Tutorial Sheet 1

1. Evaluate the terms $a_0, a_1, a_2, a_3, a_{10}, a_{100}$ in the following sequences, where possible.

   i) $a_n = 3n^2 + 2n + 1$

   ii) $a_n = \frac{n^2+7}{2n+3}$

   iii) $a_n = (-1)^n n$

   iv) $a_n = 3^n + (-3)^n + \frac{6n^4}{3n^4},\ n \neq 0$

   v) $a_n = 2^n.3^n$

   vi) $a_n = 6^n$

   vii) $a_n = \frac{1}{n+1} + \frac{1}{n+2}$

   viii) $a_n = \frac{2n+3}{n^2+3n+2}$

   ix) $a_n = 3n^7 + 2n^5 + n^5 + 2$

   x) $a_n = n^n,\ n \neq 0$

   xi) $a_n = n!$

   xii) $a_n = \frac{2^n+n^2+3n!}{3n!+n^2+2^n}$

   In addition, prove that the sequences vii) and viii) are equivilent. Write programs to evaluate the first 100 terms in these sequences.

2. Evaluate the terms a_0, a_1, a_2, a_3 in the following sequences and write programs to evaluate the first 100 terms in each sequence.

   i)
   $$a_n = \begin{cases} 2 + \frac{1}{2^n} & , \quad n \text{ even} \\ -2^n & , \quad n \text{ odd} \end{cases}$$

   ii)
   $$b_n = \begin{cases} 3^n - 14n - 7 & , \quad n \text{ even} \\ 3^n - 2n & , \quad n \text{ odd} \end{cases}$$
   $$a_n = \begin{cases} n^2 + 1 & , \quad b_n > 0 \\ 3^n - 2n & , \quad b_n \leq 0 \end{cases}$$

   iii)
   $$a_n = \begin{cases} (n+10)^6 & , \quad n < 2 \\ n^3 & , \quad \text{otherwise} \end{cases}$$

   iv)
   $$a_n = \begin{cases} (-1)^{n-1} \frac{\pi^{2n-1}}{(2n-1)!} & , \quad n \text{ even} \\ (-1)^{n+1} \frac{\pi^{2n+1}}{(2n+1)!} & , \quad n \text{ odd} \end{cases}$$

3. Evaluate the term a_7 in the following recursive sequences and write programs to evaluate the first 100 terms in each sequence. In addition, for sequence iii), evaluate $a_{12000003}$.

   i)
   $$\begin{cases} a_0 & = 1 \\ a_n & = n^2 a_{n-1} \end{cases}$$

   ii)
   $$\begin{cases} a_0 & = \frac{1}{2} \\ a_n & = a_{n-1}\left(2 - a_{n-1}\sqrt{2}\right) \end{cases}$$

   iii)
   $$\begin{cases} a_0 & = -\frac{1}{4} \\ a_1 & = 2 \\ a_n & = \frac{a_{n-1}}{a_{n-2}} \end{cases}$$

4. The factorial function is defined *recursively* as

$$n! = a_n, \text{ where } \begin{cases} a_0 & = 1 \\ a_n & = n a_{n-1} \end{cases}$$

The factorial function cas equivilently be defined *iteratively* as the product of the first n positive integers

$$n! = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times (n-1) \times (n-1) \times n$$

Write a program which computes $n!$ in this way without using recursion.

5. Evaluate the terms $a_1, a_2, a_3, a_4, a_{1000}$ in the following sequences

i) $a_n = \frac{(n+1)!}{n!}$

ii) $a_n = \frac{(n+2)!}{n!}$

iii) $a_n = \frac{(n+1)!}{(n+2)!}$

iv) $a_n = (-1)^n \frac{(n+3)!}{(n+2)!}$

6. For the following strictly positive sequences, evaluate the ratio $\frac{a_{n+1}}{a_n}$ and hence say whether the sequence is increasing, decreasing or neither. In addition, say whether the sequence is bounded or unbounded.

i) $a_n = n$

ii) $a_n = \frac{n+4}{n+5}$

iii) $a_n = \frac{n+5}{n+4}$

iv) $a_n = n!$

v) $a_n = \frac{n!}{n^n}, \ n \neq 0$

vi) $a_n = \frac{2^n+1}{2^n-1}$

7. Evaluate the limits of the following sequences as $n \to \infty$.

i) $a_n = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n!}$

ii) $a_n = \frac{2n+2}{3n+7}$

iii) $a_n = \frac{4n-3}{2n^2+2n+1}$

iv) $a_n = \frac{3n+2n^2+1}{5n^2+6}$

v) $a_n = \frac{7n^2}{20000n+n^2}$

vi) $a_n = \frac{10^{10^{10}}n+n^2}{n^2}$

vii) $a_n = \frac{3n}{n!}$

viii) $a_n = \frac{10n!}{n^n}$

ix) $a_n = \frac{5n^4+6n^2+4}{2^n+1}$

x) $a_n = \left(\frac{1}{4}\right)^n$

8. Assumming that the following recursive sequence has a limit $\lim_{n\to\infty} a_n = L$, find this limit in terms of D.

$$\begin{cases} a_0 & = 1 \\ a_{n+1} & = \frac{2}{3} a_n + \frac{D}{3a_n^2} \end{cases}$$

What is the limit of the sequence defined by

$$\begin{cases} a_0 & = 1 \\ a_{n+1} & = \frac{k-1}{k} a_n + \frac{D}{k a_n^{k-1}} \end{cases}$$

Solutions

1. i) $1, 6, 17, 34, 321, 30201$
   ii) $\frac{7}{3}, \frac{8}{5}, \frac{11}{7}, \frac{16}{9}, \frac{107}{23}, \frac{10007}{203}$
   iii) $0, -1, 2, -3, 10, 100$
   iv) $4, 2, 20, 2, 118100,\ 1.030755 \times 10^{48}$
   v) $1, 6, 36, 216, 60466176,$ $6.5331862350007 \times 10^{77}$
   vi) $1, 6, 36, 216, 60466176,$ $6.5331862350007 \times 10^{77}$
   vii) $\frac{3}{2}, \frac{5}{6}, \frac{7}{12}, \frac{9}{20}, \frac{23}{132}, \frac{203}{10302}$
   viii) $2, 8, 482, 7292, 30300002,$ 300030000000002
   ix) $1, 1, 4, 27, 10^{10}, 100^{100}$
   x) $1, 1, 2, 6, 3628800,$ $9.33262154439476 \times 10^{157}$ see footnote[1]
   xi) $1, 1, 1, 1, 1, 1$

2. i) $3, -2, \frac{9}{4}, -8$
   ii) $1, 2, 5, 10$
   iii) $1000000, 1771561,$ $8, 27$
   iv) $-, \frac{\pi^3}{6}, -\frac{\pi^3}{6}, \frac{pi^7}{5040}$

3. i) $1, 1, 4, 36, 576,$ $14400, 518400,$ 25401600
   ii) $0.5, 0.64645, 0.70190,$ $0.70707, 0.70711, 0.70711,$ $0.70711, 0.70711$
   iii) $-\frac{1}{4}, 2, -8, -4,$ $\frac{1}{2}, -\frac{1}{8}, -\frac{1}{4}, 2$

5. i) $2, 3, 4, 5, 1001$
   ii) $6, 12, 20, 30, 1003002$
   iii) $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{1002}$
   iv) $-4, 5, -6, 7, 1003$

6. i) Increasing,unbounded
   ii) Increasing,bounded
   iii) Decreasing,bounded
   iv) Increasing,unbounded
   v) Decreasing,bounded
   vi) Decreasing,bounded

7. i) 0
   ii) $\frac{2}{3}$
   iii) 0
   iv) $\frac{2}{5}$
   v) 7
   vi) 1
   vii) 0
   viii) 0
   ix) 0
   x) 0

8.

$$L = \frac{2}{3}L + \frac{D}{3L^2}, \quad \Rightarrow L^3 = D, \quad \Rightarrow L = \sqrt[3]{D}$$

$$L = \frac{k-1}{k}L + \frac{D}{kL^{k-1}}, \quad \Rightarrow L^k = D, \quad \Rightarrow L = \sqrt[k]{D}$$

[1] 100!=93326215443944152681699238856266700490715968264381621468592963895217599993-229915608941463976156518286253697920827223758251185210916864000000000000000000000000