

Weekly Homework 39

Math Gecks

December 04, 2024

Exercise 1

Let $\frac{x^2+y^2}{x^2-y^2} + \frac{x^2-y^2}{x^2+y^2} = k$. Compute the following expression in terms of k :

$$E(x, y) = \frac{x^8 + y^8}{x^8 - y^8} - \frac{x^8 - y^8}{x^8 + y^8}.$$

Source: 1997 JBMO Problem 2

Solution. *To start, we add the two fractions and simplify.*

$$\begin{aligned} k &= \frac{(x^2 + y^2)^2 + (x^2 - y^2)^2}{x^4 - y^4} \\ &= \frac{2x^4 + 2y^4}{x^4 - y^4}. \end{aligned}$$

Dividing both sides by two yields

$$\frac{k}{2} = \frac{x^4 + y^4}{x^4 - y^4}.$$

That means

$$\begin{aligned} \frac{x^4 + y^4}{x^4 - y^4} + \frac{x^4 - y^4}{x^4 + y^4} &= \frac{k}{2} + \frac{2}{k} \\ \frac{(x^4 + y^4)^2 + (x^4 - y^4)^2}{x^8 - y^8} &= \frac{k^2 + 4}{2k} \\ \frac{2x^8 + 2y^8}{x^8 - y^8} &= \frac{k^2 + 4}{2k}. \end{aligned}$$

Dividing both sides by two yields

$$\frac{x^8 + y^8}{x^8 - y^8} = \frac{k^2 + 4}{4k}.$$

That means

$$\begin{aligned}\frac{x^8 + y^8}{x^8 - y^8} - \frac{x^8 - y^8}{x^8 + y^8} &= \frac{k^2 + 4}{4k} - \frac{4k}{k^2 + 4} \\ &= \frac{k^4 + 8k^2 + 16 - 16k^2}{4k(k^2 + 4)} \\ &= \frac{k^4 - 8k^2 + 16}{4k(k^2 + 4)} \\ &= \boxed{\frac{(k^2 - 4)^2}{4k(k^2 + 4)}}.\end{aligned}$$