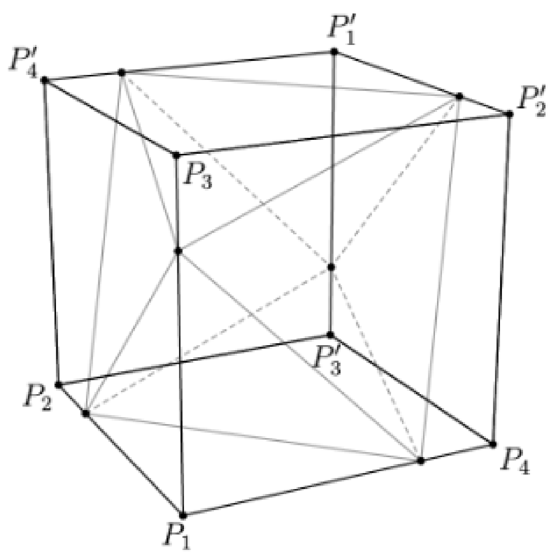
Weekly Homework 2

Math Gecs

December 30, 2023

Exercise 1

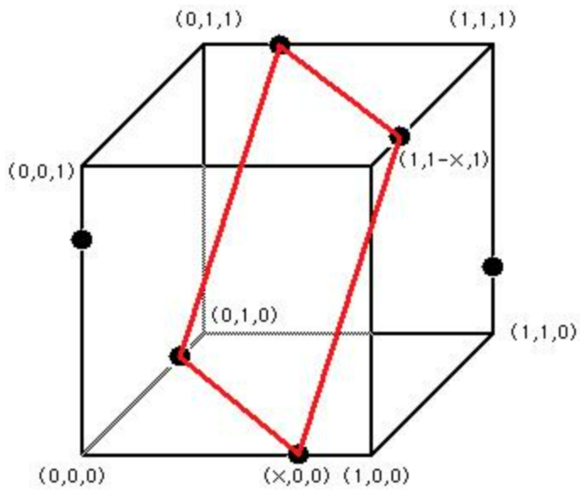
A unit cube has vertices $P_1, P_2, P_3, P_4, P_1', P_2', P_3'$, and P_4'. Vertices P_2, P_3, and P_4 are adjacent to P_1, and for $1 \leq i \leq 4$, vertices P_i and P_i' are opposite to each other. A regular octahedron has one vertex in each of the segments P_1P_2, P_1P_3, P_1P_4, $P_1'P_2'$, $P_1'P_3'$, and $P_1'P_4'$. What is the octahedron's side length?



Source: AMC12 (2012 - Problem 19)

Answer 1. $\frac{3\sqrt{2}}{4}$

Solution 1. *Observe the diagram below. Each dot represents one of the six vertices of the regular octahedron. Three dots have been placed exactly x units from the $(0,0,0)$ corner of the unit cube. The other three dots have been placed exactly x units from the $(1,1,1)$ corner of the unit cube. A red square has been drawn connecting four of the dots to provide perspective regarding the shape of the octahedron. Observe that the three dots that are near $(0,0,0)$ are*



each $(x)(\sqrt{2})$ from each other. The same is true for the three dots that are near $(1,1,1)$. There is a unique x for which the rectangle drawn in red becomes a square. This will occur when the distance from $(x,0,0)$ to $(1,1-x,1)$ is $(x)(\sqrt{2})$.

Using the distance formula we find the distance between the two points to be:

$$\sqrt{(1-x)^2+(1-x)^2+1} = \sqrt{2x^2-4x+3}$$

Equating this to $(x)(\sqrt{2})$ and squaring both sides, we have the equation:

$$2x^2-4x+3 = 2x^2$$
$$-4x+3 = 0$$
$$x = \frac{3}{4}$$

Since the length of each side is $(x)(\sqrt{2})$, we have a final result of $\frac{3\sqrt{2}}{4}$. Thus, Answer is $\frac{3\sqrt{2}}{4}$.

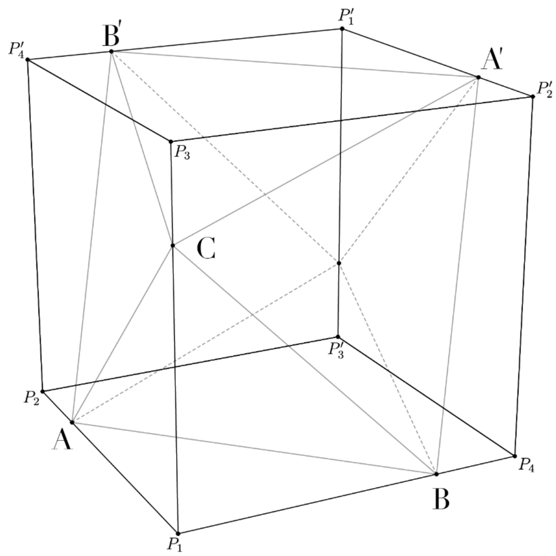
Solution 2. *Standard 3D geometry, no coordinates.*

Let the tip of the octahedron on side P_1P_3 be K_1 and the opposite vertex be K_2. Our key is to examine the trapezoid $P_1K_1K_2P_3$.

Let the side length of the octahedron be s. Then $P_1K_1 = \frac{s}{\sqrt{2}}$ and $P_3K_2 = 1 - \frac{2}{\sqrt{2}}$. Then, we have $P_1P_3 = \sqrt{2}$. Finally, we want to find K_1K_2, which is just double the height of half the octahedron. We can use Pythagorean Theorem to find that height as $\sqrt{2}s$. Now, we use the Pythagorean Theorem on the trapezoid. We get

$$(\sqrt{2})^2 + (2\sqrt{2}-1)^2 = (s\sqrt{2})^2$$
$$s = \frac{3\sqrt{2}}{4}$$

Solution 3. *Let the length of* $P_1A = a$, $P_1B = b$



$$AB = a^2 + b^2, AB' = (1-b)^2 + (1-a)^2 + 1, AB = AB'$$

$$a^2 + b^2 = (1-b)^2 + (1-a)^2 + 1, a^2 + b^2 = 1 - 2b + b^2 + 1 - 2a + a^2 + 1, a + b = \frac{3}{2}$$

$$AC = BC, a^2 + P_1C^2 = b^2 + P_1C^2, a = b, a = \frac{3}{4}$$

$$AB = \boxed{\frac{3\sqrt{2}}{4}}$$

Exercise 2

Let $\{a_n\}_{n\geq 0}$ be a non-decreasing, unbounded sequence of non-negative integers with $a_0 = 0$. Let the number of members of the sequence not exceeding n be b_n. Prove that for all positive integers m and n, we have

$$a_0 + a_1 + \cdots + a_m + b_0 + b_1 + \cdots + b_n \geq (m+1)(n+1).$$

Source: 1994 Balkan MO; Problem 4

Proof:

Note that for arbitrary nonnegative integers i, j, the relation $j \leq a_i$ is equivalent to the relation $i \geq b_{j-1}$. It then follows that

$$\sum_{i=0}^{m} a_i = \sum_{i=0}^{m}\sum_{j=1}^{a_i} 1 = \sum_{j=1}^{a_m}\sum_{i=b_{j-1}}^{m} 1 = \sum_{j=1}^{a_m}(m+1-b_{j-1}) = \sum_{j=0}^{a_m-1}(m+1-b_j).$$

Note that if $j \leq a_m - 1$, then there are at most m terms of $\{a_k\}_{k\geq 0}$ which do not exceed j, i.e., $b_j \leq m$; it follows that every term of the last summation is positive.
Now, if $a_m \geq n+1$, then we have

$$\begin{aligned}\sum_{i=0}^{m} a_i + \sum_{j=0}^{n} b_j &= \sum_{j=n+1}^{a_m-1}(m+1-b_j) + \sum_{j=0}^{n}(m+1-b_j+b_j)\\ &= \sum_{j=n+1}^{a_m-1}(m+1-b_j) + (n+1)(m+1) \geq (n+1)(m+1),\end{aligned}$$

as desired. On the other hand, if $a_m < n+1$, then for all $j \geq a_m$, $b_j \geq m+1$. It then follows that

$$\begin{aligned}\sum_{i=0}^{m} a_j + \sum_{j=0}^{n} b_j &= \sum_{j=0}^{a_m-1}(m+1-b_j+b_j) + \sum_{j=a_m}^{n} b_j\\ &= (a_m)(m+1) + \sum_{j=a_m}^{n} b_j\\ &\geq (a_m)(m+1) + (n+1-a_m)(m+1) = (n+1)(m+1),\end{aligned}$$

as desired. Therefore the problem statement is true in all cases.