# Integration

A function $F$ is called an **antiderivative** of the function $f$ if $F'(x) = f(x)$. The set of all antiderivatives of $f$ is called the **indefinite integral of** $f$ **with respect to** $x$ and is denoted by

$$\int f(x)dx.$$

**Example** Find $\int x^3 dx$.

The derivative of $x^4$ is $4x^3$, so if we take $F(x) = \frac{1}{4}x^4$ then $F'(x) = x^3$. Thus $F(x) = \frac{1}{4}x^4$ is an antiderivative of $f$ and

$$\int x^3 dx = \frac{1}{4}x^4 + C.$$

Every derivative gives an integral. For example, since

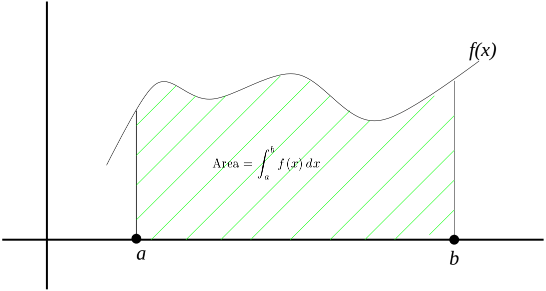
$$\frac{d}{dx}x^{n+1} = (n+1)x^n$$

we have that $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$ (as long as $n \neq -1$) and so

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

for $n \neq -1$. In particular, $\int 1\,dx = x + C$.

If $f$ is a positive-valued function on a closed interval whose end-points are $a$ and $b$, then we define the **definite integral of $f$ from $a$ to $b$**, written as $\int_a^b f(x)\,dx$, to be the area underneath the graph of $f$ between $x = a$ and $x = b$.




**Example:** Take $f(x) = 2x$. Then $\int_3^5 f(x)dx = 16$.

# The Fundamental Theorem of Calculus

Let $f$ be a function. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an anti-derivative of $f$ (that is, $F(x) = \int f(x)dx$).

We also use the notation $F(x)\Big|_a^b$ to mean $F(b) - F(a)$.

So, to find the area under the graph of $f$ between two points, you need to find an antiderivative of $f$ and calculate its values at the two points.

# Double Integrals

In the case of functions of two variables, $f(x,y)$, the integral is taken over a region of the two-dimensional plane. The simplest example is that of a rectangle, defined by the inequalities $a < x < b$, $c < y \leq d$ so we write

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dydx = \int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy$$

Geometrically, this integral represents the volume below the surface $z = f(x,y)$, bounded below by the rectangle $[a,b] \times [c,d]$.

Consider first the following example involving simple integrals:

**Example:** Calculate the integrals:

$$\int_0^1 (x^2 + y^3 + 1)\,dx \qquad \text{and} \qquad \int_0^1 (x^2 + y^3 + 1)\,dy$$

To calculate a double integral, we integrate first with respect to $y$ from $c$ to $d$ holding $x$ constant. We obtain an integral of a function of $x$ only which we calculate next. Alternatively, we could also integrate with respect to $x$ from $a$ to $b$ first and then with respect to $y$.

# Examples:

Calculate the following double integrals

$$(i)\ \int_2^4 \int_1^2 6xy^2\,dydx$$

$$(ii)\ \int_1^2 \int_0^3 (2x - 4y^3)\,dydx$$

$$(iii)\ \int_0^2 \int_1^3 (xy + x^2y^3)\,dxdy$$

$$(iv)\ \int_0^1 \int_{-2}^{-1} (\cos(\pi x) + \sin(\pi y))\,dxdy$$

The double integrals in the examples above are also called iterated integrals. When we calculate an iterated integral, the order or integration doesn't matter although sometimes it is easier to integrate in one particular order first.

**Example:** Calculate the following integral and show that it is easier to integrate with respect to $y$ first and then $x$ rather than the other way around.

$$\int_{-1}^{2}\int_{0}^{1} x\mathrm{e}^{xy}\,dydx$$

# Iterated integrals written as a product

If $f(x,y) = F(x)G(y)$ (so if $f$ can be written as a product of a function depending only on $x$ and a function depending only on $y$) then in order to evaluate the double integral of $f(x,y)$ we can evaluate the integral of $F(x)$ and the integral of $G(y)$ and then multiply the two integrals together

$$\begin{aligned} \int_a^b \int_c^d f(x,y)\,dydx &= \int_a^b \int_c^d F(x)G(y)\,dydx \\ &= \left(\int_a^b F(x)\,dx\right) \cdot \left(\int_c^d G(y)\,dy\right) \end{aligned}$$

**Examples:**

$$(i)\ \int_{-2}^{3}\int_{0}^{\frac{\pi}{2}} x\cos(y)\,dydx$$

$$(ii)\ \int_{0}^{1}\int_{0}^{1} \mathrm{e}^{x+y}\,dxdy$$

$$(iii)\ \int_{0}^{1}\int_{0}^{1} \frac{x+1}{y+1}\,dxdy$$

# Double integrals over non-rectangular regions

If $R$ is a region in the plane defined by the inequalities $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ then an integral over $R$ can be evaluated as follows

$$\iint_R f(x,y)\,dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$

If $R$ is a region in the plane defined by the inequalities $c \leq y \leq d$ and $h_1(x) \leq x \leq h_2(x)$ then an integral over $R$ can be evaluated as follows

$$\iint_R f(x,y)\,dxdy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

# Example 1: Triangle

Calculate the integral

$$\iint_R (3 - x - y)\, dxdy$$

where $R$ is the two-dimensional region bounded by the $x$-axis and the lines $y = x$ and $x = 1$.

We see that the region of integration is the triangle with vertices at the points (0,0), (1,0) and (1,1).

This region is characterised by the inequalities $0 \leq x \leq 1$ and $0 \leq y \leq x$ so the integral is written as

$$\iint_R (3-x-y)\,dxdy = \int_0^1 \int_0^x (3-x-y)\,dydx$$

The inner integral (with respect to $y$) is given by

$$\int_0^x (3-x-y)\,dy = (3y - xy - \frac{y^2}{2}) \Big|_0^x = 3x - \frac{3x^2}{2}$$

The outer integral becomes

$$\int_0^1 (3x - \frac{3x^2}{2})\,dx = (\frac{3x^2}{2} - \frac{x^3}{2}) \Big|_0^1 = 1$$

# Example 2: Quadrilateral

Integrate the function $f(x,y) = \frac{x}{y}$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$ and $x = 2$.

This region is described by the inequalities $1 \leq x \leq 2$ and $x \leq y \leq 2x$ so the integral is written as

$$\iint_R \frac{x}{y}\, dxdy = \int_1^2 \int_x^{2x} \frac{x}{y}\, dydx = \int_1^2 x \ln(y) \Big|_x^{2x} \, dx$$

$$= \int_1^2 x \ln(2)\, dx = \frac{x^2}{2} \ln(2) \Big|_1^2 = \frac{3}{2} \ln(2)$$

# Example 3: Curved region

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dydx$$

and then write an equivalent integral with the order of integration reversed.

# Double Integrals in Polar Coordinates

If $R$ is a region in the plane defined by the inequalities $a \leq \theta \leq b$ and $g_1(\theta) \leq r \leq g_2(\theta)$ then an integral over $R$ can be evaluated as follows

$$\iint_R f(r,\theta)\,dA = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

**Example:** Calculate the integral of $f(x,y) = x^2 + y^2$ over the area enclosed by the circle of radius 2 centred at the origin.

# Double integrals as areas

The area of a bounded plane region $R$ is given by the double integral

$$\text{Area} = \iint_R dxdy$$

**Example:** Find the area of the region $R$ bounded by $y = x$ and $y = x^2$ in the first quadrant.

In polar coordinates, the area is given by

$$\text{Area} = \iint_R r\, drd\theta$$

**Example:** Find the area of the region inside the circle $r = 3$ and outside the circle $r = 1$ in the third quadrant.

# Triple integrals

Triple integrals over simple (rectangular) regions in 3 dimensions are calculated by repeated integration, in the same way as double integrals.

**Examples:**

$$
\begin{aligned}
&\text{(i)} \quad \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dxdydz \\
&\text{(ii)} \quad \int_1^e \int_1^e \int_1^e \frac{1}{xyz}\, dxdydz \\
&\text{(iii)} \quad \int_0^1 \int_0^\pi \int_0^\pi y \sin(z)\, dxdydz
\end{aligned}
$$

# Calculating volumes

The volume of a closed bounded region $D$ in the space is given by

$$\iiint_D dxdydz$$

**Example:** Find the volume of the solid prism contained in the first octant, between the planes $x = 0$, $x = 2$ and $y + z = 1$.

**Answer:** The limits of integration can be determined as

$$0 \leq x \leq 2, \qquad 0 \leq y \leq 1, \quad 0 \leq z \leq 1 - y$$

so the volume is

$$V = \int_0^2 \int_0^1 \int_0^{1-y} dzdydx = 1$$

# Calculating the mass of a solid

If $\delta(x,y,z)$ is the density of an object which occupies a bounded region $D$ in space, then the mass of the object is given by

$$M = \iiint_D \delta(x,y,z)\,dxdydz$$

**Example:** Calculate the mass of the object which occupies the cube bounded by the coordinate planes and by the planes $x = 1$, $y = 1$ and $z = 1$, if the density of the object is given by the function $\delta(x,y,z) = x + y + z + 1$.