Kraft inequality

- ***Kraft's inequality*** gives a necessary and sufficient condition for the existence of a uniquely decodable code for a given set of codeword lengths (more so variable length codes)
- More specifically, Kraft's inequality limits the lengths of codewords in a prefix code, and can be thought of in terms of a constrained budget to be spent on codewords, with shorter codewords being more expensive.

Kraft inequality

- Let X be a DMS with alphabet ($x_i = \{1, 2, \ldots, m\}$).
- Assume that the length of the assigned binary code word corresponding to x_i is n_i.
- Kraft inequality is given as

$$K = \sum_{i=1}^{m} 2^{-n_i} \leq 1$$

Kraft inequality

- If Kraft's inequality holds with strict inequality (i.e. $K < 1$), the code has some redundancy.
- If Kraft's inequality holds with strict equality (i.e. $K = 1$), the code in question is a complete code.
- The closer K is to 1, the more efficient the code is.
- If Kraft's inequality does not hold, the code is not uniquely decodable.

Kraft inequality

- Note that the Kraft inequality assures us of the existence of an instantaneously decodable code with code word lengths that satisfy the inequality.
- However it does not show us how to obtain these code words, nor does it say that any code that satisfies the inequality is automatically uniquely decodable

Kraft inequality

Compute the value for K in each case, and determine whether Kraft's Inequality is observed.

- **code 1 and 2** K = 1 so Kraft's inequality is obeyed.

$$K = \sum_{i=1}^{m} 2^{-n_i} = (2^{-2} \times 4) = 1$$

- **code 3** K = 1.5 so Kraft's inequality is not obeyed.

$$K = \sum_{i=1}^{m} 2^{-n_i} = (2^{-1} + 2^{-1} + 2^{-2} + 2^{-2}) = 1.5$$

- **code 4** K = 1 so Kraft's inequality is obeyed.

$$K = \sum_{i=1}^{m} 2^{-n_i} = (2^{-1} + 2^{-2} + 2^{-3} + 2^{-3}) = 1$$

- **code 5 and 6** K = 0.9375 so Kraft's inequality is obeyed.

$$K = \sum_{i=1}^{m} 2^{-n_i} = (2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}) = 0.9375$$

Kraft inequality: Fixed length codes.

- Consider a 6 symbol alphabet: $\{x_1, x_2, x_3, x_4, x_5, x_6\}$
- If a fixed length code is to be used, how many digits are required for each symbol.
- Each codeword must be distinct.
- Answer: We must have 3 digits in every codeword.
- Compute K for this alphabet, and use Kraft's Inequality to appraise this code.
- $K = 6 \times 2^{-3} = 0.750$. This code is uniquely decodable, but not very efficient.