OPERATIONS RESEARCH 2 - ALGORITHMS

4.4.1 O-notation

Let X be a subset of $\mathbb{R}$ and suppose that $f : \mathbb{X} \to \mathbb{R}$ and $g : \mathbb{X} \to \mathbb{R}$ are two functions of a real variable x. If, for all large values of $x \in X$, the graph of the function f lies closer to the x-axis than the graph of some fixed positive multiple of the function g, then we say that f is *of order* g and we write "$f(x)$ is $O(g(x))$".
The distance of the point $(x, f(x))$ on the graph $y = f(x)$ from the x-axis is the absolute value of $f(x)$, which we denote by $|f(x)|$ (see Example 4.9). Thus we can express the definition of *order of a function* more formally as follows.

Definition 4.25 *Let $f : \mathbb{X} \to \mathbb{R}$ and $g : \mathbb{X} \to \mathbb{R}$ be two functions with a common domain $X \subseteq \mathbb{R}$. Then we say that f* **is of order** g*, written "$f(x)$ is $O(g(x))$", if we can find a positive real number M and a real number x_0 such that*

$$|f(x)| \leq M|g(x)|,$$

for all $x > x_0$.

4.4.2 Power functions

In using the O-notation, we often compare a given function f with one of the following family of functions.

Definition 4.26 *Let s be any positive rational number. Then the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = x^s$$

is called the **power function** *of x with exponent s.*

The following table compares the values of the power functions $f(x) = x^s$, where $s = 0.5, 1, 1.5, 2, 3$ (values of $f(x)$ have been entered correct to 2 decimal places).

x	0	0.5	1	2	3	4
$x^{0.5}$	0	0.71	1	1.41	1.73	2
x^1	0	0.5	1	2	3	4
$x^{1.5}$	0	0.35	1	2.83	5.20	8
x^2	0	0.25	1	4	9	16
x^3	0	0.13	1	8	27	64

The graphs of the power functions $y = x^r$, when $0 \leq x \leq 2$ and $r = 1/3, 1/2, 1, 2, 3$ are compared in Figure 4.3. They illustrate the following result.

Result 4.27 *Let r, s be any rational numbers such that $r < s$. Then when $x > 1$, we have*

$$x^r < x^s. \ \square$$

We see that for any pair of rational numbers r, s with $r < s$, the graph of $y = x^r$ lies closer than the graph of $y = x^s$ to the x-axis, for all values of $x > 1$. Reasoning analytically, when $x > 1$, x^r and x^s are both positive, and hence $|x^r| = x^r$ and $|x^s| = x^s$. Taking $x_0 = 1$ and $M = 1$ in the defintion of the O-notation, we have:

Result 4.28 *Let r, s be any rational numbers such that $r < s$. Then*

$$x^r \text{ is } O(x^s). \ \square$$

Example 4.39 Let $f(x) = x\sqrt{x}$. Then since $x\sqrt{x} = x^{1.5}$, we have

$$f(x) < x^2$$

for all $x > 1$. We can say that $f(x)$ is $O(x^2)$. $\square$

4.4.3 Orders of polynomial functions

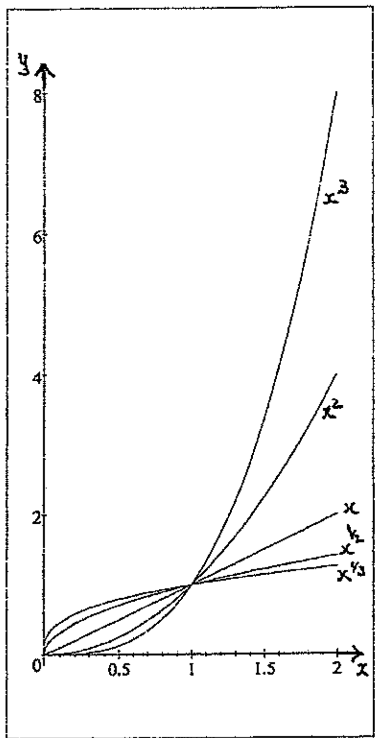

Figure 4.3.

Example 4.40 We show that $f(x) = 5x^2 + 2x + 9 < 16x^2$, for all $x > 1$.

First note that to find the value of $f(x)$ for any given x, we have to sum multiples of the three power functions x^2, x^1 and x^0. Now suppose that $x > 1$. Then from Result 4.27, we have

$$x^1 < x^2, \text{ and } x^0 < x^2.$$

Multiplying the first inequality by 2 and the second by 9, we have

$$2x^1 < 2x^2, \text{ and } 9x^0 < 9x^2.$$

Thus summing the terms gives

$$f(x) = 5x^2 + 2x + 9 < 5x^2 + 2x^2 + 9x^2 = 16x^2.$$

Hence $f(x) < 16x^2$. □

Example 4.41 We show that $f(x) = 5x^2 + 2x + 9$ is $O(x^2)$.

Suppose that $x > 1$. Then each of the terms in the polynomial $f(x)$ is positive. Hence the absolute value of $f(x)$ is given by the sum of the terms. Thus

$$|f(x)| = |5x^2 + 2x + 9| = 5x^2 + 2x + 9.$$

But in the previous example, we showed that

$$5x^2 + 2x + 9 < 5x^2 + 2x^2 + 9x^2 = 16x^2.$$

Since x^2 is positive, $|x^2| = x^2$. Thus we have

$$|f(x)| < 16|x^2|,$$

for all $x > 1$. Hence $f(x)$ is $O(x^2)$, where in the formal definition, we have $M = 16$, $g(x) = x^2$ and $x_0 = 1$. □

Example 4.42 We show that $f(x) = 2x^3 - 5x^2 + 27$ is $O(x^3)$.

Suppose that $x > 1$. This polynomial contains a negative coefficient. In this case we can say that the absolute value of $f(x)$, calculated at any given value of x, is at most the value we would get by *adding* all the terms, instead of adding some and subtracting others. Thus we have

$$|f(x)| \leq |2x^3| + |5x^2| + |27|.$$

Now each of the terms $2x^3$, $5x^2$, 27 is positive when $x > 1$. Hence

$$|2x^3| + |5x^2| + |27| = 2x^3 + 5x^2 + 27,$$

and using Result 4.27 again, we have

$$2x^3 + 5x^2 + 27 < 2x^3 + 5x^3 + 27x^3 = 34x^3,$$

for all $x > 1$. Thus $|f(x)| < 34|x^3|$, for all $x > 1$, and hence $f(x)$ is $O(x^3)$. □

Following the method of the previous two examples, we can prove the following result.

Result 4.29 *Suppose that m is the highest exponent of x present in a polynomial function $f(x)$, then $f(x)$ is $O(x^m)$.* □

4.4.4 Comparing the exponential and logarithmic functions with the power functions

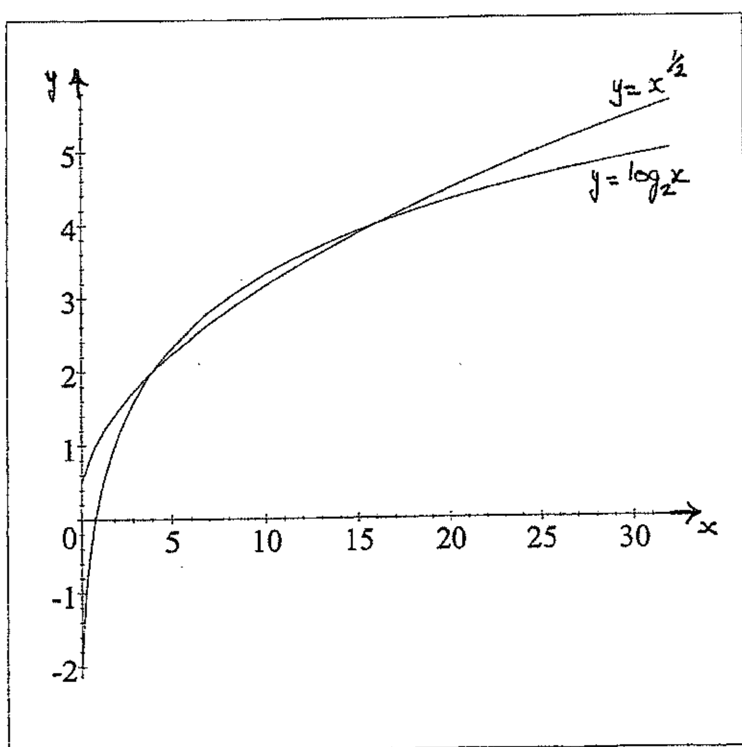

Figure 4.4.

It remains to compare the sizes of the exponential functions and the logarithmic functions with the power functions for large values of x. The following result can be established by calculus.

Result 4.30 *Let b be any real number such that $b > 1$ and r be any positive rational number. Then for all sufficiently large values of x, we have*

$$\log_b x < x^r \text{ and } x^r < b^x. \ \square$$

Because of the importance of the base 2 in computer science, we interpret this result with $b = 2$. Suppose we want to compare the behaviour of $\log_2 x$ with a power function of x, say $x^{\frac{1}{2}}$ for example. Then Result 4.30 says that we can find a real number x_0 such that $\log_2 x < x^{\frac{1}{2}}$, whenever $x > x_0$. In Figure 4.4, the graph of $y = \log_2 x$ is compared with the graph of $y = x^{\frac{1}{2}}$, when $0 < x \leq 32$. You can see that when $x > 16$, then $x^{\frac{1}{2}} > \log_2 x$, so that in this case the least value for x_0 is 16. Even if we compare the value of $\log_2 x$ with the power function $x^{\frac{1}{100}}$, for example, it would still be possible to find a real number x_0 such that $\log_2 x < x^{\frac{1}{100}}$ for all $x > x_0$.

In particular, we have the following result.

Result 4.31 $\log_2 x < x$, *for all* $x > 0$. $\square$

In a similar way, if we compare the behaviour of 2^x with any power function of x, say with x^{10} for example, then Result 4.30 says that we can find a real number x_0 such that $2^x > x^{10}$, for all $x > x_0$. The graph of $y = 2^x$ is compared with the graphs of $y = x$ and $y = x^2$, for $0 \leq x \leq 5$, in Figure 4.5. You can see from these graphs that $2^x > x$ for all $x > 0$ and $2^x > x^2$ when $x > 4$.

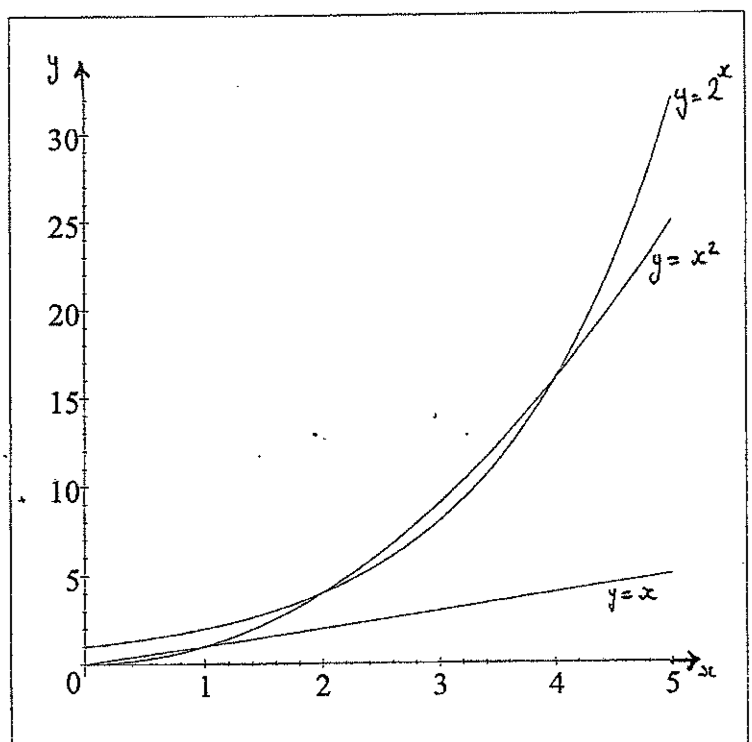

Figure 4.5.

4.4.5 Comparison of algorithms

When studying how long a given computer algorithm will take to perform some task, an important factor is the size of the data set that will be input into the algorithm. Since this will be measured as a positive integer, the domain X of a function measuring the time the algorithm takes to run will be a subset of $\mathbb{Z}^+$.

Suppose, for example, that a measurement of the running time of an algorithm is given for input size n by the formula $f(n) = 5n^2 + 8n + 3$. Then when n is *large*, terms such as $8n + 3$ are tiresome and irrelevant compared with the size of the term in n^2. Now for $n \geq 1$, we have $f(n) \leq 5n^2 + 8n^2 + 3n^2 = 16n^2$, and thus

$$5n^2 + 8n + 3 = O(n^2).$$

The use of the O-notation strips a rather complicated expression for the running time down to big-oh of a simple expression (such as a power or log function), enabling easy comparison between algorithms to be made.

You might imagine that because each operation performed by a modern digital computer takes only a tiny fraction of a second, it will not be very important whether we are running an $O(n)$ algorithm or an $O(n^2)$ or even an $O(n^3)$ algorithm. Unfortunately that is not the case, as the table below will show you. It compares the approximate time to perform $f(n)$ operations for the functions $f(n)$ with which algorithms are commonly compared, assuming that each operation takes about one microsecond.

Approximate time to perform $f(n)$ operations

$f(n)$	$n = 10^2$	$n = 10^3$	$n = 10^4$	$n = 10^5$	$n = 10^6$
$\log_2 n$	7×10^{-6}s	1×10^{-5}s	1.3×10^{-5}s	1.7×10^{-5}s	2×10^{-5}s
n	0.0001s	0.001s	0.01s	0.1s	1s
$n \log_2 n$	0.0007s	0.01s	0.13s	1.7s	20s
n^2	0.01s	1s	1.67 mins	2.78 hrs	11.6 days
n^3	1s	16.7 mins	11.6 days	31.7 yrs	31,710 yrs

There are some important conclusions to be drawn from this table.

1. There is not much difference between an $O(n)$ algorithm and an $O(n \log_2 n)$ algorithm, but the latter takes slightly longer.
2. An $O(n^2)$ algorithm takes significantly longer to run than an $O(n \log_2 n)$ algorithm and this difference will be important if the data set is large.
3. An $O(n^3)$ algorithm takes significantly longer to run than an $O(n^2)$ algorithm and is completely impractical for input sizes of much more than 1000.

Although an $O(n^3)$ algorithm takes a long time to run when $n > 1000$, this is as nothing compared with an exponential algorithm. The following table gives the times to perform 2^n operations, again assuming one operation per microsecond.

Approximate time to perform 2^n operations

n	10	10^2	10^3	10^4
2^n	0.001s	4×10^{17}yrs	3.4×10^{287}yrs	6.3×10^{2996}yrs

It goes without saying that an exponential algorithm should be avoided at all costs! However, there are tasks for which no polynomial algorithm is known and some for which a polynomial algorithm exists but is $O(n^3)$. These tasks typically arise in areas such as operational research, where an optimal (that is, very best) solution to a particular problem is sought. An important aspect of current research by mathematicians and computer scientists is in developing for this kind of problem $O(n)$ or $O(n \log_2 n)$ algorithms that find a nearly optimal solution, but one that is not necessarily the very best.

4.5 Exercises 4

Ignore

1. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3, \ldots, 10\}$. A function $f : X \to Y$ is defined by the following table.

x	1	2	3	4
$f(x)$	1	4	7	10

Find

(a) the domain of f; (b) the codomain of f; (c) $f(2)$;

(d) the ancestor of 10; (e) the range of f.

Illustrate f by drawing an arrow diagram.

11. Let X denote the set of non-negative real numbers. Then the power function $f : X \to X$ defined by $f(x) = x^s$ is a one-to-one correspondence for every positive rational number s. Hence each power function has an inverse function. For example, the inverse function of $f(x) = x^2$ is $v(x) = \sqrt{x} = x^{\frac{1}{2}}$. Find the inverse function of

$$f(x) = x^3; \qquad f(x) = x^{\frac{1}{4}}; \qquad f(x) = x^{\frac{3}{2}}.$$

12. On the same graph and with the same axes, sketch the graphs of $f(x) = \log_2 x$ for $\frac{1}{4} \leq x \leq 32$, and $g(x) = \sqrt{x}$ for $0 \leq x \leq 32$. Why can we say that $\log_2 x = O(x^{\frac{1}{2}})$?

13. Let x be a positive real number and let $f(x) = 2x\sqrt{x} + 5x + 24$. Show that $f(x) < 31x\sqrt{x}$ when $x > 1$. Deduce that $f(x) = O(x^{\frac{3}{2}})$.

14. Let k be a positive integer. Find a relationship between the number of bits in the binary representation of k and $\log_2 k$ (hint: the number of bits is always an integer, so you might think about using the floor or ceiling function). Show that your formula will give the correct number of bits for any $k \in \mathbb{Z}^+$.

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