1. The random variable X has the Poisson distribution with mean 1.5, so that

$$\Pr(X = x) = \frac{1.5^x e^{-1.5}}{x!},$$

Draw a graph of $P(X = x) versus x$, showing all probabilities greater than 0.02, and write down the mean, mode and variance of this distribution.

2. Assume that, for any t ¿ 0, the number N of incoming telephone calls to a 24-hour international call centre in a time t minutes follows a Poisson distribution with mean $3t$.

   (i) Find the probability that exactly 2 calls arrive in the next minute.

   (ii) Find the probability that no calls arrive in the next 1.5 minutes.

   (iii) State the exact distribution of N in the case t = 60. Staffing levels at the call centre are based on the assumption that not more than 200 calls will be made in an hour. Use a suitable approximation to the distribution of N to calculate the approximate probability that this assumption is violated in a given hour.

   (iv) Comment critically on the assumption made at the beginning of this part of the question.

3. The independent random variables X and Y each follow a discrete uniform distribution on the integers 1, 2, 3, 4, 5.

   (a) Write down the probability mass function (pmf) p(x) of X. Find E(X) and show that $Var(X) = 2$.

   (b) The random variable Z is defined by Z = XY.

      (i Write down a list of all the values that Z can take.

      (ii Find the pmf of Z and deduce the mode of Z.

      (iii Find E(Z) and Var(Z).

4. (a) A sequence of three plus (+) signs and two minus () signs is written in a straight line. Write down all 10 possible sequences (different orders) of these signs.

   (b) A "run" of one sign is a series of one or more consecutive instances of this sign, terminated at each end either by the opposite sign or by the beginning or end of the sequence. Assume that sequences of these signs are generated randomly, so that each possible sequence has probability 0.1 of occurring, and let the random variable X denote the number of runs in a randomly chosen sequence. For example, if the sequence + + + is observed then X = 4: one run of one , one run of one +, one run of one , one run of ++.

      (i) Write down the values taken by X for each of the 10 sequences you have listed in part (a), and hence obtain the probability distribution of X.

      (ii) Find E(X) and Var(X).

5. The Poisson random variable X with parameter $\lambda > 0$ has probability mass function

$$P_X(x) = \Pr(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where $x = \{0, 1, 2, 3 \ldots\}$.

(a) Show that, for any integer $x = 0$,

$$P_X(x+1) = \frac{\lambda}{x+1} P_X(x)$$

(b) Obtain $E(X)$ and $E(X(X-1))$. Hence show that $E(X) = Var(X) = \lambda$.

(c) Suppose that Y is a Poisson random variable with mean μ, that X and Y are independent and that $W = X + Y$. Use the relation

$$P_W(w) = \sum_{x=0}^{w} \Pr(X = x) \Pr(Y = w - x)$$

to show that W is a Poisson random variable with mean $\lambda + \mu$.

6. A manufacturer has two conveyor belts, one of type C and the other of type D.

   The numbers of breakdowns per day on these belts, X and Y, are independent Poisson random variables with means 1.5 and 0.5 for belts C and D respectively.

   (i) Find the conditional probability that if there is exactly one breakdown on a given day on either belt C or belt D, but not both, then it is conveyor belt C that fails.

   (ii) What is the probability that the factory will experience at most one breakdown in a 5-day period?

7. The random variable X has the geometric distribution with probability mass function (pmf)

$$\Pr(X = k) = (1-p)^k p$$

or, with $q = 1 - p$,.

$$\Pr(X = k) = (q)^k p$$

for $k = \{0, 1, 2, 3, \ldots\}$

(a) Find $P(X \geq x)$ and show that $P(X \leq 3) = 1q^4$.

(b) Explain why $P(X \text{ is odd } (= 1, 3, 5, 7, ...)) = qP(X \text{ is even } (= 0, 2, 4, 6, ...))$ and hence or otherwise show that

$$P(X \text{ is odd } = \frac{q}{1+q}.$$

(c) Find $P(X \text{ is odd}|X \leq 3)$ as a function of q, and, given that $P(X \text{ is odd } |X \leq 3) = 1/3$ deduce the value of q.

(d) The random variable Y is independent of X and has pmf

$$p_Y(y) = q \times p^y$$

. Write down $P(X = kandY = k)$ for an arbitrary non-negative integer k and hence show that

$$P(X = Y) = \frac{p(1-p)}{(1-p)(1-p)}.$$

(e) Using calculus, find the value of p that maximises this expression and hence deduce that the maximum possible value of $P(X = Y)$ is 1/3.

[Note. You are not required to show that any turning point that you locate is a maximum.]

8. A sequence of independent Bernoulli trials with probability of success p is performed. Let the random variable X be the number of failures before the first success.

   (a) Find the probability mass function of X, confirming that the sum over all possible values of X is one.

   (b) Obtain $E(X)$.

   (c) The probability of an enemy aircraft penetrating friendly airspace is 0.01.

      (i) What is the probability that the first penetration of friendly airspace is accomplished by the 80th aircraft to attempt the penetration, assuming penetration attempts are independent?

      (ii) What is the probability that it will take more than 80 attempts to penetrate friendly airspace?

   (d) Consider Y, the total number of failures before the second success. Find the probability mass function of Y. By considering the mean of X, show that

$$E(Y) = 2\left(\frac{1-p}{p}\right).$$

9. The following questions related to the approximations of probability distributions.

   (a) Explain carefully why a continuity correction is needed when a discrete random variable is approximated by a continuous random variable in order to calculate a probability for the discrete random variable.

   (b) Explain when a Poisson approximation or a Normal approximation to a binomial probability may be used, carefully distinguishing between the two.

   (c) A fair die is thrown 300 times. Find an approximation to the probability that there are fewer than 45 sixes.

   (d) The number of accidents in a factory in one working week has a Poisson distribution with mean 0.2. What is the distribution of the number of accidents in this factory in three years, comprising 150 working weeks (regarded as independent)? Find an approximation to the probability that in three years there are 35 or more accidents.