



**FACULTY OF SCIENCE AND ENGINEERING**  
**DEPARTMENT OF MATHEMATICS AND STATISTICS**

**END OF SEMESTER EXAMINATION PAPER 2016**

MODULE CODE: MS4131

SEMESTER: Spring 2016

MODULE TITLE: Linear Algebra 1

DURATION OF EXAM: 2.5 hours

LECTURER: Mr. Kevin O'Brien

GRADING SCHEME: 80 marks

100% of module grade

EXTERNAL EXAMINER: Prof. J. King

**INSTRUCTIONS TO CANDIDATES**

Scientific calculators approved by the University of Limerick can be used.  
Students must attempt any 4 questions from 5.

## Question 1

### Part A. Matrix Multiplication (4 Marks)

Given the matrices

$$A = \begin{pmatrix} 5 & -1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 7 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix},$$

calculate the products  $AB$  and  $CA$ .

### Part B. Matrix Multiplication (5 Marks)

Suppose  $A$  is a lower triangular matrix of the form;

$$A = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

- (i) (1 Mark) State the transpose of  $A$ .
- (ii) (3 Marks) Compute  $B$  where  $B = A \times A^T$ .
- (iii) (1 Mark)  $B$  is a symmetric matrix. What is meant by this?

*Please Turn Over For Parts D and E*

### Part C. Diagonal Matrices (3 Marks)

Consider the following diagonal matrix  $D$ . Provide answers for the following questions in terms of the values  $a$ ,  $b$  and  $c$ .

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

- (i) (1 Mark) Write an expression for the trace of the matrix  $D$ .
- (ii) (1 Mark) State the inverse of  $D$ , i.e.  $D^{-1}$ .
- (iii) (1 Mark) State the matrix  $D^5$ .

**Part D. Invertible Matrices (4 Marks)**

Show that if  $A$  is an  $n \times n$  invertible matrix that satisfies

$$9A^3 + A^2 - 3A = 0$$

where  $A^n = \underbrace{A \dots A}_{n \text{ times}}$ ,  $I$  is the  $n \times n$  identity matrix and  $0$  is the  $n \times n$  zero matrix, then the inverse of  $A$  is given by

$$A^{-1} = \frac{1}{3}I + 3A.$$

**Part E. Matrix Transposition (4 Marks)**

Let  $A$  and  $B$  be  $m \times n$  matrices.

$$(AB)^T = B^T \times A^T$$

(i) (4 Marks) Prove this identity for  $A$  and  $B$ .

*A proof that is provided on the basis that  $A$  and  $B$  are both  $2 \times 2$  matrices will be sufficient for full marks.*

## Question 2

### Part A. Fundamental Theorem of Invertible Matrices (4 Marks)

The Fundamental Theorem of Invertible Matrices states that a set of mathematical expressions concerning an  $n \times n$  matrix  $A$  are each equivalent to one another.

- (i) ( $4 \times 1$  Mark) State any four of these expressions.

### Part B. Inverting a Matrix with Elementary Row Operations (6 Marks)

In this question, you are required to find the inverse of the following matrix using elementary row operations.

$$A = \begin{pmatrix} 1 & -4 & -3 \\ 1 & 3 & 5 \\ -2 & 0 & -4 \end{pmatrix}$$

- (i) (2 Mark) Write down the augmented matrix of this system.
- (ii) (4 Marks) Find the inverse of the matrix, using elementary row operations. Show your workings for each stage of the calculation.

*Please Turn Over For Parts C and D*

**Part C. Inverting a Matrix with Co-Factor Method (9 Marks)**

$$B = \begin{pmatrix} 4 & 3 & -1 \\ -5 & -3 & 1 \\ -2 & 3 & -2 \end{pmatrix}$$

- (i) (5 Marks) For each element of  $B$ , calculate the corresponding minor. Show your workings for each calculation. State the matrix of minors.
- (ii) (2 Marks) Hence or otherwise, compute the determinant of  $B$  i.e.  $\det(B)$ .
- (iii) (1 Mark) Compute the cofactor matrix for  $B$  i.e.  $\text{cof}(B)$ .
- (iv) (1 Marks) State the inverse matrix of  $B$ , given by

$$B^{-1} = \frac{1}{\det(B)} \text{cof}(B)^T.$$

**Part D. Inverting Multiples of a Matrix (1 Mark)**

Suppose that the inverse of the following matrix  $M$

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 0 & -2 \\ -6 & -2 & 2 \end{pmatrix}$$

is given as

$$M^{-1} = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ -0.25 & -1.0 & -0.75 \\ 0.50 & 0.5 & 0.50 \end{pmatrix}$$

- (i) (1 Mark) State the inverse of the matrix  $N$  where  $N = 2M$ .

$$N = 2M = \begin{pmatrix} 4 & 4 & 4 \\ 8 & 0 & -4 \\ -12 & -4 & 4 \end{pmatrix}$$

### Question 3

#### Part A. Vector Calculations (7 Marks)

Consider the three vectors in  $\mathbb{R}^3$ :

$$u = (1, 2, 4), \quad v = (0, 4, 3), \quad w = (-4, 2, 3).$$

- (i) (2 Marks) Evaluate  $\|u\|$  and  $\|v\|$ ,
- (ii) (3 Marks) Evaluate  $u \cdot v$ ,  $u \times v$  and the angle between  $u$  and  $v$ .
- (iii) (2 Marks) Calculate the scalar triple product  $u \cdot (v \times w)$ .

#### Part B. Orthonormal Projections (7 Marks)

If

$$u = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad a = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix},$$

- (i) (3 Marks) Find the vector component of  $u$  along  $a$ ,  $\text{proj}_a u$
- (ii) (2 Marks) Find the vector orthogonal component to  $a$ ;
- (iii) (2 Marks) Calculate the norm of  $\text{proj}_a u$  and the norm of  $u - \text{proj}_a u$ ;

#### Part C. Proofs for Vector Products (6 Marks)

- (i) (3 Marks) Prove that, for any  $u, v \in \mathbb{R}^3$

$$(u + v) \times w = u \times w + v \times w.$$

- (ii) (3 Marks) Prove that, for any  $u, v \in \mathbb{R}^3$

$$(u \times v) = (ku) \times v = u \times (kv).$$

## Question 4

### Part A. System of Linear Equations (7 Marks)

Consider the linear system

$$\begin{aligned}x_1 + x_3 &= 4 \\2x_1 + 4x_2 + x_3 &= -3 \\x_2 + 3x_3 &= 7.\end{aligned}$$

- (i) (2 Marks) Write down the coefficient matrix and the augmented matrix of this system.
- (ii) (5 Marks) Solve the system of equations, using any appropriate method. Show your workings for each stage of the calculation.

### Part B. Proof of Vector Identities (7 Marks)

- (i) (1 Marks) Let  $u$  and  $v$  be two vectors in  $\mathbb{R}^3$  and let  $\theta$  be the angle between them. Define the scalar product  $u \cdot v$  in terms of  $\theta$
- (i) (2 Marks) Hence or otherwise, prove the so-called “*Cauchy-Schwarz Inequality*”:

$$\|u \cdot v\| \leq \|u\| \times \|v\|$$

- (ii) (4 Marks) Hence or otherwise, prove the so-called “*Triangular Inequality*”

$$\|u + v\| \leq \|u\| + \|v\|$$

### Part C. Distance from Planes (6 Marks)

- (i) (3 Marks) Give the general form of the equation of the plane  $\pi$  in  $\mathbb{R}^3$  passing through the point  $P_0 = (1, 0, 2)$  with the vector  $n = (-5, 5, 2)$  as the normal.
- (ii) (3 Marks) Show that the point  $Q = (1, -1, 1)$  does not lie in the plane  $\pi$  and find its distance from  $\pi$ .

## Question 5

### Part A. Eigenvalues and Eigenvectors (12 Marks)

Consider the following matrix  $A$

$$A = \begin{pmatrix} 0.5 & 1 & 1.5 \\ -3 & -5 & -3 \\ 3.5 & 5 & 2.5 \end{pmatrix}$$

- (i) (2 Marks) Determine the Characteristic Equation for  $A$ .
- (ii) (2 Marks) Determine the eigenvalues for  $A$ .
- (iii) (4 Marks) For each eigenvalues, compute the corresponding eigenvectors of  $A$ .
- (iv) (2 Marks) Diagonalise  $A$ ; i.e, give a matrix  $P$  and a diagonal matrix  $D$ , such that  $A = PDP^{-1}$ .
- (v) (2 Marks) Hence, evaluate  $A^4$ .

### Part B. Determinants of Matrices (4 Marks)

You are given the following piece of information concerning a matrix  $A$ .

$$|A| = \begin{vmatrix} 0 & 4 & 7 \\ -1 & -1 & 7 \\ 1 & 5 & 1 \end{vmatrix} = 4$$

- (i) (2 Marks) Hence or otherwise, state the determinant of matrices  $B$  and  $C$ . Provide a brief justification for your answer.

$$B = \begin{pmatrix} 0 & 4 & 7 \\ -3 & -1 & 7 \\ 3 & 5 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 8 & 14 \\ -2 & -2 & 14 \\ 2 & 10 & 2 \end{pmatrix}$$

- (ii) (2 Marks) State the determinant of matrices  $D$  and  $E$ . Provide a brief justification for your answer.

$$D = \begin{pmatrix} 0 & 4 & 8 \\ -3 & -1 & -2 \\ 3 & 5 & 10 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 8 & 14 \\ 0 & -2 & 14 \\ 0 & 10 & 2 \end{pmatrix}$$



**Part C. Row-Echelon Form of a Matrix (4 Marks)**

Consider the matrices  $U, V, W$  and  $X$  presented below. For each matrix state one reason why that matrix is not in row-echelon form. Provide distinct answers for each of the four matrices.

$$U = \begin{pmatrix} 1 & 2 & 6 & 3 & 5 \\ 0 & 1 & 4 & 0 & 6 \\ 0 & 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 1 & 6 & 8 & 5 \\ 0 & 2 & 6 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$W = \begin{pmatrix} 1 & 1 & 6 & 8 & 5 \\ 0 & 1 & 6 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 6 & 8 & 5 \\ 0 & 1 & 6 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

**Marking Scheme:**  $4 \times 1$  Marks where 1 Mark is awarded for each valid and distinct reason.

*Please Turn Over For Part C*