Weekly Homework 12

Math Gecs

April 14, 2024

Exercise 1

Let (x, y, z) be an ordered triplet of real numbers that satisfies the following system of equations:

$$x + y^2 + z^4 = 0,$$
$$y + z^2 + x^4 = 0,$$
$$z + x^2 + y^4 = 0.$$

If m is the minimum possible value of $\lfloor x^3 + y^3 + z^3 \rfloor$, find the modulo 2007 residue of m.

Source: 2007 iTest Problem 46

Answer. $\boxed{2006}$

Solution. *Rearrange the terms to get*

$$y^2 + z^4 = -x$$

$$z^2 + x^4 = -y$$

$$x^2 + y^4 = -z$$

Since the left hand side of all three equations is greater than or equal to 0, $x, y, z \leq 0$. *Also, note that the equations have symmetry, so WLOG, let* $0 \geq x \geq y \geq z$. *By substitution, we have*

$$y^2 + z^4 \leq z^2 + x^4 \leq x^2 + y^4$$

Note that $0 \leq x^2 \leq y^2 \leq z^2$ *and* $0 \leq x^4 \leq y^4 \leq z^4$. *That means* $x^2 + y^4 \leq x^2 + z^4$. *Since* $y^2 + z^4 \leq x^2 + y^4$,

$$y^2 + z^4 \leq x^2 + z^4$$
$$y^2 \leq x^2$$

Since $x^2 \leq y^2$, *then* $x^2 = y^2$. *Because* x *and* y *are nonpositive,* $x = y$.

Using substitution in the original system,

$$x^2 + z^4 = z^2 + x^4$$

$$(z^2 + x^2)(z^2 - x^2) - (z^2 - x^2) = 0$$
$$(z^2 - x^2)(x^2 + z^2 - 1) = 0$$

To find the real solutions, we use casework and the Zero Product Property.
Case 1: $z^2 = x^2$
If $z^2 = x^2$*, then since* z *and* x *are nonpositive, then* $z = x$*. Substitution results in*

$$x + x^2 + x^4 = 0$$

$$x(1 + x + x^3) = 0$$

That means $x = 0$ *or* $x^3 + x + 1 = 0$*. For the first equation,* $m = 0$*. For the second equation, note that* $x^3 = -x - 1$*, and since* $x = y = z$*,* $m = \lfloor -3x - 3 \rfloor$*, where* x *is a real number. Since* $-\frac{1}{3}^3 - \frac{1}{3} + 1 = \frac{16}{27}$ *and* $-\frac{2}{3}^3 - \frac{2}{3} + 1 = \frac{1}{27}$*, the root of* x *is less than* $-\frac{2}{3}$ *but more than* -1*, so*

$$0 > -3x - 3 > -1$$

$$m = \lfloor -3x - 3 \rfloor = -1$$

Case 2: $x^2 + z^2 = 1$
Because $x^2 \leq z^2$*,* $x^2 \leq \frac{1}{2}$*. From one of the original equations,*

$$z^2 + x^4 = -y$$

$$1 - x^2 + x^4 + x = 0$$

Using the Rational Root Theorem,

$$(x + 1)(x^3 - x^2 + 1) = 0$$

Note that if $x = -1$*, then* $x^2 \geq \frac{1}{2}$*, so that won't work. Let* $x = -a$ *(where* $a \geq 0$ *since* $x \leq 0$*), so*

$$a^3 + a^2 = 1$$

If $a \leq \frac{\sqrt{2}}{2}$*, then*

$$a^3 + a^2 \leq \frac{\sqrt{2}}{4} + \frac{1}{2}$$

$$a^3 + a^2 \leq \frac{\sqrt{2} + 2}{4} < 1$$

Thus, there are no solutions in this case.
From the two cases, the smallest possible value of m *is* -1*, so the modulo* 2007 *residue of* m *is* $\boxed{2006}$.

Exercise 2

Determine all pairs of distinct real numbers (x, y) such that both of the following are true:

$$x^{100} - y^{100} = 2^{99}(x - y)$$
$$x^{200} - y^{200} = 2^{199}(x - y)$$

Source: 2017 Indonesia MO Problem 4

Answer. $\boxed{(0, 2), (2, 0)}$.

Solution. *Let $x = 2a$ and $y = 2b$, resulting in the below system.*

$$a^{100} - b^{100} = a - b$$
$$a^{200} - b^{200} = a - b$$

Substitution and factoring of difference of squares results in

$$a^{100} - b^{100} = a^{200} - b^{200}$$
$$a^{100} - b^{100} = (a^{100} - b^{100})(a^{100} + b^{100})$$
$$0 = (a^{100} - b^{100})(a^{100} + b^{100} - 1)$$

By the [Zero Product Property, either $a^{100} = b^{100}$ or $a^{100} + b^{100} = 1$. If $a^{100} = b^{100}$ and $a \neq b$, then $a = -b$. However, a quick check reveals this to be an extraneous solution.

Thus, $a^{100} + b^{100} = 1$. If $a = 0$, then $b = 1$, and if $b = 0$, then $a = 1$. Now assume $|a|, |b| < 1$. Because $a^{100} - a = b^{100} - b$, we have $\frac{a}{b} = \frac{b^{99}-1}{a^{99}-1}$. Since $\frac{b^{99}-1}{a^{99}-1}$ is positive, a and b must have the same sign.

Note that $\frac{a^{100}-b^{100}}{a-b} = 1 = \sum_{i=0}^{99} a^i b^{99-i}$. If a, b are negative, then the sum is negative since each term is negative, so a, b must both be positive. That means $a^{99} + b^{99} < 1$, so

$$a^{99} + b^{99} < a^{100} + b^{100}$$
$$a^{99} + b^{99} < a \cdot a^{99} + b \cdot b^{99}.$$

Since $a, b < 1$, we have $a^{99} + b^{99} < a^{99} + b^{99}$. However, that can not happen, so there are no more solutions.

This means that the only solutions are $\boxed{(0, 2), (2, 0)}$.