Canonical Form

$$\max_{\mathbf{x} \in \mathbb{R}^d} \mathbf{c}^{\mathbf{T}} \mathbf{x}$$

$$\text{subject to:} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\text{where} \quad \mathbf{x} \in \mathbb{R}^d, \ \mathbf{A} \in \mathbb{R}^{m \times d}, \ \mathbf{b} \in \mathbb{R}^m, \ \mathbf{c} \in \mathbb{R}^d$$

- The feasible region, is defined by the set of inequalities $Ax \leq b$. In two dimensions, i.e. if d $=$ 2, this problem can be represented and solved graphically as polyhedron in the Euclidean plane.
- The idea is to represent the cost vector c as a level set in the plane.
- Then, moving this level set towards the highest possible value so that still intersects our feasible region, i.e. the polyhedron P defined by A, yields the optimal solution.

Divisibility

- Divisibility is one of the conventional LP assumptions.
- Divisibility allowed us to consider activities in fractions: We could produce 7.8 units of a product, buy 12500.33 liters of oil, hire 12.123 people for full time, etc.
- Divisibility assumption is very defensible at times but not always.
- We can easily buy 12500.33 liters of oil but can not employ 12.123 people.

Divisibility

- Clearly some activities cannot be done in fractions and must be specified in integers for implementation.
- As soon as some of the activities are set to be integers, we are in **Integer Programming** domain.
- Formally, in an integer program some decision variables are forced to be integers.

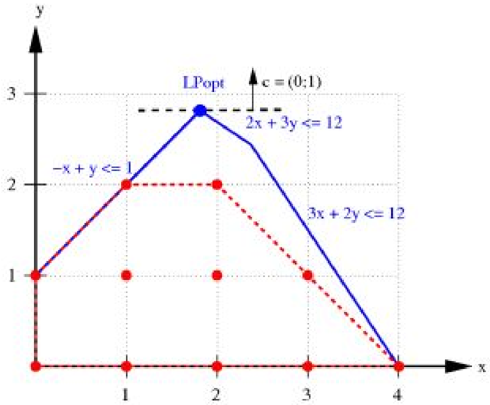


Figure:

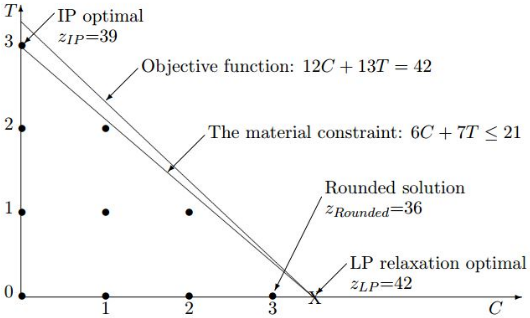
IP Example: Tables and Chairs

- Suppose we consider producing chairs and tables using only 21 m^2 of wood. Each chair (table) requires 6 (7) m^2 of wood.
- Each chair is sold at \$12 (10) and each table is sold at \$13 ($\times 10$).
- Let C and T denote the number of tables and chairs produced.
- The IP formulation below maximizes the revenue:

IP Example: Tables and Chairs

$$\begin{array}{lllllll}
\text{Maximize :} & 12C & +13T & & & \\
\text{Subject to} & & & & & \\
& 6C & +7T & \leq & 21 & (1) \\
& C & , \; T & \geq & 0 & (2) \\
& C & , \; T & int & & (3)
\end{array}$$

IP Example: Tables and Chairs



LP Relaxation

- Solving an IP can be as straightforward as solving the associated LP and rounding the solution, but only in some cases. (i.e. by coincidence rather than by definition)
- To understand what can wrong with this approach, we will first solve the IP removing constraint and round down the optimal values of C and T to satisfy the integer constraint.
- (Question: why not to round up?)
- When the integer constraints are removed from an IP formulation, we obtain an LP formulation. This LP formulation is called the **LP relaxation**.

LP Relaxation

- LP solution is (7/2,0) and is not integer so we round it down to (3,0).
- The objective value at (3,0) is 36.
- The optimal solution to IP is (0,3) with the objective value 39.
- 3 units of difference between objective value of the IP optimal and the rounded solution can be significantly higher in more complex problems.
- As a summary we cannot use rounded solutions of LP relaxations.

LP Relaxation

- Given an integer program (IP), there is an associated lenear program (LR) called the **linear relaxation**.
- It is formed by dropping (relaxing) the integrality restrictions.
- Since (LR) is less constrained than (IP), the following are immediate:

LP Relaxation

1. If (IP) is a minimization problem, the optimal objective value of (LR) is less than or equal to the optimal objective value of (IP).
2. If (IP) is a maximization problem, the optimal objective value of (LR) is greater than or equal to the optimal objective value of (IP),
3. If (LR) is infeasible, then so is (IP).
4. If all the variables in an optimal solution of (LR) are integer-valued, then that solution is optimal for (IP) too.

5 If the objective function coefficients are integer-valued, then for minimization problems, the optimal objective value of (IP) is greater than or equal to the ceiling of the optimal objective value of (LR). For maximization problems, the optimal objective value of (IP) is less than or equal to the floor of the optimal objective value of (LR).

- For simple problems one can evaluate all the integer solutions in the feasible region and pick the best.
- However, for real problems this approach will take practically infinite amount of time.
- The solution procedures for IPs are still under development.
- Two approaches are common: Branch and Bound technique, and Cutting planes.
- Cutting Planes are outside the scope of our module.

Divide and Conquer

1. Partition the problem into subproblems.

2. Solve the subproblems.

3. Combine the solutions to solve the original one.

Branch-and-bound

Branch-and-bound

- Branch-and-bound is essentially a strategy of divide and conquer. The idea is to partition the feasible region into more manageable subdivisions and then, if required, to further partition the subdivisions.
- In general, there are a number of ways to divide the feasible region, and as a consequence there are a number of branch-and-bound algorithms.
- For historical reasons, the technique that will be described next usually is referred to as the branch-and-bound procedure.

Enumeration Tree

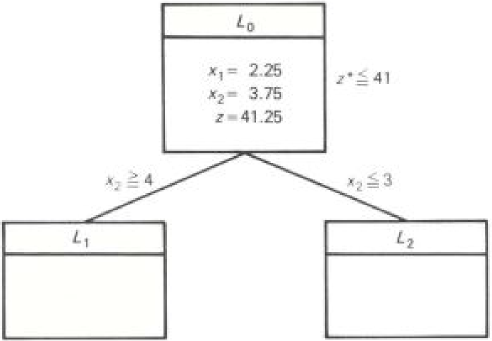


Figure:

Enumeration Tree

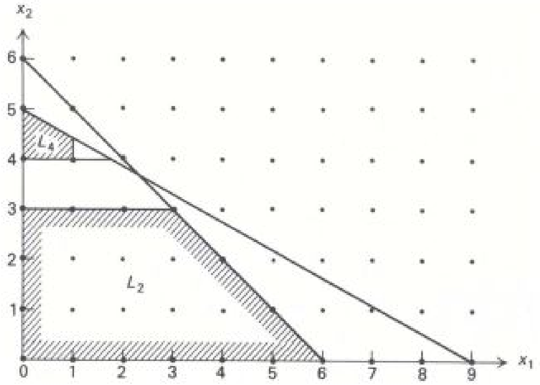


Figure:

- It is not always the case that branch and bound quickly solves integer programs.
- In particular, it is possible that the bounding aspects of branch and bound are not invoked, and the branch and bound algorithm can then generate a huge number of subproblems.
- In the worst case, a problem with n binary variables (variables that have to take on the value 0 or 1) can have 2^n subproblems.
- This exponential growth is inherent in any algorithm for integer programming,

Branch and Bound

Consider pure IP

```
Maximize
obj: x1 + x2
Subject to
c1: -x1 + x2 <= 2
c2: 8 x1 + 2 x2 <= 19
Bounds
x1 x2 >= 0
Integer
x1 x2
End
```

Branch and Bound

LP-relaxation

```
Maximize
obj: x1 + x2
Subject to
c1: -x1 + x2 <= 2
c2: 8 x1 + 2 x2 <= 19
Bounds
x1 x2 >= 0
End
```

Solution of LP-relaxation

- $x_1 = 1.5$, $x_2 = 3.5$
- Value: $x = 5$

Branch and Bound

LP-relaxation

```
Maximize
obj: x1 + x2
Subject to
c1: -x1 + x2 <= 2
c2: 8 x1 + 2 x2 <= 19
Bounds
x1 x2 >= 0
End
```

Solution of LP-relaxation

- $x_1 = 1.5$, $x_2 = 3.5$
- Value: $z = 5$

Branch and Bound

Create two sub-problems:

Left sub-problem

```
Maximize
obj: x1 + x2
Subject to
c1: -x1 + x2 <= 2
c2: 8 x1 + 2 x2 <= 19
Bounds
x1 x2 >= 0
x1 <= 1
End
```

Solution of left subproblem

- $x_1 = 1, x_2 = 3$ (integral feasible)
- Value: $z = 4$

Branch and Bound

Right sub-problem

```
Maximize
obj: x1 + x2
Subject to
c1: -x1 + x2 <= 2
c2: 8 x1 + 2 x2 <= 19
Bounds
x1 x2 >= 0
x1 >= 2
End
```

Solution of right subproblem

- $x_1 = 2$, $x_2 = 1.5$ (integral infeasible)
- Value: $z = 3.5$

Branch and Bound

- Each integer feasible solution of right sub-problem has value bounded by 3.5.
- Since value of integer feasible solution $x_1 = 1$, $x_2 = 3$ is 4, we can prune the right sub-problem
- Since integer feasible solution $x_1 = 1$, $x_2 = 3$ is also optimal solution of left sub-problem, each integer feasible solution of left-subproblem has value at most 4.
- Thus $x_1 = 1$ and $x_2 = 3$ is optimum solution to integer program.