## Question 1

a) "The number of heads in 35 flips" is $X \sim \text{Binomial}(n = 4, p = 0.5)$ where $x \in \{0, 1, 2, 3, 4\}$.

- $$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{4}{x} 0.5^x \, 0.5^{4-x}$$
- $E(X) = n\,p = 4(0.5) = 2$ heads
- $$Sd(X) = \sqrt{n\,p\,(1-p)} = \sqrt{4(0.5)(0.5)} = 1 \text{ head}$$

b) "The number of individuals with the disease per square mile" is $X \sim \text{Poisson}(\lambda = 3)$ where $x \in \{0, 1, 2, \dots, \infty\}$.

- $$\Pr(X = x) = \frac{\lambda^x}{x\,!} e^{-\lambda} = \frac{3^x}{x\,!} e^{-3}$$
- $E(X) = \lambda = 3$ individuals
- $Sd(X) = \sqrt{\lambda} = \sqrt{3} = 1.73$ individuals

c) "The number of defective bulbs in a group of 100" is $X \sim \text{Binomial}(n = 100, p = 0.03)$ where $x \in \{0, 1, 2, \dots, 100\}$.

- $$\Pr(X = x) = \binom{100}{x} 0.03^x \, 0.97^{100-x}$$
- $E(X) = 100(0.03) = 3$ bulbs
- $Sd(X) = \sqrt{100(0.03)(0.97)} = 1.71$ bulbs

d) We are given $E(T) = \frac{1}{\lambda} = 15$ minutes $\Rightarrow \lambda = \frac{1}{E(T)} = \frac{1}{15}$ customers per minute. We may prefer to work in hours $\Rightarrow \lambda = \frac{1}{15} \times 60 = 4$ customers per hour.

"The time (in hours) between customers" is $T \sim \text{Exponential}(\lambda = 4)$ where $t \in [0, \infty)$.

- $\Pr(T > t) = e^{-\lambda t} = e^{-4t}$
- $E(T) = \frac{1}{\lambda} = \frac{1}{4} = 0.25$ hours (i.e., 15 minutes)
- $$Sd(X) = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda} = \frac{1}{4} = 0.25 \text{ hours}$$

e) "The number of individuals with the disease in a group of 35" is $X \sim \text{Binomial}(n = 35, p = 0.05)$ where $x \in \{0, 1, 2, \dots, 35\}$.

- $$\Pr(X = x) = \binom{35}{x} 0.05^x \, 0.95^{35-x}$$
- $E(X) = 35(0.05) = 1.75$ individuals
- $Sd(X) = \sqrt{35(0.05)(0.95)} = 1.29$ individuals

f) We are given the average time between cars passing. This is the exponential average, i.e., $E(T) = \frac{1}{\lambda} = 2$ minutes $\Rightarrow \lambda = \frac{1}{E(T)} = \frac{1}{2} = 0.5$ cars per minute.

Thus, for one hour $\lambda = 0.5(60) = 30$ cars per hour.

"The number of cars per hour" is $X \sim \text{Poisson}(\lambda = 30)$ where $x \in \{0, 1, 2, \dots, \infty\}$.

- $$\Pr(X = x) = \frac{30^x}{x\,!} e^{-30}$$
- $E(X) = 30$ cars
- $Sd(X) = \sqrt{30} = 5.48$ cars

g) Flaws occur at a rate of $\lambda = 0.1$ per square metre $\Rightarrow \lambda = 0.1(20) = 2$ flaws per 20 square metres.

"The number of flaws per 20 square metres" is $X \sim \text{Poisson}(\lambda = 2)$ where $x \in \{0, 1, 2, \dots, \infty\}$.

- $$\Pr(X = x) = \frac{2^x}{x\,!} e^{-2}$$
- $E(X) = 2$ flaws
- $Sd(X) = \sqrt{2} = 1.41$ flaws

h) "The time (in hours) between texts" is $T \sim \text{Exponential}(\lambda = 4)$ where $t \in [0, \infty)$.

- $\Pr(T > t) = e^{-4t}$
- $E(T) = \frac{1}{4} = 0.25$ hours
- $Sd(X) = \frac{1}{4} = 0.25$ hours

## Question 1 continued

i) "The number of correctly guessed answers in 15 questions" is $X \sim \text{Binomial}(n = 15, p = 0.25)$ where $x \in \{0, 1, 2, \ldots, 15\}$.

- $\Pr(X = x) = \binom{15}{x} 0.25^x \, 0.75^{15-x}$
- $E(X) = 15(0.25) = 3.75$ answers
- $Sd(X) = \sqrt{15(0.25)(0.75)} = 1.68$ answers

j) "The time (in years) between failures" is $T \sim \text{Exponential}(\lambda = 6)$ where $t \in [0, \infty)$.

- $\Pr(T > t) = e^{-6\,t}$
- $E(T) = \frac{1}{6}$ years
- $Sd(X) = \frac{1}{6}$ years

k) Failures occur at a rate of $\lambda = 6$ per year $\Rightarrow$ $\lambda = \frac{6}{12} = 0.5$ failures per month.

"The number of failures per month" is $X \sim \text{Poisson}(\lambda = 0.5)$ where $x \in \{0, 1, 2, \ldots, \infty\}$.

- $\Pr(X = x) = \dfrac{0.5^x}{x\,!} e^{-0.5}$
- $E(X) = 0.5$ failures
- $Sd(X) = \sqrt{0.5} = 0.71$ failures

## Question 2

a) $\lambda = \frac{1}{300}$ per metre $\Rightarrow \lambda = \frac{1000}{300} = \frac{10}{3}$ per 1km.

$$\Pr(X = 0) = \frac{(\frac{10}{3})^0}{0\,!} e^{-\frac{10}{3}} = 0.0357.$$

b) $\lambda = \frac{6000}{300} = 20$ per 6km.

$$\Pr(X \geq 15) = 0.8951 \text{ (using tables)}.$$

c) $\lambda = \frac{3000}{300} = 10$ per 3km.

$$\begin{aligned}
\Pr(10 \geq X \geq 12) &= p(10) + p(11) + p(12) \\
&= \frac{10^{10}}{10\,!} e^{-10} + \frac{10^{11}}{11\,!} e^{-10} + \frac{12^{10}}{12\,!} e^{-10} \\
&= 0.1251 + 0.1137 + 0.0948 \\
&= 0.3336.
\end{aligned}$$

This can be done using tables also:

$$\begin{aligned}
\Pr(10 \geq X \geq 12) &= \Pr(X \geq 10) - \Pr(X \geq 13) \\
&= 0.5421 - 0.2084 = 0.3337.
\end{aligned}$$

d) We are given the probability and have to work out the $x$ value, i.e., using the tables in reverse. We find that:

- $\Pr(X \geq 15) = 0.0835$

is the closest probability to 0.1 $\Rightarrow x = 15$, i.e., there is an 8.35% chance of seeing 15 or more potholes on a 3km stretch.

e) $T$ represents the distance between potholes.

$$\begin{aligned}
\Pr(T < 100) &= 1 - \Pr(T > 100) \\
&= 1 - e^{-\frac{1}{300}(100)} \\
&= 1 - 0.7165 = 0.2835.
\end{aligned}$$

f)
$$\begin{aligned}
\Pr(T > 1000) &= e^{-\frac{1}{300}(1000)} \\
&= 0.0357.
\end{aligned}$$

g)
$$\begin{aligned}
&\Pr(300 < T < 1200) \\
&= \Pr(T > 300) - \Pr(T > 1200) \\
&= e^{-\frac{1}{300}(300)} - e^{-\frac{1}{300}(1200)} \\
&= 0.3679 - 0.0183 = 0.3496.
\end{aligned}$$

h)
$$E(T) = \frac{1}{\lambda} = \frac{1}{\frac{1}{300}} = 300 \text{ metres}$$

$$Sd(T) = \sqrt{\frac{1}{\lambda^2}} = E(T) = 300 \text{ metres}$$

## Question 3

a) $E(T) = 2$ years $\Rightarrow \lambda = \frac{1}{2} = 0.5$ failures / year.

b) $Sd(T) = \sqrt{\frac{1}{\lambda^2}} = E(T) = 2$ years.

c) $$\Pr(T > 1) = e^{-0.5(1)} = 0.6065.$$

d) $$\begin{aligned} \Pr(T < 5) &= 1 - \Pr(T > 5) \\ &= 1 - e^{-0.5(5)} \\ &= 1 - 0.0821 \\ &= 0.9179. \end{aligned}$$

e) $$\begin{aligned} \Pr(2 < T < 5) &= \Pr(T > 2) - \Pr(T > 5) \\ &= e^{-0.5(2)} - e^{-0.5(5)} \\ &= 0.3679 - 0.0821 \\ &= 0.2858. \end{aligned}$$

f) $$\begin{aligned} \Pr(T > t) &= 0.2 \\ e^{-0.5\,t} &= 0.2 \\ \ln e^{-0.5\,t} &= \ln 0.2 \\ -0.5\,t &= \ln 0.2 \\ t &= \frac{1}{-0.5} \ln 0.2 \\ &= 3.22 \text{ years,} \end{aligned}$$

i.e., 20 % of hard disks last longer than 3.22 years or, similarly, 80% fail before this time.

## Question 4

a) $$\Pr(H) = \Pr(T > 1) = e^{-0.5(1)} = 0.6065.$$

$$\Pr(H^c) = \Pr(T < 1) = 1 - 0.6065 = 0.3935.$$

**Note:** As $H_1$ and $H_2$ are *independent* we can calculate the joint probabilities via $\Pr(H_1 \cap H_2) = \Pr(H_1)\Pr(H_2)$ and $\Pr(H_1^c \cap H_2^c) = \Pr(H_1^c)\Pr(H_2^c)$.

b) $$\begin{aligned} \Pr(\text{R-0 fails within 1yr}) &= \Pr(\text{at least one fails}) \\ &= \Pr(H_1^c \cup H_2^c) \\ &= 1 - \Pr(H_1 \cap H_2) \\ &= 1 - \Pr(H_1)\Pr(H_2) \\ &= 1 - (0.6065)^2 \\ &= 0.6322. \end{aligned}$$

c) $$\begin{aligned} \Pr(\text{R-1 fails within 1yr}) &= \Pr(\text{both fail}) \\ &= \Pr(H_1^c \cap H_2^c) \\ &= \Pr(H_1^c)\Pr(H_2^c) \\ &= (0.3935)^2 \\ &= 0.1548. \end{aligned}$$

d) We want $\Pr(\text{R-1 fails within 1yr}) = 0.05$.
Note that:

$$\begin{aligned} \Pr(\text{R-1 fails within 1yr}) &= \Pr(H_1^c)\Pr(H_2^c) \\ &= (1 - e^{-\lambda(1)})^2 \\ &= (1 - e^{-\lambda})^2 \end{aligned}$$

Thus, we set the above equal to 0.05 and solve for $\lambda$ (from which we can calculate $E(T)$).

$$\begin{aligned} \Rightarrow (1 - e^{-\lambda})^2 &= 0.05 \\ 1 - e^{-\lambda} &= \sqrt{0.05} \\ -e^{-\lambda} &= -1 + \sqrt{0.05} \\ e^{-\lambda} &= 1 - \sqrt{0.05} \\ \ln e^{-\lambda} &= \ln(1 - \sqrt{0.05}) \\ -\lambda &= \ln(1 - \sqrt{0.05}) \\ \lambda &= -\ln(1 - \sqrt{0.05}) \\ &= 0.253. \end{aligned}$$

$\Rightarrow E(T) = \frac{1}{0.253} = 3.95$ years, i.e., if the two hard disks have an average life of 3.95 years then the RAID-1 system has a 5% chance of failing within 1 year.

e) Another option is to use $k$ of the original hard disks where $\Pr(H^c) = 0.3935$:

$$\begin{aligned} &\Pr(\text{R-1 fails within 1yr}) \\ &\quad = \Pr(H_1^c)\Pr(H_2^c)\cdots\Pr(H_k^c) \\ &\quad = (0.3935)^k. \end{aligned}$$

$$\begin{aligned} \Rightarrow (0.3935)^k &= 0.05 \\ \ln(0.3935)^k &= \ln 0.05 \\ k \ln(0.3935) &= \ln 0.05 \\ k &= \frac{\ln 0.05}{\ln(0.3935)} \\ &= 3.21 \text{ hard disks.} \end{aligned}$$

$\Rightarrow$ In practice we can use 3 or 4 hard disks: 3 gives a probability above 0.05 and 4 gives a probability below 0.05.

## Question 5

The solution to this question is in Lecture9 solutions (i.e., Q1 of Lecture9).

## Question 6

a) $E(T) = 5$ minutes and $\lambda_a = 2$ per minute.

$$\begin{aligned} \Pr(N) &= \lambda_a\, E(T) \\ &= 2(5) \\ &= 10 \text{ cars on the road.} \end{aligned}$$

b) $\lambda_a = 4$ per hour and $E(T) = 30$ minutes, i.e., $E(T) = 0.5$ hours.

$$\begin{aligned} \Pr(N) &= \lambda_a\, E(T) \\ &= 4(0.5) \\ &= 2 \text{ jobs in the system.} \end{aligned}$$

c) $\lambda_a = 20$ per hour and $E(N) = 10$ people.

$$\begin{aligned} \Pr(N) &= \lambda_a\, E(T) \\ 10 &= 20\, E(T) \\ \frac{10}{20} &= E(T) \\ \Rightarrow E(T) &= 0.5 \text{ hours} \\ &= 30 \text{ minutes.} \end{aligned}$$

## Question 7

a) $\lambda_a = 40$ per hour and $E(T) = 0.5$ hours.

$$\begin{aligned} \Pr(N) &= \lambda_a\, E(T) \\ &= 40(0.5) \\ &= 20 \text{ people.} \end{aligned}$$

b) $\lambda_a = 60$ per hour.

$$\begin{aligned} \Pr(N) &= \lambda_a\, E(T) \\ &= 60(0.5) \\ &= 30 \text{ people.} \end{aligned}$$

c) We want $E(N) = 20$ people while still maintaining $\lambda_a = 60$ per hour.

$$\begin{aligned} \Pr(N) &= \lambda_a\, E(T) \\ 20 &= 60\, E(T) \\ \frac{20}{60} &= E(T) \\ \Rightarrow E(T) &= \frac{1}{3} \text{ hours} \\ &= 20 \text{ minutes.} \end{aligned}$$