Weekly Homework 17

Math Gecs

May 21, 2024

Exercise 1

Let S be the set of all polynomials of the form z^3+az^2+bz+c, where a, b, and c are integers. Find the number of polynomials in S such that each of its roots z satisfies either $|z|=20$ or $|z|=13$.

Source: 2013 AIME II Problem 12

Answer. $\boxed{540}$

Solution. *Every cubic with real coefficients has to have either three real roots or one real and two nonreal roots which are conjugates. This follows from Vieta's formulas.*

Case 1: $f(z)=(z-r)(z-\omega)(z-\omega^*)$, *where* $r\in\mathbb{R}$, ω *is nonreal, and* ω^* *is the complex conjugate of omega (note that we may assume that* $\Im(\omega)>0$*). The real root* r *must be one of* -20, 20, -13, *or* 13. *By Viète's formulas,* $a=-(r+\omega+\omega^*)$, $b=|\omega|^2+r(\omega+\omega^*)$, *and* $c=-r|\omega|^2$. *But* $\omega+\omega^*=2\Re(\omega)$ *(i.e., adding the conjugates cancels the imaginary part). Therefore, to make* a *an integer,* $2\Re(\omega)$ *must be an integer. Conversely, if* $\omega+\omega^*=2\Re(\omega)$ *is an integer, then* $a,b,$ *and* c *are clearly integers. Therefore* $2\Re(\omega)\in\mathbb{Z}$ *is equivalent to the desired property. Let* $\omega=\alpha+i\beta$.

Subcase 1.1: $|\omega|=20$. *In this case,* ω *lies on a circle of radius* 20 *in the complex plane. As* ω *is nonreal, we see that* $\beta\neq 0$. *Hence* $-20<\Re(\omega)<20$, *or rather* $-40<2\Re(\omega)<40$. *We count* 79 *integers in this interval, each of which corresponds to a unique complex number on the circle of radius* 20 *with positive imaginary part.*

Subcase 1.2: $|\omega|=13$. *In this case,* ω *lies on a circle of radius* 13 *in the complex plane. As* ω *is nonreal, we see that* $\beta\neq 0$. *Hence* $-13<\Re(\omega)<13$, *or rather* $-26<2\Re(\omega)<26$. *We count* 51 *integers in this interval, each of which corresponds to a unique complex number on the circle of radius* 13 *with positive imaginary part.*

Therefore, there are $79+51=130$ *choices for* ω. *We also have* 4 *choices for* r, *hence there are* $4\cdot 130=520$ *total polynomials in this case.*

Case 2: $f(z) = (z-r_1)(z-r_2)(z-r_3)$, *where* r_1, r_2, r_3 *are all real. In this case, there are four possible real roots, namely* $\pm 13, \pm 20$. *Let* p *be the number of times that* 13 *appears among* r_1, r_2, r_3, *and define* q, r, s *similarly for* $-13, 20$, *and* -20, *respectively. Then* $p+q+r+s = 3$ *because there are three roots. We wish to find the number of ways to choose nonnegative integers* p, q, r, s *that satisfy that equation. By balls and urns, these can be chosen in* $\binom{6}{3} = 20$ *ways.*

Therefore, there are a total of $520 + 20 = \boxed{540}$ *polynomials with the desired property.*