Weekly Homework 6

Math Gecs

February 26, 2024

Exercise 1

Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Source: 2013 USAJMO Problem 6

The key Lemma is:

$$\sqrt{a-1} + \sqrt{b-1} \leq \sqrt{ab}$$

for all $a, b \geq 1$. Equality holds when $(a-1)(b-1) = 1$.
This is proven easily.

$$\sqrt{a-1} + \sqrt{b-1} = \sqrt{a-1}\sqrt{1} + \sqrt{1}\sqrt{b-1} \leq \sqrt{(a-1+1)(b-1+1)} = \sqrt{ab}$$

by Cauchy.
Equality then holds when $a - 1 = \frac{1}{b-1} \implies (a-1)(b-1) = 1$.
Now assume that $x = \min(x, y, z)$. Now note that, by the Lemma,

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x-1} + \sqrt{yz} \leq \sqrt{x(yz+1)} = \sqrt{xyz+x}$$

So equality must hold in order for the condition in the problem statement to be met. So $(y-1)(z-1) = 1$ and $(x-1)(yz) = 1$. If we let $z = c$, then we can easily compute that $y = \frac{c}{c-1}, x = \frac{c^2+c-1}{c^2}$. Now it remains to check that $x \leq y, z$.
But by easy computations, $x = \frac{c^2+c-1}{c^2} \leq c = z \iff (c^2-1)(c-1) \geq 0$, which is obvious.
Also $x = \frac{c^2+c-1}{c^2} \leq \frac{c}{c-1} = y \iff 2c \geq 1$, which is obvious, since $c \geq 1$.

So all solutions are of the form $\boxed{\left(\frac{c^2+c-1}{c^2}, \frac{c}{c-1}, c\right)}$, and all permutations for $c > 1$.

Remark: An alternative proof of the key Lemma is the following: By AM-GM,

$$(ab - a - b + 1) + 1 = (a-1)(b-1) + 1 \geq 2\sqrt{(a-1)(b-1)}$$

$$ab \geq (a-1)+(b-1)+2\sqrt{(a-1)(b-1)}$$

Now taking the square root of both sides gives the desired. Equality holds when $(a-1)(b-1)=1$.

WLOG, assume that $x = \min(x,y,z)$. Let $a=\sqrt{x-1}$, $b=\sqrt{y-1}$ and $c=\sqrt{z-1}$. Then $x=a^2+1$, $y=b^2+1$ and $z=c^2+1$. The equation becomes

$$(a^2+1)+(a^2+1)(b^2+1)(c^2+1)=(a+b+c)^2.$$

Rearranging the terms, we have

$$(1+a^2)(bc-1)^2+[a(b+c)-1]^2=0.$$

Therefore $bc=1$ and $a(b+c)=1$. Express a and b in terms of c, we have $a=\frac{c}{c^2+1}$ and $b=\frac{1}{c}$. Easy to check that a is the smallest among a, b and c. Then $x=\frac{c^4+3c^2+1}{(c^2+1)^2}$, $y=\frac{c^2+1}{c^2}$ and $z=c^2+1$. Let $c^2=t$, we have the solutions for (x,y,z) as follows: $(\frac{t^2+3t+1}{(t+1)^2},\frac{t+1}{t},t+1)$ and permutations for all $t>0$.

Exercise 2

Suppose $P(x)$ is a polynomial with real coefficients, satisfying the condition $P(\cos\theta+\sin\theta) = P(\cos\theta - \sin\theta)$, for every real θ. Prove that $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(1-x^2)^2 + a_2(1-x^2)^4 + \cdots + a_n(1-x^2)^{2n}$$

for some real numbers $a_0, a_1, \ldots, a_n$ and non-negative integer n.

Source: 2020 INMO Problem 2

Solution. *Assume to the contrary. Suppose P satisfies $P(\cos\theta + \sin\theta) = P(\cos\theta - \sin\theta)$ for all real θ, and is of minimal degree and not of the prescribed form.*

Claim: *For some $c \in \mathbb{R}$, we have $(1-x^2)^2 \mid P(x) - c$.*

Proof. *Note that $\theta = \frac{\pi}{2} \implies P(1) = P(-1)$. Set $c = P(1)$. Then $(1-x^2) \mid P(x) - c$ as the latter vanishes at both ± 1. Now let $P(x) - c = (1-x^2)Q(x)$ for some $Q \in \mathbb{R}[x]$.*

Then $Q(\cos\theta + \sin\theta) = -Q(\cos\theta - \sin\theta)$ holds for all θ, by plugging $P(x) = (1-x^2)Q(x)$ in the original equation, since we have the identities $1 - (\cos\theta + \sin\theta)^2 = -\sin 2\theta$ and $1 - (\cos\theta - \sin\theta)^2 = \sin 2\theta$.

(Subtlety for beginners: while the equation in Q only holds for θ away from roots of $\sin 2\theta = 0$, since these form a discrete subset of $\mathbb{R}$, the equation extends to these as Q is continuous.)

In particular, plugging $\theta = 0, \pi$ we get $Q(1) = -Q(1)$ and $Q(-1) = -Q(-1)$ so $Q(\pm 1) = 0$, hence $(1-x^2) \mid Q(x)$. Thus, $(1-x^2)^2 \mid P(x) - c$ as desired.

Finally, we see that $P(x) = c + (1-x^2)^2 h(x)$ and $deg h < deg P$ so h has the prescribed form. But then P also has the prescribed form, and our result follows.