Weekly Homework 60

Math Gecs

May 17, 2025

Exercise 1

Let x and y be positive reals such that

$$x^3 + y^3 + (x + y)^3 + 30xy = 2000.$$

Show that $x + y = 10$.

Source: 2000 JBMO Problem 1

Proof. Rearranging the equation yields

$$x^3 + y^3 + (x + y)^3 + 30xy - 2000 = 0.$$

If $x + y = 10$ in the large equation, then $x + y - 10$ must be a [[factor]] of the large equation. Note that we can rewrite the large equation as

$$\begin{aligned} 0 &= (x + y)^3 - 1000 + x^3 + 3x^2y - 3x^2y + 3xy^2 - 3xy^2 + y^3 - 1000 + 30xy \\ &= 2[(x + y)^3 - 1000] - 3x^2y - 3xy^2 + 30xy. \end{aligned}$$

We can factor the difference of cubes in the first part and factor $3xy$ in the second part, resulting in

$$0 = 2(x + y - 10)((x + y)^2 + 10x + 10y + 100) - 3xy(x + y - 10)$$

Finally, we can factor by grouping, which results in

$$\begin{aligned} 0 &= (x + y - 10)(2(x + y)^2 + 20x + 20y + 200 - 3xy) \\ &= (x + y - 10)(2x^2 + xy + 2y^2 + 20x + 20y + 200). \end{aligned}$$

By the Zero Product Property, either $x + y = 10$ or $2x^2 + xy + 2y^2 + 20x + 20y + 200 = 0$. However, since x and y are both positive, $2x^2 + xy + 2y^2 + 20x + 20y + 200$ can not equal zero, so we have proved that $x + y = 10$.

$\square$

Exercise 2

Let $a, b, c \geq 0$ and satisfy

$$a^2 + b^2 + c^2 + abc = 4.$$

Show that

$$0 \leq ab + bc + ca - abc \leq 2.$$

Source: 2001 USAMO Problem 3

Proof. First we prove the lower bound.
Note that we cannot have a, b, c all greater than 1. Therefore, suppose $a \leq 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \geq 0.$$

Note that, by the Pigeonhole Principle, at least two of a, b, c are either both greater than or less than 1. Without loss of generality, let them be b and c. Therefore, $(b - 1)(c - 1) \geq 0$. From the given equation, we can express a in terms of b and c as

$$a = \frac{\sqrt{(4 - b^2)(4 - c^2)} - bc}{2}$$

Thus,

$$ab + bc + ca - abc = -a(b - 1)(c - 1) + a + bc \leq a + bc = \frac{\sqrt{(4 - b^2)(4 - c^2)} + bc}{2}$$

From the Cauchy-Schwarz Inequality,

$$\frac{\sqrt{(4 - b^2)(4 - c^2)} + bc}{2} \leq \frac{\sqrt{(4 - b^2 + b^2)(4 - c^2 + c^2)}}{2} = 2.$$

This completes the proof. $\square$