## Question 6

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix}$$

- Last Class - We found the Eigenvalues of A
- Characteristic Equation $(\lambda - 1)(\lambda + 1)(\lambda + 2) = 0$
- Eigenvalues $\lambda = \{-1, -2, 1\}$
- To find Eigenspaces
- (Tutorial 6 Question 5 is a good example for this question)

$$A = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 2/3 & 1 \\ 0 & -2/3 & -3 \end{pmatrix}$$

- Find the Eigenspaces : Solve $(\lambda I - A)e = 0$

- $(\lambda I - A)$ is computed below. First $\lambda = -2$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

- The Eigenspace weightings are the solutions to the following. (you can let $e_1 = 1$, then re-weight if necessary)

$$\begin{pmatrix} -3 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- The solution is $e_1 = 1$, $e_2 = -3$ $e_3 = 6$

- For $\lambda = -1$ and $\lambda = 1$. The matrices are

$$\begin{pmatrix} -3 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- The P matrix is therefore

$$\begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & 0 \\ 6 & -2 & 0 \end{pmatrix}$$

- Normalising the columns

$$\begin{pmatrix} 1/\sqrt{46} & -1/\sqrt{9} & 1/\sqrt{1} \\ -3/\sqrt{46} & 2/\sqrt{9} & 0/\sqrt{1} \\ 6/\sqrt{46} & -2/\sqrt{9} & 0/\sqrt{1} \end{pmatrix} = \begin{pmatrix} 0.147 & -0.333 & 1 \\ -0.442 & 0.666 & 0 \\ 0.885 & -0.666 & 0 \end{pmatrix}$$

- Normalization : divide each element by magnitude of column vector $(\sqrt{1^2 + (-3)^2 + 6^2} = \sqrt{46})$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Boundary conditions

$$X(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Linear Transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \left( \begin{pmatrix} 1/\sqrt{46} & -1/3 & 1 \\ -3/\sqrt{46} & 2/3 & 0 \\ 6/\sqrt{46} & -2/3 & 0 \end{pmatrix} \right) \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix}$$

## Question 7

$$L = \begin{pmatrix} 1 & 0 & 0 \\ X & 1 & 0 \\ Y & Z & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{pmatrix}$$

$$A = LU = \begin{pmatrix} A & B & C \\ AX & BX + D & CX + E \\ AY & BY + DZ & CY + EZ + F \end{pmatrix}$$

- A=1
- B=-3
- C=1
- $AX = 2 \therefore X = 2$
- $BX + D = -5 \therefore D = 1($ Recall$BX = -6)$
- $AY = 2 \therefore Y = 2$
- $BY + DZ = 2(BY = -6. \therefore DZ = 4 \therefore Z = 4)$
- $CX + E = 4$ Because $CX = 2$ necessarily $E = 2$
- $CY + EZ + F$ $(CY = 2$, $EZ = 8)$ therefore F=1

$$A = L \times U = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -5 & 4 \\ 2 & -2 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Ax = L \times U \times x = b$$

Required to solve for x. Break it into two steps.

- Let Ux = Y
- Let Ly = B

Firstly, do the second one $Ly = B$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 22 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ 2y_1 + y_2 \\ 2y_1 + 4y_2 + y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 22 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

Now do Ux=y to solve for x

$$\begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 - 3x_2 + x_3 \\ x_2 + 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

- Let $\|A\|$ represent the norm of A.
- The condition number is defined as $\|A\| \times \|A^{-1}\|$
- Condition number is the measure of how well conditioned a matrix is. The smaller the value of $\kappa(A)$, the more accurate the solution of Ax=b
- Inverse of A found yesterday, using Elementary Row Operations.
- Can quickly compute the inverse of L and U by the same method.
- Recall $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -5 & 4 \\ 2 & -2 & 11 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} -47 & 31 & -7 \\ -14 & 9 & -2 \\ 6 & -4 & 1 \end{pmatrix}$$

- $\|A\| = \text{Max}\{(1+|-3|+1, 2+|-5|+4, 2+|-2|+11\} = \max\{5, 11, 15\} = 15$
- $\|A^-1\| = \text{Max}\{(|-47|+31+|-7|, |-14|+9+|-2|, 6+|-4|+1\} = \max\{85, 25, 11\}$

- $\kappa(A) = 15 \times 85 = 1275$

## Q7b - Inverting a Matrix

$$\begin{pmatrix} 1 & -3 & 1 & : & 1 & 0 & 0 \\ 2 & -5 & 4 & : & 0 & 1 & 0 \\ 2 & -2 & 11 & : & 0 & 0 & 1 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix}$$

$$\begin{pmatrix} 1 & -3 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 2 & : & -2 & 1 & 0 \\ 0 & 4 & 9 & : & -2 & 0 & 1 \end{pmatrix} \begin{matrix} r4 = r1 \\ r5 = r2 - 2r1 \\ r6 = r2 = 2r1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 7 & : & -5 & 3 & 0 \\ 0 & 1 & 2 & : & -2 & 1 & 0 \\ 0 & 0 & 1 & : & 6 & -4 & 1 \end{pmatrix} \begin{matrix} r7 = r4 + 3r5 \\ r8 = r5 \\ r9 = r6 - 4r5 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & : & -47 & 31 & -7 \\ 0 & 1 & 0 & : & -14 & 9 & -2 \\ 0 & 0 & 1 & : & 6 & -4 & 1 \end{pmatrix} \begin{matrix} r10 = r7 - 7r9 \\ r11 = r9 - 2r9 \\ r12 = r9 \end{matrix}$$

## Question 4

$$a_o + a_1x + a_2x^2 = \alpha_1(1+2x) + \alpha_2(2x+x^2) + \alpha_3(4+2x-3X^2)$$

$$= (\alpha_1 + 4\alpha_3) + (2\alpha_1 + 2\alpha_2 + 2\alpha_3)x + (\alpha_2 - 3\alpha_3)x^2$$

- $a_o = \alpha_1 + 4\alpha_3$
- $a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3$
- $a_2 = \alpha_2 - 3\alpha_3$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 2 & 2 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 & 4 \\ 2 & 2 & 2 \\ 0 & 1 & -3 \end{vmatrix} = 0?\text{Yes}$$

- To obtain an orthonormal basis for an inner product space V, use the Gram-Schmidt algorithm to construct an orthogonal basis.

- Then simply normalize each vector in the basis.

$$w = \begin{bmatrix} \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \\ \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 \\ \vdots \\ \frac{\langle \mathbf{w}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \mathbf{v}_n \end{bmatrix}$$

**Part(a)**

Let $P_1$ be the space of polynomials of degree at most one. Use the Gram-Schmidt process to transform the standard basis $\{1, x\}$ for $P_1$ to an orthonormal one defined by the inner product

$$< p, q >= \sum_{i=1}^{5} p(x_i)q(x_i)$$

where $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$, $x_5 = 4$.

**Part (b)**

Let f be a function on [0; 4], taking the value $y_i = f(x_i)$ at $x = x_i$; i = 1, 2, 3, 4, 5 as given in the following table

| $y_i$ | $y_i$ |
|---|---|
| 0 | 4 |
| 1 | 3 |
| 2 | 1 |
| 3 | 0 |
| 4 | -1 |

Find the least squares approximation to f in $P_1$ using the inner product defined in part (a).

- Step 1: Let $\mathbf{v}_1 = \mathbf{u}_1$.

- Step 2: Let

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

  where $W_1$ is the space spanned by $\mathbf{v}_1$, and $\text{proj}_{W_1}\mathbf{u}_2$ is the orthogonal projection of $\mathbf{u}_2$ on $W_1$.

- Step 3 Let

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

  where $W_2$ is the space spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

- Step 4 Let

$$\mathbf{v}_4 = \mathbf{u}_4 - \text{proj}_{W_3}\mathbf{u}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$$

  where $W_3$ is the space spanned by $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

  $\vdots$

$$\mathbf{U}_1 = \frac{\mathbf{V}_1}{\|\mathbf{V}_1\|}$$

$$\mathbf{V}_1 = 1$$

$$\|\mathbf{V}_1\| = \sqrt{< \mathbf{V}_1, \mathbf{V}_1 >} = \sqrt{\sum_{i=1}^{5} \mathbf{V}_1 \times \mathbf{V}_1}$$

$$\|\mathbf{V}_1\| = \sqrt{\sum_{i=1}^{5} (1)^2} = \sqrt{5}$$

$$\mathbf{U}_1 = \frac{\mathbf{V}_1}{\|\mathbf{V}_1\|} = \frac{1}{\sqrt{5}}$$

Line 2A

$$\mathbf{U}_2 = \frac{\mathbf{V}_2 - < \mathbf{V}_2, \mathbf{U}_1 > \mathbf{U}_1}{\|\mathbf{V}_2 - < \mathbf{V}_2, \mathbf{U}_1 > \mathbf{U}_1\|}$$

Line 2B

$$< \mathbf{V}_2, \mathbf{U}_1 >= \sum_{i=1}^{5} \mathbf{V}_2 \times \mathbf{U}_1 = \sum_{i=1}^{5} \left( x_i \times \frac{1}{\sqrt{5}} \right)$$

Line 2C

$$< \mathbf{V}_2, \mathbf{U}_1 >= \frac{0}{\sqrt{5}} + \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{10}{\sqrt{5}}$$

Line 2D

$$< \mathbf{V}_2, \mathbf{U}_1 > \mathbf{U}_1 = \frac{10}{\sqrt{5}} \times \frac{1}{\sqrt{5}} = 2$$

Line 2E

$$\mathbf{V}_2 - < \mathbf{V}_2, \mathbf{U}_1 > \mathbf{U}_1 = x - 2$$

Line 2F

$$\|\mathbf{V}_2 - < \mathbf{V}_2, \mathbf{U}_1 > \mathbf{U}_1\| = \sqrt{\sum_{i=1}^{5} (x-2)^2}$$

Line 2G

$$\sqrt{\sum_{i=1}^{5} (x-2)^2} = \sqrt{(0-2)^2 + (1-2)^2 + \ldots + (4-2)^2}$$

Line 2H

$$\|\mathbf{V}_2 - <\mathbf{V}_2, \mathbf{U}_1> \mathbf{U}_1\| = \sqrt{10}$$

Line 2I

$$\mathbf{U}_2 = \frac{x-2}{\sqrt{10}}$$

Line 2J **Orthonormal Basis**

$$\left\{\frac{1}{\sqrt{5}}, \frac{x-2}{\sqrt{10}}\right\}$$

Line 3A The best fit P(x) is given by

$$P^*(x) =< f, \mathbf{U}_1 > \mathbf{U}_1 + < f, \mathbf{U}_2 > \mathbf{U}_2$$

Line 3B

$$< f, \mathbf{U}_1 >= \sum_{i=1}^{5} f(x_i)\mathbf{U}_1$$

Line 3C

$$< f, \mathbf{U}_1 >= \frac{4}{\sqrt{5}} + \frac{3}{\sqrt{5}} + \ldots + \frac{-1}{\sqrt{5}} = \frac{7}{\sqrt{5}}$$

Line 3D

$$< f, \mathbf{U}_1 > \mathbf{U}_1 = \frac{7}{\sqrt{5}} \times \frac{1}{\sqrt{5}} = 7/5 = 14/10$$

Line 3E

$$< f, \mathbf{U}_2 >= \sum_{i=1}^{5} f(x_i)\mathbf{U}_2 = \sum_{i=1}^{5} \frac{f(x_i) \times (x_i - 2)}{\sqrt{10}}$$

Line 3F

$$= \frac{4(0-2)}{\sqrt{10}} + \frac{3(1-2)}{\sqrt{10}} + \frac{1(2-2)}{\sqrt{10}} + \frac{0(3-2)}{\sqrt{10}} + \frac{-1(4-2)}{\sqrt{10}}$$

Line 3G

$$< f, \mathbf{U}_2 >= \frac{-13}{\sqrt{10}}$$

Line 3H

$$< f, \mathbf{U}_2 > \mathbf{U}_2 = \frac{-13}{\sqrt{10}} \frac{(x-2)}{\sqrt{10}} = \frac{-13x+26}{10}$$

Line 3I

$$P^*(x) = \frac{14}{10} + \frac{-13x+26}{10} = \frac{-13x+40}{10}$$