Weekly Homework 22

Math Gecs

June 30, 2024

Exercise 1

Determine the remainder obtained when the expression

$$2004^{2003^{2002^{2001}}}$$

is divided by 1000.

Source: Mock AIME 2 Pre 2005 Problems/Problem 8

Solution. *We note that* $2004^{2003^{2002^{2001}}} \equiv 4^{2003^{2002^{2001}}} \pmod{1000}$*. The remainder of the RHS modulo* 8 *is trivially zero, but the remainder of the RHS modulo* 125 *depends on the remainder of the exponent modulo* $\phi(125) = 50$*, so we defer the calculation until later.*

We compute $2003^{2002^{2001}}$ *modulo* 50*; again noting that this is equivalent to* $3^{2002^{2001}}$ *modulo* 50*. The remainder is trivially one modulo two, but the remainder modulo* 25 *depends on the remainder of the second exponent modulo* $\phi(25) = 20$*.*

Now we start to unroll the recursion: We have $2002^{2001} \equiv 2^{2001} \pmod{20}$*. Modulo four, the remainder is trivially zero; modulo five, the remainder is* $2^{2001 \pmod 5} \equiv 2^1 \equiv 2 \pmod 5$*, so we have* $2002^{2001} \equiv 12 \pmod{20}$*.*

Then $2003^{2002^{2001}} \equiv 3^{12} \equiv 16 \pmod{25}$*, so that* $2003^{2002^{2001}} \equiv 41 \pmod{50}$ *by CRT.*

Then $2004^{2003^{2002^{2001}}} \equiv 4^{41} \equiv 78 \pmod{125}$*, so that* $2004^{2003^{2002^{2001}}} \equiv 704 \pmod{1000}$ *by CRT, and we are done.*