Chapter 4

Duality

Corresponding to every linear programming problem, there is another closely related linear programming problem called its **dual**. The original problem is known as the **primal**. In this chapter, we show how to formulate the dual problem and discuss its significance.

4.1 Formulating the dual problem

We shall assume that the *primal* problem is in standard form. Thus, if we are given a general l.p. problem as our primal, we would first convert it to standard form by changing a minimization problem to maximization, replacing each equality constraint by a pair of " $\geq$ " and " $\leq$ " constraints (or eliminating all equality constraints from the problem, as described in section 3.2) and multiplying every " $\geq$ " functional constraint by (-1) to convert it to a " $\leq$ " constraint.

Definition 4.1 *Given a standard linear programming problem P1:*

> *Find* $\mathbf{x} \in \mathbf{R}^n$ *to maximize* $z_1 = \mathbf{c}^T\mathbf{x}$, *subject to* $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, *where* $\mathbf{A}$ *is an* $m \times n$ *real matrix,* $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{b} \in \mathbf{R}^m$,

we define the dual problem P2 as follows.

> *Find* $\mathbf{y} \in \mathbf{R}^m$ *to minimize* $z_2 = \mathbf{b}^T\mathbf{y}$, *subject to* $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$.

Example 4.1 We construct the dual of the Boatbuilder's problem, with the primal objective function $z_1 = 6x_1 + 5x_2$. This problem is already in standard form. Expressed in vector and matrix notation (see Example 2.2), it becomes:

Find $\mathbf{x} \in \mathbf{R}^2$ to maximize

$$z_1 = [6, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 24 & 6 \\ 6 & 4 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 2400 \\ 720 \\ 1000 \end{bmatrix} \text{ and } \mathbf{x} \geq \mathbf{0}.$$

The dual problem P2, expressed in vector and matrix notation, is therefore:

Find $\mathbf{y} \in \mathbf{R}^3$ to minimize

$$z_2 = [2400, 720, 1000]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 24 & 6 & 2 \\ 6 & 4 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 6 \\ 5 \end{bmatrix} \text{and } \mathbf{y} \geq \mathbf{0}.$$

Equivalently, P2 can be stated as:
Find $y_1, y_2, y_3 \in \mathbf{R}$ to minimize $z_2 = 2400y_1 + 720y_2 + 1000y_3$, subject to

$$\begin{aligned} 24y_1 + 6y_2 + 2y_3 &\geq 6 \qquad (4.1) \\ 6y_1 + 4y_2 + 8y_3 &\geq 5 \qquad (4.2) \\ y_1, y_2, y_3 &\geq 0. \qquad (4.3) \end{aligned}$$

□

Example 4.2 We construct the dual of the following l.p. problem.

Find $x_1, x_2, x_3 \in \mathbf{R}$ to minimize $z_1 = x_1 - 2x_2 + 4x_3$, subject to

$$\begin{aligned} x_1 - x_2 + 2x_3 &\geq 3 \\ x_1 + x_2 - 4x_3 &\leq 6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Converting the problem to standard form, P1 becomes:

Find $x_1, x_2, x_3 \in \mathbf{R}$ to maximize $(-z_1) = -x_1 + 2x_2 - 4x_3$, subject to

$$\begin{aligned} -x_1 + x_2 - 2x_3 &\leq -3 \\ x_1 + x_2 - 4x_3 &\leq 6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

P1 is now in the standard form

Find $\mathbf{x} \in \mathbf{R}^n$ to maximize $z_1 = \mathbf{c}^T\mathbf{x}$, subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,

where

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & -2 \\ 1 & 1 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} \text{and } \mathbf{c} = \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}.$$

The dual problem P2 is therefore:

Find $y_1, y_2 \in \mathbf{R}$ to minimize $z_2 = -3y_1 + 6y_2$, subject to

$$\begin{aligned} -y_1 + y_2 &\geq -1 \\ y_1 + y_2 &\geq 2 \\ -2y_1 - 4y_2 &\geq -4 \\ y_1, y_2 &\geq 0. \end{aligned}$$

□

4.2 Finding an optimal solution to the dual problem

We shall see that there is a close relationship between feasible solutions for the primal and dual problems. Given a feasible solution $\mathbf{x}$ for P1, we shall use $z_1(\mathbf{x})$ to denote the value of z_1 evaluated at $\mathbf{x}$. Thus $z_1(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$.

Before proving the following theorem, we recall two key facts on transposing matrices. Suppose that $\mathbf{A}, \mathbf{B}$ are two compatible matrices, so that their product $\mathbf{AB}$ can be defined. Then

1. $(\mathbf{A}^T)^T = \mathbf{A}$;
2. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Theorem 4.2 (The Weak Duality Theorem). *For every feasible solution* $\mathbf{x}$ *for P1 and for every feasible solution* $\mathbf{y}$ *for P2, we have*

$$z_1(\mathbf{x}) \le z_2(\mathbf{y}).$$

Proof. Since $\mathbf{x}$ is a feasible solution to P1, we have $\mathbf{Ax} \le \mathbf{b}$ and hence $\mathbf{b}^T \ge \mathbf{x}^T\mathbf{A}^T$. Since $\mathbf{y} \ge \mathbf{0}$, we can multiply this inequality from the right by $\mathbf{y}$ to give

$$z_2(\mathbf{y}) = \mathbf{b}^T\mathbf{y} \ge \mathbf{x}^T\mathbf{A}^T\mathbf{y}. \tag{4.4}$$

Similarly, since $\mathbf{y}$ is a feasible solution to P2, we have $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$ and hence $\mathbf{c}^T \le \mathbf{y}^T\mathbf{A}$. Since $\mathbf{x} \ge \mathbf{0}$, we can multiply this inequality from the right by $\mathbf{x}$ to give

$$z_1(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \le \mathbf{y}^T\mathbf{Ax}. \tag{4.5}$$

Finally, since $\mathbf{y}^T\mathbf{Ax}$ is a 1×1 matrix, we have

$$\mathbf{y}^T\mathbf{Ax} = (\mathbf{y}^T\mathbf{Ax})^T = \mathbf{x}^T\mathbf{A}^T\mathbf{y}. \tag{4.6}$$

Hence from equations (4.5), (4.6) and (4.4), we have

$$z_1(\mathbf{x}) \le \mathbf{y}^T\mathbf{Ax} = \mathbf{x}^T\mathbf{A}^T\mathbf{y} \le z_2(\mathbf{y}),$$

so that $z_1(\mathbf{x}) \le z_2(\mathbf{y})$. □

Theorem 4.2 tells us that the value of the primal objective function z_1 evaluated at any feasible solution of P1 is always less than or equal to the value of the dual objective function z_2 evaluated at any feasible solution of P2. In particular, it tells us that when both problems have optimal solutions, then the *maximum* value of z_1 is always less than or equal to the *minimum* value of z_2.

If we are fortunate enough to find feasible solutions $\mathbf{x}^*$ for P1 and $\mathbf{y}^*$ for P2 and that

$$z_1(\mathbf{x}^*) = z_2(\mathbf{y}^*),$$

then the following result tells us that we can conclude that $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions of P1 and P2 respectively.

Corollary 4.3 *Suppose* $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are feasible solutions for P1 and P2 respectively and that* $z_1(\mathbf{x}^*) = z_2(\mathbf{y}^*)$. *Then* $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are optimal solutions for P1 and P2 respectively.*

Proof. Let $\mathbf{x}$ be any feasible solution for P1. By Lemma 4.2, we have

$$z_1(\mathbf{x}) \le z_2(\mathbf{y}^*).$$

But we are given that $z_2(\mathbf{y}^*) = z_1(\mathbf{x}^*)$. Hence

$$z_1(\mathbf{x}) \leq z_1(\mathbf{x}^*).$$

Since this holds for all feasible solutions $\mathbf{x}$ for P1, we deduce that $z_1(\mathbf{x}^*)$ is the maximum value of z_1 and hence $\mathbf{x}^*$ is an optimal solution for P1. Similarly $\mathbf{y}^*$ is an optimal solution for P2. □

Now suppose that P1 has an optimal solution. We shall show how the simplex algorithm for solving P1 also provides us with an optimal solution for P2. We demonstrate this first of all by finding an optimal solution to the dual of the Boatbuilder's problem from the final tableau in the solution of P1 (see Section 2.5).

Example 4.3 Consider the objective row (row 0) of the final tableau in the solution of the Boatbuilder's problem, shown below with a division between the columns containing the coefficients of the decision variables and those containing the coefficients of the slack variables (the reason for this will become apparent in a moment).

		decision		*slack*			
Eqn	z	x_1	x_2	x_3	x_4	x_5	RS
0	1	0	0	0	19/20	3/20	834

Recollect that x_3, x_4, x_5 are the slack variables in the first, second and third constraints, respectively. For $i = 1, 2, 3$, we put y_i^* equal to the coefficient of the ith slack variable in the final objective row. Thus, we put $y_1^* = 0$ (the coefficient of x_3), $y_2^* = 19/20 = 0.95$ (the coefficient of x_4) and $y_3^* = 3/20 = 0.15$ (the coefficient of x_5).

We first verify that $\mathbf{y}^* = [y_1^*, y_2^*, y_3^*]^T$ is a *feasible* solution for the dual problem P2. Substituting $y_i = y_i^*$, $i = 1, 2, 3$, in the constraints (4.1) and (4.2) for the dual problem, we have

$$\begin{aligned} 24y_1^* + 6y_2^* + 2y_3^* &= 6 \geq 6 \\ 6y_1^* + 4y_2^* + 8y_3^* &= 5 \geq 5, \end{aligned}$$

and since also these values of y_1^*, y_2^* and y_3^* are all non-negative, we see that $\mathbf{y}^* = [y_1^*, y_2^*, y_3^*]^T$ satisfies all the constraints for the dual problem and is thus a feasible solution for P2.

Further, substituting $y_i = y_i^*$, $i = 1, 2, 3$, in the dual objective function, gives

$$z_2 = 2400y_1^* + 720y_2^* + 1000y_3^* = 834.$$

Thus we have found feasible solutions $\mathbf{x}^* = [44, 114]^T$ and $\mathbf{y}^* = [0, 0.95, 0.15]^T$ such that $z_1(\mathbf{x}^*) = z_2(\mathbf{y}^*)$ and hence by Corollary 4.3, $\mathbf{y}^*$ is the optimal solution of P2. □

We shall now show why the coefficients of the slack variables in the objective row of the final tableau for the solution of P1 gave us an optimal solution for P2 in the previous example. We reproduce below just the *initial* tableau together with the *final* objective row for the solution of the Boatbuilder's problem.

			decision		*slack*			
Basis	*Eqn*	z	x_1	x_2	x_3	x_4	x_5	RS
-	0	1	−6	−5	0	0	0	0
x_3	1	0	24	6	1	0	0	2400
x_4	2	0	6	4	0	1	0	720
x_5	3	0	2	8	0	0	1	1000
-	0	1	0	0	0	19/20	3/20	834

The final objective row has been obtained from the rows of the initial tableau by a sequence of elementary row operations. These consisted of *adding* multiples of some or all of the rows 1, 2 and 3 to row 0 until all the coefficients in the objective row had became non-negative. However, the columns belonging to each of the *slack* variables in the initial tableau are columns of the identity matrix. Thus, in order to change the coefficient of x_4 from its initial value of 0 to its final value of 19/20, the cumulative effect of the row operations has been to add altogether 19/20 times row 2 to row 0. Similarly, we have added altogether 3/20 times row 3 to row 0 in order to obtain 3/20 as the coefficient of x_5 in the final objective row. Finally, we have added altogether 0 times row 1 to row 0, because the coefficient of x_3 in the final objective row is 0, its initial value. We summarize this discussion below.

> The coefficients of the slack variables in the objective row in the final tableau provide a record of the cumulative effect of the elementary row operations that were performed while solving the primal problem.

In this case, we have

$$\text{Final row } 0 = \text{initial row } 0 + 0 \times \text{row } 1 + 0.95 \times \text{row } 2 + 0.15 \times \text{row } 3. \tag{4.7}$$

Since the final tableau represents the *optimal* solution to the primal problem, all the entries in the final objective row are non-negative. In particular, the coefficients of the slack variables in this row will satisfy the non-negativity constraints. To see that they also satisfy the functional constraints, we consider the cumulative effect of the elementary row operations on the coefficients of the decision variables. Let us trace what has happened to the coefficient of x_1. Applying (4.7) to the x_1 column, the coefficient of x_1 in the final objective row is

$$-6 + 0 \times 24 + 0.95 \times 6 + 0.15 \times 2.$$

However, the coefficients of the decision variables are also non-negative in the final objective row, and thus we have

$$-6 + 24(0) + 6(0.95) + 2(0.15) \geq 0$$

and hence

$$24(0) + 6(0.95) + 2(0.15) \geq 6. \tag{4.8}$$

But (4.8) is just the condition that $\mathbf{y}^* = [0, 0.95, 0115]^T$ satisfies the first functional constraint (4.1) for P2.

Repeating this argument for the coefficient of x_2, we obtain

$$-5 + 6(0) + 4(0.95) + 8(0.15) \geq 0$$

and hence

$$24(0) + 6(0.95) + 2(0.15) \geq 5. \tag{4.9}$$

However, (4.9) is just the condition that $\mathbf{y}^* = [0, 0.95, 0.15]^T$ satisfies the second functional constraint (4.2) for P2.

Finally, we apply (4.7) to the right side in the final tableau. This gives the final value of z_1 to be

$$0 + 0 \times 2400 + 0.95 \times 720 + 0.15 \times 1000.$$

Thus we have

$$2400(0) + 720(0.95) + 1000(0.15) = 834. \tag{4.10}$$

But the left side of equation (4.10) is precisely the condition that $z_2(\mathbf{y}^*) = 834 = z_1(\mathbf{x}^*)$, where $\mathbf{x}^*$ is the optimal solution to P1.

Suppose now that we have been given a primal problem P1 that has an optimal solution, which we have found by the simplex algorithm. The method that we used in this example can be generalised to prove the following result.

Result 4.4 *Let* $\mathbf{y}^* = [y_1^*, y_2^*, ..., y_m^*]^T$*, where* y_i^* *is the value of the coefficient of the ith primal slack (or surplus) variable* x_{i+n} *in the objective equation (row 0) in the final tableau for P1. Then* $\mathbf{y}^*$ *is an optimal solution to the dual problem P2.*

This fact is used to prove the following fundamental theorem. A formal proof of it will not be asked in the examination (although you are expected to show understanding of the method explained above applied to a particular example). The interested reader can find the formal proof in the recommended textbook.

Theorem 4.5 (The Duality Theorem.) *Suppose that P1 has an optimal solution* $\mathbf{x}^*$*. Then P2 has an optimal solution* $\mathbf{y}^*$ *and* $z_1(\mathbf{x}^*) = z_2(\mathbf{y}^*)$.

4.3 Shadow prices

Suppose that the primal problem P1 represents the kind of manufacturing problem typified by the Boatbuilder's problem, where the objective is to maximize the firm's profit, each of the functional constraints represents an *activity* in the manufacturing process, and each parameter b_i represents the maximum level of resource allocated to activity i, $i = 1, 2, ..., m$. Suppose further that the primal problem has an optimal solution $\mathbf{x}^*$. Then, by the Duality Theorem, the dual problem will also have an optimal solution $\mathbf{y}^*$ and

$$z_1(\mathbf{x}^*) = z_2(\mathbf{y}^*) = b_1y_1^* + b_2y_2^* + \cdots + b_my_m^*.$$

Now $z_1(\mathbf{x}^*)$ is the firm's maximum profit and hence we can think of the term $b_iy_i^*$ as representing the *contribution to maximum profit made by having* b_i *units of resource* i *available* in the primal problem. Thus each unit of b_i contributes y_i^* to the maximum profit, so that y_i^* represents the contribution of each unit of b_i to the maximum profit. Put another way, if we could increase b_i by 1 unit, then (provided that this increase did not alter which constraints were binding at the optimal solution) the corresponding increase in the profit would be y_i^*.

Definition 4.6 *The value* y_i^* *of the dual variable* y_i *at the optimal solution is called the* **shadow price** *of resource* i*. It measures the rate at which* z *can be increased by increasing* b_i.

Frequently, the value of the parameter b_i represents management's initial decision on the maximum amount of resource i that it is willing to commit to activity i (in competition, perhaps, with other calls on this resource by the firm). Knowledge of the shadow prices will be used to determine whether the corresponding increase in profit will make it worth increasing the amount of resource i available for this activity. Interest is normally focussed on those activities for which the shadow price is highest at the optimal solution.

Example 4.4 The shadow prices for the Boatbuilder's problem are $y_1^* = 0$, $y_2^* = 19/20 = 0.95$ and $y_3^* = 3/20 = 0.15$. These values reflect the fact that we have already observed in Example 3.6, that the constraints on machine time and hand labour are the only constraints that are *binding* at the optimal solution and increasing the maximum amount of aluminium available would not increase the profit. The new information that the values of the shadow prices give us is that increasing the machine time by t mins would increase the maximum value of z by $0.95t$, while increasing the hand labour time by t hours would increase z by $0.15t$ (for small values of t). This would indicate that they might give priority to trying to increase the number of minutes of machine time available.

4.4 Symmetry between primal and dual problems

So far in this chapter, we have assumed that we are given a "primal" problem, from which we deduce a "dual" problem. In fact, the relationship between these two problems is *symmetric*, so that the roles of the primal and dual can always be reversed. This is established in the following result, where we prove that the dual of the dual problem is the primal.

Theorem 4.7 *Let P1 be a given l.p. problem and P2 be its dual. Then the dual of P2 is P1.*

Proof. Suppose that P1 and P2 are as in definition 4.1. Expressed in standard form, P2 is as follows.

Find $\mathbf{y} \in \mathbf{R}^m$ to maximize $(-z_2) = -\mathbf{b}^T\mathbf{y}$, subject to $-\mathbf{A}^T\mathbf{y} \leq -\mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$.

The dual of P2 is therefore:

Find $\mathbf{x} \in \mathbf{R}^n$ to minimize $z = -\mathbf{c}^T\mathbf{x}$, subject to $-\mathbf{A}\mathbf{x} \geq -\mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Rewriting this problem in standard form, this becomes:

Find $x \in \mathbf{R}^n$ to maximize $(-z) = \mathbf{c}^T\mathbf{x}$, subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,

which is the problem P1, with $z_1 = (-z)$. □

This theorem shows that given an l.p. problem and its dual, it does not matter which problem we regard as the *primal* and which the *dual*. It therefore follows from Theorem 4.5 that if either the primal or the dual has an optimal solution, then so does the other and we can find the solutions to *both* problems from the final tableau for the solution of either one of them.

Example 4.5 Consider the Cattle Feed problem formulated in Example 1.2. This is a minization problem in which all the constraints are "$\geq$". Changing the variable to $\mathbf{y}$ and simplifying the third and fourth functional constraints by multiplying through by 100, P1 can be expressed as:

Find $y_1, y_2 \in \mathbf{R}$ to minimize $z = 8y_1 + 21y_2$, subject to

$$\begin{aligned} 13.0y_1 + 32.5y_2 &\geq 65.0 \\ 4.2y_1 + 5.6y_2 &\geq 14.0 \\ 2y_1 + 8y_2 &\geq 12 \\ 5y_1 + 9y_2 &\geq 15 \\ y_1, y_2 &\geq 0. \end{aligned}$$

The dual P2 is then:

Find $x_1, x_2, x_3, x_4 \in \mathbf{R}$ to maximize $z_1 = 65x_1 + 14x_2 + 12x_3 + 15x_4$, subject to

$$\begin{aligned} 13.0x_1 + 4.2x_2 + 2x_3 + 5x_4 &\leq 8 \\ 32.5x_1 + 5.6x_2 + 8x_3 + 9x_4 &\leq 21 \\ x_1, x_2, x_3, x_4 &\geq 0. \ \square \end{aligned}$$

As we have seen in section 3.1, problems containing "$\geq$" constraints with non-negative values of the parameters b_i on the right side can be solved by the Big M method. If, however, *all* the functional constraints are of this type, and also the objective coefficients c_j are all non-negative, then the *dual* (maximization) problem can be solved by the ordinary simplex method and the solution to the primal problem can be read off from the final tableau in the solution of its dual. Thus we can find the optimal solution to the Cattle Feed problem, for example, by solving the dual problem formulated in Example 4.5 and reading off the optimal solution to the primal from the final tableau in the solution of the dual. This can shorten the working for small problems of this type that are being solved "by hand". We use this method extensively in the next chapter. However, there are l.p. problems where both the primal and the dual require the use of artificial variables for their solution and in these cases, the use of the Big M (or the Two Phase) method cannot be avoided.

4.5 Exercises

Exercise 4.1
Write the following l.p. problem (see Exercise 3.1) in standard form and hence construct its dual problem.

Find $x_1, x_2, x_3 \in \mathbf{R}$ to minimize $z_1 = 3x_1 - x_2 + 2x_3$, subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 6 \\ 4x_1 + 5x_2 + 4x_3 &\leq 24 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Exercise 4.2
Let P1 denote the following l.p. problem.

Find $x_1, x_2 \in \mathbf{R}$ to maximize $z_1 = 4x_1 + 10x_2$ subject to:

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leq & 3 \\ x_1 & + & 4x_2 & \leq & 5 \\ 2x_1 & + & 5x_2 & \leq & 10 \\ x_1, & & x_2 & \geq & 0. \end{array}$$

(a) Formulate the dual problem P2.

(b) The following tableau is obtained in the solution to the problem P1 by the simplex algorithm, where x_3, x_4, x_5 are the slack variables added to the first, second and third constraints respectively.

Eqn	z_1	x_1	x_2	x_3	x_4	x_5	RS
0	1	0	0	2	2	0	16
1	0	1	0	4/3	$-1/3$	0	7/3
2	0	0	1	$-1/3$	1/3	0	2/3
3	0	0	0	-1	-1	1	2

(i) Give the optimal solution for the decision variables of the problem P1 and the optimal value of z_1.

(ii) Find from this tableau the optimal solution for the decision variables for the dual problem P2 and the optimal value of the dual objective function.

(iii) Check that the values you obtained above for the dual decision variables give a *feasible* solution for P2 and use them to calculate the corresponding value of z_2. Does the value obtained agree with the value you gave in the previous part of the question?

(iv) Which constraints in the primal problem P1 are *binding* at the optimal solution? Explain your answer.

(v) What are the values of the shadow prices?

Exercise 4.3
Repeat Exercise 4.2 for the following l.p. problem P1.

Find $x_1, x_2, x_3 \in \mathbf{R}$ to minimize $z = 3x_1 + 2x_2 + 4x_3$, subject to:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & \geq & 5 \\ 5x_1 & + & 4x_2 & + & x_3 & \geq & 12 \\ & & x_1, & x_2, & x_3 & \geq & 0, \end{array}$$

given that the final tableau in the solution of P1 is as shown below, where x_4 and x_5 are the surplus variables subtracted from the first and second constraints respectively.

Eqn	$-z_1$	x_1	x_2	x_3	x_4	x_5	RS
0	1	1	0	2	2	0	−10
1	0	−1	0	3	−4	1	8
2	0	1	1	1	−1	0	5

Exercise 4.4
Solve the problem P2 (the maximization problem) obtained in Example 4.5 by the simplex algorithm and hence find the optimal solution to the Cattle Feed problem.