

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

Queue: {}

Incumbent: none

Best cost  $Z^*$ : - inf

• Initialize

# Example: B&B for BIPs



$\{ \}$

Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_1 \leq 1, x_1 \geq 0, x_1$  **integer**

Queue:  ~~$\{ \}$~~

Incumbent: none

Best cost  $Z^*$ : - inf

• Dequeue  $\{ \}$

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 16.5, x = \langle 0.8333, 1, 0, 1 \rangle$$

Queue:

Incumbent: none

Best cost  $Z^*$ : - inf

- Bound {}
  1. Constrain  $x_i$  by {}
  2. Relax to LP
  3. Solve LP

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_1 \leq 1, x_1 \geq 0, x_1 \text{ integer}$$

$$Z = 16.5, x = \langle 0.8333, 1, 0, 1 \rangle$$

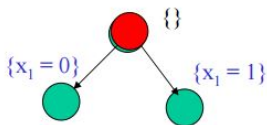
Queue:

Incumbent: none

Best cost  $Z^*$ : - inf

- Try to fathom:
  1. infeasible?
  2. worse than incumbent?
  3. integer solution?

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- x_1 + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 16.7, x = \langle 0.8333, 1, 0, 1 \rangle$$

Queue:  $\{x_1 = 0\} \{x_1 = 1\}$

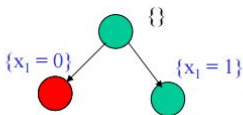
Incumbent: none

Best cost  $Z^*$ : - inf

- Branch:

- select unassigned  $x_i$ 
  - pick non-integer ( $x_1$ )
- Split on  $x_i$

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_1 \leq 1, x_1 \geq 0, x_1$  ~~integer~~

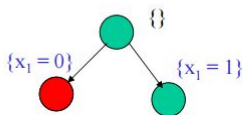
Queue:  ~~$\{x_1 = 0\}$~~   $\{x_1 = 1\}$

Incumbent: none

Best cost  $Z^*$ : - inf

- Dequeue:
  - depth first or
  - best first

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_1 \leq 1, x_1 \geq 0, x_1$  ~~integer~~

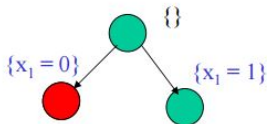
Queue:  $\{x_1 = 1\}$

Incumbent: none

Best cost  $Z^*$ : - inf

- Bound  $\{x_1 = 0\}$ 
  - constrain x by  $\{x_1 = 0\}$

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9 \mathbf{0} + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6 \mathbf{0} + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- \mathbf{0} + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ ~~integer~~}$$

Queue:  $\{x_1 = 1\}$

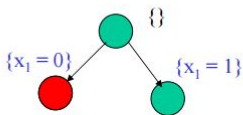
Incumbent: none

Best cost  $Z^*$ : - inf

- Bound  $\{x_1 = 0\}$ 
  - constrain  $x$  by  $\{x_1 = 0\}$



# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$3x_2 + 5x_3 + 2x_4 \leq 10$$

$$-x_3 + x_4 \leq 1$$

$$+x_3 \leq 0$$

$$-x_2 + x_4 \leq 0$$

$$-x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 9, \quad x = \langle 0, 1, 0, 1 \rangle$$

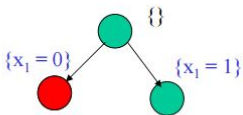
Queue:  $\{x_1 = 1\}$

Incumbent: none

Best cost  $Z^*$ : - inf

- Bound  $\{x_1 = 0\}$ 
  - constrain  $x$  by  $\{x_1 = 0\}$
  - relax to LP
  - solve LP

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$3x_2 + 5x_3 + 2x_4 \leq 10$$

$$-x_3 + x_4 \leq 1$$

$$+x_3 \leq 0$$

$$-x_2 + x_4 \leq 0$$

$$-x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 9, \quad x = \langle 0, 1, 0, 1 \rangle$$

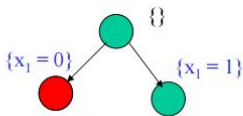
Queue:  $\{x_1 = 1\}$

Incumbent: none

Best cost  $Z^*$ : - inf

- Try to fathom:
  1. infeasible?
  2. worse than incumbent?
  3. integer solution?

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$3x_2 + 5x_3 + 2x_4 \leq 10$$

$$-x_3 + x_4 \leq 1$$

$$+x_3 \leq 0$$

$$-x_2 + x_4 \leq 0$$

$$-x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 9, \quad x = \langle 0, 1, 0, 1 \rangle$$

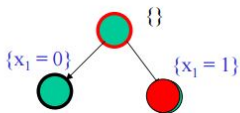
Queue:  $\{x_1 = 1\}$

Incumbent:  $x = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

- Try to fathom:
  - infeasible?
  - worse than incumbent?
  - integer solution?

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_i \leq 1, x_i \geq 0, x_i \text{ integer}$

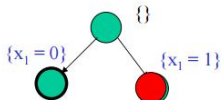
Queue:  ~~$\{x_1 = 1\}$~~

Incumbent:  $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

• Dequeue

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
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- $x_1 \leq 1, x_i \geq 0, x_i \text{ integer}$

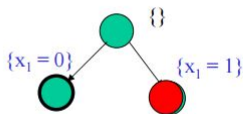
Queue:

Incumbent:  $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : **9**

- Bound  $\{x_1 = 1\}$

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = \mathbf{9} + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- \mathbf{6} + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- \mathbf{-1} + x_3 \leq 0$$

$$- x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ ~~integer~~}$$

$$Z = \mathbf{16.7}, x = \langle \mathbf{1}, .8, 0, .8 \rangle$$

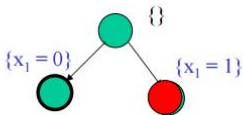
Queue:

Incumbent:  $x = \langle \mathbf{0}, 1, 0, 1 \rangle$

Best cost  $Z^*$ :  $\mathbf{9}$

- Bound  $\{x_1 = 1\}$

## Example: B&B for BIPs



**Solve:**

$$\text{Max } Z = 9 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$-6 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$-x_3 + x_4 \leq 1$$

$$-1 + x_3 \leq 0$$

$$-x_2 + x_4 \leq 0$$

- $x_i \leq 1, x_i \geq 0, x_i$  ~~integer~~

$$Z = 16.2, x = \langle 1, .8, 0, .8 \rangle$$

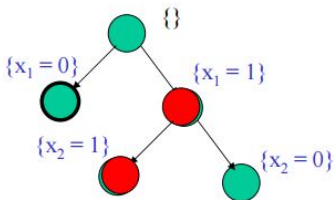
Queue:

Incumbent:  $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

- Try to fathom:
  - infeasible?
  - worse than incumbent?
  - integer solution?

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_1 \leq 1, x_i \geq 0, x_i \text{ integer}$

$$Z = 16.7, x = \langle 1, .8, 0, .8 \rangle$$

Queue:  ~~$\{x_1=1, x_2=1\}$~~   $\{x_1=1, x_2=0\}$

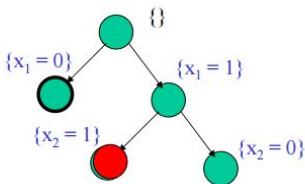
Incumbent:  $x = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

- Branch
- Dequeue



# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9 + 5 + 6x_3 + 4x_4$$

Subject to:

$$- 6 + 3 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -1 + x_3 \leq 0$$

$$- -1 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 16, x = \langle 1, 1, 0, .5 \rangle$$

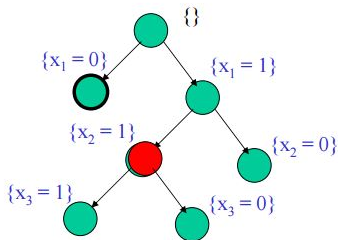
Queue:  $\{x_1=1, x_2=0\}$

Incumbent:  $x = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

- Try to fathom:
  - infeasible?
  - worse than incumbent?
  - integer solution?

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_1 \leq 1, x_1 \geq 0, x_1 \text{ integer}$

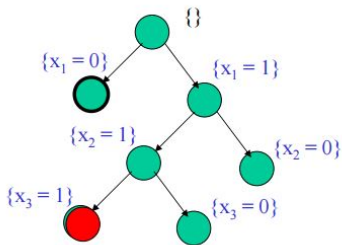
$$Z = 16, x = \langle 1, 1, 0, 5 \rangle$$

Queue:  $\{\dots, x_3=0\} \{\dots, x_3=0\} \{\dots, x_2=0\}$  • Branch

Incumbent:  $x = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

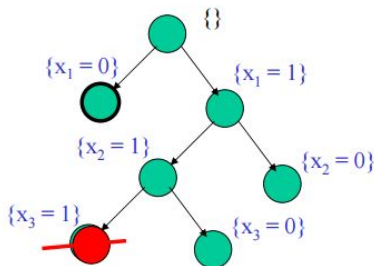
- $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
- $x_3 + x_4 \leq 1$
- $-x_1 + x_3 \leq 0$
- $-x_2 + x_4 \leq 0$
- $x_i \leq 1, x_i \geq 0, x_i \text{ integer}$

Queue:  ~~$\{\dots, x_3=1\}$~~   $\{\dots, x_3=0\}$   $\{\dots, x_2=0\}$  • Dequeue  
 • Bound  $\{x_1=1, x_2=1, x_3=1\}$

Incumbent:  $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9 + 5 + 6 + 4x_4$$

Subject to:

$$- 6 + 3 + 5 + 2x_4 \leq 10$$

$$- 1 + x_4 \leq 1$$

$$- -1 + 1 \leq 0$$

$$- -1 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

**No Solution**

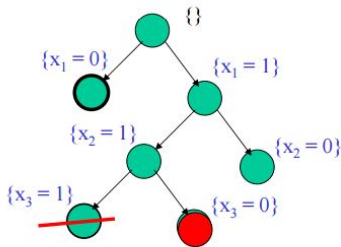
- Try to fathom:
  - infeasible?

Queue:  $\{\dots, x_3 = 0\} \{\dots, x_2 = 0\}$

Incumbent:  $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to:

$$- 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -x_1 + x_3 \leq 0$$

$$- -x_2 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

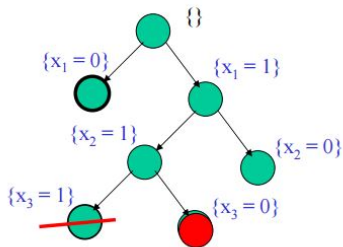
Queue:  ~~$\{\dots, x_3 = 0\}$~~   $\{\dots, x_2 = 0\}$

Incumbent:  $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

- Dequeue
- Bound  $\{x_1=1, x_2=1, x_3=0\}$

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = \mathbf{9} + \mathbf{5} + \quad + 4x_4$$

Subject to:

$$\begin{aligned} - \mathbf{6} + \mathbf{3} + \quad + 2x_4 &\leq 10 \\ \quad + x_4 &\leq 1 \end{aligned}$$

$$- \mathbf{-1} \leq 0$$

$$- \mathbf{-1} + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$\mathbf{Z = 16, x = \langle 1, 1, 0, .5 \rangle}$$

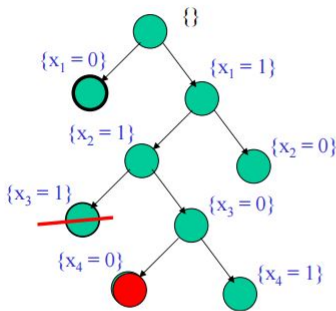
Queue:  $\{\dots, x_2 = 0\}$

Incumbent:  $\mathbf{x = \langle 0, 1, 0, 1 \rangle}$

Best cost  $Z^*$ :  $\mathbf{9}$

- Try to fathom:
  - infeasible?
  - worse than incumbent?
  - integer solution?

# Example: B&B for BIPs



Solve:

$$\text{Max } Z = 9x_1 + 5x_2$$

Subject to:

$$-6x_1 + 3x_2 \leq 10$$

$$x_1 \leq 1$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

$$-x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$Z = 14, x = \langle 1, 1, 0, 0 \rangle$$

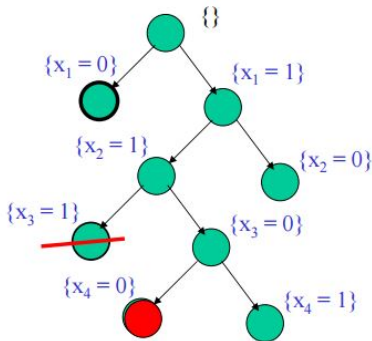
Queue:  $\{ \dots, x_2=0 \} \{ \dots, x_4=1 \} \{ \dots, x_2=0 \}$

- Branch
- Dequeue
- Bound

Incumbent:  $x = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

# Example: B&B for BIPs



Queue:  $\{\dots, x_4=1\} \{\dots, x_2=0\}$

Incumbent:  $\mathbf{x} = \langle 0, 1, 0, 1 \rangle$

Best cost  $Z^*$ : 9

Solve:

$$\text{Max } Z = 9 \quad + 5$$

Subject to:

$$- 6 \quad + 3 \quad \leq 10$$

$$\leq 1$$

$$- -1 \quad \leq 0$$

$$- -1 \quad \leq 0$$

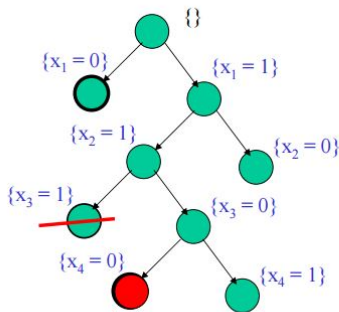
$$- x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$\mathbf{Z = 14, x = \langle 1, 1, 0, 0 \rangle}$$

- Try to fathom:
  - infeasible?
  - worse than incumbent?
  - integer solution?



# Example: B&B for BIPs



Queue:  $\{\dots, x_4=1\} \{\dots, x_2=0\}$

Incumbent:  $\mathbf{x} = \langle 1, 1, 0, 0 \rangle$

Best cost  $Z^*$ : **14**

Solve:

$$\text{Max } Z = 9x_1 + 5x_2$$

Subject to:

$$-6x_1 + 3x_2 \leq 10$$

$$\leq 1$$

$$-1x_1 \leq 0$$

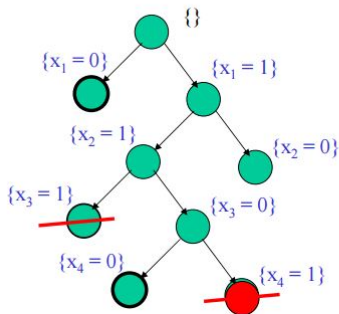
$$-1x_2 \leq 0$$

$$-x_i \leq 1, x_i \geq 0, x_i \text{ integer}$$

$$\mathbf{Z = 14, x = \langle 1, 1, 0, 0 \rangle}$$

- Try to fathom:
  - infeasible?
  - worse than incumbent?
  - integer solution?

# Example: B&B for BIPs



Queue:  $\{\dots, x_4=1\} \{\dots, x_2=0\}$

Incumbent:  $\mathbf{x} = \langle 1, 1, 0, 0 \rangle$

Best cost  $Z^*$ : 14

Solve:

$$\text{Max } Z = 9 \quad + 5 \quad + 4$$

Subject to:

$$- 6 \quad + 3 \quad + 2 \leq 10$$

$$+ 1 \leq 1$$

$$- -1 \leq 0$$

$$- -1 + 1 \leq 0$$

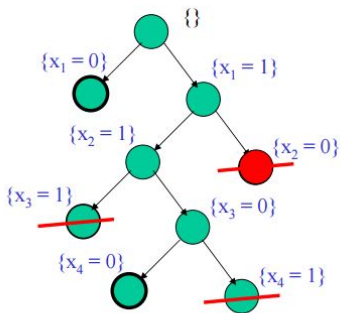
$$- x_1 \leq 1, x_1 \geq 0, \text{ ~~x}_1 \text{ integer}~~$$

**No Solution,  $\mathbf{x} = \langle 1, 1, 0, 1 \rangle$**

• Try to fathom:

- infeasible?
- worse than incumbent?
- integer solution?

# Example: B&B for BIPs



Queue: ~~{ $\dots, x_2=0$ }~~

Incumbent:  $\mathbf{x} = \langle 1, 1, 0, 0 \rangle$

Best cost  $Z^*$ : 14

Solve:

$$\text{Max } Z = 9 \quad + 6x_3 + 4x_4$$

Subject to:

$$- 6 \quad + 5x_3 + 2x_4 \leq 10$$

$$- x_3 + x_4 \leq 1$$

$$- -1_1 + x_3 \leq 0$$

$$- -1 + x_4 \leq 0$$

$$- x_i \leq 1, x_i \geq 0, \text{  ~~$x_i$  integer~~}$$

$$Z = 13.8, \mathbf{x} = \langle 1, 0, .8, 0 \rangle$$

- Try to fathom:
  - infeasible?
  - worse than incumbent?
  - integer solution?