Contents

0.1 Example

Find $h(t)$ when $h(t) = f * g(t)$, with $f(t) = t$ and $g(t) = t^2$.

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\begin{aligned}
f(t) = t \quad &\Leftrightarrow \quad F(S) = \frac{1}{S^2} \\
g(t) = t^2 \quad &\Leftrightarrow \quad G(S) = \frac{2}{S^3} \\
H(S) = F(S) &\times G(S) = \frac{2}{S^5} \\
(H(S) \text{ is in} &\text{ form } k\frac{n!}{S^{n+1}})
\end{aligned}
$$

With $n = 4$, $n! = 4! = 24$. Solving for k, $k \times n! = 2$. Therefore $k = \frac{1}{12}$. The solution is $\mathcal{L}^{-\infty}[\mathcal{H}(\mathcal{S})]$

1 Period of a trigonomteric function

Period of a function is denoted $2l$. (Sometimes it is denoted as L, with $L = 2l$).

When given a trigonometric function in form $f(t) = Cos(kx)$ or $f(t) = Sin(kx)$, the period of the function can be calculated as follows:

$$2l = \frac{2\pi}{k}$$

1.1 Example

$$f(t) = Cos(\frac{2\pi x}{3})$$
$$2l = \frac{2\pi}{(\frac{2\pi}{3})} = \frac{1}{(\frac{1}{3})} = \mathbf{3}$$

1.2 Example

$$f(t) = Sin(\frac{5x}{2})$$
$$2l = \frac{2\pi}{(\frac{5}{2})} = \frac{4\pi}{5}$$

2 Curl

In vector calculus, the curl is a vector operator that describes the infinitesimal rotation of a 3-dimensional vector field. At every point in the field, the curl of that field is represented by a vector. The attributes of this vector (length and direction) characterize the rotation at that point.

3 Laplacian Operator

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$$

4 Notation

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

5 Example

$$f(x,y,z) = \ 2x + 3y^2 - \sin(z)$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = 2\mathbf{i} + 6y\mathbf{j} - \cos(z)\mathbf{k}$$

6 Stoke's Theorem

$$\iint_{\Sigma} \nabla \times \mathbf{F} \cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r},$$

7 Laplace Transforms

If $g(t) = k \times f(t)$ then $G(S) = k \times F(S)$ where k is a constant. $\{(\sqcup) = F(S)$.

$$\begin{aligned} f(t) &= (t+1)^2 \qquad (1) \\ &= t^2 + 2t + 1 \end{aligned}$$

8 Inverse Laplace Transforms 2

The denominator has form $S^2 - 2aS + a^2 + k$ which is equivalent to $(S-a)^2 + k$. Therefore $G(S)$ will have form $F(S-a)$

The function $G(S)$ may have the form $\frac{S+D}{S^2+(C+D)S+CD}$, where C and D are constants. This expression simplifies $\frac{S+D}{(S+C)(S+D)}$ and again to $\frac{1}{S+C}$. The inverse laplace transform $g(t)$ can be easily determined.

9 Convolution

We are asked to find a function h(t) which is the convolution of two given functions $f(t)$ and $g(t)$. i.e $h(t) = h * g(t)$.

Importantly $H(S) = F(S) \times G(S)$. We determine the laplace transforms, F(S) and G(S), and multiply them to determine H(S). We then find the inverse Laplace transform of H(S) to yield our solution.

9.1 Example

Find $h(t)$ when $h(t) = f * g(t)$, with $f(t) = e^t$ and $g(t) = e^{-t}$.

$$f(t) = e^t \quad \Leftrightarrow \quad F(S) = \frac{1}{S-1}$$
$$g(t) = e^{-t} \quad \Leftrightarrow \quad G(S) = \frac{1}{S+1}$$
$$H(S) = F(S) \times G(S) = \frac{1}{(S-1)(S+1)}$$

10 Inverse Laplace Transforms (2 questions)

Partial fraction expansion is used in questions 4 and 5.

11 Even and Odd Function

Even Functions: $Cos(X)$,$|X|$ (i.e absolute value of X) and X^2, X^4 etc

Odd Functions: $Sin(X)$, X, X^3 etc

Functions that are products of two even functions are also **even** functions.

Functions that are products of two odd functions are **even** functions. (e.g $X \times X^3 = X^4$)

Functions that are products of an even function and an odd function are **odd** functions.

12 Fourier Series - determining the arguments

Given a period $2l$, we must determine the form of the fourier series. $sin(\frac{nx\pi}{l})$

13 Fourier Series

X

MA4005 Syllabus

- Functions of several variables and partial differentiation.
- The indefinite integral. Integration techniques: of standard functions, by substitution, by parts and using partial fractions.
- The definite integral. Finding areas, lengths, surface areas, volumes, and moments of inertial.
- Numerical integration: trapezoidal rule, Simpson's rule.
- Ordinary differential equations.

- First order including linear and separable. Linear second order equations with constant coefficients.
- Numerical solution by Runge-Kutta. The Laplace transform: tables and theorems and solution of linear ODEs.
- Fourier series: functions of arbitary period, even and odd functions, half-range expansions.
- Application of Fourier series to solving ODEs.
- Matrix representation of and solution of systems of linear equations.
- Matrix algebra: invertibility, determinants.
- Vector spaces: linear independence, spanning, bases, row and column spaces, rank.
- Inner products: norms, orthogonanality. Eigenvalues and eignenvectors.
- Numerical solution of systems of linear equations. Gauss elimination, LU decomposition, Cholesky decomposition, iterative methods. Extension to non-linear systems using Newton's method.

13.1 convolution

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = F(s) \cdot G(s)$$

14 Laplace Transforms Using 1st Shifting Theorem

$$g(t) = e^{at}f(t) \quad \Leftrightarrow \quad G(S) = F(S-a)$$

The function $g(t)$ is presented in a form whereby a and $f(t)$ are easily identifiable. First determine $F(S)$ by finding the Laplace transform of $f(t)$. Then replace all S terms with $S - a$.

15 Laplace Transforms Using 2nd Shifting Theorem

$$g(t) = u^a f(t-a) \quad \Leftrightarrow \quad G(S) = e^{-aS} F(S)$$

The function $g(t)$ is presented in a form whereby a and $f(t-a)$ are easily identifiable. ($U_a(t)$ is called the unit step function). First determine $f(t)$ by replace all $t-a$ terms in $f(t-a)$ with t. Then calculate the laplace transform of $f(t)$ i.e. $F(S)$. The solutions is in form $G(S) = e^{-aS}F(S)$.

16 Inverse Laplace Transforms 2

The denominator has form $S^2 - 2aS + a^2 + k$ which is equivalent to $(S-a)^2 + k$. Therefore $G(S)$ will have form $F(S-a)$

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G(S), and multiply them to determine H(S). We then find the inverse Laplace transform of H(S) to yield our solution.

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$$\begin{aligned} f(t) = e^t \quad &\Leftrightarrow \quad F(S) = \frac{1}{S-1} \\ g(t) = e^{-t} \quad &\Leftrightarrow \quad G(S) = \frac{1}{S+1} \\ H(S) = F(S) \times G(S) &= \frac{1}{(S-1)(S+1)} \end{aligned}$$

17.2 Example

Find $h(t)$ when $h(t) = f * g(t)$, with $f(t) = t$ and $g(t) = t^2$.

$$\begin{aligned} f(t) = t \quad &\Leftrightarrow \quad F(S) = \frac{1}{S^2} \\ g(t) = t^2 \quad &\Leftrightarrow \quad G(S) = \frac{2}{S^3} \\ H(S) = F(S) \times G(S) &= \frac{2}{S^5} \\ (H(S) \text{ is in form } k&\frac{n!}{S^{n+1}}) \end{aligned}$$

With $n = 4$, $n! = 4! = 24$. Solving for k, $k \times n! = 2$. Therefore $k = \frac{1}{12}$. The solution is $\mathcal{L}^{-\infty}[\mathcal{H}(\mathcal{S})]$

18 Fourier Series

X

19 Laplacian Operator

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$$

Laplacian Analysis: convolution

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau = F(s) \cdot G(s)$$

20 Notation

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

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$$f(x, y, z) = \ 2x + 3y^2 - \sin(z)$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = 2\mathbf{i} + 6y\mathbf{j} - \cos(z)\mathbf{k}$$

22 Laplace Transforms

If $g(t) = k \times f(t)$ then $G(S) = k \times F(S)$ where k is a constant. $\{(\sqcup) = F(S)$.

$$\begin{aligned} f(t) &= (t+1)^2 \qquad (2) \\ &= t^2 + 2t + 1 \end{aligned}$$

23 Laplace Transforms Using 2nd Shifting Theorem

$$g(t) = u^a f(t-a) \quad \Leftrightarrow \quad G(S) = e^{-aS}F(S)$$

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H(S) = F(S) \times G(S) &= \frac{1}{(S-1)(S+1)}
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25.2 Example

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Laplace Transforms

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$\begin{aligned}
\mathcal{L}[4] &= \int_0^\infty t^2 e^{-st}dt && (3)\\
&= 4\int_0^\infty e^{-st}dt && (4)\\
&= 4\left[\frac{e^{-st}}{-s}\right]_0^\infty && (5)\\
&= 4\left[\left(\frac{e^{-\infty}}{-s}\right) - \left(\frac{e^{-0}}{-s}\right)\right] && (6)\\
&= \frac{4}{s} && (7)
\end{aligned}$$

Fourier Series

$$f(x) = \frac{a_o}{2} + sum_1^\infty \left(a_n cos(nx) + b_n sin(nx)\right)$$

$$a_0 = \int_{-\pi}^{\pi} f(x)dx$$

Fourier Series

$$f(x) = \frac{a_o}{2} + sum_1^\infty \left(a_n cos(nx) + b_n sin(nx)\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi sin(nx)dx + \int_0^\pi \pi sin(nx)dx\right]$$

$$b_n = \frac{1}{\pi}\left(\left[\frac{pi}{n}cos(nx)\right]_{-\pi}^{0} - \left[\frac{pi}{n}cos(nx)\right]_0^{\pi}\right)$$

$$b_n = \frac{\pi}{n\pi}\left(cos(0) - cos(-n\pi) - cos(n\pi) + cos(0)\right)$$

$$b_n = \frac{\pi}{n\pi}\left(2 - 2cos(n\pi)\right)$$

Heaviside function

$u_1(t)$

$$[U_a(t) - U_b(t)] \times f(t)$$

Special Cases:

- $U_0(t) = 1$
- $U_\infty = 0$

26 Notation

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

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28 Laplace Transforms

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$$\begin{aligned} f(t) &= (t+1)^2 \\ &= t^2 + 2t + 1 \end{aligned} \tag{8}$$

Numerical Methods: Syllabus

- Numerical Differentiation and Integration Approximation formulae for derivatives. Trapezoidal rule, Simpsons rule, Use of error estimates.
- Numerical Linear Algebra Linear least squares approximation. The above algorithms will be used to solve problems in mathematics and science using the Matlab and Derive computer packages.
- Solving Systems of Linear Equations Gaussian and Gauss/Jordan elimination, error accumulation, introduction to iterative techniques (Jacobi method). LU decomposition.
- Solution of Non-Linear Equations Bracketing methods, linear interpolation technique, fixed point iteration, the Newton-Raphson method. Error analysis of iterative methods.
- Mathematical Preliminaries Computer representation of numbers, types of computational error. Condition and stability of numerical algorithms.
- Interpolation Piecewise-linear interpolation and Lagrange interpolating polynomial.

The Secant method

Use the secant method to evaluate a root for each of the equations on Sheet 1, subject to the required accuracy restrictions. Compare the secant method with the previous methods in each case.

Conservative Vector Fields

A vector field A is called conservative if any of the following equivalent conditions holds

- The line integral of A between two points is independent of the path
- The line integral of A over any closed curve C is equal to zero, that is

The exists a scalar field, called the potential, such that

The divergence theorem

Stokes Theorem

The line integral of a vector A taken around a simple closed curve (that is, a non-intersecting closed curve), C, is equal to the surface integral of the curl of A taken over any surface S having C as a boundary.