Confidence Intervals
Confidence Intervals for the Mean

www.Stats-Lab.com

Confidence Intervals for the Mean

From the properties of the Normal Distribution we know that

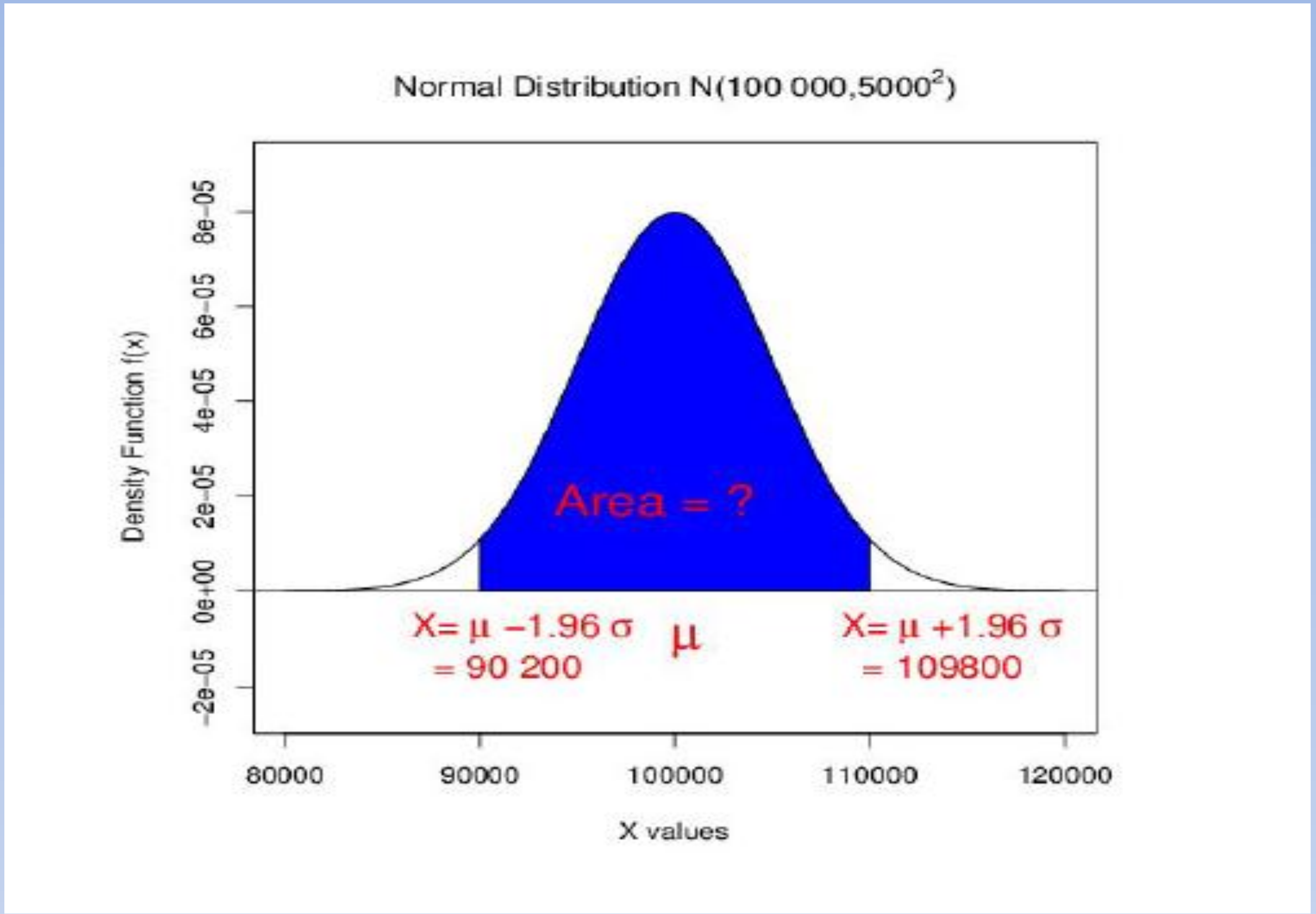
$$P[\mu - 1.96\sigma \leq X \leq \mu + 1.96\sigma] = 0.95$$

i.e. 95% of the values of a normal random variable X lie within 1.96 standard deviations of the mean.

Confidence Intervals for the Mean

Example: The average annual remuneration package in euro for experienced chartered accountants working in the industrial sector is known to follow a Normal distribution with mean μ=100,000 and standard deviation σ=5000.
Show that 95% of the annual salaries lie within within 1.96 standard deviations of the mean, i.e. in the interval $\mu \pm 1.96\sigma$.

Confidence Intervals for the Mean



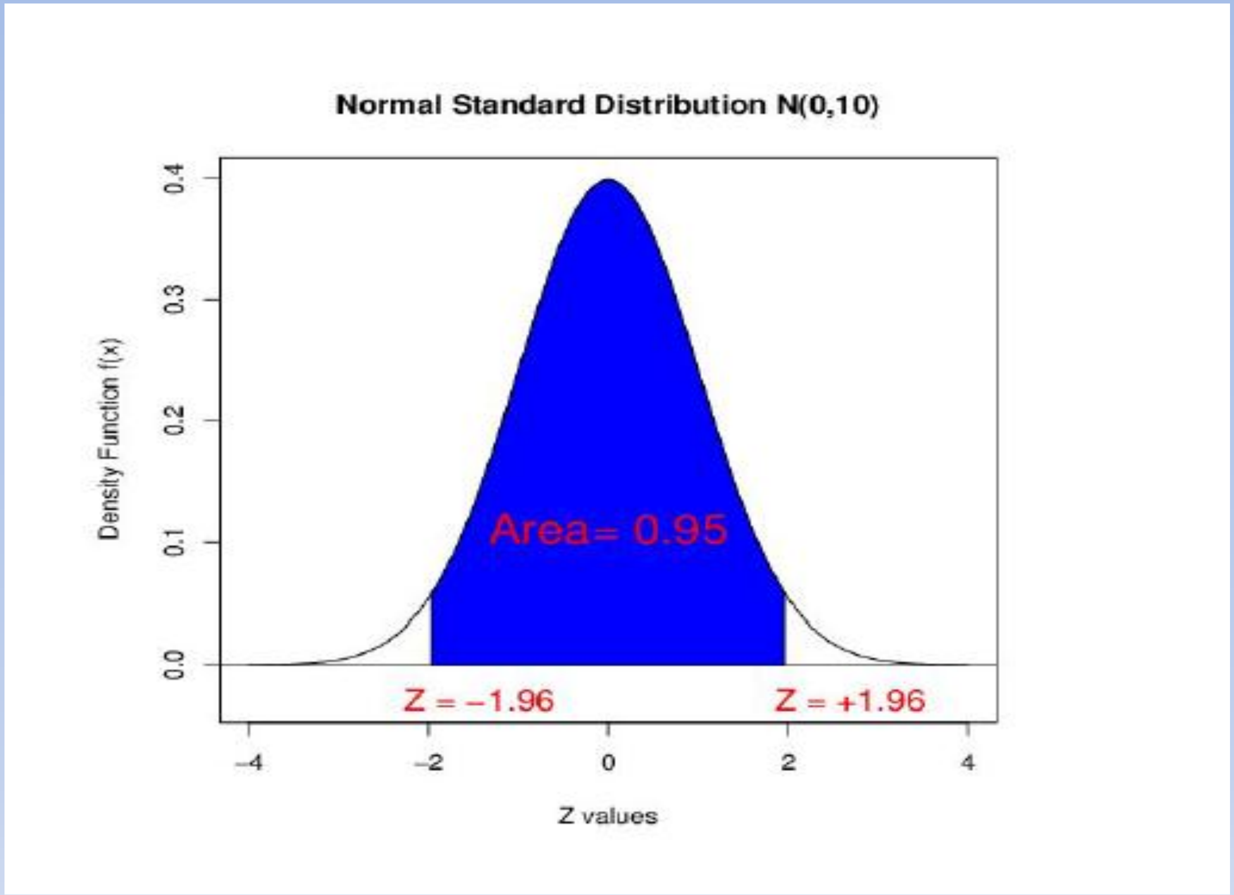
Confidence Intervals for the Mean

$$
\begin{aligned}
& P[\mu - 1.96\sigma \leq X \leq \mu + 1.96\sigma] \\
= \; & P[100000 - 1.96(5000) \leq X \leq 100000 + 1.96(5000)] \\
= \; & P[100000 - 9800 \leq X \leq 100000 + 9800] \\
= \; & P[90200 \leq X \leq 109800] \\
= \; & P[-1.96 \leq Z \leq +1.96]
\end{aligned}
$$

Confidence Intervals for the Mean

$$
\begin{aligned}
&= P[90200 \leq X \leq 109800] \\
&= P[-1.96 \leq Z \leq +1.96] \\
&= 1 - 2 \cdot P[Z \geq 1.96] \\
&= 1 - 2(0.025) \\
&= 1 - 0.05 \\
&= 0.95
\end{aligned}
$$

Confidence Intervals for the Mean



Confidence Intervals for the Mean

Using the properties of the Normal distribution, we know that 95% of the **sample means**$(\overline{x})$, will lie within the range of the population mean $\mu \pm$ 1.96 standard errors of the mean i.e.

$$P[\mu - 1.96SE \leq \overline{x} \leq \mu + 1.96SE] = 0.95$$

or 95% of the sample means lie in the interval

$$\mu \pm 1.96SE(\overline{x}) = [\mu - 1.96SE(\overline{x}), \mu + 1.96SE(\overline{x})]$$

where $SE(\overline{x}) = \frac{\sigma}{\sqrt{n}}$ from the Central Limit Theorem.

Confidence Intervals for the Mean

Since in practice $\overline{x}$ (sample mean) is known and μ (population mean) is unknown, the equations above can be rearranged to obtain a range that contains the values of the true parameter μ with 95% confidence.

$$\overline{x} - 1.96SE(\overline{x}) < \mu < \overline{x} + 1.96SE(\overline{x})$$

The 95% CI of the mean μ = [$\overline{x} \pm 1.96$ SE($\overline{x}$)]

Confidence Intervals for the Mean

This range is called a 95% confidence interval for μ .
It is a range of values which contains the true population mean with a probability of 0.95 or 95%. We can expect that a 95% confident interval will not include the true mean 5% of the time.
From the **Central Limit Theorem** we know that the standard error is

$$SE(\overline{x}) = \frac{\sigma}{\sqrt{n}}$$

When σ is unknown (most likely) use s, the sample standard deviation

$$SE(\overline{x}) = \frac{s}{\sqrt{n}}$$

Confidence Intervals for the Mean

A company wishes to estimate the mean age of all of its employees(μ). A random sample of 25 employees gives a sample mean $\overline{x}$ of 40 years.
It is known that the standard deviation σ is 10 years.

Determine a 95% confidence interval for the population mean.

Confidence Intervals for the Mean

μ= unknown population mean.

$\overline{x}$ = known sample mean, $\overline{x}$ =40.

n=25

σ=10

The 95% Confidence Interval(CI) for μ is $\overline{x} \pm 1.96SE(\overline{x})$

$$= [\overline{x} - 1.96SE(\overline{x}), \overline{x} + 1.96SE(\overline{x})]$$

Confidence Intervals for the Mean

μ= unknown population mean.
$\overline{x}$ = known sample mean, $\overline{x}$ =40.
n=25
σ=10

The 95% Confidence Interval(CI) for μ is $\overline{x} \pm 1.96SE(\overline{x})$

$$
\begin{aligned}
&= [\overline{x} - 1.96SE(\overline{x}), \overline{x} + 1.96SE(\overline{x})] \\
&= [\overline{x} - 1.96\frac{\sigma}{\sqrt{n}}, \overline{x} + 1.96\frac{\sigma}{\sqrt{n}}] \\
&= [40 - 1.96\frac{10}{\sqrt{25}},\ 40 + 1.96\frac{10}{\sqrt{25}}] \\
&= [40 - 1.96 * 2, \quad 40 + 1.96 * 2] \\
&= [36.08, 43.92]
\end{aligned}
$$

Confidence Intervals for the Mean

What if we wanted a range of values for which there is a 99% chance the true population mean lies within?

$$\overline{x} \pm 2.58SE(\overline{x})$$

Example:
Give a 99% confidence interval for the mean age for the previous example.

Confidence Intervals for the Mean

The 99% Confidence Interval(CI) for μ is $\overline{x} \pm 2.58 SE(\overline{x})$

$$
\begin{aligned}
&= [\overline{x} - 2.58 SE(\overline{x}), \overline{x} + 2.58 SE(\overline{x})] \\
&= [\overline{x} - 2.58 \frac{\sigma}{\sqrt{n}}, \overline{x} + 2.58 \frac{\sigma}{\sqrt{n}}]
\end{aligned}
$$

Confidence Intervals for the Mean

The 99% Confidence Interval(CI) for μ is $\overline{x} \pm 2.58 SE(\overline{x})$

$$
\begin{aligned}
&= [\overline{x} - 2.58 SE(\overline{x}), \overline{x} + 2.58 SE(\overline{x})] \\
&= [\overline{x} - 2.58 \frac{\sigma}{\sqrt{n}}, \overline{x} + 2.58 \frac{\sigma}{\sqrt{n}}] \\
&= [40 - 2.58 \frac{10}{\sqrt{25}}, \ 40 + 2.58 \frac{10}{\sqrt{25}}] \\
&= [40 - 2.58 * 2, \quad 40 + 2.58 * 2] \\
&= [33.84, 45.16]
\end{aligned}
$$

Note: The 99% confidence interval is wider than the 95% confidence interval i.e. as you increase the confidence level, you also increase the width of the interval.