Integration

MA4602

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What is Integration?

Integration is the reverse process of differentiation.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

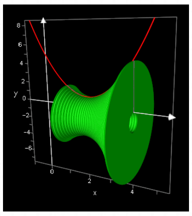
Add one to the power   Divide by new power

"c" represents the constant that may be present (to be explained later).

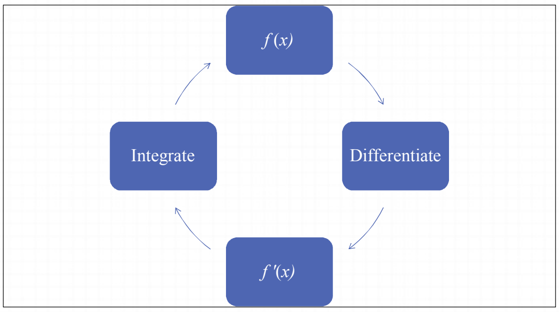
Applications of Integration

Integration is used to calculate the following:

- Moments of Inertia.
- Volume of a solid revolution.
- Electric Charges.
- Force by a liquid pressure.
- Area under a curve.
- Area between two curves.
- Work by a variable force.
- displacement, Velocity, Acceleration.



Integration and Differentiation



Integration and Differentiation

If we start with the function $f(x) = 3x^2 + 4x - 10$

Differentiate $f(x)$

$$\frac{df}{dx} = 6x + 4$$

Integrate to reverse this process:

$$\int df = \int 6x + 4\, dx$$

$$f(x) = \frac{6x^2}{2} + 4x + c$$

$$f(x) = 3x^2 + 4x + c$$

Integration and Differentiation

We started with

$$f(x) = 3x^2 + 4x - 10$$

Integration reversed the process of differentiation and we ended up with

$$f(x) = 3x^2 + 4x + c$$

The constant (-10) was eliminated during differentiation thus the reason for including "c" when integrating.

Rules for Integration

$\int k\,dx = kx + c$ [k is a constant]

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ [$n \neq -1$]

$\int k\,f(x) = k \int f(x)$ [k is a constant]

$\int [f(x) \pm g(x)]dx = \int f(x)\,dx \pm \int g(x)\,dx$

COMMON INTEGRALS	
$\int k\,dx = kx + C$	$\int \sec^2 x\ dx = \tan x + C$
$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$	$\int \sec x \tan x\ dx = \sec x + C$
$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int \csc x \cot x\ dx = -\csc x + C$
$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$	$\int \csc^2 x\ dx = -\cot x + C$
$\int \ln(x)\ dx = x\ln(x) - x + C$	$\int \tan x\ dx = \ln\lvert\sec x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int \sec x\ dx = \ln\lvert\sec x + \tan x\rvert + C$
$\int \cos x\ dx = \sin x + C$	$\int \frac{1}{a^2+u^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \sin x\ dx = -\cos x + C$	$\int \frac{1}{\sqrt{a^2-u^2}}\,dx = \sin^{-1}\left(\frac{u}{a}\right) + C$

Basic Examples

$$\int x^2 + 2\cos x\ dx$$

$$= \frac{x^3}{3} + 2\sin x + c$$

$$\int 4e^x + \frac{1}{x} - 20$$

$$= 4e^x + \ln x - 20x + c$$

Remember…

When differentiating the product of two functions we used the product rule:

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

When differentiating the quotient of two functions we used the quotient rule:

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Remember…

When finding the derivative of the composition of two functions, we use the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Substitution and Integration by Parts

Integration requires a similar type of approach.

When integrating the product, quotient, or composition of two functions, we can integrate using one of the following:

1. Substitution.
2. Integration by parts.

We'll explore substitution first.

Substitution – Ex. 1

Evaluate:

$$\int \frac{2x}{(x^2+4)^{10}}\,dx$$

Solution:

Let $u = x^2 + 4$

From here, we will replace all "x parts" with "u parts"

$$u = x^2 + 4$$

$$\frac{du}{dx} = 2x$$

$$\frac{du}{2x} = dx$$

Now we can replace dx with what it is in terms of u.

Substitution – Ex. 1

$$\int \frac{2x}{(x^2+4)^{10}}\,dx$$

Replace with "u parts":

$$\int \frac{2x}{u^{10}}\frac{du}{2x}$$

$$\int \frac{1}{u^{10}}\,du$$

Notice how it's simplified into one function – now we can integrate.

$$\int u^{-10}\,du$$

$$= \frac{u^{-9}}{-9} + c$$

$$= -\frac{1}{9u^9} + c$$

$$= -\frac{1}{9(x^2+4)^9} + c$$

Substitution – Ex. 2

Evaluate:

$$\int x\sin x^2\,dx$$

We have the product of two functions so we must use substitution to integrate.

Let $u = x^2$

Note: knowing what to let u equal may take some trial and error.

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{du}{2x} = dx$$

Now we can replace dx with what it is in terms of u.

Substitution – Ex. 2

$$\int x\sin x^2\,dx$$

$$= \int x\sin u\,\frac{du}{2x}$$

$$= \int \frac{\sin u}{2}\,du$$

$$= \frac{1}{2}\int \sin u\,du$$

Notice how it's simplified into one function – now we can integrate.

$$= \frac{1}{2}(-\cos u) + c$$

$$= -\frac{1}{2}\cos u + c$$

$$= -\frac{1}{2}\cos x^2 + c$$

Steps for Integrating by Substitution

1. Decide what to let u equal (this may take some trial and error).
2. Find $\frac{du}{dx}$ and adjust it to get a value for dx
3. Replace dx and substitute in u wherever possible.
4. At this point, there should be no x values in the integral.
5. Integrate the function then, at the end, replace u with what it was originally.

Substitution – Ex. 3

Evaluate:

$$\int (x+2)^{20}\,dx$$

Let $u = x + 2$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Now we can replace dx with what it is in terms of u.

$$\int (x+2)^{20}\,dx$$

$$= \int u^{20}\,du$$

$$= \frac{u^{21}}{21} + c$$

$$= \frac{(x+2)^{21}}{21} + c$$

Substitution – Ex. 4

Evaluate:

$$\int e^{3x+5}\,dx$$

Let $u = 3x + 5$

$$\frac{du}{dx} = 3$$

$$\frac{du}{3} = dx$$

$$\int e^{3x+5}\,dx$$

$$= \int e^{u}\frac{du}{3}$$

$$= \frac{1}{3}\int e^{u}\,du$$

$$= \frac{1}{3}e^{u} + c$$

$$= \frac{1}{3}e^{3x+5} + c$$

Substitution – Ex. 5

Evaluate:

$$\int \sin^4 x \cos x\;dx$$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$\frac{du}{\cos x} = dx$$

Now we can replace dx with what it is in terms of u.

$$\int \sin^4 x \cos x\;dx$$

$$= \int u^4 \cos x \frac{du}{\cos x}$$

$$= \int u^4\,du$$

Substitution – Ex. 5

$$\int u^4\,du$$

$$= \frac{u^5}{5} + c$$

$$= \frac{(\sin x)^5}{5} + c$$

$$= \frac{\sin^5 x}{5} + c$$

$$\int \sin^4 x \cos x\;dx = \frac{\sin^5 x}{5} + c$$

Applying Integration to Dynamics

When we differentiate w.r.t. time a function representing the displacement of an object, we obtain an equation for the velocity of that object. If we differentiate again, we get an equation for the acceleration of that object.

Displacement → Velocity → Acceleration

As integration is the opposite of differentiation, the same process can be applied through integration but in reverse:

Acceleration → Velocity → Displacement

Applying Integration to Dynamics

Acceleration: $a(t)$

Velocity: $v(t)$

Displacement: $s(t)$

$$\int a(t)dt = v(t)$$

$$\int v(t)dt = s(t)$$

Dynamics – Ex. 1

Q. A car has an acceleration:

$$a(t) = 2 + 6t$$

The car starts from rest at time $t = 0$ from position $s = 10$. Find its position and velocity at all times t.

Find velocity first:

$$v(t) = \int a(t)\,dt$$

$$v(t) = \int (2 + 6t)dt$$

$$v(t) = 2t + 3t^2 + c$$

Dynamics – Ex. 1

We know that velocity was zero at $t = 0$. This info will help us find out what c is:

$$v(t) = 2t + 3t^2 + c$$

$$v(0) = 2(0) + 3(0)^2 + c$$

$$v(0) = c$$

Velocity was zero at $t = 0$

$$v(0) = 0$$

Thus, $c = 0$

$$v(t) = 2t + 3t^2$$

This is the velocity of the car at all times t.

Next, find the displacement of the car $s(t)$

$$s(t) = \int v(t)\,dt$$

$$s(t) = \int (2t + 3t^2)dt$$

Dynamics – Ex. 1

$$s(t) = \int 2t + 3t^2$$

$$s(t) = t^2 + t^3 + c$$

We know that the car started from rest at time $t = 0$ from position $s = 10$. This info will help us find c.

$$s(0) = (0)^2 + (0)^3 + c$$

$$s(0) = c$$

We know $s(0) = 10$

Thus, $c = 10$

$$s(t) = t^2 + t^3 + 10$$

This is the position of the car at all times t.

Dynamics – Ex. 2

Q. A boat has a velocity:

$$v(t) = 3t^2 + 4t$$

The car starts from rest at time $t = 0$ from position $s = 4$. Find its position at all times t.

$$s(t) = \int v(t)dt$$

$$s(t) = \int (3t^2 + 4t\,)dt$$

$$s(t) = \frac{3t^3}{3} + \frac{4t^2}{2} + c$$

$$s(t) = t^3 + 2t^2 + c$$

Dynamics – Ex. 2

$$s(t) = t^3 + 2t^2 + c$$

Remember: the car starts from rest at time $t = 0$ from position $s = 4$.

Thus:

$$s(0) = 4$$

$$s(t) = t^3 + 2t^2 + c$$

$$s(0) = (0)^3 + 2(0)^2 + c$$

$$s(0) = c$$

We know: $s(0) = 4$

$$c = 4$$

$$s(t) = t^3 + 2t^2 + 4$$

Dynamics – Ex. 3

Q. When an object is dropped, its acceleration (ignoring air resistance) is constant at 9.8 m/s^2

$$a(t) = -9.8$$

The negative sign is used because the object is falling.

Suppose an object is thrown from the 381 metre high rooftop of the Empire State Building in New York. If the initial velocity of the object is $-6\ m/s$, find out how long it takes the object to hit the ground and the velocity at which the object is travelling just before it hits the ground.

Dynamics – Ex. 3

$$a(t) = -9.8$$

To find the velocity of the object we must integrate the acceleration:

$$v(t) = \int a(t)\,dt$$

$$v(t) = \int (-9.8)\,dt$$

$$v(t) = -9.8t + c$$

We know the initial velocity was $-6\ m/s$ so we can find “c” using this information:

$$v(0) = -9.8(0) + c$$

$$v(0) = c$$

But we know:

$$v(0) = -6$$

Thus

$$c = -6$$

$$v(t) = -9.8t - 6$$

Dynamics – Ex. 3

$v(t) = -9.8t - 6$

To find the distance of the object from the ground, we must integrate the velocity:

$$s(t) = \int v(t)\,dt$$

$$s(t) = \int (-9.8t - 6)\,dt$$

$$s(t) = -4.9t^2 - 6t + c$$

We know the initial distance from the ground was 381 metres so we can find "c" using this information:

$$s(0) = -9.8(0)^2 - 6(0) + c$$

$$s(0) = c$$

$$s(0) = 381$$

Thus

$$c = 381$$

$$s(t) = -4.9t^2 - 6t + 381$$

Dynamics – Ex. 3

When the object hits the ground, its distance from the ground will be zero:

$$s(t) = -4.9t^2 - 6t + 381$$

$$-4.9t^2 - 6t + 381 = 0$$

Solve for t using quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(-4.9)(381)}}{2(-4.9)}$$

$$= \frac{6 \pm \sqrt{7503.6}}{-9.8}$$

$$t = -9.45 \text{ or } 8.23$$

We take our value for time as 8.23 seconds, because a negative value would indicate going backwards in time.

The object takes 8.23 seconds to reach the ground.

Dynamics – Ex. 3

To find the velocity of the object just before it hits the ground, substitute 8.23 instead of *t* in the equation for the velocity of the object as it falls:

$$v(t) = -9.8t - 6$$

$$v(8.23) = -9.8(8.23) - 6$$

$$v(8.23) = -86.654$$

This indicates that the object was travelling at a velocity of 86.654 m/s just as it hit the ground.

This is equivalent to a velocity of 311.95 km/h

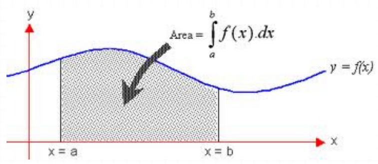
The minus indicates that it was moving in a downward direction i.e. towards the ground.

Note: air resistance would alter this velocity considerably.

The Definite Integral

To determine the area under a curve $f(x)$ between $x = a$ and $x = b$ we use the definite integral:

$$\int_a^b f(x)\,dx$$



Definite Integral – Ex. 1

Evaluate:

$$\int_0^2 x^3 + 1$$

$$= \left[\frac{x^4}{4} + x\right]_0^2$$

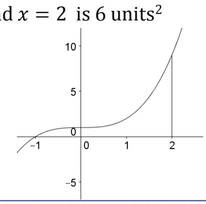
Sub in the limits 2 and 0

$$= \left(\frac{(2)^4}{4} + 2\right) - \left(\frac{(0)^4}{4} + 0\right)$$

$$= 6 - 0$$

$$= 6$$

The area under the curve $f(x) = x^3 + 1$ between $x = 0$ and $x = 2$ is 6 units^2



Definite Integral – Ex. 1

Note: when the function was integrated, the "c" that is normally added on was left out.

The reason for this is that when calculating the definite integral, "c" will be cancelled out every time:

$$\int_0^2 (x^3+1)dx$$

$$= \left[\frac{x^4}{4} + x + c\right]_0^2$$

Sub in the limits 2 and 0

$$= \left(\frac{(2)^4}{4} + 2 + c\right) - \left(\frac{(0)^4}{4} + 0 + c\right)$$

$$= 6 + c - 0 - c = 6$$

"c" always cancels so we just leave it out at the start.

Definite Integral – Ex. 2

Find the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = \pi$

Solution:

$$\int_0^{\pi} \sin x \, dx$$

$$= [-\cos x]_0^{\pi}$$

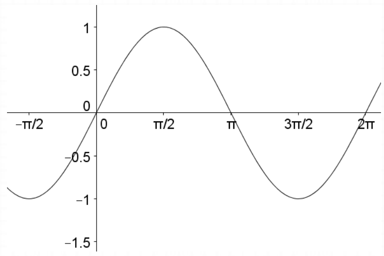
$$= (-\cos \pi) - (-\cos 0)$$

$$= (1) - (-1)$$

$$= 2$$

The area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = \pi$ is 2 units^2

Definite Integral – Ex. 2



Definite Integral – Ex. 3

Find the area between $f(x) = \sin x$ and the x axis between $x = \pi$ and $x = 2\pi$

Solution:

$$\int_{\pi}^{2\pi} \sin x \, dx$$

$$= [-\cos x]_{\pi}^{2\pi}$$

$$= (-\cos 2\pi) - (-\cos \pi)$$

$$= (-1) - (1)$$

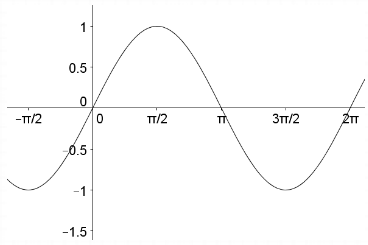
$$= -2$$

Because it is an area that we are calculating, we ignore the minus.

The area between $f(x) = \sin x$ and the x axis between $x = \pi$ and $x = 2\pi$ is 2 units^2

Definite Integral – Ex. 3

When the area is below the x axis, the calculation of the area between the curve and the x axis will produce a negative value as it did in this example.



Definite Integral – Ex. 4

Find the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$

Solution:

$$\int_0^{2\pi} \sin x \, dx$$

$$= [-\cos x]_0^{2\pi}$$

$$= (-\cos 2\pi) - (-\cos 0)$$

$$= (-1) - (-1)$$

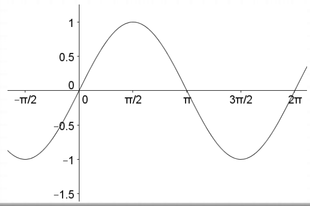
$$= 0$$

This suggest that there is no area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$

But if we look at the graph of $f(x) = \sin x$ we can see that this is obviously not true…

Definite Integral – Ex. 4

The positive area between $x = 0$ and $x = \pi$ was cancelled out by the negative area between $x = \pi$ and $x = 2\pi$. As such, the area between the curve and the x axis above the x axis needs to be calculated separately to the area between the curve and the x axis below the x axis, then combined to find the full area.



Definite Integral – Ex. 4

In Example 2, we found:

$$\int_0^{\pi} \sin x \, dx = 2 \text{ units}^2$$

In Example 3, we found:

$$\int_{\pi}^{2\pi} \sin x \, dx = 2 \text{ units}^2$$

This means that the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$ is:

$$2 + 2 = 4 \text{ units}^2$$

Definite Integral – Ex. 5

Find the area between $f(x) = x^2 + 4x$ and the x axis between $x = -4$ and $x = 3$

Solution:

We must check if the curve crosses the x axis and how that might affect our calculations.

$$f(x) = x^2 + 4x$$

Crosses x axis when $y = 0$

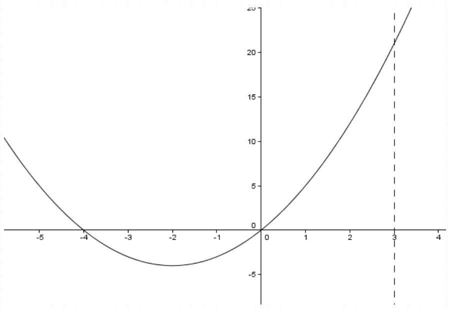
$$0 = x^2 + 4x$$

$$x(x+4) = 0$$

$$x = 0 \qquad or \qquad x = -4$$

Crosses x axis at 0 and −4

Definite Integral – Ex. 5



Definite Integral – Ex. 5

As such, we need to find the area below the x axis and the area above the x axis separately.

We'll start with the area below the x axis:

$$\int_{-4}^{0} (x^2 + 4x) dx$$

$$= \left[\frac{x^3}{3} + 2x^2\right]_{-4}^{0}$$

$$= \left(\frac{(0)^3}{3} + 2(0)^2\right) - \left(\frac{(-4)^3}{3} + 2(-4)^2\right)$$

$$= 0 - \left(-\frac{64}{3} + 32\right)$$

$$= -\frac{32}{3}$$

The area is $\frac{32}{3}$ units^2

Definite Integral – Ex. 5

Next we need to find the area above the x axis:

$$\int_0^3 (x^2 + 4x) dx$$

$$= \left[\frac{x^3}{3} + 2x^2\right]_0^3$$

$$= \left(\frac{(3)^3}{3} + 2(3)^2\right) - \left(\frac{(0)^3}{3} + 2(0)^2\right)$$

$$= (9 + 18) - (0)$$

$$= 27 \text{ units}^2$$

The area above the x axis is 27 units^2

Definite Integral – Ex. 5

To find the area between $f(x) = x^2 + 4x$ and the x axis between $x = -4$ and $x = 3$

We add the two areas we found:

$$Total\ Area = \frac{32}{3} + 27 = 37\frac{2}{3}$$

Answer: Area is $37\frac{2}{3}$ units^2

Definite Integrals – Ex. 6

Q. Calculate the following:

$$\int_0^1 \frac{4x^3}{x^4+1}\,dx$$

Integrating the quotient of two functions requires us to use substitution.

Let $u = x^4 + 1$

$$\frac{du}{dx} = 4x^3$$

$$\frac{du}{4x^3} = dx$$

Now, we must also change the limits:

$x = 1 \qquad u = (1)^4 + 1 = 2$

$x = 0 \qquad u = (0)^4 + 1 = 1$

Definite Integrals – Ex. 6

$$\int_0^1 \frac{4x^3}{x^4+1}\,dx$$

$$= \int_1^2 \frac{4x^3}{u}\frac{du}{4x^3}$$

$$= \int_1^2 \frac{1}{u}\,du$$

$$= [ln|u|]_1^2$$

$$= ln|2| - ln|1|$$

$$= 0.693 - 0$$

$$= 0.693$$

Definite Integrals – Ex. 7

Q. Evaluate:

$$\int_0^{\frac{\pi}{2}} \cos^4 x \sin x\,dx$$

Solution:
Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$\frac{du}{-\sin x} = dx$$

Now, we must also change the limits:

$x = \frac{\pi}{2} \qquad u = \cos\left(\frac{\pi}{2}\right) = 0$

$x = 0 \qquad u = \cos(0) = 1$

Definite Integrals – Ex. 7

$$\int_0^{\frac{\pi}{2}} \cos^4 x \sin x\,dx$$

$$= \int_1^0 u^4 \sin x \frac{du}{-\sin x}$$

$$= \int_1^0 -u^4\,du$$

$$= \left[-\frac{u^5}{5}\right]_1^0$$

$$= \left(-\frac{(0)^5}{5}\right) - \left(-\frac{(1)^5}{5}\right)$$

$$= 0 - \left(-\frac{1}{5}\right)$$

$$= \frac{1}{5}$$

Dynamics & Def. Integral – Example

Q. Find the distance travelled $s(t)$ by a car in metres with velocity $v(t) = t^3 + 2$ in the first 4 seconds from take off.

Solution:

$$s(t) = \int v(t)dt$$

$$s(t) = \int_0^4 (t^3 + 2)dt$$

$$s(t) = \left[\frac{t^4}{4} + 2t\right]_0^4$$

$$s(t) = \left(\frac{(4)^4}{4} + 2(4)\right) - \left(\frac{(0)^4}{4} + 2(0)\right)$$

$$s(t) = (64 + 8) - 0 = 72$$

The car travelled 72 metres in the first 4 seconds.

Explaining Integration

If we know the rate at which a quantity is changing, we can find the total change over a period of time by integrating.

For example, if we know the equation for the velocity of an object, which is the rate of change in displacement over time, then integrating it will give us the displacement over a certain period of time.

Another example: If we know the increase in length per year of the horn of a bighorn ram then integrating the function that represents this growth will allow us to calculate the total growth of the horn over a number of years.

Rams

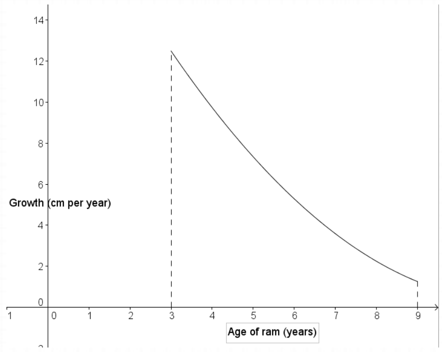
The average annual increase in the horn length (in cm) of bighorn rams can be approximated by

$$y = 0.1762x^2 - 3.986x + 22.68$$

Where x is the ram's age in years for $3 \leq x \leq 9$

Integrate this function to find the total increase in the length of a ram's horn during this time.

Rams



Rams

$$y = 0.1762x^2 - 3.986x + 22.68$$

Total growth between the ages of 3 and 9:

$$\int_3^9 0.1762x^2 - 3.986x + 22.68$$

$$= \frac{0.1762x^3}{3} - \frac{3.986x^2}{2} + 22.68x \Big|_3^9$$

$$= 0.0587x^3 - 1.993x^2 + 22.68x|_3^9$$

Rams

$$= 0.0587x^3 - 1.993x^2 + 22.68x|_3^9$$

$$= [0.0587(9)^3 - 1.993(9)^2 + 22.68(9)]$$
$$- [0.0587(3)^3 - 1.993(3)^2 + 22.68(3)]$$

$$= 85.4793 - 51.6879$$

$$= 33.79$$

The total increase in the length of a ram's horn between the ages of 3 and 9 was 33.79 cm

Area Problems – Ex. 1

Q. Find the area enclosed by the curve $f(x) = x^2 - x$ and the x axis.

Solution:

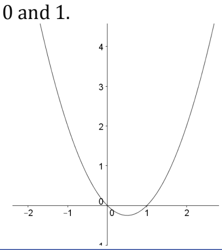
First, find where the curve cuts the x axis:

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0 \quad or \quad x = 1$$

$f(x) = x^2 - x$ cuts the x axis at 0 and 1.



Area Problems – Ex. 1

This gives us the limits of our definite integral. To find the area between the curve and the x axis, we must calculate the following:

$$\int_0^1 (x^2 - x)\, dx$$

$$\int_0^1 (x^2 - x)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

$$= \left(\frac{(1)^3}{3} - \frac{(1)^2}{2}\right) - \left(\frac{(0)^3}{3} - \frac{(0)^2}{2}\right)$$

$$= -\frac{1}{6}$$

Area Problems – Ex. 1

$$\int_0^1 (x^2 - x)\,dx = -\frac{1}{6}$$

This means that the area enclosed by the curve $f(x) = x^2 - x$ and the x axis is $\frac{1}{6}$ $units^2$

Area Problems – Ex. 2

Q. Find the area enclosed by the curve $f(x) = x^3 - 4x$ and the x axis.

Solution:

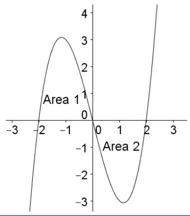
First, find where the curve cuts the x axis:

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$x = 0$ | $x^2 = 4$, $x = \pm 2$

Curve cuts the x axis at -2, 0 and 2



Area Problems – Ex. 2

We must find Area 1 and Area 2 separately.

Area 1:

$$\int_{-2}^{0} (x^3 - 4x)\,dx$$

$$\left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}$$

$$= \left(\frac{(0)^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right)$$

$$= 0 - (-4)$$

$$= 4\ units^2$$

Area Problems – Ex. 2

Area 2:

$$\int_0^2 (x^3 - 4x)\,dx$$

$$\left[\frac{x^4}{4} - 2x^2\right]_0^2$$

$$= \left(\frac{(2)^4}{4} - 2(2)^2\right) - \left(\frac{(0)^4}{4} - 2(0)^2\right)$$

$$= -4$$

Area 1: 4 $units^2$

Area 2: 4 $units^2$

Total Area = 8 $units^2$

The area enclosed by the curve $f(x) = x^3 - 4x$ and the x axis is 8 $units^2$

Improper Integrals

Improper Integrals are integrals involving infinity.

Example: The current flowing into a capacitor at time t is e^{-t} amps.

Find the total charge (as $t \to \infty$) which builds up on the capacitor which is initially uncharged.

Improper Integrals – Ex. 1

Solution:

Find the total charge (q) by integrating the function representing current ($i(t)$)

$$q = \int_0^{\infty} e^{-t}\,dt$$

$$q = \left[-e^{-t}\right]_0^{\infty}$$

$$q = (-e^{-\infty}) - (-e^{0})$$

$$q = \left(-\frac{1}{e^{\infty}}\right) - (-1)$$

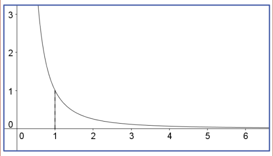
$$q = \left(-\frac{1}{\infty}\right) + 1$$

$$q = 0 + 1 = 1$$

The total charge is 1 coulomb.

Improper Integrals – Ex. 2

Q. Find the area enclosed between $y = \frac{1}{x^2}$ and the x axis for $x \geq 1$



Solution: Evaluate

$$\int_1^\infty \frac{1}{x^2}\,dx$$

$$= \int_1^\infty x^{-2}\,dx$$

$$= \left[\frac{x^{-1}}{-1}\right]_1^\infty$$

Improper Integrals – Ex. 2

$$= \left[\frac{x^{-1}}{-1}\right]_1^\infty$$

$$= \left[\frac{-1}{x}\right]_1^\infty$$

$$= \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{1}\right)$$

$$= 0 + 1 = 1$$

The area enclosed between $y = \frac{1}{x^2}$ and the x axis for $x \geq 1$ is $1\ unit^2$

Finding the Area between two curves

To find the area between two curves $g(x)$ and $f(x)$ with points of intersection at $x = a$ and $x = b$, we use the following approach:

$$\int_a^b [g(x) - f(x)]\,dx$$

Area between two curves – Ex. 1

Find the area of the region bounded by the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$

Solution:

1. Sketch the curve and the line then find the points of intersection of the curve and the line.
2. Use integration methods to find the area between the line and curve:

$$\int_a^b [g(x) - f(x)]\,dx$$

Area between two curves – Ex. 1

$y = -x^2 + 5x - 4$

x intercepts:

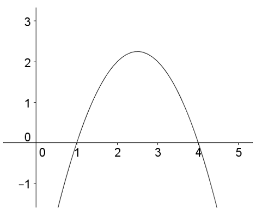
$$-x^2 + 5x - 4 = 0$$

$$x^2 - 5x + 4 = 0$$

$$(x-4)(x-1) = 0$$

$$x = 4 \qquad x = 1$$

Curve cuts x axis at 1 and 4.



Area between two curves – Ex. 1

Find the points of intersection of the line and the curve using simultaneous equations:

$$y = -x^2 + 5x - 4$$
$$y = x - 1$$

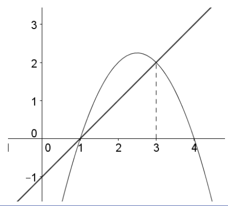
$$-x^2 + 5x - 4 = x - 1$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

$$x = 3 \qquad x = 1$$

The line and the curve intersect at $x = 3$ and at $x = 1$



Area between two curves – Ex. 1

We now know the equation for the two curves:

$$y = -x^2 + 5x - 4$$
$$y = x - 1$$

And the x values at which they intersect: 1 and 3.

So we can apply:

$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_1^3 [(-x^2 + 5x - 4) - (x - 1)]dx$$

$$\int_1^3 (-x^2 + 4x - 3)dx$$

$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$

Area between two curves – Ex. 1

$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$

$$= \left(-\frac{(3)^3}{3} + 2(3)^2 - 3(3)\right) - \left(-\frac{(1)^3}{3} + 2(1)^2 - 3(1)\right)$$

$$= (-9 + 18 - 9) - \left(-\frac{1}{3} - 1\right)$$

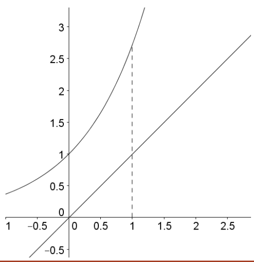
$$= 0 - \left(-\frac{4}{3}\right)$$

$$= \frac{4}{3}\ units^2$$

The area of the region bounded by the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$ is $\frac{4}{3}\ units^2$

Area between two curves – Ex. 2

Q. Find the area between $y = e^x$ and $y = x$ for $0 \leq x \leq 1$



$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_0^1 (e^x - x)\, dx$$

$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$

Area between two curves – Ex. 2

$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$

$$= \left(e^1 - \frac{(1)^2}{2}\right) - \left(e^0 - \frac{(0)^2}{2}\right)$$

$$= e - \frac{1}{2} - 1$$

$$= 1.218\ units^2$$

The area between $y = e^x$ and $y = x$ for $0 \leq x \leq 1$ is

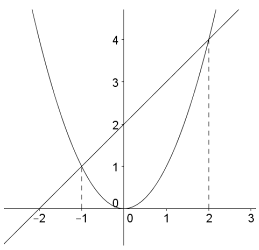
$1.218\ units^2$

Area between two curves – Ex. 3

Find the area between $y = x^2$ and $y = x + 2$

Solution:

Sketch the curves and find where they intersect.



Area between two curves – Ex. 3

Use simultaneous equations to find where they intersect:

$$y = x^2$$
$$y = x + 2$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2 \qquad x = -1$$

The curve and the line intersect at $x = 2$ and $x = -1$

Now we can apply:

$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_{-1}^2 [(x^2) - (x + 2)\ dx$$

Area between two curves – Ex. 3

$$\int_{-1}^{2} [(x^2) - (x+2)\ dx$$

$$\int_{-1}^{2} (x^2 - x - 2)\ dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}$$

$$= \left(\frac{(2)^3}{3} - \frac{(2)^2}{2} - 2(2)\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2} - 2(-1)\right)$$

$$= \left(\frac{8}{3} - 6\right) - \left(-\frac{5}{6} + 2\right)$$

$$= -4\frac{1}{2}$$

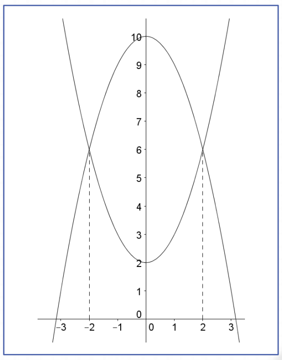
$\text{Area} = 4\frac{1}{2}\ \text{units}^2$

Area between two curves – Ex. 4

Find the area enclosed by the curves $y = x^2 + 2$ and $y = 10 - x^2$

Solution:

Find the points at which the curves intersect.



Area between two curves – Ex. 4

Point of Intersection:

$y = x^2 + 2$

$y = 10 - x^2$

$x^2 + 2 = 10 - x^2$

$2x^2 = 8$

$x = \pm 2$

Curves intersect at $x = 2$ and at $x = -2$

Now we can find the area between the curves using:

$$\int_{a}^{b} [g(x) - f(x)]\ dx$$

$$\int_{-2}^{2} [(x^2 + 2) - (10 - x^2)]\ dx$$

Area between two curves – Ex. 4

$$\int_{-2}^{2} [(x^2 + 2) - (10 - x^2)]\ dx$$

$$= \int_{-2}^{2} (2x^2 - 8) dx$$

$$= \left[\frac{2x^3}{3} - 8x\right]_{-2}^{2}$$

$$= \left(\frac{2(2)^3}{3} - 8(2)\right) - \left(\frac{2(-2)^3}{3} - 8(-2)\right)$$

$$= \left(\frac{16}{3} - 16\right) - \left(-\frac{16}{3} + 16\right)$$

$$= -21\frac{1}{3}$$

$\text{Area} = 21\frac{1}{3}\mathit{units}^2$

Finding the Volume of a Sphere

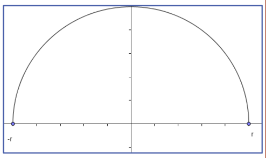
We can use integration to find the volume of objects.

When calculating the volume of a solid generated by revolving a region bounded by a given function about an axis, follow these steps…

1. Sketch the area and determine the axis of revolution, (this determines the variable of integration).
2. Sketch the cross-section, (disk, shell, washer) and determine the appropriate formula.
3. Determine the boundaries of the solid,
4. Set up the definite integral, and integrate.

Finding the Volume of a Sphere

Sketch the area:



The centre is (0, 0) and we will call the radius r, so the equation of this circle would be $x^2 + y^2 = r^2$

We know that to find the volume of a solid generated by revolving a region bounded by a given function about the x-axis, we use the following:

$$\pi \int_{a}^{b} y^2\ dx$$

Where a and b are the points of intersection of the curve and the x-axis.

Finding the Volume of a Sphere

If we apply this to our graph then

$$\pi \int_a^b y^2\,dx$$

Becomes

$$\pi \int_{-r}^{r} r^2 - x^2\,dx$$

Complete our integration:

$$r^2x - \frac{x^3}{3}\Big|_{-r}^{r}$$

$$= \left[r^2(r) - \frac{(r)^3}{3}\right] - \left[r^2(-r) - \frac{(-r)^3}{3}\right]$$

$$= r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}$$

$$= 2r^3 - \frac{2r^3}{3} = \frac{4r^3}{3}$$

Finding the Volume of a Sphere

So

$$\int_{-r}^{r} r^2 - x^2\,dx = \frac{4r^3}{3}$$

Thus

$$\pi \int_{-r}^{r} r^2 - x^2\,dx = \frac{4\pi r^3}{3}$$

This is the volume of a sphere:

$$\frac{4\pi r^3}{3}$$

Integration by Parts

- Used if the substitution method does not work.
- Typically used for the following types of integrals:

$$\int x\cos x\,dx$$

$$\int xe^x dx$$

$$\int x\ln x\,dx$$

Integration by Parts

The formula (can be found in log tables):

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts – Ex. 1

Q. Evaluate:

$$\int x\cos x\,dx$$

$$\int u\,dv = uv - \int v\,du$$

First, identify u and dv, then use them to find v and du

We can see that u corresponds to x and dv corresponds to $\cos x\,dx$

Now, find du and v

$$u = x$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Integration by Parts – Ex. 1

$$dv = \cos x\,dx$$

$$\int dv = \int \cos x\,dx$$

$$v = \sin x$$

Now we know:

$$u = x$$
$$du = dx$$
$$dv = \cos x\,dx$$
$$v = \sin x$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos x\,dx$$

$$= (x)(sinx) - \int \sin x\,dx$$

$$= x\sin x - (-\cos x) + c$$

$$= x\sin x + \cos x\ + c$$

Integration by Parts – Ex. 2

Q. Evaluate:

$$\int_0^1 xe^x dx$$

$$\int u\,dv = uv - \int v\,du$$

$u = x$ | $dv = e^x dx$

$\frac{du}{dx} = 1$ | $v = e^x$

$du = dx$

$$\int u\,dv = uv - \int v\,du$$

$$\int xe^x dx = x(e^x) - \int e^x dx$$

$$= xe^x - e^x + c$$

$$\int_0^1 xe^x dx = [xe^x - e^x]_0^1$$

Integration by Parts – Ex. 2

$$= [xe^x - e^x]_0^1$$

$$= (1(e^1) - e^1) - (0(e^0) - e^0)$$

$$= (0) - (-1)$$

$$= 1$$

$$\int_0^1 xe^x dx = 1$$

Integration by Parts – Ex. 3

Q. Evaluate:

$$\int_{-\pi}^{\pi} x \sin x \, dx$$

$$\int u\,dv = uv - \int v\,du$$

$u = x$ | $dv = \sin x \, dx$

$\frac{du}{dx} = 1$ | $v = -\cos x$

$du = dx$

$$\int u\,dv = uv - \int v\,du$$

$$\int x \sin x \, dx$$

$$= x(-\cos x) - \int (-\cos x) dx$$

$$= -x \cos x + \sin x + c$$

Integration by Parts – Ex. 3

$$\int_{-\pi}^{\pi} x \sin x \, dx$$

$$= [-x \cos x + \sin x]_{-\pi}^{\pi}$$

$$= (-\pi \cos \pi + \sin \pi) - (-(-\pi)\cos(-\pi) + \sin(-\pi))$$

$$= (-\pi(-1) + 0) - (\pi(-1) + 0)$$

$$= 2\pi$$

$$\int_{-\pi}^{\pi} x \sin x \, dx = 2\pi$$

Steps for Integrating by Parts

1. Identify u and dv
2. Differentiate u to get a value for du
3. Integrate dv to get a value for v
4. Apply the values found to the equation:

$$\int u\,dv = uv - \int v\,du$$

5. Solve the equation and simplify your answer.