Contents

Chapter 1

Introduction to Linear Programming

In order to use mathematics to solve a practical problem, we must first formulate the problem in mathematical language. In this chapter, we introduce some typical optimization problems of the kind that can be solved by *linear programming*, and discuss how the mathematical model can be derived. We shall then see how to obtain an optimal solution to these examples by a graphical method.

1.1 Formulating linear programming models

We first consider a problem where the requirement is to find a production schedule which *maximizes* a firm's profit, subject to an *upper* limit on the resourses available.

Example 1.1 A boat building company makes two kinds of boats for the sports and leisure trade: a small rowing dinghy and a sports canoe. The boats are moulded from aluminium by means of a large pressing machine and finished by hand. Each dinghy requires 24kg of aluminium and takes 6 minutes of machine time and 2 hours of hand labour to finish. Each canoe requires 6kg of aluminium and takes 4 minutes of machine time and 8 hours of hand labour. The company can allocate per month a maximum of 2400kg of aluminium, 12 hours machine time and 1000 hours labour to the production of these boats.

The company reckons to make a profit of £60 on each dinghy and £50 on each canoe. They would like a production schedule that will give them the maximum profit under the assumption that they can sell all the boats of these types that they can manufacture. They would also like to know the maximum profit that can be achieved using this schedule. (Note that the optimum schedule may include fractions of boats, representing boats begun in one month and finished in the following month.)

The following steps illustrate the procedure for formulating a mathematical model for this type of problem.

Step 1. We first summarize the information. It is helpful to do this in the form of a table, as shown below.

Resource	*per dinghy*	*per canoe*	*max availability*
aluminium	24 kg	6 kg	2400kg
machine time	6 mins	4 mins	12 hrs
hand labour	2 hrs	8 hrs	1000 hrs
profit	£60	£50	

Step 2. We define the *decision variables.*

As the name suggests, these variables measure the unknown quantities that our client requires us to determine. So, in this case, we state:

> Let x_1 be the number of dinghies and x_2 be the number of canoes scheduled per month.

Step 3. We specify an *objective function.*

The objective in this example is to maximize the company's profit. From the information given, the profit per month is $£(60x_1 + 50x_2)$. It is sometimes convenient to scale the units, to make the numbers easier to handle. In this example, we could conveniently work in units of £10 and define:

$$z = 6x_1 + 5x_2,$$

where the company's profit per month is $£10z$.

The linear function $z = 6x_1 + 5x_2$ is called the *objective function* for this linear programming problem.

Step 4. We specify the *constraints.*

There are two types of constraint in a typical linear programming problem. *Functional* constraints arise from conditions explicitly stated in the problem, such as the availability of resources. In this case, we have three functional constraints, which state the *maximum* amount of each of aluminium, machine time and hand labour available per month. Using the table we made in Step 1 and taking care to make the units consistent throughout each inequality, we have:

$$\begin{aligned} 24x_1 + 6x_2 &\leq 2400 \\ 6x_1 + 4x_2 &\leq 720 \\ 2x_1 + 8x_2 &\leq 1000. \end{aligned}$$

There are two other constraints, however, that are not explicitly stated in the problem. These arise from the fact that it is not possible to make a *negative* number of dinghies or canoes. Hence in addition to the funtional constraints, we have the two *non-negativity* constraints:

$$x_1 \geq 0, x_2 \geq 0.$$

We can now make a formal statement of this problem as follows.

We require to find $x_1, x_2 \in \mathbf{R}$ to maximize $z = 6x_1 + 5x_2$, subject to

$$\begin{aligned} 24x_1 + 6x_2 &\leq 2400 \\ 6x_1 + 4x_2 &\leq 720 \\ 2x_1 + 8x_2 &\leq 1000 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Our second example illustrates a problem in which the requirement is to find a mixture that *minimizes* the total cost, subject to a *lower* limit on the amount of each ingredient used.

Example 1.2 During the winter months, a farmer feeds his cattle on a mixture of oats and hay. The mixture must supply the nutritional requirements of the cows in terms of digestible protein, carbohydrates, calcium and phosphorus. The amount of each of these ingredients supplied by a unit of oats and by a unit of hay, and the minimum daily requirement of each ingredient per cow is shown in the table below.

	hay/unit	*oats/unit*	*min requirement*
protein	13.0	32.5	65.0
carbohydrates	4.2	5.6	14.0
calcium	0.02	0.08	0.12
phosphorus	0.05	0.09	0.15

The cost of one unit of hay is \$0.80 and of one unit of oats is \$2.10. The farmer wants to find a feeding plan that supplies the cows' nutritional requirements at minimum cost.

We can start modelling this problem at Step 2, since the information has been provided in tabular form.

Step 2. The decision variables.
To specify a *feeding plan*, we need to determine the number of units of hay and oats that each cow must receive each day. Thus we state:

> Let x_1 be the number of units of hay and x_2 be the number of units of oats given to each cow per day.

Step 3. The objective function.
Our objective in this case is to minimize the cost of the feeding plan. The cost of x_1 units of hay and x_2 units of oats is $\$(0.80x_1 + 2.10x_2)$. Thus, working in units of 10 cents, we can define the objective function to be:

$$z = 8x_1 + 21x_2,$$

where the cost of the feed is $\$0.1z$.

Step 4. The constraints.
The functional constraints express the fact that the diet must contain *at least* the minimum requirements of protein, carbohydrates, calcium and phosphorous. Thus we have:

$$\begin{aligned} 13.0x_1 + 32.5x_2 &\geq 65.0 \\ 4.2x_1 + 5.6x_2 &\geq 14.0 \\ 0.02x_1 + 0.08x_2 &\geq 0.12 \\ 0.05x_1 + 0.09x_2 &\geq 0.15 \end{aligned}$$

Then, as in Example 1.1, we have the two non-negativity constraints:

$$x_1 \geq 0, x_2 \geq 0.$$

Thus we have the following statement of this problem:

> We require to find $x_1, x_2 \in \mathbf{R}$ to minimize $z = 8x_1 + 21x_2$, subject to
>
> $$\begin{aligned} 13.0x_1 + 32.5x_2 &\geq 65.0 \\ 4.2x_1 + 5.6x_2 &\geq 14.0 \\ 0.02x_1 + 0.08x_2 &\geq 0.12 \\ 0.05x_1 + 0.09x_2 &\geq 0.15 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Definition 1.1 *A problem in which we are asked to find the "best" or optimal solution subject to given conditions is called an* **optimization** *problem.*

Definition 1.2 *An optimization problem in which the objective function can be expressed as a* linear *function of the variables and in which the constraints can be expressed as* linear *equations or* linear *inequalities is called a* **linear programming problem**. *We shall frequently refer to such a problem in the sequel as an* **l.p. problem**.

Note that in any optimization problem, the objective may be to find values of the variables which give the *maximum* value of the objective function, as in Example 1.1, or it may be to find values that give the *minimum* of the objective function, as in Example 1.2.

1.2 Assumptions of the model

Any mathematical model of a real world problem is to some extent idealised. In obtaining a linear programming model, some assumptions are made that you should be aware of.

1.2.1 Proportionality

In obtaining the objective function as a *linear* function of the decision variables, we assume that the total profit on each product the company makes (or cost associated with each unit it buys) is proportional to the number of articles produced (or the number of units bought). For example, in Example 1.1, it was assumed that the profit per dinghy and the profit per canoe were each *constant*, regardless of the number of each type of boat produced. Similarly, in Example 1.2, it was assumed that the cost per unit of hay and oats remained the same however many units were bought.

Similar assumptions were made in formulating the functional constraints in both these problems. We assumed that the contribution made by each article to the left side of the constraint is constant for that type of article, and does not vary with the number of articles produced.

These assumptions are known collectively as the *proportionality assumption*. In any practical situation, we have to ask whether this assumption is reasonable before formulating the linear programming model. Factors which might make the assumption invalid if their affect were sufficiently large are discussed very briefly below.

Startup costs If one of the products considered in the model is a new line, there may be costs associated with launching it. These could arise, for example, from investment in new equipment, in hiring additional labour or in retraining workers, and in advertising and marketing the new product. This will tend to make the incremental costs higher for a product at the beginning of the production run than when it has become established.

Economies of scale Sometimes unit costs decrease for longer production runs. This might happen if, for example, production facilities get used more intensively; or if the production workers become faster and more skilfull with experience; or if it becomes possible to use raw materials more efficiently. Bulk buying of raw materials or components may also reduce unit costs. These considerations tend to *reduce* the incremental cost as production increases.

Expenses of scale In contrast to the previous item, there are some costs which may increase when production of a product line increases beyond a certain point. For example, storage costs might suddenly increase if it became necessary to store more items than can be accommodated in existing facilities. Marketing might become more expensive if it became necessary to undertake a more intensive advertising campaign to increase the demand for the product to match output. Such considerations tend to *increase* the incremental cost as production increases.

1.2.2 Additivity

In formulating the objective function and the constraints as linear functions of the variables, we have also assumed that the total effect on the objective function and the total amount of resources used is the *sum* of the contributions made by each of the products. This is called the *additivity assumption.*

In practice, this assumption may not be valid, because there may be *cross-product* effects. We consider below a few examples of the way in which these can arise.

Complementary effects These effects arise when there is a production cost which is not proportionately increased by increasing the number of product lines. One area in which this may arise is marketing. For example, it may not cost more for the Boatbuilders to advertise two types of boat than it does to advertise one type.

Complementary effects can also affect the validity of the additivity assumption for the constraints. Sometimes the production of different product lines can be made to "dovetail" in such a way that machine time or other resources are used more efficiently for two or more product lines than for only one.

Competing effects In other ways, having more product lines may tend to increase the unit cost of each product. For example, if two or more product lines share the same machinery or equipment for their production, time may be lost and other costs incurred by having to switch between product lines.

Before formulating a practical problem as a linear programming problem, we would need to be sure that any cross-product effects were not so substantial as to invalidate the additivity assumption.

1.2.3 Divisibility

In some problems, the decision variables may only take integer values. However, in order to apply linear programming, the decision variables must be capable of being divided into fractional parts, however small. This is called the *divisibility assumption.*

In the case of the Cattle Feed problem described in Example 1.2, fractional units of hay and oats are reasonable, and give no problems of interpretation in the constraints. However, when we come to consider the costs in the objective function, we might suspect that the farmer will have to buy *whole* units of hay and oats in the marketplace. This does not invalidate the model, however, due to the overall scale of the problem. The farmer is likely to have a number of cows and he is also likely to stock several weeks supply of feed. Thus having decided on the fraction of units required by each cow per day, he will scale this up many times before buying the raw materials to mix the feed.

Similar considerations arise in the Boatbuilder's problem (Example 1.1). Linear programming may give us fractional values for the optimal number of dinghies and canoes to be manufactured each month. Clearly the boatbuilder can only realise a profit on each boat after it has been completed. However, over a sufficiently long production period, all the boats will be completed for sale. It is the on-going nature of the production schedule that makes the divisibility assumption valid in this type of problem.

Problems in which the divisibility assumption is invalid are solved by *integer programming* techniques. These are outside the scope of this half unit.

1.2.4 Certainty

Another assumption we make when we solve a linear programming problem is that the coefficients in the constraints and the objective function are known with certainty at the point at which we solve the problem. This is called the *certainty* assumption.

In practice it is almost never satisfied, because it requires a company to predict, in advance of knowing the production schedule, exactly what its profits, costs and other requirements are going to be. In any case, changes in demand, in prices of raw materials and other costs are certain to occur over time, and these may change the optimal solution.

What is done in practice is to solve the problem with estimated values of these coefficients. Having found all optimal solutions, a technique known as *sensitivity analysis* is applied to determine over what range of values of the coefficients this solution remains optimal. This activity is part of the final phase in the solution of an l.p. problem, known as *post-optimal analysis.* We shall return to this in a later chapter.

1.3 Graphical solution for problems with two variables

Once we have formulated a practical problem as a linear programming problem, the next step is to explore whether a solution exists satisfying all the constraints and if so, to find all solutions that give the optimal value of the objective function.

Definition 1.3 *A solution that satisfies ALL the constraints on the variables is called a* **feasible** *solution.*

Definition 1.4 *A feasible solution that gives the optimum value of the objective function is called an* **optimal** *solution.*

Definition 1.5 *If an l.p. problem has a feasible solution, then the set of all feasible solutions is called the* **feasible region** *for the problem. If no feasible solutions exist, then the feasible region is the empty set.*

For problems with only two decision variables, x_1, x_2, such as those discussed in Example 1.1 and Example 1.2, we can explore the existence of a feasible solution and, if the feasible region is non-empty, find all the optimal solutions by a graphical method. We do this by plotting all points (x_1, x_2) that satisfy all the constraints.

Example 1.3 We illustrate the method by solving the Boatbuilder's problem (Example 1.1). In this case, we want to find $x_1, x_2 \in \mathbf{R}$ to maximize $z = 6x_1 + 5x_2$, subject to

$$
\begin{aligned}
24x_1 + 6x_2 &\leq 2400 &(1.1)\\
6x_1 + 4x_2 &\leq 720 &(1.2)\\
2x_1 + 8x_2 &\leq 1000 &(1.3)\\
x_1, x_2 &\geq 0. &(1.4)
\end{aligned}
$$

We consider each constraint in turn, starting with the two non-negativity constraints (1.4).

The line $x_1 = 0$ divides the plane into two regions: the points to the left of the line, with coordinates satisfying $x_1 < 0$; and the points to the right of the line, with coordinates satisfying $x_1 > 0$. Thus the points satisfying the constraint $x_1 \geq 0$ are just those lying on and to the right of the line $x_1 = 0$.

Similarly, the set of points satisfying the constraint $x_2 \geq 0$ all lie on and above the line $x_2 = 0$. Thus if a feasible solution exists, it will lie in the positive quadrant of the x_1x_2-plane.

Now consider the set of points satisfying constraint (1.1). The line

$$24x_1 + 6x_2 = 2400$$

also divides the plane into two regions. The points on one side of the line satisfy the inequality $24x_1 + 6x_2 < 2400$ and those on the other side of the line satisfy the inequality $24x_1 + 6x_2 > 2400$. To decide which region is which, we test a sample point (a, b) not on the line. For example, $(0, 0)$ does not lie on the line. Substituting $x_1 = 0$ and $x_2 = 0$, gives $24x_1 + 6x_2 = 0 < 2400$. Thus the points satisfying the first constraint are those that lie on the line $24x_1 + 6x_2 = 2400$ and on the side of this line containing the point (0,0).

The set of points satisfying the first constraint and the non-negativity constraints are those in the shaded region of the diagram shown in Figure 1.1 below.

Repeating this procedure for constraints (1.2) and (1.3), we obtain the feasible region for this l.p. problem. It is the shaded region on the diagram illustrated in Figure 1.2. You will see that it is the boundary and interior of a polygon with five vertices, O,A,B,C,D. These points are called the **corner-points** of the feasible region.
(Note: many authors call these points the **extreme points** of the feasible region, but we shall follow the terminology used by Hillier and Lieberman.)

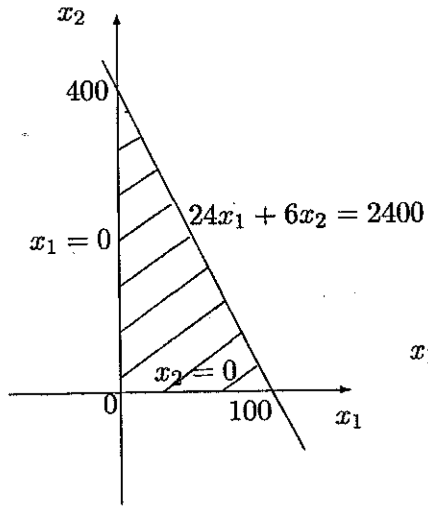

Figure 1.1.

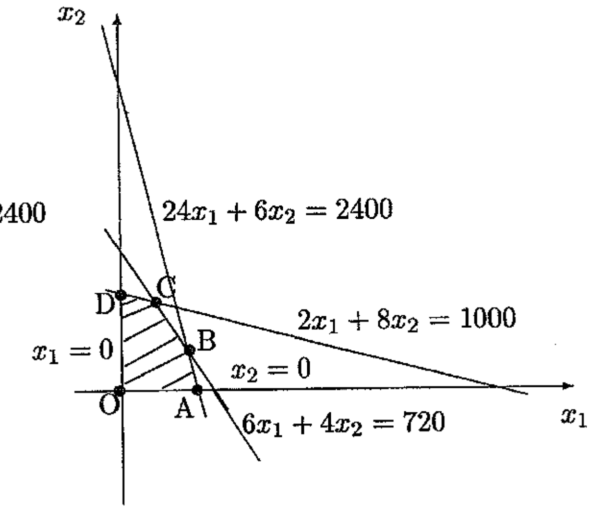

Figure 1.2.

We now determine the point or points in the feasible region that give the maximum value of the objective function

$$z = 6x_1 + 5x_2.$$

To do this, we consider the line

$$6x_1 + 5x_2 = c,$$

where c is a constant. By choosing a range of values for the number c we get a family of parallel lines, each with gradient $-6/5$. We first find a value of c that will give a line of this family intersecting the feasible region. To do this, note that the intercept of the line $6x_1 + 5x_2 = c$ on the x_1-axis is $c/6$ and on the x_2-axis is $c/5$. Suppose we choose $c = 360$, for example, giving intercepts of 60 and 72 respectively. Now any of the points of this line that lie in the feasible region give $z = 360$. Thus we are certain that we can make z at least 360.

Can we do better? Try $c = 480$. This gives a line parallel to the line $c = 360$, intersecting the feasible region. Thus there are values of x_1 and x_2 that give $z = 480$.

Noting that as we *increase* the value of c, the line $z = c$ moves *further away from* the origin, we see that the maximum value of z is given by the *greatest* value of c for which the line $6x_1 + 5x_2 = c$ has a non-empty intersection with the feasible region. Moving a ruler parallel to the line $c = 480$ and away from the origin, we leave the feasible region at the point C.

Now C is at the intersection of the lines determined by the second and third constraints. Solving the corresponding pair of simultaneous equations, gives C at the point (44,114). Thus the optimal solution to the l.p. problem formulated in Example 1.3 is

$$x_1 = 44,\ x_2 = 114,$$

and the maximum value of z is given by $z = 6(44) + 5(114) = 834$.

To solve Example 1.1, it remains to interpret this solution as a monthly production schedule for the Boatyard and calculate the maximum profit from the optimal value of z. From the definion of the decision variables and the objective function, we obtain the following solution:

> Schedule 44 dinghies and 114 canoes per month. The maximum profit that can be obtained is £ 8340 per month. □

Example 1.4 Suppose that in Example 1.1, the profit alters to £60 per dinghy and £40 per canoe. Does the optimal schedule change? What is the maximum profit in this case?

The only change in the formulation of the problem solved in Example 1.3 is that the objective function becomes

$$z = 6x_1 + 4x_2,$$

where the profit is again £$10z$.

In particular, since the constraints remain unchanged, the feasible region also remains unchanged (see Figure 1.2). Thus, all that is required to obtain the optimal solution is to plot a line representing the new objective function. However, in this case, you will find that the family of lines representing $z = c$ are parallel to the line representing the second constraint. Thus the maximum value of c for which the line $z = c$ has a non-empty intersection with the feasible region is when $z = c$ coincides with the line $6x_1 + 4x_2 = 720$. Hence the maximum value of z is now 720. However, this time, instead of finding a *unique* optimal solution, we have an *infinite number* of optimal soltutions, represented by all the points (x_1, x_2) of the line segment BC.

In order to find all the optimal solutions, it is first necessary to find both end-points of the line segment. We already know that C is the point (44,114). The point B is at the intersection of the lines determined by the first and second constraints. Solving the equations of these lines simultaneously, gives B at the point (88,48). Thus on the line segment BC, $44 \leq x_1 \leq 88$. We may choose the value of x_1 arbitrarily in this range, so we put $x_1 = r$, where $r \in \mathbf{R}$ and $44 \leq r \leq 88$. Then $4x_2 = 720 - 6x_1$, giving $x_2 = 180 - 3r/2$. Thus the set of all optimal solutions is given by

$$\{(r, 180 - 3r/2) : r \in \mathbf{R}, 44 \leq r \leq 88\},$$

and the maximum profit is £7200. □

1.4 Exercises

Exercise 1.1
Formulate a new objective function for the Boatyard problem described in Example 1.1 for the case when the profit on each dinghy is £70 and on each canoe is £35. Find graphically the optimum schedule and the maximum profit in this case.

Exercise 1.2
Find graphically the optimal solution to Example 1.2.

You can see from the diagram that in this problem, one of the constraints does not form part of the boundary of the feasible region. What interpretation can you put on this with regard to choosing a feed that satisfies the dietary requirements of the cows?

Exercise 1.3
The Aylesbury brickyard produces three types of brick in each of its kilns. Kiln A can produce 2 000 standard, 1 000 oversize and 500 glazed bricks per day; while kiln B can produce 1 000 standard, 1 500 oversize and 2 500 glazed bricks per day. The daily operating cost of kiln A is £400 and of kiln B is £320. The brickyard has an order for 10 000 standard, 9 000 oversize and 7 500 glazed bricks. Both kilns can be operated immediately.

(a) Formulate the problem of finding the number of days each kiln should be operated in order to meet the demand at minimum cost as a linear programming problem.

(b) Find graphically the optimum schedule. Hence find the shortest time in which the order can be completed and the minimum production cost.

(c) Suppose that there is at most 2 days available on kiln A and 9 days available on kiln B to meet this order. What constraints must be added to the problem formulated in (a) to model this new problem? Add the new constraints to the diagram and find the revised optimal solution. What extra cost (if any) is incurred by satisfying these additional constraints?

Exercise 1.4
A dietician is planning a hospital meal using two main foods A and B. She wants each portion to contain at most 20.5 units of fat and to provide at least 14 units of carbohydrates and at least 17 units of protein. The number of units of fat, carbohydrates and protein in 100 grams of A and of B is shown in the table below. The cost of 100 grams of A is 30 pence and of B is 40 pence.

	fat	*carbohydrate*	*protein*
A	4	2	6
B	6	8	4

(a) Formulate the problem of determining the amount (in grams) of each food per portion that should be served to provide a meal meeting these requirements at minimum cost as a linear programming problem.

(b) Comment on the assumptions of proportionality, additivity and divisibility that you have made in formulating this model.

(c) Solve this problem graphically, giving the optimum solution and the cost per portion of the meal.

Exercise 1.5
Not all linear programming problems have an optimal solution. Draw a diagram to show the constraints in each of the following problems and say why each has no optimal solution.

(a) Find $x_1, x_2 \in \mathbf{R}$ to maximize $z = x_1 + 2x_2$ subject to

$$\begin{aligned} x_1 - 5x_2 &\leq 5 \\ -x_1 + x_2 &\leq 4 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(b) Find $x_1, x_2 \in \mathbf{R}$ to maximize $z = x_1 + x_2$ subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 2 \\ 3x_1 + 4x_2 &\geq 12 \\ x_1, x_2 &\geq 0. \end{aligned}$$