Numerical Methods

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Practical Information

- There will be one Continuous Assessment Test (date to be arranged) which counts for 30% of the total mark for this module.
- The typed notes will be available online at
  `http://www.maths.dit.ie/~dmackey/lectures.html`
- Suggested reading:
  1. Bradie B., *A Friendly Introduction to Numerical Analysis*, 2006;
  2. Burden R.J. & Faires J.D., *Numerical Analysis*, 2004;
  3. Atkinson K.E. & Han W., *Elementary Numerical Analysis*, 2004.

Contents

Numerical analysis is concerned with finding approximate solutions to problems from mathematics or other sciences.

Topics studied this year:

- Root finding techniques (solutions of nonlinear equations)
- Interpolating methods
- Systems of linear equations
- Numerical differentiation and integration

Example

The number $\sqrt{2} = 1.414213562...$ is defined as the number which satisfies the equation

$$x^2 - 2 = 0.$$

Question: How can we find an approximate value for this number?

Answer: The sequence constructed with the algorithm

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = 1.5 \\
&\vdots \\
x_{n+1} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)
\end{aligned}$$

gives successive approximations to $\sqrt{2}$.

Example: root finding

- The roots (solutions) of the quadratic equation

$$x^2 - 3x + 2 = 0$$

can be calculated using the formula

$$x = \frac{3 \pm \sqrt{3^2 - 4 \cdot 2}}{2} \qquad \text{so} \quad x_1 = 1,\ x_2 = 2.$$

- The roots of the equation

$$x - \sin(x) = 0$$

cannot be calculated explicitly so we need to find methods for approximating them.

Example

Calculate an approximate value of the expression

$$E = \sqrt{2}\sin\frac{\pi}{3} + \sqrt{3}\sin\frac{\pi}{4}$$

- Method 1:

$$E = (1.414)\cdot(0.8660) + (1.732)\cdot(0.707) = 2.450.$$

- Method 2:

$$E = \sqrt{2}\,\frac{\sqrt{3}}{2} + \sqrt{3}\,\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{2} = \sqrt{6} = 2.449.$$

Conclusion: Hold on to symbolic notation for as long as possible!

Number systems

The **natural** numbers (or the counting numbers) are $1, 2, 3, 4, 5, \ldots$.
The **integers** are $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$.
The **rational** numbers are those numbers that can be written as the ratio a/b of two integers ($b \neq 0$).

An easy way to visualize all these numbers is to represent them on the **number line** (an unbroken and endless straight line, with an origin and a positive unit of length).

The set of all numbers on the number line is called the set of **real** numbers. Each real number represents a length.

There are, however, many points on this number line which are not rational numbers: for example, $\sqrt{2}$ and π.

Properties of rational and real numbers:

The numbers which are not rational (like $\sqrt{2}$ and π) are called **irrational**. Irrational numbers may seem "rare" but, in fact, there are infinitely more irrational numbers than there are rational numbers.

- The real numbers are **dense:** between any two real numbers there is always another one!
- The rational numbers are also dense on the real line: between any two real numbers we can always find a rational one!
- The rational numbers are **countable** (this means that there is a one-to-one correspondence between the rational numbers and the natural ones - such number systems are also called **discrete**).
- The irrational (and real) numbers are **not countable**.

The floating point number system

Computers represent numbers in a floating point number system. A floating point number has the form

$$\pm 0.d_1 d_2 d_3 \ldots d_k \times 10^n$$

which consists of a sign (plus or minus), a number between 0 and 1 (the **significant digits**) and a power of 10 (the **exponent**).

Example: Write the following numbers in the floating point form described above:

$$3.2; \quad 20,000; \quad -6.5 \times 10^4; \quad 0.3; \quad 5/2.$$

Note: Unlike the real numbers, the floating point number system is a **discrete** set.

Rounding and chopping

There are two ways of converting a real number to its floating point equivalent: **rounding** and **chopping**.

Example: Convert the real number $\sqrt{5} \approx 2.236067977...$ to a floating point representation with 3 significant digits.

We have $\sqrt{5} = +(0.2236067977...) \times 10$.
Chopping produces $\sqrt{5} \approx 0.223 \times 10 = 2.23$ while rounding gives $\sqrt{5} \approx 0.224 \times 10 = 2.24$.

The error introduced by rounding or chopping a real number to produce its floating point equivalent is called **roundoff error**.

Absolute and relative errors

If x^* is an approximation of the value x then the **absolute error** is defined as

$$|x^* - x|$$

The **relative error** is a measure of the error in relation to the size of the true value being sought:

$$\text{relative error} = \frac{\text{absolute error}}{\text{true value}} = \left|\frac{x^* - x}{x}\right|$$

Exercise: Calculate the absolute and relative errors when rounding and chopping the number $\sqrt{5}$ to 3 significant digits as above.

Loss of significance

Loss of significance is an undesirable consequence of floating point arithmetic and denotes the amplification in error which occurs when we subtract two numbers close to each other.

Example: Consider the numbers $x = 0.3721478693$ and $y = 0.3720230572$. Then $x - y = 0.0001248121$. Now assume the numbers x and y have been rounded to 5 digits before subtracting, so $x^* = 0.37215$ and $y^* = 0.37202$.

Calculate the relative errors for x, y and for the difference $x - y$.

Note: Loss of significance can usually be avoided through careful computation (or programming). For example, what happens when we try to calculate values of the function

$$f(x) = \sqrt{x^2 + 1} - 1$$

for small values of x? Suggest a way to avoid loss of significance.