Question 1

y	1	2
$\Pr(Y = y)$	$\frac{1}{2}$	$\frac{1}{2}$

a)
$$E(Y) = \left(1 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{2}\right)$$
$$= \frac{1}{2} + \frac{2}{2} = 1.5 \text{ unique faces.}$$

b)
$$E(Y^2) = \left(1^2 \times \frac{1}{2}\right) + \left(2^2 \times \frac{1}{2}\right)$$
$$= \frac{1}{2} + \frac{4}{2} = 2.5.$$

Question 2

a)
$$\begin{aligned} Var(Y) &= E(Y^2) - [E(Y)]^2 \\ &= 2.5 - (1.5)^2 \\ &= 2.5 - 2.25 \\ &= 0.25 \text{ unique faces}^2. \end{aligned}$$

b)
$$\begin{aligned} Sd(Y) &= \sqrt{Var(Y)} \\ &= \sqrt{0.25} \\ &= 0.5 \text{ unique faces.} \end{aligned}$$

Question 3

Rolling two dice, let X = "the sum of the two numbers".

The value of X for each outcome is given below.

	X		X		X		X		X		X
(1,1)	2	(2,1)	3	(3,1)	4	(4,1)	5	(5,1)	6	(6,1)	7
(1,2)	3	(2,2)	4	(3,2)	5	(4,2)	6	(5,2)	7	(6,2)	8
(1,3)	4	(2,3)	5	(3,3)	6	(4,3)	7	(5,3)	8	(6,3)	9
(1,4)	5	(2,4)	6	(3,4)	7	(4,4)	8	(5,4)	9	(6,4)	10
(1,5)	6	(2,5)	7	(3,5)	8	(4,5)	9	(5,5)	10	(6,5)	11
(1,6)	7	(2,6)	8	(3,6)	9	(4,6)	10	(5,6)	11	(6,6)	12

a) From the above we can construct the *probability distribution* of X:

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b)
$$\begin{aligned} E(X) &= \left(2 \times \frac{1}{36}\right) + \left(3 \times \frac{2}{36}\right) + \left(4 \times \frac{3}{36}\right) + \left(5 \times \frac{4}{36}\right) + \left(6 \times \frac{5}{36}\right) + \left(7 \times \frac{6}{36}\right) + \\ &\quad \left(8 \times \frac{5}{36}\right) + \left(9 \times \frac{4}{36}\right) + \left(10 \times \frac{3}{36}\right) + \left(11 \times \frac{2}{36}\right) + \left(12 \times \frac{1}{36}\right) \\ &= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36} \\ &= \frac{252}{36} \\ &= 7. \end{aligned}$$

Question 3 continued

c)

$$
\begin{aligned}
E(X^2) &= \left(2^2 \times \frac{1}{36}\right) + \left(3^2 \times \frac{2}{36}\right) + \left(4^2 \times \frac{3}{36}\right) + \left(5^2 \times \frac{4}{36}\right) + \left(6^2 \times \frac{5}{36}\right) + \left(7^2 \times \frac{6}{36}\right) + \\
&\quad \left(8^2 \times \frac{5}{36}\right) + \left(9^2 \times \frac{4}{36}\right) + \left(10^2 \times \frac{3}{36}\right) + \left(11^2 \times \frac{2}{36}\right) + \left(12^2 \times \frac{1}{36}\right) \\
&= \left(4 \times \frac{1}{36}\right) + \left(9 \times \frac{2}{36}\right) + \left(16 \times \frac{3}{36}\right) + \left(25 \times \frac{4}{36}\right) + \left(36 \times \frac{5}{36}\right) + \left(49 \times \frac{6}{36}\right) + \\
&\quad \left(64 \times \frac{5}{36}\right) + \left(81 \times \frac{4}{36}\right) + \left(100 \times \frac{3}{36}\right) + \left(121 \times \frac{2}{36}\right) + \left(144 \times \frac{1}{36}\right) \\
&= \frac{4}{36} + \frac{18}{36} + \frac{48}{36} + \frac{100}{36} + \frac{180}{36} + \frac{294}{36} + \frac{320}{36} + \frac{324}{36} + \frac{300}{36} + \frac{242}{36} + \frac{144}{36} \\
&= \frac{1974}{36} \\
&= 54.83.
\end{aligned}
$$

d)

$$Var(X) = E(X^2) - [E(X)]^2 = 54.83 - 7^2 = 54.83 - 49 = 5.83.$$

e)

$$Sd(X) = \sqrt{Var(X)} = \sqrt{5.83} = 2.42.$$

Question 4

a) Since the probabilities have to sum to 1, $\Pr(X = 1 \cap Y = 1) = 0.3$.

$\Rightarrow 0.4 + 0.2 + 0.1 + 0.3 = 1.$

b) Sum across the rows / columns:

	X = 0	X = 1	$p(y)$
Y = 0	0.4	0.2	0.6
Y = 1	0.1	0.3	0.4
$p(x)$	0.5	0.5	1

The distribution of X is

x	0	1
$\Pr(X = x)$	0.5	0.5

The distribution of Y is

y	0	1
$\Pr(Y = y)$	0.6	0.4

c) If X and Y were independent then the joint distribution would be

	X = 0	X = 1	$p(y)$
Y = 0	$0.6(0.5) = 0.3$	$0.6(0.5) = 0.3$	0.6
Y = 1	$0.4(0.5) = 0.2$	$0.4(0.5) = 0.2$	0.4
$p(x)$	0.5	0.5	1

We can see that this is *not* the joint distribution. Therefore, X and Y are not independent.

d)

$$
\begin{aligned}
\Pr(X = 1 \mid Y = 0) &= \frac{\Pr(X = 1 \cap Y = 0)}{\Pr(Y = 0)} \\
&= \frac{0.2}{0.6} = 0.33.
\end{aligned}
$$

e)

$$E(X) = 0(0.5) + 1(0.5) = 0.5.$$

$$E(X^2) = (0^2)(0.5) + (1^2)(0.5) = 0.5.$$

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)^2] \\
&= 0.5 - (0.5)^2 = 0.25.
\end{aligned}
$$

$$Sd(x) = \sqrt{Var(X)} = \sqrt{0.25} = 0.5.$$