Weekly Homework 14

Math Gecs

April 30, 2024

happy bday to me. holy shit im 18 rn

Exercise 1

Find the largest real number C such that for any positive integer n and any real numbers $x_1, x_2, \ldots, x_n$, the following inequality holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(n-|j-i|)x_ix_j \geq C\sum_{i=1}^{n}x_i^2$$

Source: 2023 CMO(CHINA) Problem 2

Solution. *Define the $n \times n$ matrix A_n as follows:*

$$A_n = \begin{pmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ 1 & n & n-1 & \cdots & 3 & 2 \\ 2 & 1 & n & \cdots & 4 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-2 & n-3 & n-4 & \cdots & n & n-1 \\ n-1 & n-2 & n-3 & \cdots & 1 & n \end{pmatrix}$$

The problem simplifies to finding the smallest real number C such that for all $n \in \mathbb{N}^*$ and any vector $x \in \mathbb{R}^n$, the following inequality holds:

$$x^TA_nx \geq Cx^Tx$$

In other words, find the real number C such that:

$$C \leq \inf_{n\in\mathbb{N}^*}\inf_{\substack{x\in\mathbb{R}^n \\ x\neq 0}}\frac{x^TA_nx}{x^Tx}$$

Given that A_n is not easily invertible directly, but is invertible (as it is a sparse matrix):

$$A_n^{-1} = \begin{pmatrix} \frac{n+2}{2n+2} & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2n+2} \\ -\frac{1}{2} & 1 & \frac{1}{2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2n+2} & 0 & \cdots & 0 & -\frac{1}{2} & \frac{n+2}{2n+2} \end{pmatrix}$$

Since the inverse has non-zero entries:

$$\inf_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A_n x}{x^T x} = \left(\sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A_n^{-1} x}{x^T x} \right)^{-1}$$

For the eigenvalues μ of A_n :

$$\frac{1}{2} + \frac{1}{2n+2} \leq \mu - 1 \leq \frac{1}{2} + \frac{1}{2n+2}$$

Thus:

$$0 \leq \mu \leq 2$$

Given that A_n^{-1} is invertible, $\mu \neq 0$, therefore:

$$0 < \mu \leq 2$$

$$\sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A_n^{-1} x}{x^T x} \leq 2$$

For a specific $y \in \mathbb{R}^n, y = [1, -1, 1, -1, \ldots]^T \in \mathbb{R}^n$, we have:

$$y^T y = n$$

$$y^T A_n^{-1} y = \begin{cases} 2(n-2) + \frac{2(n+2)}{n+1} & \textit{if } n \textit{ is even} \\ 2(n-1) & \textit{if } n \textit{ is odd} \end{cases}$$

$$\sup_n \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A_n x}{x^T x} \geq \sup_n \frac{y^T A_n y}{y^T y} \geq 2$$

In conclusion:

$$\inf_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A_n x}{x^T x} = \inf_n \left(\sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A_n^{-1} x}{x^T x} \right)^{-1} = \left(\sup_{\substack{n \\ n \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A_n^{-1} x}{x^T x} \right)^{-1} = \frac{1}{2}$$

Therefore, the maximum value of the real number C is $\frac{1}{2}$.

Exercise 2

Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

Source: 2014 USAMO Problem 2

Answer. $\boxed{f(x) = 0}$ *and* $\boxed{f(x) = x^2}$

Solution. *Lemma 1: $f(0) = 0$. Proof: Assume the opposite for a contradiction. Plug in $x = 2f(0)$ (because we assumed that $f(0) \neq 0$), $y = 0$. What you get eventually reduces to:*

$$4f(0) - 2 = \left(\frac{f(2f(0))}{f(0)}\right)^2$$

which is a contradiction since the LHS is divisible by 2 but not 4.
Then plug in $y = 0$ into the original equation and simplify by Lemma 1. We get:

$$x^2 f(-x) = f(x)^2$$

Then:

$$\begin{aligned} x^6 f(x) &= x^4 \big(x^2 f(x)\big) \\ &= x^4 \big((-x)^2 f(-(-x))\big) \\ &= x^4 (-x)^2 f(-(-x)) \\ &= x^4 f(-x)^2 \\ &= f(x)^4 \end{aligned}$$

Therefore, $f(x)$ must be 0 or x^2.
Now either $f(x)$ is x^2 for all x or there exists $a \neq 0$ such that $f(a) = 0$. The first case gives a valid solution. In the second case, we let $y = a$ in the original equation and simplify to get:

$$xf(-x) + a^2 f(2x) = \frac{f(x)^2}{x}$$

But we know that $xf(-x) = \frac{f(x)^2}{x}$, so:

$$a^2 f(2x) = 0$$

Since a is not 0, $f(2x)$ is 0 for all x (including 0). Now either $f(x)$ is 0 for all x, or there exists some $m \neq 0$ such that $f(m) = m^2$. Then m must be odd. We can let $x = 2k$ in the original equation, and since $f(2x)$ is 0 for all x, stuff cancels and we get:

$$y^2 f(4k - f(y)) = f(yf(y))$$

¡b¿for $\mathbf{k \neq 0}$.*¡/b¿ Now, let* $y = m$ *and we get:*

$$m^2 f(4k - m^2) = f(m^3)$$

Now, either both sides are 0 or both are equal to m^6. *If both are* m^6 *then:*

$$m^2 (4k - m^2)^2 = m^6$$

which simplifies to:

$$4k - m^2 = \pm m^2$$

Since $k \neq 0$ *and* m *is odd, both cases are impossible, so we must have:*

$$m^2 f(4k - m^2) = f(m^3) = 0$$

Then we can let k *be anything except 0, and get* $f(x)$ *is 0 for all* $x \equiv 3 \pmod 4$ *except* $-m^2$. *Also since* $x^2 f(-x) = f(x)^2$, *we have* $f(x) = 0 \Rightarrow f(-x) = 0$, *so* $f(x)$ *is 0 for all* $x \equiv 1 \pmod 4$ *except* m^2. *So* $f(x)$ *is 0 for all* x *except* $\pm m^2$. *Since* $f(m) \neq 0$, $m = \pm m^2$. *Squaring,* $m^2 = m^4$ *and dividing by* m, $m = m^3$. *Since* $f(m^3) = 0$, $f(m) = 0$, *which is a contradiction for* $m \neq 1$. *However, if we plug in* $x = 1$ *with* $f(1) = 1$ *and* y *as an arbitrary large number with* $f(y) = 0$ *into the original equation, we get* $0 = 1$ *which is a clear contradiction, so our only solutions are* $f(x) = 0$ *and* $f(x) = x^2$.

Solution. *Given that the range of* f *consists entirely of integers, it is clear that the LHS must be an integer and* $f(yf(y))$ *must also be an integer, therefore* $\frac{f(x)^2}{x}$ *is an integer. If* x *divides* $f(x)^2$ *for all integers* $x \neq 0$, *then* x *must be a factor of* $f(x)$, *therefore* $f(0) = 0$. *Now, by setting* $y = 0$ *in the original equation, this simplifies to* $xf(-x) = \frac{f(x)^2}{x}$. *Assuming* $x \neq 0$, *we have* $x^2 f(-x) = f(x)^2$. *Substituting in* $-x$ *for* x *gives us* $x^2 f(x) = f(-x)^2$. *Substituting in* $\frac{f(x)^2}{x^2}$ *in for* $f(-x)$ *in the second equation gives us* $x^2 f(x) = \frac{f(x)^4}{x^4}$, *so* $x^6 f(x) = f(x)^4$. *In particular, if* $f(x) \neq 0$, *then we have* $f(x)^3 = x^6$, *therefore* $f(x)$ *is equivalent to* 0 *or* x^2 *for every integer* x. *Now, we shall prove that if for some integer* $t \neq 0$, *if* $f(t) = 0$, *then* $f(x) = 0$ *for all integers* x. *If we assume* $f(y) = 0$ *and* $y \neq 0$ *in the original equation, this simplifies to* $xf(-x) + y^2 f(2x) = \frac{f(x)^2}{x}$. *However, since* $x^2 f(-x) = f(x)^2$, *we can rewrite this equation as* $\frac{f(x)^2}{x} + y^2 f(2x) = \frac{f(x)^2}{x}$, $y^2 f(2x)$ *must therefore be equivalent to* 0. *Since, by our initial assumption,* $y \neq 0$, *this means that* $f(2x) = 0$, *so, if for some integer* $y \neq 0$, $f(y) = 0$, *then* $f(x) = 0$ *for all integers* x. *The contrapositive must also be true, i.e. If* $f(x) \neq 0$ *for all integers* x, *then there is no integral value of* $y \neq 0$ *such that* $f(y) = 0$, *therefore* $f(x)$ *must be equivalent for* x^2 *for every integer* x, *including* 0, *since* $f(0) = 0$. *Thus,* $f(x) = 0, x^2$ *are the only possible solutions.*

Solution. *Let's assume $f(0) \neq 0$. Substitute $(x, y) = (2f(0), 0)$ to get*

$$2f(0)^2 = f(2f(0))^2/2f(0) + f(0)$$

$$2f(0)^2(2f(0) - 1) = f(2f(0))^2$$

This means that $2(2f(0) - 1)$ is a perfect square. However, this is impossible, as it is equivalent to $2 \pmod 4$. Therefore, $f(0) = 0$. Now substitute $x \neq 0, y = 0$ to get

$$xf(-x) = \frac{f(x)^2}{x} \implies x^2 f(-x) = f(x)^2.$$

Similarly,

$$x^2 f(x) = f(-x)^2.$$

From these two equations, we can find either $f(x) = f(-x) = 0$, or $f(x) = f(-x) = x^2$. Both of these are valid solutions on their own, so let's see if there are any solutions combining the two.
Let's say we can find $f(x) = x^2, f(y) = 0$, and $x, y \neq 0$. Then

$$xf(-x) + y^2 f(2x) = f(x)^2/x.$$

$$y^2 f(2x) = x - x^3.$$

(NEEDS FIXING: $f(x)^2/x = x^4/x = x^3$, so the RHS is 0 instead of $x - x^3$.)
If $f(2x) = 4x^2$, then $y^2 = \frac{x-x^3}{4x^2} = \frac{1-x^2}{4x}$, which is only possible when $y = 0$. This contradicts our assumption. Therefore, $f(2x) = 0$. This forces $x = \pm 1$ due to the right side of the equation. Let's consider the possibility $f(2) = 0, f(1) = 1$. Substituting $(x, y) = (2, 1)$ into the original equation yields

$$0 = 2f(0) + 1f(2) = 0 + f(1) = 1,$$

which is impossible. So $f(2) = f(-2) = 4$ and there are no solutions "combining" $f(x) = x^2$ and $f(x) = 0$.
Therefore our only solutions are $\boxed{f(x) = 0}$ and $\boxed{f(x) = x^2.}$

Solution. *Let the given assertion be $P(x, y)$. We try $P(x, 0)$ and get $xf(2c-x) = f(x)^2/x + c$, where $f(0) = c$. We plug in $x = c$ and get $cf(c) = f(c)^2/c + c$. Rearranging and solving for c^2 gives us $c^2 = \frac{f(c)^2}{f(c)-1}$. Obviously, the only c that works such that the RHS is an integer is $c = 0$, and thus $f(0) = 0$.*

We use this information on assertion $P(x, 0)$ and obtain $xf(-x) = f(x^2)/x$, or $f(-x) = \frac{f(x)^2}{x^2}$. Thus, $f(x)$ is an even function. It follows that $f(x) = 0, x^2$ for each x. We now prove that $f(x) = x^2$, f(x)=0aretheonlysolutions.