Weekly Homework 5

Math Gecs

February 18, 2024

Exercise 1

Let $f(x)$ be a a quotient of two quadratic polynomials. Given that $f(n) = n^3$ for all $n \in 1, 2, 3, 4, 5$, compute $f(0)$.

Source: Harvard-MIT Mathematics Tournament February 17, 2024

Answer. $\boxed{\dfrac{27}{14}}$

Solution. *Let $f(x) = \frac{p(x)}{q(x)}$. Then, $x^3q(x) - p(x)$ has $1, 2, 3, 4, 5$ as roots. Therefore, WLOG, let*

$$x^3q(x) - p(x) = (x-1)(x-2)(x-3)(x-4)(x-5) = x^5 - 15x^4 + 85x^3 - \cdots$$

Thus, $q(x) = x^2 - 15x + 85$, so $q(0) = 85$. Plugging $x = 0$ in the above equation also gives $-p(0) = -120$. Hence, the answer is $\frac{120}{85} = \frac{24}{17}$.

Remark: *From the solution above, it is not hard to see that the unique f that satisfies the problem is*

$$f(x) = \frac{225x^2 - 274x + 120}{x^2 - 15x + 85}.$$

Exercise 2

The country of HMMTLand has 8 cities. Its government decides to construct several two-way roads between pairs of distinct cities. After they finish construction, it turns out that each city can reach exactly 3 other cities via a single road, and from any pair of distinct cities, either exactly 0 or 2 other cities can be reached from both cities by a single road. Compute the number of ways HMMTLand could have constructed the roads.

Source: Harvard-MIT Mathematics Tournament February 17, 2024

Answer. $\boxed{875}$

Solution. *Let the cities be numbered 1, 2, 3, 4, 5, 6, 7, 8. WLOG, 1 is connected to 2, 3, and 4. First suppose 2 and 3 are connected; then 3 and 1 share a second common neighbor, which must be 4 (as 1 is not connected to anything else). Likewise 2 and 4 are connected, and so 5, 6, 7, 8 are pairwise connected as well, so the graph consists of two disjoint copies of K_4:*
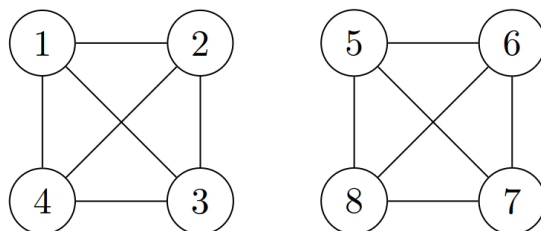
There are $\frac{1}{2}\binom{8}{4} = 35$ ways to partition the 8 vertices into two groups of 4, so there are 35 such graphs. Otherwise, none of 2, 3, 4 are connected to each other. Then 2 and 3 must share a common neighbor, as must 3 and 4, and 2 and 4. If these are the same neighbor, this vertex would share all three neighbors with 1, so they must be pairwise distinct. The last vertex must then be connected to these three, creating a cube graph.
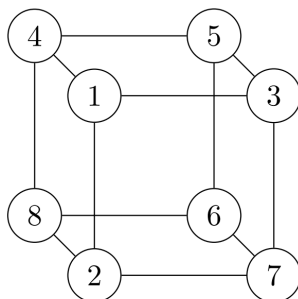
A cube has 48 symmetries, so the number of such graphs is $\frac{8!}{48} = 840$.
The total is $35 + 840 = \boxed{875}$.