Question 1

In all of the multiplications below: the first position corresponds to the shirt, the second position is the jacket and the third is the trousers.

a) Altogether there are $4(2)(2) = 16$ outfits.

b) If the shirt *must* be red then we only have one option for the shirt $\Rightarrow 1(2)(2) = 4$ outfits.

c) Shirt can be green *or* black so we have two options for the shirt $\Rightarrow 2(2)(2) = 8$ outfits.

d) Both the shirt and jacket are specified so we only have one option for both of these $\Rightarrow 1(1)(2) = 2$ outfits.

e) We have 3 non-black shirts, 1 non-black jacket and 1 non-black pair of trousers $\Rightarrow 3(1)(1) = 3$ outfits.

f) There are 3 outfits with no black item and 16 outfits altogether $\Rightarrow 16 - 3 = 13$ outfits where there is at least one black item of clothing.

g) The key thing here is that the trousers must be a different colour to *both* the jacket and the shirt (but the jacket and shirt can be the same colour).

If we choose brown trousers, we have 3 shirt options (green, red, black) and 2 jacket options (blue, black) $\Rightarrow 3(2)(1) = 6$.

If we choose black trousers, we have 3 shirt options (green, red, brown) and 1 jacket option (blue) $\Rightarrow 3(1)(1) = 3$.

In total there are $6 + 3 = 9$ outfits.

Question 2

a) No repetitions $\Rightarrow 10(9)(8)(7) = 5040$.

b) We decide the position for the full stop. We then have 9 choices for the other 3 positions since we cannot use the full stop again.

Full stop in position 2 $\Rightarrow 9(1)(9)(9) = 729$.
Full stop in position 3 $\Rightarrow 9(9)(1)(9) = 729$.
Full stop in position 4 $\Rightarrow 9(9)(9)(1) = 729$.

In total: $729+729+729 = 2187$ passwords with one full stop but not in position 1.

c) No upper case so we only have 7 characters to choose from $\Rightarrow 7(7)(7)(7) = 2401$.

d) In total there are $10(10)(10)(10) = 10000$ passwords. We have just found that 2401 of these have no upper case letters.

$\Rightarrow 10000 - 2401 = 7599$ passwords have at least one upper case letter.

e) Using only lower case and numbers, there are 6 characters to choose from $\Rightarrow 6(6)(6)(6) = 1296$.

f) The hacker knows that there are 2 possibilities for the first character, $\{a, A\}$, and 3 possibilities for the last, $\{1, 2, 3\}$. The middle two characters may be any of the 10.

$\Rightarrow$ The hacker can narrow it down to $2(10)(10)(3) = 600$ possibilities.

f) In addition to the above constraints, the hacker now also knows that either:

the second character is a full stop $\Rightarrow 2(1)(10)(3) = 60$ or
the third character is a full stop $\Rightarrow 2(10)(1)(3) = 60$.

$\Rightarrow$ The hacker can narrow it down to $60 + 60 = 120$ possibilities.

Question 3

$$\binom{8}{6} = \frac{8!}{6!\,2!} = \frac{8 \times 7 \times \cancel{6}!}{\cancel{6}!\,2!} = \frac{8 \times 7}{2 \times 1} = \frac{\cancel{8}^{\,4} \times 7}{\cancel{2} \times 1}$$
$$= 28 \text{ choices.}$$

There are 28 possible groups of 6 objects chosen from a total of 8.

$$\binom{8}{2} = \frac{8!}{2!\,6!} = \frac{8 \times 7 \times \cancel{6}!}{2!\,\cancel{6}!} = \frac{8 \times 7}{2 \times 1} = 28.$$

There are 28 possible groups of 2 objects chosen from a total of 8.

Notice that this is the same as $\binom{8}{6}$ since choosing 6 objects from 8 is the same as choosing 2 to leave behind.

Question 3 continued

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = \frac{n \times (n-1)!}{1!\,(n-1)!} = \frac{n}{1} = n.$$

There are n groups of 1 object chosen from a total of n objects.

$$\binom{n}{n} = \frac{n!}{n!\,0!} = \frac{n!}{n! \times 1} = \frac{n!}{n!} = 1.$$

There is only 1 group of n objects chosen from a total of n objects, i.e., we choose all of them.

$$\binom{n}{0} = \frac{n!}{0!\,n!} = \frac{n!}{1 \times n!} = \frac{n!}{n!} = 1.$$

There is only 1 group of 0 objects chosen from a total of n objects, i.e., we choose none of them.

$\binom{8}{11}$ is the number of groups of 11 objects chosen from a total of 8. Clearly there are no such groups. We should set $\binom{8}{11} = 0$.

Question 4

a) There are 10 people from which we must choose 5 people $\Rightarrow \binom{10}{5} = 252$.

b) We must select one of the men. Thus, we need 4 more people to form our team and we have 9 people to choose from $\Rightarrow \binom{9}{4} = 126$.

c) The easiest way to do this is to first consider the teams which contain *both* of these individuals. If these two *are* on our team then we need to choose 3 more people from the remaining 8 people $\Rightarrow \binom{8}{3} = 56$.

   Since there are 252 possible teams altogether there are $252 - 56 = 196$ teams where these two individuals are not present at the same time.

d) We must choose 3 women from 7 and 2 men from 3.
   $\Rightarrow \binom{7}{3}\binom{3}{2} = 35(3) = 105$.

e) 3 women and 2 men: $\binom{7}{3}\binom{3}{2} = 35(3) = 105$.

   4 women and 1 man: $\binom{7}{4}\binom{3}{1} = 35(3) = 105$.

   5 women and 0 men: $\binom{7}{5}\binom{3}{0} = 21(1) = 21$.

   Total: $105 + 105 + 21 = 231$ teams with more women than men.

f) 3 men and 2 women: $\binom{3}{3}\binom{7}{2} = 1(21) = 21$.

   Note that we can't have teams with 4 or 5 men since there are only 3 men to choose from.