Chapter 1

6. Discrete Probability Distributions

Discrete probability distributions

The discrete probability distributions that described in this course are

- the binomial distribution,
- the geometric distribution,
- the hypergeometric distribution,
- the Poisson distributions.

1.1 Discrete Probability Distributions

- Over the next set of lectures, we are now going to look at two important discrete probability distributions
- The first is the ***binomial*** probability distribution.
- The second is the Poisson probability distribution.
- In `R`, calculations are performed using the `binom` family of functions and `pois` family of functions respectively.

  * Poisson
  * Binomial
  * Geometric
  * Hypergeometric

1.1.1 The Cumulative Distribution Function

- The Cumulative Distribution Function, denoted $F(x)$, is a common way that the probabilities of a random variable (both discrete and continuous) can be summarized.
- The Cumulative Distribution Function, which also can be described by a formula or summarized in a table, is defined as:
$$F(x) = P(X \leq x)$$

- The notation for a cumulative distribution function, F(x), entails using a capital "F". (The notation for a probability mass or density function, f(x), i.e. using a lowercase "f'. The notation is not interchangeable.

Useful Results for Discrete Random Variables

- $P(X \leq 1) = P(X = 0) + P(X = 1)$
- $P(X \leq r) = P(X = 0) + P(X = 1) + \ldots P(X = r)$
- $P(X \leq 0) = P(X = 0)$
- $P(X = r) = P(X \geq r) - P(X \geq r + 1)$
- **Complement Rule**: $P(X \leq r - 1) = P(X < r) = 1 - P(X \geq r)$
- **Interval Rule**:$P(a \leq X \leq b) = P(X \geq a) - P(X \geq b + 1)$.

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1.3 Bernouilli Trial

- Now consider an experiment with only two outcomes. Independent repeated trials of such an experiment are called ***Bernoulli trials***, named after the Swiss mathematician Jacob Bernoulli (16541705).
- The term ***independent trials*** means that the outcome of any trial does not depend on the previous outcomes (such as tossing a coin).
- We will call one of the outcomes the "success" and the other outcome the "failure".
- Let p denote the probability of success in a Bernoulli trial, and so $q = 1 - p$ is the probability of failure. A binomial experiment consists of a fixed number of Bernoulli trials.
- A binomial experiment with n trials and probability p of success will be denoted by

$$B(n, p)$$

- a probability mass function (pmf) is a function that gives the probability that a discrete random variable is exactly equal to some value.
- The probability mass function is often the primary means of defining a discrete probability distribution

1.3.1 Crooked die

Crooked die

- Consider the random experiment of rolling a 'crooked' six-sided die, i.e. the outcomes of the throw occur with different probabilities.
- Suppose we have a die with which an outcome '5' or '6' is twice as likely to occur compared to the other numbers.
- What is the probability of each outcome?
  - Remark: The ratio of outcomes is 1:1:1:1:2:2
- The probability distributiion can be tabulated as follows

x_i	1	2	3	4	5	6
$p(x_i)$	1/8	1/8	1/8	1/8	2/8	2/8

- What is expected value and variance of the outcomes?

Variance of the crooked die

Recall the formula for computing the variance of a discrete random variable:

$$V(x) = E(X^2) - E(X)^2$$

We must compute $E(X^2)$

x_i	1	2	3	4	5	6
x_i^2	1	4	9	16	25	36
$p(x_i)$	1/8	1/8	1/8	1/8	2/8	2/8

$$E(X) = (0 \times 1/8) + (1 \times 1/8) + \ldots + (25 \times 2/8) + (36 \times 2/8) = \frac{32}{8} = 4$$

Expected value of the crooked die

What is the variance?

Bernoulli Distribution: The coin toss

There is no more basic random event than the flipping of a coin. Heads or tails. It's as simple as you can get! The "Bernoulli Trial" refers to a single event which can have one of two possible outcomes with a fixed probability of each occurring. You can describe these events as "yes or no" questions. For example:

- Will the coin land heads?
- Will the newborn child be a girl?
- Are a random person's eyes green?
- Will a mosquito die after the area was sprayed with insecticide?
- Will a potential customer decide to buy my product?
- Will a citizen vote for a specific candidate?
- Is an employee going to vote pro-union?

Will this person be abducted by aliens in their lifetime? The Bernoulli Distribution has one controlling parameter: the probability of success. A "fair coin" or an experiment where success and failure are equally likely will have a probability of 0.5 (50

If a random variable X is distributed with a Bernoulli Distribution with a parameter p we write its probability mass function as:

$$f(x) = \begin{cases} p, & \text{if } x = 1 \\ 1-p, & \text{if } x = 0 \end{cases} \quad 0 \leq p \leq 1$$

Where the event X=1 represents the "yes."

This distribution may seem trivial, but it is still a very important building block in probability. The Binomial distribution extends the Bernoulli distribution to encompass multiple "yes" or "no" cases with a fixed probability. Take a close look at the examples cited above. Some similar questions will be presented in the next section which might give an understanding of how these distributions are related.

Mean

The mean (E[X]) can be derived:

$$\mathrm{E}[X] = \sum_i f(x_i) \cdot x_i$$

$$\mathrm{E}[X] = p \cdot 1 + (1-p) \cdot 0$$

$$\mathrm{E}[X] = p$$

Variance

$$\mathrm{Var}(X) = \mathrm{E}[(X - \mathrm{E}[X])^2] = \sum_i f(x_i) \cdot (x_i - \mathrm{E}[X])^2 \; \mathrm{Var}(X) = p \cdot (1-p)^2 + (1-p) \cdot (0-p)^2$$

$$\mathrm{Var}(X) = [p(1-p) + p^2](1-p) \; \mathrm{Var}(X) = p(1-p)$$

```
Bernoulli
Parameters 0<p<1, p\in\R
Support k=\{0,1\}\,
PMF
\begin{cases}
q=(1-p) & \text{for }k=0 \\ p & \text{for }k=1
\end{cases}

CDF
\begin{cases}
0 & \text{for }k<0 \\ q & \text{for }0\leq k<1 \\ 1 & \text{for }k\geq 1
\end{cases}

Mean p\,
Median \begin{cases}
0 & \text{if } q > p\\
0.5 & \text{if } q=p\\
1 & \text{if } q<p
\end{cases}
Mode \begin{cases}
0 & \text{if } q > p\\
0, 1 & \text{if } q=p\\
1 & \text{if } q < p
\end{cases}
Variance p(1-p)\,
Skewness \frac{q-p}{\sqrt{pq}}
Ex. kurtosis \frac{1-6pq}{pq}
Entropy -q\ln(q)-p\ln(p)\,
MGF q+pe^t\,
CF q+pe^{it}\,
PGF q+pz\,
Fisher information  \frac{1}{p(1-p)}
```


```
Bernoulli
Parameters 0<p<1, p\in\R
Support k=\{0,1\}\,
PMF
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q=(1-p) & \text{for }k=0 \\ p & \text{for }k=1
\end{cases}
```


CDF

$$\begin{cases} 0 & \text{for } k<0 \\ q & \text{for } 0\leq k<1 \\ 1 & \text{for } k\geq 1 \end{cases}$$

- Mean

$$E(X) = p$$

- Median

$$\begin{cases} 0 & \text{if } q > p \\ 0.5 & \text{if } q = p \\ 1 & \text{if } q < p \end{cases}$$

- Mode

$$\begin{cases} 0 & \text{if } q > p \\ 0, 1 & \text{if } q = p \\ 1 & \text{if } q < p \end{cases}$$

Variance $p(1-p)$ Skewness $\frac{q-p}{\sqrt{pq}}$ Ex. kurtosis $\frac{1-6pq}{pq}$

Binomial Experiment A binomial experiment (also known as a Bernoulli trial) is a statistical experiment that has the following properties:

- The experiment consists of n repeated trials.
- Each trial can result in just two possible outcomes. We call one of these outcomes a success and the other, a failure.
- The probability of success, denoted by P, is the same on every trial.
- The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.

Consider the following statistical experiment. You flip a coin 2 times and count the number of times the coin lands on heads. This is a binomial experiment because:

- The experiment consists of repeated trials. We flip a coin 2 times.
- Each trial can result in just two possible outcomes - heads or tails.
- The probability of success is constant - 0.5 on every trial.
- The trials are independent; that is, getting heads on one trial does not affect whether we get heads on other trials.

1.4 Question 4 : Poisson Distribution

- Poisson mean for 1 hours: 16 per hour.

  $m = 16$

- Poisson mean for 30 minutes: 8 per 30 minutes.

  $m = 8$

- Poisson mean for 45 minutes: 12 per 45 minutes. $m = 12$

1.4.1 Question 5 : Poisson Distribution

Emission rate = 1.2 per minute

In the notes the Poisson mean is denoted λ . However in the Murdoch Barnes tables it is denoted m.

$\lambda = 1.2$

$$P(X = 0) = 1 - P(X \geq 1)$$

From tables $P(X \leq 1) = 0.699$

$P(X \leq 4) = 0.0338$

1.5 Question D2 - Binomial Distribution (2 Marks)

A biased coin yields 'Tails' on 48% of throws. Consider an experiment that consists of throwing this coin 11 times.

a. (1 Mark) Evaluate the following term ${}^{11}C_2$.

b. (1 Mark) Compute the probability of getting two 'Tails' in this experiment.

1.6 Question D1 - Binomial Distribution

An inspector of computer parts selects a random sample of components from a large batch to decide whether or not to audit the full batch.

(i) lf 20% or more of the sample is defective, the entire batch is inspected, Calculate the probability of this happening if it is thought that the population contains 4% defective components and a sample of 20 is selected.

(ii) lf 10% or more of the sample is defective, the entire batch is inspected. Calculate the probability of this happening if it is thought that the population contains 4% defective components and a sample of 50 is selected. (10 marks)

1.7 Question D2 - Binomial Distribution (2 Marks)

Under what circumstances is it appropriate to use the binomial distribution when calculating probabilities? (1 mark)

(b) Flextronics supply PCB boards to Dell. You are a production manager with Dell. There is a constant probability of 0.01 that a board will be defective. You select 20 boards at random. What is the probability that:

(i) 0 boards will be defective

(ii) 1 or more boards will be defective

(iii) 2 or less boards will be defective (6 marks)

1.7.1 Example: Poisson

A computer server breaks down on average once every three months.

- What is the probability that the server breaks down three times in a quarter?
- What is the probability that a server breaks down exactly five times in one year?

1.7.2 Poisson Distribution (Power Failures Example)

- Suppose that electricity power failures occur according to a Poisson distribution with an average of 2 outages every twenty weeks.
- Calculate the probability that there will not be more than one power outage during a particular week.

Solution:

- The average number of failures per week is: $m = 2/20 = 0.10$
- "Not more than one power outage" means we need to compute and add the probabilities for "0 outages" plus "1 outage".

Recall:

$$P(X = k) = e^{-m}\frac{m^k}{k!}$$

- $P(X = 0)$

$$P(X = 0) = e^{-0.10}\frac{0.10^0}{0!} = e^{-0.10} = 0.9048$$

- $P(X = 1)$

$$P(X = 1) = e^{-0.10}\frac{0.10^1}{1!} = e^{-0.10} \times 0.1 = 0.0905$$

- $P(X \leq 1)$

$$P(X \leq 1) = P(X = 0) + P(X = 1) = 0.9048 + 0.0905 = 0.995$$

1.7.3 The Poisson Distribution

Statistical records for road traffic accidents on a particular stretch of road state that the average number of accidents per week is 2.

- Four accidents during a randomly selected week
- No accidents