Continuous Assessment Test - Solutions

Q1

$$x^4 - 18x^2 + 45 = 0$$

has a root α between 1 and 2.

(a) Let $f(x) = x^4 - 18x^2 + 45$, then $f'(x) = 4x^3 - 36x$

The algorithm for Newton's method is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 18x_n^2 + 45}{4x_n^3 - 36x_n}$$

Use the initial guess $x_0 = 1.5$ and construct the table:

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1.5	9.5625	—
1	1.736111	−0.168783	0.236111
2	1.732051	$9.3 \cdot 10^{-7}$	−0.004060
3	1.732051	$3 \cdot 10^{-8}$	$2 \cdot 10^{-8}$

We obtained the desired accuracy after 3 steps. Recall that the last column is a measure for the solution error, since

$$x_n - x_{n-1} \approx \alpha - x_{n-1}$$

(b) The secant method algorithm is

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

We take $x_0 = 1, x_1 = 2$

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1	28	—
1	2	−11	1
2	1.717949	0.586193	−0.282051
3	1.732219	−0.006986	0.014270
4	1.732051	$2.2 \cdot 10^{-7}$	−0.000168
5	1.732051	$3 \cdot 10^{-8}$	$5 \cdot 10^{-9}$

Q2 $e^x - x - 2 = 0$

a) Let $f(x) = e^x - x - 2$. Easy to show $f(1) < 0$ and $f(2) > 0$. Since $f(x)$ has a sign change between 1 and 2, there is a root α in the interval $(1,2)$.

b) Recall the inequality:

$$n \geq \frac{\log \frac{b-a}{\varepsilon}}{\log 2}$$

, which gives the number of iterations required for the bisection method to approximate a root $\alpha \in (a,b)$ with an accuracy ε.

In our case $n \geq \dfrac{\log(10^3)}{\log 2} = 9.96$ hence we need at least 10 iterations.

c) Construct the table:

n	a	b	c	$b-c$	$f(c)$
1	1	2	1.5	0.5	0.981689
2	1	1.5	1.25	0.25	0.240343
3	1	1.25	1.125	0.125	−0.044783
4	1.125	1.25	1.1875	0.0625	0.091374
5	1.125	1.1875	1.1562	0.0312	0.021634
6	1.125	1.1562	1.1406	0.0156	−0.01195
7	1.1406	1.1562	1.1484	0.0078	0.004744
8	1.1406	1.1484	1.1445	0.0039	−0.003629
9	1.1445	1.1484	1.14645	0.00195	0.000551
10	1.1445	1.14645	1.145475	0.000975	−0.0015

Note: The algorithm is stopped when $b-c \leq \varepsilon = 10^{-3}$ which happens after 10 iterations.

Q3 $f(x) = x^3 - 2x^2 - 3 = 0$ has a root α between 2 and 3

a) $f(x) = 0 \Rightarrow x^3 = 2x^2 + 3 \Rightarrow x = 2 + \dfrac{3}{x^2} \equiv g_1(x)$

The fixed point iteration algorithm associated with $g_1(x)$ converges to α if and only if $|g_1'(\alpha)| < 1$.

Since $g_1'(x) = -\dfrac{6}{x^3}$, we have $|g_1'(\alpha)| = \dfrac{6}{\alpha^3} < \dfrac{6}{2^3} < 1$ (since $\alpha > 2$). Hence, the algorithm converges.

b) Two other algorithms:

$x^3 - 2x^2 - 3 = 0 \Rightarrow 2x^2 = x^3 - 3 \Rightarrow x = \dfrac{x^2}{2} - \dfrac{3}{2x} \equiv g_2(x)$

$x^3 - 2x^2 - 3 = 0 \Rightarrow x^3 = 2x^2 + 3 \Rightarrow x = \sqrt[3]{2x^2 + 3} \equiv g_3(x)$

c) Construct table of iterations for the 3 algorithms:

n	$x_n = g_1(x_{n-1})$	$x_n = g_2(x_{n-1})$	$x_n = g_3(x_{n-1})$
1	2	2	2
2	2.75	1.25	2.22398
3	2.396694	-0.41875	2.344816
4	2.522271	3.669765	2.409931
5	2.471561	6.324843	2.444959
6	2.491110	19.764661	2.463779
7	2.483432	195.245	2.47388
8	2.486426	19060.30	2.47930
9	2.485255	$1.8 \cdot 10^8$	2.482216
10	2.485712	–	2.483777

Note that algorithms g_1 and g_3 converge to α while algorithm g_2 diverges!

Q4 | a) Easy to check $x_1 = x_2 = x_3 = 1$ is an exact solution.

b) Gaussian elimination without pivoting leads to

$$\begin{pmatrix} 0.002 & 1.231 & 2.471 \\ 0 & -732.9 & -1475 \\ 0 & 0 & -1.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3.704 \\ -2208 \\ -2.0 \end{pmatrix}$$

so the approximate solution is

$x_1 = 4.000$ (Relative error $= 300\%$)

$x_2 = -1.012$ (Relative error $\simeq 200\%$)

$x_3 = 2.000$ (Relative error $= 100\%$)

With partial pivoting we get:

$$\begin{pmatrix} 0.002 & 1.231 & 2.471 & 3.704 \\ 1.196 & 3.165 & 2.543 & 6.904 \\ 1.475 & 4.271 & 2.142 & 7.888 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1.475 & 4.271 & 2.142 & 7.888 \\ 1.196 & 3.165 & 2.543 & 6.904 \\ 0.002 & 1.231 & 2.471 & 3.704 \end{pmatrix}$$

$$\begin{pmatrix} R_2 - R_1 \cdot 0.8108 \\ R_3 - R_1 \cdot 0.001356 \end{pmatrix}$$

$$\begin{pmatrix} 1.475 & 4.271 & 2.142 & 7.888 \\ 0 & -0.298 & 0.806 & 0.508 \\ 0 & 1.225 & 2.468 & 3.693 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1.475 & 4.271 & 2.142 & 7.888 \\ 0 & 1.225 & 2.468 & 3.693 \\ 0 & -0.298 & 0.806 & 0.508 \end{pmatrix}$$

$$R_3 + R_2 \cdot 0.2433$$

$$\begin{pmatrix} 1.475 & 4.271 & 2.142 & 7.888 \\ 0 & 1.225 & 2.468 & 3.693 \\ 0 & 0 & 1.406 & 1.406 \end{pmatrix}$$

Hence $\boxed{\begin{aligned} x_3 &= 1.00 \\ x_2 &= 1.00 \\ x_1 &= 1.00 \end{aligned}}$

All relative errors $= 0$!

Q5 a) $AXB = A + B + C$

We need to calculate A^{-1} and B^{-1} and then get X as:

$$X = A^{-1}(A+B+C)B^{-1}.$$

Calculate A^{-1}:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & 4 & 0 & 1 & 0 \\ 4 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 4 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -1 & 0 \\ 0 & 1 & 11 & 1 & 2 & 0 \\ 0 & -2 & 19 & 0 & 4 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -1 & 0 \\ 0 & 1 & 11 & 1 & 2 & 0 \\ 0 & 0 & 41 & 2 & 8 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & -1 & 0 \\ 0 & 1 & 11 & 1 & 2 & 0 \\ 0 & 0 & 1 & \frac{2}{41} & \frac{8}{41} & \frac{1}{41} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{41} & -\frac{9}{41} & \frac{4}{41} \\ 0 & 1 & 0 & \frac{19}{41} & -\frac{6}{41} & -\frac{11}{41} \\ 0 & 0 & 1 & \frac{2}{41} & \frac{8}{41} & \frac{1}{41} \end{array}\right)$$

hence $A^{-1} = \frac{1}{41}\begin{pmatrix} 8 & -9 & 4 \\ 19 & -6 & -11 \\ 2 & 8 & 1 \end{pmatrix}$

Similarly, we find that $B^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 & 1 \\ -2 & -4 & -1 \\ 1 & -3 & -2 \end{pmatrix}$

$A+B+C = \begin{pmatrix} 4 & 5 & 4 \\ 0 & -3 & 5 \\ 6 & 0 & 2 \end{pmatrix}$ hence $X = \frac{1}{205}\begin{pmatrix} -27 & -309 & -1 \\ -182 & -534 & -151 \\ 106 & -108 & -72 \end{pmatrix}$

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b) $AY = D \Rightarrow Y = A^{-1}D = \frac{1}{41}\begin{pmatrix} 29 \\ -8 \\ -3 \end{pmatrix}$

Q6 Matrix form : $AX = B$, where

$$A = \begin{pmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} -3 \\ 0 \\ 15 \end{pmatrix}$$

a) If we write

$$\begin{pmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

we obtain : $l_{21} = \frac{2}{3}$, $l_{31} = -\frac{4}{3}$, $l_{32} = 9$

$u_{11} = 3$, $u_{12} = 6$, $u_{13} = -9$, $u_{22} = 1$, $u_{23} = 3$

$u_{33} = -29$

$AX = B \Rightarrow LUX = B$. Let $Y = UX$ and solve $LY = B$

to get $y_1 = -3$, $y_2 = 2$, $y_3 = -7$.

Then solve $UX = Y$ and get $x_1 = -\frac{82}{29}$, $x_2 = \frac{37}{29}$

$x_3 = \frac{7}{29}$

b) Write

$$\begin{pmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

to get: $l_{11} = 3$, $l_{21} = 2$, $l_{22} = 1$, $l_{31} = -4$, $l_{32} = 9$,

$l_{33} = -29$.

$u_{12} = 2$, $u_{13} = -3$, $u_{23} = 3$.

Follow the same procedure as in part (a) to get :

$y_1 = -1$, $y_2 = 2$, $y_3 = \frac{7}{29}$ and then

$x_1 = -\frac{82}{29}$, $x_2 = \frac{37}{29}$, $x_3 = 7/29$