Weekly Homework 31

Math Gecs

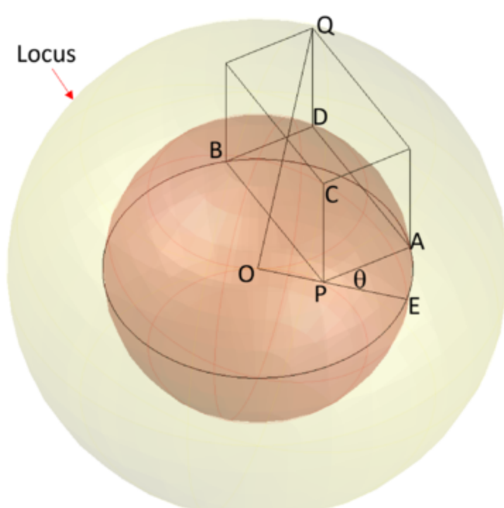
October 06, 2024

Exercise 1

We consider a fixed point P in the interior of a fixed sphere. We construct three segments PA, PB, PC, perpendicular two by two, with the vertexes A, B, C on the sphere. We consider the vertex Q which is opposite to P in the parallelepiped (with right angles) with PA, PB, PC as edges. Find the locus of the point Q when A, B, C take all the positions compatible with our problem.

Source: 1978 IMO Problem 2

Solution. *Let R be the radius of the given fixed sphere.*



Let point O be the center of the sphere.
Let point D be the 4th vertex of the face of the parallelepiped that contains points P, A, and B.
Let point E be the point where the line that passes through OP intersects the circle on the side nearest to point A
Let $\alpha = \angle AOP$, $\beta = \angle BPD$, $\theta = \angle APE$

We start the calculations as follows:
$|AB| = |PD|$
$|AB|^2 = |PA|^2 + |PB|^2$
Therefore, $|PD|^2 = |PA|^2 + |PB|^2$ *[Equation 1]*
Using law of cosines:
$R^2 = |OP|^2 + |PB|^2 - 2|OP||PB| cos(\angle OPB)$
$R^2 = |OP|^2 + |PB|^2 - 2|OP||PB| cos\left(\frac{\pi}{2} - \theta\right)$
$R^2 = |OP|^2 + |PB|^2 - 2|OP||PB| sin(\theta)$
$|PB|^2 = R^2 - |OP|^2 + 2|OP||PB| sin(\theta)$ *[Equation 2]*
Using law of cosines again we also get:
$|PA|^2 = R^2 + |OP|^2 - 2|OP| Rcos(\alpha)$
Since $Rcos(\alpha) = |PA| cos(\theta) + |OP|$*, then*
$|PA|^2 = R^2 + |OP|^2 - 2|OP| [|PA| cos(\theta) + |OP|]$
$|PA|^2 = R^2 - |OP|^2 - 2|OP||PA| cos(\theta)$ *[Equation 3]*
Substituting [Equation 2] and [Equation 3] into [Equation 1] we get:
$|PD|^2 = 2R^2 - 2|OP|^2 + 2|OP| [|PB| sin(\theta) - |PA| cos(\theta)]$ *[Equation 4]*
Now we apply the law of cosines again:
$|OD|^2 = |OP|^2 + |PD|^2 - 2|OP||PD| cos(\angle OPD)$
$|OD|^2 = |OP|^2 + |PD|^2 - 2|OP||PD| cos(\angle OPB + \angle BPD)$
$|OD|^2 = |OP|^2 + |PD|^2 - 2|OP||PD| cos\left(\frac{\pi}{2} - \theta + \beta\right)$
$|OD|^2 = |OP|^2 + |PD|^2 - 2|OP||PD| sin(\theta - \beta)$
$|OD|^2 = |OP|^2 + |PD|^2 - 2|OP||PD| [sin(\theta)cos(\beta) - sin(\beta)cos(\theta)]$
Since, $sin(\beta) = \frac{|PA|}{|PD|}$ *and* $cos(\beta) = \frac{|PB|}{|PD|}$ *then,*
$|OD|^2 = |OP|^2 + |PD|^2 - 2|OP||PD| \left[\frac{|PB|}{|PD|} sin(\theta) - \frac{|PA|}{|PD|} cos(\theta)\right]$
$|OD|^2 = |OP|^2 + |PD|^2 - 2|OP| [|PB| sin(\theta) - |PA| cos(\theta)]$ *[Equation 5]*
Substituting [Equation 4] into [Equation 5] we get:
$|OD|^2 = |OP|^2 + 2R^2 - 2|OP|^2 + 2|OP| [|PB| sin(\theta) - |PA| cos(\theta)] - 2|OP| [|PB| sin(\theta) - |PA| cos(\theta)]$
Notice that all of the terms with θ *cancel and thus we're left with:*
$|OD|^2 = 2R^2 - |OP|^2$ *regardless of* θ*. [Equation 6]*
Now we need to find $|PC|$
Since points O*,* P*, and* C *are on the plane perpendicular to the plane with points* O*,* P*, and* A*, then these points lie on the big circle of the sphere. Therefore the distance* $|PC|$ *can be found using the formula:*
$R^2 = |OP|^2 + |PC|^2$
Solving for $|PC|^2$ *we get:*
$|PC|^2 = R^2 - |OP|^2$ *[Equation 7]*
Now we need to get $|OQ|^2$ *which will be using the formula:*
$|OQ|^2 = |OD|^2 + |PC|^2$ *[Equation 8]*
Substituting [Equation 6] and [Equation 7] into [Equation 8] we get:
$|OQ|^2 = 2R^2 - |OP|^2 + R^2 - |OP|^2$
This results in:
$|OQ|^2 = 3R^2 - 2|OP|^2$

which is constant regardless of θ and constant regardless of where points A, B, and C are located as long as they're still perpendicular to each other.
In space, this is a sphere with radius $|OQ|$ which is equal to $\sqrt{3R^2 - 2\left|OP\right|^2}$
Therefore, the locus of vertex Q is a sphere of radius $\sqrt{3R^2 - 2\left|OP\right|^2}$ with center at O, where R is the radius of the given sphere and $|OP|$ the distance from the center of the given sphere to point P