Weekly Homework 18

Math Gecs

May 30, 2024

Exercise 1

Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$.

Source: The 84th William Lowell Putnam Mathematical Competition Saturday, December 2, 2023 Problem A2

Solution. *The only other real numbers with this property are* $\pm 1/n!$. *(Note that these are indeed* other *values than* $\pm 1, \ldots, \pm n$ *because* $n > 1$.*)*
Define the polynomial $q(x) = x^{2n+2} - x^{2n} p(1/x) = x^{2n+2} - (a_0 x^{2n} + \cdots + a_{2n-1} x + 1)$. *The statement that* $p(1/x) = x^2$ *is equivalent (for* $x \neq 0$*) to the statement that* x *is a root of* $q(x)$. *Thus we know that* $\pm 1, \pm 2, \ldots, \pm n$ *are roots of* $q(x)$, *and we can write*

$$q(x) = (x^2 + ax + b)(x^2 - 1)(x^2 - 4) \cdots (x^2 - n^2)$$

for some monic quadratic polynomial $x^2 + ax + b$. *Equating the coefficients of* x^{2n+1} *and* x^0 *on both sides gives* $0 = a$ *and* $-1 = (-1)^n (n!)^2 b$, *respectively. Since* n *is even, we have* $x^2 + ax + b = x^2 - (n!)^{-2}$. *We conclude that there are precisely two other real numbers* x *such that* $p(1/x) = x^2$, *and they are* $\pm 1/n!$.