1. The continuous random variable X has probability density function (pdf) $f(x)$ given by

$$f(x) = \frac{k}{x^4}, \text{ where} x =\geq \theta$$

where θ is a positive constant.

(a) Find k in terms of θ. For the case $\theta = 1$, sketch the graph of $f(x)$, marking the value of $f(1)$ on your graph. (6)

(b) Show that $E(X) = \frac{3}{2}\theta$ and $Var(X) = \frac{3}{4}\theta^2$.

(c) Let X denote the mean of n independent random variables, $X_1, X_2, \ldots X_n$, each of which has the pdf $f(x)$.

(i) Write down an approximation to the distribution of X based on the central limit theorem. How would you expect the success of the approximation to vary with n?

(ii) Use this distribution to show that

$$P\left(|X - E(X)| \ \leq \ 1.96\theta\sqrt{\frac{3}{4n}}\right)$$

approximately. Hence find an approximation to the least value of n such that

$$P\left(|X - E(X)| \ \leq \ 0.1\theta\right) \geq 0.95.$$

2. The random variable Y has probability density function

$$f(y) = k(y - y^3) \qquad \text{where} \quad 0 \leq y \leq 2,$$

and zero otherwise, where k is a positive constant.

(a) Show that $k = 1/6$.

(b) Show that the cumulative distribution function is

$$f(y) = \begin{cases} 0, & \text{for } y \leq 0 \\ \frac{y^2}{12}\left(\frac{y^2+2}{2}\right) & \text{for } 0 < y < 2 \\ 1, & \text{for } y \geq 2 \end{cases}$$

(c) Hence find $P(1/2 < Y < 3/2)$?

(d) Find the variance of Y.

3. The continuous random variable X has probability density function given by

$$f_X(x) = c(1 - x^2), \text{ where } 1 \leq x \leq 1,$$

where

$$c$$

is a suitable constant.

(a) Show that $c = 3/4$ and plot the graph of $f_X(x)$ against x.

(b) Show that the cumulative distribution function of X is given by

$$f(x) = \begin{cases} 0, & \text{for } x \leq -1 \\ \frac{2+3x-x^2}{4} & \text{for } -1 < x < 1 \\ 1, & \text{for } x \geq 1 \end{cases}$$

Also find $P(-1/2 \leq X \leq 1/2)$.

(c) Obtain the standard deviation of X, giving your answer correct to 3 significant figures.

4. The continuous random variable X has probability density function

$$f(x) = \alpha(1-x)^{\alpha-1}, \quad 0 < x < 1, \alpha > 0.$$

(a) Find the cumulative distribution function, $F(x)$, of X.

(b) Find $P(0.25 < X < 0.75)$.

(c) Use $F(x)$ to obtain the median of X.

5. The continuous random variable Y has probability density function given by

$$f(y) = 9ye^{-3y}, \quad y \geq 0.$$

(a) Obtain $E(Y)$.

(b) What is $P(Y < 3)$?