Question 1

a) $S = \{H1,\ H2,\ H3,\ H4,\ H5,\ H6$
$T1,\ T2,\ T3,\ T4,\ T5,\ T6\}$

b) "head & any no." $=$ $\{H1, H2, H3, H4, H5, H6\}$.

$\Rightarrow$ Pr("head & any no.") $= \frac{6}{12} = \frac{1}{2} = 0.5$.

c) "head & six" $= \{H6\}$.

$\Rightarrow$ Pr("head & six") $= \frac{1}{12} = 0.0833$.

d) "head & even" $= \{H2, H4, H6\}$.

$\Rightarrow$ Pr("head & even") $= \frac{3}{12} = \frac{1}{4} = 0.25$.

e) "tail & greater than four" $= \{T5, T6\}$.

$\Rightarrow$ Pr("tail & greater than four.") $= \frac{2}{12} = \frac{1}{6} =$ 0.167.

f) Pr("two heads & six") $= 0$ as it is not possible to get $HH6$.

Question 2

a) $A = \{H2, H4, H6\}$.
$B = \{H1, H2, H3, H4, H5, H6\}$.
$C = \{H5, T5\}$.

$\Pr(A) = \frac{3}{12} = \frac{1}{4} = 0.25$.
$\Pr(B) = \frac{6}{12} = \frac{1}{2} = 0.5$.
$\Pr(C) = \frac{2}{12} = \frac{1}{6} = 0.1667$.

b) $A \cap B = \{H2, H4, H6\}$.
$A \cap C = \{\}$.
$B \cap C = \{H5\}$.

$\Pr(A \cap B) = \frac{3}{12} = \frac{1}{4} = 0.25$.
$\Pr(A \cap C) = \frac{0}{12} = 0$.
$\Pr(B \cap C) = \frac{1}{12} = 0.0833$.

c) $\Pr(A \cap C) = 0 \Rightarrow A$ and C are mutually exclusive.

d)
$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= \frac{3}{12} + \frac{6}{12} - \frac{3}{12}\\ &= \frac{6}{12} = \frac{1}{2} = 0.5.\end{aligned}$$

$$\begin{aligned}\Pr(A \cup C) &= \Pr(A) + \Pr(C) - \Pr(A \cap C)\\ &= \frac{3}{12} + \frac{2}{12} - \frac{0}{12}\\ &= \frac{5}{12} = 0.4167.\end{aligned}$$

$$\begin{aligned}\Pr(B \cup C) &= \Pr(B) + \Pr(C) - \Pr(B \cap C)\\ &= \frac{6}{12} + \frac{2}{12} - \frac{1}{12}\\ &= \frac{7}{12} = 0.5833.\end{aligned}$$

e)
$$\begin{aligned}\Pr(A \text{ nor } B) = \Pr(A^c \cap B^c) &= 1 - \Pr(A \cup B)\\ &= 1 - 0.5\\ &= 0.5.\end{aligned}$$

Question 3

a) $\Pr(\text{RAID-0 works}) = \Pr(H_1 \cap H_2) = 0.81$.

b)
$$\begin{aligned}\Pr(\text{RAID-0 fails}) &= 1 - \Pr(\text{RAID-0 works})\\ &= 1 - 0.81\\ &= 0.19.\end{aligned}$$

c)
$$\begin{aligned}\Pr(\text{RAID-1 works}) &= \Pr(H_1 \text{ or } H_2 \text{ or both})\\ &= \Pr(H_1 \cup H_2)\\ &= \Pr(H_1) + \Pr(H_2)\\ &\qquad - \Pr(H_1 \cap H_2)\\ &= 0.9 + 0.9 - 0.81\\ &= 0.99.\end{aligned}$$

d)
$$\begin{aligned}\Pr(\text{RAID-1 fails}) &= 1 - \Pr(\text{RAID-1 works})\\ &= 1 - 0.99\\ &= 0.01.\end{aligned}$$

Question 4

a)

$$\begin{aligned}\Pr(A)\Pr(B) &= \frac{1}{4} \times \frac{1}{2}\\ &= \frac{1}{8}\\ &\neq \Pr(A \cap B) = \frac{1}{4}\end{aligned}$$

$\Rightarrow A$ and B are *dependent.*

$$\begin{aligned}\Pr(A)\Pr(C) &= \frac{1}{4} \times \frac{1}{6}\\ &= \frac{1}{24}\\ &\neq \Pr(A \cap C) = 0\end{aligned}$$

$\Rightarrow A$ and C are *dependent.*

$$\begin{aligned}\Pr(B)\Pr(C) &= \frac{1}{2} \times \frac{1}{6}\\ &= \frac{1}{12}\\ &= \Pr(B \cap C) = \frac{1}{12}\end{aligned}$$

$\Rightarrow B$ and C are *independent.*

Question 5

a)

$$\begin{aligned}\Pr(H_1 \cap H_2) = \Pr(H_1)\Pr(H_2) &= 0.9(0.9)\\ &= 0.81.\end{aligned}$$

b) $\Pr(H_1 \cup H_2) = 0.99$ (calculated in Q3).

c) $\Pr(H_1^c) = 1 - \Pr(H_1) = 0.1.$
$\Pr(H_2^c) = 1 - \Pr(H_2) = 0.1.$

d)

$$\begin{aligned}\Pr(H_1^c \cap H_2^c) = \Pr(H_1^c)\Pr(H_2^c) &= 0.1(0.1)\\ &= 0.01.\end{aligned}$$

e) In this case $\Pr(H) = 0.6$. So $\Pr(H^c) = 0.4$.

We want $\Pr(\text{fail}) = 0.01$ to match performance above.

Two cheap disks: $\Pr(\text{fail}) = 0.4 \times 0.4 = 0.16$.
Three cheap disks: $\Pr(\text{fail}) = 0.4^3 = 0.064$.
Four cheap disks: $\Pr(\text{fail}) = 0.4^4 = 0.0256$.
Five cheap disks: $\Pr(\text{fail}) = 0.4^5 = 0.01024$.
Six cheap disks: $\Pr(\text{fail}) = 0.4^6 = 0.0041$.

$\Rightarrow$ So five cheap disks provide similar performance level to two expensive disks. Six cheap disks provide *superior* performance.

Question 5(e) - alternative method

We can see from above that the general form of Pr(fail) for k cheap disks is:

$\Pr(\text{fail}) = 0.4^k$

We can set the above expression equal to the 0.01 (or any desired level) and solve for k.

$$\begin{aligned}0.4^k &= 0.01\\ \log 0.4^k &= \log 0.01\\ k \log 0.4 &= \log 0.01\\ k &= \frac{\log 0.01}{\log 0.4}\\ &\approx 5.026,\end{aligned}$$

i.e., roughly five disks are required (as found previously using the more laborious approach).

Question 6

a) Mutually exclusive.

b) Dependent.

c) Independent.

d) Dependent.

e) Mutually exclusive.

f) Dependent.

g) Dependent.

h) Independent.