Weekly Homework 35

Math Gecs

November 05, 2024

Exercise 1

In the polynomial $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$, the product of 2 of its roots is -32. Find k.

Source: 1984 USAMO Problem 1

Answer. $\boxed{86}$

Solution. *Using Vieta's formulas, we have:*

$$\begin{aligned} a+b+c+d &= 18, \\ ab+ac+ad+bc+bd+cd &= k, \\ abc+abd+acd+bcd &= -200, \\ abcd &= -1984. \end{aligned}$$

From the last of these equations, we see that $cd = \frac{abcd}{ab} = \frac{-1984}{-32} = 62$. *Thus, the second equation becomes* $-32 + ac + ad + bc + bd + 62 = k$, *and so* $ac + ad + bc + bd = k - 30$. *The key insight is now to factor the left-hand side as a product of two binomials:* $(a+b)(c+d) = k - 30$, *so that we now only need to determine* $a+b$ *and* $c+d$ *rather than all four of* a, b, c, d.

Let $p = a + b$ *and* $q = c + d$. *Plugging our known values for* ab *and* cd *into the third Vieta equation,* $-200 = abc + abd + acd + bcd = ab(c+d) + cd(a+b)$, *we have* $-200 = -32(c+d) + 62(a+b) = 62p - 32q$. *Moreover, the first Vieta equation,* $a + b + c + d = 18$, *gives* $p + q = 18$. *Thus we have two linear equations in* p *and* q, *which we solve to obtain* $p = 4$ *and* $q = 14$.

Therefore, we have $(\underbrace{a+b}_{4})(\underbrace{c+d}_{14}) = k - 30$, *yielding* $k = 4 \cdot 14 + 30 = \boxed{86}$.