Question 1

a) $\Pr(X > 45) = \Pr(Z > \frac{45-40}{3}) = \Pr(Z > 1.67)$
$= 0.0475.$

b) $$\begin{aligned}\Pr(32 < X < 42) &= \Pr(X > 32) - \Pr(X > 42)\\ &= \Pr(Z > \tfrac{32-40}{3}) - \Pr(Z > \tfrac{42-40}{3})\\ &= \Pr(Z > -2.67) - \Pr(Z > 0.67)\\ &= \Pr(Z < 2.67) - \Pr(Z > 0.67)\\ &= 1 - \Pr(Z > 2.67) - \Pr(Z > 0.67)\\ &= 1 - 0.00379 - 0.2514\\ &= 1 - 0.24761 = 0.74481.\end{aligned}$$

c) The sum of two normal variables:

$$\begin{aligned}Y = X_1 + X_2 &\sim \text{Normal}\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)\\ &\sim \text{Normal}\left(40 + 40, \sqrt{9+9}\right)\\ &\sim \text{Normal}\left(80, \sqrt{18}\right)\\ &\sim \text{Normal}\left(\mu_Y = 80, \sigma_Y = 4.2426\right)\end{aligned}$$

$$\begin{aligned}\Pr(Y > 85) &= \Pr(Z > \tfrac{85-80}{4.2426})\\ &= \Pr(Z > 1.18)\\ &= 0.1190.\end{aligned}$$

d) 99% limits $\Rightarrow$ 1% left over, i.e., $\alpha = 0.01$ and $\alpha/2 = 0.005$ probability in each tail.

$$\begin{aligned}&\mu_Y \ \pm\ z_{0.005}\ \sigma_Y\\ &80 \ \pm\ 2.58\ (4.2426)\\ &80 \ \pm\ 10.9459\\ \Rightarrow\ &[69.05,\ 90.95]\end{aligned}$$

95% of the time, the sum of the two attacks will be in the interval $[69.05, 90.95]$.

e) The difference between two normal variables:

$$\begin{aligned}D = X_1 - X_2 &\sim \text{Normal}\left(\mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)\\ &\sim \text{Normal}\left(\mu_D = 0, \sigma_D = 4.2426\right)\end{aligned}$$

$$\begin{aligned}\Pr(|D| > 5) &= \Pr(D < -5) + \Pr(D > 5)\\ &= \Pr(D < \tfrac{-5-0}{4.2426}) + \Pr(D > \tfrac{5-0}{4.2426})\\ &= \Pr(D < -1.18) + \Pr(D > 1.18)\\ &= \Pr(D > 1.18) + \Pr(D > 1.18)\\ &= 2\Pr(D > 1.18)\\ &= 2(0.1190) = 0.238.\end{aligned}$$

Question 2

a) This is the sum of a Normal(40, 3) variable and a Normal(5, 1) variable:

$$\begin{aligned}X \mid S^c &\sim \text{Normal}(40+5, \sqrt{3^2+1^2})\\ &\sim \text{Normal}(45, \sqrt{10})\\ &\sim \text{Normal}(45,\ 3.1623)\end{aligned}$$

b) $$\begin{aligned}\Pr(X < 43 \mid S) &= \Pr(Z < \tfrac{43-40}{3})\\ &= \Pr(Z < 1)\\ &= 1 - \Pr(Z > 1)\\ &= 1 - 0.1587 = 0.8413.\end{aligned}$$

$$\begin{aligned}\Pr(X < 43 \mid S^c) &= \Pr(Z < \tfrac{43-45}{3.1623})\\ &= \Pr(Z < -0.63)\\ &= \Pr(Z > 0.63)\\ &= 0.2643.\end{aligned}$$

c) $\Pr(S) = 0.75 \quad \Pr(X < 43 \mid S) = 0.8413$

$\Pr(S^c) = 0.25 \quad \Pr(X < 43 \mid S^c) = 0.2643$

Using the law of total probability:

$$\begin{aligned}\Pr(X < 43) &= \Pr(X < 43 \cap S)\\ &\quad + \Pr(X < 43 \cap S^c)\\ &= \Pr(S)\Pr(X < 43 \mid S)\\ &\quad + \Pr(S^c)\Pr(X < 43 \mid S^c)\\ &= 0.75(0.8413) + 0.25(0.2643)\\ &= 0.6310 + 0.0661\\ &= 0.6971.\end{aligned}$$

d) $$\begin{aligned}\Pr(S^c \mid X < 43) &= \frac{\Pr(S^c \cap X < 43)}{\Pr(X < 43)}\\ &= \frac{0.0661}{0.6971}\\ &= 0.0948.\end{aligned}$$

Question 3

a) This is the probability that $X_1 - X_2 < 0$, i.e., $X_1 < X_2$. First note the difference between two normal variables:

$$\begin{aligned} X_1 - X_2 &\sim \text{Normal}\left(\mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right) \\ &\sim \text{Normal}\left(30 - 40, \sqrt{(2)^2 + (3.5)^2}\right) \\ &\sim \text{Normal}\left(-10, \sqrt{4 + 12.25}\right) \\ &\sim \text{Normal}\left(-10, \sqrt{16.25}\right) \\ &\sim \text{Normal}\left(-10,\ 4.031\right) \end{aligned}$$

$$\begin{aligned} \Pr(X_1 - X_2 < 0) &= \Pr(Z < \tfrac{0-(-10)}{4.031}) \\ &= \Pr(Z < \tfrac{10}{4.031}) \\ &= \Pr(Z < 2.48) \\ &= 1 - \Pr(Z > 2.48) \\ &= 1 - 0.00657 = 0.99343. \end{aligned}$$

b) 90% limits $\Rightarrow$ 10% left over, i.e., $\alpha = 0.1$ and $\alpha/2 = 0.05$ probability in each tail.

$$\begin{aligned} (\mu_1 - \mu_2) \ &\pm\ z_{0.05} \sqrt{\sigma_1^2 + \sigma_2^2} \\ -10 \ &\pm\ 1.64\ (4.031) \\ -10 \ &\pm\ 6.6108 \\ \Rightarrow [-16.61,\ &-3.39] \end{aligned}$$

90% of the time, engineers earn between $3,390$ and $16,610$ more than technicians.

c) This pertains to the sample mean of $n = 25$ technicians $\Rightarrow$ *central limit theorem.*

$$\begin{aligned} \overline{X}_1 &\sim \text{Normal}\left(\mu_1, \frac{\sigma_1}{\sqrt{n}}\right) \\ &\sim \text{Normal}\left(30, \frac{2}{\sqrt{25}}\right) \\ &\sim \text{Normal}\left(\mu_1 = 30, \sigma(\overline{X}_1) = 0.4\right) \end{aligned}$$

$$\begin{aligned} \Pr(\overline{X}_1 < 30.5) &= \Pr(Z < \tfrac{30.5-30}{0.4}) \\ &= \Pr(Z < 1.25) \\ &= 1 - \Pr(Z > 1.25) \\ &= 1 - 0.1056 = 0.8944. \end{aligned}$$

d) For one engineer:

$$\begin{aligned} \Pr(X_2 > 45) &= \Pr(Z > \tfrac{45-40}{3.5}) \\ &= \Pr(Z > 1.43) = 0.0764. \end{aligned}$$

Now, consider the number of engineers who earn more than 45,000 in a group of 10. This is $X \sim \text{Binomial}(n = 10, p = 0.0764)$

$$\begin{aligned} \Pr(X \geq 2) &= 1 - \Pr(X < 2) \\ &= 1 - (p(0) + p(1)) \\ &= 1 - \left(\binom{10}{0} 0.0764^0\, 0.9236^{10} \right. \\ &\qquad \left. + \binom{10}{1} 0.0764^1\, 0.9236^9 \right) \\ &= 1 - (0.4517 + 0.3736) \\ &= 1 - 0.8253 = 0.1747. \end{aligned}$$

e) The difference between sample means:

$$\begin{aligned} &\overline{X}_1 - \overline{X}_2 \\ &\sim \text{Normal}\left(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right) \\ &\sim \text{Normal}\left(30 - 40, \sqrt{\frac{(2)^2}{30} + \frac{(3.5)^2}{35}}\right) \\ &\sim \text{Normal}\left(-10,\ 0.6952\right) \end{aligned}$$

80% limits $\Rightarrow$ 20% left over, i.e., $\alpha = 0.2$ and $\alpha/2 = 0.1$ probability in each tail.

$$\begin{aligned} (\mu_1 - \mu_2) \ &\pm\ z_{0.1}\ \sigma(\overline{X}_1 - \overline{X}_2) \\ -10 \ &\pm\ 1.28\ (0.6952) \\ -10 \ &\pm\ 0.8899 \\ \Rightarrow [-10.89,\ &-9.11] \end{aligned}$$

In 80% of samples, the difference $\overline{X}_1 - \overline{X}_2$ will lie in the above range.

Question 4

For an exponential variable we have that

$$\mu = E(X) = \frac{1}{\lambda} = \frac{1}{0.02} = 50.$$

$$\sigma = Sd(X) = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda} = 50.$$

a)

$$\begin{aligned}\overline{X} &\sim \text{Normal}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)\\ &\sim \text{Normal}\left(50, \frac{50}{\sqrt{100}}\right)\\ &\sim \text{Normal}\big(50,\ \sigma(\overline{X}) = 5\big)\end{aligned}$$

$$\begin{aligned}\Pr(\overline{X} > 55) &= \Pr(Z > \tfrac{55-50}{5})\\ &= \Pr(Z > 1)\\ &= 0.1587.\end{aligned}$$

b)

$$\begin{aligned}\overline{X} &\sim \text{Normal}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)\\ &\sim \text{Normal}\left(50, \frac{50}{\sqrt{40}}\right)\\ &\sim \text{Normal}\big(50,\ \sigma(\overline{X}) = 7.906\big)\end{aligned}$$

$$\begin{aligned}\Pr(\overline{X} < 53) &= \Pr(Z < \tfrac{53-50}{7.906})\\ &= \Pr(Z < 0.38)\\ &= 1 - \Pr(Z > 0.38)\\ &= 1 - 0.3520 = 0.6480.\end{aligned}$$

c)

$$\begin{aligned}\overline{X} &\sim \text{Normal}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)\\ &\sim \text{Normal}\left(50, \frac{50}{\sqrt{65}}\right)\\ &\sim \text{Normal}\big(50,\ \sigma(\overline{X}) = 6.202\big)\end{aligned}$$

$$\begin{aligned}\Pr(\overline{X} > \bar{x}) &= 0.1\\ \Pr(Z > \tfrac{\bar{x}-50}{6.202}) &= 0.1.\end{aligned}$$

But from tables:

$$\Pr(Z > 1.28) = 0.1003 \approx 0.1.$$

$$\begin{aligned}\Rightarrow \frac{\bar{x}-50}{6.202} &= 1.28\\ \bar{x} - 50 &= 1.28(6.202)\\ \bar{x} &= 50 + 1.28(6.202)\\ &= 57.94.\end{aligned}$$

d)

$$\begin{aligned}\Pr(\overline{X} < 49) &= 0.1\\ \Pr(Z < \tfrac{49-50}{50/\sqrt{n}}) &= 0.1\\ \Pr(Z < -\tfrac{1}{50/\sqrt{n}}) &= 0.1\\ \Pr(Z > \tfrac{1}{50/\sqrt{n}}) &= 0.1\end{aligned}$$

But from tables:

$$\Pr(Z > 1.28) = 0.1003 \approx 0.1.$$

$$\begin{aligned}\Rightarrow \tfrac{1}{50/\sqrt{n}} &= 1.28\\ 1 &= 1.28\left(\frac{50}{\sqrt{n}}\right)\\ \sqrt{n} &= 1.28(50)\\ &= 64.\end{aligned}$$

$$\Rightarrow n = 64^2 = 4096.$$