Weekly Homework 29

Math Gecs

September 22, 2024

Exercise 1

Compute the *number* of ordered quadruples (w, x, y, z) of complex numbers (not necessarily nonreal) such that the following system is satisfied:

$$\begin{aligned} wxyz &= 1 \\ wxy^2 + wx^2z + w^2yz + xyz^2 &= 2 \\ wx^2y + w^2y^2 + w^2xz + xy^2z + x^2z^2 + ywz^2 &= -3 \\ w^2xy + x^2yz + wy^2z + wxz^2 &= -1 \end{aligned}$$

Source: 2006 iTest Problem 35

Solution. *as we are given* $xyzw = 1$ *, so from this we get second equation as* $\frac{y}{z}+\frac{x}{y}+\frac{w}{x}+\frac{z}{w} = 2$. *so say* $a = \frac{y}{z}, b = \frac{x}{y}, \frac{w}{x} = c, \frac{z}{w} = d$. *so we get* $a + b + c + d = 2$. *from fourth equation we get* $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = -1$. *so we get* $abc + abd + acd + bcd = -1$. *also from third equation we get* $ab + bc + cd + ad + w^2y^2 + x^2z^2 = -3$. *notice we want* ac *and* bd. *so* $ac = \frac{1}{x^2z^2}$. *so this gives* $ab+bc+cd+ad+ac+bd = -3$. *and* $abcd = 1$. *so we get a equation* $\alpha^4-2\alpha^3-3\alpha^2+\alpha+1 = 0$ *whose roots are* a, b, c, d. *so we get* $(\alpha + 1)(\alpha^3 - 3\alpha^2 + 1) = 0$. *this gives* $\alpha = -1$. *and three distinct complex (not necessarily non real) solutions. so as* $\alpha = -1$. *we get any one pair say* $\frac{x}{y} = -1$. *so* $x = -y = k$ *for some* $k \in \mathbb{C}$. *so as* z, w, *will be distinct we will get* 4 *quadruples from* $-k, k, w, z$ *solution so we can have such* $4 \cdot 4 = \boxed{16}$ *quadruples.*