2006

MA4702 Solutions

Section A

1 (a) (i) $f(x) = \sqrt{8-2x}$

$f(4-2x^2) = \sqrt{8-2(4-2x^2)}$

$= \sqrt{8-8+4x^2}$

$= \sqrt{4x^2}$

$= 2x$

(ii) Domain : $8-2x \geq 0$

$-2x \geq -8$

$x \leq 4$

$\Rightarrow$ ~~[illegible]~~ $(-\infty, 4]$

Range : $[0, \infty)$

(iii) $g: y = \log_e(2x+1)$

$\Rightarrow$ ~~[illegible]~~

$g^{-1}: \quad x = \log_e(2y+1)$

$\Rightarrow e^x = 2y+1$

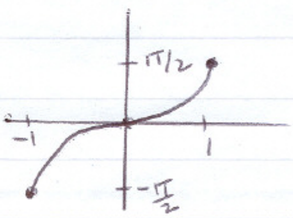
$e^x - 1 = 2y$

$\frac{e^x-1}{2} = y$

$\Rightarrow g^{-1}(x) = \frac{e^x-1}{2}$

(b) (i) $\sin^{-1}(-\frac{1}{4}) \Rightarrow -14.47^\circ$ $(-0.2526$ Rad$)$

(ii)



(c)

$\cosh^2 x$

$\Rightarrow \left(\frac{e^x+e^{-x}}{2}\right)^2$

$\Rightarrow \frac{(e^x+e^{-x})(e^x+e^{-x})}{4}$

$\Rightarrow \frac{e^{2x}+e^0+e^0+e^{-2x}}{4}$

$\Rightarrow \frac{e^{2x}+2e^0+e^{-2x}}{4}$

$\Rightarrow \frac{e^{2x}+2+e^{-2x}}{4}$

$\frac{1}{2}(1+\cosh 2x)$

$\frac{1}{2}\left(1+\frac{e^{2x}+e^{-2x}}{2}\right)$

$\Rightarrow \frac{1}{2}\left(\frac{2+e^{2x}+e^{-2x}}{2}\right)$

$\Rightarrow \frac{e^{2x}+2+e^{-2x}}{4}$

Proved.

2 $\quad y = f(x) = \frac{1}{x-4}$

(i) $x=0 \Rightarrow y = -\frac{1}{4} \Rightarrow (0, -\frac{1}{4})$

(ii) $y = \frac{1}{x-4}$

$\frac{dy}{dx} = \frac{(x-4).0 - 1(1)}{(x-4)^2}$

$= \frac{-1}{(x-4)^2}$

for Turning points :

$-\frac{1}{(x-4)^2} = 0$

$\Rightarrow$ not true

$\Rightarrow$ no solution

$\Rightarrow$ no turning points

(iii) $\frac{dy}{dx} = -\frac{1}{(x-4)^2} < 0$ for all x $(x \neq 4)$

$\Rightarrow$ decreasing function

(iv) $y = \frac{1}{x-4}$

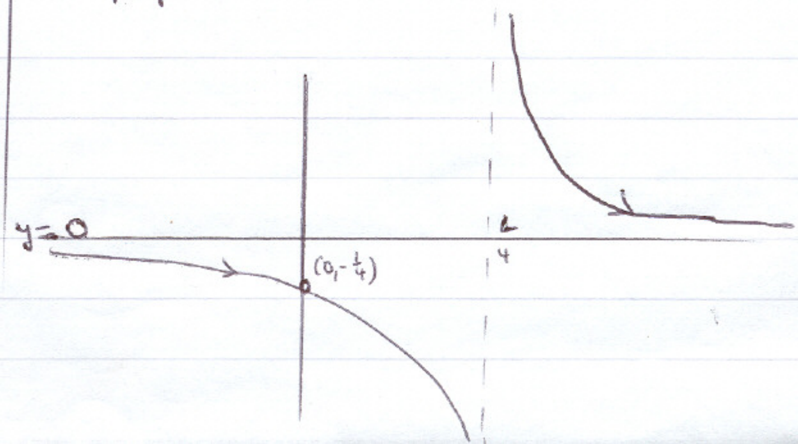
$\Rightarrow x-4=0$

$\Rightarrow x = 4$ $\quad \Rightarrow$ line $x=4$ is vertical asymptote

(v) $\lim_{x \to \infty}\left(\frac{1}{x-4}\right) = \frac{1}{\infty} = 0$

$\Rightarrow$ line $y=0$ is horizontal asymptote

(vi) graph.



Section B

3 (a) (i) $\int_0^{\pi/2} \cos x(2+\sin x)^2 dx$

let $u = 2+\sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

$\frac{du}{\cos x} = dx$

$\int \cos x \cdot u^2 \cdot \frac{du}{\cos x}$

$\Rightarrow \int u^2 du$

$\Rightarrow \frac{u^3}{3} \Rightarrow \left.\frac{(2+\sin x)^3}{3}\right|_{x=0}^{x=\pi/2}$

$\Rightarrow \frac{(2+\sin\frac{\pi}{2})^3}{3} - \frac{(2+\sin 0)^3}{3}$

$\Rightarrow \frac{27}{3} - \frac{8}{3} \Rightarrow \boxed{\frac{19}{3}}$

(ii) $\int_1^4 \frac{2x+1}{x^2+x+1} dx$

let $u = x^2+x+1$

$\frac{du}{dx} = 2x+1$

$\Rightarrow du = (2x+1)dx \Rightarrow \frac{du}{2x+1} = dx$

$\Rightarrow \int \frac{(2x+1)}{u} \cdot \frac{du}{(2x+1)}$

$\Rightarrow \int \frac{1}{u} du$

$\Rightarrow \ln u \Rightarrow \left.\ln(x^2+x+1)\right|_{x=1}^{x=4}$

$\Rightarrow \ln 21 - \ln 3$

$\Rightarrow \boxed{\ln\left(\frac{21}{3}\right) \Rightarrow \ln 7}$

(iii) $\int x \ln x\, dx$

$u = \ln x$	$dv = x\, dx$
$du = \frac{1}{x} dx$	$v = \int x\, dx \Rightarrow \frac{x^2}{2}$

$\Rightarrow \int x\ln x\, dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$\Rightarrow \int x\ln x\, dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx$

$= \frac{x^2\ln x}{2} - \frac{1}{2}\int x\, dx$

$= \frac{x^2\ln x}{2} - \frac{1}{2}\cdot\frac{x^2}{2} + C$

$\Rightarrow \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$

(b) $i(t) = 3 + 2\sin 4t$

$q(t) = \int 3 + 2\sin 4t$

$\Rightarrow q(t) = 3t + 2\left(\frac{-\cos 4t}{4}\right) + C$

$q = 0$ when $t = 0$

$\Rightarrow 0 = 0 + 2\left(\frac{-\cos 0}{4}\right) + C$

$\Rightarrow 0 = -\frac{2(1)}{4} + C$

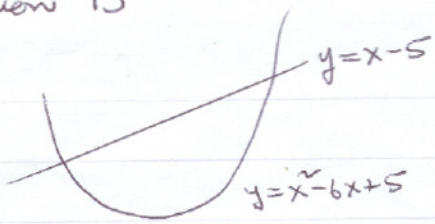
$\Rightarrow \frac{2}{4} = C$

$\Rightarrow C = \frac{1}{2}$

$\Rightarrow q(t) = 3t - \frac{\cos 4t}{2} + \frac{1}{2}$

Section B

4 (a)



Solve $y=x^2-6x+5$

$\underline{y=x-5}$

$\Rightarrow x^2-6x+5 = x-5$

$\Rightarrow x^2-7x+10=0$

$\Rightarrow (x-5)(x-2)=0$

$\Rightarrow x=5,\ x=2$

$$\Rightarrow A = \int_2^5 f(x)-g(x)\,dx$$

$$= \int_2^5 x-5-(x^2-6x+5)\,dx$$

$$= \int_2^5 x-5-x^2+6x-5\ dx$$

$$= \int -x^2+7x-10\ dx$$

$$\Rightarrow \frac{-x^3}{3}+\frac{7x^2}{2}-10x \Bigg|_{x=2}^{x=5}$$

$$\Rightarrow \left(-\frac{125}{3}+\frac{175}{2}-50\right)-\left(-\frac{8}{3}+14-20\right)$$

$$\Rightarrow \frac{-25}{6}+\frac{26}{3}$$

$$\Rightarrow \boxed{\frac{9}{2}}$$

4(B) $h = \frac{3-1}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$

1 1.5 2 2.5 3

$y = \sqrt{1+\ln x}$

$x=$ 1	1.5	2	2.5	3
$y=$ 1	1.185	1.301	1.384	1.448

Simpsons Rule Page 42

$\Rightarrow$ Approx

$$= \frac{.5}{3}\left[(1+1.448)+4(1.185+1.384)+2(1.301)\right]$$

$\Rightarrow$ 2.554.

Section C

5 (a)(i) $S_\infty = \frac{a}{1-r}$

$= \frac{1}{(1-\frac{2}{3})}$

$= \frac{1}{(\frac{1}{3})} \Rightarrow \boxed{3}$

(ii) $a_n = \frac{2}{(2n-1)(2n+1)}$

$\Rightarrow a_n = \frac{1}{2n-1} - \frac{1}{2n+1}$

$a_1 = 1 - \cancel{\frac{1}{3}}$

$a_2 = \cancel{\frac{1}{3}} - \frac{1}{5}$

$a_3 = \cancel{\frac{1}{5}} - \frac{1}{7}$

$\vdots$

$a_n = \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}$

$S_n = 1 - \frac{1}{2n+1}$

$\Rightarrow \lim_{n\to\infty} S_n = 1 - \frac{1}{\infty}$

$\Rightarrow 1 - 0$

$\Rightarrow \cancel{S_\infty} =$

$\Rightarrow \boxed{S_\infty = 1}$

(b) $\cancel{\sum \frac{2n-1}{2n+1}}$ $\sum_{n=1}^{\infty} \frac{2n-1}{n+2}$

(i)

$\lim_{n\to\infty} \frac{2n-1}{n+2}$ divide by n

$\Rightarrow \lim_{n\to\infty} \left(\frac{2-\frac{1}{n}}{1+\frac{2}{n}}\right)$

$\Rightarrow \frac{2-0}{1+0}$

$\Rightarrow 2$

$2 \neq 0$

$\Rightarrow$ by necessary condition test Series is divergent

(ii) $\sum_{n=1}^{\infty} \frac{n-2}{2n^2+3n+4}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

$\Rightarrow \frac{\frac{n-2}{(2n^2+3n+4)}}{\frac{1}{n}}$

$\Rightarrow \frac{(n-2)}{(2n^2+3n+4)} \cdot \frac{n}{1}$

$\Rightarrow \frac{n^2-2n}{2n^2+3n+4}$ divide by n^2

$\Rightarrow \frac{1 - \frac{2}{n}}{2+\frac{3}{n}+\frac{4}{n^2}}$

$\Rightarrow \lim_{n\to\infty} \frac{1-\frac{2}{n}}{2+\frac{3}{n}+\frac{4}{n^2}} = \frac{1-0}{2+0+0}$

$= \frac{1}{2}$

as $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{n-2}{2n^2+3n+4}$ is divergent

(iii) $\sum_{n=1}^{\infty} \frac{x^n}{n.2^n}$

$a_n = \frac{x^n}{n.2^n}$, $a_{n+1} = \frac{x^{n+1}}{(n+1).2^{n+1}}$

$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1).2^{n+1}} \cdot \frac{n.2^n}{x^n}$

$= \frac{xn}{2(n+1)}$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{nx}{2n+2}\right|$ divide by n

$= \lim_{n\to\infty} \left|\frac{x}{2+\frac{2}{n}}\right|$

$\Rightarrow \left|\frac{x}{2+0}\right| = \left|\frac{x}{2}\right|$

if Convergent $\left|\frac{x}{2}\right| < 1$

$\Rightarrow \boxed{|x| < 2}$

Section c

6 (a) $f(x) = e^{2x}$

$\boxed{f(0) = e^0 = 1}$

$f'(x) = 2e^{2x}$

$\boxed{f'(0) = 2}$

$f''(x) = 2e^{2x}.2 \Rightarrow 4e^{2x}$

$\boxed{f''(0) = 4}$

$f'''(x) = 4e^{2x}.2 \Rightarrow 8e^{2x}$

$\boxed{f'''(0) = 8}$

Maclaurin series

$f(x) = f(0) + f'(0).x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$

$\Rightarrow \boxed{e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!}}$

$e^{0.2} \Rightarrow x = 0.1$

$\Rightarrow e^{0.2} = 1 + 2(.1) + \frac{4(.1)^2}{2!} + \frac{8(.1)^3}{3!}$

$= 1 + .2 + .02 + .0013$

$= 1.22$

6(b)

$z = x^3 + 2x^2y^2 + y^3.$

~~δz/δ~~ $\frac{\partial z}{\partial x} = 3x^2 + 4xy^2$

(i) $\frac{\partial^2 z}{\partial x^2} = 6x + 4y^2$

$\frac{\partial^2 z}{\partial y \partial x} = 8xy$

(ii) $z = \sin(x+3t)$

$\frac{\partial z}{\partial x} = \cos(x+3t).1$

$= \cos(x+3t)$

$\frac{\partial^2 z}{\partial x^2} = -\sin(x+3t).1$

$\boxed{\frac{\partial^2 z}{\partial x^2} = -\sin(x+3t)}$

$\frac{\partial z}{\partial t} = \cos(x+3t).3$

$\frac{\partial^2 z}{\partial t^2} = -\sin(x+3t).3.3$

$\boxed{\frac{\partial^2 z}{\partial t^2} = -9\sin(x+3t)}$

$\frac{\partial^2 z}{\partial t^2} - 9\frac{\partial^2 z}{\partial x^2}$

$\Rightarrow -9\sin(x+3t) - 9.(-\sin(x+3t))$

$= -9\sin(x+3t) + 9\sin(x+3t)$

$\Rightarrow 0$

$\Rightarrow$ $z = \sin(x+3t)$ satisfies

eq. $\frac{\partial^2 z}{\partial t^2} - 9\frac{\partial^2 z}{\partial x^2} = 0.$

Section D

7 (a) `evalf(((Pi+2^6)/sqrt(7))^21, 18);`

(b) `subs(x=(1+sqrt(12))/sqrt(11), 3*x^2+2*x);`

(c) `factor(x^3-3*x^2+2);`

(d) `plot(arcsin(x), x=-1..1);`

(e) `diff(sqrt((x-2)/x), x);`

(f) `diff(sqrt((x-2)/x), x $2);`

(g) `int(3*ln(x)/((sin(x))^2+2), x=0..1);`

8 (a) (i) x intercepts (-1, 0) (1, 0) (3, 0)
y intercept (0, 3)

(ii) local max turning point (-0.154700539, 3.079201436)
min turning point (2.154700539, -3.079201439)

(iii) inflection point (1, 0)

(iv) as $x \to -\infty$, $y \to -\infty$
as $x \to +\infty$, $y \to +\infty$

(b) graph: