Mathematical Functions

Functions are used to describe the rules which define the ways in which a *change* occurs.

This rule operates on an input to produce an output.

Converting Degrees Fahrenheit to Degrees Celsius

$$C = \frac{5}{9}(F - 32)$$

Input: 95°F

Function: $C = \frac{5}{9}(95 - 32) = \frac{5}{9}(63) = 35°\text{C}$

Output: 35°C

The Domain

The Domain of a function is the set of all possible input values.

Basically, this is the set of input values or x values which will produce a defined answer or output.

Rules for the Domain of a Function

To determine the Domain of a function, we use the following two conditions:

1. The denominator of a function cannot equal zero.
2. The Radicand (value inside a square root) must be positive or equal zero.

Domain Example 1

Find the domain of the function $f(x) = \sqrt{x - 10}$

Solution: The Radicand cannot be negative.

$$x - 10 \geq 0$$

$$x \geq 10$$

Domain: $[10, \infty)$

Domain Example 2

Find the domain of the function $f(x) = \sqrt{3x+5}$

Solution: The Radicand cannot be negative.

$$3x + 5 \geq 0$$

$$3x \geq -5$$

$$x \geq -\frac{5}{3}$$

Domain: $[-\frac{5}{3}, \infty)$

Domain Example 3

Find the domain of the function $f(x) = \frac{3}{2x+6}$

Solution: The denominator $\neq 0$

$$2x + 6 \neq 0$$

$$2x \neq -6$$

$$x \neq -3$$

Domain: $(-\infty, -3) \cup (-3, \infty)$

Domain Example 4

Find the domain of the function $f(x) = \frac{-7}{2-3x}$

Solution: The denominator $\neq 0$

$$2 - 3x \neq 0$$

$$-3x \neq -2$$

$$x \neq \frac{2}{3}$$

Domain: $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Domain Example 5

Find the domain of the function $f(x) = \frac{3}{\sqrt{3x-12}}$

Solution: Here we have both a denominator and a square root. Firstly, the Radicand cannot be negative:

$$3x - 12 \geq 0$$

$$x \geq 4$$

Domain Example 5 Continued

Secondly, the denominator $\neq 0$

$$\sqrt{3x-12} \neq 0$$

$$3x - 12 \neq 0 \qquad \text{[Square both sides]}$$

$$3x \neq 12$$

$$x \neq 4$$

Domain Example 5 Continued

Now we have two conditions:

$$x \geq 4 \quad \text{and} \quad x \neq 4$$

These two *combine* to produce the overall condition:

$$x > 4$$

Domain: $(4, \infty)$

Domain Example 6

Find the domain of the function $f(x) = 7 + \sqrt{x^2 - x - 2}$

Solution: The Radicand cannot be negative.

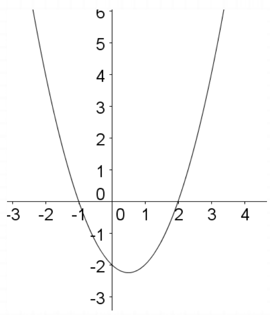
$$x^2 - x - 2 \geq 0$$

$$(x-2)(x+1) \geq 0$$

Roots are $x = 2$ and $x = -1$

Sketch the graph of the function to check where it is ≥ 0

Domain Example 6



When is $x^2 - x - 2 \geq 0$?

Domain Example 6

It is clear from the graph that $x^2 - x - 2 \geq 0$ when:

$$x \leq -1 \text{ and } x \geq 2$$

This produces the following domain:

$$(-\infty, -1] \cup [2, \infty)$$

The Domain – why can't the radicand be negative?

Example: $f(x) = \sqrt{x}$

If we let $x = -1$ then:

$$f(-1) = \sqrt{-1} = i$$

Thus, the output value is an *imaginary number* so it is not defined on the Cartesian plane.

Therefore, our x values for this particular function cannot be negative. Domain: $[0, \infty)$

The Domain – why can't the denominator equal zero?

Example: $f(x) = \frac{1}{x}$

If we let $x = 0$ then:

$$f(x) = \frac{1}{0} = \infty$$

Again, this is not a defined point on the Cartesian plane, thus we cannot include 0 in our domain for this function.

Domain: $(-\infty, 0) \cup (0, \infty)$

The Range

The Range of a function is the set of all possible output values.

In other words, it is the range of possible answers you can obtain from a particular function.

How can we determine the Range?

1. First analyse whether it's possible to obtain answers that are positive, negative, and/or equal to zero.
   $+, -, 0$

2. Analyse the function for any specific features which will affect the type of answer obtained.

Range Example 1

Find the range of the function $f(x) = \sqrt{x - 10}$

Domain: $[10, \infty)$

Solution: $+$ answers: Yes
$-$ answers: No
Can it equal zero: Yes
Anything else?: No.

Range: $[0, \infty)$

Range Example 2

Find the range of the function $f(x) = \sqrt{3x + 5}$

Domain: $[-\frac{5}{3}, \infty)$

Solution: $+$ answers: Yes
$-$ answers: No
Can it equal zero: Yes
Anything else?: No.

Range: $[0, \infty)$

Range Example 3

Find the range of the function $f(x) = \frac{3}{2x+6}$

Domain: $(-\infty, -3) \cup (-3, \infty)$

Solution: $+$ answers: Yes
$-$ answers: Yes
Can it equal zero: No
Anything else?: No.

Range: $(-\infty, 0) \cup (0, \infty)$

Range Example 4

Find the range of the function $f(x) = \frac{-7}{2-3x}$

Domain: $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$

Solution: $+$ answers: Yes
$-$ answers: Yes
Can it equal zero: No
Anything else?: No.

Range: $(-\infty, 0) \cup (0, \infty)$

Range Example 5

Find the range of the function $f(x) = \frac{3}{\sqrt{3x-12}}$

Domain: $(4, \infty)$

Solution: $+$ answers: Yes
$-$ answers: No
Can it equal zero: No
Anything else?: No.

Range: $(0, \infty)$

Range Example 6

Find the range of the function $f(x) = 7 + \sqrt{x^2 - x - 2}$

Domain: $(-\infty, -1] \cup [2, \infty)$

Solution: $+$ answers: Yes
$-$ answers: No
Can it equal zero: No
Anything else?: All answers will be ≥ 7

Range: $[7, \infty)$

Domain and Range – Special Cases

There are certain functions that must be considered a little more carefully.

These include log, sine, cosine, and exponential functions.

Domain and Range – Special Cases – Ex. 1

Q. Find the domain and range of the following function:

$$f(x) = e^{3x}$$

Domain:
Notice that there is no <u>denominator</u> or <u>radicand</u>, so any value for x can be used within this function

Domain: $(-\infty, \infty)$

Domain and Range – Special Cases – Ex. 1

$$f(x) = e^{3x}$$

Range:

$+$ answers: Yes
$-$ answers: No
Can it equal zero: No
Anything else?: No.

Range: $(0, \infty)$

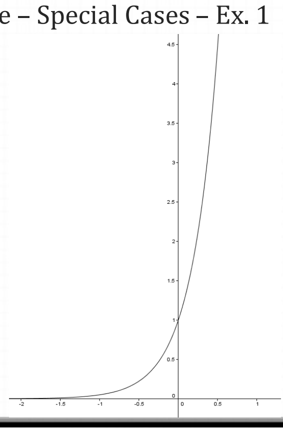
Domain and Range – Special Cases – Ex. 1

$$f(x) = e^{3x}$$

- There will be no negative answers as e raised to any power (either positive or negative) will produce a positive value.
- The answer will never be zero as e raised to any defined power will never equal zero. Remember $e^0 = 1$ so even if the power is zero this still will not produce the answer zero.

Domain and Range – Special Cases – Ex. 1

Graph of $f(x) = e^{3x}$



Domain and Range – Special Cases – Ex. 2

Q. Find the domain and range of the following function:

$$f(x) = e^{-7x}$$

Domain:

Notice that there is no denominator or radicand, so any value for x can be used within this function

Domain: $(-\infty, \infty)$

Domain and Range – Special Cases – Ex. 2

$$f(x) = e^{-7x}$$

Range:

$+$ answers: Yes

$-$ answers: No

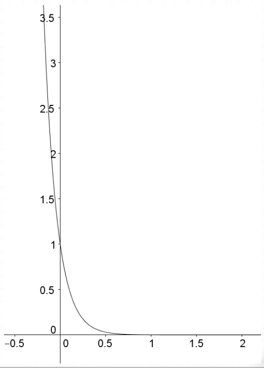
Can it equal zero: No

Anything else?: No.

Range: $(0, \infty)$

Domain and Range – Special Cases – Ex. 2

Graph of $f(x) = e^{-7x}$



Domain and Range – Special Cases – Ex. 3

Q. Find the domain and range of the following function:

$$f(x) = \log_e 3x$$

Domain:

Notice that there is no denominator or radicand BUT you cannot calculate the log of a negative number or zero therefore $3x > 0$ and thus $x > 0$

Domain: $(0, \infty)$

Why can't you find the log of a negative value?

We know:

$$\log_a c = b$$

Converts to:

$$a^b = c$$

Lets take the log of a negative number and consider what type of answer we will find:

$$\log_{10}(-100) = x$$

If we convert that to its index form:

$$\log_{10}(-100) = x$$

Converts to:

$$10^x = -100$$

So, 10 raised to what power will give us an answer of -100?

Answer: there is no such power, which means the original equation is invalid. This is why we can never find the log of a negative value.

Domain and Range – Special Cases – Ex. 3

$$f(x) = \log_e 3x$$

Range:

$+$ answers: Yes

$-$ answers: Yes

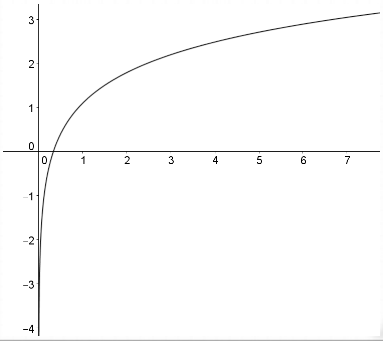
Can it equal zero: Yes

Anything else?: No.

Range: $(-\infty, \infty)$

Domain and Range – Special Cases – Ex. 3

Graph of
$f(x) = \log_e 3x$



Domain and Range – Special Cases – Ex. 4

Q. Find the domain and range of the following function:

$$f(x) = 4\sin x$$

Domain:

Notice that there is no <u>denominator</u> or <u>radicand</u> and there are no restrictions on the input value for the sine function.

Domain: $(-\infty, \infty)$

Domain and Range – Special Cases – Ex. 4

$$f(x) = 4\sin x$$

Range:

$+$ answers: Yes
$-$ answers: Yes
Can it equal zero: Yes
Anything else?: Must be between -4 and 4

Range: $[-4, 4]$

Note: The sine of any value will produce an answer between -1 and 1.

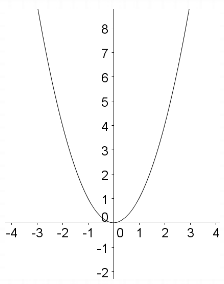
Odd and Even Functions

Determining whether a function is odd, even, or neither, will give an indication of what the graph of the function will look like.

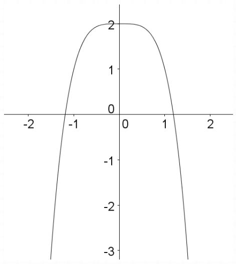
When a function is odd or even, it has a certain type of symmetry.

Even Functions

$f(x) = x^2$

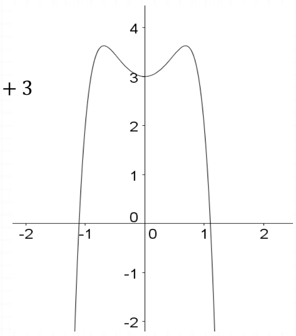


$f(x) = -x^4 + 2$



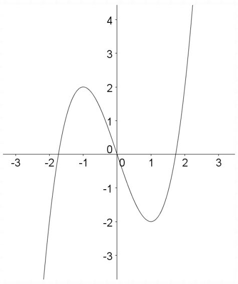
Even Function

$f(x) = -3x^6 + 2x^2 + 3$

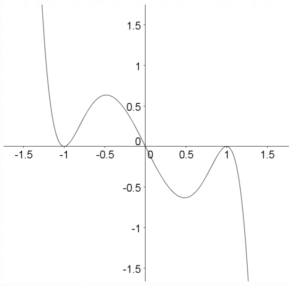


Odd Functions

$f(x) = x^3 - 3x$



$f(x) = -x^7 + 3x^3 - 2x$



When is a function even, odd, or neither?

Function is even if $f(x) = f(-x)$

Function is odd if $-f(x) = f(-x)$

Even/Odd Functions Example 1

Determine whether the following function is odd, even, or neither:

$$f(x) = 3x^2 + 7$$

Step 1: Find $f(-x)$

$$f(-x) = 3(-x)^2 + 7$$

$$f(-x) = 3x^2 + 7$$

Even/Odd Functions Example 1

$$f(x) = f(-x)$$

Thus, the function is even. If it is even then it can't be odd so no need to check.

Answer: $f(x) = 3x^2 + 7$ is an <u>even</u> function

Even/Odd Functions Example 2

Determine whether the following function is odd, even, or neither:

$$f(x) = x^3 + 5x$$

Step 1: Find $f(-x)$

$$f(-x) = (-x)^3 + 5(-x)$$

$$f(-x) = -x^3 - 5x$$

Even/Odd Functions Example 2

$$f(x) \neq f(-x)$$

Thus, the function is **not** even.

Step 2: Check if it is odd by finding $-f(x)$

$$f(x) = x^3 + 5x$$

$$-f(x) = -(x^3 + 5x)$$

Even/Odd Functions Example 2

$$-f(x) = -(x^3 + 5x)$$

$$-f(x) = -x^3 - 5x$$

We obtained the same answer for both $f(-x)$ *and* $-f(x)$

Thus $f(-x) = -f(x)$

This indicates that $f(x) = x^3 + 5x$ is an odd function

Even/Odd Functions Example 3

Determine whether the following function is odd, even, or neither:

$$f(x) = -4x^3 + 7$$

Step 1: Find $f(-x)$

$$f(-x) = -4(-x)^3 + 7$$

$$f(-x) = 4x^3 + 7$$

Even/Odd Functions Example 3

$$f(x) \neq f(-x)$$

Thus, the function is **not** even.

Step 2: Check if it is odd by finding $-f(x)$

$$f(x) = -4x^3 + 7$$

$$-f(x) = -(-4x^3 + 7)$$

Even/Odd Functions Example 3

$$-f(x) = -(-4x^3 + 7)$$

$$-f(x) = 4x^3 - 7$$

We obtained different answers for $f(-x)$ *and* $-f(x)$

Thus $f(-x) \neq -f(x)$

This indicates that $f(x) = -4x^3 + 7$ is neither odd nor even.

Even/Odd Functions - Example 4

Determine whether the following function is odd, even, or neither:

$$f(x) = x^2 \cos x$$

Step 1: Find $f(-x)$

$$f(-x) = (-x)^2 \cos(-x)$$

$$f(-x) = x^2 \cos x$$

Even/Odd Functions Example 4

$$f(x) = f(-x)$$

Thus, the function is even.

Even/Odd Functions – Example 5

Determine whether the following function is odd, even, or neither:

$$f(x) = \sin x$$

Step 1: Find $f(-x)$

$$f(-x) = \sin(-x)$$

$$f(-x) = -\sin x$$

Thus, the function is not even as $f(x) \neq f(-x)$

Even/Odd Functions – Example 5

Step 2: Check if the function is odd by finding $-f(x)$

$$f(x) = \sin x$$

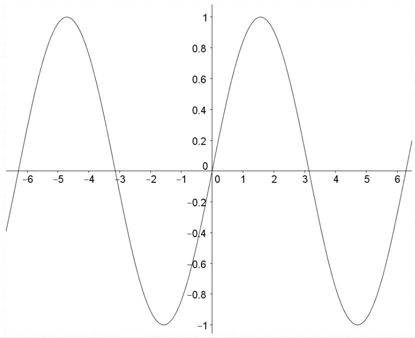
$$-f(x) = -\sin x$$

The function is odd as $-f(x) = f(-x)$

Answer: $f(x) = \sin x$ is an odd function

Even/Odd Functions – Example 5

$$f(x) = \sin x$$



Composite Functions

Composition of Functions is the process of combining two functions where one function is substituted in place of each x in the other function.

$f \circ g(x)$ is a composition of the individual functions $f(x)$ and $g(x)$

Applications of Composite Functions: http://www.youtube.com/watch?v=97P6p9Gxyw8

Composite Functions Example

Given $f(x) = x^2 + 2$ and $g(x) = \sqrt{x}$ Find $f \circ g(x)$

$$\boxed{f \circ g(x) = f(g(x)) = f(\sqrt{x})}$$

$$f(x) = x^2 + 2$$

$$f(\sqrt{x}) = (\sqrt{x})^2 + 2$$

$$f(\sqrt{x}) = x + 2$$

Answer: $f \circ g(x) = x + 2$

Composite Functions Example

Given $f(x) = x^2 + 2$ and $g(x) = \sqrt{x}$ Find $g \circ f(x)$

$$\boxed{g \circ f(x) = g(f(x)) = g(x^2 + 2)}$$

$$g(x) = \sqrt{x}$$

$$g(x^2 + 2) = \sqrt{x^2 + 2}$$

Answer: $g \circ f(x) = \sqrt{x^2 + 2}$

The Inverse of a Function

Input:
Bruce Banner

Function:
Gets angry

Output:
The Incredible Hulk

Inverse Function:
Calms Down

What does the Inverse do?

Function: $f(x) = \frac{2x+3}{5}$

It's Inverse: $f^{-1}(x) = \frac{5x-3}{2}$

If we let $x = 6$

$$f(6) = \frac{2(6)+3}{5} = 3$$

Input of 6 produces an output of 3.

What does the Inverse do?

It's Inverse: $f^{-1}(x) = \frac{5x-3}{2}$

If we take the output and use it as the input of the inverse then…

$$f^{-1}(3) = \frac{5(3)-3}{2} = 6$$

We get back to where we started. The inverse reverses the process.

Finding the Inverse: Ex. 1

Find the inverse of the function: $f(x) = 2x + 1$

$$y = 2x + 1$$

$$y - 1 = 2x$$

$$\frac{y-1}{2} = x$$

$$f^{-1}(x) = \frac{x-1}{2}$$

Steps for Finding the Inverse

1. Replace $f(x)$ with y.
2. Find x in terms of y.
3. Replace y with x. This will result in the inverse.

Finding the Inverse: Ex. 2

Find the inverse of the function: $f(x) = \sqrt{x-7}$

$$y = \sqrt{x-7}$$

$$y^2 = x - 7 \quad \text{[Square both sides]}$$

$$y^2 + 7 = x$$

$$x = y^2 + 7$$

$$f^{-1}(x) = x^2 + 7$$

Finding the Inverse: Ex. 3

Find the inverse of the function: $f(x) = \sqrt{x^2+1}$

$$y = \sqrt{x^2+1}$$

$$y^2 = x^2 + 1$$

$$y^2 - 1 = x^2$$

$$x = \pm\sqrt{y^2-1}$$

Conclusion: **No Inverse**. There can only be one solution to the inverse.

Finding the Inverse: Ex. 4

Find the inverse of the function: $f(x) = e^{3x}$

Solution: $y = e^{3x}$

$3x = log_e\, y$ [Convert to log form]

$$x = \frac{1}{3} log_e\, y$$

$$f^{-1}(x) = \frac{1}{3}\log_e x$$

Finding the Inverse: Ex. 5

Find the inverse of the function:

$$f(x) = 3\sin(2x+1)$$

Solution:

$y = 3\sin(2x+1)$

$$\frac{y}{3} = \sin(2x+1)$$

$$\sin^{-1}\left(\frac{y}{3}\right) = 2x + 1$$

$$\sin^{-1}\left(\frac{y}{3}\right) - 1 = 2x$$

$$x = \frac{1}{2}\left[\sin^{-1}\left(\frac{y}{3}\right) - 1\right]$$

$$f^{-1}(x) = \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - 1\right]$$

Blood Alcohol Percent

The number of drinks and the resulting blood alcohol percent for a man weighing 13 stone (182 lbs) is given by the table below:

No. of Drinks	3	4	5	6	7	8	9	10
Blood Alcohol %	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.21

Note: One drink is equal to 1.25 oz of 80-proof liqour, 12 oz of regular beer (just over half a pint), or 5 oz of wine.

Write the equation of the function that models the blood alcohol percent as a function of the number of drinks.

Blood Alcohol Percent

No. of Drinks	3	4	5	6	7	8	9	10
Blood Alcohol %	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.21

We can see that the percentage increases by 0.02 for every drink so the function is linear:

$$f(x) = 0.02x + 0.01$$

Where x is the number of drinks consumed.

This function is determined by finding the slope using two of the above points, then using the slope and a point on the line to find the equation of the line.

Blood Alcohol Percent

- The drink driving limit in Ireland is 0.05 percent of alcohol in the blood for drivers with a full license and 0.02 percent for learner, novice, and professional drivers.
- If a 13-stone male driver with a full license wished to find out how many drinks he would be allowed consume before he reaches the limit of 0.05 percent of alcohol in the blood, how would he do this? What about a 13-stone male learner driver?

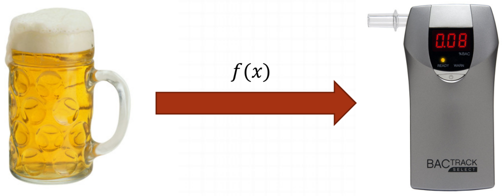
Blood Alcohol Percent

- We know we can calculate the percent of alcohol in the blood of a 13-stone man when given the number of drinks consumed by using:

$$f(x) = 0.02x + 0.01$$

- But we want to figure out the number of drinks that can be consumed when given the set percent of alcohol in the blood.

Blood Alcohol Percent



Blood Alcohol Percent

Solution: Find the inverse of the function.

$$f(x) = 0.02x + 0.01$$

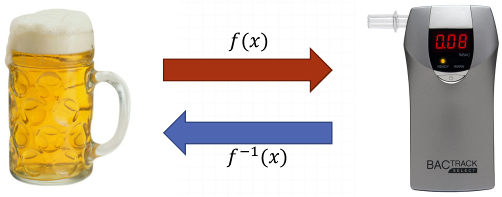
$$y = 0.02x + 0.01$$

$$0.02x = y - 0.01$$

$$x = 50y - 0.5$$

$$f^{-1}(x) = 50x - 0.5$$

Blood Alcohol Percent



Blood Alcohol Percent

$$f^{-1}(x) = 50x - 0.5$$

Now our input will be the percent of alcohol in the blood and the output will be the number of drinks.

0.05 percent of alcohol in the blood:

$$f^{-1}(0.05) = 50(0.05) - 0.5$$

$$f^{-1}(0.05) = 2.5 - 0.5 = 2$$

A 13 stone male driver with a full license could consume up to two drinks before reaching the limit of 0.05 percent of alcohol in the blood.

Blood Alcohol Percent

$$f^{-1}(x) = 50x - 0.5$$

0.02 percent of alcohol in the blood:

$$f^{-1}(0.02) = 50(0.02) - 0.5$$

$$f^{-1}(0.02) = 1 - 0.5 = 0.5$$

A 13 stone male driver with a learner license could consume up to half a standard drink before reaching the limit of 0.02 percent of alcohol in the blood.