MA4402

2009

Section A

1 (a) $f(x) = \sqrt{2x-4}$, $g(x) = 8x^2+2$

(i) $fg(x)$

$\Rightarrow f(8x^2+2)$

$\Rightarrow \sqrt{2(8x^2+2)-4}$

$\Rightarrow \sqrt{16x^2+4-4}$

$\Rightarrow \sqrt{16x^2}$

$\Rightarrow 4x$

(ii) $g(x) = \dfrac{2x^3}{x^2-1}$

$g(-x) = \dfrac{2(-x)^3}{(-x)^2-1}$

$= \dfrac{-2x^3}{x^2-1}$

$= -g(x)$

$\Rightarrow$ odd function

(iii) $f(x) = y = e^{x+1}$

$f^{-1}: \ x = e^{y+1}$

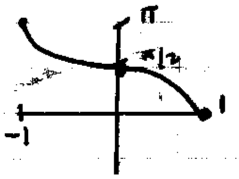
$\Rightarrow \log_e x = y+1$

$\Rightarrow -1 + \log_e x = y$

$\Rightarrow f^{-1}(x) = -1+\ln x$

(b) $\tan^{-1}(\frac{1}{2}) = 26.56^\circ$

or 0.4636 Radians



(c) $\cosh^2 x - \sinh^2 x$

$\Rightarrow \left(\dfrac{e^x+e^{-x}}{2}\right)^2 - \left(\dfrac{e^x-e^{-x}}{2}\right)^2$

$\Rightarrow \dfrac{e^{2x}+2e^0+e^{-2x}}{4} - \dfrac{(e^{2x}-2e^0+e^{-2x})}{4}$

$\Rightarrow \dfrac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}-\cancel{e^{2x}}+2-\cancel{e^{-2x}}}{4}$

$\Rightarrow \dfrac{4}{4} \Rightarrow 1$

2 $y = f(x) = 2x^4-4x^2+1$

(i) $f(0) = 1 \quad \Rightarrow (0,1)$

(ii) $f'(x) = 8x^3-8x$

$\Rightarrow 8x^3-8x=0$

$\Rightarrow x^3-x=0$

$\Rightarrow x(x^2-1)=0$

$\Rightarrow x=0 \ \Big| \ x^2-1=0$

$x^2=1$

$x=\pm 1$

$x=0$	$x=1$	$x=-1$
$y=1$	$y=-1$	$y=-1$

$\Rightarrow (0,1)\ (1,-1)\ (-1,-1)$ are critical points

) Classification: $f''(x) = 24x^2-8$

$f''(0) = -8 < 0 \Rightarrow$ max

$f''(1) = 16 > 0 \Rightarrow$ min.

$f''(-1) = 16 > 0 \Rightarrow$ min

$\Rightarrow (0,1)$ max turning pt. $(1,-1)\ (-1,-1)$ are min. turning points

(iii) $f''(x) = 24x^2-8 = 0$

$\Rightarrow 24x^2=8$

$\Rightarrow 3x^2=1$

$\Rightarrow x^2=\frac{1}{3}$

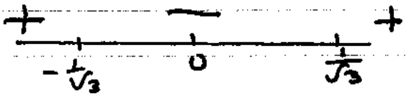
$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$

$f(\frac{1}{\sqrt{3}}) = 2(\frac{1}{\sqrt{3}})^2 - 4(\frac{1}{\sqrt{3}})^2+1 = \frac{2}{9}-\frac{4}{3}+1 = -\frac{1}{9}$.

$f(-\frac{1}{\sqrt{3}}) = -\frac{1}{9}$.

$\Rightarrow$) $(-\frac{1}{\sqrt{3}}, -\frac{1}{9})$ $(\frac{1}{\sqrt{3}}, -\frac{1}{9})$

(iii)



$\frac{1}{\sqrt{3}} \approx .6$

$f''(x) = 24x^2-8$

$f''(-1) = 16 > 0$

$f''(0) = -8 < 0$

$f''(1) = 16 > 0$

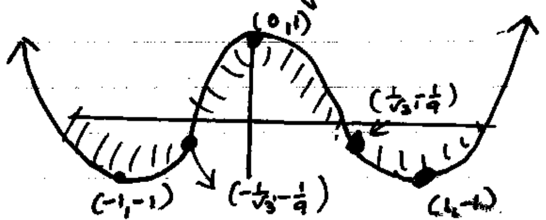
$\Rightarrow$ Concave up for $x < \frac{1}{\sqrt{3}}$, $x > \frac{1}{\sqrt{3}}$

concave down $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

(iv) $x \to +\infty, \Rightarrow y \to +\infty$

$x \to -\infty, \Rightarrow y \to +\infty$

(v)



Section B

3 (a) (i) $\int \sin x(4+\cos x)^3 dx$

let $u = 4+\cos x$

$\frac{du}{dx} = -\sin x$

$\Rightarrow du = (-\sin x)dx$

$\Rightarrow \frac{du}{-\sin x} = dx$

$\Rightarrow \int \cancel{\sin x} . u^3 . \frac{du}{-\cancel{\sin x}}$

$\Rightarrow -\int u^3 du$

$\Rightarrow -\frac{u^4}{4} + c$

$\Rightarrow -\frac{(4+\cos x)^4}{4} + C$

(ii) $\int_0^1 (2x+1)e^{x^2+x} dx$

let $u = x^2+x$

$\frac{du}{dx} = 2x+1$

$\Rightarrow du = (2x+1)dx$

$\Rightarrow \frac{du}{(2x+1)} = dx$

$\Rightarrow \int \cancel{(2x+1)} e^u \frac{du}{\cancel{(2x+1)}} \Rightarrow \int e^u du$

$\Rightarrow e^u$

$\Rightarrow e^{x^2+x}\Big|_{x=0}^{x=1}$

$\Rightarrow e^2 - e^0 = 6.389$

(iii) $\int x \cosh x \, dx$

let $u = x$ | $dv = \cosh x \, dx$

$\frac{du}{dx} = 1$ | $v = \int \cosh x \, dx$

$du = dx$ | $v = \sinh x$

$\Rightarrow \int u dv = uv - \int v du$

$\Rightarrow \int x \cosh x dx = x \sinh x - \int \sinh x dx$

$= x \sinh x - \cosh x + C$

3b

$a = 8e^{-2t}$

$v = \int 8e^{-2t} dt$

$v = 8\int e^{-2t} dt$

$v = 8\frac{e^{-2t}}{-2} + c$

$v = -4e^{-2t} + c$

$t = 0, \Rightarrow v = 0$

$\Rightarrow 0 = -4e^0 + c$

$\Rightarrow 0 = -4(1) + c$

$\Rightarrow 4 = c$

$\Rightarrow v = -4e^{-2t} + 4.$

4 (a)

$y = 5x - x^2$

$y = 2x$

limits : $2x = 5x - x^2$

$\Rightarrow x^2 - 5x + 2x = 0$

$\Rightarrow x^2 - 3x = 0$

$\Rightarrow x(x-3) = 0$

$x = 0$ | $x - 3 = 0$, $x = 3$

$A = \int_0^3 f(x) - g(x) dx$

$= \int_0^3 5x - x^2 - 2x \, dx$

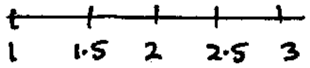
$= \int_0^3 3x - x^2 dx$

$\Rightarrow \frac{3x^2}{2} - \frac{x^3}{3}\Big|_{x=0}^{x=3}$

$\Rightarrow (\frac{27}{2} - \frac{27}{3}) - 0$

$\Rightarrow \frac{9}{2} = 4\frac{1}{2}.$

(b) $h = \frac{3-1}{4} = \frac{2}{4} = 0.5$



$y = \sqrt{\ln(x^2+1)}$

x =	1	1.5	2	2.5	3
y =	·832	1·085	1·268	1·407	1·517
	y_1	y_2	y_3	y_4	y_5

$A = \frac{.5}{3}\left[(.832+1.517)+4(1.085+1.407) + 2(1.268)\right]$

$= \frac{1}{6}\left[2.349 + 4(2.492) + 2(1.268)\right]$

$= \frac{1}{6}(14.853)$

$= 2.475$

Section C

5 (a) $\sum_{n=1}^{\infty} \frac{3}{(3n+1)(3n+4)}$

$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{3n+1} - \frac{1}{3n+4}\right)$

$\Rightarrow \frac{1}{4} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{10}} + \cancel{\frac{1}{10}} - \cancel{\frac{1}{13}} + - - - -$

$\Rightarrow \frac{1}{4}$

b (i) $\sum_{n=1}^{\infty} \frac{2n-1}{4n+3}$

$\Rightarrow \lim_{n\to\infty} \frac{2n-1}{4n+3} = \frac{\infty}{\infty}$

$\Rightarrow \lim_{n\to\infty} \left(\frac{2-\frac{1}{n}}{4+3/n}\right) = \frac{2-0}{4+0} = \frac{1}{2}$

$\Rightarrow \frac{1}{2} \neq 0 \Rightarrow$ Series is divergent.

(ii) $\sum_{n=1}^{\infty} \frac{n-2}{2n^4+3n}$ Compare with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$\Rightarrow \lim_{n\to\infty} \frac{\left(\frac{n-2}{2n^4+3n}\right)}{\left(\frac{1}{n^3}\right)}$

$\Rightarrow \lim_{n\to\infty} \frac{n^4-2n^3}{2n^4+3n} = \frac{\infty}{\infty}$

$\Rightarrow \lim_{n\to\infty} \left(\frac{1-\frac{2}{n}}{2+\frac{3}{n^3}}\right) = \frac{1-0}{2+0} = \frac{1}{2}$

as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is cgt.

$\Rightarrow \sum_{n=1}^{\infty} \frac{n-2}{2n^4+3n}$ is cgt.

(iii) $\sum_{n=1}^{\infty} \frac{n+1}{2^n}$

$a_n = \frac{n+1}{2^n}$

$a_{n+1} = \frac{n+2}{2^{n+1}}$

$\frac{a_{n+1}}{a_n} = \frac{n+2}{\underset{2}{\cancel{2^{n+1}}}} \cdot \frac{\cancel{2^n}}{n+1}$

$= \frac{n+2}{2n+2}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$\Rightarrow \lim_{n\to\infty} \left|\frac{n+2}{2n+2}\right|$

$\Rightarrow \lim_{n\to\infty} \left|\frac{1+\frac{2}{n}}{2+2/n}\right)$

$\Rightarrow \left|\frac{1+0}{2+0}\right| = \left|\frac{1}{2}\right|$

$\Rightarrow \frac{1}{2} < 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n+1}{2^n}$ is cgt.

6 (a) $f(x) = \sinh x$

$f(0) = \sinh 0 = 0$

$f'(x) = \cosh x$

$f'(0) = \cosh 0 = 1$

$f''(x) = \sinh x$

$f''(0) = \sinh 0 = 0$

$f'''(x) = \cosh x$

$f'''(0) = \cosh 0 = 1$

$f^{IV}(x) = \sinh x$

$f^{IV}(0) = \sinh 0 = 0$

$f^{V}(x) = \cosh x$

$f^{V}(0) = \cosh 0 = 1$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{IV}(0)x^4}{4!} + \frac{f^{V}(0)x^5}{5!}$$

$$\Rightarrow \sinh x = 0 + 1x + \cancel{\frac{0x^2}{2!}} + \frac{1x^3}{3!} + \cancel{\frac{0x^4}{4!}} + \frac{1x^5}{5!}$$

$$\Rightarrow \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!}$$

(i) $$\frac{d(\sinh x)}{dx} = 1 + \frac{3x^2}{3!} + \frac{5x^4}{5!}$$

$$\Rightarrow \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$$

(ii) $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sinh(\cdot 2) = \cdot 2 + \frac{(\cdot 2)^3}{3!} + \frac{(\cdot 2)^5}{5!}$$

$$= \cdot 2 + \cdot 00133 + \cdot 000\cancel{0026}$$

$$= \cdot 2013$$

(b)

(i) $Z = 3x^3y + 4xy^2$

$$\frac{\partial z}{\partial x} = 9x^2y + 4y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 18xy + 0$$

$$= 18xy$$

$$\frac{\partial^2 z}{\partial y \partial x} = 9x^2 + 8y$$

(ii) $z = e^{-2y}\cos 2x$

$$\frac{\partial z}{\partial x} = e^{-2y}(-\sin 2x \cdot 2)$$

$$\frac{\partial z}{\partial x} = -2e^{-2y}\sin 2x$$

$$\frac{\partial^2 z}{\partial x^2} = -2e^{-2y}\cos 2x \cdot 2$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = -4e^{-2y}\cos 2x}$$

$$z = e^{-2y}\cos 2x$$

$$\frac{\partial z}{\partial y} = e^{-2y}(-2)\cos 2x$$

$$= -2e^{-2y}\cos 2x$$

$$\frac{\partial^2 z}{\partial y^2} = $$

$$= -2e^{-2y}(-2)\cos 2x$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial y^2} = 4e^{-2y}\cos 2x}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow -4\cancel{e^{-2y}\cos 2x} + 4e^{-2y}\cos 2x$$

$$\Rightarrow 0$$

$\Rightarrow$ proved q.E.D.

Solutions 2009

Section D

Q7

a)
```
evalf(((5^3+sqrt(18)+42)/sqrt(3))^3,15);
```

b)
```
subs(x=2/3,2*x*exp(3*x));
```

c)
```
factor(x^3-2*x^2-10+4*x);
```

d)
```
plot(sin(x),x=-1..1);
```

or

```
y:=sin(x);
plot(y,x=-1..1);
```

e)
```
y:=((sin(x))^2)/(5+ln(x));
diff(y,x);
simplify(%);
```

or

```
diff(((sin(x))^2)/(5+ln(x)),x);
simplify(%);
```

or

```
simplify(diff(((sin(x))^2)/(5+ln(x)),x));
```

f)
```
y:=((sin(x))^2)/(5+ln(x));
y1:=(diff(%,x));
y2:=diff(%,x);
simplify(%);
```

or

```
y:=((sin(x))^2)/(5+ln(x));
diff(y,x$2);
simplify(%);
```

or

```
diff(((sin(x))^2)/(5+ln(x)),x$2);
simplify(%);
```

or

```
simplify(diff(((sin(x))^2)/(5+ln(x)),x$2));
```

g) `int(sin(x)*cosh(x),x=1..3);`

Q8 a)

(i) x-intercepts = (-5/2, 0) ; (2, 0)
y-intercept = (0, 20)

(ii) min [redacted] = (2,0)
max [redacted] = (-1, 27)

(iii) pt. of inflection = (0.5, 13.5)

(iv) As $x \to +\infty \quad f(x) \to +\infty$
As $x \to -\infty \quad f(x) \to 0$