Chapter 3

General linear programming problems

In the previous chapter, we saw how to use the simplex algorithm to solve linear programming problems expressed in standard form with $\mathbf{b} \geq \mathbf{0}$. In this chapter, we show how other formulations can be adapted to give a problem in standard form, which can then be solved by the methods of the previous chapter.

3.1 Converting to standard form

1. **Minimizing the objective function**
   We have seen that some problems require us to *minimize* z. Suppose that z has a finite minimum value m. Then $z \geq m$. Multiplying through by (-1), gives $(-z) \leq (-m)$. Thus z has a finite minimum value m if and only if $(-z)$ has a finite maximum value $(-m)$. Thus we can change the problem of finding the *minimum* value of z into the problem of finding the *maximum* value of $(-z)$: both problems have the same optimal solution.

2. **Functional constraints with "$\geq$"**
   We multiply through any constraint that is expressed with a "$\geq$" sign by (-1), to change the sense of the inequality to "$\leq$". Note that this may result in a negative number occurring on the right side of the inequality; we explain how to deal with this below.

3. **Equality constraints**
   There are two ways of dealing with equality constraints. One method is to replace an equality constraint by two inequality constraints, where the "$=$" sign is replaced by "$\leq$" in one constraint and by "$\geq$" in the other. Thus, for example, we would replace the constraint
   $$x_1 + x_2 + x_3 = 1,$$
   by the two constraints
   $$\begin{aligned} x_1 + x_2 + x_3 &\leq 1 \\ x_1 + x_2 + x_3 &\geq 1. \end{aligned}$$
   We would then multiply through the second constraint by (-1) to change it into a "$\leq$" constraint, as explained above. Thus, we would obtain the constraints
   $$\begin{aligned} x_1 + x_2 + x_3 &\leq 1 \\ -x_1 - x_2 - x_3 &\leq -1. \end{aligned}$$

We discuss an alternative method of dealing with equality constraints in the next section of this chapter.

Following these steps, any l.p. problem can be expressed in standard form (see Definition 2.1). In particular, all the functional constraints can be expressed as "$\leq$" constraints, although some of them may have negative values on the right side as a result.

Once the problem is expressed in standard form, we prepare it for solution by the simplex algorithm as follows.

1. We add a new non-negative variable to the *left* side of each functional constraint, to change it into an equation.
2. We multipy through any of the resulting equations that have a negative right side by (-1).

Example 3.1 Suppose that in a problem with just two decision variables, x_1, x_2, we have the functional constraint

$$x_1 + 4x_2 \geq 8. \tag{3.1}$$

We first multiply through (3.1) by (-1) to change it to a "$\leq$" constraint. This gives

$$-x_1 - 4x_2 \leq -8. \tag{3.2}$$

Next, we add a surplus variable x_3 to the left side of (3.2) to convert this inequality to the equation

$$-x_1 - 4x_2 + x_3 = -8, \tag{3.3}$$

and add the non-negativity condition $x_3 \geq 0$ to the system.

Finally, we multiply through equation (3.3) by (-1) to obtain

$$x_1 + 4x_2 - x_3 = 8. \ \square \tag{3.4}$$

You will see that if we start with a "$\geq$" constraint which has a *positive* number on the right side, then the net effect of these operations is the same as converting the original inequality constraint to an equation by *subtracting* a new non-negative variable from the left side of the inequality. Thus when the inequality is strict, the positive value of this new variable represents the difference between the greater value on the left side and the smaller value on the right side of the constraint. For this reason, it is often called a *surplus* variable as, at any given feasible solution, it represents the surplus provided by that solution over the minimum amount required of a given resource. You can, when you feel confident to do so, go straight from the inequality in (3.1) to the final equation (3.4).

Example 3.2 We convert the following general linear programming problem to standard form.

Find $x_1, x_2 \in \mathbf{R}$ to minimize $z = 2x_1 + x_2$, subject to

$$\begin{aligned} x_1 - x_2 &\geq -3 && (3.5)\\ x_1 + 4x_2 &\geq 8 && (3.6)\\ 2x_1 + 3x_2 &\geq 12 && (3.7)\\ x_1, x_2 &\geq 0. && (3.8) \end{aligned}$$

1. We convert the objective function to maximization form.

$$\text{Minimize } z = 2x_1 + x_2 \to \text{Maximize } (-z) = -2x_1 - x_2.$$

2. We convert the functional constraints to "$\leq$" constraints by multiplying through each of them by (-1). This gives

$$\begin{aligned} -x_1 + x_2 &\leq 3 \\ -x_1 - 4x_2 &\leq -8 \\ -2x_1 - 3x_2 &\leq -12. \end{aligned}$$

3. We convert the functional constraints to equations by adding a new non-negative variable to the left side of each, to give

$$-x_1 + x_2 + x_3 = 3 \tag{3.9}$$
$$-x_1 - 4x_2 + x_4 = -8 \tag{3.10}$$
$$-2x_1 - 3x_2 + x_5 = -12. \tag{3.11}$$

4. We multiply through equations (3.10) and (3.11) by (-1), to make their right sides positive, giving

$$x_1 + 4x_2 - x_4 = 8 \tag{3.12}$$
$$2x_1 + 3x_2 - x_5 = 12. \tag{3.13}$$

5. We update the non-negativity constaints, to give

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0.$$

Then we can restate the problem as follows:

Find $x_1, x_2 \in \mathbf{R}$ to maximize $(-z) = -2x_1 - x_2$, subject to

$$\begin{array}{rcrcrcrcrcr} -x_1 & + & x_2 & + & x_3 & & & & & = & 3 \\ x_1 & + & 4x_2 & & & - & x_4 & & & = & 8 \\ 2x_1 & + & 3x_2 & & & & & - & x_5 & = & 12 \end{array} \tag{3.14}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0. \ \square \tag{3.15}$$

3.2 The Big M method

In solving Example 3.2 by the simplex algorithm, we run into a difficulty which does not occur in the l.p. problems we solved in the previous chapter. Putting the decision variables x_1 and x_2 equal to zero in the system (3.14) does *not* give us an initial basic feasible solution. This is because the solution $[0,0]^T$ does not satisfy the constraints (3.6) and (3.7). In terms of the simplex algorithm, the second and third equations in the system (3.14) do not contain a basic variable, because the surplus variables x_4 and x_5 each have coefficient (-1). In fact, we do not even know whether this problem possesses a feasible solution.

Introducing artificial variables

To initiate the simplex algorithm, however, we need a basic feasible solution. We therefore contrive an artificial one, *by adding a new non-negative variable to the left side of every equation that does not already contain a basic variable.*

These variables are called **artificial** variables, because they have no physical significance in the problem. In order to distinguish them from the other variables, we denote artificial variables with a bar over the x..

Example 3.3 In (3.14), we add artificial variables $\overline{x}_6$ and $\overline{x}_7$ to the second and third constraints respectively. Note that we do *not* add an artificial variable to the first equation in (3.14); this is because this equation already contains the basic variable x_3. Thus (3.14) becomes

$$\begin{array}{rcrcrcrcrcrcrcl} -x_1 & + & x_2 & + & x_3 & & & & & & & & & = & 3 \\ x_1 & + & 4x_2 & & & - & x_4 & & & + & \overline{x}_6 & & & = & 8 \\ 2x_1 & + & 3x_2 & & & & & - & x_5 & & & + & \overline{x}_7 & = & 12 \end{array} \tag{3.16}$$

and (3.15) becomes

$$x_1, x_2, x_3, x_4, x_5, \overline{x}_6, \overline{x}_7 \geq 0. \ \square$$

The revised system (3.16) now contains a total of seven variables. Only three of these can be basic (because there are just three functional constraints) and hence four variables are non-basic. Putting $x_1 = x_2 = x_4 = x_5 = 0$, gives the initial basic feasible solution

$$x_3 = 3,\ \overline{x}_6 = 8,\ \overline{x}_7 = 12.$$

Now suppose that $[x_1, x_2, x_3, x_4, x_5, 0, 0]^T$ is an augmented feasible solution to the revised problem, so that these values of the variables satisfy the system (3.16). Then $[x_1, x_2, x_3, x_4, x_5]^T$ satisfies the system (3.14), and hence is an augmented feasible solution to the original problem. Conversely, any augmented feasible solution to the original problem corresponds to an augmented feasible solution to the revised problem in which the values of $\overline{x}_6$ and $\overline{x}_7$ are both zero.

Thus any optimal solution to the original problem must give an optimal solution to the revised problem in which $\overline{x}_6 = \overline{x}_7 = 0$, and vice-versa. We must therefore adapt the objective function so that in the revised problem, the simplex algorithm will seek an optimal solution at which $\overline{x}_6 = \overline{x}_7 = 0$.

Formulating the revised objective function z_0

To form the objective function for the revised problem, we *subtract from $(-z)$ a very large multiple of each of the artificial variables.* Thus the revised objective function becomes

$$(-z_0) = -2x_1 - x_2 - M\overline{x}_6 - M\overline{x}_7,$$

where M is a huge (but unspecified) positive number. Then since we want to *maximize* $(-z_0)$, the simplex algorithm will automatically seek a solution in which $\overline{x}_6 = \overline{x}_7 = 0$. If at the optimal solution the artificial variables $\overline{x}_6$ and $\overline{x}_7$ are *not* both zero, we can conclude that the original problem is infeasible.

Expressing z_0 in non-basic variables

Before we start the simplex algorithm, there is one more step to complete. We have noted that at the initial basic feasible solution to the revised problem, both the artificial variables are *basic*. However, to be in proper form for the simplex algorithm, the objective function must be expressed in terms of non-basic variables only. Thus we must eliminate $\overline{x}_6$ and $\overline{x}_7$ from the expression for $(-z_0)$. Rewriting the objective function with all variables on the left side, in the usual way and subtracting $M\times$ the second and third constraint equations gives the following expression for $(-z_0)$ in terms of non-basic variables.

$$\begin{aligned} (-z_0) + 2x_1 + x_2 + M\overline{x}_6 + M\overline{x}_7 &= 0 \\ -M(x_1 + 4x_2 - x_4 + \overline{x}_6 &= 8) \\ -M(2x_1 + 3x_2 - x_5 + \overline{x}_7 &= 12) \end{aligned}$$

$$(-z_0) + (2-3M)x_1 + (1-7M)x_2 + Mx_4 + Mx_5 \quad = \quad -20M.$$

Solving the revised problem

The first three tableaux in the solution by the simplex algorithm are shown below. Note that we have used a double row to record the entries in equation 0, with the multiples of M in the upper half of the row and the part of the coefficient that does not involve M in the lower half. It is not necessary to do this, but you may find it helpful in the selection of the entering variable and also in the Gaussian elimination step, as you can then deal with the upper and lower parts of row 0 independently.

Since M is very large compared to all other coefficients in the system, our rule for entering variable amounts to this: scan the upper half of row 0 for the largest multiple of $-M$; in the event of a tie, take the numbers in the lower half of row 0 into consideration to choose the negative entry with greatest absolute value; if there are no negative multiples of M, then choose the negative entry of largest absolute value from the lower half of row 0; if there are no negative entries in the objective row, then the maximum value of $(-z_0)$ has been reached, and we stop. Ties for absolute value are settled arbitrarily.

In this example, the first entering variable is x_2, which has entry $(-7M)$ in the top half of row 0.

Note, also, that since row (0) is *never* a pivot row during the Gaussian elimination step, multiples of M *only* occur in row (0).

Basis	*Eqn*	$-z_0$	x_1	x_2	x_3	x_4	x_5	$\bar{x}_6$	$\bar{x}_7$	RS	*Limit on EV*
			$-3M$	$-7M$		$1M$	$1M$			$-20M$	-
-	0	1	$+2$	$+1$	0	$+0$	$+0$	0	0	$+0$	-
x_3	1	0	-1	1	1	0	0	0	0	3	$x_2 \leq 3$
$\bar{x}_6$	2	0	1	4	0	-1	0	1	0	8	$x_2 \leq 2 \leftarrow$
$\bar{x}_7$	3	0	2	3	0	0	-1	0	1	12	$x_2 \leq 4$
			$-1.25M$			$-0.75M$	M	$1.75M$		$-6M$	-
-	0	1	$+1.75$	0	0	$+0.25$	$+0$	-0.25	0	-2	-
x_3	1	0	-1.25	0	1	0.25	0	-0.25	0	1	no limit
x_2	2	0	0.25	1	0	-0.25	0	0.25	0	2	$x_1 \leq 8$
$\bar{x}_7$	3	0	1.25	0	0	0.75	-1	-0.75	1	6	$x_1 \leq 4.8 \leftarrow$
								$1M$	$1M$		
-	0	1	0	0	0	-0.8	1.4	$+0.8$	-1.4	-10.4	
x_3	1	0	0	0	1	1	-1	-1	1	7	
x_2	2	0	0	1	0	-0.4	0.2	0.4	-0.2	0.8	
x_1	3	0	1	0	0	0.6	-0.8	-0.6	0.8	4.8	

We pause here to analyse the solution so far. The augmented basic feasible solution to the revised problem given by the second tableau is $[0,2,1,0,0,0,6]^T$ and the corresponding augmented solution to the original problem is therefore $[0,2,1,0,0]^T$. However, this is not a *feasible* solution to the original problem, because it does not satisfy the third functional constraint. We can tell from the tableau that this solution will not satisfy (3.16), because the artificial variable $\bar{x}_7$ has a positive value.

In the third tableau, however, both artificial variables are non-basic and hence have value 0. Note that, simultaneaously, the big Ms have disappeared from the coefficient of every variable *except* for $\bar{x}_6$ and $\bar{x}_7$ in the objective row. The part of these coefficients involving M will now remain the same during any further iterations. The augmented basic feasible solution to the revised problem is now $[4.8, 0.8, 7, 0, 0, 0, 0]^T$ and the corresponding augmented solution to the original problem is therefore $[4.8, 0.8, 7, 0, 0]^T$. It is easily checked that this gives a basic feasible solution to the original problem.

Solution of the original problem

We obtain the tableau corresponding to this basic feasible solution for the original problem, by replacing $(-z_0)$ by $(-z)$, and deleting the columns corresponding to $\overline{x}_6$ and $\overline{x}_7$ (this corresponds to putting $\overline{x}_6 = \overline{x}_7 = 0$). We illustrate this below, where the third tableau is repeated for continuity. We now complete the solution of the problem exactly as in the examples of the previous chapter.

Basis	*Eqn*	$-z$	x_1	x_2	x_3	x_4	x_5	$\overline{x}_6$	$\overline{x}_7$	RS	*limit on EV*
-	0	1	0	0	0	-0.8	1.4	$1M$ $+0.8$	$1M$ -1.4	-10.4	- -
x_3	1	0	0	0	1	1	-1	-1	1	7	$x_4 \leq 7 \leftarrow$
x_2	2	0	0	1	0	-0.4	0.2	0.4	-0.2	0.8	No limit
x_1	3	0	1	0	0	0.6	-0.8	-0.6	0.8	4.8	$x_4 \leq 8$
-	0	1	0	0	0.8	0	0.6			-4.8	
x_4	1	0	0	0	1	1	-1			7	
x_2	2	0	0	1	0.4	0	-0.2			3.6	
x_1	3	0	1	0	-0.6	0	-0.2			0.6	

The solution is now optimal. The augmented basic feasible solution given by the last tableau is $[0.6, 3.6, 0, 7, 0]^T$ and the maximum value of $(-z)$ is -4.8. Hence the minimum value of z is 4.8 and the optimal solution to the original problem is $x_1 = 0.6$ and $x_2 = 3.6$.

The method we have described in the course of solving this example is known as the **Big M method** for solving problems for which there is no obvious initial basic feasible solution. There is an alternative, known as the **Two Phase method**, for solving this kind of problem. This method also uses artificial variables to create a new problem in standard form with an obvious initial basic feasible solution. The method of solution then parallels the Big M method, but divides up the problem into two separate subproblems. We will not describe the details of the method here, but interested readers will find it fully explained in the recommended course text book by Hillier and Lieberman. Either method is acceptible for solving examination questions, but you should be prepared to explain the logic behind the steps of whichever method you use.

3.3 Problems with equality constraints

Problems with an equality constraint often arise in practice from the requirement to find a *blend* or *mixture* at minimum cost, where the amounts of each constituent of the blend are expressed in terms of *percentages* or, equivalently, as *parts of one unit* of the blend.

Example 3.4 A firm selling coking coal to power stations requires to formulate a blend of coal with a phosphorous content of at most 0.04% and an ash impurity of at most 4.5%. Three different grades of coal are available to blend; the phosphorous and ash content and the price of each grade is given in the table below.

Grade	*%Phos*	*%Ash*	*\$/tonne*
1	0.02	4.0	90
2	0.05	5.0	70
3	0.07	3.0	60

We formulate the problem of finding a blend satisfying the requirements at minimum cost as an l.p. problem.

Suppose that 1 tonne of the blend is formed by mixing x_j tonnes of grade j coal, for $j = 1, 2, 3$. Then the cost of 1 tonne of the blend is $\$(90x_1 + 70x_2 + 60x_3)$. We define the objective function to be $z = 9x_1 + 7x_2 + 6x_3$, so that the cost of 1 tonne of the blend is \$$10z$.

We next formulate the constraints.

1. Since 100 tonnes of blend can contain at most 0.04 tonnes of phosphorous, we have:

$$0.02x_1 + 0.05x_2 + 0.07x_3 \leq 0.04.$$

The coefficients in this constraint can be simplified by multiplying through by 100, to give:

$$2x_1 + 5x_2 + 7x_3 \leq 4.$$

2. Since 100 tonnes of blend can contain at most 4.5 tonnes of ash, we have:

$$4x_1 + 5x_2 + 3x_3 \leq 4.5.$$

3. Since 1 tonne of blend is produced by mixing x_j tonnes of grade j coal, j=1,2,3, we have:

$$x_1 + x_2 + x_3 = 1.$$

4. The decision variables are non-negative.

Thus the problem can be formulated as an l.p problem as follows.
Find $x_1, x_2, x_3 \in \mathbf{R}$ to minimize $z = 9x_1 + 7x_2 + 6x_3$, subject to

$$\begin{aligned} 2x_1 + 5x_2 + 7x_3 &\leq 4 \\ 4x_1 + 5x_2 + 3x_3 &\leq 4.5 \\ x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The next step is to express the problem in standard form. We start by changing to a maximization problem.

$$\text{Minimize } z = 9x_1 + 7x_2 + 6x_3 \to \text{Maximize } (-z) = -9x_1 - 7x_2 - 6x_3.$$

As we have seen in Section 3.1, we can express the equality constraint in standard form by replacing it with a pair of inequalities with opposite senses. However, there are neater ways of dealing with this situation computationally. One such method is to use the equality constraint to eliminate one of the variables from the objective function and the remaining constraints. However, this is not usually a practicable method in large systems. A simpler method which can be used in all cases is to provide a basic variable in the equality constraint by adding an *artificial variable* to the left side. We illustrate this method below. The revised problem becomes:

Find $x_1, x_2, x_3 \in \mathbf{R}$ to maximize $(-z) = -9x_1 - 7x_2 - 6x_3$, subject to

$$\begin{aligned} 2x_1 + 5x_2 + 7x_3 + x_4 &\leq 4 \\ 4x_1 + 5x_2 + 3x_3 + x_5 &\leq 4.5 \\ x_1 + x_2 + x_3 + \overline{x}_6 &= 1 \\ x_1, x_2, x_3, x_4, x_5, \overline{x}_6 &\geq 0. \end{aligned}$$

We now have three equations in six unknowns. Choosing x_1, x_2, x_3 as non-basic variables, we achieve the intial basic feasible solution $\mathbf{x} = [0, 0, 0, 4, 4.5, 1]^T$.

The objective function for the revised problem is

$$(-z_0) = (-z) - M\overline{x}_6 = -9x_1 - 7x_2 - 6x_3 - M(1 - x_1 - x_2 - x_3),$$

using the equation to substitute for $\overline{x}_6$. This gives

$$(-z_0) = (M-9)x_1 + (M-7)x_2 + (M-6)x_3 - M.$$

The initial solution is clearly not optimal, since we can increase $(-z_0)$ by increasing any one of x_1, x_2 or x_3 from zero. In fact, because the value of M is assumed to be very large compared with the coefficients in the original objective function, we have effectively a tie for entering variable. We choose EV=x_1, arbitrarily. The initial tableau and the tableau obtained after the first iteration of the simplex algorithm are shown below.

Basis	*Eqn*	$-z_0$	x_1	x_2	x_3	x_4	x_5	$\overline{x}_6$	RS	*limit on EV*
			$-1M$	$-1M$	$-1M$				$-1M$	
-	0	1	9	7	6	0	0	0	0	
x_4	1	0	2	5	7	1	0	0	4	$x_1 \leq 2$
x_5	2	0	4	5	3	0	1	0	4.5	$x_1 \leq 1.125$
$\overline{x}_6$	3	0	1	1	1	0	0	1	1	$x_1 \leq 1 \leftarrow$
								$1M$		
-	0	1	0	-2	-3	0	0	-9	-9	
x_4	1	0	0	3	5	1	0	-2	2	
x_5	2	0	0	1	-1	0	1	-4	0.5	
x_1	3	0	1	1	1	0	0	1	1	

At the end of the first iteration, the artificial variable $\overline{x}_6$ is non-basic and hence zero. We therefore have a basic feasible solution to the original problem, with $x_1 = 1$, $x_2 = x_3 = 0$, $x_4 = 2$ and $x_5 = 0.5$. We can now eliminate $\overline{x}_6$ from the problem, delete its column and replace $(-z_0)$ by $(-z)$. This gives $-z = -9 + 2x_2 + 3x_3$. Thus this tableau does not represent an optimal solution. We choose x_3 as the next entering variable as it has the highest postive objective coefficient. Continuing for two further iterations, we obtain the following tableau.

Basis	*Eqn*	$-z$	x_1	x_2	x_3	x_4	x_5	RS
-	0	1	0	0	0	0.625	0.125	-7.6875
x_3	1	0	0	0	1	0.125	-0.375	0.0625
x_2	2	0	0	1	0	0.125	0.625	0.5625
x_1	3	0	1	0	0	-0.25	-0.25	0.3750

The basic feasible solution given by this tableau is optimal. Hence the optimal solution is $x_1 = 0.375$, $x_2 = 0.5625$, $x_3 = 0.0625$ and the maximum value of $(-z)$ is -7.6875, so that the minimum value of z is 7.6875.

Thus to produce 1 tonne of the blend satisfying the requirements at minimum cost, we mix 0.375 tonnes of grade 1, 0.5625 tonnes of grade 2 and 0.0625 tonnes of grade 3 coal. The cost of this mixture is \$76.88 per tonne. □

3.4 Problems with no feasible solution

In this section, we consider what happens when we try to solve an infeasible problem by the simplex algorithm. As an illustration, we apply the simplex algorithm to the l.p. problem that you explored graphically in Exercise 1.5(b),

Example 3.5 Find $x_1, x_2 \in \mathbf{R}$ to maximize $z = x_1 + x_2$, subject to

$$2x_1 + x_2 \leq 2 \tag{3.17}$$

$$3x_1 + 4x_2 \geq 12 \tag{3.18}$$

$$x_1, x_2 \geq 0. \tag{3.19}$$

To prepare this problem for the simplex algorithm, we add slack variable x_3 to the left side of (3.17) and subtract a surplus variable x_4 from the left side of (3.18). Then since (3.18) contains no basic variable, we add an artificial variable $\overline{x}_5$ to the left side of (3.18). Now a feasible solution to the original problem exists if and only if we can find a feasible solution to the revised problem in which the artificial variable $\overline{x}_5$ is zero. We therefore form a new objective function z_0 by subtracting the term $M\overline{x}_5$ from z, where M is a very large number. Then the maximum value of z_0 will occur when $\overline{x}_5$ is as small as possible. The revised problem is as follows.

Find $x_1, x_2 \in \mathbf{R}$ to maximize $z_0 = x_1 + x_2 - M\overline{x}_5$, subject to

$$\begin{array}{rcrcrcrcrcl} 2x_1 & + & x_2 & + & x_3 & & & & & = & 2 \\ 3x_1 & + & 4x_2 & & & - & x_4 & + & \overline{x}_5 & = & 12 \end{array}$$

$$x_1, x_2, x_3, x_4, \overline{x}_5 \geq 0.$$

The revised problem has an initial basic feasible solution at which $\overline{x}_5$ is the basic variable in the second constraint. Hence we must eliminate $\overline{x}_5$ from the objective function. To do this, we rewrite the objective function with all variables on the left side and subtract M times the second constraint.

$$\begin{array}{rcl} z_0 - x_1 - x_2 + M\overline{x}_5 & = & 0 \\ -M(3x_1 + 4x_2 - x_4 + \overline{x}_5 & = & 12) \\ \hline z_0 + (-1 - 3M)x_1 + (-1 - 4M)x_2 + Mx_4 & = & -12M. \end{array}$$

The first two tableau in the solution by the simplex algorithm are shown below.

Basis	*Eqn*	z_0	x_1	x_2	x_3	x_4	$\overline{x}_5$	RS	*limit on EV*
			$-3M$	$-4M$		$1M$		$-12M$	-
-	0	1	-1	-1	0	$+0$	0	$+0$	-
x_3	1	0	2	1	1	0	0	2	$x_2 \leq 2 \leftarrow$
$\overline{x}_5$	2	0	3	4	0	-1	1	12	$x_2 \leq 12/4$
			$5M$		$4M$	$1M$		$-4M$	
-	0	1	$+1$	0	$+1$	$+0$	0	$+2$	
x_2	1	0	2	1	1	0	0	2	
$\overline{x}_5$	2	0	-5	0	-4	-1	1	4	

Since all entries in the objective row in the second tableau are *positive*, the basic feasible solution given by this tableau is an optimal solution for the revised problem. However, at this solution $\overline{x}_5 = 4$. Hence the minimum value of the $\overline{x}_5$ is 4 and the original problem is infeasible. □

3.5 Post optimal analysis

Modelling an l.p. problem that arises from a practical situation and finding an optimal solution are only the first two stages in the study. The model will need to be tested and the solution debugged. Then management will need to consider the implications of the soltution. In this section, we consider some simple cases of issues that will need to be considered.

3.5.1 Interpretation of slack and surplus variables

The augmented optimal solution gives more useful information than just the optimal values of the decision variables. It also gives the values of the slack and surplus variables at the optimal solution. Suppose we had added a *slack* variable to the left side of constraint i.

Then the value of this slack variable at any basic feasible solution represents the amount of resource i left unused in that solution.

Example 3.6 The augmented optimal solution of the Boatbuilder's problem (Example 2.2) is $\mathbf{x} = [44, 114, 660, 0, 0]^T$. Since $x_4 = 0$ and $x_5 = 0$, this solution uses all the available supply of machine time and hand labour. However $x_3 = 660$, so that 660kg of the available aluminium is not used by this solution. Thus, if the company wants to expand production beyond the proposed optimum schedule, it should consider the possibility of increasing the amount of machine time or hand labour. □

A similar analysis can be applied to the value of the surplus variables at the optimal solution. If we subtracted a surplus variable from the left side of constraint j, then this variable represents the amount in excess of the minimum requirement of ingredient j that is provided by that solution (hence the reason for the term "surplus"). Thus any surplus variables that are zero at the optimal solution indicate where this solution provides only the minimum requirements. Constraints for which the corresponding slack or surplus variable is zero at the optimum solution are said to be **binding** on the solution.

3.5.2 Reoptimization

As we noted in our discussion in Chapter 1 on the certainity assumptions, the coefficients in the constraints and objective function are usually based on predictions and some will almost certainly need to be modified in the light of experience. Small changes in just some of these coefficients may lead to a change in the optimal solution, and the possibility of making changes to the model to achieve a more desirable outcome will also have to be analysed. The coefficients in the objective function and the constraints are collectively known as the **parameters** of the model.

When a change in one of the paramenters is made, we do not have to begin finding an optimal solution all over again from the beginning. If we have made only a small change in a parameter, it is likely that the new optimal is at a feasible corner-point very near to (or even the same as) the old solution. Thus we take the optimal solution we have already found and modify the expressions for the objective function and the variables in the *final* tableau to take account of the changes in the parameters. Then we use the amended optimal solution as the initial basic feasible solution and start the optimization process by the simplex algorithm from this point. This process is called **reoptimization**. In this section, we consider only the effect of changes to the objective coefficients c_j, as an illustration of reoptimization. Changes to the b_i are discussed in the next chapter.

Example 3.7 Suppose that in the Boatbuilder's problem (Example 1.1), the profit on each dinghy is increased from £60 to £80. Does this change affect the optimal solution?

The change in the profit on making x_1 dinghies and x_2 canoes has increased by an amount £$20x_1$. Working in units of £10, the new objective function z_0 can be expressed in terms of the original objective function z as

$$z_0 = z + 2x_1. \tag{3.20}$$

The final tableau for the solution of this problem by the simplex algorithm is given in section 2.5. We repeat it here for convenience.

Basis	*Eqn*	z	x_1	x_2	x_3	x_4	x_5	RS
-	0	1	0	0	0	19/20	3/20	834
x_1	1	0	1	0	0	1/5	−1/10	44
x_2	2	0	0	1	0	−1/20	3/20	114
x_3	3	0	0	0	1	−9/2	3/2	660

Fron this tableau, we see that at the optimal solution

$$z = 834 - 19/20x_4 - 3/20x_5 \tag{3.21}$$
$$x_1 = 44 - 1/5x_4 + 1/10x_5. \tag{3.22}$$

Thus from (3.20), we obtain the new objective function z_0 in terms of the non-basic variables x_4 and x_5 by adding 2 times equation (3.22) to equation (3.21), giving

$$z_0 = 922 - 27/20x_4 + 1/20x_5.$$

Thus the optimal solution for the original problem is not optimal for the new problem, because we can increase the value of z_0 by increasing x_5. We start the simplex algorithm to find the optimal solution for z_0 by replacing z by z_0 in the objective row in the tableau shown above. This gives the first tableau shown below. Choosing x_5 as entering variable and performing one iteration of the simplex algorithm, we obtain the second tableau shown below.

Basis	*Eqn*	z	x_1	x_2	x_3	x_4	x_5	RS	*upper bound on EV*
-	0	1	0	0	0	27/20	−1/20	922	-
x_1	1	0	1	0	0	1/5	−1/10	44	no limit
x_2	2	0	0	1	0	−1/20	3/20	114	$x_5 \leq 760$
x_3	3	0	0	0	1	−9/2	3/2	660	$x_5 \leq 440 \leftarrow$
-	0	1	0	0	1/30	6/5	0	944	
x_1	1	0	1	0	1/15	−1/10	0	88	
x_2	2	0	0	1	−1/10	2/5	0	48	
x_5	3	0	0	0	2/3	−3	1	440	

The basic feasible solution given by this tableau is optimal. Hence with the revised coefficient for x_1, the optimal solution is $x_1 = 88$, $x_2 = 48$, and the maximum value of z_0 is 944, giving a profit of £9440. □

In terms of the graphical solution for this problem (see Example 1.3 and Figure 1.2), the optimal solution for z_0 occurs at the feasible corner-point B, whereas the optimal solution for z occurs at C. This change occurs because altering the coefficient of x_1 changes the gradient of the objective line.

3.5.3 Sensitivity analysis

We have seen in the previous example that a change in just one parameter can change the optimal solution. However, we considered quite a large percentage change in the parameter and a smaller one might not have had any effect on the optimal solution. Part of the post optimal analysis is to determine those parameters in which any change will cause a change in the optimal solution. These parameters are called **sensitive** and the analysis required to decide which parameters are sensitive and the amount by which the other parameters can vary without changing the model is called **sensitivity analysis**.

The most sensitive parameters in the model are the coefficients a_{ij} and b_i in the constraints that are *binding* at the optimal solution. Any alteration in any one of these will change the optimal solution. To understand this, think of the graphical solution of a problem with two decision variables. The optimal corner-point solution is determined by the intersection of two (or more) lines representing the constraints that are binding at the optimal solution. An alteration in any coefficient in a binding constraint line will change its points of intersection with every other constraint line and hence change the optimal solution.

By contrast, the coefficients in the constraints that are not binding at the optimal solution and the objective coefficients are usually not sensitive. Because of the uncertainity concerning the exact values of some of the parameters when the model is constructed, it is helpful to determine the range over which each parameter can vary without changing the optimum solution. Again, we limit ourselves in this section to a change in an objective coefficient.

Example 3.8 Suppose in the Boatbuilder's problem, solved by the simplex algorithm in Section 2.5, we change the coefficient of x_1 in the objective function $z = 6x_1 + 5x_2$ from 6 to $6 + t$, where $t \in \mathbf{R}$. We find the range of values of t for which the current optimal solution remains optimal.

We follow the same method as we used in Example 3.7. The new objective function z_0 is given by $z_0 = (6+t)x_1 + 5x_2$, and hence

$$z_0 = z + tx_1.$$

Thus we can express z_0 in terms of non-basic variables by adding t times equation (3.22) to equation (3.21), giving

$$z_0 = (834 + 44t) - (19/20 + 1/5t)x_4 - (3/20 - 1/10t)x_5.$$

The current optimal solution remains optimal while the coefficients of x_4 and x_5 in this expression are negative or zero. Thus from the coefficient for x_4, we require

$$\begin{aligned} 19/20 + 1/5t &\geq 0 \\ t &\geq -4.75; \end{aligned}$$

and considering the coefficient of x_5, we require

$$\begin{aligned} 3/20 - 1/10t &\geq 0 \\ t &\leq 1.5. \end{aligned}$$

Thus when $-4.75 \leq t \leq 1.5$, the current optimal solution remains optimal. Thus the coefficient $(6+t)$ can vary between 1.25 and 7.5 without affecting the optimal solution. In the original problem, this would indicate that the optimal monthly schedule to produce 44 dinghies and 114 canoes remains optimal while the profit made on each dinghy varies between £12.50 and £75. □

3.6 Computer implementation

Linear programming packages are widely available for most modern computer systems and continue to be improved. The computer code for these systems does not follow the tableau method exactly, but uses the **revised simplex method**. This works by constructing at each iteration only the row corresponding to the objective function and the column corresponding to the entering variable. This cuts out much of the redundancy of the ordinary tableau method. Using the revised simplex method, recent mainframe computers can now solve problems with as many as 5000 functional constraints and 10000 variables in less than an hour. Packages for use with microcomputers are also widely available.

Although the revised simplex algorithm solves most l.p. problems that arise in practice in polynomial time, examples of l.p. problems have been constructed that require an exponential amount of time to solve them (such problems typically have many degenerate basic feasible solutions). Thus the simplex algorithm is *not* a polynomial algorithm. A different approach to solving l.p. problems is provided by the **interior-point algorithms**. Whereas the simplex algorithm goes from corner-point to corner-point round the boundary of the feasible region, the interior-point algorithms cut through the interior of the feasible region to reach an optimal solution. The first interior-point algorithm, known as the *ellipsoid method*, was published in 1979 by L.G. Khachiyan, who showed that it solves all l.p. problems in polynomial time. Recently a powerful new version has been developed by N. Karmarkar. Using Karmarker's algorithm, or one of its variants, it has been shown possible to solve really huge l.p. problems with tens of thousands of functional constraints. However, compared with the simplex algorithm, Karmarker's algorithm cannot handle most aspects of sensitivity analysis efficiently. Thus it is likely that the simplex algorithm will continue to be used for some time to come, on its own or in tandem with an interior point algorithm. The interested reader is referred to Hillier and Lieberman for a fuller discussion of these issues. This material will not be examined, however.

3.7 Exercises

Exercise 3.1
Rewrite the following problem as a *maximization* problem, introducing slack, surplus and artificial variables into the constraints, as appropriate.

Find $x_1, x_2, x_3 \in \mathbf{R}$ to minimize $z = 3x_1 - x_2 + 2x_3$, subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 6 \\ 4x_1 + 5x_2 + 4x_3 &\leq 24 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

(a) State the initial basic feasible solution.

(b) Give a revised objective function and rewrite it as a constraint expressed in terms on non-basic variables.

(c) Use the simplex algorithm to find an optimal solution and the minimum value of z by the Big M method.

Exercise 3.2
Use the simplex algorithm to decide for each of the following problems whether it has a feasible solution. If so, find an optimal solution by the Big M method and say whether it is unique.

(a) Find $x_1, x_2 \in \mathbf{R}$ to minimize $z = 3x_1 - x_2$, subject to

$$\begin{aligned} x_1 - x_2 &\geq -1 \\ x_1 + x_2 &\geq 4 \\ 3x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(b) Find $x_1, x_2 \in \mathbf{R}$ to minimize $z = x_1 + x_2$, subject to

$$\begin{aligned} x_1 - x_2 &\leq -1 \\ x_1 + x_2 &\geq 3 \\ 2x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Check your conclusions by investigating the feasible region for each of these problems graphically.

Exercise 3.3
Use the simplex algorithm to solve the following l.p. problem by adding an artificial variable to the equality constraint and using the Big M method.

Find $x_1, x_2, x_3 \in \mathbf{R}$ to minimize $z = x_1 + 2x_2 + 4x_3$, subject to

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 5x_1 + 3x_2 + x_3 &\leq 4 \\ 4x_1 + 5x_2 + 2x_3 &\leq 6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Exercise 3.4
Suppose that the coefficient of x_3 in the objective function for the l.p. problem of Exercise 3.1 is changed from 2 to $2 + t$. Use the final tableau of your solution to Exercise 3.1 to find the range of values of t for which the current optimal solution remains optimal.