1 Set Theory

A set, in mathematics, is a collection of distinct entities, called its elements, considered as a whole. The early study of sets led to a family of paradoxes and apparent contradictions. It therefore became necessary to abandon "nave" conceptions of sets, and a precise definition that avoids the paradoxes turns out to be a tricky matter. However, some unproblematic examples from nave set theory will make the concept clearer. These examples will be used throughout this article:

- A = the set of the numbers 1, 2 and 3.
- B = the set of primary light coloursred, green and blue.
- C = the empty set (the set with no elements).
- D = the set of all books in the British Library.
- E = the set of all positive integers, 1, 2, 3, 4, and so on.

Note that the last of these sets is infinite.

A set is the collection of its elements considered as a single, abstract entity. Note that this is different from the elements themselves, and may have different properties. For example, the elements of D are flammable (they are books), but D itself is not flammable, since abstract objects cannot be burnt.

1.1 Notation - Listing Method

By convention, a set can be written by **listing** its elements, separated by commas, between braces.

Using the sets defined above:

- A = $\{1, 2, 3\}$
- B = $\{red, green, blue\}$
- C = $\{\}$

This is impractical for large sets (D), and impossible for infinite ones (E).

1.2 Notation - Building Method

Thus a set can also be described by naming a particular property that is shared by all its elements and only by them. A common notation uses a bar (—) to separate a variable name (e.g. "x") from a property of the variable that elements of the set must have. For example:

- D = $\{x|$ x is a book and x is in the British Library $\}$
- E = $\{x|$ x is a positive integer$\}$

A simple translation of this notation is that "*D is the set of all x, where x is a book and x is in the British Library*" or "*E is the set of all x, where x is a positive integer*".

2 Definition of a set

A set is defined completely by its elements. Formally, sets X and Y are the same set if they have the same elements; that is, if every element of X is also an element of Y, and vice versa. For example, suppose we define:

$$F = \{x|(\text{x is an integer}) \text{ and } 0 < x < 4)\}$$

The equivalence of empty sets has a ***metaphysical*** consequence for some theories of the metaphysics of properties that define the property of being x as simply the set of all x, then if the two properties are uninstantiated or coextensive they are equivalent - under this theory, because there are no unicorns and there are no pixies, the property of being a unicorn and being a pixie are the same - but if there were a unicorns and pixies, we could tell them apart. (See Universals for more on this.)

2.1 Elements and subsets

The $\in$ sign indicates set membership. If x is an element (or "member") of a set X, we write $\in X$; e.g. $3 \in A$. (We may also say "X contains x" and "A contains 3")

A very important notion is that of a subset. X is a subset of Y, written $X \subseteq Y$ (sometimes simply as$X \subset Y$), if every element of X is also an element of Y. From before $C \subseteq A \subseteq E$.

2.2 Sets containing Sets

Sets can of course be elements of other sets; for example we can form the set $G = \{A, B, C, D, E\}$, whose five elements are the sets we considered earlier. Then, for instance, $A \in G$. (Note that this is very different from saying $A \subseteq G$)

2.3 Set operations

Suppose X and Y are sets. Various operations allow us to build new sets from them.

Union The union of X and Y, written $X \cup Y$, contains all the elements in X and all those in Y. Thus $A \cup B = \{1, 2, 3, red, green, blue\}$. As A is a subset of E, the set $A \cup E$ is just E.

Intersection The intersection of X and Y, written XY, contains all the elements that are common to both X and Y. Thus 1,2,3,red,green,blue 2,4,6,8,10 = 2.

Set difference The difference X minus Y, written XY or $X \| Y$, contains all those elements in X that are not also in Y. For example, EA contains all integers greater than 3. AB is just A; red, green and blue were not elements of A, so no difference is made by excluding them.

2.4 Complement and universal set

The universal set (if it exists), usually denoted U, is a set of which everything conceivable is a member. In pure set theory, normally sets are the only objects considered (unlike here, where we have also considered numbers, colours and books, for example); in this case U would be the set of all sets. (Non-set objects, where they are allowed, are called 'urelemente' and are discussed below.)

In the presence of a universal set we can define X, the complement of X, to be UX. X contains everything in the universe apart from the elements of X. We could alternatively have defined it as

$$X = \{x|\tilde{(}x \in X)\}$$

and U as the complement of the empty set.

2.5 Cardinality

The cardinality $\|X\|$ of X is, roughly speaking, its size. The empty set has size 0, while B = $\{red, green, blue\}$ has size 3.

This method of expressing cardinality works for finite sets but is not helpful for infinite ones. A more useful notion is that of two sets having the same size: if a direct one-to-one correspondence can be found between the elements of X and those of Y, they have the same size.

Consider the sets, both infinite, of positive integers $\{1, 2, 3, 4, \ldots\}$ and of even positive integers $\{2, 4, 6, 8, \ldots\}$. One is a subset of the other. Nevertheless they have the same cardinality, as is shown by the correspondence mapping n in the former set to 2n in the latter. We can express this by writing $\|X\| = \|Y\|$, or by saying that X and Y are "equinumerous".

2.6 Power set

The power set of X, $P(X)$, is the set whose elements are all the subsets of X. Thus

$$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

. The power set of the empty set $P(\{\}) = \{\{\}\}$.

Note that in both cases the cardinality of the power set is strictly greater than that of base set: No one-to-one correspondence exists between the set and its power set.