The typed notes will be available online at

`http://www.maths.dit.ie/˜dmackey/lectures.html`

Partial Differentiation

In science and engineering many functions depend on two or more variables, rather than a single one. For example,

$$z = f(x, y) = x^2 + y^2,$$
$$w = f(x, y, z) = xy + yz + zx$$

Assume we have a function of two variables $f(x, y)$. The partial derivative with respect to x at a point (x_0, y_0) is defined as the rate of change of the function with respect to x at that point. To calculate this derivative, we let $y = y_0$ and calculate the “ordinary” derivative of the resulting function $f(x, y_0)$.

Example: Calculate the partial derivatives with respect to x and y of the function $f(x,y) = x^3 + y^2 x$ at the point $(1,2)$.

To calculate the derivative with respect to x let $y = 2$ so $f(x,2) = x^3 + 4x$ and $\frac{df}{dx} = 3x^2 + 4$. The value of the derivative at the point $(1,2)$ is now obtained by letting $x = 1$ so $\frac{df}{dx}(1,2) = 7$.

Example: Calculate the partial derivatives with respect to x and y of the function $f(x,y) = y\exp(x+y)$ at the point $(3,3)$.

In general (when the point is not specified) the partial derivative of $f(x,y)$ with respect to x is calculated by treating y as a constant. Also, the partial derivative of $f(x,y)$ with respect to y is calculated by treating x as a constant.

The notation used for partial derivatives is the following:

$$\frac{\partial f}{\partial x}(x_0,y_0), \quad \frac{\partial f}{\partial y}(x_0,y_0), \quad f_x(x_0,y_0), \quad f_y(x_0,y_0), \quad \frac{\partial z}{\partial x}\bigg|_{(x_0,y_0)}, \quad \frac{\partial z}{\partial y}\bigg|_{(x_0,y_0)}$$

or

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad f_x, \quad f_y, \quad \frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}$$

Examples:

Find all the partial derivatives for the following functions

$$f(x,y) = x^2 + 3xy + y - 4$$
$$f(x,y) = y\sin(xy)$$
$$f(x,y,z) = \ln(x + 2y + 3z)$$
$$f(x,y,z) = xy + yz + xz \qquad \text{at} \quad (1,2,3)$$

Second-order partial derivatives

The second order partial derivatives are obtained by differentiating the function twice. For example, if $z = f(x, y)$,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \text{also denoted by } f_{xx} \text{ or } \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \qquad \text{also denoted by } f_{yy} \text{ or } \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad \text{also denoted by } f_{xy} \text{ or } \frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad \text{also denoted by } f_{yx} \text{ or } \frac{\partial^2 z}{\partial y \partial x}$$

The mixed derivatives are equal!

When calculating a mixed second order partial derivative, the order of differentiation does not matter (so we can differentiate first with respect to x then with respect to y or the other way around). That is

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Examples: Find all second-order partial derivatives for the functions

$$f(x,y) = xy^2 + x^2y^3 + x^3y^4$$
$$z = e^x + x\ln(y) + y\ln(x)$$

Verify that all mixed partial derivatives are equal.

Maximum and minimum points for functions of two variables

Definition:
A point (a,b) is called a **local maximum point** for a function of two variables f if

$$f(a,b) \geq f(x,y)$$

for all points (x,y) close to (a,b).

Recall that, in the case of one-variable functions, the first step towards finding maxima or minima was finding the critical points of the function. A critical point was a point at which the derivative of the function was equal to 0.

A similar definition holds for two-variable functions.

Definition: A point (a,b) is called a **critical point** for the function $f(x,y)$ if both partial derivatives of f (with respect to x and y) at the point (a,b) are equal to 0, that is

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$$

Example: Find the critical points for the function

$$f(x,y) = x^2 + y^2 + 2x - 4y - 4$$

We have

$$\frac{\partial f}{\partial x} = 2x + 2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2y - 4.$$

Hence the only critical point is $x = -1$ and $y = 2$, or $(x,y) = (-1,2)$.

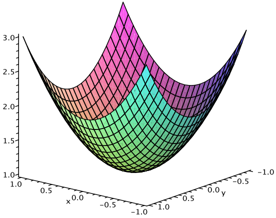
How do we decide whether a critical point (a,b) is a maximum or minimum point for a function $f(x,y)$?

Example 1: Find the critical points of $f(x,y) = 1 + x^2 + y^2$ and decide whether they are maximum or minimum points.

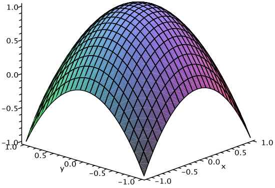
Example 2: Find the critical points of $f(x,y) = 1 - x^2 - y^2$ and decide whether they are maximum or minimum points.

Example 3: Find the critical points of $f(x,y) = x^2 - y^2$ and decide whether they are maximum or minimum points.

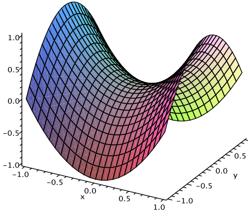
The graphs of the functions in Examples 1,2, 3 are given below.



(a) Minimum point



(b) Maximum point



(c) Saddle point

The procedure for finding the maximum, minimum or saddle points for a function $f(x,y)$ is the following:

Step 1: Find all the critical points of $f(x,y)$ by solving $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$.

Step 2: Suppose (a,b) is a critical point. Find all second order derivatives of $f(x,y)$ at (a,b) and arrange them in a matrix as below:

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix}$$

(This matrix is called the **Hessian** of the function $f(x,y)$).

Step 3: Calculate the determinant of H_f.

Step 4: If $\det(H_f) < 0$ then (a,b) is a **saddle point.** If $\det(H_f) > 0$ then we have to look at $\frac{\partial^2 f}{\partial x^2}$:

- If $\frac{\partial^2 f}{\partial x^2} > 0$ then (a,b) is a **minimum point**
- If $\frac{\partial^2 f}{\partial x^2} < 0$ then (a,b) is a **maximum point**

Exercise: Find the maximum, minimum and saddle points for the following functions of two variables.

$$f(x,y) = 2x^3 - 6xy + 3y^2$$
$$f(x,y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2$$
$$f(x,y) = xy(x^2 + y^2 - 1)$$

Maximum and minimum values under constraints

Consider the problem of finding maximum and minimum values for a function $f(x,y)$ in the case when the variables x and y are not independent but are related by an equation. (We say that x and y satisfy a constraint.)

Example: The temperature of a point (x,y) is given by $T(x,y) = 1 + xy$. Find the maximum value of this function if the point is constrained to move on the unit circle, $x^2 + y^2 = 1$.

The method for maximizing or minimizing functions subject to constraints is called **the method of Lagrange multipliers**.

Lagrange multipliers

To find the extreme values of a function $f(x,y)$ subject to the constraint $g(x,y) = 0$ we solve the equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$
$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

where λ is an unknown parameter called a **Lagrange multiplier**.

Together with the constraint equation $g(x,y) = 0$ we have a system of 3 equations with 3 unknowns (x, y and λ). The values of x and y will give us the extreme points.

The case where f is a function of 3 variables, $f(x,y,z)$ is similar (we have 4 equations with 4 unknowns instead!)

Example

In the temperature example we have

$$f(x,y) = T(x,y) = 1 + xy \qquad \text{and} \qquad g(x,y) = x^2 + y^2 - 1$$

We get the equations

$$y = \lambda \cdot 2x, \qquad x = \lambda \cdot 2y, \qquad x^2 + y^2 = 1$$

which give

$$\lambda = \pm\frac{1}{2} \qquad \text{and} \qquad y = \mp x$$

There are 4 extreme points given by
$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

More examples

Exercise 1: Find the maximum value of $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

Exercise 2: Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is 16π cm^3.

Exercise 3: Find the maximum volume of a box such that the sum of the lengths of the edges of the box is equal to 6.