Tutorial Sheet 1

1. Evaluate the terms $a_0, a_1, a_2, a_3, a_{10}, a_{100}$ in the following sequences, where possible.

    i) $a_n = 3n^2 + 2n + 1$

    ii) $a_n = \frac{n^2+7}{2n+3}$

    iii) $a_n = (-1)^n n$

    iv) $a_n = 3^n + (-3)^n + \frac{6n^4}{3n^4},\ n \neq 0$

    v) $a_n = 2^n.3^n$

    vi) $a_n = 6^n$

    vii) $a_n = \frac{1}{n+1} + \frac{1}{n+2}$

    viii) $a_n = \frac{2n+3}{n^2+3n+2}$

    ix) $a_n = 3n^7 + 2n^5 + n^5 + 2$

    x) $a_n = n^n,\ n \neq 0$

    xi) $a_n = n!$

    xii) $a_n = \frac{2^n+n^2+3n!}{3n!+n^2+2^n}$

    In addition, prove that the sequences vii) and viii) are equivilent. Write programs to evaluate the first 100 terms in these sequences.

2. Evaluate the terms a_0, a_1, a_2, a_3 in the following sequences and write programs to evaluate the first 100 terms in each sequence.

    i)
    $$a_n = \begin{cases} 2 + \frac{1}{2^n} & , \quad n \text{ even} \\ -2^n & , \quad n \text{ odd} \end{cases}$$

    ii)
    $$b_n = \begin{cases} 3^n - 14n - 7 & , \quad n \text{ even} \\ 3^n - 2n & , \quad n \text{ odd} \end{cases}$$

    $$a_n = \begin{cases} n^2 + 1 & , \quad b_n > 0 \\ 3^n - 2n & , \quad b_n \leq 0 \end{cases}$$

    iii)
    $$a_n = \begin{cases} (n+10)^6 & , \quad n < 2 \\ n^3 & , \quad \text{otherwise} \end{cases}$$

    iv)
    $$a_n = \begin{cases} (-1)^{n-1} \frac{\pi^{2n-1}}{(2n-1)!} & , \quad n \text{ even} \\ (-1)^{n+1} \frac{\pi^{2n+1}}{(2n+1)!} & , \quad n \text{ odd} \end{cases}$$

3. Evaluate the term a_7 in the following recursive sequences and write programs to evaluate the first 100 terms in each sequence. In addition, for sequence iii), evaluate $a_{12000003}$.

    i)
    $$\begin{cases} a_0 & = 1 \\ a_n & = n^2 a_{n-1} \end{cases}$$

    ii)
    $$\begin{cases} a_0 & = \frac{1}{2} \\ a_n & = a_{n-1}\left(2 - a_{n-1}\sqrt{2}\right) \end{cases}$$

    iii)
    $$\begin{cases} a_0 & = -\frac{1}{4} \\ a_1 & = 2 \\ a_n & = \frac{a_{n-1}}{a_{n-2}} \end{cases}$$

4. The factorial function is defined *recursively* as

$$n! = a_n, \text{ where } \begin{cases} a_0 & = 1 \\ a_n & = n a_{n-1} \end{cases}$$

The factorial function cas equivilently be defined *iteratively* as the product of the first n positive integers

$$n! = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times (n-1) \times (n-1) \times n$$

Write a program which computes $n!$ in this way without using recursion.

5. Evaluate the terms $a_1, a_2, a_3, a_4, a_{1000}$ in the following sequences

i) $a_n = \frac{(n+1)!}{n!}$

ii) $a_n = \frac{(n+2)!}{n!}$

iii) $a_n = \frac{(n+1)!}{(n+2)!}$

iv) $a_n = (-1)^n \frac{(n+3)!}{(n+2)!}$

6. For the following strictly positive sequences, evaluate the ratio $\frac{a_{n+1}}{a_n}$ and hence say whether the sequence is increasing, decreasing or neither. In addition, say whether the sequence is bounded or unbounded.

i) $a_n = n$

ii) $a_n = \frac{n+4}{n+5}$

iii) $a_n = \frac{n+5}{n+4}$

iv) $a_n = n!$

v) $a_n = \frac{n!}{n^n}, \; n \neq 0$

vi) $a_n = \frac{2^n+1}{2^n-1}$

7. Evaluate the limits of the following sequences as $n \to \infty$.

i) $a_n = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n!}$

ii) $a_n = \frac{2n+2}{3n+7}$

iii) $a_n = \frac{4n-3}{2n^2+2n+1}$

iv) $a_n = \frac{3n+2n^2+1}{5n^2+6}$

v) $a_n = \frac{7n^2}{20000n+n^2}$

vi) $a_n = \frac{10^{10^{10}}n+n^2}{n^2}$

vii) $a_n = \frac{3n}{n!}$

viii) $a_n = \frac{10n!}{n^n}$

ix) $a_n = \frac{5n^4+6n^2+4}{2^n+1}$

x) $a_n = \left(\frac{1}{4}\right)^n$

8. Assumming that the following recursive sequence has a limit $\lim_{n\to\infty} a_n = L$, find this limit in terms of D.

$$\begin{cases} a_0 & = 1 \\ a_{n+1} & = \frac{2}{3}a_n + \frac{D}{3a_n^2} \end{cases}$$

What is the limit of the sequence defined by

$$\begin{cases} a_0 & = 1 \\ a_{n+1} & = \frac{k-1}{k}a_n + \frac{D}{ka_n^{k-1}} \end{cases}$$

9. The Fibonacci sequence f_n is defined recursively by the rule

$$\begin{cases} f_0 & = 0 \\ f_1 & = 1 \\ f_n & = a_{n-1} + a_{n-2} \end{cases}$$

i) Write a program to evaluate the Fibonacci sequence and hence evaluate f_{50}.

ii) Let the sequence g_n be defined as the ratio

$$g_n = \frac{f_{n+1}}{f_n}$$

Write a program to evaluate the first 50 terms of the sequence g_n.

iii) Assumming that the sequence g_n has a limit ϕ, find this limit.