Statistics for Computing MA4413

Lecture 3

Probability: Definitions, Set Notation, Complement Rule, Addition Rule and Multiplication Rule

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Definitions

- **Experiment**: any process which generates various outcomes.
- **Outcome**: one possible result of the experiment.
- **Sample Space**: the set of *all* possible outcomes, denoted by S.
- **Event**: any subset of the sample space.

Example: Flipping Two Coins

- Experiment: flipping two coins.
- Outcome: the results showing on the coins.
- Sample Space: $S = \{HH, TH, HT, TT\}$.
- Event (examples):
  - "At least one head showing" $= \{HH, TH, HT\}$.
  - "Getting two tails" $= \{TT\}$.
  - "The same result on both coins" $= \{HH, TT\}$.
  - "The coins show different results" $= \{TH, HT\}$.

Probability

- **Probability**: the proportion (i.e., relative frequency) of times that a particular event occurs in the *long run* (after repeating the experiment many times).

Let A be the symbol for the event in question. The probability of A is

$$\boxed{\Pr(A) = \frac{\#(A)}{\#(S)}}$$

where $\#(A)$ is the number of outcomes contained in A and $\#(S)$ is the number of *all possible outcomes*, i.e., the number of outcomes in the sample space.

Probability

Probability *must be* **between zero and one**:

- $\Pr(A) = 0$: the event A is impossible.
- $\Pr(A) = 1$: the event A is certain.

The probability value is a measure of *how likely* the event is, e.g.,

- $\Pr(A) = 0.01$: highly unlikely.
- $\Pr(A) = 0.3$: unlikely.
- $\Pr(A) = 0.5$: there is a 50-50 chance.
- $\Pr(A) = 0.7$: likely.
- $\Pr(A) = 0.99$: highly likely.

Example: Flipping Two Coins

Here $S = \{HH, TH, HT, TT\} \Rightarrow \#(S) = 4$.

Consider the following events:

- A = "At least one head showing" $= \{HH, TH, HT\}$.
  $\#(A) = 3 \Rightarrow \Pr(A) = \frac{3}{4} = 0.75$.

- B = "Getting two tails" $= \{TT\}$.
  $\#(B) = 1 \Rightarrow \Pr(B) = \frac{1}{4} = 0.25$.

- C = "The same result on both coins" $= \{HH, TT\}$.
  $\#(C) = 2 \Rightarrow \Pr(C) = \frac{2}{4} = \frac{1}{2} = 0.5$.

- D = "The coins show different results" $= \{TH, HT\}$.
  $\#(D) = 2 \Rightarrow \Pr(D) = \frac{2}{4} = \frac{1}{2} = 0.5$.

Complement Rule

Either the event A happens or it does not happen.

In the latter case, A^c (pronounced “A-complement”) occurs instead - A^c the event which is the opposite of A.

The **complement rule** states that

$$\boxed{\Pr(A^c) = 1 - \Pr(A)}.$$

For example, if $\Pr(\text{light bulb fails}) = 0.05$, then
$\Pr(\text{light bulb works}) = 1 - \Pr(\text{light bulb fails}) = 1 - 0.05 = 0.95$.

Question 1

Consider the experiment where both a die is rolled and a coin is flipped.

a) What is the sample space for this experiment?

b) Calculate Pr(getting a head and any number).

c) Calculate Pr(getting a head and a six).

d) Calculate Pr(getting a head and an *even* number).

e) Calculate Pr(getting a tail and a number *greater than* four).
(greater than four $\Rightarrow$ five or six)

f) What is the probability of getting *two* heads and a six?

R Code

The `expand.grid` function in R is useful for finding all outcomes in a sample space:

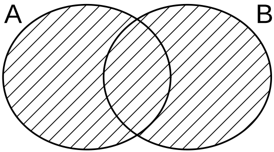
```
coin = c("H","T")
die = c(1,2,3,4,5,6)

expand.grid(coin,coin) # two coins
expand.grid(coin,coin,coin) # three coins
expand.grid(coin,coin,coin,coin) # four coins
expand.grid(coin,die) # coin and a die
expand.grid(die,die) # two dice
```

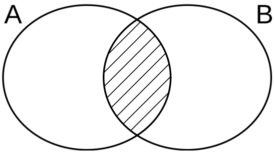
Set Notation

If A and B are two events then we have

- $\boldsymbol{A \cup B}$: "A union B" represents A or B or *both together*, i.e., $A \cup B =$ at least one of the two occurs.



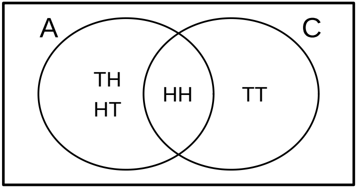
- $\boldsymbol{A \cap B}$: "A intersection B" represents A *and* B, i.e., $A \cap B =$ both occur together.



Example: Flipping Two Coins

A = "At least one head showing" $= \{HH, TH, HT\}$.

C = "The same result on both coins" $= \{HH, TT\}$.



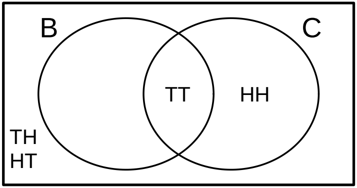
$A \cup C = \{HH, TH, HT, TT\} \Rightarrow \Pr(A \cup C) = \frac{4}{4} = 1.$

$A \cap C = \{HH\} \Rightarrow \Pr(A \cap C) = \frac{1}{4}.$

Example: Flipping Two Coins

B = "Getting two tails" $= \{TT\}$.

C = "The same result on both coins" $= \{HH, TT\}$.



$B \cup C = \{HH, TT\} \Rightarrow \Pr(B \cup C) = \frac{2}{4} = \frac{1}{2}$.

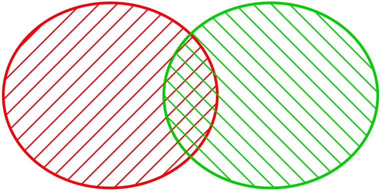
$B \cap C = \{TT\} \Rightarrow \Pr(B \cap C) = \frac{1}{4}$.

Addition Rule: Two Events

For two events, A and B, the probability of *at least one* occurring is

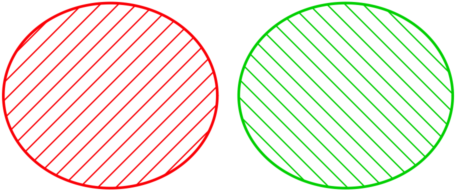
$$\boxed{\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)}.$$

Note that the intersection probability is subtracted once since it was added twice:



Addition Rule: Two Mutually Exclusive Events

A and B are **mutually exclusive** events if they *cannot* occur simultaneously, i.e., the presence of one *excludes* the presence of the other $\Rightarrow \boxed{\Pr(A \cap B) = 0}$.



In this case the previous formula simplifies to

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$

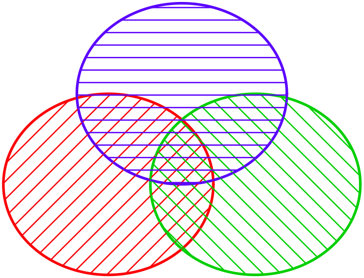
since $\Pr(A \cap B) = 0$.

More than Two Events

For *three* events, A, B and C, the probability of *at least one* occurring is

$$\begin{aligned}\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) \\ - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) \\ + \Pr(A \cap B \cap C).\end{aligned}$$

Can you see why from the diagram?



(clearly the situation becomes complicated for more than three events - we will not deal with such situations)

More than Two Mutually Exclusive Events

For *three* mutually exclusive events, the previous formula simplifies to

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C)$$

as *all* intersections have zero probability since the events cannot happen simultaneously.

For ***k* mutually exclusive** events, $E_1, E_2, E_3, \ldots E_k$, it should be clear that the probability of at least one occurring is:

$$\boxed{\Pr(E_1 \cup E_2 \cup \cdots \cup E_k) = \Pr(E_1) + \Pr(E_2) + \cdots + \Pr(E_k)}.$$

At Least One Vs None of the Events

$\Pr(A \cup B)$ is the probability of *at least one* of A or B occurring.

Recall from earlier that the *complement rule* allows us to evaluate the probability of a complementary (i.e., opposite) event.

The complement of "at least one" is "none". In mathematical notation: $(A \cup B)^c = A^c \cap B^c$, i.e., simultaneously not A *and* not B.

Thus, the probability of *neither A nor B* is

$$\boxed{\Pr(A^c \cap B^c) = 1 - \Pr(A \cup B)}.$$

Conversely, if we have $\Pr(A^c \cap B^c)$ and wish for $\Pr(A \cup B)$, we can use

$$\boxed{\Pr(A \cup B) = 1 - \Pr(A^c \cap B^c)}.$$

At Least One Vs None of the Events

Of course the ideas on the previous slide hold for more than two events, i.e.,

$$\boxed{\Pr(E_1^c \cap E_2^c \cap \cdots \cap E_k^c) = 1 - \Pr(E_1 \cup E_2 \cup \cdots \cup E_k)},$$

and, conversely,

$$\boxed{\Pr(E_1 \cup E_2 \cup \cdots \cup E_k) = 1 - \Pr(E_1^c \cap E_2^c \cap \cdots \cap E_k^c)}.$$

Example: Flipping Two Coins

In the example of flipping two coins we had:

$B =$ "Getting two tails" $= \{TT\} \Rightarrow \Pr(B) = \frac{1}{4} = 0.25.$

$C =$ "The same result on both coins" $= \{HH, TT\} \Rightarrow \Pr(B) = \frac{2}{4} = 0.5.$

These events are clearly *not* mutually exclusive since $B \cap C = \{TT\} \Rightarrow \Pr(B \cap C) = \frac{1}{4} = 0.25.$

To calculate the probability of "getting two tails" *or* "the same result" *or* both (i.e., at least one of B or C), we can use the addition rule:

$$\Pr(B \cup C) = \Pr(B) + \Pr(C) - \Pr(B \cap C) = 0.25 + 0.5 - 0.25 = 0.5.$$

(note that we found this probability already using the venn diagram - see slide 12)

Example: Flipping Two Coins

If we want the probability that *neither B nor C* occurs we use the complement rule:

$$\Pr(B^c \cap C^c) = 1 - \Pr(B \cup C) = 1 - 0.5 = 0.5.$$

(Note that the above probability can be found using a more manual approach since $B^c = \{HH, TH, HT\}$ and $C^c = \{TH, HT\} \Rightarrow B^c \cap C^c = \{TH, HT\}$)

Example: Rolling One Die

Consider the experiment of rolling a die which has $S = \{1, 2, 3, 4, 5, 6\}$.

We then define the events:

- E_1 = "the result is a one" $= \{1\} \Rightarrow \Pr(E_1) = \frac{1}{6}$.
- E_2 = "the result is a two" $= \{2\} \Rightarrow \Pr(E_2) = \frac{1}{6}$.
- E_3 = "the result is a three" $= \{3\} \Rightarrow \Pr(E_3) = \frac{1}{6}$.
- E_4 = "the result is a four" $= \{4\} \Rightarrow \Pr(E_4) = \frac{1}{6}$.
- E_5 = "the result is a five" $= \{5\} \Rightarrow \Pr(E_5) = \frac{1}{6}$.
- E_6 = "the result is a six" $= \{6\} \Rightarrow \Pr(E_6) = \frac{1}{6}$.

These events are all *mutually exclusive* since, for example, the result cannot *simultaneously* be a one *and* a two.

Example: Rolling One Die

We can add the event probabilities without worrying about intersections since they are mutually exclusive events, e.g.:

$$\begin{aligned}\Pr(\text{“result between three and five”}) &= \Pr(E_3 \cup E_4 \cup E_5) \\ &= \Pr(E_3) + \Pr(E_4) + \Pr(E_5) \\ &= \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{3}{6} = \tfrac{1}{2}.\end{aligned}$$

$$\begin{aligned}\Pr(\text{“greater than four”}) = \Pr(\text{“a five or a six”}) &= \Pr(E_5 \cup E_6) \\ &= \Pr(E_5) + \Pr(E_6) \\ &= \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{2}{6} = \tfrac{1}{3}.\end{aligned}$$

Finally, an example of the usefulness of the complement rule:

$$\begin{aligned}\Pr(\text{“less than or equal to four”}) &= 1 - \Pr(\text{“greater than four”}) \\ &= 1 - \tfrac{1}{3} = \tfrac{2}{3}.\end{aligned}$$

Exhaustive Events

It is worth noting that in the last example

$$
\begin{aligned}
\Pr(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6) \\
&= \Pr(E_1) + \Pr(E_2) + \Pr(E_3) + \Pr(E_4) + \Pr(E_5) + \Pr(E_6) \\
&= \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} \\
&= 1.
\end{aligned}
$$

Such events are called **exhaustive** as they cover the whole sample space, i.e., they *exhaust all possibilities*.

(this is important later for the *law of total probability*)

Question 2

Consider the experiment where both a die is rolled and a coin is flipped (considered earlier in Question 1). Let $A =$ "head & even number", $B =$ "head & any number", $C =$ "any face & a five".

a) $A = \{H2, H4, H6\}$. Write out the set of outcomes in B and C. Hence, calculate $\Pr(A)$, $\Pr(B)$ and $\Pr(C)$.

b) Write down $A \cap B$, $A \cap C$ and $B \cap C$. Hence, calculate the probabilities $\Pr(A \cap B)$, $\Pr(A \cap C)$ and $\Pr(B \cap C)$.

c) Which events are mutually exclusive?

d) Calculate $\Pr(A \cup B)$, $\Pr(A \cup C)$ and $\Pr(B \cup C)$ using the addition rule.

e) What is the probability that the result is neither A nor B?

Question 3

Consider a RAID (redundant array of inexpensive disks) system where multiple hard disks are used simultaneously.

Let's assume that we have two hard disks. Define the events $H_1 =$ "hard disk one works" and $H_2 =$ "hard disk two works" and also assume that $\Pr(H_1) = \Pr(H_2) = 0.9$. If the hard disks work *independently* (more on this later), the probability that they both work is $\Pr(H_1 \cap H_2) = 0.81$.

a) RAID-0 is a system which increases performance but only works if *both* hard disks work. What is Pr(RAID-0 works)?

b) Calculate Pr(RAID-0 fails).

c) RAID-1 is a system which does not increase performance but still works with only one working hard disk. What is Pr(RAID-1 works)?

d) Calculate Pr(RAID-1 fails).

Note on Question 3

It may interest you to find out more about RAID systems (if you don't already know).

You can have a look at this video: http://youtu.be/RYBtmVMtH1g.

The first four minutes provide a good discussion of how RAID-0 and RAID-1 work. The rest of the video describes how you can set up these systems in practice.

Also see wikipedia for information on various RAID systems: http://en.wikipedia.org/wiki/RAID.

Multiplication Rule: Two Events

Although we have found $\Pr(A \cap B)$ in the previous section manually, there is also a formula to calculate it:

$$\boxed{\Pr(A \cap B) = \Pr(A)\ \Pr(B \mid A)}.$$

What is $B \mid A$? This is the event of B *given* that A has happened.

$\Pr(B \mid A)$ is a *conditional probability* - it is the probability that B occurs under the condition that A has already occurred.

We can change the order of multiplication if we like:

$$\Pr(A \cap B) = \Pr(B)\ \Pr(A \mid B).$$

Multiplication Rule: Independent Events

Events are described as **independent** if the occurrence of one has *no effect* on the other.

In this case $\boxed{\Pr(B \mid A) = \Pr(B)}$, i.e., it does not matter if A has happened or not.

For two independent events the multiplication formula simplifies to

$$\Pr(A \cap B) = \Pr(A) \Pr(B).$$

Of course this extends to $\boldsymbol{k}$ **independent events**:

$$\boxed{\Pr(E_1 \cap E_2 \cap \cdots \cap E_k) = \Pr(E_1) \Pr(E_2) \cdots \Pr(E_k)}.$$

(for more than two *non-independent* events there is no simple multiplication formula - much like the addition rule for *non-mutually exclusive* events)

Checking Independence

Assuming we have calculated $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cap B)$ then we can check:

- If $\Pr(A) \times \Pr(B) = \Pr(A \cap B) \Rightarrow$ independent events.
- If $\Pr(A) \times \Pr(B) \neq \Pr(A \cap B) \Rightarrow$ non-independent events.

Question 4

In Question 2 we calculated: $\Pr(A)$, $\Pr(B)$, $\Pr(C)$, $\Pr(A \cap B)$, $\Pr(A \cap C)$ and $\Pr(B \cap C)$.

a) Check if any of the events A, B or C are independent.

Question 5

In Question 3 we looked at RAID systems and defined the events $H_1 =$ "hard disk one works" and $H_2 =$ "hard disk two works" with $\Pr(H_1) = \Pr(H_2) = 0.9$. We assume that H_1 and H_2 are *independent*.

a) Show that the probability of both hard disks working is $\Pr(H_1 \cap H_2) = 0.81$.

b) Calculate $\Pr(\text{RAID-1 works}) = \Pr(H_1 \cup H_2)$.

c) What is the probability that hard disk one *fails* (i.e., $\Pr(H_1^c)$)? What is the value of $\Pr(H_2^c)$?

d) $\Pr(H_1^c \cap H_2^c)$ is the probability that *both* fail simultaneously, i.e., $\Pr(\text{RAID-1 fails})$. Calculate this probability using the answer to part (c) and the fact these events are independent.

e) Cheap hard disks exist with $\Pr(\text{cheap hard disk works}) = 0.6$. How many of these are needed to match the performance of the two-hard disk system described above?

Independence Vs Mutual Exclusion

Do not mix up the ideas of independence and mutual exclusion.

- **Independent events**
  - Have *no effect* on each other.
  - *Can* happen at the same time (but work *independently* of each other).
  - Allow us to simplify the **multiplication rule**.
- **Mutually exclusive events**
  - *Cannot* happen at the same time.
  - Certainly *affect* each other since the presence of one excludes the presence of the other.
  - Allow us to simplify the **addition rule**.

Bottom line: If events are independent they are not mutually exclusive. If events are mutually exclusive they are not independent.

Question 6

Classify the following pairs of events as being mutually exclusive, independent or dependent (but not mutually exclusive).

	Event A	Event B
a)	A coin shows a head	The same coin shows a tail
b)	You work hard	You get promoted
c)	You are Irish	It rains in Japan
d)	Anti-virus out of date	Laptop is virus-free
e)	You are in this lecture	You are in the Stables
f)	You are in this lecture	You are on Facebook
g)	An individual is not wealthy	He/she drives an expensive car
h)	One coin shows a head	Another coin shows a head