Weekly Homework 38

Math Gecs

November 26, 2024

Exercise 1

An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where m, n, and p are integers, and $m \le n \le p$. What is the largest possible value of p?

Source: 1998 AIME Problem 14

Solution.

$$2mnp = (m+2)(n+2)(p+2)$$

Let's solve for p:

$$(2mn)p = p(m+2)(n+2) + 2(m+2)(n+2)$$
$$[2mn - (m+2)(n+2)]p = 2(m+2)(n+2)$$
$$p = \frac{2(m+2)(n+2)}{mn-2n-2m-4} = \frac{2(m+2)(n+2)}{(m-2)(n-2)-8}$$

Clearly, we want to minimize the denominator, so we test $(m-2)(n-2)-8 = 1 \implies (m-2)(n-2) = 9$. *The possible pairs of factors of* 9 *are* $(1,9)(3,3)$. *These give* $m = 3, n = 11$ *and* $m = 5, n = 5$ *respectively. Substituting into the numerator, we see that the first pair gives* 130*, while the second pair gives* 98. *We now check that* 130 *is optimal, setting* $a = m-2$, $b = n-2$ *in order to simplify calculations. Since*

$$0 \le (a-1)(b-1) \implies a+b \le ab+1$$

We have

$$p = \frac{2(a+4)(b+4)}{ab-8} = \frac{2ab+8(a+b)+32}{ab-8} \le \frac{2ab+8(ab+1)+32}{ab-8} = 10 + \frac{120}{ab-8} \le 130$$

Where we see $(m,n) = (3,11)$ *gives us our maximum value of* $\boxed{130}$.
Note that $0 \le (a-1)(b-1)$ *assumes* $m, n \ge 3$, *but this is clear as* $\frac{2m}{m+2} = \frac{(n+2)(p+2)}{np} > 1$ *and similarly for* n.