UNIVERSITY *of* LIMERICK

OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4315

SEMESTER: Autumn 2016

MODULE TITLE: Operations Research 2

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. M. Burke & Mr. K. O'Brien

PERCENTAGE OF TOTAL MARKS: 95%

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES:

Answer four questions correctly for full marks.

Answer two questions from Q1–Q3 and two questions from Q4–Q6.

45% of the marks are for the two questions from Q1–Q3.

50% for the two questions from Q4–Q6..

Question 4

(a) (i) Explain briefly why the following strategy for the solution of Integer Linear Programs (IPs) is not useful: "Solve the LP relaxation then round off the components of the solution to the nearest integers". 3 %

(ii) Given an LP (the *Primal* problem) we can write a closely related LP, its *Dual*:

$$\begin{aligned} z &= \max\{c^T x : Ax \leq b, x \in \mathbb{R}^n, x \geq 0\} \quad \textit{Primal} \\ w &= \min\{b^T y : A^T y \geq c, y \in \mathbb{R}^m, y \geq 0\}. \quad \textit{Dual} \end{aligned}$$

Prove the Weak Duality Theorem: for *any* primal feasible point x and *any* dual feasible point y, $b^T y \geq c^T x$. 4 %

(b) ERBIZAKIP Investments is considering investments into 6 projects: A, B, C, D, E and F.

Each project has an initial cost, an expected profit rate (one year from now) expressed as a percentage of the initial cost, and an associated risk of failure. These numbers are given in the table below:

	A	B	C	D	E	F
Initial Cost	1.8	1.5	1.1	1.8	2.1	3.2
Profit Rate	11%	13%	10%	12%	11%	9%
Failure Risk	6%	4%	5.5%	5%	4.5%	4.5%

(i) Provide a formulation to choose the projects that maximize total expected profit, such that ERBIZAKIP Investments does not invest more than 5M dollars and its average failure risk is not over 5%.

You may assuming equal weighting for each project when determining average risk. For example, if ERBIZAKIP Investments invests only into A,B and C, it invests only 4.4M dollars and its average failure risk is $(6\% + 4\% + 5\%)/3 = 5\%$. 4 %

(ii) Suppose that if C is chosen, D must be chosen. Modify your formulation. 2 %

(iii) Suppose that if A and C are chosen, D must be chosen. Modify your formulation. 2 %

(iv) Suppose that only two projects, at most, can be chosed from A, B and C. Modify your formulation. 2 %

(c) (i) Provide a short description of the Dynamic Programing Paradigm. 4 %

(ii) What is a Greedy Algorithm? Support your answer with a simple example, and discuss the advantages and disadvantages of using Greedy Algorithms. 2 %

(iii) In the context of the design of algorithms, describe the Divide and Conquer paradigm. 2 %

Question 5

(a) (i) Big O-notation is used to classify algorithms according to their relative complexity. Compare the complexity of algorithms of order $O(\log n)$, $O(n)$, $O(n \log n)$, $O(2^n)$and $O(n!)$. Illustrate your answer with a sketch. 3 %

(ii) Compare and contract Big O-Notation, Big Omega Notation and Big Theta Notation. 2 %

(iii) What is meant by Combinatorial Explosion? Why is it relevant for Binary Integer Problems? 2 %

(b) Consider the Integer Linear Program (IP):

$$\begin{aligned} \max 5x_1 + 4x_2 & \\ x_1 + x_2 &\leq 5 \\ 10x_1 + 6x_2 &\leq 45 \\ x_1, x_2 \leq 0 \,\text{and integer.} & \end{aligned}$$

The corresponding Simplex Tableau (transforming the max problem into a min problem) is:

0	-5	-4	0	0
5	1	1	1	0
45	10	6	0	1

N.B. The Simplex Method and the Dual Simplex Method are stated on the last page of this paper.

(i) Apply one iteration of the Simplex Method and show that the Simplex Tableau now takes the form: 4 %

22.5	0	-1	0	0.5
0.5	0	0.4	1	-0.1
4.5	1	0.6	0	0.1

(ii) After a second iteration of the Simplex Method the Simplex Tableau now takes the form: (**N.B.do not perform the arithmetic!**)

23.75	0	0	2.5	0.25
1.25	0	1	2.5	-0.25
3.75	1	0	-1.5	0.25

Explain why this Tableau is optimal. 1 %

(iii) The solution to the LP Relaxation of the IP is $x_1 = 3.75, x_2 = 1.25$. Suppose that we decide to branch on x_1. The two branches are $S_0 : x_1 \leq 3$ and $S_1 : x_1 \geq 4$.

Consider the branch $S_0 : x_1 \leq 3$.

(a) First show that the basic variable x_1 may be expressed in terms of the non-basic variables x_3 & x_4 as: $x_1 = 3.75 + 1.5x_3 - 0.25x_4$. 2 %

(b) Substitute this expression for x_1 into the S_0 branch constraint and show that it takes the form $1.5x_3 - 0.25x_4 + s = -0.75$. (The variable s is the slack variable for the constraint $x_1 \leq 3$.) 1 %

(c) Show that the Simplex Tableau with the addition of this constraint takes the form: 1 %

23.75	0	0	2.5	0.25	0
1.25	0	1	2.5	-0.25	0
3.75	1	0	-1.5	0.25	0
-0.75	0	0	1.5	-0.25	1

(d) Explain why this tableau is optimal but infeasible. 1 %

(e) Apply **one** iteration of the Dual Simplex Method to this tableau and show that the Simplex Tableau now takes the form: 4 %

23	0	0	4	0	1
2	0	1	1	0	-1
3	1	0	0	0	1
3	0	0	-6	1	-4

(f) This tableau is LP optimal and integer feasible. Explain why. What is the solution to the IP? 2 %

(g) Finally, **suppose** that we had started with the branch $S_1 : x_1 \geq 4$, expressed x_1 in terms of the non-basic variables x_3 & x_4 as $x_1 = 3.75 + 1.5x_3 - 0.25x_4$ as above and applied the Dual Simplex method to the resulting tableau.
We would have found (**N.B.do not perform the arithmetic!**)

23.33	0	0	0	0.67	1.67
0.83	0	1	0	0.17	1.67
4	1	0	0	0	-1
0.17	0	0	1	-0.17	-0.67

i. This tableau is now optimal. Is the solution integer? 1 %

ii. What would be the next branch & bound step? (**N.B.do not perform the arithmetic!**) 1 %

Question 6

(a) Maximize P = 12x+7y

subject to

5x + 3y ≤ 175
5x + 4y ≤ 185
x ≤ 35
x ≤ 27
x,y ≥ 0
x,y integers

This Integer Program is to be solved using the tabular Branch and Bound method.

Use the solution grids below to solve the problem. Each node is referenced by its tree level, ordered from left to right so that the annotation **Node XY** is the node at level **X** at position **Y** where **Y** = A is the left-most position in level **X**, where **Y** = B is the 2nd from the left in level **X**, and so on.

You must draw an enumeration tree/diagram to keep track of your progress. Draw the enumeration tree on an otherwise blank page. 25%

	Node 0		Node 1A		Node 1B
(i)	$x = 27.50, y = 12.50$	(i)	$x = 27.50, y = 12.50$	(i)	$x = 26.60, y = 13.00$
(ii)	$x = 27.00, y = 12.50$	(ii)	$x = 27.00, y = 13.50$	(ii)	$x = 26.60, y = 13.50$
(iii)	$x = 28.20, y = 11.40$	(iii)	$x = 25.60, y = 12.70$	(iii)	$x = 25.70, y = 14.50$
(iv)	$x = 28.50, y = 13.50$	(iv)	$x = 27.00, y = 12.00$	(iv)	$x = 28.50, y = 13.75$
(v)	$x = 27.00, y = 13.50$	(v)	$x = 28.00, y = 11.00$	(v)	$x = 27.00, y = 11.00$
	Node 2A		Node 2B		Node 2C
(i)	$x = 26.00, y = 13.75$	(i)	$x = 26.25, y = 11.30$	(i)	$x = 27.00, y = 14.75$
(ii)	$x = 27.25, y = 13.50$	(ii)	$x = 27.25, y = 11.50$	(ii)	$x = 26.75, y = 14.00$
(iii)	$x = 25.75, y = 11.00$	(iii)	$x = 24.25, y = 12.25$	(iii)	$x = 25.75, y = 13.50$
(iv)	$x = 24.50, y = 11.60$	(iv)	$x = 28.60, y = 12.80$	(iv)	$x = 25.80, y = 16.00$
(v)	$x = 28.25, y = 13.25$	(v)	$x = 28.00, y = 11.00$	(v)	$x = 26.00, y = 13.75$

	Node 2D		Node 3A		Node 3B
(i)	$x = 26.90$, $y = 15.125$	(i)	$x = 27.125$, $y = 14.75$	(i)	$x = 26.30$, $y = 11.50$
(ii)	$x = 27.00$, $y = 12.50$	(ii)	$x = 26.50$, $y = 14.50$	(ii)	$x = 27.25$, $y = 15.50$
(iii)	$x = 27.00$, $y = 13.40$	(iii)	$x = 27.40$, $y = 14.00$	(iii)	$x = 27.25$, $y = 12.50$
(iv)	$x = 27.00$, $y = 12.90$	(iv)	$x = 26.00$, $y = 12.90$	(iv)	$x = 27.10$, $y = 13.50$
(v)	$x = 27.50$, $y = 14.10$	(v)	$x = 26.90$, $y = 14.125$	(v)	$x = 26.00$, $y = 11.25$
	Node 3C		Node 3D		Node 3E
(i)	$x = 25.20$, $y = 16.10$	(i)	$x = 28.25$, $y = 13.60$	(i)	$x = 27.00$, $y = 12.00$
(ii)	$x = 25.75$, $y = 18.50$	(ii)	$x = 30.40$, $y = 17.50$	(ii)	$x = 27.750$, $y = 15.80$
(iii)	$x = 28.50$, $y = 28.125$	(iii)	$x = 23.00$, $y = 18.00$	(iii)	$x = 26.00$, $y = 13.00$
(iv)	$x = 28.125$, $y = 14.5$	(iv)	$x = 25.25$, $y = 15.333$	(iv)	$x = 25.00$, $y = 15.80$
(v)	$x = 28.60$, $y = 11.20$	(v)	$x = 25.00$, $y = 12.60$	(v)	$x = 22.60$, $y = 16.70$
	Node 3F		Node 3G		Node 3H
(i)	$x = 25.80$, $y = 14.00$	(i)	$x = 25.40$, $y = 14.25$	(i)	$x = 24.60$, $y = 14.80$
(ii)	$x = 25.80$, $y = 14.60$	(ii)	$x = 27.50$, $y = 15.50$	(ii)	$x = 22.30$, $y = 14.10$
(iii)	$x = 25.25$, $y = 14.50$	(iii)	$x = 27.40$, $y = 13.50$	(iii)	$x = 22.30$, $y = 14.50$
(iv)	$x = 26.10$, $y = 11.60$	(iv)	$x = 27.30$, $y = 13.50$	(iv)	$x = 22.60$, $y = 14.60$
(v)	$x = 25.40$, $y = 14.90$	(v)	$x = 23.90$, $y = 16.50$	(v)	$x = 21.70$, $y = 14.50$