Weekly Homework 30

Math Gecs

September 30, 2024

Exercise 1

Let α denote $\cos^{-1}(\frac{2}{3})$. The recursive sequence $a_0, a_1, a_2, \ldots$ satisfies $a_0 = 1$ and, for all positive integers n,

$$a_n = \frac{\cos(n\alpha) - (a_1 a_{n-1} + \cdots + a_{n-1} a_1)}{2a_0}.$$

Suppose that the series

$$\sum_{k=0}^{\infty} \frac{a_k}{2^k}$$

can be expressed uniquely as $\frac{p\sqrt{q}}{r}$, where p and r are coprime positive integers and q is not divisible by the square of any prime. Find the value of $p + q + r$.

Source: 2006 iTest Problem 36

Answer. $\boxed{23}$

Solution. *We write* $\sum_{k=0}^{n} a_k a_{n-k} = \cos(n\alpha)$ *by rearranging the defining equation and using* $a_0 = 1$. *Summing this with weights* $\frac{1}{2^n}$ *from zero to infinity, we get* $\sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{n}^{k=0} a_k a_{n-k} = \sum_{n=0}^{\infty} \frac{\cos(n\alpha)}{2^n}$. *We can rewrite this as* $(\sum_{n=0}^{\infty} \frac{a_n}{2^n})^2 = \sum_{n=0}^{\infty} \frac{\cos(n\alpha)}{2^n}$.

Next, we compute$\sum_{n=0}^{\infty} \frac{\cos(n\alpha)}{2^n} = \frac{1}{2}(\sum_{n=0}^{\infty} \frac{e^{in\alpha}}{2^n} + \sum_{n=0}^{\infty} \frac{e^{-in\alpha}}{2^n}) = \frac{1}{2}(\frac{1}{1-\frac{e^{i\alpha}}{2}} + \frac{1}{1-\frac{e^{-i\alpha}}{2}})$, *which simplifies to* $\frac{1-\frac{e^{i\alpha}+e^{-i\alpha}}{4}}{\frac{5}{4}-\frac{e^{i\alpha}+e^{-i\alpha}}{2}}$. *Since* $\frac{e^{i\alpha}+e^{-i\alpha}}{2} = \cos\alpha = \frac{2}{3}$, *the entire expression becomes* $\frac{1-\frac{1}{2}\cdot\frac{2}{3}}{\frac{5}{4}-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{7}{12}} = \frac{8}{7}$.

Taking square roots, we get $\sum_{n=0}^{\infty} \frac{a_n}{2^n} = \sqrt{\frac{8}{7}} = \frac{2\sqrt{14}}{7}$, *so our answer is* $2 + 14 + 7 = \boxed{23}$ *and we are done.*