## Question 1

a)
$$\begin{aligned} p(0) + p(0.5) + p(1) + p(2) &= 0.4 + 0.2 + 0.15 + 0.15 \\ &= 0.9. \end{aligned}$$

$$\Rightarrow p(3) = 1 - 0.9 = 0.1.$$

Since the probabilities must add to one in a probability distribution.

b)
$$\begin{aligned} E(X) &= 0(0.4) + 0.5(0.2) + 1(0.15) + \\ &\quad 2(0.15) + 3(0.1) \\ &= 0.85. \end{aligned}$$

$$\begin{aligned} E(X^2) &= 0^2(0.4) + 0.5^2(0.2) + 1^2(0.15) + \\ &\quad 2^2(0.15) + 3^2(0.1) \\ &= 1.7. \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= 1.7 - (0.85^2) = 0.9775. \end{aligned}$$

$$Sd(X) = \sqrt{Var(X)} = \sqrt{0.9775} = 0.9887.$$

c) Since $E(X) = 0.85$, the player will win €0.85 on average per game. We should charge €0.95 for a play of this game so that the average amount won is then € -0.10, i.e., a *loss* of €0.10.

d) We are charging €0.95 for a play of the game. If the player wins less than this, we make a profit. Thus, if the player wins €0.00 or €0.50 we make a profit.

$$\begin{aligned} \Pr(\text{we profit}) &= p(0) + p(0.5) \\ &= 0.4 + 0.2 = 0.6. \end{aligned}$$

$\Rightarrow$ If somebody plays our game, there is a 0.6 probability that we profit.

e) Each play of the game is a Bernoulli trial where $p = 0.6$ is the probability that we profit. 10 games $\Rightarrow$ 10 Bernoulli trials $\Rightarrow$ Binomial$(n = 10, p = 0.6)$.

Let $Y$ = "the number of times we profit":

$$\begin{aligned} \Pr(Y = 8) &= \binom{10}{8} (0.6^8)(0.4^2) \\ &= 0.1209. \end{aligned}$$

## Question 2

a) The sample space is $S = \{HHH, THH, HTH, TTH, HHT, THT, HTT, TTT\}$.

| Outcome | $HHH$ | $THH$ | $HTH$ | $TTH$ | $HHT$ | $THT$ | $HTT$ | $TTT$ |
|---|---|---|---|---|---|---|---|---|
| $X$ = "no. of heads" | 3 | 2 | 2 | 1 | 2 | 1 | 1 | 0 |
| $Y$ = "no. of unique faces" | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 1 |

b) From the above we get the joint distribution:

| $Y$ \ $X$ | 0 | 1 | 2 | 3 | $p(y)$ |
|---|---|---|---|---|---|
| 1 | $\frac{1}{8}$ | 0 | 0 | $\frac{1}{8}$ | $\frac{2}{8}$ |
| 2 | 0 | $\frac{3}{8}$ | $\frac{3}{8}$ | 0 | $\frac{6}{8}$ |
| $p(x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ | 1 |

c) The marginal distributions for $X$ and $Y$ are shown in the margins of the table above.

d) If $X$ and $Y$ were independent then:

| $Y$ \ $X$ | 0 | 1 | 2 | 3 | $p(y)$ |
|---|---|---|---|---|---|
| 1 | $\frac{2}{64}$ | $\frac{6}{64}$ | $\frac{6}{64}$ | $\frac{2}{64}$ | $\frac{2}{8}$ |
| 2 | $\frac{6}{64}$ | $\frac{18}{64}$ | $\frac{18}{64}$ | $\frac{6}{64}$ | $\frac{6}{8}$ |
| $p(x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ | 1 |

Clearly this is *not* the joint distribution as in part (b) $\Rightarrow$ $X$ and $Y$ are dependent.

## Question 2 continued

e)
$$E(Y) = 1\left(\frac{2}{8}\right) + 2\left(\frac{6}{8}\right)$$
$$= 1.75 \text{ unique faces.}$$

$$E(Y^2) = 1^2\left(\frac{2}{8}\right) + 2^2\left(\frac{6}{8}\right)$$
$$= 3.25.$$

$$Var(Y) = 3.25 - (1.75^2)$$
$$= 0.1875 \text{ unique faces}^2.$$

$$Sd(X) = \sqrt{0.1875}$$
$$= 0.433 \text{ unique faces.}$$

f)
$$\Pr(Y = 2 \mid X = 2) = \frac{\Pr(Y = 2 \cap X = 2)}{\Pr(X = 2)}$$
$$= \frac{3/8}{3/8} = 1.$$

Given that there are two heads ($X = 2$), we are certain that there are two unique faces ($Y = 2$). Hence $\Pr(Y = 2 \mid X = 2) = 1$.

Note that $\Pr(Y = 2) = \frac{6}{8}$ when we do not have any information.

## Question 3

a)
$$E(X) = 0(0.2) + 100(0.75) + 300(0.05)$$
$$= 90.$$

$$E(Y) = 0(0.1) + 80(0.6) + 200(0.3)$$
$$= 108.$$

b) P1 has an average attack power of $E(X) = 90$ and P2 has 1000 hit points.
$\Rightarrow$ On average it takes $\frac{1000}{90} = 11.11$ attacks to defeat P2.

P2 has an average attack power of $E(Y) = 108$ and P1 has 1000 hit points.
$\Rightarrow$ On average it takes $\frac{1000}{108} = 9.25$ attacks to defeat P1.

Thus, P2 will win on average (note: this does *not* mean that P1 cannot win - it is just less likely).

c) We split this into two parts: the first turn and then the battle after the first turn.

On the first turn, P1 casts a spell and does not attack $\Rightarrow$ after turn 1, P2's life is still 1000.

On the first turn, P2 has an average attack power of 108 $\Rightarrow$ P1's life is 1000 - 108 = 892 after the first turn (on average).

From turn two onwards, P2 can no longer deal a critical blow, i.e., $p(200) = 0$. We distribute the remaining 0.3 probability evenly across the other outcomes $\Rightarrow p(0) = 0.1 + 0.15 = 0.25$ and $p(80) = 0.6 + 0.15 = 0.75$.

Thus, P2's average attack is $E(Y) = 0(0.25) + 80(0.75) = 60$ from turn 2 onwards.

We can think of the rest of the battle as a new battle. P1 has 892 hit points and average attack $E(X) = 90$. P2 has 1000 hit points and average attack $E(Y) = 60$.

On average it takes $\frac{1000}{90} = 11.11$ attacks for P1 to defeat P2 (after the first turn).

On average it takes $\frac{892}{60} = 14.87$ attacks for P2 to defeat P1 (after the first turn).

Therefore, P1 is more likely to win the fight in this scenario.

## Question 4

$X \sim \text{Binomial}(n = 10, p = 0.5)$.

$\Rightarrow$ The *probability function* is:
$p(x) = \binom{10}{x}\, 0.5^x\, 0.5^{10-x}$,

and, since $0.5^x\, 0.5^{10-x} = 0.5^{x+10-x} = 0.5^{10}$, we can simplify: $p(x) = \binom{10}{x}\, 0.5^{10}$.

Note: this only possible when $p = 0.5$.

a) $$\Pr(X = 2) = \binom{10}{2}\, 0.5^{10} = 0.0439.$$

b) $$\Pr(X = 0) = \binom{10}{0}\, 0.5^{10} = 0.00098.$$

**To save space I will write $p(x)$ from now on. In the exam, please show your work as in parts (a) and (b) above.**

c) $$\begin{aligned}\Pr(X > 2) &= 1 - \Pr(X \le 2)\\ &= 1 - [p(0) + p(1) + p(2)]\\ &= 1 - [0.00098 + 0.0098 + 0.0439]\\ &= 1 - 0.0547 = 0.9453.\end{aligned}$$

d) $$\begin{aligned}\Pr(X \le 3) &= p(0) + p(1) + p(2) + p(3)\\ &= 0.00098 + 0.0098 + 0.0439 + 0.1172\\ &= 0.1719.\end{aligned}$$

e) $$\begin{aligned}\Pr(5 \le X \le 7) &= p(5) + p(6) + p(7)\\ &= 0.2461 + 0.2051 + 0.1172\\ &= 0.5684.\end{aligned}$$

f) $$E(X) = n\,p = 10(0.5) = 5 \text{ heads.}$$

$$\begin{aligned}Var(X) &= n\,p\,(1-p)\\ &= 10(0.5)(0.5) = 2.5 \text{ heads}^2.\end{aligned}$$

$$Sd(X) = \sqrt{2.5} = 1.58 \text{ heads.}$$

g) Here $X$ is now Binomial$(n = 20, p = 0.5)$. Calculating $\Pr(X \le 10) = p(0) + p(1) + \ldots + p(9) + p(10)$ is tedious using the probability function and $1 - \Pr(X > 10)$ is no easier using this approach.

With the tables we must rewrite the problem in terms of "$\ge$" probabilities.

$$\begin{aligned}\Pr(X \le 10) &= 1 - \Pr(X > 10)\\ &= 1 - \Pr(X \ge 11)\\ &= 1 - 0.4119 = 0.5581.\end{aligned}$$

h) $E(X) = n\,p = 50(0.5) = 25$ heads.

## Question 5

Same as above but now using the binomial tables. Go to column $p = 0.5$ and row $n = 10$.

a) $$\begin{aligned}\Pr(X = 2) &= \Pr(X \ge 2) - \Pr(X \ge 3)\\ &= 0.9893 - 0.9453\\ &= 0.0440.\end{aligned}$$

b) $$\begin{aligned}\Pr(X = 0) &= \Pr(X \ge 0) - \Pr(X \ge 1)\\ &= 1.0000 - 0.9990\\ &= 0.001.\end{aligned}$$

c) $$\Pr(X > 2) = \Pr(X \ge 3) = 0.9453.$$

d) $$\begin{aligned}\Pr(X \le 3) &= 1 - \Pr(X > 3)\\ &= 1 - \Pr(X \ge 4)\\ &= 1 - 0.8281 = 0.1719.\end{aligned}$$

e) $$\begin{aligned}\Pr(5 \le X \le 7) &= \Pr(X \ge 5) - \Pr(X \ge 8)\\ &= 0.6230 - 0.0547\\ &= 0.5683.\end{aligned}$$

We can see that these are the same as before apart from small differences due to rounding.

## Question 6

Let $X$ = "the number of errors in a string of bits".

a) $X \sim \text{Binomial}(n = 20, p = 0.1)$.

$$\Pr(X = 0) = \binom{20}{0} 0.1^0\ 0.9^{20} = 0.1216.$$

b) $X \sim \text{Binomial}(n = 10, p = 0.1)$.

$$\begin{aligned}\Pr(X < 3) &= \Pr(X \le 2)\\ &= p(0) + p(1) + p(2)\\ &= \binom{10}{0} 0.1^0\ 0.9^{10} + \binom{10}{1} 0.1^1\ 0.9^9 + \binom{10}{2} 0.1^2\ 0.9^8\\ &= 0.3487 + 0.3874 + 0.1937 = 0.9298.\end{aligned}$$

Note: (a) and (b) can also be done using tables.

It is essential to use the tables for part (c) since there is far too much work involved if we do not use the tables.

c) (i) $X \sim \text{Binomial}(n = 50, p = 0.1)$
$\Rightarrow$ tables: column $p = 0.1$, row $n = 50$.

$$\Pr(X > 10) = \Pr(X \ge 11) = 0.0094.$$

(ii) $X \sim \text{Binomial}(n = 100, p = 0.1)$
$\Rightarrow$ tables: column $p = 0.1$, row $n = 100$.

$$\Pr(X > 10) = \Pr(X \ge 11) = 0.4168.$$

d) $$E(X) = n\,p = 100(0.1) = 10 \text{ errors.}$$

$$\begin{aligned}Var(X) &= n\,p\,(1-p)\\ &= 100(0.1)(0.9) = 9 \text{ errors}^2.\end{aligned}$$

$$Sd(X) = \sqrt{9} = 3 \text{ errors.}$$

## Question 7

Let $Y$ = "the number of errors in three replicates".
$\Rightarrow Y \sim \text{Binomial}(n = 3, p = 0.1)$

a) The bit will be assigned the wrong value if there are two or three errors:

$$\begin{aligned}\Pr(\text{error}) &= \Pr(Y \ge 2)\\ &= p(2) + p(3)\\ &= \binom{3}{2} 0.1^2\ 0.9^1 + \binom{3}{3} 0.1^3\ 0.9^0\\ &= 0.027 + 0.001\\ &= 0.028.\end{aligned}$$

b) $X$ = "the number of errors in a string of bits" as before. However, since each bit is replicated three times, the probability of error is $p = 0.028$.

$\Rightarrow X \sim \text{Binomial}(n = 20, p = 0.028)$.

$$\Pr(X = 0) = \binom{20}{0} 0.028^0\ 0.972^{20} = 0.5667.$$

c) Here $Y \sim \text{Binomial}(n = 5, p = 0.1)$ and a bit will be assigned the wrong value if there are *three or more* errors:

$$\begin{aligned}\Pr(\text{error}) &= \Pr(Y \ge 3) = p(3) + p(4) + p(5)\\ &= \binom{5}{3} 0.1^3\ 0.9^2 + \binom{5}{4} 0.1^4\ 0.9^1 + \binom{5}{5} 0.1^5\ 0.9^0\\ &= 0.0081 + 0.00045 + 0.00001\\ &= 0.00856.\end{aligned}$$

(it should be clear that by increasing replication we can reduce the error probability to any desired level)

Thus, with this error probability, the number of errors in a 20-bit string is has the following distribution:

$X \sim \text{Binomial}(n = 20, p = 0.00856)$

$$\Rightarrow \Pr(X = 0) = \binom{20}{0} 0.00856^0\ 0.99144^{20} = 0.8420.$$