Question 1

The information we are given is as follows:

$$\Pr(R_1) = 0.75 \qquad \Pr(T \mid R_1) = 0.9$$
$$\Pr(R_2) = 0.2 \qquad \Pr(T \mid R_2) = 0.5$$
$$\Pr(R_3) = 0.05 \qquad \Pr(T \mid R_3) = 0.7$$

We can also calculate (multiplication rule):

$$\begin{aligned}\Pr(T \cap R_1) &= \Pr(R_1)\Pr(T \mid R_1)\\ &= 0.75(0.9) = 0.675.\end{aligned}$$

$$\begin{aligned}\Pr(T \cap R_2) &= \Pr(R_2)\Pr(T \mid R_2)\\ &= 0.2(0.5) = 0.1.\end{aligned}$$

$$\begin{aligned}\Pr(T \cap R_3) &= \Pr(R_3)\Pr(T \mid R_3)\\ &= 0.05(0.7) = 0.035.\end{aligned}$$

a) T^c = "not on time", i.e., "late".

b) You cannot simultaneously travel two different routes. These events are *mutually exclusive* $\Rightarrow \Pr(R_1 \cap R_2) = 0$.

c) Using the law of total probability:

$$\begin{aligned}\Pr(T) &= \Pr(T \cap R_1) + \Pr(T \cap R_2) + \Pr(T \cap R_3)\\ &= 0.675 + 0.1 + 0.035\\ &= 0.81.\end{aligned}$$

In words, the individual is on time on 81% of occasions.

d) We know the individual is on time, i.e., *given* the individual is on time:

$$\begin{aligned}\Pr(R_1 \mid T) &= \frac{\Pr(T \cap R_1)}{\Pr(T)}\\ &= \frac{0.675}{0.81} = 0.833.\end{aligned}$$

$$\begin{aligned}\Pr(R_2 \mid T) &= \frac{\Pr(T \cap R_2)}{\Pr(T)}\\ &= \frac{0.1}{0.81} = 0.123.\end{aligned}$$

$$\begin{aligned}\Pr(R_3 \mid T) &= \frac{\Pr(T \cap R_3)}{\Pr(T)}\\ &= \frac{0.035}{0.81} = 0.043.\end{aligned}$$

$\Rightarrow$ Given that the individual is on time, it is most likely that he/she came via R_1.

e) We require:

$$\begin{aligned}\Pr(R_1 \mid T^c) &= \frac{\Pr(T^c \cap R_1)}{\Pr(T^c)}\\ &= \frac{\Pr(R_1)\Pr(T^c \mid R_1)}{\Pr(T^c)}.\end{aligned}$$

Thus, in order to make progress we first need to calculate:

$$\begin{aligned}\Pr(T^c \mid R_1) &= 1 - \Pr(T \mid R_1)\\ &= 1 - 0.9 = 0.1.\end{aligned}$$

and also

$$\begin{aligned}\Pr(T^c) &= 1 - \Pr(T)\\ &= 1 - 0.81 = 0.19.\end{aligned}$$

$$\begin{aligned}\Rightarrow \Pr(R_1 \mid T^c) &= \frac{\Pr(R_1)\Pr(T^c \mid R_1)}{\Pr(T^c)}\\ &= \frac{0.75(0.1)}{0.19} = 0.395.\end{aligned}$$

If the individual is late, there is a 39.5% chance that he/she travelled via R_1.

Question 2

a) Let S = "the email is spam" and, therefore, S^c = "the email is not spam".

Also, let F_S = "folder: spam".

Using the above notation, the information given in the question is as follows:

$$\Pr(S) = 0.4 \qquad \Pr(F_S \mid S) = 0.9$$

$$\Pr(S^c) = 0.6 \qquad \Pr(F_S \mid S^c) = 0.01$$

We can also calculate (multiplication rule):

$$\begin{aligned}\Pr(F_S \cap S) &= \Pr(S)\Pr(F_S \mid S)\\ &= 0.4(0.9) = 0.36.\end{aligned}$$

$$\begin{aligned}\Pr(F_S \cap S^c) &= \Pr(S^c)\Pr(F_S \mid S^c)\\ &= 0.6(0.01) = 0.006.\end{aligned}$$

b) Using the law of total probability:

$$\begin{aligned}\Pr(F_S) &= \Pr(F_S \cap S) + \Pr(F_S \cap S^c)\\ &= 0.36 + 0.006\\ &= 0.366.\end{aligned}$$

In words, 36.6% of all emails are put into the spam folder

c) We know the email is in the spam folder, i.e., this is *given*:

$$\begin{aligned}\Pr(S \mid F_S) &= \frac{\Pr(S \cap F_S)}{\Pr(F_S)}\\ &= \frac{0.36}{0.366} = 0.984.\end{aligned}$$

$\Rightarrow$ If the email is in the spam folder, we are quite sure that it is spam.

d) We require:

$$\begin{aligned}\Pr(S \mid F_S^c) &= \frac{\Pr(F_S^c \cap S)}{\Pr(F_S^c)}\\ &= \frac{\Pr(S)\Pr(F_S^c \mid S)}{\Pr(F_S^c)}\\ &= \frac{\Pr(S)[1 - \Pr(F_S \mid S)]}{1 - \Pr(F_S)}\\ &= \frac{0.4(1 - 0.9)}{1 - 0.366}\\ &= \frac{0.4(0.1)}{0.634}\\ &= \frac{0.04}{0.634} = 0.063.\end{aligned}$$

In words, 6.3% of emails in our inbox (i.e., the non-spam folder) are spam emails.

Question 3

The solution to this question is in Lecture5 solutions (i.e., Q1 of Lecture5).

Question 4

There are 8 characters in total and the password is of length 5.

In the multiplications below, the first position corresponds to the first character in the password, the second position corresponds to the second character etc.

a) $8(8)(8)(8)(8) = 8^5 = 32768$.

b) Using letters only, there are 4 options for each character of the password
$\Rightarrow 4(4)(4)(4)(4) = 4^5 = 1024$.

c) We have only 4 options for the first character as it cannot be a number
$\Rightarrow 4(8)(8)(8)(8) = 16384$.

d) In part (a) we calculated all possible passwords. In part (b) we calculated those with no numbers
$\Rightarrow 32768 - 16384 = 31744$ passwords have at least one number.

e) We have 8 options for the first character, then 7 for the second (as we've chosen one), 6 for the third (as we've chosen two), etc.
$\Rightarrow 8(7)(6)(5)(4) = 6720$.

Question 5

In the multiplications below: the first position corresponds to the character class, the second corresponds to gender and the third corresponds to the difficulty level.

a) $5(2)(3) = 30$.

b) Only one choice for the character class
$\Rightarrow 1(2)(3) = 6$.

c) Only one choice for the gender
$\Rightarrow 5(1)(3) = 15$.

d) Only one choice for the difficulty level
$\Rightarrow 5(2)(1) = 10$.

For the following multiplications we have additionally: the fourth position which corresponds to the character class of the second player and the fifth corresponds to the gender of the second character.

Note how we do not have a sixth position for the difficulty level for player two since both player one and player two are within the same game.

e) $5(2)(3)(5)(2) = 300$.

f) Here player two only has 4 options for the character class since player one has chosen 1 already
$\Rightarrow 5(2)(3)(4)(2) = 240$.

Question 6

a) Arranging 6 objects. Thus, we have 6 objects we can put in the first position, 5 in the second, 4 in the third etc.
$\Rightarrow 6(5)(4)(3)(2)(1) = 6! = 720$.

b) We must choose 3 items from 6 $\Rightarrow \binom{6}{3}$.

$$\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{6(5)(4)(\cancel{3!})}{3!\ \cancel{3!}} = \frac{\cancel{6}(5)(4)}{\cancel{3}(\cancel{2})(1)} = 20.$$

c) We must bring the pen. Thus, we only need to choose 2 more items from the remaining 5 $\Rightarrow \binom{5}{2}$.

$$\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5(4)(\cancel{3!})}{2!\ \cancel{3!}} = \frac{5(\cancel{4}^{\,2})}{\cancel{2}(1)} = 10.$$

d) We must bring the pen. Thus, we must choose 2 more items from the 3 remaining options (since the pen is gone and we won't choose an apple or a laptop) $\Rightarrow \binom{3}{2}$.

$$\binom{3}{2} = \frac{3!}{2!\,1!} = \frac{3(\cancel{2!})}{\cancel{2!}\,1!} = \frac{3}{1} = 3.$$

Question 7

The solution to this question is in Lecture5 solutions (i.e., Q4 of Lecture5).