MA4005 Syllabus

- Functions of several variables and partial differentiation.
- The indefinite integral. Integration techniques: of standard functions, by substitution, by parts and using partial fractions.
- The definite integral. Finding areas, lengths, surface areas, volumes, and moments of inertial.
- Numerical integration: trapezoidal rule, Simpson's rule.
- Ordinary differential equations.
- First order including linear and separable. Linear second order equations with constant coefficients.
- Numerical solution by Runge-Kutta.
- The Laplace transform: tables and theorems and solution of linear ODEs.
- Fourier series: functions of arbitary period, even and odd functions, half-range expansions.
- Application of Fourier series to solving ODEs.
- Matrix representation of and solution of systems of linear equations.
- Matrix algebra: invertibility, determinants.
- Vector spaces: linear independence, spanning, bases, row and column spaces, rank.
- Inner products: norms, orthogonanality. Eigenvalues and eignenvectors.

- Numerical solution of systems of linear equations. Gauss elimination, LU decomposition, Cholesky decomposition, iterative methods. Extension to non-linear systems using Newton's method.

Trigonometric Subsititution

$$\int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$x = a\sin(\theta), \quad dx = a\cos(\theta)\,d\theta, \quad \theta = \arcsin\left(\frac{x}{a}\right)$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2 - a^2\sin^2(\theta)}} && (1)\\
&= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2(1 - \sin^2(\theta))}}\\
&= \int \frac{a\cos(\theta)\,d\theta}{\sqrt{a^2\cos^2(\theta)}}\\
&= \int d\theta = \theta + C && (2)\\
&= \arcsin\left(\tfrac{x}{a}\right) + C
\end{aligned}$$

Augmented Matrices

Given the matrices A and B, where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 5 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix},$$

the augmented matrix $(A|B)$ is written as

$$(A|B) = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 5 & 2 & 2 & 1 \end{array}\right].$$

This is useful when solving systems of linear equations.

$$x + 2y + 3z = 0$$

$$3x + 4y + 7z = 2$$

$$6x + 5y + 9z = 11$$

the coefficients and constant terms give the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \\ 6 & 5 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ 11 \end{bmatrix},$$

and hence give the augmented matrix

$$(A|B) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 4 & 7 & 2 \\ 6 & 5 & 9 & 11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Reduced Row Echelon Form

Specifically, a matrix is in row echelon form if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix).
- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

- All entries in a column below a leading entry are zeroes (implied by the first two criteria).[1]

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 1 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \end{bmatrix}$$

Newton-Raphson Method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Numerical Integration

Numerical integration constitutes a broad family of algorithms for calculating the numerical value of a definite integral, and by extension, the term is also sometimes used to describe the numerical solution of differential equations.

Numerical Integration: Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx (b-a)\frac{f(a)+f(b)}{2}$$

Laplacian Analysis: convolution

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = F(s)\cdot G(s)$$

0.1 ODEs: Integrating factor

The integrating factor is a function that is chosen to facilitate the solving of a given equation involving differentials. It is commonly used to solve ordinary differential equations.

$$y' + P(x)y = Q(x)$$

the integration factor is

$$M(x) = e^{\int P(x')dx'}$$

ODEs: Example

Solve the differential equation

$$y' - \frac{2y}{x} = 0.$$

We can see that in this case

$$P(x) = \frac{-2}{x}$$

$$M(x) = e^{\int P(x)\, dx}$$

$$M(x) = e^{\int \frac{-2}{x}\, dx} = e^{-2\ln x} = \left(e^{\ln x}\right)^{-2} = x^{-2}$$

(Note we do not need to include the integrating constant - we need only a solution, not the general solution)

$$M(x) = \frac{1}{x^2}.$$

Multiplying both sides by

$$M(x)$$

we obtain

$$\frac{y'}{x^2} - \frac{2y}{x^3} = 0$$

$$\frac{y'x^3 - 2x^2y}{x^5} = 0$$

$$\frac{x(y'x^2 - 2xy)}{x^5} = 0$$

$$\frac{y'x^2 - 2xy}{x^4} = 0.$$

0.2 Partial Derivatives: Volume of a Cone

The volume "V" of a cone depends on the cone's height "h" and its radius 'r' according to the formula

$$V(r, h) = \frac{\pi r^2 h}{3}.$$

The partial derivative of "V" with respect to 'r' is

$$\frac{\partial V}{\partial r} = \frac{2\pi r h}{3},$$

which represents the rate with which a cone's volume changes if its radius is varied and its height is kept constant. The partial derivative with respect to "h" is

$$\frac{\partial V}{\partial h} = \frac{\pi r^2}{3},$$

which represents the rate with which the volume changes if its height is varied and its radius is kept constant.

1 Fundamental Theorem of Calculus

The fundamental theorem of calculus states that the integral of a function f over the interval $[a, b]$ can be calculated by finding an antiderivative F of f:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

2 Curl

In vector calculus, the curl is a vector operator that describes the infinitesimal rotation of a 3-dimensional vector field. At every point in the field, the curl of that field is represented by a vector. The attributes of this vector (length and direction) characterize the rotation at that point.

3 Laplacian Operator

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$$

4 Notation

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

5 Example

$$f(x, y, z) = \ 2x + 3y^2 - \sin(z)$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = 2\mathbf{i} + 6y\mathbf{j} - \cos(z)\mathbf{k}$$

6 Stoke's Theorem

$$\iint_{\Sigma} \nabla \times \mathbf{F} \cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r},$$

7 Laplace Transforms

If $g(t) = k \times f(t)$ then $G(S) = k \times F(S)$ where k is a constant. $\{(\sqcup) = F(S)$.

$$\begin{aligned} f(t) &= (t+1)^2 \qquad (3) \\ &= t^2 + 2t + 1 \end{aligned}$$

8 Laplace Transforms Using 1st Shifting Theorem

$$g(t) = e^{at} f(t) \quad \Leftrightarrow \quad G(S) = F(S-a)$$

The function $g(t)$ is presented in a form whereby a and $f(t)$ are easily identifiable. First determine $F(S)$ by finding the Laplace transform of $f(t)$. Then replace all S terms with $S - a$.

9 Laplace Transforms Using 2nd Shifting Theorem

$$g(t) = u^a f(t-a) \quad \Leftrightarrow \quad G(S) = e^{-aS} F(S)$$

The function $g(t)$ is presented in a form whereby a and $f(t-a)$ are easily identifiable. ($U_a(t)$ is called the unit step function). First determine $f(t)$ by replace all $t - a$ terms in $f(t-a)$ with t. Then calculate the laplace transform of $f(t)$ i.e. $F(S)$. The solutions is in form $G(S) = e^{-aS} F(S)$.

10 Inverse Laplace Transforms (2 questions)

Partial fraction expansion is used in questions 4 and 5.

11 Inverse Laplace Transforms 2

The denominator has form $S^2 - 2aS + a^2 + k$ which is equivalent to $(S-a)^2 + k$. Therefore $G(S)$ will have form $F(S-a)$

The function $G(S)$ may have the form $\frac{S+D}{S^2+(C+D)S+CD}$, where C and D are constants. This expression simplifies $\frac{S+D}{(S+C)(S+D)}$ and again to $\frac{1}{S+C}$. The inverse laplace transform $g(t)$ can be easily determined.

12 Convolution

We are asked to find a function h(t) which is the convolution of two given functions $f(t)$ and $g(t)$. i.e $h(t) = h * g(t)$.

Importantly $H(S) = F(S) \times G(S)$. We determine the laplace transforms, F(S) and G(S), and multiply them to determine H(S). We then find the inverse Laplace transform of H(S) to yield our solution.

12.1 Example

Find $h(t)$ when $h(t) = f * g(t)$, with $f(t) = e^t$ and $g(t) = e^{-t}$.

$$
\begin{aligned}
f(t) = e^t \quad &\Leftrightarrow \quad F(S) = \frac{1}{S-1} \\
g(t) = e^{-t} \quad &\Leftrightarrow \quad G(S) = \frac{1}{S+1} \\
H(S) = F(S) \times G(S) &= \frac{1}{(S-1)(S+1)}
\end{aligned}
$$

12.2 Example

Find $h(t)$ when $h(t) = f * g(t)$, with $f(t) = t$ and $g(t) = t^2$.

$$\begin{aligned} f(t) = t \quad &\Leftrightarrow \quad F(S) = \frac{1}{S^2} \\ g(t) = t^2 \quad &\Leftrightarrow \quad G(S) = \frac{2}{S^3} \\ H(S) = F(S) \times G(S) &= \frac{2}{S^5} \\ (H(S) \text{ is in form } k&\frac{n!}{S^{n+1}}) \end{aligned}$$

With $n = 4$, $n! = 4! = 24$. Solving for k, $k \times n! = 2$. Therefore $k = \frac{1}{12}$. The solution is $\mathcal{L}^{-\infty}[\mathcal{H}(\mathcal{S})]$

13 Period of a trigonomteric function

Period of a function is denoted $2l$. (Sometimes it is denoted as L, with $L = 2l$).

When given a trigonometric function in form $f(t) = Cos(kx)$ or $f(t) = Sin(kx)$, the period of the function can be calculated as follows:

$$2l = \frac{2\pi}{k}$$

13.1 Example

$$f(t) = Cos(\frac{2\pi x}{3})$$

$$2l \quad = \frac{2\pi}{(\frac{2\pi}{3})} \quad = \frac{1}{(\frac{1}{3})} \quad = \mathbf{3}$$

13.2 Example

$$f(t) = Sin(\frac{5x}{2})$$
$$2l \quad = \frac{2\pi}{(\frac{5}{2})} \quad = \frac{4\pi}{5}$$

14 Even and Odd Function

Even Functions: $Cos(X)$,$|X|$ (i.e absolute value of X) and X^2, X^4 etc

Odd Functions: $Sin(X)$, X, X^3 etc

Functions that are products of two even functions are also **even** functions.

Functions that are products of two odd functions are **even** functions. (e.g $X \times X^3 = X^4$)

Functions that are products of an even function and an odd function are **odd** functions.

15 Fourier Series - determining the arguments

Given a period $2l$, we must determine the form of the fourier series. $sin(\frac{nx\pi}{l})$

16 Fourier Series

X