DUBLIN INSTITUTE OF TECHNOLOGY
KEVIN STREET, DUBLIN 8

B.Sc. in Mathematics (Ordinary)

Autumn Examinations 2010

Numerical Methods 2

Dr. Colum Watt
Prof. J.M. Golden
Dr. M. Khurram Wadee

Wednesday, September 1 9.30a.m. to 12.30p.m.

Attempt **FIVE** questions.
Each question carries 20 marks.

Unless stated otherwise, use at least three significant digits

Show your calculations clearly.

Graph Paper, Department of Education Mathematics Tables

1.(a) Consider the Cholesky factorisation $A = LL^T$ of a $p \times p$ matrix A.
What conditions must the matrices A and L satisfy?
In the case of a 3×3 matrix A, derive the six equations satisfied by the entries of L.
Indicate the order in which these equations should be solved. [**8 marks**]

(b) Using the equations from part (a), determine the lower triangular matrix L for which $LL^T = B$, where B is the matrix [**5 marks**]

$$B = \begin{bmatrix} 9 & 6 & -6 \\ 6 & 5 & -3 \\ -6 & -3 & 30 \end{bmatrix}$$

(c) Use your answer to part (b) to solve the following system of equations. [**7 marks**]

$$\begin{array}{rcrcrcr} 9x & + & 6y & - & 6z & = & 12 \\ 6x & + & 5y & - & 3z & = & 7 \\ -6x & - & 3y & + & 30z & = & -9 \end{array}$$

2.(a) Reorder the following system of equations so that the resulting coefficient matrix B is strictly diagonally dominant. Keep the unknowns in the order w, x, y, z. [**2 marks**]

$$\begin{array}{rcrcrcr} x & + & y & - & 3z & = & -3 \\ & & 3w & - & 2z & = & -1 \\ & & w & + & 2y & = & 2 \\ & & -2x & + & z & = & 5 \end{array}$$

(b)(i) Starting with $X_0 = [w_0, x_0, y_0, z_0]^T = [0, 0, 0, 0]^T$, perform two iterations of the Gauss-Seidel method on your reordered system from part (b).

(b)(ii) Calculate the relative percentage change from the first iterate X_1 to the second X_2. [**8 marks**]

(c) Derive the matrix formulation $X_{n+1} = (L + D)^{-1}(b - UX_n)$ for the Gauss-Seidel method for the following strictly diagonally dominant system of equations. Your solution should include a brief explanation of each of the terms X_{n+1}, X_n, L, D, U and b. [**7 marks**]

$$\begin{array}{rcrcrcr} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_1 \end{array}$$

(d) Determine whether the sequence of vectors $Y_n = \left(\frac{n}{2n+1}, \frac{(-1)^{n+1}}{n^2+3}, 3 + (-2)^{-n} \right)$ converges and, if so, determine $\lim_{n \to \infty} Y_n$. Give reasons for your answer.
[**3 marks**]

3.(a)(i) Calculate the table of divided differences for the points $(-2, 24), (-1, 2), (0, 0), (2, 8)$.

(a)(ii) Use your table from part (a)(i) to determine the Newton interpolating polynomial $p(x)$ of degree two or less for $(-2, 24), (-1, 2), (0, 0)$.

(a)(iii) Use your table from part (a)(i) to determine the Newton interpolating polynomial $q(x)$ of degree three or less for $(-2, 24), (-1, 2), (0, 0), (2, 8)$.

[**7 marks**]

(b) Assume that the four points $(-2, 24), (-1, 2), (0, 0), (2, 8)$ were obtained from a function $f(x)$ which satisfies $|f^{(n)}(x)| \leq \dfrac{135}{n^2 - 5n + 8}$ for $-2 \leq x \leq 2$ and $1 \leq n \leq 7$. Determine a bound for the error in approximating $f(-0.5)$ by $q(-0.5)$.

$$\text{Error formula:} \quad f(x) - q(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)(x - x_1)\dots(x - x_n).$$

[**6 marks**]

(c) Let $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$ denote the Newton interpolating polynomial which passes through the three points (x_0, y_0), (x_1, y_1) and (x_2, y_2). Derive formulae for a_0, a_1 and a_2 and show that a_2 can be written in the form

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

[**7 marks**]

4.(a) Determine the values of α, β, γ and δ for which $f(x)$ is a clamped cubic spline with $f'(-1) = 2$ and $f'(1) = -4$ where

$$f(x) = \begin{cases} (x+1)^3 - (x+1)^2 + \alpha(x+1) - 1 & \text{if} \quad -1 \leq x \leq 0 \\ -2x^3 + \beta x^2 + 3x + 1 & \text{if} \quad 0 \leq x \leq 1 \\ \gamma(x-1)^3 - 4(x-1)^2 + \delta(x-1) + 4 & \text{if} \quad 1 \leq x \leq 2. \end{cases}$$

[**6 marks**]

(b) Follow steps (i), (ii) and (iii) below to calculate the clamped cubic spline $g(x)$ which passes through the points $(-2, -4)$, $(0, 8)$, $(1, 10)$ and $(2, 6)$ and which satisfies the conditions $g'(-2) = 2$ and $g'(2) = 4$.

(i) Set up the system of equations (in matrix form) which determines m_0, m_1, m_2, m_3. (Note: The nodes are not equally spaced.)

(ii) Verify that the solution to the system of equations in (i) is $m_0 = 8$, $m_1 = -4$, $m_2 = -16$, and $m_3 = 32$.

(iii) Using the values from (ii), complete the construction of the spline.

[**14 marks**]

5.(a) Define the Chebyshev polynomials, $T_n(x)$, for $n \geq 0$ and for $x \in [-1, 1]$.
Show that $|T_n(x)| \leq 1$ for all $n \geq 0$ and for all $x \in [-1, 1]$.
Is $|T_n(x)| \leq 1$ for all $n \geq 0$ and for all x? Give a reason for your answer.

[**4 marks**]

(b) Prove each of the following statements.

(i) $T_{n+m}(x) + T_{n-m}(x) = 2T_n(x)T_m(x)$ for all $n \geq m \geq 0$ and $x \in [-1, 1]$.
Hint: Substitute $A = \cos^{-1}(x)$ and use a formula from trigonometry.

(ii) $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$ for all $n \geq 1$.

[**6 marks**]

(c) Calculate $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$ and $T_5(x)$ by using the definition of $T_n(x)$ and the recurrence relation in part (a)(ii).

[**5 marks**]

(d) Economise the polynomial $p(x)$ to a polynomial $q(x)$ of degree four or less and then to a polynomial $r(x)$ of degree three or less, where

$$p(x) = -3x^5 - \frac{x^4}{4} + x^3 + 4x^2 - 2x - 3.$$

Give an upper bound for the total error in each of these approximations and state the interval of values of x over which these bounds are valid. [**5 marks**]

6.(a) Suppose that (X, λ) is an eigenpair for a $p \times p$ matrix A and that s is a number. Show that $(X, \lambda - s)$ is an eigenpair for the shifted matrix B given by $B = A - sI$. [**3 marks**]

(b) Let A be the matrix

$$A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -3 & -1 \end{bmatrix}$$

Sketch the Gerschgorin discs for A. [**3 marks**]

(c) Without doing further calculations, what can be said about the eigenvalues of A? Give reasons for your answer. [**3 marks**]

(d) Let B be the matrix obtained by shifting the matrix A by the amount $s = -3$. Perform two iterations of the inverse power method to estimate the eigenvalue of B with smallest absolute value. (Use exact arithmetic in finding B^{-1}. Start with $X_0 = [1, 1, 1]^T$ when applying the power method.) [**9 marks**]

(e) Use your answer to part (d) to estimate the eigenvalue of A which is closest to -3 and give the corresponding (approximate) eigenvector. [**2 marks**]

7.(a) Starting from Simpson's 1/3 rule (given below), derive the **composite form of Simpson's 1/3-rule** for approximating integrals by subdividing the interval of integration into six equal sub-intervals. Your solution should explain the notation used.

$$\text{Simpson's 1/3 Rule:} \qquad \int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(a) + 4f(a+h) + f(b)\right)$$

Using this composite formula and at least three decimal places, approximate the following integral.

$$\int_{-3}^{3} \frac{x^2}{\sqrt{1+x^2}}\,dx$$

[**10 marks**]

(b) By making a suitable change of variables, derive the three-point Gaussian quadrature formula for approximating integrals of the form $\displaystyle\int_{-3}^{3} f(x)\,dx$. You may assume the three point Gaussian quadrature formula for the interval $[-1, 1]$ which is given by

$$\int_{-1}^{1} f(x)\,dx \approx \frac{1}{9}\left(5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right).$$

Using the three point formula derived above and at least 5 decimal places, approximate the following integral.

$$\int_{-3}^{3} \frac{x^2}{\sqrt{1+x^2}}\,dx$$

Calculate the relative percentage error in this approximation given that the correct value of this integral is 7.6683865.

[**10 marks**]

Interpolation Formulae

1. Lagrange Polynomials: Data $(x_0, y_0), \dots, (x_n, y_n)$. Lagrange interpolating polynomial $p(x)$.

$$p(x) = y_0 L_0(x) + \cdots + y_n L_n(x) \quad \text{where} \quad L_i(x) = \prod_{j \neq i} \left(\frac{x - x_j}{x_i - x_j} \right)$$

2. Divided Differences:

$$f[x_i, x_j, x_k] = \frac{f[x_j, x_k] - f[x_i, x_j]}{x_k - x_i} \quad \text{etc.}$$

3. Newton Interpolating Polynomial: Data $(x_0, y_0), \dots, (x_n, y_n)$.

$$p_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$$

4. Cubic Splines: Data $(x_0, y_0), \dots, (x_n, y_n)$. The required spline $f(x)$ is given by

$$f(x) = f_k(x) = a_k(x - x_{k-1})^3 + b_k(x - x_{k-1})^2 + c_k(x - x_{k-1}) + d_k \quad \text{for} \quad x \in [x_{k-1}, x_k]$$

where

$$a_k = \frac{m_k - m_{k-1}}{6h_k} \qquad b_k = \frac{m_{k-1}}{2} \qquad c_k = f[x_{k-1}, x_k] - h_k \frac{2m_{k-1} + m_k}{6} \qquad d_k = y_{k-1}$$

with $h_k = x_k - x_{k-1}$.

The value m_k is equal to $f''(x_k)$. The values $m_0, \dots, m_n$ satisfy the system of equations.

$$h_k m_{k-1} + 2(h_{k+1} + h_k)m_k + h_{k+1} m_{k+1} = 6(f[x_k, x_{k+1}] - f[x_{k-1}, x_k]) \quad \text{for } k = 1, \dots, n-1$$

Clamped Cubic Splines:

Here the values $f'(x_0)$ and $f'(x_1)$ are also specified. In addition to the above system of $n - 1$ linear equations, the following two equations are also satisfied.

$$\begin{aligned} 2h_1 m_0 + h_1 m_1 &= 6(f[x_0, x_1] - f'(x_0)) \\ h_n m_{n-1} + 2h_n m_n &= 6(f'(x_n) - f[x_{n-1}, x_n]) \end{aligned}$$