Weekly Homework 21

Math Gecs

June 23, 2024

Exercise 1

What is the maximum value of n for which there is a set of distinct positive integers $k_1, k_2, ...k_n$ for which

$$k_1^2 + k_2^2 + ... + k_n^2 = 2002?$$

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

Source: 2002 AMC 12P Problem 13

Answer. $\boxed{\textbf{\textit{(D)}}\ 17}$

Solution. *Note that* $k_1^2 + k_2^2 + ... + k_n^2 = 2002 \geq \frac{n(n+1)(2n+1)}{6}$

When $n = 17$, $\frac{n(n+1)(2n+1)}{6} = \frac{(17)(18)(35)}{6} = 1785 < 2002$.

When $n = 18$, $\frac{n(n+1)(2n+1)}{6} = 1785 + 18^2 = 2109 > 2002$.

Therefore, we know $n \leq 17$.

Now we must show that $n = 17$ *works. We replace some integer* b *within the set* $\{1, 2, ...17\}$ *with an integer* $a > 17$ *to account for the amount under* 2002*, which is* $2002 - 1785 = 217$.

Essentially, this boils down to writing 217 *as a difference of squares. Assume there exist positive integers* a *and* b *where* $a > 17$ *and* $b \leq 17$ *such that* $a^2 - b^2 = 217$.

We can rewrite this as $(a + b)(a - b) = 217$. *Since* $217 = 7 \cdot 31$, *either* $a + b = 217$ *and* $a - b = 1$ *or* $a + b = 31$ *and* $a - b = 7$. *We analyze each case separately.*

Case 1: $a + b = 217$ *and* $a - b = 1$

Solving this system of equations gives $a = 109$ *and* $b = 108$. *However,* $108 > 17$, *so this case* *does not yield a solution.*

Case 2: $a+b=31$ *and* $a-b=7$

Solving this system of equations gives $a = 19$ *and* $b = 12$*. This satisfies all the requirements of the problem.*

The list $1, 2...11, 13, 14...17, 19$ *has* 17 *terms whose sum of squares equals* 2002*. Since* $n \geq 18$ *is impossible, the answer is* $\boxed{\textbf{\textit{(D)}}\ 17}$.