Weekly Homework 23

Math Gecs

August 08, 2024

Exercise 1

For each positive integer n let S_n denote the set of positive integers k such that $n^k - 1$ is divisible by 2006. Define the function $P(n)$ by the rule

$$P(n) := \begin{cases} \min(s)_{s \in S_n} & \text{if } S_n \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let d be the least upper bound of $\{P(1), P(2), P(3), \ldots\}$ and let m be the number of integers i such that $1 \leq i \leq 2006$ and $P(i) = d$. Compute the value of $d + m$.

Source: 2006 iTest Problems/Problem 34

Answer. 688

Solution. *We find that the prime factorization of* 2006 *is* $2 * 17 * 59$.

Then we can compute $d = \lambda(2006)$ *(where* λ *is the Carmichael function) by Carmichael's theorem: it is* $lcm(\lambda(2), \lambda(17), \lambda(59)) = lcm(1, 16, 58) = 2^4 * 29 = 464$.

As for solving $P(i) = d$*, we must have* i *odd (otherwise it would not be coprime to* 2*), and we must also have* i *be a primitive root modulo* 17 *as well as a primitive root modulo* 59*. There are* $\phi(\phi(17)) = \phi(16) = 8$ *primitive roots modulo* 17 *(where* ϕ *is the Euler totient function) and* $\phi(\phi(59)) = \phi(58) = (2-1)*(29-1) = 28$ *primitive roots modulo* 59*. Then we have* $m = 1 * 8 * 28 = 224$ *by the Chinese Remainder Theorem, so our answer is* $d + m = 464 + 224 = 688$ *and we are done.*