Weekly Homework 16

Math Gecs

May 14, 2024

Exercise 1

How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

Source: 2019 AMC 10B Problem 25

Answer. $\boxed{65}$

Solution. *Let $f(n)$ be the number of valid sequences of length n (satisfying the conditions given in the problem).*
We know our valid sequence must end in a 0. Then, since we cannot have two consecutive 0s, it must end in a 10. Now, we only have two cases: it ends with 010, or it ends with 110 which is equivalent to 0110. Thus, our sequence must be of the forms $0\ldots010$ or $0\ldots0110$. In the first case, the first $n-2$ digits are equivalent to a valid sequence of length $n-2$. In the second, the first $n-3$ digits are equivalent to a valid sequence of length $n-3$. Therefore, it must be the case that $f(n) = f(n-3) + f(n-2)$, with $n \geq 3$ (because otherwise, the sequence would contain only 0s and this is not allowed due to the given conditions).

It is easy to find $f(3) = 1$ since the only possible valid sequence is 010. $f(4) = 1$ since the only possible valid sequence is 0110. $f(5) = 1$ since the only possible valid sequence is 01010.

The recursive sequence is then as follows:

$$f(3) = 1$$
$$f(4) = 1$$
$$f(5) = 1$$
$$f(6) = 1 + 1 = 2$$
$$f(7) = 1 + 1 = 2$$
$$f(8) = 1 + 2 = 3$$

$$f(9) = 2 + 2 = 4$$
$$f(10) = 2 + 3 = 5$$
$$f(11) = 3 + 4 = 7$$
$$f(12) = 4 + 5 = 9$$
$$f(13) = 5 + 7 = 12$$
$$f(14) = 7 + 9 = 16$$
$$f(15) = 9 + 12 = 21$$
$$f(16) = 12 + 16 = 28$$
$$f(17) = 16 + 21 = 37$$
$$f(18) = 21 + 28 = 49$$
$$f(19) = 28 + 37 = 65$$

So, our answer is $\boxed{65}$.