Weekly Homework 24

Math Gecs

August 15, 2024

Exercise 1

Find all pairs of positive integers (x, y) such that

$$x^y = y^{x-y}.$$

Source: 1998 JBMO Problem 3

Solution. *Note that x^y is at least one. Then y^{x-y} is at least one, so $x \geq y$.*

Write $x = a^{b+c}, y = a^c$, where $\gcd(b, c) = 1$. (We know that b is nonnegative because $x \geq y$.) Then our equation becomes $a^{(b+c)*a^c} = a^{c*(a^{b+c}-a^c)}$. Taking logarithms base a and dividing through by a^c, we obtain $b + c = c * (a^b - 1)$.

Since c divides the RHS of this equation, it must divide the LHS. Since $\gcd(b, c) = 1$ by assumption, we must have $c = 1$, so that the equation reduces to $b + 1 = a^b - 1$, or $b + 2 = a^b$. This equation has only the solutions $b = 1, a = 3$ and $b = 2, a = 2$.

Therefore, our only solutions are $x = 3^{1+1} = 9, y = 3^1 = 3$, and $x = 2^{2+1} = 8, y = 2^1 = 2$, and we are done.