Weekly Homework 44

Math Gecs

January 1, 2025

Exercise 1

An unfair coin has a 2/3 probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even?

(A) $25\left(\frac{2}{3}\right)^{50}$ (B) $\frac{1}{2}\left(1-\frac{1}{3^{50}}\right)$ (C) $\frac{1}{2}$ (D) $\frac{1}{2}\left(1+\frac{1}{3^{50}}\right)$ (E) $\frac{2}{3}$

Source: 1992 AHSME Problem 29

Solution. *Doing casework on the number of heads (0 heads, 2 heads, 4 heads...), we get the equation*

$$P=\left(\frac{1}{3}\right)^{50}+\binom{50}{2}\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{48}+\cdots+\left(\frac{2}{3}\right)^{50}$$

This is essentially the expansion of $\left(\frac{2}{3}+\frac{1}{3}\right)^{50}$ *but without the odd power terms. To get rid of the odd power terms in* $\left(\frac{2}{3}+\frac{1}{3}\right)^{50}$*, we add* $\left(\frac{2}{3}-\frac{1}{3}\right)^{50}$ *and then divide by* 2 *because the even power terms that were not canceled were expressed twice. Thus, we have*

$$P=\frac{1}{2}\cdot\left(\left(\frac{1}{3}+\frac{2}{3}\right)^{50}+\left(\frac{2}{3}-\frac{1}{3}\right)^{50}\right)$$

Or

$$\frac{1}{2}\left(1+\left(\frac{1}{3}\right)^{50}\right)$$

which is equivalent to answer choice $\boxed{D}$.