Question 4

4) Vectors / Systems of Linear Equations

Addition of Vectors

Prove that for any $u,\ v,\ \in \mathbb{R}^3$ and any $k \in \mathbb{R}$ we have

1. $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$;
2. $(u + v) \times w = u \times w + v \times w$;
3. $k(u \times v) = (ku) \times v = u \times (kv)$.

1. Prove that, for any $u,\ v\ \in \mathbb{R}^3$,

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u.$$

   (It is sufficient to verify this property for one component.)

2. Consider the three vectors in $\mathbb{R}^3$:

$$u = (1, 2, 0), \quad v = (0, 1, 3), \quad w = (1, 2, 3).$$

   (a) Evaluate $\|u\|$, $\|v\|$, $u \cdot v$, $u \times v$ and the angle between u and v.

   (b) Calculate the scalar triple product $u \cdot (v \times w)$.

3. (a) Find the general form of the equation of the plane π in $\mathbb{R}^3$ which passes through the point $P = (3, 1, 6)$ and is orthogonal to the vector $n = (1, 7, -2)$.

   (b) Show that the point $Q = (1, -1, 1)$ does not lie in the plane π and find its distance from π.

Part A. Cross Product abd Scalar Triple Product

Calculate the scalar triple product

$$u \cdot (v \times w)$$

for

1. $u = (1, 3, 5)$; $v = (0, 5, 3)$; $w = (3, 0, 7)$;
2. $u = (0, 1, 2)$; $v = (5, 0, 1)$; $w = (2, 2, 2)$.

Part B. Orthonormal Projections

If

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix},$$

$$u = (1, 0, 3); \qquad a = (3, 5, 7,)$$

i) find the vector component of u along a, $proj_a u$ and the vector orthogonal component to a;

ii) calculate the norm of $proj_a u$ and the norm of $u - proj_a u$;

iii) draw $proj_a u$ showing its direction and orientation.

Calculate the cross products $u \times u'$, $v \times v'$, $w \times w'$, where u, u', v, v', w, w' are given in Question 6.

Part A. Triangle Inequality

Cauchy Schwarz Identity

Triangle Identity

1. For the vectors given below, evaluate the following expressions where it is possible.

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{y} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -5 \end{bmatrix}, \vec{z} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

   (a) $2\vec{u} + 3\vec{v}$

   (b) $3\vec{u} - \vec{v}$

   (c) $\vec{x} + 3\vec{v}$

   (d) $2\vec{z} - \vec{w}$

   (e) $\vec{u} + \vec{x}$

   (f) $\vec{v} + \vec{w}$

   (g) $\vec{u} \cdot \vec{v}$

   (h) $(2\vec{u}) \cdot (3\vec{v})$

   (i) $\vec{x} \cdot \vec{y}$

   (j) $\vec{w} \cdot \vec{z}$

   (k) $\vec{w} \cdot (\vec{z} + \vec{w})$

   (l) $|\vec{x}|$

   (m) $|\vec{w}|$

   (n) $|\vec{y}| + |\vec{w}|$

2. Calculate the angles between the pairs $\vec{u}, \vec{v}$, $\vec{x}, \vec{y}$, and $\vec{w}, \vec{z}$ from the previous question. Give your answers in both radians and degrees.

3. Calculate the area of the parallelogram spanned by the vectors $\vec{x}$ and $\vec{y}$.

4. Show that the volume of the parallelopiped spanned by the vectors $\vec{u}, \vec{v}$ and $2\vec{u} + 3\vec{v}$ is zero.

5.

1. For each of the following systems of linear equations, write down the corresspond-ing coefficient matrix A, vector of unknowns $\vec{x}$, and vector of right hand sides $\vec{b}$ so that the system can be expressed in the form $A\vec{x} = \vec{b}$

(a)

$$2x + 3y = 1$$
$$5x + 7y = 3$$

(b)

$$2x + 3y + 4z = 1$$
$$x - 2y + 2z = 7$$
$$3x + 2y + z = 0.2$$

(c)

$$3x + y + z = 1$$
$$y + 4z = -4$$
$$x - y = 2$$

2.

Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given by

$$\vec{a} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{c} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

These vectors span a three dimensional parallelopiped as shown in the figure to the right.

i) Find the area of the parallelogram $S_{\vec{a}\vec{b}}$ which is spanned by the vectors $\vec{a}$ and $\vec{b}$. Hence state the area of the parallelogram $S'_{\vec{a}\vec{b}}$ on the opposite side of the parrallelopiped.

ii) Find the areas of the parallelograms $S_{\vec{b}\vec{c}}$ and $S_{\vec{a}\vec{c}}$ spanned by the relevant pairs of vectors and hence find the total surface area of the parrallelopiped.

iii) Find the signed volume of the parrallelopiped.

3. Rotate the vector $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ anti-clockwise $\frac{\pi}{4}$ radians about the origin.

4. Rotate the point $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ anti-clockwise $\frac{\pi}{4}$ radians about the point $\vec{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

5. Rotate the line segment with endpoints $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ anti-clockwise $\frac{\pi}{2}$ radians about the point $\vec{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Give the new endpoints $\vec{x}'$ and $\vec{y}'$ of the rotated line segment.