Weekly Homework 41

Math Gecs

December 17, 2024

Exercise 1

Real numbers x and y are chosen independently and uniformly at random from the interval $(0, 1)$. What is the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Source: 2017 AMC 12B Problem 20

Answer. $\boxed{\textbf{\textit{(D)}}\frac{1}{3}}$

Solution. *First let us take the case that* $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -1$. *In this case, both* x *and* y *lie in the interval* $[\frac{1}{2}, 1)$. *The probability of this is* $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. *Similarly, in the case that* $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -2$, x *and* y *lie in the interval* $[\frac{1}{4}, \frac{1}{2})$, *and the probability is* $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. *Recall that the probability that* A *or* B *is the case, where case* A *and case* B *are mutually exclusive, is the sum of each individual probability. Symbolically that's* $P(A \text{ or } B \text{ or } C...) = P(A) + P(B) + P(C)...$. *Thus, the probability we are looking for is the sum of the probability for each of the cases* $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -1, -2, -3....$ *It is easy to see that the probabilities for* $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = n$ *for* $-\infty < n < 0$ *are the infinite geometric series that starts at* $\frac{1}{4}$ *and with common ratio* $\frac{1}{4}$. *Using the formula for the sum of an infinite geometric series, we get that the probability is* $\frac{\frac{1}{4}}{1-\frac{1}{4}} = \boxed{\textbf{\textit{(D)}}\frac{1}{3}}$.