Problem 4

October 11, 2019

1 4.1

Let Σ be a finite set of alphabets. Then Σ^* is the set of all words that can be created out of the symbols in Σ i.e. Σ^* is the kleene closure of Σ

a):
Let $\preceq \subseteq \Sigma^* \times \Sigma^* : \{p, w \in \Sigma^*, \exists q \in \Sigma^* : p \preceq w \Rightarrow w = pq \vee w = p\}$
For the relation $\preceq$ on Σ^* to be a partial order it must be reflexive, antisymmetric and transitive on Σ^*

Checking for reflexivity:
for $p \preceq p$
This signifies that:
$p = p\epsilon$ where ϵ is an empty set.
Which is true because we allow such cases to occour in this relation. $(w = p)$
Thus the relation is reflexive.

Checking for antisymmetry:
if $(p, q) \in \preceq q = pa$
if $(q, p) \in \preceq p = qb$

The only way this can be true is if a, b is $\emptyset$ and $p = q$.
$p = q$ is allowed for this relation. Thus The relation is antisymmetric.

Checking for transitivity:

for $(p, q) \in \preceq q = pa$
for $(q, r) \in \preceq r = qb$
if $(p, q) \in \preceq \wedge (q, r) \in \preceq$
$r = qb$
$r = (pa)b$
$r = pab$
Thus $(p, r) \in \preceq^*$
b)
Let $\prec \subset \Sigma^* \times \Sigma^* : \{p, w \in \Sigma^*, \exists q \in \Sigma^* : p \prec w \Rightarrow w = pq \vee w \neq p\}$

For the relation $\prec$ on Σ^* to be a partial order it must be reflexive, antisymmetric and transitive on Σ^*

Checking for reflexivity:
for $p \preceq p$
This signifies that:
$p = pa$
where a is $\emptyset$ Which is false because we do not allow such cases to occour in this relation. $(w \neq p)$
Thus the relation is ireflexive.

Checking for asymmetry:

if $(p, q) \in \preceq q = pa$
if $(q, p) \in \preceq p = qb$

The only way this can be true is if a, b is $\emptyset$ and $p = q$. but $p = q$ is explicitely not allowed for this relation. Thus The relation is asymmetric.
Checking for transitivity:

for $(p, q) \in \prec q = pa$
for $(q, r) \in \prec r = qb$
if $(p, q) \in \prec \wedge (q, r) \in \prec$
$r = qb$
$r = (pa)b$
$r = pab$
Thus $(p, r) \in \prec^*$

c)
The relation $\preceq$ is not total because for $(a, b) \in \preceq$ and $(b, a) \in \preceq$ if and only if $a = b$ The relation does not allow for all other cases of a and b thus it is not total.
The relation $\prec$ is not total because it does not allow for the case where $a = b\epsilon$ where ϵ is an empty set.

2 4.2

Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$ be two functions
a)
For $g \circ f$ if f is not injective, it means that: $f(a) = f(b)$
which implies that $g \circ f(a) = g \circ f(b)$
Thus $g \circ f$ is not injective thus it is also not bijective.

For $g \circ f$ if g is not surjective, it means that:
$\forall a \ \nexists g \circ f(a)$

Because if $f(a)$ exists for a, if g is not
surjective then $\exists y$ such that $y = f(a)$ $\nexists g(y)$

b)

Let $f : A \to B\{\forall a \in A, a = f(a)\}$
Let $g : B \to C\{\forall b \in B, b^2 = g(b)\}$
$g \circ f(a) = g(f(a))$
$g \circ f(a) = g(b)$
$g \circ f(a) = b^2$

Here, f is injective and g is surjective but $g \circ f$ is not a bijective.

c)

Let $f : A \to B\{\forall a \in A, \sqrt{a} = f(a)\}$
Let $g : B \to C\{\forall b \in B, b^2 = g(b)\}$
$g \circ f(a) = g(f(a))$
$g \circ f(a) = g(\sqrt{a})$
$g \circ f(a) = (\sqrt{a})^2$
$g \circ f(a) = a$

Here f is not surjective and g is not injective but $g \circ f$ is bijective.