§9.3 Householder's method

In this section, we will use Householder's transformation to find a symmetric **tridiagonal** matrix T that is similar to a given symmetric matrix A.

In next section, we will discuss how to find all eigenvalues of a symmetric tridiagonal matrix.

Definition 9.16: Let $w \in \mathbf{R}^n$ with $w^t w = 1$. Then the $n \times n$ matrix

$$P = I - 2ww^t$$

is called a Householder transformation (or Householder matrix).

Theorem 9.17: If $P = I - 2ww^t$ is a householder's matrix, then P is symmetric and orthogonal. So $P^{-1} = P^t = P$.

Proof: The symmetry is from

$$P^t = (I - 2ww^t)^t = I^t - (2ww^t)^t = I - 2(w^t)^t(w)^t = I - 2ww^t = P.$$

The orthogonality is from

$$PP^t = (I - 2ww^t)(I - 2ww^t) = I - 4ww^t + 4ww^tww^t$$

$$= I - 4ww^t + 4w(w^tw)w^t = I - 4ww^t + 4ww^t = I.$$

Some basics about the Householder matrix:

(a) $Pw = -w$. For any x orthogonal to w (i.e., $w^t x = 0$), then $Px = x$. These imply that P has eigenvalues 1 (multiplicity $n-1$) and -1.

(b) For any two distinct vectors x and y in $\mathbf{R}^n$ with **same length**, the Householder matrix

$$P = I - 2ww^t, \qquad w = (x - y)/\|x - y\|_2$$

satisfies

$$Px = y.$$

Proof of part (b): Let's figure out how to prove it. If $Px = y$, then

$$x - 2\frac{x - y}{\|x - y\|_2}\frac{(x - y)^t}{\|x - y\|_2}x = y.$$

$$(x - y) - 2\frac{(x - y)(x - y)^t x}{\|x - y\|_2^2} = 0.$$

$$\frac{x - y}{\|x - y\|_2^2}\left(\|x - y\|_2^2 - 2(x - y)^t x\right) = 0.$$

$$\frac{x - y}{\|x - y\|_2^2}\left((x - y)^t(x - y) - 2(x - y)^t x\right) = 0.$$

$$\frac{x - y}{\|x - y\|_2^2}\left(x^t x - x^t y - y^t x + y^t y - 2x^t x + 2y^t x\right) = 0.$$

The last equation is true since $x^t y = y^t x$ and $x^t x = y^t y$. For the final proof, you write above statements backwards.

By (b) above, for any $x \in \mathbf{R}^n$, let $y = (\pm\|x\|_2, 0, \ldots, 0)^t \in \mathbf{R}_n$, then $\|x\|_2 = \|y\|_2$. There is a Householder matrix P such that $Px = y$. This is the key step of Householder method. You may pick "+" or "−" in y.

Ex. If $x = (3,0,4)^t$, choose $y = (\|x\|_2, 0, 0)^t = (5,0,0)^t$. Let $w = (x-y)/\|x-y\|_2$ and $P = I - 2ww^t$. Then $Px = y$. (You can verify it.)

Ex. If $x = (1,1,3,0,4)^t$, find a Householder matrix to change x to the form $y = (1,1,*,0,0)$.

Sol. To make $\|x\|_2 = \|y\|_2$, $y = (1,1,5,0,0)$. Choose $w = (x-y)/\|x-y\|_2$ and $P = I - 2ww^t$ as before. You can verify that $Px = y$.

Just to remind you, for a Householder matrix P, if $Px = y$, then $x^tP = y^t$ since P is symmetric.

We use the following example to explain the Householder method for changing a symmetric matrix to a **similar** triangular matrix.

Ex. Use Householder matrices to transform

$$A = \begin{bmatrix} 4 & 1 & -2 & 2 \\ 1 & 2 & 0 & 1 \\ -2 & 0 & 3 & -2 \\ 2 & 1 & -2 & -1 \end{bmatrix}$$

to a **similar symmetric** tridiagonal matrix.

Sol. Let $x = (4,1,-2,2)^t$. We want to change x to $y = (4,*,0,0)^4$. So, let $y = (4,3,0,0)^t$ and $w = (x-y)/\|x-y\|_2 = (0,-2,-2,2)^t/\sqrt{12} = (0,-1,-1,1)/\sqrt{3}$. Let

$$P_1 = I - 2ww^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & -2/3 & 2/3 \\ 0 & -2/3 & 1/3 & 2/3 \\ 0 & 2/3 & 2/3 & 1/3 \end{bmatrix}.$$

Then

$$P_1 \begin{bmatrix} 4 \\ 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4,1,-2,2 \end{bmatrix} P_1 = \begin{bmatrix} 4,3,0,0 \end{bmatrix}$$

Then,

$$P_1AP_1 = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 10/3 & -4/3 & -1 \\ 0 & -4/3 & -1 & -4/3 \\ 0 & -1 & -4/3 & 5/3 \end{bmatrix}.$$

Now, for $x = (3, 10/3, -4/3, -1)$, choose $y = (3, 10/3, 5/3, 0)$ and $w = (x-y)^t/\|x-y\|_2$. Then

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4/5 & -3/5 \\ 0 & 0 & -3/5 & 4/5 \end{bmatrix}.$$

and

$$P_2P_1AP_1P_2 = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 10/3 & 5/3 & 0 \\ 0 & 5/3 & -33/25 & -68/75 \\ 0 & 0 & -68/75 & 149/75 \end{bmatrix} = T.$$

This matrix T is tridiagonal. Note that

$$T = P_2 P_1 A P_1 P_2 = P_2^t P_1^t A P_1 P_2 = (P_1 P_2)^t A P_1 P_2.$$

Hence, T and A are similar and have same eigenvalues.

You can easily generalize this method to deal with any symmetric matrices.

Remark: This method can NOT be applied to transforming a symmetric matrix to a **similar** diagonal matrix D. (which step doesn't work in the method?) (If can, the diagonal elements of D will be the eigenvalues of A!)

Definition: A matrix $H = (h_{ij})$ is called an upper Hessenberg if $h_{ij} = 0$ for all $i \geq j + 2$.

Remark: If we apply the method above to any matrix, the result will be an upper Hessenberg matrix. You will have a question in your homework.

Remark: Householder method can also be used for solving systems of linear equations. The method is much more stable than Gaussian elimination method. But this method takes more time to get the solution than Gaussian method does.