Question 1

We have the arrival rate $\lambda_a = 10$ per hour.

We calculate the service rate λ_s from the average service time $E(T_s) = 4$ minutes $= \frac{4}{60} = \frac{1}{15}$ hours.

Since the service time has an exponential distribution we know $E(T_s) = \frac{1}{\lambda_s}$. Manipulating this formula gives $\lambda_s = \frac{1}{E(T_s)} = \frac{1}{1/15} = 15$ per hour.

a) The result of the $M/M/1$ assumptions (i.e., Poisson arrivals with exponential service times) is that T, the total time in the system, has an exponential distribution with parameter $\lambda = \lambda_s - \lambda_a = 15 - 10 = 5$, i.e.,

$$T \sim \text{Exponential}(\lambda = 5).$$

b) $$E(T) = \frac{1}{\lambda} = \frac{1}{5} = 0.2 \text{ hours.} \quad \text{(i.e., 12 minutes)}$$

$$Sd(T) = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda} = 0.2 \text{ hours.}$$

c) $$\begin{aligned} \Pr(N) &= \lambda_a\, E(T) = 10(0.2) \\ &= 2 \text{ jobs in the system.} \end{aligned}$$

d) This relates to the total time in the system, $T \sim \text{Exponential}(\lambda = 5)$.
(note: 15 minutes $= \frac{15}{60} = 0.25$ hours)

$$\Pr(T > 0.25) = e^{-5\,(0.25)} = 0.2865.$$

e) This relates to the service time, $T_s \sim \text{Exponential}(\lambda_s = 15)$.

$$\Pr(T_s > 0.25) = e^{-15\,(0.25)} = 0.0235.$$

f) Burke's theorem says departures (i.e., completed jobs) have the same Poisson distribution as arrivals. Thus, the departure rate is $\lambda_d = \lambda_a = 10$ per hour.

For a 3 hour period, $\lambda_d = 10(3) = 30$.

$\Rightarrow X_d \sim \text{Poisson}(\lambda_d = 30)$ and the average number of departures is $E(X_d) = \lambda_d = 30$.

g) Since $X_d \sim \text{Poisson}(\lambda_d = 30)$

$$\Pr(X_d > 40) = \Pr(X_d \geq 41) = 0.0323.$$

(Poisson tables: column $m = 30$, row $r = 41$)

Question 2

We have $\lambda_a = 3$ per minute and $\lambda_s = 4$ per minute. Thus, the total time in the system is $T \sim \text{Exponential}(\lambda = 4 - 3 = 1)$

a) $$E(T) = \frac{1}{\lambda} = \frac{1}{1} = 1 \text{ minute.}$$

b) Let T_q represent the time spent in the queue component. Clearly the time in the queue is the total time minus the time spent being served.

Since $E(T) = 1$ and $E(T_s) = \frac{1}{\lambda_s} = \frac{1}{4} = 0.25$:

$$\begin{aligned} E(T_q) &= E(T) - E(T_s) \\ &= 1 - 0.25 \\ &= 0.75 \text{ minutes.} \end{aligned}$$

c) $$\begin{aligned} E(N) &= \lambda_a\, E(T) \\ &= 3(1) \\ &= 3 \text{ individuals in the system.} \end{aligned}$$

d) $$\begin{aligned} E(N_q) &= \lambda_a\, E(T_q) \\ &= 3(0.75) \\ &= 2.25 \text{ individuals in the queue.} \end{aligned}$$

e) $$\rho = \frac{\lambda_a}{\lambda_s} = \frac{3}{4} = 0.75.$$

This means that the service component is working 75% of the time, i.e., it is idle 25% of the time.

f) Total time $T \sim \text{Exponential}(\lambda = 1)$.

$$\Pr(T > 2) = e^{-1\,(2)} = 0.1353.$$

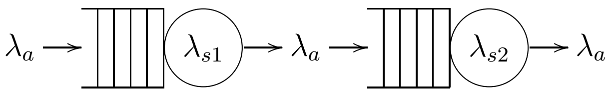
g) Burke's theorem: $X_d \sim \text{Poisson}(\lambda_d = 3)$

$$\begin{aligned} \Pr(X_d < 3) &= \Pr(X_d \leq 2) \\ &= p(0) + p(1) + p(2) \\ &= \frac{3^0}{0\,!}e^{-3} + \frac{3^1}{1\,!}e^{-3} + \frac{3^2}{2\,!}e^{-3} \\ &= 0.0498 + 0.1494 + 0.2240 \\ &= 0.4232. \end{aligned}$$

We could also do this using tables $m = 3$:

$$\begin{aligned} \Pr(X_d < 3) &= 1 - \Pr(X_d \geq 3) \\ &= 1 - 0.5768 \\ &= 0.4232. \end{aligned}$$

Question 3



- $\lambda_a = 12$ per hour.
- $\lambda_{s1} = \frac{1}{E(T_{s1})} = \frac{1}{3}$ per minute

  $\Rightarrow \lambda_{s1} = \frac{1}{3}(60) = 20$ per hour.
- $\lambda_{s2} = \frac{1}{E(T_{s2})} = \frac{1}{1} = 1$ per minute

  $\Rightarrow \lambda_{s2} = 1(60) = 60$ per hour.

a) The time spent in the deli system is $T_1 \sim \text{Exponential}(\lambda_1 = \lambda_{s1} - \lambda_a = 8)$

$$\Rightarrow E(T_1) = \frac{1}{8} \text{ hours.} \quad \text{(i.e., 7.5 minutes)}$$

The time spent in the paying system is $T_2 \sim \text{Exponential}(\lambda_2 = \lambda_{s2} - \lambda_a = 48)$

$$\Rightarrow E(T_2) = \frac{1}{48} \text{ hours.} \quad \text{(i.e., 1.25 minutes)}$$

b) The total time in the system is

$$\begin{aligned} E(T) &= E(T_1) + E(T_2) \\ &= \frac{1}{8} + \frac{1}{48} \\ &= \frac{7}{48} \text{ hours.} \quad \text{(i.e., 8.75 minutes)} \end{aligned}$$

c) $E(N) = \lambda_a E(T) = 12\left(\frac{7}{48}\right) = 1.75$ customers.

d)
$$\rho_1 = \frac{\lambda_a}{\lambda_{s1}} = \frac{12}{20} = 0.6.$$
$$\rho_2 = \frac{\lambda_a}{\lambda_{s2}} = \frac{12}{60} = 0.2.$$

e)
$$\begin{aligned} E(T_q) &= E(T) - [E(T_{s1}) + E(T_{s2})] \\ &= E(T) - \left[\frac{1}{\lambda_{s1}} + \frac{1}{\lambda_{s2}}\right] \\ &= \frac{7}{48} - \left(\frac{1}{20} + \frac{1}{60}\right) \\ &= \frac{7}{48} - \frac{1}{15} \\ &= \frac{19}{240} \\ &\approx 0.07917 \text{ hours.} \\ &\text{(i.e., 4.75 minutes)} \end{aligned}$$

f) Burke's theorem: $X_d \sim \text{Poisson}(\lambda_d = 12)$.

$$\Pr(X_d \geq 20) = 0.0213.$$

(Poisson tables: column $m = 12$, row $r = 20$)

Question 4

Note that:
$\lambda_1 = \frac{1}{0.25} = 4$ and $\lambda_2 = \frac{1}{0.5} = 2$.

$$\Pr(R_1) = 0.8 \qquad \Pr(T > t \,|\, R_1) = e^{-4t}$$
$$\Pr(R_2) = 0.2 \qquad \Pr(T > t \,|\, R_2) = e^{-2t}$$

a)
$$\Pr(T > 0.5 \,|\, R_1) = e^{-4(0.5)} = 0.1353.$$
$$\Pr(T > 0.5 \,|\, R_2) = e^{-2(0.5)} = 0.3679.$$

b) First calculate

$$\begin{aligned} \Pr(T > 0.5 \cap R_1) &= \Pr(R_1)\Pr(T > 0.5 \,|\, R_1) \\ &= 0.8(0.1353) = 0.1082. \end{aligned}$$

$$\begin{aligned} \Pr(T > 0.5 \cap R_2) &= \Pr(R_2)\Pr(T > 0.5 \,|\, R_2) \\ &= 0.2(0.3679) = 0.0736. \end{aligned}$$

Using the law of total probability:

$$\begin{aligned} \Pr(T > 0.5) &= \Pr(T > 0.5 \cap R_1) \\ &\quad + \Pr(T > 0.5 \cap R_2) \\ &= 0.1082 + 0.0736 \\ &= 0.1818. \end{aligned}$$

c)
$$\begin{aligned} \Pr(R_1 \,|\, T > 0.5) &= \frac{\Pr(R_1 \cap T > 0.5)}{\Pr(T > 0.5)} \\ &= \frac{0.1082}{0.1818} \\ &= 0.5952. \end{aligned}$$

Question 4 continued

d) In parts (a) - (c) we consider the specific case $T > 0.5$. Here we work with $T > t$. Thus,

$$\begin{aligned}
\Pr(T > t) &= \Pr(T > t \cap R_1) \\
&\quad + \Pr(T > t \cap R_2) \\
&= \Pr(R_1)\Pr(T > t \mid R_1) \\
&\quad + \Pr(R_2)\Pr(T > t \mid R_2) \\
&= 0.8\,e^{-4t} + 0.2\,e^{-2t}.
\end{aligned}$$

.

$$\begin{aligned}
\Rightarrow \Pr(R_1 \mid T > t) &= \frac{\Pr(R_1 \cap T > t)}{\Pr(T > t)} \\
&= \frac{0.8\,e^{-4t}}{0.8\,e^{-4t} + 0.2\,e^{-2t}}.
\end{aligned}$$

Now we have a general formula for $\Pr(R_1 \mid T > t)$ that we can evaluate at any t value.

$$\begin{aligned}
\Pr(R_1 \mid T > 0.25) &= \frac{0.8\,e^{-4(0.25)}}{0.8\,e^{-4(0.25)} + 0.2\,e^{-2(0.25)}} \\
&= 0.7081
\end{aligned}$$

$$\begin{aligned}
\Pr(R_1 \mid T > 1) &= \frac{0.8\,e^{-4(1)}}{0.8\,e^{-4(1)} + 0.2\,e^{-2(1)}} \\
&= 0.3512
\end{aligned}$$

$$\begin{aligned}
\Pr(R_1 \mid T > 2) &= \frac{0.8\,e^{-4(2)}}{0.8\,e^{-4(2)} + 0.2\,e^{-2(2)}} \\
&= 0.06826
\end{aligned}$$

The longer the journey has taken you, the less likely it is that you used R_1.

Question 5

Here $X \sim \text{Normal}(\mu = 10, \sigma = 2)$ so we use

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 10}{2}$$

to convert to a z score (for the tables).

a)
$$\begin{aligned}
\Pr(X > 10) = \Pr(Z > \tfrac{10-10}{2}) &= \Pr(Z > 0) \\
&= 0.5.
\end{aligned}$$

b)
$$\begin{aligned}
\Pr(X < 3) = \Pr(Z < \tfrac{3-10}{2}) &= \Pr(Z < -3.5) \\
&= \Pr(Z > 3.5) \\
&= 0.00023.
\end{aligned}$$

c)
$$\begin{aligned}
\Pr(X > 8.4) &= \Pr(Z > \tfrac{8.4-10}{2}) \\
&= \Pr(Z > -0.8) \\
&= \Pr(Z < 0.8) \\
&= 1 - \Pr(Z > 0.8) \\
&= 1 - 0.2119 = 0.7881.
\end{aligned}$$

d)
$$\begin{aligned}
\Pr(6 < X < 14) &= \Pr(X > 6) - \Pr(X > 14) \\
&= \Pr(Z > \tfrac{6-10}{2}) - \Pr(Z > \tfrac{14-10}{2}) \\
&= \Pr(Z > -2) - \Pr(Z > 2) \\
&= \Pr(Z < 2) - \Pr(Z > 2) \\
&= 1 - \Pr(Z > 2) - \Pr(Z > 2) \\
&= 1 - 2\ \Pr(Z > 2) \\
&= 1 - 2(0.02275) = 0.9545.
\end{aligned}$$

e)
$$\begin{aligned}
\Pr(X > x) &= 0.3 \\
\Pr(Z > \tfrac{x-10}{2}) &= 0.3
\end{aligned}$$

From tables: $\Pr(Z > 0.52) = 0.3015 \approx 0.3$

$$\begin{aligned}
\Rightarrow \frac{x-10}{2} &= 0.52 \\
x - 10 &= 0.52(2) \\
x &= 10 + 0.52(2) \\
x &= 11.04.
\end{aligned}$$

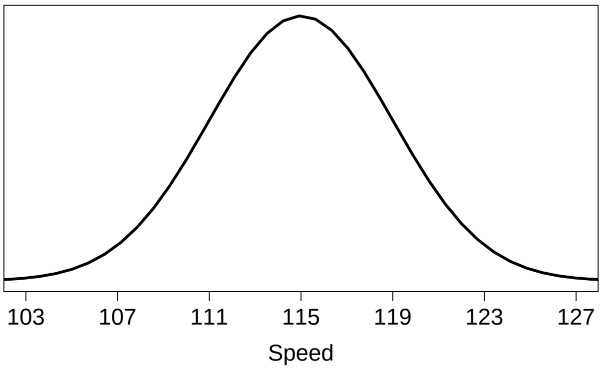
f)
$$\begin{aligned}
\Pr(X > x) &= 0.8 \\
\Pr(Z > \tfrac{x-10}{2}) &= 0.8 \\
\Pr(Z < \tfrac{x-10}{2}) &= 0.2 \\
\Pr(Z > -\tfrac{x-10}{2}) &= 0.2
\end{aligned}$$

From tables: $\Pr(Z > 0.84) = 0.2005 \approx 0.2$

$$\begin{aligned}
\Rightarrow -\frac{x-10}{2} &= 0.84 \\
\frac{x-10}{2} &= -0.84 \\
x - 10 &= -0.84(2) \\
x &= 10 - 0.84(2) \\
x &= 8.32.
\end{aligned}$$

Question 6

a)



b) $\Pr(X > 120) = \Pr(Z > \frac{120-115}{4}) = \Pr(Z > 1.25)$
$= 0.1056$

c) $\Pr(X < 100) = \Pr(Z < \frac{100-115}{4}) = \Pr(Z < -3.75)$
$= \Pr(Z > 3.75)$
$= 0.000088.$

d) $\Pr(100 < X < 110)$

$$
\begin{aligned}
&= \Pr(X > 100) - \Pr(X > 110) \\
&= \Pr(Z > \tfrac{100-115}{4}) - \Pr(Z > \tfrac{110-115}{4}) \\
&= \Pr(Z > -3.75) - \Pr(Z > -1.25) \\
&= \Pr(Z < 3.75) - \Pr(Z < 1.25) \\
&= (1 - 0.000088) - (1 - 0.1056) \\
&= 0.999912 - 0.8944 \\
&= 0.1055.
\end{aligned}
$$

e)

$$
\begin{aligned}
\Pr(X > x) &= 0.01 \\
\Pr(Z > \tfrac{x-115}{4}) &= 0.01
\end{aligned}
$$

From tables: $\Pr(Z > 2.33) = 0.0099 \approx 0.1$

$$
\begin{aligned}
\Rightarrow \frac{x-115}{4} &= 2.33 \\
\frac{x-115}{4} &= 2.33 \\
x - 115 &= 2.33(4) \\
x &= 115 + 2.33(4) \\
x &= 124.32.
\end{aligned}
$$

Question 7

a) $\Pr(\mu - 3\sigma < X < \mu + 3\sigma)$

$$
\begin{aligned}
&= \Pr(X > \mu - 3\sigma) - \Pr(X > \mu + 3\sigma) \\
&= \Pr(Z > \tfrac{\mu-3\sigma-\mu}{\sigma}) - \Pr(Z > \tfrac{\mu+3\sigma-\mu}{\sigma}) \\
&= \Pr(Z > -3) - \Pr(Z > 3) \\
&= \Pr(Z < 3) - \Pr(Z > 3) \\
&= 1 - \Pr(Z > 3) - \Pr(Z > 3) \\
&= 1 - 2\ \Pr(Z > 3) \\
&= 1 - 2(0.00135) \\
&= 0.9973.
\end{aligned}
$$

b) Note that the workings are the same as in part (a) above except that we have k instead of 3.

$$\Pr(\mu - k\sigma < X < \mu + k\sigma) = 0.95$$

$$\vdots$$

$$
\begin{aligned}
\Rightarrow 1 - 2\ \Pr(Z > k) &= 0.95 \\
-2\ \Pr(Z > k) &= -1 + 0.95 \\
2\ \Pr(Z > k) &= 1 - 0.95 \\
\Pr(Z > k) &= \frac{1 - 0.95}{2} \\
\Pr(Z > k) &= 0.025
\end{aligned}
$$

$$\Rightarrow k = 1.96.$$

c) This is the same as part (b) except we have 0.99 rather than 0.95.

$$\vdots$$

$$\Rightarrow \Pr(Z > k) = \frac{1 - 0.99}{2} = 0.005$$

$$\Rightarrow k = 2.58.$$

d)

$$
\begin{aligned}
\Pr(X > \mu + 1.64\sigma) &= \Pr(Z > \tfrac{\mu+1.64\sigma-\mu}{\sigma}) \\
&= \Pr(Z > 1.64) \\
&= 0.0495 \approx 0.05.
\end{aligned}
$$