indefinite integral:

$$\int 3x^2 + 2e^x - 1dx$$

(ii) Evaluate the following definite integral:

$$\int_4^9 \frac{1}{\sqrt{x}}dx$$

(iii) Find

$$\int \left(e^{4x} + \cos(3) + \frac{1}{x^2}\right) dx$$

(iv) The area enclosed between the curve $y = cos(x)$ and the x-axis between $x = 0$ and $x = \frac{\pi}{3}$

Question 47 : Definite Integrals (Worked Example)

Consider the integral

$$\int_0^2 x\cos(x^2+1)\,dx$$

By using the substitution $u = x^2 + 1$, we obtain $du = 2xdx$ and

$$\begin{aligned} \int_0^2 x\cos(x^2+1)\,dx &= \frac{1}{2}\int_0^2 \cos(x^2+1)2x\,dx \\ = \frac{1}{2}\int_1^5 \cos(u)\,du &= \frac{1}{2}\Big(\sin(5) - \sin(1)\Big). \end{aligned}$$

Question 48 : Area Between Curves and Lines

(i) (2005) Find the area bounded by the curves $y = x^2$ and $y = 2–x^2$.

(ii) (2006) Find the area bounded by the curve $y = x^2–6x + 5$ and the line $y = x–5$.

(iii) (2007) Find the area bounded by the curves defined by $y = x^2–4$ and $y = 4–x^2$.

(iv) (2008) Find the area bounded by the curve $y = x^2 - 1$ and the line $y = 4x–1$.

(v) (2009) Find the area bounded by the curve $y = 5x–x^2$ and the line $y = 2x$.

(vi) (2010) Find the area enclosed by the curve $y = 4–x^2$ and the line $y = x + 2$.

Question 49 : Integration : Partial Fraction Expansion Questions

- Suppose we have to integrate the following expression.

$$\int \frac{1}{x+1} + \frac{1}{x-2} dx$$

- The integral of both individual terms are $ln(x + 1)$ and $ln(x - 2)$
- The overall answer is therefore

$$\int \left(\frac{1}{x+1} + \frac{1}{x-2} \right) \, dx \;=\; \ln(x+1) + \ln(x-2) + c$$

- Further simplifcations are possible, but you will get full marks once you get to there.
- As we covered this extensively at the start of the semester, You should expect a question like this.

Question 50a : Partial Derivatives

(i) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$ of the function $z = 2x^2y + 4xy^3 + 5x^2$.

- (Rest of Question removed)

Question 50b : Partial Derivatives

Solve for $\partial f/\partial x$ and $\partial f/\partial y$ if

$$f(x, y) = ln(xy) + sin(x) = ln(x) + ln(y) + sin(x)$$

.

Exercise 51 : (Worked Example)

Solve for $\partial f/\partial x$ and $\partial f/\partial y$ if

$$f(x, y) = -x^2 + y$$

Solution :

$$\partial f/\partial x = -2x$$

$$\partial f/\partial y = 1$$

Question 52 : Partial Derivatives

Determine all first order and second order partial derivatives of each of the following.

(i) $f(x, y) = 3x + 4y$

(ii) $f(x, y) = xy^3 + x^2y^2$

(iii) $f(x, y) = x^3y + e^x$

Question 53 : Curve Sketching

Ex. 1 (i) Find the y intercept of the function $y = f(x)$.

(ii) Show that $(\sqrt{3}, 1)$ is a stationary point of the function. Find the other two stationary points and classify all three points as local maxima or minima.

(iii) Find the two inflection pints of $f(x)$.

(iv) Find the x values for which is $y = f(x)$ is concave up/down.

(v) Determine the behaviour of y as $x \to +\infty$ and as $x \to -\infty$

(vi) Sketch the graph of $y = f(x)$ indicating clearly the features of the curve obtained in aprts (i) to (v) of this quesiton.

Ex. 2 Find α and β so that the function

$$f(x) = \alpha x^3 + \beta x^2 + 1$$

has a point of inflection at $(-1, 2)$

Question 54 : Curve Sketching

Past Paper Question - Summer 04/05

2 Consider the function $y = f(x) = x^4 - 6x^2 + 2$

(i)	Find the y intercept of $f(x)$.	**1**
(ii)	Find and classify the critical points of $f(x)$ as local maxima or local minima or points of inflection.	**6**
(iii)	Find all points of inflection.	**4**
(iv)	Find the x values for which $y = f(x)$ is concave up/down.	**2**
(v)	Determine the behaviour of y as $x \to +\infty$ and as $x \to -\infty$.	**2**
(vi)	Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – v).	**5**

Question 56 : Curve Sketching

Past Paper Question - Summer 05/06

2 Consider the function $y = f(x) = \frac{1}{x-4}$ $(x \neq 4)$

(i)	Find the y intercept of $f(x)$.	**2**
(ii)	Show that the function has no local maximum or local minimum point.	**5**
(iii)	Explain why the function is decreasing for all values of x.	**2**
(iv)	Find the equation of the vertical asymptote.	**2**
(v)	Find the equation of the horizontal asymptote.	**3**
(vi)	Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – v).	**6**

Question 56 : Curve Sketching

This is Curve Sketching Example 3 from the lectures. The entirety of the material covered by this question is examinable in the End-Of-Semester exam.

Curve Sketching – Complete Ex. 3

Consider the function $f(x) = xe^{-x}$

i. Find the x and y intercepts of $f(x)$.
ii. Find and classify the critical points of $f(x)$ as local maxima or local minima.
iii. Find all points of inflection.
iv. Determine the behaviour of y as $x \to -\infty$ and as $x \to +\infty$
v. Sketch the graph of $y = f(x)$ illustrating clearly the features of the curve obtained in parts (i – iv)

Question 57 : Curve Sketching

This is Curve Sketching Example 4 from the lectures. The entirety of the material covered by this question is examinable in the End-Of-Semester exam.

Curve Sketching – Complete Ex. 4

The concentration of a drug in a patient's bloodstream h hours after it was injected is given by

$$A(h) = \frac{0.17h}{h^2 + 2}$$

i. Find the axis intercepts of $A(h)$.

ii. Find and classify the critical points of $A(h)$ as local maxima or local minima.

iii. Determine the behaviour of $A(h)$ as $h \to +\infty$

iv. Sketch the graph of $y = A(h)$ for $h \geq 0$ illustrating clearly the features of the curve obtained in parts (i – iii)

Question 58 : Curve Sketching

Consider the function $y = f(x) = x^4 – 6x^2 + 2$

(i) Find the y intercept of f(x).

(ii) Find and classify the critical points of f(x) as local maxima or local minima or points of inflection.

(iii) Find all points of inflection.

(iv) Find the x values for which y = f(x) is concave up/down.

(v) Determine the behaviour of y as $x \to +\infty$ and as $x \to -\infty$.

(vi) Sketch the graph of y = f(x) illustrating clearly the features of the curve obtained in parts (i – v).