Weekly Homework 9

Math Gecs

March 23, 2024

Exercise 1

For what real values of x is

$$\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=A,$$

given (a) $A=\sqrt{2}$, (b) $A=1$, (c) $A=2$, where only non-negative real numbers are admitted for square roots?

Source: 1959 IMO Problem 2

Answer. $x=\frac{3}{2}$

Solution. *The square roots imply that* $x \geq \frac{1}{2}$.
Square both sides of the given equation:

$$A^2=\left(x+\sqrt{2x-1}\right)+2\sqrt{x+\sqrt{2x-1}}\sqrt{x-\sqrt{2x-1}}+\left(x-\sqrt{2x-1}\right)$$

Add the first and the last terms to get:

$$A^2=2x+2\sqrt{x+\sqrt{2x-1}}\sqrt{x-\sqrt{2x-1}}$$

Multiply the middle terms, and use $(a+b)(a-b)=a^2-b^2$ *to get:*

$$A^2=2x+2\sqrt{x^2-2x+1}$$

Since the term inside the square root is a perfect square, and by factoring 2 out, we get

$$A^2=2(x+\sqrt{(x-1)^2})$$

Use the property that $\sqrt{x^2}=|x|$ *to get*

$$A^2=2(x+|x-1|)$$

Case I: *If $x \leq 1$, then $|x-1| = 1-x$, and the equation reduces to $A^2 = 2$. This is precisely part (a) of the question, for which the valid interval is now $x \in \left[\frac{1}{2}, 1\right]$*

Case II: *If $x > 1$, then $|x-1| = x-1$ and we have*

$$x = \frac{A^2+2}{4} > 1$$

which simplifies to

$$A^2 > 2$$

This tells there that there is no solution for (b), since we must have $A^2 \geq 2$
For (c), we have $A = 2$, which means that $A^2 = 4$, so the only solution is $x = \frac{3}{2}$.

Solution. *Note that the equation can be rewritten to*

$$\sqrt{(\sqrt{2x-1}+1)^2} + \sqrt{(\sqrt{2x-1}-1)^2} = A\sqrt{2}$$

i.e., $\sqrt{2x-1} + 1 + |\sqrt{2x-1} - 1| = A\sqrt{2}$.

Case I: *when $2x-1 \geq 1$ (i.e., $x \geq 1$), the equation becomes $2\sqrt{2x-1} = \sqrt{2}A$. For (a), we have $x = 1$; for (b) we have $x = \frac{3}{4}$; for (c) we have $x = \frac{3}{2}$. Since $x \geq 1$, (b) $x = \frac{3}{4}$ is not what we want.*

Case II: *when $0 \leq 2x-1 < 1$ (i.e., $1/2 \leq x < 1$), the equation becomes $2 = \sqrt{2}A$, which only works for (a) $A = \sqrt{2}$.*

In summary, any $x \in \left[\frac{1}{2}, 1\right]$ is a solution for (a); there is no solution for (b); there is one solution for (c), which is $x = \frac{3}{2}$.

Exercise 2

Let $\{a_n\}_{n\geq 0}$ be a non-decreasing, unbounded sequence of non-negative integers with $a_0 = 0$. Let the number of members of the sequence not exceeding n be b_n. Prove that for all positive integers m and n, we have

$$a_0 + a_1 + \cdots + a_m + b_0 + b_1 + \cdots + b_n \geq (m+1)(n+1).$$

Source: 1999 BMO Problem 4

Proof. Note that for arbitrary nonnegative integers i, j, the relation $j \leq a_i$ is equivalent to the relation $i \geq b_{j-1}$. It then follows that

$$\sum_{i=0}^{m} a_i = \sum_{i=0}^{m}\sum_{j=1}^{a_i} 1 = \sum_{j=1}^{a_m}\sum_{i=b_{j-1}}^{m} 1 = \sum_{j=1}^{a_m}(m+1-b_{j-1}) = \sum_{j=0}^{a_m-1}(m+1-b_j).$$

Note that if $j \leq a_m - 1$, then there are at most m terms of $\{a_k\}_{k\geq 0}$ which do not exceed j, i.e., $b_j \leq m$; it follows that every term of the last summation is positive.
Now, if $a_m \geq n+1$, then we have

$$\begin{aligned}\sum_{i=0}^{m} a_i + \sum_{j=0}^{n} b_j &= \sum_{j=n+1}^{a_m-1}(m+1-b_j) + \sum_{j=0}^{n}(m+1-b_j+b_j)\\ &= \sum_{j=n+1}^{a_m-1}(m+1-b_j) + (n+1)(m+1) \geq (n+1)(m+1),\end{aligned}$$

as desired. On the other hand, if $a_m < n+1$, then for all $j \geq a_m$, $b_j \geq m+1$. It then follows that

$$\begin{aligned}\sum_{i=0}^{m} a_j + \sum_{j=0}^{n} b_j &= \sum_{j=0}^{a_m-1}(m+1-b_j+b_j) + \sum_{j=a_m}^{n} b_j\\ &= (a_m)(m+1) + \sum_{j=a_m}^{n} b_j\\ &\geq (a_m)(m+1) + (n+1-a_m)(m+1) = (n+1)(m+1),\end{aligned}$$

as desired. Therefore the problem statement is true in all cases.

$\square$