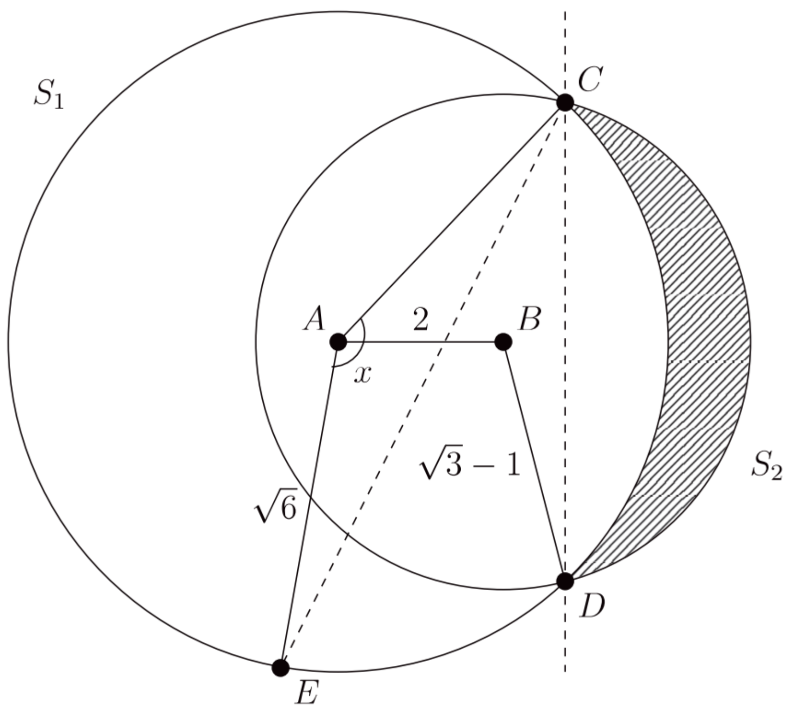
Weekly Homework 7

Math Gecs

March 8, 2024

Exercise 1

Consider two circles S_1 and S_2 centered at A and B and with radii $\sqrt{6}$ and $\sqrt{3}-1$, respectively. Suppose that the two circles intersect at two distinct points C and D. Suppose further that the two centers A and B are of distinct 2 apart. The sketch below is not to scale.



i Find the angle $\measuredangle CBA$, and deduce that A and B lie on the same side of the line CD.

ii Show that CD has length $3-\sqrt{3}$ and hence calculate the angle $\measuredangle CAD$

iii Show that the area of the region lying inside the circle S_2 and outside of the circle S_1 (that is the shaded region in the picture) is equal to.

iv Suppose that a line through C is drawn such that the total area covered by S_1 and S_2 is split into two equal areas. Let E be the intersection of this line with S_1 and x denote the angle $\measuredangle CAE$. You may assume that E lies on the larger arc CD of S_1. Write down an equation which x satisfies and explain why there is a unique solution x.

Source: Oxford MAT 2018 (Problem 4)

Solution. *(Type your solutions here.)*

Exercise 2

How many integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and 999 ? For example, both 121 and 211 have this property.
(A) 226 **(B)** 243 **(C)** 270 **(D)** 469 **(E)** 486

Source: 2017 AMC 10A Problem 25

Answer. *A*

Solution. *There are* 81 *multiples of* 11 *between* 100 *and* 999 *inclusive. Some have digits repeated twice, making* 3 *permutations.*

Others that have no repeated digits have 6 *permutations, but switching the hundreds and units digits also yield a multiple of* 11*. Switching shows we have overcounted by a factor of* 2*, so assign* $6 \div 2 = 3$ *permutations to each multiple.*

There are now $81 * 3 = 243$ *permutations, but we have overcounted*. Some multiples of 11 have* 0 *as a digit. Since* 0 *cannot be the digit of the hundreds place, we must subtract a permutation for each.*

There are $110, 220, 330 ... 990$*, yielding* 9 *extra permutations*

Also, there are $209, 308, 407...902$*, yielding* 8 *more permutations.*

Now, just subtract these 17 *from the total* (243) *to get* 226*.* $\boxed{\textbf{\textit{(A)}}\ 226}$

**If short on time, observe that* 226 *is the only answer choice less than* 243*, and therefore is the only feasible answer.*

Solution. *We note that we only have to consider multiples of* 11 *and see how many valid permutations each has. We can do casework on the number of repeating digits that the multiple of* 11 *has:*

Case 1: *All three digits are the same. By inspection, we find that there are no multiples of* 11 *here.*

Case 2: *Two of the digits are the same, and the third is different.*

Case 2a: *There are* 8 *multiples of* 11 *without a zero that have this property:* 121, 242, 363, 484, 616, 737, 858, 979*. Each contributes* 3 *valid permutations, so there are* $8 \cdot 3 = 24$ *permutations in this subcase.*

Case 2b: *There are* 9 *multiples of* 11 *with a zero that have this property:* 110, 220, 330, 440, 550, 660, 770, 880, 990*. Each one contributes* 2 *valid permutations (the first digit can't be zero), so there are* $9 \cdot 2 = 18$ *permutations in this subcase.*

Case 3: *All the digits are different. Since there are* $\frac{990-110}{11} + 1 = 81$ *multiples of* 11 *between* 100 *and* 999*, there are* $81 - 8 - 9 = 64$ *multiples of* 11 *remaining in this case. However,* 8 *of them contain a zero, namely* 209*,* 308*,* 407*,* 506*,* 605*,* 704*,* 803*, and* 902*. Each of those multiples of* 11 *contributes* $2 \cdot 2 = 4$ *valid permutations, but we overcounted by a factor of* 2*; every permutation of* 209*, for example, is also a permutation of* 902*. Therefore, there are* $8 \cdot 4/2 = 16$*. Therefore, there are* $64 - 8 = 56$ *remaining multiples of* 11 *without a* 0 *in this case. Each one contributes* $3! = 6$ *valid permutations, but once again, we overcounted by a factor of* 2 *(note that if a number* ABC *is a multiple of* 11*, then so is* CBA*). Therefore, there are* $56 \cdot 6/2 = 168$ *valid permutations in this subcase.*

Adding up all the permutations from all the cases, we have $24 + 18 + 16 + 168 = \boxed{\textbf{\textit{(A)}}\ 226}$.

Solution. *We can first overcount and then subtract. We know that there are* 81 *multiples of* 11*.*

We can then multiply by 6 *for each permutation of these multiples. (Yet some multiples do not have six distinct permutations.)*

Now divide by 2*, because if a number* abc *with digits* a*,* b*, and* c *is a multiple of* 11*, then* cba *is also a multiple of* 11 *so we have counted the same permutations twice.*

Basically, each multiple of 11 *has its own* 3 *permutations (say abc has abc acb and bac whereas cba has cba cab and bca). We know that each multiple of* 11 *has at least* 3 *permutations because it cannot have* 3 *repeating digits.*

Hence we have 243 *permutations without subtracting for overcounting. Now note that we overcounted cases in which we have* 0*'s at the start of each number. So, in theory, we could just answer* A *and then move on.*

If we want to solve it, then we continue.

We overcounted cases where the middle digit of the number is 0 *and the last digit is* 0*.*

Note that we assigned each multiple of 11 *three permutations.*

The last digit is 0 *gives* 9 *possibilities where we overcounted by* 1 *permutation for each of* $110, 220, ..., 990$.

The middle digit is 0 *gives* 8 *possibilities where we overcount by* 1. $605, 704, 803, 902$ *and* $506, 407, 308, 209$

Subtracting 17 *gives* $\boxed{\textbf{\textit{(A)}}\ 226}$.

Now, we may ask if there is further overlap (i.e if two of abc and bac and acb were multiples of 11*). Thankfully, using divisibility rules, this can never happen, as taking the divisibility rule mod* 11 *and adding, we get that* $2a$*,* $2b$*, or* $2c$ *is congruent to* $0 \pmod{11}$*. Since* a, b, c *are digits, this can never happen as none of them can equal* 11 *and they can't equal* 0 *as they are the leading digit of a three-digit number in each of the cases.*