Tutorial Sheet 2

1. For each of the following sequences a_n, evaluate the corressponding series $s_n = \sum_{k=0}^{n} a_k$, up to the term s_5. In addition, find s_{100}.

   i) $a_n = \frac{9}{10^n}$

   ii) $a_n = \frac{1}{2^n}$

   iii) $a_n = 2^n$

   iv) $a_n = \frac{2^n}{3^n}$

   v) $a_n = 3 + 4n$

   vi) $a_n = \frac{1}{4} + \frac{5n}{4}$

   vii) $a_n = 1 + 2n + 2^n$

   viii) $a_n = 30 + 40n + 8\frac{2^n}{3^n}$

2. Use the ratio test to determine whether the following infinite series are convergent or divergent, or say whether the test is inconclusive.

   i) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$

   ii) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{5^n}$

   iii) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{n^3}$

   iv) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

   v) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+1}$

   vi) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

   vii) $\displaystyle\sum_{n=0}^{\infty} x^n$

   viii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

   ix) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(n+1)!}$

3. The exponential function $e^x = \exp(x)$ can be defined using an infinite series as

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Given that this infinite series is convergent for all $x \in \mathbb{R}$, use the series and the following table to estimate the value of $e^{0.6}$ to 3 decimal places.

n		a_n		s_n
0		1.000000000		1.000000000
1		.		.
2		.		.
3		.		.
4		.		.
5		.		.
6		.		.
7		.		.
8		.		.

Solutions

1\. i)

n	a_n	s_n
0	9.00000	9.00000
1	0.90000	9.90000
2	0.09000	9.99000
3	0.00900	9.99900
4	0.00090	9.99990
5	0.00009	9.99999

$s_n = 9\frac{1-\frac{1}{10^{n+1}}}{1-\frac{1}{10}}$

$s_{100} = 10\left(1 - 10^{-101}\right) \cong 10$

v)

n	a_n	s_n
0	3	3
1	7	10
2	11	21
3	15	36
4	19	55
5	23	78

$s_n = \frac{n+1}{2}\left(2(3) + 4n\right)$

$s_{100} = 20503$

viii) $s_n = \frac{n+1}{2}\left(2(30) + 40n\right) + 8\frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}$

$s_{100} = 205030 + 8\frac{1-\left(\frac{2}{3}\right)^{101}}{\frac{1}{3}} \cong 205030 + 24 = 205054$

2\.

i) $R = \frac{1}{3} < 1$ cgt.

ii) $R = \frac{1}{5} < 1$ cgt.

iii) $R = 5 > 1$ dgt.

iv) $R = 1$ Inconclusive[1].

v) $R = 1$ inconclusive.

vi) $R = 0 < 1$ cgt for all x.

vii) $R = |x|$ cgt if $|x| < 1$.

viii) $R = |x|$ cgt if $|x| < 1$.

ix) $R = 0 < 1$ cgt for all x.

3\.

n		a_n		s_n
0		1.000000000		1.000000000
1		0.600000000		1.600000000
2		0.180000000		1.780000000
3		0.036000000		1.816000000
4		0.005400000		1.821400000
5		0.000648000		1.822048000
6		0.000064800		1.822112800
7		0.000005554		1.822118354
8		.		.

$\Rightarrow e^{0.6} \cong 1.822$

[1] However, this series is the harmonic series, which is in fact divergent.