Multithreaded Programming in *Cilk*

LECTURE 2

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Minicourse Outline

- **LECTURE 1**
  Basic Cilk programming: Cilk keywords, performance measures, scheduling.
- **LECTURE 2**
  Analysis of Cilk algorithms: matrix multiplication, sorting, tableau construction.
- **LABORATORY**
  Programming matrix multiplication in Cilk — Dr. Bradley C. Kuszmaul
- **LECTURE 3**
  Advanced Cilk programming: inlets, abort, speculation, data synchronization, & more.

LECTURE 2

- Recurrences (Review)
- Matrix Multiplication
- Merge Sort
- Tableau Construction
- Conclusion

The Master Method

The ***Master Method*** for solving recurrences applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,^{*}$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

IDEA: Compare $n^{\log_b a}$ with $f(n)$.

*The unstated base case is $T(n) = \Theta(1)$ for sufficiently small n.

Master Method — Case 1

$$T(n) = a\,T(n/b) + f(n)$$

$$n^{\log_b a} \gg f(n)$$

Specifically, $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.

Solution: $T(n) = \Theta(n^{\log_b a})$.

Master Method — CASE 2

$$T(n) = a\,T(n/b) + f(n)$$

$$n^{\log_b a} \approx f(n)$$

Specifically, $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.

Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

Master Method — CASE 3

$$T(n) = a\,T(n/b) + f(n)$$

$$n^{\log_b a} \ll f(n)$$

Specifically, $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$ *and* $f(n)$ satisfies the ***regularity condition*** that $a\,f(n/b) \le c\,f(n)$ for some constant $c < 1$.

Solution: $T(n) = \Theta(f(n))$.

Master Method Summary

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$, constant $\varepsilon > 0$

$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, constant $k \geq 0$

$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$, constant $\varepsilon > 0$, and regularity condition

$\Rightarrow T(n) = \Theta(f(n))$.

Master Method Quiz

- $T(n) = 4\,T(n/2) + n$

  $n^{\log_b a} = n^2 \gg n \Rightarrow$ **CASE 1:** $T(n) = \Theta(n^2)$.

- $T(n) = 4\,T(n/2) + n^2$

  $n^{\log_b a} = n^2 = n^2 \lg^0 n \Rightarrow$ **CASE 2:** $T(n) = \Theta(n^2 \lg n)$.

- $T(n) = 4\,T(n/2) + n^3$

  $n^{\log_b a} = n^2 \ll n^3 \Rightarrow$ **CASE 3:** $T(n) = \Theta(n^3)$.

- $T(n) = 4\,T(n/2) + n^2/\lg n$

  Master method does not apply!

LECTURE 2

- Recurrences (Review)
- **Matrix Multiplication**
- **Merge Sort**
- **Tableau Construction**
- **Conclusion**

Square-Matrix Multiplication

$$\begin{array}{ccccc} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} & = & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} & \times & \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \\ C & & A & & B \end{array}$$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

Assume for simplicity that $n = 2^k$.

Recursive Matrix Multiplication

***Divide and conquer* —**

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{bmatrix} + \begin{bmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

8 multiplications of $(n/2) \times (n/2)$ matrices.
1 addition of $n \times n$ matrices.

Matrix Multiply in Pseudo-Cilk

```
cilk void Mult(*C, *A, *B, n) {
  float *T = Cilk_alloca(n*n*sizeof(float));
  ⟨ base case & partition matrices ⟩
  spawn Mult(C11,A11,B11  n/2);
  spawn Mult(C12,A11,B12,  /2);
  spawn Mult(C22,A21,B12,  /2);
  spawn Mult(C21,A21,B11,    );
  spawn Mult(T11,A12,B21,n    );
  spawn Mult(T12,A12,B22,n     ;
  spawn Mult(T22,A22,B22,n/
  spawn Mult(T21,A22,B21,n/
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

$C = A \cdot B$

Absence of type declarations.

Matrix Multiply in Pseudo-Cilk

```
cilk void Mult(*C, *A, *B, n) {
  float *T = Cilk_alloca(n*n*sizeof(float));
  ⟨ base case & partition matrices ⟩
  spawn Mult(C11,A11,B11,n/2);
  spawn Mult(C12,A11,B12,n/2);
  spawn Mult(C22,A21,B12,n/2);
  spawn Mult(C21,A21,B11,n/2);
  spawn Mult(T11,A12,B21,n/2);
  spawn Mult(T12,A12,B22,n/2);
  spawn Mult(T22,A22,B22,n/2);
  spawn Mult(T21,A22,B21,n/2);
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

$$C = A \cdot B$$

Coarsen base cases for efficiency.

Matrix Multiply in Pseudo-Cilk

```
cilk void Mult(*C, *A, *B, n) {
  float *T = Cilk_alloca(n*n*sizeof(float));
  ⟨ base case & partition matrices ⟩
  spawn Mult(C11,A11,B11,n/2);
  spawn Mult(C12,A11,B12,n/2)
  spawn Mult(C22,A21,B12,n/2)
  spawn Mult(C21,A21,B  ,n/2)
  spawn Mult(T11,A12,B2  /2)
  spawn Mult(T12,A12,B22,    )
  spawn Mult(T22,A22,B22,n
  spawn Mult(T21,A22,B21,n/
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

Also need a row-size argument for array indexing.

Submatrices are produced by pointer calculation, not copying of elements.

$$C = A \cdot B$$

Matrix Multiply in Pseudo-Cilk

```
cilk void Mult(*C, *A, *B, n) {
  float *T = Cilk_alloca(n*n*sizeof(float));
  ⟨ base case & partition matrices ⟩
  spawn Mult(C11,A11,B11,n/2);
  spawn Mult(C12,A11,B12,n/2);
  spawn Mult(C22,A21,B12,n/2);
  spawn Mult(C21,A21,B11,n/2);
  spawn Mult(T11,A12,B21,n/2);
  spawn Mult(T12,A12,B22,n/2);
  spawn Mult(T22,A22,B22,n/2);
  spawn Mult(T21,A22,B21,n/2);
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

$C = A \cdot B$

```
cilk void Add(*C, *T, n) {
  ⟨ base case & partition matrices ⟩
  spawn Add(C11,T11,n/2);
  spawn Add(C12,T12,n/2);
  spawn Add(C21,T21,n/2);
  spawn Add(C22,T22,n/2);
  sync;
  return;
}
```

$C = C + T$

Work of Matrix Addition

```
cilk void Add(*C, *T, n) {
  ⟨ base case & partition matrices ⟩
   spawn Add(C11,T11,n/2);
   spawn Add(C12,T12,n/2);
   spawn Add(C21,T21,n/2);
   spawn Add(C22,T22,n/2);
   sync;
   return;
}
```

Work: $A_1(n) = 4A_1(n/2) + \Theta(1)$
$= \Theta(n^2)$ — **CASE 1**

$$n^{\log_b a} = n^{\log_2 4} = n^2 \gg \Theta(1) \,.$$

Span of Matrix Addition

maximum

```
cilk void Add(*C, *T, n) {
   ⟨ base case & partition matrices ⟩
   spawn Add(C11,T11,n/2);
   spawn Add(C12,T12,n/2);
   spawn Add(C21,T21,n/2);
   spawn Add(C22,T22,n/2);
   sync;
   return;
}
```

Span: $A_\infty(n) = A_\infty(n/2) + \Theta(1)$

$= \Theta(\lg n)$ — **CASE 2**

$n^{\log_b a} = n^{\log_2 1} = 1 \Rightarrow f(n) = \Theta(n^{\log_b a} \lg^0 n)$.

Work of Matrix Multiplication

```
cilk void Mult(*C, *A, *B, n) {
  float *T = Cilk_alloca(n*n*sizeof(float));
  ⟨ base case & partition matrices ⟩
  spawn Mult(C11,A11,B11,n/2);
  spawn Mult(C12,A11,B12,n/2);
  ⋮
  spawn Mult(T21,A22,B21,n/2);
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

8

Work: $M_1(n) = 8\,M_1(n/2) + A_1(n) + \Theta(1)$

$$= 8\,M_1(n/2) + \Theta(n^2)$$

$$= \Theta(n^3) \quad \text{— \textbf{CASE 1}}$$

$$n^{\log_b a} = n^{\log_2 8} = n^3 \gg \Theta(n^2)\,.$$

Span of Matrix Multiplication

```
cilk void Mult(*C, *A, *B, n) {
  float *T = Cilk_alloca(n*n*sizeof(float));
  ⟨ base case & partition matrices ⟩
  spawn Mult(C11,A11,B11,n/2);
  spawn Mult(C12,A11,B12,n/2);
  ⋮
  spawn Mult(T21,A22,B21,n/2);
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

8 (the eight spawn Mult calls)

Span: $M_\infty(n) = M_\infty(n/2) + A_\infty(n) + \Theta(1)$
$= M_\infty(n/2) + \Theta(\lg n)$
$= \Theta(\lg^2 n)$ — **CASE 2**

$$n^{\log_b a} = n^{\log_2 1} = 1 \Rightarrow f(n) = \Theta(n^{\log_b a} \lg^1 n) .$$

Parallelism of Matrix Multiply

Work: $M_1(n) = \Theta(n^3)$

Span: $M_\infty(n) = \Theta(\lg^2 n)$

Parallelism: $$\frac{M_1(n)}{M_\infty(n)} = \Theta(n^3/\lg^2 n)$$

For 1000×1000 matrices,
parallelism $\approx (10^3)^3/10^2 = 10^7$.

Stack Temporaries

```
cilk void Mult(*C, *A, *B, n) {
  float *T = Cilk_alloca(n*n*sizeof(float));
  ⟨ base case & partition matrices ⟩
  spawn Mult(C11,A11,B11,n/2);
  spawn Mult(C12,A11,B12,n/2);
  ⋮
  spawn Mult(T21,A22,B21,n/2);
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

In hierarchical-memory machines (especially chip multiprocessors), memory accesses are so expensive that minimizing storage often yields higher performance.

IDEA: Trade off parallelism for less storage.

No-Temp Matrix Multiplication

```
cilk void MultA(*C, *A, *B, n) {
  // C = C + A * B
  ⟨ base case & partition matrices ⟩
  spawn MultA(C11,A11,B11,n/2);
  spawn MultA(C12,A11,B12,n/2);
  spawn MultA(C22,A21,B12,n/2);
  spawn MultA(C21,A21,B11,n/2);
  sync;
  spawn MultA(C21,A22,B21,n/2);
  spawn MultA(C22,A22,B22,n/2);
  spawn MultA(C12,A12,B22,n/2);
  spawn MultA(C11,A12,B21,n/2);
  sync;
  return;
}
```

Saves space, but at what expense?

Work of No-Temp Multiply

```
cilk void MultA(*C, *A, *B, n) {
  // C = C + A * B
  ⟨ base case & partition matrices ⟩
  spawn MultA(C11,A11,B11,n/2);
  spawn MultA(C12,A11,B12,n/2);
  spawn MultA(C22,A21,B12,n/2);
  spawn MultA(C21,A21,B11,n/2);
  sync;
  spawn MultA(C21,A22,B21,n/2);
  spawn MultA(C22,A22,B22,n/2);
  spawn MultA(C12,A12,B22,n/2);
  spawn MultA(C11,A12,B21,n/2);
  sync;
  return;
}
```

Work: $M_1(n) = 8\,M_1(n/2) + \Theta(1)$

$= \Theta(n^3)$ — **CASE 1**

Span of No-Temp Multiply

```
cilk void MultA(*C, *A, *B, n) {
  // C = C + A * B
  ⟨ base case & partition matrices ⟩
  spawn MultA(C11,A11,B11,n/2);
  spawn MultA(C12,A11,B12,n/2);
  spawn MultA(C22,A21,B12,n/2);
  spawn MultA(C21,A21,B11,n/2);
  sync;
  spawn MultA(C21,A22,B21,n/2);
  spawn MultA(C22,A22,B22,n/2);
  spawn MultA(C12,A12,B22,n/2);
  spawn MultA(C11,A12,B21,n/2);
  sync;
  return;
}
```

maximum

maximum

Span: $M_\infty(n) = 2\,M_\infty(n/2) + \Theta(1)$
$= \Theta(n)$ — **CASE 1**

Parallelism of No-Temp Multiply

Work: $M_1(n) = \Theta(n^3)$

Span: $M_\infty(n) = \Theta(n)$

Parallelism: $$\frac{M_1(n)}{M_\infty(n)} = \Theta(n^2)$$

For 1000×1000 matrices, parallelism $\approx (10^3)^3/10^3 = 10^6$.

Faster in practice!

Testing Synchronization

Cilk language feature: A programmer can check whether a Cilk procedure is "synched" (without actually performing a **`sync`**) by testing the pseudovariable **`SYNCHED`**:

- **`SYNCHED`** = 0 ⇒ some spawned children might not have returned.
- **`SYNCHED`** = 1 ⇒ all spawned children have definitely returned.

Best of Both Worlds

```
cilk void Mult1(*C, *A, *B, n) {// multiply & store
  ⟨ base case & partition matrices ⟩
  spawn Mult1(C11,A11,B11,n/2); // multiply & store
  spawn Mult1(C12,A11,B12,n/2);
  spawn Mult1(C22,A21,B12,n/2);
  spawn Mult1(C21,A21,B11,n/2);
  if (SYNCHED) {
    spawn MultA1(C11,A12,B21,n/2); // multiply & add
    spawn MultA1(C12,A12,B22,n/2);
    spawn MultA1(C22,A22,B22,n/2);
    spawn MultA1(C21,A22,B21,n/2);
  } else {
    float *T = Cilk_alloca(n*n*sizeof(float));
    spawn Mult1(T11
    spawn Mult1(T1
    spawn Mult1(T2
    spawn Mult1(T2
    sync;
    spawn Add(C,T,
  }
  sync;
  return;
}
```

This code is just as parallel as the original, but it only uses more space if runtime parallelism actually exists.

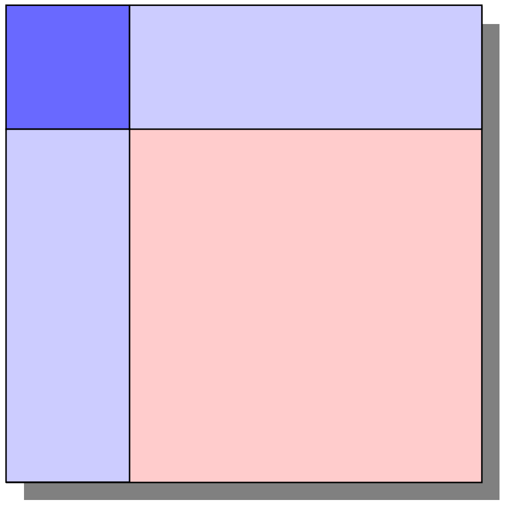
Ordinary Matrix Multiplication

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

IDEA: Spawn n^2 inner products in parallel. Compute each inner product in parallel.

Work: $\Theta(n^3)$
Span: $\Theta(\lg n)$
Parallelism: $\Theta(n^3/\lg n)$

BUT, this algorithm exhibits poor locality and does not exploit the cache hierarchy of modern microprocessors, especially CMP's.



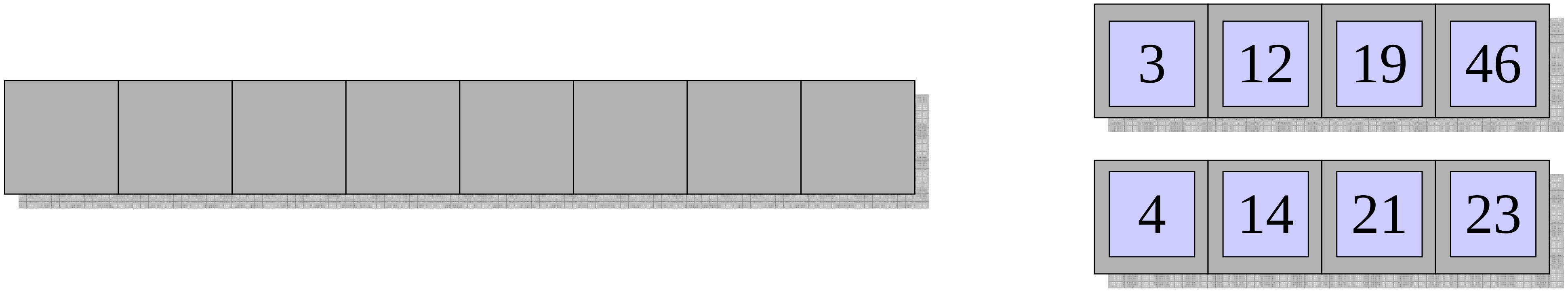
LECTURE 2

- Recurrences (Review)
- Matrix Multiplication
- **Merge Sort**
- **Tableau Construction**
- **Conclusion**

Merging Two Sorted Arrays

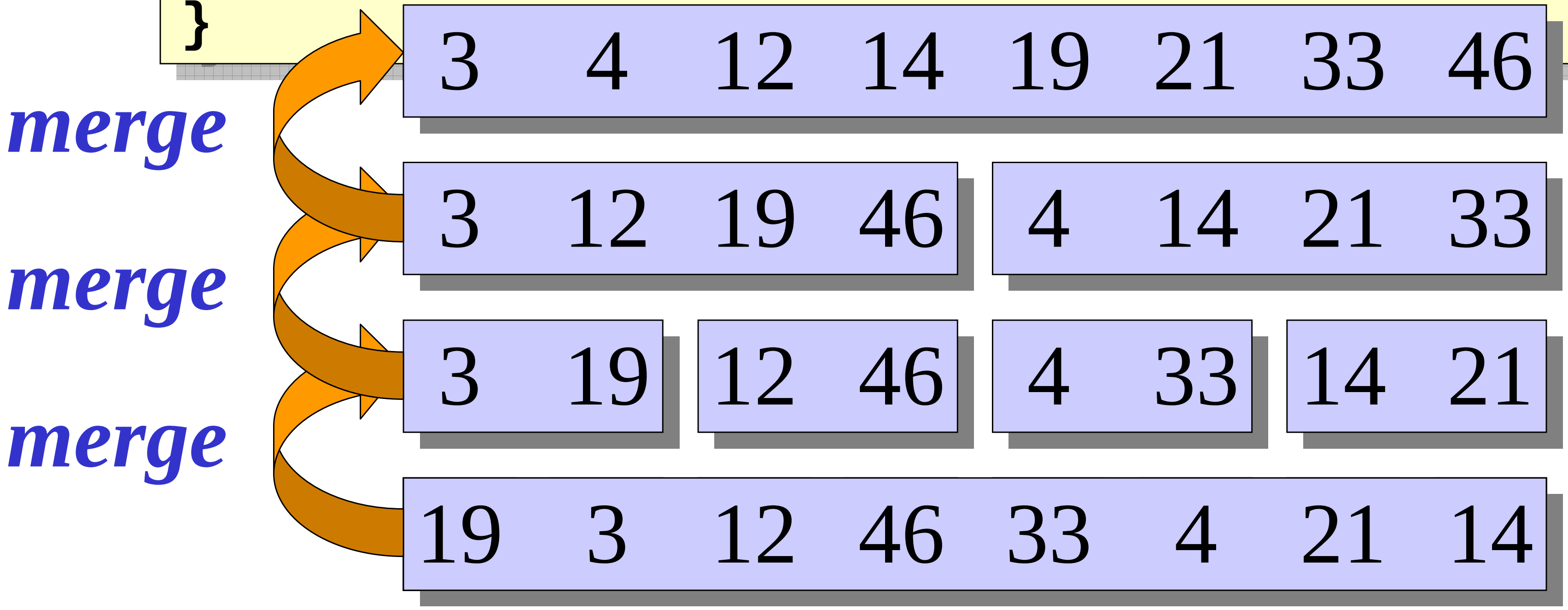
```
void Merge(int *C, int *A, int *B, int na, int nb) {
  while (na>0 && nb>0) {
    if (*A <= *B) {
      *C++ = *A++; na--;
    } else {
      *C++ = *B++; nb--;
    }
  }
  while (na>0) {
    *C++ = *A++; na--;
  }
  while (nb>0) {
    *C++ = *B++; nb--;
  }
}
```

Time to merge n elements = $\Theta(n)$.



Merge Sort

```
cilk void MergeSort(int *B, int *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    int *C;
    C = (int*) Cilk_alloca(n*sizeof(int));
    spawn MergeSort(C, A, n/2);
    spawn MergeSort(C+n/2, A+n/2, n-n/2);
    sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
}
```



Work of Merge Sort

```
cilk void MergeSort(int *B, int *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    int *C;
    C = (int*) Cilk_alloca(n*sizeof(int));
    spawn MergeSort(C, A, n/2);
    spawn MergeSort(C+n/2, A+n/2, n-n/2);
    sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
}
```

Work: $T_1(n) = 2\,T_1(n/2) + \Theta(n)$

$= \Theta(n \lg n)$ — **CASE 2**

$$n^{\log_b a} = n^{\log_2 2} = n \Rightarrow f(n) = \Theta(n^{\log_b a} \lg^0 n) .$$

Span of Merge Sort

```
cilk void MergeSort(int *B, int *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    int *C;
    C = (int*) Cilk_alloca(n*sizeof(int));
    spawn MergeSort(C, A, n/2);
    spawn MergeSort(C+n/2, A+n/2, n-n/2);
    sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
}
```

Span: $T_\infty(n) = T_\infty(n/2) + \Theta(n)$

$= \Theta(n)$ — **CASE 3**

$$n^{\log_b a} = n^{\log_2 1} = 1 \ll \Theta(n) .$$

Parallelism of Merge Sort

Work: $T_1(n) = \Theta(n \lg n)$

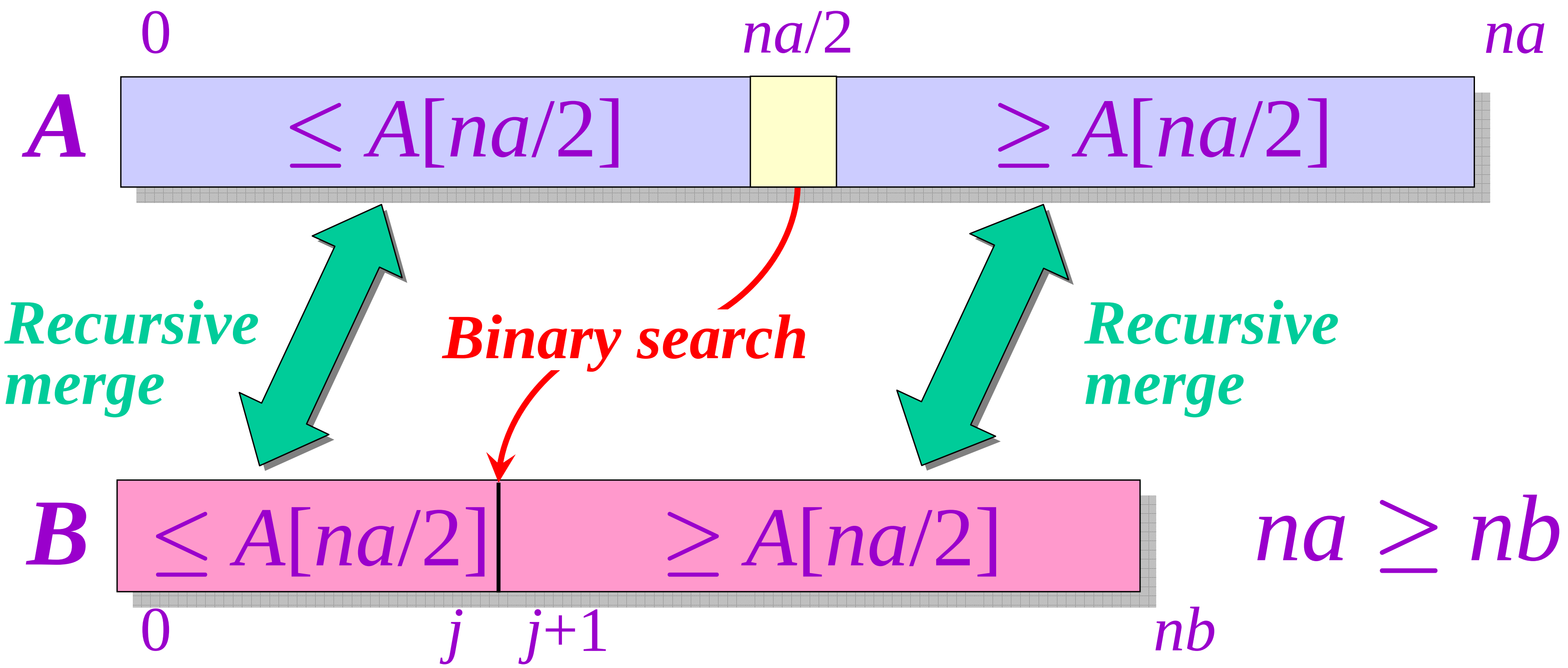
Span: $T_\infty(n) = \Theta(n)$

PUNY!

Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(\lg n)$$

We need to parallelize the ***merge****!*

Parallel Merge



KEY IDEA: If the total number of elements to be merged in the two arrays is $n = na + nb$, the total number of elements in the larger of the two recursive merges is at most $(3/4)\,n$.

Parallel Merge

```
cilk void P_Merge(int *C, int *A, int *B,
                          int na, int nb) {
  if (na < nb) {
    spawn P_Merge(C, B, A, nb, na);
  } else if (na==1) {
    if (nb == 0) {
      C[0] = A[0];
    } else {
      C[0] = (A[0]<B[0]) ? A[0] : B[0]; /* minimum */
      C[1] = (A[0]<B[0]) ? B[0] : A[0]; /* maximum */
    }
  } else {
    int ma = na/2;
    int mb = BinarySearch(A[ma], B, nb);
    spawn P_Merge(C, A, B, ma, mb);
    spawn P_Merge(C+ma+mb, A+ma, B+mb, na-ma, nb-mb);
    sync;
  }
}
```

Coarsen base cases for efficiency.

Span of `P_Merge`

```
cilk void P_Merge(int *C, int *A, int *B,
                          int na, int nb) {
  if (na < nb) {
  ⋮
  } else {
    int ma = na/2;
    int mb = BinarySearch(A[ma], B, nb);
    spawn P_Merge(C, A, B, ma, mb);
    spawn P_Merge(C+ma+mb, A+ma, B+mb, na-ma, nb-mb);
    sync;
  }
}
```

Span: $T_\infty(n) = T_\infty(3n/4) + \Theta(\lg n)$

$= \Theta(\lg^2 n)$ — **CASE 2**

$$n^{\log_b a} = n^{\log_{4/3} 1} = 1 \Rightarrow f(n) = \Theta(n^{\log_b a} \lg^1 n) .$$

Work of P_Merge

```
cilk void P_Merge(int *C, int *A, int *B,
                          int na, int nb) {
  if (na < nb) {
  ⋮
  } else {
    int ma = na/2;
    int mb = BinarySearch(A[ma], B, nb);
    spawn P_Merge(C, A, B, ma, mb);
    spawn P_Merge(C+ma+mb, A+ma, B+mb,          );
    sync;
  }
}
```

HAIRY!

Work: $T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n)$,
where $1/4 \le \alpha \le 3/4$.

CLAIM: $T_1(n) = \Theta(n)$.

Analysis of Work Recurrence

$T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n)$,
where $1/4 \le \alpha \le 3/4$.

Substitution method: Inductive hypothesis is $T_1(k) \le c_1 k - c_2 \lg k$, where $c_1, c_2 > 0$. Prove that the relation holds, and solve for c_1 and c_2.

$$\begin{aligned} T_1(n) &= T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n) \\ &\le c_1(\alpha n) - c_2 \lg(\alpha n) \\ &\quad + c_1((1-\alpha)n) - c_2 \lg((1-\alpha)n) + \Theta(\lg n) \end{aligned}$$

Analysis of Work Recurrence

$T_1(n) = T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n)$,
where $1/4 \le \alpha \le 3/4$.

$$\begin{aligned} T_1(n) &= T_1(\alpha n) + T_1((1-\alpha)n) + \Theta(\lg n) \\ &\le c_1(\alpha n) - c_2\lg(\alpha n) \\ &\quad + c_1(1-\alpha)n - c_2\lg((1-\alpha)n) + \Theta(\lg n) \end{aligned}$$

Analysis of Work Recurrence

$T_1(n) = T_1(\alpha n) + T_1((1–\alpha)n) + \Theta(\lg n)$,

where $1/4 \le \alpha \le 3/4$.

$$\begin{aligned}
T_1(n) &= T_1(\alpha n) + T_1((1–\alpha)n) + \Theta(\lg n) \\
&\le c_1(\alpha n) – c_2\lg(\alpha n) \\
&\quad + c_1(1–\alpha)n – c_2\lg((1–\alpha)n) + \Theta(\lg n) \\
&\le c_1 n – c_2\lg(\alpha n) – c_2\lg((1–\alpha)n) + \Theta(\lg n) \\
&\le c_1 n – c_2 (\lg(\alpha(1–\alpha)) + 2 \lg n) + \Theta(\lg n) \\
&\le c_1 n – c_2 \lg n \\
&\quad – (c_2(\lg n + \lg(\alpha(1–\alpha))) – \Theta(\lg n)) \\
&\le c_1 n – c_2 \lg n
\end{aligned}$$

by choosing c_1 and c_2 large enough.

Parallelism of `P_Merge`

Work: $T_1(n) = \Theta(n)$

Span: $T_\infty(n) = \Theta(\lg^2 n)$

Parallelism: $$\frac{T_1(n)}{T_\infty(n)} = \Theta(n/\lg^2 n)$$

Parallel Merge Sort

```
cilk void P_MergeSort(int *B, int *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    int *C;
    C = (int*) Cilk_alloca(n*sizeof(int));
    spawn P_MergeSort(C, A, n/2);
    spawn P_MergeSort(C+n/2, A+n/2, n-n/2);
    sync;
    spawn P_Merge(B, C, C+n/2, n/2, n-n/2);
  }
}
```

Work of Parallel Merge Sort

```
cilk void P_MergeSort(int *B, int *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    int *C;
    C = (int*) Cilk_alloca(n*sizeof(int));
    spawn P_MergeSort(C, A, n/2);
    spawn P_MergeSort(C+n/2, A+n/2, n-n/2);
    sync;
    spawn P_Merge(B, C, C+n/2, n/2, n-n/2);
  }
}
```

Work: $T_1(n) = 2\,T_1(n/2) + \Theta(n)$

$= \Theta(n \lg n)$ — **CASE 2**

Span of Parallel Merge Sort

```
cilk void P_MergeSort(int *B, int *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    int *C;
    C = (int*) Cilk_alloca(n*sizeof(int));
    spawn P_MergeSort(C, A, n/2);
    spawn P_MergeSort(C+n/2, A+n/2, n-n/2);
    sync;
    spawn P_Merge(B, C, C+n/2, n/2, n-n/2);
  }
}
```

Span: $T_\infty(n) = T_\infty(n/2) + \Theta(\lg^2 n)$

$= \Theta(\lg^3 n)$ — **CASE 2**

$n^{\log_b a} = n^{\log_2 1} = 1 \Rightarrow f(n) = \Theta(n^{\log_b a} \lg^2 n)$.

Parallelism of Merge Sort

Work: $T_1(n) = \Theta(n \lg n)$

Span: $T_\infty(n) = \Theta(\lg^3 n)$

Parallelism: $\dfrac{T_1(n)}{T_\infty(n)} = \Theta(n/\lg^2 n)$

LECTURE 2

- Recurrences (Review)
- Matrix Multiplication
- Merge Sort
- Tableau Construction
- Conclusion

Tableau Construction

Problem: Fill in an $n \times n$ tableau A, where $A[i, j] = f\big(A[i, j-1], A[i-1, j], A[i-1, j-1]\big)$.

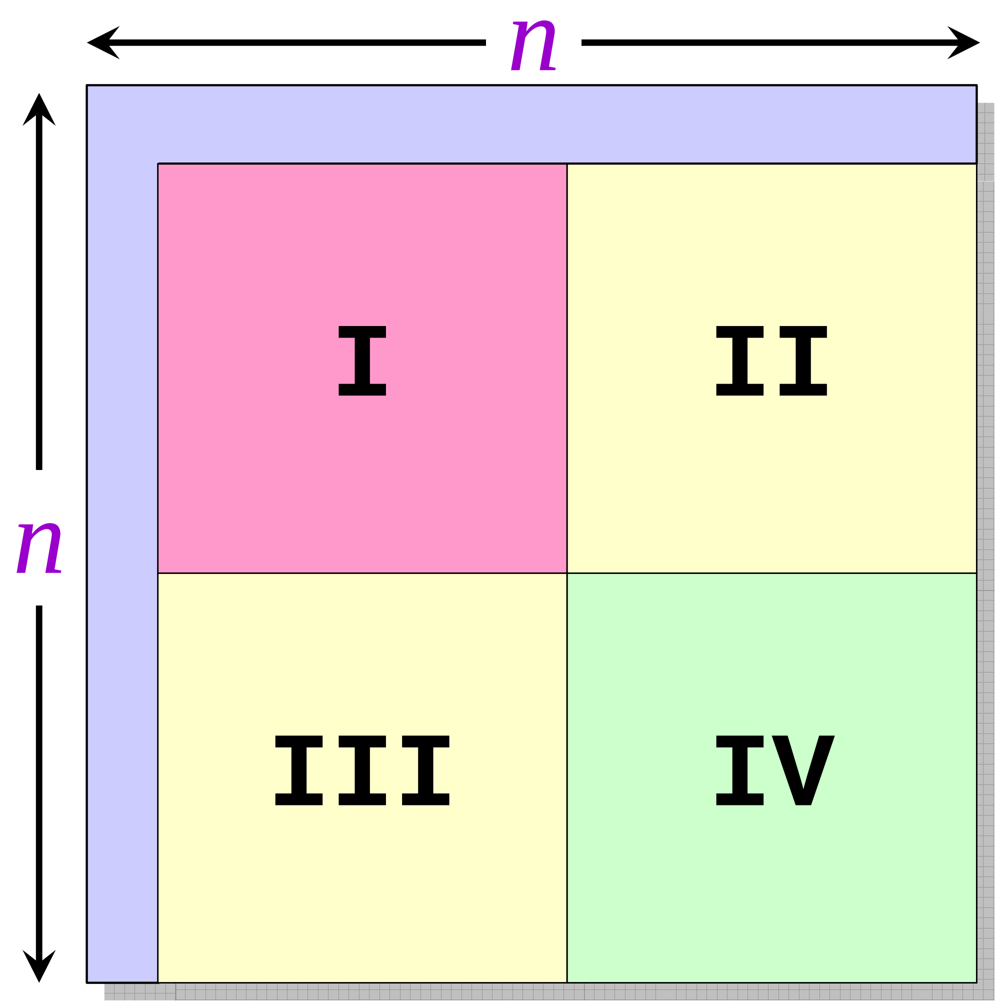
00	01	02	03	04	05	06	07
10	11	12	13	14	15	16	17
20	21	22	23	24	25	26	27
30	31	32	33	34	35	36	37
40	41	42	43	44	45	46	47
50	51	52	53	54	55	56	57
60	61	62	63	64	65	66	67
70	71	72	73	74	75	76	77

Dynamic programming

- Longest common subsequence
- Edit distance
- Time warping

Work: $\Theta(n^2)$.

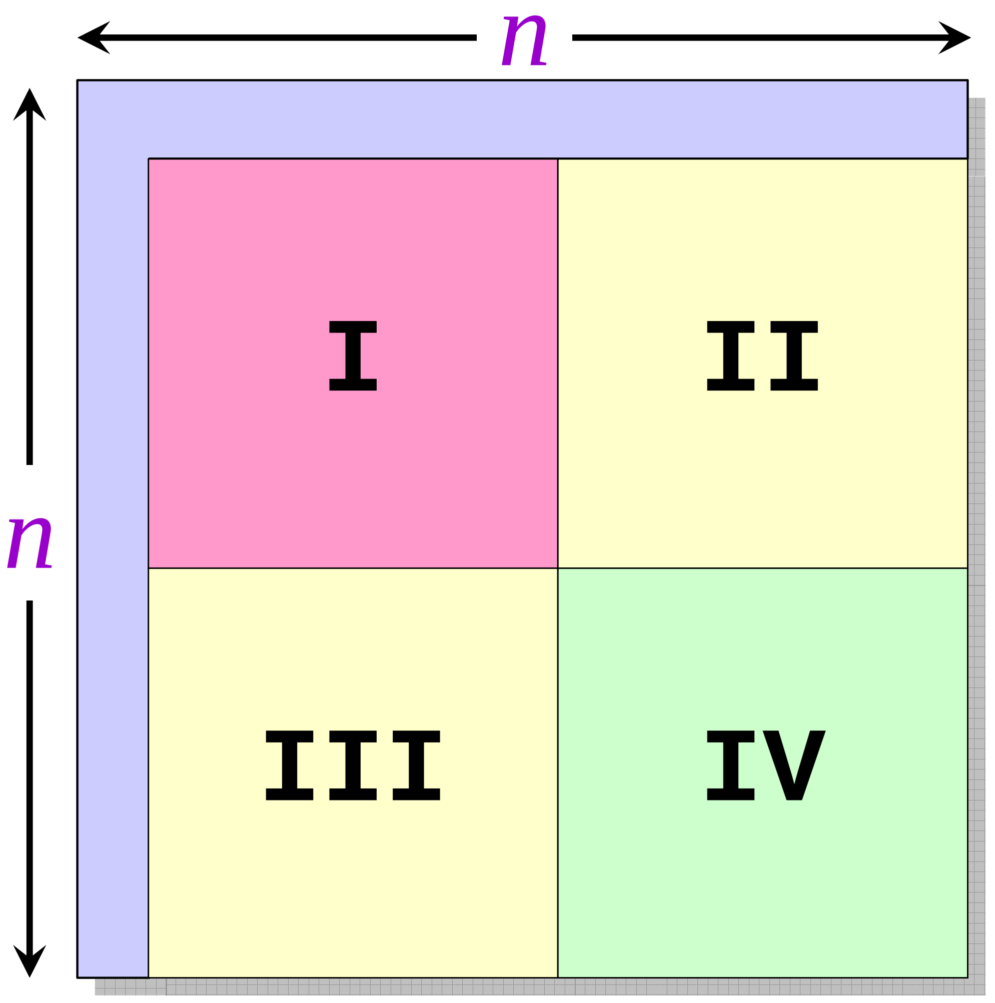
Recursive Construction



Cilk code

```
spawn I;
sync;
spawn II;
spawn III;
sync;
spawn IV;
sync;
```

Recursive Construction

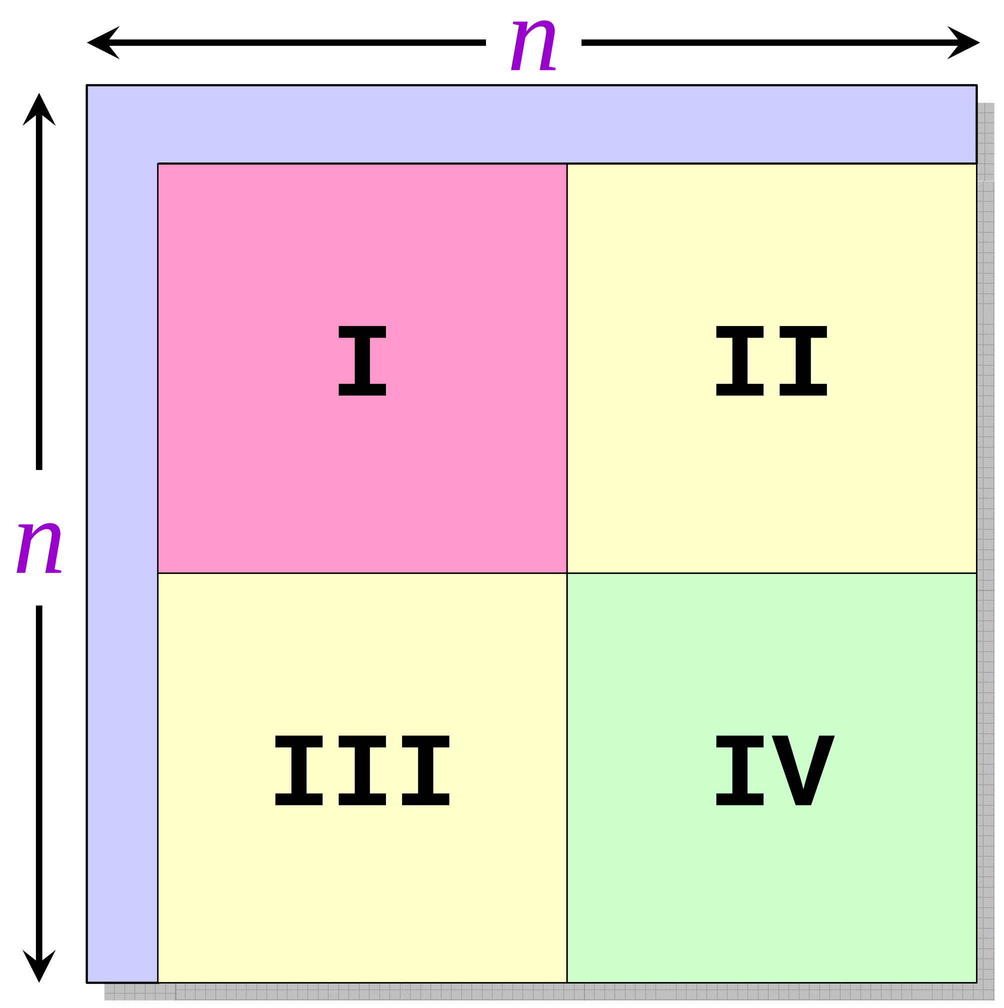


Cilk code

```
spawn I;
sync;
spawn II;
spawn III;
sync;
spawn IV;
sync;
```

Work: $T_1(n) = 4T_1(n/2) + \Theta(1)$
$= \Theta(n^2)$ — **CASE 1**

Recursive Construction



Cilk code

```
spawn I;
sync;
spawn II;
spawn III;
sync;
spawn IV;
sync;
```

Span: $T_\infty(n) = 3T_\infty(n/2) + \Theta(1)$

$= \Theta(n^{\lg 3})$ — **CASE 1**

Analysis of Tableau Construction

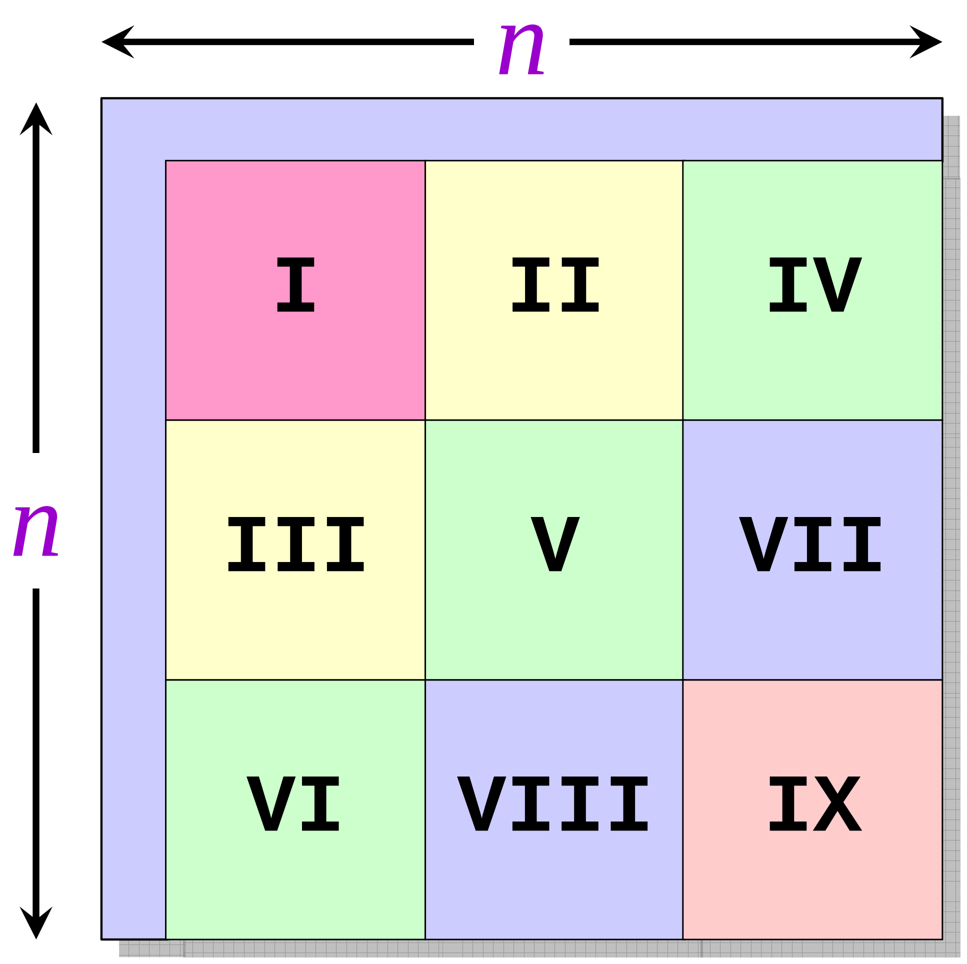
Work: $T_1(n) = \Theta(n^2)$

Span: $T_\infty(n) = \Theta(n^{\lg 3})$

$\approx \Theta(n^{1.58})$

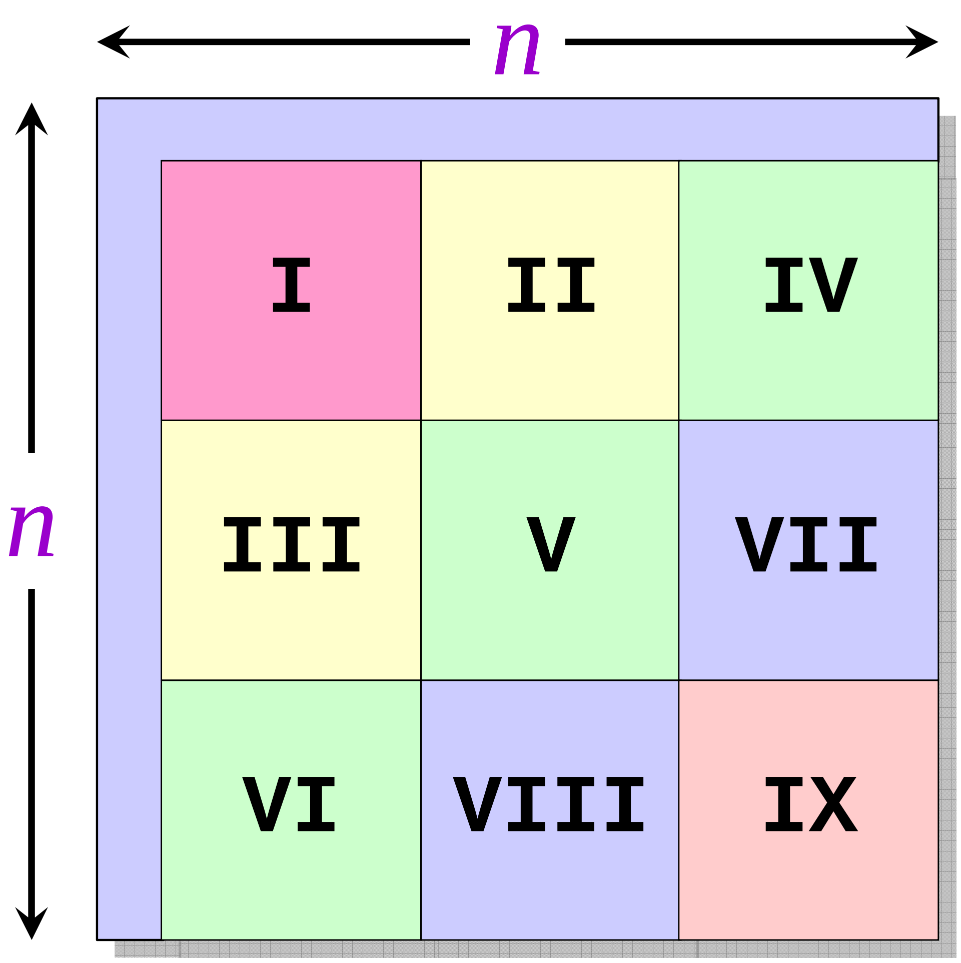
Parallelism: $\dfrac{T_1(n)}{T_\infty(n)} \approx \Theta(n^{0.42})$

A More-Parallel Construction



```
spawn I;
sync;
spawn II;
spawn III;
sync;
spawn IV;
spawn V;
spawn VI
sync;
spawn VII;
spawn VIII;
sync;
spawn IX;
sync;
```

A More-Parallel Construction

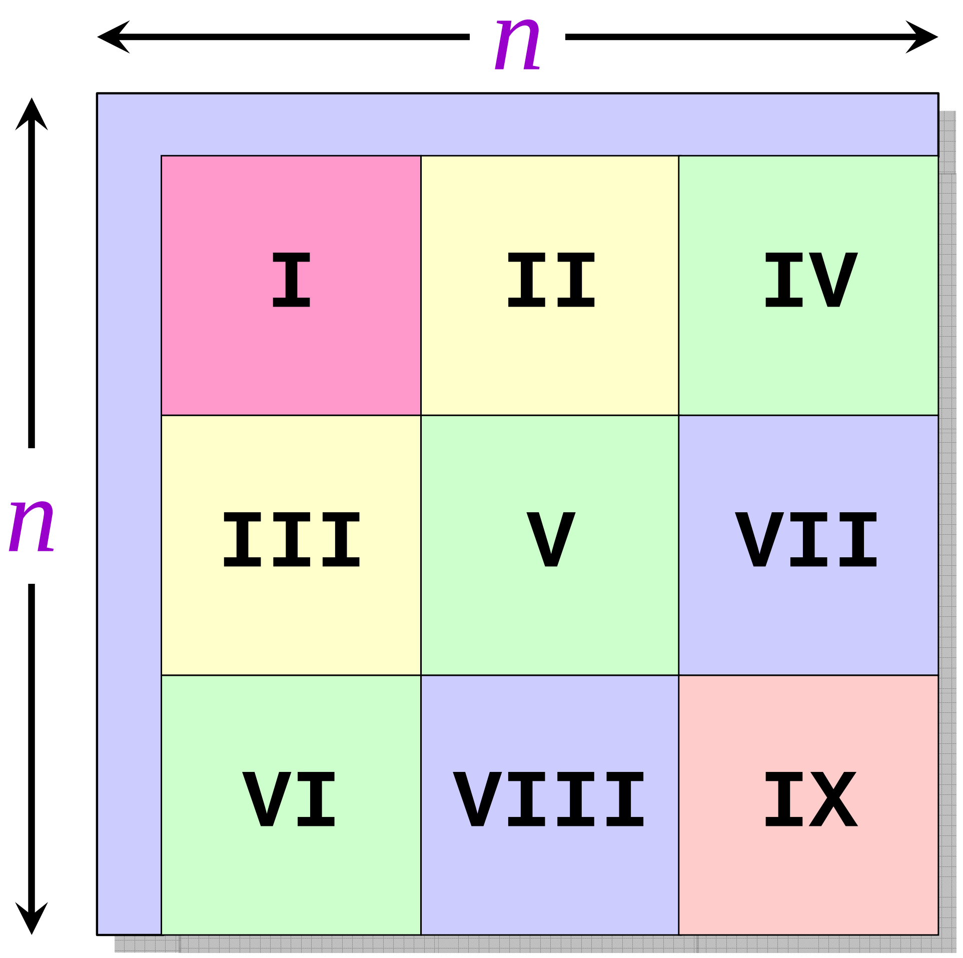


```
spawn I;
sync;
spawn II;
spawn III;
sync;
spawn IV;
spawn V;
spawn VI
sync;
spawn VII;
spawn VIII;
sync;
spawn IX;
sync;
```

Work: $T_1(n) = 9T_1(n/3) + \Theta(1)$

$= \Theta(n^2)$ — **CASE 1**

A More-Parallel Construction



```
spawn I;
sync;
spawn II;
spawn III;
sync;
spawn IV;
spawn V;
spawn VI
sync;
spawn VII;
spawn VIII;
sync;
spawn IX;
sync;
```

Span: $T_\infty(n) = 5T_\infty(n/3) + \Theta(1)$

$= \Theta(n^{\log_3 5})$ — **CASE 1**

Analysis of Revised Construction

Work: $T_1(n) = \Theta(n^2)$

Span: $T_\infty(n) = \Theta(n^{\log_3 5})$

$\approx \Theta(n^{1.46})$

Parallelism: $\dfrac{T_1(n)}{T_\infty(n)} \approx \Theta(n^{0.54})$

More parallel by a factor of

$\Theta(n^{0.54})/\Theta(n^{0.42}) = \Theta(n^{0.12})$.

Puzzle

What is the largest parallelism that can be obtained for the tableau-construction problem using Cilk?

- You may only use basic Cilk control constructs (`spawn`, `sync`) for synchronization.
- No locks, synchronizing through memory, etc.

LECTURE 2

- Recurrences (Review)
- Matrix Multiplication
- Merge Sort
- Tableau Construction
- **Conclusion**

Key Ideas

- Cilk is simple: **`cilk`**, **`spawn`**, **`sync`**, **`SYNCHED`**
- Recurrences, recurrences, recurrences, …
- Work & span
- Work & span
- Work & span
- Work & span
- Work & span
- Work & span
- Work & span
- Work & span
- Work & span
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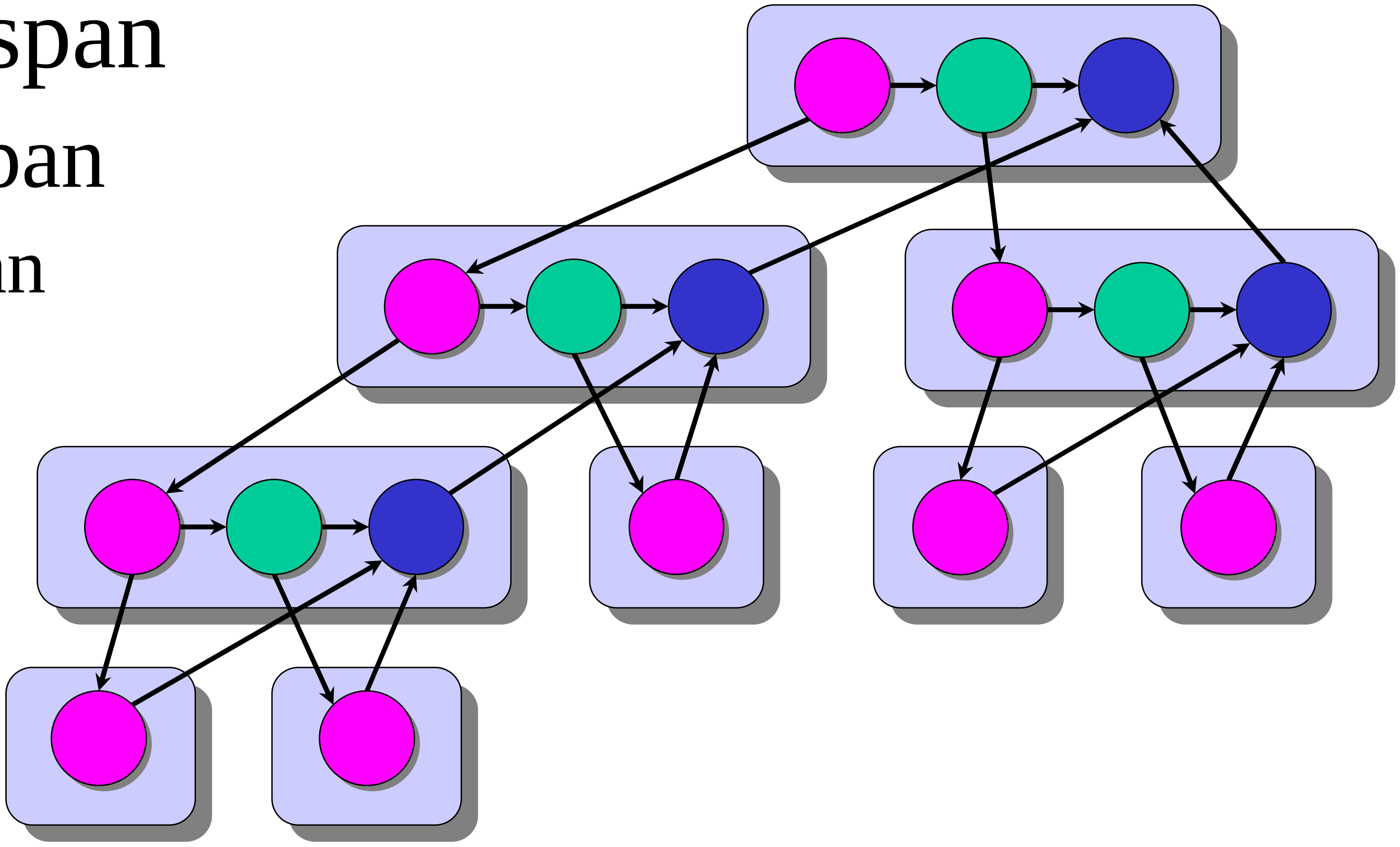
- Work & span
- Work & span
- Work & span
- Work & span



Minicourse Outline

- **LECTURE 1**
  Basic Cilk programming: Cilk keywords, performance measures, scheduling.
- **LECTURE 2**
  Analysis of Cilk algorithms: matrix multiplication, sorting, tableau construction.
- **LABORATORY**
  Programming matrix multiplication in Cilk — Dr. Bradley C. Kuszmaul
- **LECTURE 3**
  Advanced Cilk programming: inlets, abort, speculation, data synchronization, & more.