Weekly Homework 42

Math Gecs

December 24, 2024

Exercise 1

Let b be a real number randomly selected from the interval $[-17, 17]$. Then, m and n are two relatively prime positive integers such that m/n is the probability that the equation $x^4+25b^2 = (4b^2-10b)x^2$ has *at least* two distinct real solutions. Find the value of $m+n$.

Source: 2007 iTest Problem 36

Solution. *The equation has quadratic form, so complete the square to solve for x.*

$$x^4 - (4b^2 - 10b)x^2 + 25b^2 = 0$$
$$x^4 - (4b^2 - 10b)x^2 + (2b^2 - 5b)^2 - 4b^4 + 20b^3 = 0$$
$$(x^2 - (2b^2 - 5b))^2 = 4b^4 - 20b^3$$

In order for the equation to have real solutions,

$$16b^4 - 80b^3 \geq 0$$
$$b^3(b-5) \geq 0$$
$$b \leq 0 \text{ or } b \geq 5$$

Note that $2b^2 - 5b = b(2b-5)$ is greater than or equal to 0 when $b \leq 0$ or $b \geq 5$. Also, if $b = 0$, then expression leads to $x^4 = 0$ and only has one unique solution, so discard $b = 0$ as a solution. The rest of the values leads to b^2 equalling some positive value, so these values will lead to two distinct real solutions.
Therefore, in interval notation, $b \in [-17, 0) \cup [5, 17]$, so the probability that the equation has at least two distinct real solutions when b is randomly picked from interval $[-17, 17]$ is $\frac{29}{34}$.
This means that $m + n = \boxed{63}$.