Weekly Homework 3

Math Gecs

January 14, 2024

Exercise 1

For $t = 1, 2, 3, 4$, define $S_t = \sum_{i=1}^{350} a_i^t$, where $a_i \in \{1, 2, 3, 4\}$. If $S_1 = 513$ and $S_4 = 4745$, find the minimum possible value for S_2.

Source: 2009 AIME I Problem 14

Answer 1. 905

Solution 1. *Because the order of the a's doesn't matter, we simply need to find the number of* $1s$ $2s$ $3s$ *and* $4s$ *that minimize* S_2. *So let* $w, x, y,$ *and* z *represent the number of* $1s$, $2s$, $3s$, *and* $4s$ *respectively. Then we can write three equations based on these variables. Since there are a total of* 350 as, *we know that* $w+x+y+z = 350$. *We also know that* $w+2x+3y+4z = 513$ *and* $w + 16x + 81y + 256z = 4745$. *We can now solve these down to two variables:*

$$w = 350 - x - y - z$$

Substituting this into the second and third equations, we get

$$x + 2y + 3z = 163$$

and

$$15x + 80y + 255z = 4395.$$

The second of these can be reduced to

$$3x + 16y + 51z = 879.$$

Now we substitute x *from the first new equation into the other new equation.*

$$x = 163 - 2y - 3z$$

$$3(163 - 2y - 3z) + 16y + 51z = 879$$

$$489 + 10y + 42z = 879$$

$$5y + 21z = 195$$

Since y *and* z *are integers, the two solutions to this are* $(y, z) = (39, 0)$ *or* $(18, 5)$. *If you plug both these solutions in to* S_2 *it is apparent that the second one returns a smaller value. It turns out that* $w = 215$, $x = 112$, $y = 18$, *and* $z = 5$, *so* $S_2 = 215 + 4 * 112 + 9 * 18 + 16 * 5 = 215 + 448 + 162 + 80 = \boxed{905}$.

Exercise 2

Let (a, b, c) be a Pythagorean triple, "i.e.", a triplet of positive integers with $a^2 + b^2 = c^2$.

- Prove that $(c/a + c/b)^2 > 8$.
- Prove that there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.

Source: 2005 Canadian MO Problem 2

Proof:

- We have

$$\left(\tfrac{c}{a} + \tfrac{c}{b}\right)^2 = \tfrac{c^2}{a^2} + 2\tfrac{c^2}{ab} + \tfrac{c^2}{b^2} = \tfrac{a^2+b^2}{a^2} + 2\tfrac{a^2+b^2}{ab} + \tfrac{a^2+b^2}{b^2} = 2 + \left(\tfrac{a^2}{b^2} + \tfrac{b^2}{a^2}\right) + 2\left(\tfrac{a}{b} + \tfrac{b}{a}\right)$$

By AM-GM, we have

$$x + \frac{1}{x} > 2,$$

where x is a positive real number not equal to one. If $a = b$, then $c \notin \mathbb{Z}$. Thus $a \neq b$ and $\frac{a}{b} \neq 1 \implies \frac{a^2}{b^2} \neq 1$. Therefore,

$$\left(\frac{c}{a} + \frac{c}{b}\right)^2 > 2 + 2 + 2(2) = 8.$$

- Now since a, b, and c are positive integers, $c/a + c/b$ is a rational number p/q, where p and q are positive integers. Now if $p^2/q^2 = n$, where n is an integer, then p/q must also be an integer. Thus $c(a + b)/ab$ must be an integer.

Now every pythagorean triple can be written in the form $(2mn, m^2 - n^2, m^2 + n^2)$, with m and n positive integers. Thus one of a or b must be even. If a and b are both even, then c is even too. Factors of 4 can be cancelled from the numerator and the denominator(since every time one of a, b, c, and $a+b$ increase by a factor of 2, they all increase by a factor of 2) repeatedly until one of a, b, or c is odd, and we can continue from there. Thus the $m^2 - n^2$ term is odd, and thus c is odd. Now c and $a + b$ are odd, and ab is even. Thus $c(a + b)/ab$ is not an integer. Now we have reached a contradiction, and thus there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.