Weekly Homework 1

Math Gecs

December 19, 2023

Exercise 1

Define the double factorial via $(2n-1)!! = (2n-1)(2n-3)\cdots 1$. Compute the unique pair (a, c) with $c > 0$ and $a \in (0, \infty)$ such that

$$\lim_{n\to\infty} \frac{c^n(4n-1)!!}{(2n-1)!!(2n-1)!!} = a.$$

Source: Stanford Mathematics Competition (2023 - Problem 10)

Answer 1. $\left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)$

Solution 1. *We claim that* $(a, c) = \left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)$ *is the answer.*
First, rewrite

$$P_n = \frac{(4n-1)!!}{(2n-1)!!(2n-1)!!} = \prod_{k=1}^{n} \frac{(4k-1)(4k-3)}{(2k-1)^2}$$

and as $(4k-1)(4k-3) \le 4(2k-1)^2 = (2k-1)(4k-2)$, *it follows that* $P_n \le 4^n$, *so if* $c < \frac{1}{4}$, *the value of this limit would be 0.*
From the other end, we claim that $P_n \ge \left(\frac{1}{2} + \frac{1}{4n}\right) 4^n$, *implying that indeed* $c = \frac{1}{4}$.
To do so, we proceed by induction. Note that $P_1 = 3$ *which satisfies the hypothesis. Now, note that*

$$P_{n+1} = P_n \cdot \frac{(4n+3)(4n+1)}{(2n+1)^2} = 4P_n \cdot \left(1 - \frac{1}{(4n+2)^2}\right) \ge 4P_n \cdot \left(\frac{1}{2} + \frac{1}{4n}\right) \cdot \left(1 - \frac{1}{(4n+2)^2}\right) 4^{n+1}$$

and

$$\begin{aligned}\left(\frac{1}{2} + \frac{1}{4n}\right)\left(1 - \frac{1}{(4n+2)^2}\right) &= \frac{1}{2} + \frac{1}{4n} - \frac{1}{2(4n+2)^2} - \frac{1}{4n(4n+2)^2} \\ &= \frac{1}{2} + \frac{1}{4n} - \frac{3}{16n} + \frac{1}{16n+8} \\ &\ge \frac{1}{2} + \frac{16n+16}{4(4n+1)}\end{aligned}$$

so our induction is complete.
Finally, we show that $Q_n = P_n 4^{-n} \to \frac{\sqrt{2}}{2}$. *To do so, consider writing*

$$Q(z) = \prod_{n=0}^{\infty} \left(1 - \frac{z^2}{\pi^2 (n+\frac{1}{2})^2}\right) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{\pi(n+\frac{1}{2})}\right)\left(1 + \frac{z}{\pi(n+\frac{1}{2})}\right)$$

and note that our desired answer is $Q\left(\frac{\pi}{4}\right)$.
Our (surprising) claim is that in fact $Q(z) = \cos(z)$: *writing* $\cos(z)$ *as a Taylor series gives that it is a polynomial with first coefficient 1, and the zeros of* $Q(z)$ *are exactly those of* $\cos(z)$ *(with the same multiplicities, as* $\cos(z)$ *and* $\cos(z)' = \sin(z)$ *share no zeros). To show formal convergence, we appeal to the Weierstrass Factorization Theorem, which guarantees such a representation (maybe insert a more formal convergence statement).*
Now, we have $Q\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ *and we are done.*

Solution 2. *Note that* $(4n-1)!! = \frac{(4n)!}{(4n)!} = \frac{(4n)!}{(2n)!2^{2n}}$ *and similarly* $(2n-1)!! = \frac{(2n)!}{n!2^n}$. *So, we can rewrite*

$$\frac{(4n-1)!!}{(2n-1)!!(2n-1)!!} = \frac{(4n)!}{(2n)!(2n)!}$$

Define $f(n) \sim g(n)$ *if* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. *Then, we claim that*

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}$$

Indeed, by Stirling's Approximation,

$$\binom{2n}{n} \sim \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n)^2} = \frac{2^{2n}}{\sqrt{\pi n}}$$

Hence,

$$\frac{\binom{4n}{2n}}{\binom{2n}{n}} \sim \frac{\frac{2^{4n}}{\sqrt{2\pi n}}}{\frac{2^{2n}}{\sqrt{\pi n}}} = \frac{4^n}{\sqrt{2}}$$

This immediately implies $(a, c) = \left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)$.

Exercise 2

Find all positive integers n such that $3_{n-1} + 5_{n-1}$ divides $3_n + 5_n$.

Source: St.Petersburg 1996

Answer 2. 1

Solution 3. *This only occurs for* $n = 1$*. Let* $s_n = 3^n + 5^n$ *and note that*

$$s_n = (3 + 5)s_{n-1} - 3 \cdot 5 \cdot s_{n-2}$$

So s_{n-1} *must also divide* $3 \cdot 5 \cdot s_{n-2}$.

If $n > 1$*, then* s_{n-1} *is coprime to 3 and 5, then* s_{n-1} *must divide* s_{n-2}*, which is impossible since* $s_{n-1} > s_{n-2}$.