Question 1

Jobs are sent to a supercomputer at a rate of 10 per hour and take the supercomputer on average 4 minutes to process. We will assume that the number of arrivals is $X_a \sim \text{Poisson}(\lambda_a)$ and the processing (i.e., service) time is $T_s \sim \text{Exponential}(\lambda_s)$. This leads to an $M/M/1$ system.

(a) Let T be the total time in the system - what distribution has T? **(b)** What is the average time spent in the system? Calculate $Sd(T)$ also. **(c)** How many jobs are in the system on average? (hint: Little's law) **(d)** From the time the job is sent, what is the probability that it takes more than 15 minutes to complete? **(e)** From the time the job enters the processor (i.e., service component), what is the probability that it takes more than 15 minutes to complete? **(f)** What is the average number of jobs completed in a 3 hour period of operation? (hint: Burke's theorem) **(g)** What is the probability that more than 40 jobs are completed in a 3 hour period? (hint: Burke's theorem again)

Question 2

Consider an $M/M/1$ system with arrivals $X_a \sim \text{Poisson}(\lambda_a = 3 \text{ / minute})$ and service time $T_s \sim \text{Exponential}(\lambda_s = 4 \text{ / minute})$. Calculate the following:

(a) The expected time spent in the system. **(b)** The expected time spent in the queue component. **(c)** The expected number of individuals in the system. **(d)** The expected number of individuals in the queue component. **(e)** The utilisation factor. **(f)** The probability that an individual spends more than 2 minutes in the system. **(g)** The probability that less than 3 individuals exit the system in a 1 minute period.

Question 3

Customers arrive to a deli counter at a rate of 12 per hour. On average it takes 3 minutes to serve a customer at this counter. Customers then exit and head to another counter to pay. It takes 1 minute to deal with a customer at this counter. We will assume that arrivals have a $\text{Poisson}(\lambda_a)$ distribution and service times have $\text{Exponential}(\lambda_{s1})$ and $\text{Exponential}(\lambda_{s2})$ distributions respectively (hint: this is a sequence of two $M/M/1$ systems).

(a) What is the average time spent in each sub-system? **(b)** What is the average total time spent in the system? **(c)** How many customers are there (on average) in the system? **(d)** Calculate the utilisation factor for each sub-system. **(e)** What is the average total queueing time? (i.e., total time excluding service time) **(f)** Calculate the probability that at least 20 people exit the shop (i.e., the whole system) in one hour.

Question 4

(**Note**: this is not a queueing theory question. It is a generalisation of a question which appears on Tutorial2)

There are two possible routes to a particular location. You take R_1 80% of the time and R_2 20% of the time. We assume that travel time has an exponential distribution and, furthermore, the average travel time is 0.25 hours if you take R_1 and 0.5 hours if you take R_2.

(a) Calculate the probability that the journey takes more than 0.5 hours for each of the routes, i.e., $\Pr(T > 0.5 \,|\, R_1)$ and $\Pr(T > 0.5 \,|\, R_2)$ respectively. **(b)** Calculate $\Pr(T > 0.5)$. (hint: law of total probability) **(c)** Given that $T > 0.5$ hours, what is the probability that you used R_1? (i.e., calculate $\Pr(R_1 \,|\, T > 0.5)$) **(d)** Derive a general expression for $\Pr(R_1 \,|\, T > t)$ and evaluate it at $t = 0.25$, $t = 1$ and $t = 2$ respectively. Interpret the results.

Question 5

Let $X \sim \text{Normal}(\mu = 10, \sigma = 2)$. Calculate the following:

(a) $\Pr(X > 10)$. **(b)** $\Pr(X < 3)$. **(c)** $\Pr(X > 8.4)$. **(d)** $\Pr(6 < X < 14)$. **(e)** The value of x such that $\Pr(X > x) = 0.3$. **(f)** The value of x such that $\Pr(X > x) = 0.8$.

Question 6

Assume that speeds of cars on a motorway have a normal distribution with mean 115km/hr and standard deviation 4km/hr.

(a) Draw a rough sketch of the distribution. **(b)** $\Pr(X > 120) = ?$ **(c)** $\Pr(X < 100) = ?$ **(d)** $\Pr(100 < X < 110) = ?$ **(e)** 1% of drivers travel above what speed?

Question 7

For *any* normal variable $X \sim \text{Normal}(\mu, \sigma)$:

(a) Show that $\Pr(\mu - 3\,\sigma < X < \mu + 3\,\sigma) = 0.997$. **(b)** Find a value for k such that $\Pr(\mu - k\,\sigma < X < \mu + k\,\sigma) = 0.95$. **(c)** Find k such that $\Pr(\mu - k\,\sigma < X < \mu + k\,\sigma) = 0.99$. **(d)** Show that $\Pr(X > \mu + 1.64\,\sigma) = 0.05$.