Course notes for Science Mathematics 2 (MA4602)

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Chapter 4

Series and Power Series

This chapter is devoted to the study of series and power series. We divide the chapter in three parts: i. we define the concept of sequence in part 1; ii. part 2 is about infinite sums of sequences i.e. series; iii. part 3 is about power series.

4.1 Sequences

DEFINITION **4.1.1.** *A sequence is a function of the variable n, where $n = 0, 1, 2, 3, \dots$ (n can only be a natural number).*

We give below some examples of sequences.

Example **4.1.1.**

$$f(n) = \frac{1}{n}. \qquad n = 1, 2, 3, \dots$$

We denote

$$a_n = f(n) = \frac{1}{n}$$

and we also write

$$\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

Example **4.1.2.**

$$a_n = \frac{n}{n+1}. \qquad n = 0, 1, 2, 3, \dots$$

We can also denote the sequence in the following way

$$\left\{\frac{1}{n}\right\}_{n=0}^{\infty} = 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

or if we start with $n = 1$, then we have

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

Example **4.1.3.**

$$a_n = (-1)^n. \qquad n = 0, 1, 2, 3, \ldots$$

We can list the first few elements of the sequence

$$\begin{aligned}\left\{(-1)^n\right\}_{n=0}^{\infty} &= (-1)^0, (-1)^1, (-1)^2, (-1)^3, \ldots \\ &= 1, -1, 1, -1, \ldots\end{aligned}$$

or if we start with $n = 1$, then we have

$$\begin{aligned}\left\{(-1)^n\right\}_{n=1}^{\infty} &= (-1)^1, (-1)^2, (-1)^3, \ldots \\ &= -1, 1, -1, \ldots\end{aligned}$$

4.1.1 Limit of a sequence

We are interested in the behavior of a any sequence

$$\{a_n\}$$

when n is becoming larger and larger (positive), in other words, when speaking about a sequence $\{a_n\}$, we are interested in the following limit

$$\lim_{n\to+\infty} a_n$$

and we simply write

$$\lim_{n\to\infty} a_n$$

by omitting the "+" in front of infinity. This is because $n \geq 0$. When evaluating the limit

$$\lim_{n\to+\infty} a_n,$$

we just use the same techniques seen in Chapter 1 to evaluate the limit of a function $f(x)$

$$\lim_{x\to+\infty} f(x)$$

ad we just replace the variable x with n.

DEFINITION **4.1.2.** *If the limit of a sequence $\{a_n\}$ is a finite value ℓ*

$$\lim_{n\to\infty} a_n = \ell,$$

we say that the sequence converges to ℓ, otherwise we say that the sequence diverges.

Example **4.1.4.**

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

therefore the sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

Example **4.1.5.**

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{n/\div n}{n+1/\div n}\\ &= \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1\end{aligned}$$

therefore the sequence $\left\{\frac{n}{n+1}\right\}$ converges to 1.

Example **4.1.6.**

$$\begin{aligned}\lim_{n\to\infty} \frac{2n^2+3}{3n^3+3n-1} &= \lim_{n\to\infty} \frac{2n^2+3/\div n^3}{3n^3+3n-1/\div n^3}\\ &= \frac{\frac{2}{n}+\frac{3}{n^3}}{1+\frac{3}{n^2}-\frac{1}{n^3}} = \frac{0+0}{1+0-0} = 0\end{aligned}$$

therefore the sequence $\left\{\frac{2n^2+3}{3n^3+3n-1}\right\}$ converges to 0.

Example **4.1.7.**

$$\begin{aligned}\lim_{n\to\infty} \frac{5n^3+3}{3n^2+3n-1} &= \lim_{n\to\infty} \frac{5n^3+3/\div n^2}{3n^2+3n-1/\div n^2}\\ &= \frac{5n+\frac{3}{n^2}}{3+\frac{3}{n}-\frac{1}{n^2}} = \frac{+\infty+0}{3+0-0} = +\infty\end{aligned}$$

therefore the sequence $\left\{\frac{5n^3+3}{3n^2+3n-1}\right\}$ diverges.

4.2 Series or infinite series

Let us consider a sequence

$$\{a_n\} = a_1, a_2, a_3, a_4, \ldots$$

and let us consider its infinite sum

$$a_1 + a_2 + a_3 + a_4 + \ldots$$

DEFINITION **4.2.1.** *We denote its infinite sum by*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$$

and we call it series or infinite series.

The question now is: what does

$$\sum_{n=1}^{\infty} a_n$$

sum to? Let us consider the following example.

Example **4.2.1.**

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

The terms of this series are the areas shown in Figure 4.1 and their sum tends to the area of the unit square shown in the picture, which is 1.

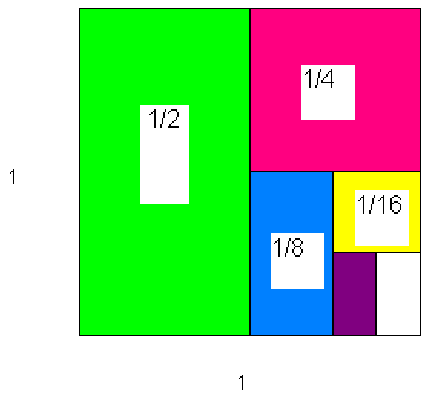

Figure 4.1: Picture showing a unit square and the values of the areas highlighted with the colors corresponds to the terms of the series we are examining.

Therefore we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

DEFINITION **4.2.2.** *If a series sums to a finite value ℓ*

$$\sum_{n=1}^{\infty} a_n = \ell,$$

we say that the series converges to ℓ. If this is not the case, then we say that the series diverges.

4.2.1 The ratio test

Consider the series

$$\sum_{n=1}^{\infty} a_n$$

and consider the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = R.$$

The so-called Ratio test says that

1. If $R < 1$, then the series converges.
2. If $R > 1$, then the series diverges.
3. If $R = 1$, then the test fails.

Example **4.2.2.** *Study the behavior of the series*

$$\sum_{n=1}^{\infty} \frac{2n-1}{2^{n-1}}.$$

Is it convergent or divergent?

Let us consider the ratio

$$\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{2(n+1)-1}{2^{n+1-1}}}{\frac{2n-1}{2^{n-1}}} \right| \\
&= \left| \frac{2n+2-1}{2^n} \frac{2^{n-1}}{2n-1} \right| \\
&= \frac{2n+1}{2^n} \frac{2^{n-1}}{2n-1} \\
&= \frac{2n+1}{2n-1} \frac{2^{n-1}}{2^n} \\
&= \frac{2n+1}{2n-1} \frac{1}{2} \\
&= \frac{1}{2} \frac{2n+1}{2n-1} = \frac{1}{2} \cdot 1 = \frac{1}{2} < 1,
\end{aligned}$$

therefore the series converges.

4.3 Power series

We start by giving a brief introduction to what a power series is. A power series centered at $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \ldots,$$

where a_0, a_1, a_2, a_3,... are numbers and x is the variable.

Remark **4.3.1.** *If the variable x is replaced by any number, then the power series becomes a series which can either converge or diverge.*

4.3.1 The Maclaurin series

Given a function $f(x)$ that is infinitely many times differentiable at $x = 0$, its Maclaurin's expansion or Maclaurin series is given by the formula

$$\begin{aligned} f(x) &= f(0) + f^{'}(0)x + \frac{f^{(2)}(0)}{2!}\, x^2 + \frac{f^{(3)}(0)}{3!}\, x^3 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}\, x^n, \end{aligned}$$

where $\frac{f^n(0)}{n!}$ is the n-th derivative of $f(x)$ (with respect to x) and

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1,$$

which is read n factorial. For example we have

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \cdot 1 \\ 3! &= 3 \cdot 2 \cdot 1 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\ \ldots \end{aligned}$$

Note that we define

$$0! = 1.$$

Remark **4.3.2.** *The Maclaurin series of a function $f(x)$ is an example of power series centered at $x = 0$.*

Remark **4.3.3.** *Note that the Maclaurin series of $f(x)$ converges to $f(0)$ at $x = 0$. The question now is to find out for which other values of x is the Maclaurin series of $f(x)$ convergent. For these values of x then it makes sense to equate $f(x)$ to its Maclaurin series:*

$$\begin{aligned} f(x) &= f(0) + f^{'}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n. \end{aligned}$$

Example **4.3.1.** *Find the Maclaurin series of function*

$$f(x) = e^x.$$

STEP 1: Recall the formula for the Maclaurin series

$$f(x)\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n.$$

We need to find the coefficients of the series i.e.

$$\frac{f^n(0)}{n!}, \qquad \text{for} \quad f(x) = e^x.$$

$$\begin{array}{rcl} f(x) = e^x & ; & f(0) = e^0 = 1 \\ f^{'}(x) = e^x & ; & f^{'}(0) = e^0 = 1 \\ f^{(2)}(x) = e^x & ; & f^{(2)}(0) = e^0 = 1 \\ \vdots & ; & \vdots \\ f^{(n)}(x) = e^x & ; & f^{(n)}(0) = e^0 = 1, \end{array}$$

therefore

$$\frac{f^n(0)}{n!} = \frac{1}{n!},$$

therefore the Maclaurin series of $f(x) = e^x$ is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Once the Maclaurin series of a function is found, one needs to check where is the series converging. We know that the series converges to $f(0) = e^0 = 1$ at $x = 0$, but we need to check now for which other values of x the series is convergent. For this second step we use the ratio test.

STEP 2: RATIO TEST

Call

$$u_n = \frac{x^n}{n!}$$

and consider the ratio

$$\begin{aligned}
\left|\frac{u_{n+1}}{u_n}\right| &= \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| \\
&= \left|\frac{x^{n+1}}{(n+1)!}\,\frac{n!}{x^n}\right| \\
&= \left|\frac{x^{n+1}}{x^n}\,\frac{n!}{(n+1)!}\right| \\
&= \left|x\,\frac{n!}{(n+1)(n!)}\right| \\
&= \left|\frac{x}{n+1}\right| = \frac{|x|}{n+1}.
\end{aligned}$$

Therefore we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = |x|\lim_{n\to\infty}\frac{1}{n+1} = |x|\cdot 0 = 0 < 1, \qquad \text{for all } x.$$

Therefore we say that the series converges for all x i.e. for all x in $(\infty, +\infty)$ which is called the interval of convergence. We also say that the radius of convergence in this case is $R{=}\infty$. We can therefore write

$$e^x = \sum_{n=0}^{\infty}\frac{1}{n!}\,x^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \qquad \text{for all } x.$$

Exercise **4.3.1.** *Find the Maclaurin series of*

$$f(x) = \sin(x).$$

Find the radius and the interval of convergence.

Note that a full solution of this exercise is give on the slides for Chapter 4. It is a good exercise for the student to try to solve the question first and then check the solution given on the slides.

Exercise **4.3.2.** *Find the Maclaurin series of*

$$f(x) = \cos(x).$$

Find the radius and the interval of convergence.

4.3.2 The Taylor series

A generalization of the Maclaurin series of a function $f(x)$ is given by the so-called Taylor series of $f(x)$ at x_0 or centered at x_0:

$$\begin{aligned} f(x) &= f(x_0) + f^{'}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n, \end{aligned}$$

which is an example of a power series centered at $x = x_0$. The Taylor series centered at x_0 provides and power series expansion of $f(x)$ centered at $x = x_0$.

Remark **4.3.4.** *Note that $f(x)$ needs to be infinitely many times differentiable at $x = x_0$ in order to be able to have a Taylor expansion centered at x_0.*

Remark **4.3.5.** *Note that if $x_0 = 0$ then the Taylor series of $f(x)$ reduces to the Maclaurin series of $f(x)$. Note also that the Taylor series of $f(x)$ centered at x_0 converges at $x = x_0$ to $f(x_0)$. Once we wrote the Taylor series of a function $f(x)$ centered at x_0, we need to check for which other values of x is the series convergent (similarly to what we did for the Maclaurin series) and for these values of x we can then equate $f(x)$ to its Taylor series*

$$f(x) \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n.$$

Example **4.3.2.** *Find the Taylor series of function*

$$f(x) = \ln(x)$$

at $x_0 = 1$. Find then the radius and the interval of convergence of the Taylor series.

STEP 1: Recall the Taylor series of $f(x)$ at $x_0 = 1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x - 1)^n;$$

we want to find the coefficients

$$\frac{f^n(1)}{n!}$$

for $f(x) = \ln(x)$.

$$
\begin{array}{rcl}
f(x) = \ln(x) & ; & f(1) = \ln(1) = 0 \\
f^{'}(x) = \dfrac{1}{x} = x^{-1} & ; & f^{'}(1) = \dfrac{1}{1} = 1 \\
f^{(2)}(x) = -x^{-2} & ; & f^{(2)}(1) = -1 \\
f^{(3)}(x) = (-1)(-2)x^{-3} = 1 \cdot 2x^{-3} & ; & f^{(3)}(1) = 1 \cdot 2 = 2! \\
f^{(4)}(x) = -1 \cdot 2 \cdot 3x^{-4} = 1 \cdot 2x^{-3} & ; & f^{(4)}(1) = -1 \cdot 2 \cdot 3 = -3! \\
\vdots & ; & \vdots
\end{array}
$$

Therefore we have

$$
\begin{aligned}
\ln(x) &= 0 + \frac{1}{1!}(x-1) + \frac{(-1)}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \frac{(-3!)}{4!}(x-1)^4 + \ldots \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots \\
&= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}.
\end{aligned}
$$

Therefore the Taylor series or $f(x) = \ln(x)$ centered at $x_0 = 1$ is given by

$$
\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n.
$$

STEP 2: RATIO TEST to find the radius and interval of convergence.

Call

$$
u_n = \frac{(-1)^{n+1}}{n} (x-1)^n
$$

and consider the ratio

$$
\begin{aligned}
\left| \frac{u_{n+1}}{u_n} \right| &= \left| \frac{\frac{(-1)^{n+1+1}}{n+1} (x-1)^{n+1}}{\frac{(-1)^n}{n} (x-1)^n} \right| \\
&= \left| \frac{(-1)^{n+2}}{n+1} (x-1)^{n+1} \frac{n}{(-1)^n (x-1)^n} \right| \\
&= \left| \frac{(-1)^{n+2}}{(-1)^n} \frac{(x-1)^{n+1}}{(x-1)^n} \frac{n}{n+1} \right| \\
&= \left| (-1)^2 (x-1) \frac{n}{n+1} \right| \\
&= |x-1| \frac{n}{n+1}.
\end{aligned}
$$

Therefore we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}|(x-1)|\,\frac{n}{n+1} = |x-1|\lim_{n\to\infty}\frac{n}{n+1} = |x-1|\cdot 1 = |x-1|.$$

Therefore the Taylor series is convergent for

$$|x-1| < 1$$

i.e.

$$-1 < x-1 < 1$$

i.e.

$$0 < x < 2.$$

Therefore the Taylor series converges on the interval of convergence $(0,2)$ and the radius of convergence is therefore $R = 1$. Therefore we can write

$$\ln(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\,\frac{(x-1)^n}{n} \qquad \text{for all } x \text{ in} \quad (0,2).$$

Note that the series may or my not converge at $x = 2$ but we do not have the tools in this course to check it. All we can say here is that the series converges on $(0,2)$.