FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF MATHEMATICS AND STATISTICS

END OF SEMESTER EXAMINATION PAPER 2016

MODULE CODE: MS4131 SEMESTER: Spring 2016

MODULE TITLE: Linear Algebra 1 DURATION OF EXAM: 2.5 hours

LECTURER: Mr. Kevin O'Brien GRADING SCHEME: 80 marks
100% of module grade

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES

Scientific calculators approved by the University of Limerick can be used.
Students must attempt any 4 questions from 5.

Question 1

Part A. Matrix Multiplication (4 Marks)

Given the matrices

$$A = \begin{pmatrix} 3 & 5 & -1 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 7 \\ -1 & 0 & \\ 7 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix},$$

calculate the products AB and CA.

Part B. Diagonal Matrices (3 Marks)

Consider the following diagonal matrix D. Provide answers for the following questions in terms of the values a, b and c.

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

(i) (1 Mark) Write an expression for the trace of the matrix D.

(ii) (1 Mark) State the inverse of D, i.e. D^{-1}.

(iii) (1 Mark) State the matrix D^3.

Part C. Matrix Multiplication (5 Marks)

Suppose A is a lower triangular matrix of the form;

$$A = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

(i) (1 Mark) State the transpose of A.

(ii) (3 Marks) Compute B where B = $A \times A^T$.

(iii) (1 Mark) B is a symmetric matrix. What is meant by this?

Please Turn Over For Parts D and E

Part D. Invertible Matrices (4 Marks)

Show that if A is an $n \times n$ invertible matrix that satisfies

$$9A^3 + A^2 - 3A = 0$$

where $A^n = \underbrace{A \ldots A}_{n \text{ times}}$, I is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix, then the inverse of A is given by

$$A^{-1} = \frac{1}{3}I + 3A.$$

Part E. Matrix Transposition (4 Marks)

Let A and B be $m \times n$ matrices.

$$(AB)^T = B^T \times A^T$$

(i) (4 Marks) Prove this identity for A and B.
A proof that is provided on the basis that A and B are both 2×2 matrices will be sufficient for full marks.

Question 2

Part A. Fundamental Theorem of Invertible Matrices (5 Marks)

The Fundamental Theorem of Invertible Matrices states that a set of mathematical expressions concerning an $n \times n$ matrix A are each equivalent to one another.

(i) (4×1 Mark) State any four of these expressions.

(ii) (1 Mark) What is the trace of a square matrix

Part B. Inverting a Matrix with Elementary Row Operations (5 Marks)

In this question, you are required to find the inverse of the following matrix using elementary row operations.

$$A = \begin{pmatrix} -2 & -2 & -2 \\ 2 & 3 & 2 \\ 3 & -2 & 5 \end{pmatrix}$$

(i) (5 Marks) Find the inverse of the matrix, using elementary row operations. Show your workings for each stage of the calculation.

Please Turn Over For Parts C and D

Part C. Inverting a Matrix with Co-Factor Method (9 Marks)

$$B = \begin{pmatrix} -4 & 3 & -2 \\ -2 & 3 & 4 \\ -1 & 1 & 0 \end{pmatrix}$$

(i) (5 Marks) For each element of B, calculate the corresponding minor. Show your workings for each calculation. State the matrix of minors.

(ii) (2 Marks) Hence or otherwise, compute the determinant of B i.e. $\det(B)$.

(iii) (1 Mark) Compute the cofactor matrix for B i.e. $\mathrm{cof}(B)$.

(iv) (1 Marks) State the inverse matrix of B, given by

$$B^{-1} = \frac{1}{\det(B)} \mathrm{cof}(B)^T.$$

Part D. Inverting Multiples of a Matrix (1 Mark)

Suppose that the inverse of the following matrix M

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 0 & -2 \\ -6 & -2 & 2 \end{pmatrix}$$

is given as

$$M^{-1} = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ -0.25 & -1.0 & -0.75 \\ 0.50 & 0.5 & 0.50 \end{pmatrix}$$

(i) (1 Mark) State the inverse of the matrix N where $N = 2M$.

$$N = 2M = \begin{pmatrix} 4 & 4 & 4 \\ 8 & 0 & -4 \\ -12 & -4 & 4 \end{pmatrix}$$

Question 3

Part A. Vector Calculations (9 Marks)

Consider the three vectors in $\mathbb{R}^3$:

$$u = (1, 2, 0), \quad v = (0, 1, 3), \quad w = (1, 2, 3).$$

(i) (3 Marks) Evaluate $u + v$, $\|u\|$ and $\|v\|$,

(ii) (3 Marks) Evaluate $u \cdot v$, $u \times v$ and the angle between u and v.

(iii) (2 Marks) Calculate the scalar triple product $u \cdot (v \times w)$.

(iv) (1 Mark) With reference to the answer for part (iii), state the scalar triple product $u \cdot (w \times v)$. Justify your answer.

Part B. Orthonormal Projections (7 Marks)

If

$$u = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \qquad a = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix},$$

(i) (3 Marks) Find the vector component of u along a, $\operatorname{proj}_a u$

(ii) (2 Marks) Find the vector orthogonal component to a;

(iii) (2 Marks) Calculate the norm of $\operatorname{proj}_a u$ and the norm of $u - \operatorname{proj}_a u$;

Part C. Proofs for Vector Products (4 Marks)

(i) (4 Marks) Prove that for any $u,\ v,\ w,\ \in \mathbb{R}^3$ and any $k \in \mathbb{R}$ we have

$$(u \times v) \times w = (u \cdot w)v - (v \cdot w)u.$$

Question 4

Part A. System of Linear Equations (7 Marks)

Consider the linear system

$$\begin{aligned} x_1 + x_3 &= 4 \\ 2x_1 + 4x_2 + x_3 &= -3 \\ x_2 + 3x_3 &= 7. \end{aligned}$$

(i) (2 Marks) Write down the coefficient matrix and the augmented matrix of this system.

(ii) (5 Marks) Solve the system of equations, using any appropriate method. Show your workings for each stage of the calculation.

Part B. Proof of Vector Identities (7 Marks)

(i) (1 Marks) Let u and v be two vectors in $\mathbb{R}^3$ and let θ be the angle between them. Define the scalar product $u \cdot v$ in terms of θ

(i) (2 Marks) Hence or otherwise, prove the so-called "*Cauchy-Schwarz Inequality*":

$$\|u \cdot v\| \leq \|u\| \times \|v\|$$

(ii) (4 Marks) Hence or otherwise, prove the so-called "*Triangular Inequality*"

$$\|u + v\| \leq \|u\| + \|v\|$$

Part C. Distance from Planes (6 Marks)

(i) (3 Marks) Give the general form of the equation of the plant π in $\mathbb{R}^3$ passing throughthe point $P_0 = (1, 0, 2)$ with the vector $n = (-5, 3, 2)$ as the normal.

(ii) (3 Marks) Show that the point $Q = (1, -1, 1)$ does not lie in the plane π and find its distance from π.

Question 5

Part A. Determinants of Matrices (4 Marks)

You are given the following piece of information concerning a matrix A.

$$|A| = \begin{vmatrix} 0 & 4 & 7 \\ -1 & -1 & 7 \\ 1 & 5 & 1 \end{vmatrix} = 4$$

(i) (2 Marks) Hence or otherwise, state the determinant of matrices B and C. Provide a brief justification for your answer.

$$B = \begin{pmatrix} 0 & 4 & 7 \\ -3 & -1 & 7 \\ 3 & 5 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 8 & 14 \\ -2 & -2 & 14 \\ 2 & 10 & 2 \end{pmatrix}$$

(ii) (2 Marks) State the determinant of matrices D and E. Provide a brief justification for your answer.

$$D = \begin{pmatrix} 0 & 4 & 8 \\ -3 & -1 & -2 \\ 3 & 5 & 10 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 8 & 14 \\ 0 & -2 & 14 \\ 0 & 10 & 2 \end{pmatrix}$$

Part B. Row-Echelon Form of a Matrix (4 Marks)

Consider the matrices U, V, W and X presented below. For each matrix state one reason why that matrix in not in row-echelon form. Provide distinct answers for each of the four matrices.

$$U = \begin{pmatrix} 1 & 2 & 6 & 3 & 5 \\ 0 & 1 & 4 & 0 & 6 \\ 0 & 1 & 2 & 1 & 7 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} \qquad V = \begin{pmatrix} 1 & 1 & 6 & 8 & 5 \\ 0 & 2 & 6 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$W = \begin{pmatrix} 1 & 1 & 6 & 8 & 5 \\ 0 & 1 & 6 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 6 & 8 & 5 \\ 0 & 1 & 6 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Marking Scheme: 4×1 *Marks where 1 Mark is awarded for each valid and distinct reason.*

Please Turn Over For Part C

Part C. Eigenvalues and Eigenvectos (12 Marks)

Consider the following matrix A

$$A = \begin{pmatrix} 0.5 & 1 & 1.5 \\ -3 & -5 & -3 \\ 3.5 & 5 & 2.5 \end{pmatrix}$$

(i) (2 Marks) Determine the Characteristic Equation for A.

(ii) (2 Marks) Determine the eigenvalues for A.

(iii) (4 Marks) For each eigenvalues, compute the corresponding eigenvectors of A.

(iv) (2 Marks) Diagonalise A; i.e, give a matrix P and a diagonal matrix D, such that $A = PDP^{-1}$.

(v) (2 Marks) Hence, evaluate A^4.