Calculus

Partial Fraction Expansion

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$$\int_a^b f(x)dx$$

Partial Fraction Expansion

$$\frac{2x+5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

- $x-3$ and $x+2$ are both polynomials of degree 1 (i.e. linear functions)
- Both A and B are therefore polynomials of degree 0 (i.e. constants)

Partial Fraction Expansion

- We will simplify the lefthandside of the previous equation using **cross -multiplication**.

$$\frac{A}{x-3}+\frac{B}{x+2} = \left(\frac{A(x+2)}{(x-3)(x+2)}\right)+\left(\frac{B(x-}{(x-3)(x}\right.$$

$$(3x + 5)/(12x)^2$$

can be decomposed in the form

$$\frac{3x + 5}{(1 - 2x)^2} = \frac{A}{(1 - 2x)^2} + \frac{B}{(1 - 2x)}.$$

Clearing denominators shows that 3x + 5 = A + B(1 2x).

Expanding and equating the coefficients of powers of x gives
$5 = A + B$ and $3x = 2Bx$
Solving for A and B yields $A = 13/2$ and $B = 3/2$. Hence,

$$\frac{3x+5}{(1-2x)^2} = \frac{13/2}{(1-2x)^2} + \frac{-3/2}{(1-2x)}.$$

Partial Fractions

Example 1

$$f(x) = \frac{1}{x^2 + 2x - 3}$$

Here, the denominator splits into two distinct linear factors:

$$q(x) = x^2 + 2x - 3 = (x + 3)(x - 1)$$

so we have the partial fraction decomposition

$$f(x) = \frac{1}{x^2 + 2x - 3} = \frac{A}{x + 3} + \frac{B}{x - 1}$$

Partial Fractions

Multiplying through by x2 + 2x 3, we have the polynomial identity

$$1 = A(x - 1) + B(x + 3)$$

Substituting x = 3 into this equation gives A = 1/4, and substituting x = 1 gives B = 1/4, so that

$$f(x) = \frac{1}{x^2 + 2x - 3} = \frac{1}{4}\left(\frac{-1}{x + 3} + \frac{1}{x - 1}\right)$$

Partial Fractions

Example 2

$$f(x) = \frac{x^3 + 16}{x^3 - 4x^2 + 8x}$$

After long-division, we have

$$f(x) = 1 + \frac{4x^2 - 8x + 16}{x^3 - 4x^2 + 8x} = 1 + \frac{4x^2 - 8x + 16}{x(x^2 - 4x + 8)}$$

Since (4)2 4(8) = 16 ¡ 0, x2 4x + 8 is irreducible, and so

$$\frac{4x^2 - 8x + 16}{x(x^2 - 4x + 8)} = \frac{A}{x} + \frac{Bx + C}{x^2 - 4x + 8}$$

Partial Fractions

Multiplying through by x3 4x2 + 8x, we have the polynomial identity

$$4x^2 - 8x + 16 = A(x^2 - 4x + 8) + (Bx + C)x$$

Taking x = 0, we see that 16 = 8A, so A = 2.
Comparing the x2 coefficients, we see that 4 = A + B = 2 + B, so B = 2.

Partial Fractions

Comparing linear coefficients, we see that $8 = 4A + C = 8 + C$, so $C = 0$. Altogether,

$$f(x) = 1 + 2\left(\frac{1}{x} + \frac{x}{x^2 - 4x + 8}\right)$$

The following example illustrates almost all the "tricks" one would need to use short of consulting a computer algebra system.

Example 3

$$f(x) = \frac{x^9 - 2x^6 + 2x^5 - 7x^4 + 13x^3 - 11x^2 + 12x - 4}{x^7 - 3x^6 + 5x^5 - 7x^4 + 7x^3 - 5x^2 + 3x - 1}$$

After long-division and factoring the denominator, we have

$$f(x) = x^2 + 3x + 4 + \frac{2x^6 - 4x^5 + 5x^4 - 3x^3 + x^2 + 3x}{(x-1)^3(x^2+1)^2}$$

Partial Fractions

The partial fraction decomposition takes the form

$$\frac{2x^6 - 4x^5 + 5x^4 - 3x^3 + x^2 + 3x}{(x-1)^3(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx + }{x^2 + }$$

Multiplying through by (x 1)3(x2 + 1)2 we have the polynomial identity

Partial Fractions

$$2x^6 - 4x^5 + 5x^4 - 3x^3 + x^2 + 3x$$
$$= A(x-1)^2(x^2+1)^2 + B(x-1)(x^2+1)^2 + C(x^2+1)^2 + (Dx+E)($$

Taking $x = 1$ gives $4 = 4C$, so $C = 1$. Similarly, taking $x = i$ gives $2 + 2i = (Fi + G)(2 + 2i)$, so $Fi + G = 1$, so $F = 0$ and $G = 1$ by equating real and imaginary parts. With $C = G = 1$ and $F = 0$, taking $x = 0$ we get $A \quad B + 1 \quad E \quad 1 = 0$, thus $E = A \quad B$. We now have the identity

Partial Fractions

$$
\begin{aligned}
& 2x^6 - 4x^5 + 5x^4 - 3x^3 \\
&= A(x-1)^2(x^2+1)^2 + B(x-1)(x^2+1)^2 + (x^2+1)^2 \\
&= A((x-1)^2(x^2+1)^2 + (x-1)^3(x^2+1)) + B((x-1)(x^2+1) - (x
\end{aligned}
$$

Expanding and sorting by exponents of x we get

Partial Fractions

$$2x^6 - 4x^5 + 5x^4 - 3x^3 + x$$
$$= (A + D)x^6 + (-A - 3D)x^5 + (2B + 4D + 1)x^4 + (-2B - 4D + 1)x$$

We can now compare the coefficients and see that

Partial Fractions

$$
\begin{aligned}
A + D &= 2 && (8)\\
-A - 3D &= -4 && (9)\\
2B + 4D + 1 &= 5 && (10)\\
-2B - 4D + 1 &= -3 && (11)\\
-A + 2B + 3D - 1 &= 1 && (12)\\
A - 2B - D + 3 &= 3, && (13)
\end{aligned}
$$

Partial Fractions

with A = 2 D and A 3 D =4 we get A = D = 1 and so B = 0, furthermore is C = 1, E = A B = 1, F = 0 and G = 1. The partial fraction decomposition of (x) is thus

$$f(x) = x^2 + 3x + 4 + \frac{1}{(x-1)} + \frac{1}{(x-1)^3} + \frac{x+1}{x^2+1} + \frac{1}{(x^2+1)^2}.$$

Alternatively, instead of expanding, one can obtain other linear dependences on the coefficients computing some derivatives at x=1 and at x=i in the above polynomial identity.

Partial Fractions

(To this end, recall that the derivative at x=a of (xa)mp(x) vanishes if m ¿ 1 and it is just p(a) if m=1.) Thus, for instance the first derivative at x=1 gives

$$2 \cdot 6 - 4 \cdot 5 + 5 \cdot 4 - 3 \cdot 3 + 2 + 3 = A \cdot (0+0) + B \cdot (2+0) + 8 + D \cdot 0$$

that is 8 = 2B + 8 so B=0.

Partial Fractions

Example 4 (residue method)

$$f(z) = \frac{z^2 - 5}{(z^2 - 1)(z^2 + 1)} = \frac{z^2 - 5}{(z+1)(z-1)(z+i)(z-i)}$$

Partial Fractions

Thus, f(z) can be decomposed into rational functions whose denominators are z+1, z1, z+i, zi. Since each term is of power one, 1, 1, i and i are simple poles. Hence, the residues associated with each pole, given by

$$\frac{P(z_i)}{Q'(z_i)} = \frac{z_i^2 - 5}{4z_i^3}$$

are

$$1, -1, \tfrac{3i}{2}, -\tfrac{3i}{2}$$

respectively, and

$$f(z) = \frac{1}{z+1} - \frac{1}{z-1} + \frac{3i}{2}\frac{1}{z+i} - \frac{3i}{2}\frac{1}{z-i}$$