1. The random variable T has the exponential distribution with rate parameter λ, so that the probability density function (pdf) of T is

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0, \\ 0 & t < 0. \end{cases}$$

   (a) Obtain the cumulative distribution function (cdf) $F_T(t)$ of T, and draw the graph of $F_T(t)$.

   (b) Show that $P(a < T \leq b) = e^{-\lambda a} - e^{-\lambda b}$.

   (c) Given that $P(0 < T \leq 1) = 2P(1 < T \leq 2)$, find the value of λ to three significant figures.

   (d) For any choice of c and t such that $t > c > 0$, find $P(T > t|T > c)$. Deduce the conditional pdf of T given that $T > c$. In a similar way, find the conditional pdf of Tc given that $T > c$, and comment briefly on your results.

2. The random variable X has the exponential distribution with probability density function

$$f_X(x) = \lambda e^{-\lambda x}, \qquad \text{where x } > 0 \;\; \& \;\; \lambda > 0$$

   (a) Show that $E(X) = 1/\lambda$.

   (b) Show that $P(X > a) = e^{-\lambda x}$ for any $a > 0$. Deduce the median of X.

   (c) For any $b > 0$, find $P(X > a + b|X > a)$ and comment on this result.

   Now consider the case where $\lambda = 1$.

   (a) Sketch the graph of $f(x)$.

   (b) State with a reason whether the distribution of X is positively or negatively skew.

   (c) Write down the mode of the distribution of X, and find the value of k such that Mean Mode = k(Mean Median).

   (d) A student has read that, for many distributions,
      - the skewness is positive if the mean is greater than the median
      - the value of k is about 3.

      Comment on the truth of each of these statements for the distribution of X.

3. The random variable X has the exponential probability density function (pdf) given by

$$f_X(x) = \lambda e^{-\lambda x}, \qquad \text{where x } > 0 \;\; \& \;\; \lambda > 0$$

   (a) Show that $E(X) = 1/\lambda$ and find the standard deviation of X.

   (b) Show that, for any $c > 0$, $P(X > c) = exp(\lambda c)$.
   Hence show that, for any $x > c$, $P(X > x|X > c) = exp(\lambda(xc))$. Deduce the conditional pdf of X given that $X > c$, and comment briefly.

(c) A random sample has been selected from a distribution that is thought to be exponential. The values obtained, arranged in ascending order, are

$$\{0.1, 0.1, 0.2, 0.4, 1.1, 2.3, 2.5, 3.4, 4.3, 5.6\}.$$

[You are given that the sum and sum of squares of these values are 20.0 and 74.38 respectively.]

Calculate the sample mean and the sample standard deviation and say with a reason whether you think the exponential model is suitable for the distribution underlying this sample.

4. (a) The continuous random variable X has the exponential distribution with probability density function (pdf) $f(x)$ given by

$$f_X(x) = \lambda e^{-\lambda x}, \qquad \text{where x } > 0 \ \ \& \ \ \lambda > 0.$$

(i) Find the cumulative distribution function (cdf) F(x) of X, and sketch the graph of $F(x)$ for the case $\lambda = 1/2$. Mark on your graph the median of X.

(ii) The continuous random variable Y is independent of X and has a distribution with pdf $g(y)$ given by

$$g_y(y) = \mu e^{-\mu y}, \qquad \text{where y } > 0 \ \ \& \ \ \mu > 0$$

Write down the cdf of Y.

(b) Striplights A and B, from two different suppliers, have lifetimes respectively distributed as X and Y in part (a), where $\lambda = 1/2$ and $\mu = 1/3$, for lifetimes measured in units of 1000 hours. Two new striplights, one from each supplier, are installed at the same time. Their lifetimes may be assumed to be independent.

(i) Find the values of $P(X \leq 2)$ and $P(Y \leq 2)$.

(ii) Find the probability that both striplights last at least 2000 hours, i.e. that $X \geq 2$ and Y ≥ 2.

(iii) Find the probability that exactly one striplight lasts at least 2000 hours.

(iv) Given that exactly one striplight lasts at least 2000 hours, find the probability that it is A.