

Weekly Homework 20

Math Gecs

June 16, 2024

Exercise 1

There are exactly K positive integers b with $5 \leq b \leq 2024$ such that the base- b integer 2024_b is divisible by 16 (where 16 is in base ten). What is the sum of the digits of K ?

Source: 2024 AMC 10A Problem 18

Answer. 20

Solution. $2b^3 + 2b + 4 \equiv 0 \pmod{16} \implies b^3 + b + 2 \equiv 0 \pmod{8}$, if b even then $b + 2 \equiv 0 \pmod{8} \implies b \equiv 6 \pmod{8}$. If b odd then $b^2 \equiv 1 \pmod{8} \implies b^3 + b + 2 \equiv 2b + 2 \pmod{8}$ so $2b + 2 \equiv 0 \pmod{8} \implies b + 1 \equiv 0 \pmod{4} \implies b \equiv 3, 7 \pmod{8}$. Now $8 \mid 2024$ so $\frac{3}{8} \cdot 2024 = 759$ but 3 is too small so $759 - 1 = 758 \implies \boxed{(D) 20}$.