Weekly Homework 34

Math Gecs

October 28, 2024

Exercise 1

Let $ABCD$ be a rhombus with $\angle A = 60^\circ$, and P is the intersection of diagonals AC and BD. Let Q, R, and S are three points on the rhombus' perimeter. If $PQRS$ is also a rhombus, show that exactly one of Q, R, and S is located on the vertices of rhombus $ABCD$.

Source: 2002 Indonesia MO Problem 7

Solution. *Firstly, all rhombi are parallelograms, so that P is the centroid of $ABCD$.*

Suppose that Q, R, S are all on one side of the rhombus. Then, in order for $PQRS$ to be a parallelogram, P should also be on that side. But this is not so, so this case is impossible.

Suppose that Q, R, S are on two sides of the rhombus; then one side is occupied by two of these points (the "majority side") and one side is occupied by only one of these points (the "minority side"). If R is on the minority side, then $PQRS$ is necessarily self-intersecting and thusly not a parallelogram. Thusly, either Q or S is on the minority side; WLOG it is Q. Then $\vec{SR}$ is parallel to the majority side, so $\vec{PQ}$ must also be parallel to the majority side, so that Q is the midpoint of the minority side. Then $\vec{PQ} = \vec{SR}$ must be exactly half the length of the majority side.

From here, we consider cases. Based on the symmetry of $ABCD$, however, we only need consider two: that where the majority side is AB and the minority side BC, and that where the majority side is AB and the minority side DA. In the first case, we find that in order to satisfy $PQ = QR$ and A, R, B collinear, we must have $R = B$ or R be outside of the segment AB, which is forbidden, so that exactly one vertex of $PQRS$ (R) is also a vertex of $ABCD$. In the second case, we find that in order to satisfy $PQ = QR$ and A, R, B collinear, we must have $R = A$ or R be the midpoint of AB, so that exactly one vertex of $PQRS$ (R or S, respectively) is also a vertex of $ABCD$.

Finally, suppose that Q, R, S are on three different sides. WLOG, suppose that $R \in AB$. If one of the other vertices is on CD (WLOG it is Q), then S must be outside the parallelogram (since $h_S = h_P - h_Q + h_R = \frac{1}{2} - 1 + 0 = -\frac{1}{2}$, where h_X is the (signed) height of X to AB, *scaled by the height of C). This is impossible, so we know that Q and S must not be on CD; WLOG, we have $Q \in DA, S \in BC$. Then the midpoint of QS is on the line halfway between lines DA and BC. Since the midpoint of QS and that of PR are the same, R is the midpoint of AB. Then, in order to satisfy $PQ = QR$ and $PS = SR$, we must have Q the midpoint of DA and $S = B$, so that exactly one vertex of $PQRS$ (that is, S) is also a vertex of $ABCD$.*

All cases having been considered, we have shown that if $PQRS$ is a rhombus, then exactly one of Q, R, and S is a vertex of $ABCD$, and we are done.